

# Fourier–Mukai transforms commuting with Frobenius over algebraic spaces

Chandranandan Gangopadhyay and Ankit Rai

**Abstract.** In this article, we give a characterization of Fourier–Mukai transforms on algebraic spaces that commute with the Frobenius thereby extending the work of Daniel Bragg [Bull. Lond. Math. Soc. 56 (2024), 3477–3483]. We also treat the case of twisted sheaves on algebraic spaces.

## 1. Introduction

Let  $X$  be a quasi-compact, and quasi-separated algebraic space defined over an algebraically closed field  $k$  of characteristic  $p$ . Let  $D(X)$ ,  $D_{\text{qcoh}}(X)$ , and  $D_{\text{perf}}(X)$  denote, respectively, the unbounded derived categories of complexes of  $\mathcal{O}_X$ -modules, quasi-coherent  $\mathcal{O}_X$  modules, and perfect  $\mathcal{O}_X$ -modules on the étale site. For any morphism  $f: X \rightarrow Y$  among quasi-compact and quasi-separated algebraic spaces, there exists well-defined pushforward and pullback functors

$$f^*: D(Y) \rightarrow D(X), \quad f_*: D(X) \rightarrow D(Y).$$

Since any morphism among quasi-compact and quasi-separated algebraic spaces is itself quasi-compact and quasi-separated [23, Tag 03KS, Tag 03KR] and  $Y$  is a quasi-separated algebraic space, [23, Tag 08FA, Tag 08F4] implies that  $f^*$  and  $f_*$  restrict to the unbounded derived category of quasi-coherent sheaves to define a pair of adjoint functors

$$f^*: D_{\text{qcoh}}(Y) \rightarrow D_{\text{qcoh}}(X), \quad f_*: D_{\text{qcoh}}(X) \rightarrow D_{\text{qcoh}}(Y).$$

In other words,  $f$  is a concentrated morphism in the sense of [11, Definition 2.4]. In this article, we only consider functors on quasi-coherent sheaves or its subcategory of perfect complexes. When  $p > 0$ , the absolute Frobenius map given by the  $p$ -power map on affine  $k$ -algebras, gives rise to the absolute Frobenius morphism  $F_X: X \rightarrow X$ . The definition of the Frobenius morphism is discussed in more generality in the appendix (Section A) for convenience. The discussion of the paragraph above applies to the Frobenius morphisms  $F_X$  and  $F_Y$ .

A functor  $\Phi: \mathbf{D}_{\text{qcoh}}(X) \rightarrow \mathbf{D}_{\text{qcoh}}(Y)$  is said to be a Fourier–Mukai transform if there exists  $K \in \mathbf{D}_{\text{qcoh}}(X \times_k Y)$  such that  $\Phi(F) = \text{pr}_{Y*}(K \otimes \text{pr}_X^*(F))$ . We may also write  $\Phi_K$  for the Fourier–Mukai transform given by  $K$ . It is known from [22, 25] that not all functors  $\Phi: \mathbf{D}_{\text{qcoh}}(X) \rightarrow \mathbf{D}_{\text{qcoh}}(Y)$  are Fourier–Mukai functors. But something very close is true as demonstrated by Toën [24] and more generally by Ben-Zvi–Francis–Nadler [2]. In another direction, under the assumption that  $\Phi$  is an equivalence of categories and  $X, Y$  are smooth projective varieties, it is a result of Orlov (see [6, Remark 2.1]) that  $\Phi$  is a Fourier–Mukai transform [20, Theorem 2.2].

When  $X$  is a smooth algebraic space there exists an equivalence of categories known as a Serre functor

$$S_X: \mathbf{D}_{\text{qcoh}}(X) \rightarrow \mathbf{D}_{\text{qcoh}}(X); \quad F \mapsto F \otimes \omega_X[\dim X].$$

It is known from [4, Lemma 4.2] that any equivalence  $\Phi$  necessarily commutes with the Serre functor. A result of Calabrese [4, Theorems 6.1 and 7.1] implies that under the assumption that  $\omega_X$  is either ample or anti-ample,  $X \simeq Y$  and  $\Phi$  is given by the composition of the following three functors: pullback along an isomorphism  $f: X \rightarrow Y$ , shifts  $[n]$ , and twist  $- \otimes L$  by a line bundle  $L$  on  $X$ . In characteristic  $p > 0$ , the Frobenius pushforwards provide another interesting endo-functor of the category  $\mathbf{D}_{\text{qcoh}}(X)$ . Bragg investigated the constraints put on  $\Phi$  if it satisfies the equation

$$\Phi \circ F_{X*} \cong F_{Y*} \circ \Phi, \tag{1.1}$$

and proves the following.

**Theorem 1.1** ([3, Theorem 1.1]). *Let  $X$  and  $Y$  be smooth projective connected varieties defined over a field  $k$  of characteristic  $p > 0$ . Let  $\Phi: \mathbf{D}_{\text{qcoh}}(X) \rightarrow \mathbf{D}_{\text{qcoh}}(Y)$  be an equivalence of categories which satisfies (1.1). Then there exists an isomorphism  $f: X \xrightarrow{\sim} Y$ , a line bundle  $L$  on  $Y$  and an integer  $n$  such that*

$$\Phi \simeq f_* \circ (- \otimes L)[n].$$

In this article we prove the following generalization of the above theorem to the context of smooth algebraic spaces.

**Theorem 1.2.** *Let  $X$  and  $Y$  be smooth proper connected algebraic spaces over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $\Phi$  be a Fourier–Mukai equivalence which satisfies (1.1). Then there exists an isomorphism  $f: X \xrightarrow{\sim} Y$ , a line bundle  $L$  on  $X$  and an integer  $n$  such that*

$$\Phi \simeq f_* \circ (- \otimes L)[n].$$

We also prove an appropriate twisted analogue of the above result. This result is new even in the case of schemes. The setup is as follows. The algebraic Brauer group  $\text{Br}(X)$  is the set of Morita equivalence classes of Azumaya  $\mathcal{O}_X$ -algebras of finite rank equipped with the monoid structure given by tensor product. Let  $\alpha \in \text{Br}(X)$  be an element in the

algebraic Brauer group of the algebraic space  $X$ . Consider the  $\mu_n$ -gerbe  $\pi_\alpha: \mathcal{X}_\alpha \rightarrow X$  associated to the element  $\alpha$  [17, Proposition 2.2.1.4]. Any quasi-coherent sheaf on  $\mathcal{X}_\alpha$  comes equipped with a right (inertial) action of  $\mu_n$ . A sheaf  $F$  on  $\mathcal{X}_\alpha$  is called  $\alpha$ -twisted if the action of  $\mu_n$  on  $F$  coincides with the natural action of  $\underline{\mathbb{G}}_m$  on  $F$  coming from the  $\mathcal{O}_{\mathcal{X}_\alpha}$ -module structure.  $\mathrm{QCoh}(X, \alpha)$  denotes the abelian category of  $\alpha$ -twisted quasi-coherent sheaves on  $X$  and  $\mathrm{D}_{\mathrm{qcoh}}(X, \alpha)$  denotes the unbounded derived category of sheaves of  $\mathcal{O}_{\mathcal{X}_\alpha}$ -modules whose cohomology sheaves lie in  $\mathrm{QCoh}(X, \alpha)$ . Let  $\beta \in \mathrm{Br}(Y)$  and  $\pi_\beta: \mathcal{Y}_\beta \rightarrow Y$  be the gerbe corresponding to the class  $\beta$ . For a morphism  $f: X \rightarrow Y$ , the gerbe  $\mathcal{Y}_\beta \times_Y X$  represents the class  $f^*\beta$  and we denote it by  $\mathcal{X}_{f^*\beta}$ . As per the discussion on concentrated morphisms, the following functors are well defined

$$\begin{aligned} f^*: \mathrm{D}_{\mathrm{qcoh}}(Y, \beta) &\rightarrow \mathrm{D}_{\mathrm{qcoh}}(X, f^*\beta), & f_*: \mathrm{D}_{\mathrm{qcoh}}(X, f^*\beta) &\rightarrow \mathrm{D}_{\mathrm{qcoh}}(Y, \beta), \\ F_{\mathcal{X}_\alpha*}: \mathrm{D}_{\mathrm{qcoh}}(X, p\alpha) &\rightarrow \mathrm{D}_{\mathrm{qcoh}}(X, \alpha). \end{aligned}$$

Note that  $\mathcal{X}_\alpha$  and  $\mathcal{Y}_\beta$  are tame stacks and hence [10, Theorem C] implies that  $\mathcal{X}_\alpha$  and  $\mathcal{Y}_\beta$  are concentrated. Any morphism among concentrated stacks is a concentrated morphism [11, Lemma 2.5 (iv)]. Assume that  $\alpha$  and  $\beta$  satisfy  $n\alpha = n\beta = 0$  for some  $n$  that is coprime to  $p$ . By the choice of  $n$  there exists  $r > 0$  such that  $p^r \equiv 1 \pmod{n}$ , hence  $(F_{X*}^*)^r \alpha = \alpha$ . Consider the  $r$ -th iterate of the Frobenius pushforward

$$(F_{X*}^*)^r: \mathrm{D}_{\mathrm{qcoh}}(X, \alpha) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(X, \alpha).$$

For any  $\beta \in \mathrm{Br}(Y)$ , and any  $K \in \mathrm{D}_{\mathrm{qcoh}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ , we get a Fourier–Mukai transform

$$\Phi_K: \mathrm{D}_{\mathrm{qcoh}}(X, \alpha) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(Y, \beta).$$

In the situation described above, there is a natural generalization of Theorem 1.2 as stated below.

**Theorem 1.3.** *Let  $X$  and  $Y$  be smooth proper connected algebraic space over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $\alpha \in \mathrm{Br}(X)$  and  $\beta \in \mathrm{Br}(Y)$  such that  $n\alpha = 0 = n\beta$  for a positive integer  $n$  coprime to  $p$ . Let  $r$  be a natural number such that  $p^r \equiv 1 \pmod{n}$ . Let  $\Phi$  be a Fourier–Mukai equivalence which satisfies*

$$\Phi \circ (F_{X*}^*)^r = (F_{Y*}^*)^r \circ \Phi.$$

*Then there exists an isomorphism  $f: X \rightarrow Y$ , a line bundle  $L$  on  $X$  and an integer  $n$  such that  $f^*\beta = \alpha$  and*

$$\Phi \simeq f_* \circ (- \otimes L)[n].$$

We briefly describe the organization of the article and our strategy for the proofs of the main theorems. In Section 2.1 we recall the general formalism of derived pushforward and pullback on the (unbounded) derived categories of quasi-coherent sheaves on concentrated stacks due to Hall and Rydh. Their formalism is especially helpful when treating the case of twisted sheaves and their derived categories. In Section 2.2 we give a brief discussion

on twisted sheaves following [17] and record facts about the derived category of twisted sheaves to be used in the subsequent sections. In Section 3 we prove Theorem 1.2 and in Section 4 we prove Theorem 1.3. The article also contains two appendices. Appendix A describes the construction of the absolute Frobenius morphism for stacks, and Appendix B discusses vanishing of higher derived functors of pushforward along proper morphisms whose fibers have bounded cohomological dimension.

Our proof depends on the same principle that underlies both the proof of the theorem of Bondal–Orlov and the theorem of Bragg. Let  $\Psi_X$  (resp.  $\Psi_Y$ ) be an endofunctor of the derived category  $D_{\text{qcoh}}(X)$  (resp.  $D_{\text{qcoh}}(Y)$ ) which moreover has the property:  $\Psi_X(E) \cong E[n]$  (resp.  $\Psi_Y(F) \cong F[m]$ ) for some  $E \in D_{\text{coh}}^b(X)$  (resp.  $F \in D_{\text{coh}}^b(X)$ ) and  $n$  (resp.  $m$ ) if and only if  $E$  (resp.  $F$ ) is a necessarily simple skyscraper sheaf placed in certain degree. If  $\Phi_K$  is a Fourier–Mukai equivalence which moreover satisfies  $\Psi_Y \circ \Phi_K = \Phi_K \circ \Psi_X$ , then  $K$  is a coherent sheaf supported on the graph of an isomorphism  $f: X \rightarrow Y$  and  $\Phi_K$  is naturally equivalent to the graph of the isomorphism  $f$  after possibly tensoring with a line bundle on  $X$  and shifting by an integer  $r$ . For the proof of the Bondal–Orlov theorem the functor  $\Psi_X$  and  $\Psi_Y$  are respective Serre functors, and for the proof of [3, Theorem 1.1] they are Frobenius pushforwards. One of the key points in *loc. cit.* is to show that objects in the derived category which are stable under Frobenius pushforward have zero-dimensional support. Bragg’s proof extends using standard techniques to show that the same result holds for Frobenius pushforwards in the context of algebraic spaces (see Lemma 3.2).

A quick look at the proof of the Bondal–Orlov theorem will convince the reader that the later half of the proof heavily depends on the arguments involving points on schemes. This approach seems to break down in the context of algebraic spaces, nonetheless the (relative!) analogue of Bondal–Orlov theorem for algebraic spaces continues to be true as shown by Calabrese [4, Theorems 6.2 and 7.1] via moduli theoretic techniques. The obvious approach of extending Calabrese’s proof to prove Theorem 1.2 seems to run into the problem of finding the correct set of ‘point-like objects’. We circumvent this problem by devising a criterion to detect if a given Fourier–Mukai transform (possibly after composing with shift functor  $[n]$  for some integer  $n$ ) induces an equivalence of the  $k$ -linear abelian category of quasi-coherent sheaves (see Proposition 3.4). We verify using this criterion that a Fourier–Mukai equivalence commuting with Frobenius pushforwards (up to shifts) gives an equivalence among the abelian categories of quasi-coherent sheaves of  $X$  and  $Y$ . Now, an application of the main theorems of [5] finishes the proof. The proof of the aforementioned criterion depends on a generalization of the theorem of formal functions to proper quasi-finite morphisms among tame stacks (Proposition B.1). A more general statement which may be of some future use and whose proof we were unable to find in the literature is also proven (see Proposition B.2).

## 2. Prerequisites

In this section, we collect all the necessary facts about algebraic spaces and the unbounded derived category of quasi-coherent sheaves on them that will be needed in the sequel.

## 2.1. Generalities on concentrated stacks

Throughout the article, sheaves on algebraic stacks are defined as in [14, 18] via the lisse-étale site. Let  $D(\mathcal{O}_{\mathcal{X}})$  denote the unbounded derived category of sheaves of  $\mathcal{O}_{\mathcal{X}}$ -modules. Any morphism  $f: \mathcal{X} \rightarrow \mathcal{Y}$  among quasi-compact and quasi-separated stacks is itself quasi-compact and quasi-separated [23, Tag 075S]. All the algebraic stacks that appear in this article are quasi-compact, quasi-separated Deligne–Mumford stacks. Recall that for such stacks the change of topology map  $\mathcal{X}_{\text{lis-ét}} \rightarrow \mathcal{X}_{\text{ét}}$  induces an equivalence of categories

$$D_{\text{qcoh}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow D_{\text{qcoh}}(\mathcal{X}_{\text{ét}}).$$

As the small étale site topos is functorial, any morphism  $f: \mathcal{X} \rightarrow \mathcal{Y}$  induces a pair of adjoint functors

$$(f_{\text{ét}}^*, f_{\text{ét}*}): D(\mathcal{X}_{\text{ét}}) \rightarrow D(\mathcal{Y}_{\text{ét}}).$$

But  $f_*$  does not necessarily restrict to the quasi-coherent derived category of sheaves. Hall–Rydh introduce the notion of concentrated stacks and concentrated morphisms to correct this defect [11, Definition 2.4].

A quasi-compact quasi-separated algebraic stack  $\mathcal{X}$  is called concentrated if the structure sheaf  $\mathcal{O}_{\mathcal{X}}$  is a compact object of  $D_{\text{qcoh}}(\mathcal{X})$ . According to [11, Lemma 2.5 (iv)] morphisms among concentrated stacks are themselves concentrated, in particular  $f_*$  (hence so does  $f_{\text{ét}}$ ) restricts to a functor  $f_*: D_{\text{qcoh}}(\mathcal{X}) \rightarrow D_{\text{qcoh}}(\mathcal{Y})$  (see [11, Theorem 2.6 (ii)]). In the setup of concentrated morphism among concentrated stacks, flat base change and projection formula have been proved by Hall–Rydh for unbounded derived category of quasi-coherent sheaves [11, Theorem 2.6 (iv), Corollary 4.12]. We will apply the discussion above to the following two scenarios

- (1)  $\mathcal{X}$  is a quasi-compact and quasi-separated algebraic space. In this case we follow the standard convention in the literature and denote algebraic spaces by standard letters  $X, Y$ , etc. It follows from [23, Tag 073G] that  $X$  is a concentrated stack.
- (2)  $\mathcal{X}$  is a  $\mu_n$ -gerbe over an algebraic space  $X$ . In particular, it is tame. It follows from [10, Theorem C] that  $\mathcal{X}$  is a concentrated stack. When  $n$  is an integer that is coprime to  $p$  [19, Remark 8.3.4] implies that  $\mathcal{X}$  is also a Deligne–Mumford stack.

Since we are only concerned with scenarios (1) and (2), we will assume and freely use the fact that the stacks that appear in Section 3 and Section 4 are quasi-compact quasi-separated concentrated Deligne–Mumford stacks. The main theorem of this article concerns smooth proper algebraic spaces over  $k$ . It follows that  $X$  is finitely presented and separated. This observation is implicitly used to apply Lemma 3.2, Lemma 3.3, and Proposition 3.4 and its twisted counterparts in Section 4 to the proofs of the main theorems of this article.

## 2.2. Derived category of twisted sheaves on algebraic spaces

We recall some facts from [17]. Let  $X$  be a quasi-compact and quasi-separated algebraic space defined over  $k$ . Let  $\text{Br}(X)$  denote the group formed by the Morita equivalence

classes of Azumaya  $\mathcal{O}_X$ -algebras  $A$  of finite rank, and  $\mathrm{Br}'(X)$  denote  $H_{\text{ét}}^2(X, \mathbb{G}_m)_{\text{tors}}$  the subgroup of torsion elements in  $H_{\text{ét}}^2(X, \mathbb{G}_m)$ . The canonical map

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$$

is injective. The group  $\mathrm{Br}(X)$  is called the algebraic Brauer group, and  $\mathrm{Br}'(X)$  is called the cohomological Brauer group.

Let  $\alpha \in \mathrm{Br}(X)$  be an element of order  $n$  which is coprime to  $p$ , and let  $p_\alpha: \mathcal{X}_\alpha \rightarrow X$  denote the associated  $\mu_n$ -gerbe. Let  $\mathcal{J}(\mathcal{X}_\alpha)$  denote the inertia stack; it follows that  $\mathcal{J}(\mathcal{X}_\alpha) \simeq \underline{\mu}_{n, \mathcal{X}_\alpha}$ . For any quasi-coherent sheaf  $\mathcal{F}$  on  $\mathcal{X}_\alpha$  we have the following map

$$\mathcal{F} \times \underline{\mu}_{n, \mathcal{X}_\alpha} \rightarrow \mathcal{F}.$$

We say that  $\mathcal{F}$  is an  $\alpha$ -twisted sheaf on  $X$ , if the above action coincides with the right action corresponding to the left  $\mathcal{O}_{\mathcal{X}_\alpha}$ -module structure on  $\mathcal{F}$ . Let  $\mathrm{QCoh}(X, \alpha)$  denote the abelian category of  $\alpha$ -twisted quasi-coherent sheaves on  $X$ , and define

$$\mathrm{D}_{\mathrm{qcoh}}(X, \alpha) = \{\mathcal{K} \in \mathrm{D}(X) \mid H^i(\mathcal{K}) \in \mathrm{QCoh}(X, \alpha) \text{ for all } i \in \mathbb{Z}\}.$$

Under the assumption that  $X$  is separated, the gerbe  $\mathcal{X}_\alpha$  has affine diagonal. It follows from [9, Theorem 1.2] that the canonical map  $\mathrm{D}(\mathrm{QCoh}(\mathcal{X}_\alpha)) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}_\alpha)$  is an equivalence of categories. This fact via an application of [15, Lemma 3.2.5] allows to upgrade the decomposition of a quasi-coherent sheaf on  $\mathcal{X}_\alpha$  as direct sum of eigensheaves for the action of  $\underline{\mu}_{n, \mathcal{X}_\alpha}$  (see [17, Proposition 3.1.1.4]) to the following decomposition of derived categories

$$\mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}_\alpha) = \bigsqcup_{i=1}^n \mathrm{D}_{\mathrm{qcoh}}(X, i\alpha). \quad (2.1)$$

Under the above decomposition,  $\mathrm{D}_{\mathrm{qcoh}}(X, 0)$  is canonically identified with the essential image of the natural pullback map  $p_\alpha^*: \mathrm{D}_{\mathrm{qcoh}}(X) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}_\alpha)$ . For any morphism  $f: X \rightarrow Y$  and  $\beta \in \mathrm{Br}(Y)$ , we have the following identification  $\mathcal{X}_{f^*\beta} = \mathcal{Y}_\beta \times_Y X$ . The pullback and pushforward maps among the derived categories of quasi-coherent sheaves induced by the natural map  $\mathcal{X}_{f^*\beta} \rightarrow \mathcal{Y}_\beta$  restricts (see [17, Lemma 3.1.1.5]) to give an adjoint pair of functors

$$f^*: \mathrm{D}_{\mathrm{qcoh}}(Y, \beta) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(X, f^*\beta), \quad f_*: \mathrm{D}_{\mathrm{qcoh}}(X, f^*\beta) \rightarrow \mathrm{D}_{\mathrm{qcoh}}(Y, \beta).$$

Let  $\alpha$  correspond to a class arising from Azumaya an  $\mathcal{O}_X$ -algebra  $A$ , and let  $\mathrm{QCoh}(X) - A$  denote the category of right  $A$ -quasi-coherent modules. There exists a locally free  $\alpha$ -twisted sheaf  $\mathcal{E}_\alpha$  of finite rank such that  $A \cong \underline{\mathrm{End}}(\mathcal{E}_\alpha)$ . Moreover, the functor

$$- \otimes \mathcal{E}_\alpha^\vee: \mathrm{QCoh}(X, \alpha) \rightarrow \mathrm{QCoh}(X) - A. \quad (2.2)$$

is an equivalence of categories [16, Proposition 3.2.2.1].

**Lemma 2.1.** *Let  $\mathcal{X}_\alpha$  (respectively  $\mathcal{Y}_\beta$ ) be two  $\mu_n$ -gerbes over quasi-compact separated algebraic spaces  $X$  (respectively  $Y$ ) corresponding to classes  $\alpha \in \mathrm{Br}(X)$  (respectively  $\beta \in \mathrm{Br}(Y)$ ), and  $\mathrm{pr}: \mathcal{X}_\alpha \times \mathcal{Y}_\beta \rightarrow \mathcal{Y}_\beta$  denote the second projection. Let  $K \in \mathrm{D}_{\mathrm{qcoh}}(X \times Y, r\alpha \boxtimes \beta)$  for some  $r$  such that  $r\alpha \neq 0$ . Then  $\mathrm{pr}_*(K) = 0$ .*

*Proof.* Using flat base change, we may reduce to the case  $Y = \mathrm{Spec}(A)$  where  $A$  is completion of localization of a affine  $k$ -algebra. Note that  $\mathrm{Br}(A) \simeq \mathrm{Br}(k) = 0$ . Thus it suffices to prove the following statement: With  $\mathcal{X}_\alpha$  as in the statement of the theorem,  $\mathcal{Y}_\beta = \mathrm{Spec}(A) \times B\mu_n$  for  $A$  as above, and  $K \in \mathrm{D}_{\mathrm{qcoh}}(X \times \mathrm{Spec}(A), \alpha \boxtimes 0)$ , the cohomology groups  $\mathrm{H}^i(\mathcal{X}_\alpha \times \mathcal{Y}_\beta, K)$  vanish for all  $i \geq 0$ . Another application of flat base change applied to the morphism  $\mathrm{Spec}(k) \rightarrow B\mu_n$  we may further reduce to proving that  $\mathrm{H}^i(\mathcal{X}_\alpha \times \mathrm{Spec}(A), K) = 0$  for all  $i \geq 0$ .

It follows from the more general statement that for any two objects

$$F_1, F_2 \in \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}_\alpha \times \mathrm{Spec}(A)),$$

the Ext groups  $\mathrm{Ext}^k(F_1, F_2)$  vanish for all  $k \geq 0$  unless both  $F_1$  and  $F_2$  both belong to the subcategory  $\mathrm{D}_{\mathrm{qcoh}}(X \times \mathrm{Spec}(A), i\alpha \boxtimes 0)$  for some  $i$ . But this is true by upgrading [17, Proposition 3.1.1.4] to the level of derived categories using [15, Lemma 3.2.5] ■

### 3. Proof of Theorem 1.2

We recall the setup for convenience.  $k$  is an algebraically closed field of char  $p > 0$ ,  $X$  is a quasi-compact, quasi-separated algebraic space over  $k$ . Any other assumptions that will be imposed will be mentioned in the statement of the lemma/proposition/theorem.

**Definition 3.1.** For  $E \in \mathrm{D}_{\mathrm{coh}}^b(X)$ , the cohomological support is defined to be the finite union

$$\mathrm{Supph}(E) := \bigcup \mathrm{Supp} \mathcal{H}^i(E),$$

where the support of an abelian sheaf  $\mathcal{H}^i(E)$  is as defined in [23, Tag 04KA].

**Lemma 3.2.** *Let  $X$  be of finite type over  $k$ , and let  $E \in \mathrm{D}_{\mathrm{coh}}^b(X)$ . Suppose  $F_{X*}E \cong E$ , then  $\mathrm{Supph}(E)$  is zero-dimensional.*

*Proof.* Since  $F_{X*}$  is a finite morphism and only finitely many cohomology sheaves of  $E$  are non-vanishing, we may assume that  $E \in \mathrm{Coh}(X)$ . Let  $Z \xrightarrow{i} X$  be the scheme theoretic support of the sheaf  $E$  [23, Tag 07U1] and let  $F \subset E$  be the unique<sup>1</sup> sheaf on  $Z$  such that  $i_*F = E$ . Now,  $F_{X*}E = E$  implies that  $i_*FZ_*F = i_*F$ , and therefore  $F_{Y_*}F = F$  [23, Tag 04CJ]. Note that  $|Z| = \mathrm{Supp}(E)$  by [23, Tag 07U0]. In conclusion, we are reduced to showing that if  $X$  is an algebraic space and  $E$  is a quasi-coherent sheaf

---

<sup>1</sup>Note that this also implies the uniqueness of the subsheaf  $F$ .

supported on  $X$  satisfying  $F_{X*}E = E$ , then  $\dim X = 0$ . Since  $X$  is quasi-separated using [23, Tag 06NH] we may choose an open dense immersion  $U \rightarrow X$  of an affine scheme  $U = \text{Spec}(R)$  in  $X$ . Since  $X$  is of finite type over  $k$ ,  $R$  is a  $k$ -algebra of finite type. Fix an embedding  $j: \text{Spec}(R) \hookrightarrow \mathbb{A}_k^N$ . Then

$$F_{\mathbb{A}_k^N*}(j_*E|_U) = j_*(F_{U*}E|_U) \cong j_*(E|_U) = j_*E|_U.$$

Let  $V \subset \text{Spec}(R)$  be an irreducible component of  $\text{Spec}(R)$  equipped with the reduced induced subscheme structure such that  $\dim(V) = \dim(\text{Spec}(R))$ . It suffices to show that  $\dim(V) = 0$ . Applying [3, Lemma 3.1] to  $j_*E|_U$  we get that

$$\text{rk}_V((F_{\mathbb{A}_k^N*})j_*E|_U) = p^{\dim(V)}\text{rk}_V(j_*E|_U),$$

But  $(F_{\mathbb{A}_k^N*})j_*E|_U \cong j_*E|_U$  and  $\text{rk}_V(j_*E|_U) > 0$  due to our assumption that  $E$  is supported on  $V$ . Hence  $\dim(V) = 0$ . This is what we wanted to show. ■

Given a morphism  $x: \text{Spec}(k) \rightarrow X$ , define the skyscraper sheaf  $k(x) := x_*k$ . If  $X$  is separated  $x$  is a closed immersion, hence  $k(x)$  is a coherent sheaf on  $X$ . We recall a lemma from [12]. Note that in *loc. cit.* the statement was proved for schemes, but the same proof works in our context.

**Lemma 3.3** ([12, Lemma 4.5]). *Let  $X$  be smooth and proper over  $k$ . Suppose that  $E \in \mathcal{D}_{\text{coh}}^b(X)$  satisfies the following three conditions*

- (1)  *$\text{Supp}(E)$  is zero-dimensional,*
- (2)  *$\text{Hom}_{\mathcal{D}_{\text{coh}}(X)}(E, E[i]) = 0$  for  $i < 0$ ,*
- (3)  *$\text{Hom}_{\mathcal{D}_{\text{coh}}(X)}(E, E) = k$ .*

*Then  $E \cong k(x)[m]$  for some  $x \in X(k)$  and an integer  $m$ .*

Recall that the Fourier–Mukai transform with kernel  $K \in \mathcal{D}_{\text{qcoh}}(X \times Y)$  is defined to be the functor

$$\Phi_K: \mathcal{D}_{\text{qcoh}}(X) \rightarrow \mathcal{D}_{\text{qcoh}}(Y); \quad F \mapsto \text{pr}_{Y*}(K \otimes \text{pr}_X^*F). \quad (3.1)$$

We state a proposition which will be needed later.

**Proposition 3.4.** *Let  $X, Y$  be smooth proper connected algebraic spaces over  $k$ . Let  $K \in \mathcal{D}_{\text{perf}}(X \times Y)$  be such that  $\Phi_K: \mathcal{D}_{\text{qcoh}}(X) \rightarrow \mathcal{D}_{\text{qcoh}}(Y)$  is an equivalence of categories. Then the following are true.*

- (1) *Assume that  $\Phi_K(k(x))$  is placed in a single degree for all  $x \in X(k)$ . Then  $K$  is a (possibly shifted) quasi-coherent sheaf on  $X \times Y$  which is flat over  $X$ .*
- (2) *Assume that  $\Phi_K(k(x))$  is a sheaf placed in degree zero whose support is zero dimensional. Then  $\Phi_K$  restricts to a  $k$ -linear exact functor*

$$\Phi_K: \text{QCoh}(X) \rightarrow \text{QCoh}(Y).$$

*Proof.* The projection map  $\mathrm{pr}_X: X \times Y \rightarrow X$  is flat. Let  $p: U \rightarrow X$  be an étale cover of  $X$  with  $U$  a finite type  $k$ -scheme. For any  $u \in U(k)$  we have the following commutative diagram where the squares are cartesian

$$\begin{array}{ccccccc}
 & & & & r & & \\
 & & & & \curvearrowright & & \\
 u \times Y & \xrightarrow{i_u \times \mathrm{id}} & U \times Y & \xrightarrow{p \times \mathrm{id}} & X \times Y & \xrightarrow{\mathrm{pr}_Y} & Y \\
 \downarrow q & & \downarrow \mathrm{pr}_U & & \downarrow \mathrm{pr}_X & & \\
 u & \xrightarrow{i_u} & U & \xrightarrow{p} & X & & 
 \end{array}$$

It suffices to prove that  $(p \times \mathrm{id})^* K$  is flat over  $U$  and is concentrated in a single fixed degree. Since  $U$  is a scheme, [4, Lemma 4.1]<sup>2</sup> (see also [12, Lemma 3.31]) implies that it suffices to prove  $((p \times \mathrm{id})^* K)|_{u \times Y}$  is concentrated in a single degree. This is what we will show now. We have the following series of equations

$$\begin{aligned}
 & ((p \times \mathrm{id})^* K)|_{u \times Y} \\
 &= \mathrm{pr}'_{Y*}(((p \times \mathrm{id})^* K)|_{u \times Y}) && \text{use the isomorphism } \mathrm{pr}'_Y \circ r = \mathrm{pr}'_Y: u \times Y \rightarrow Y \\
 &= \mathrm{pr}_{Y*}(r_* r^* K) && r = (p \times \mathrm{id}) \circ (i_u \times \mathrm{id}), \text{ and } \mathrm{pr}'_Y = \mathrm{pr}_Y \circ r \\
 &= \mathrm{pr}_{Y*}(K \otimes r_*(\mathcal{O}_{u \times Y})) && \text{since } K \text{ is perfect, the projection formula holds} \\
 & && \text{for the map } r \\
 &= \mathrm{pr}_{Y*}(K \otimes r_* q^* k(u)) && \text{use that } q^* k(u) = \mathcal{O}_{u \times Y} \\
 &= \mathrm{pr}_{Y*}(K \otimes \mathrm{pr}_X^*(p \circ i_u)_* k(u)) && \text{use flat base change in the relevant diagram} \\
 &= \Phi_K((p \circ i_u)_* k(u)) && \text{use definition of Fourier–Mukai transforms.}
 \end{aligned}$$

Therefore, our hypothesis implies that  $((p \times \mathrm{id})^* K)|_{u \times Y}$  is concentrated in a single degree. If  $U$  were a connected scheme, we would be done. Otherwise let  $U = U_0 \sqcup U_1$  be disjoint union of two nonempty open and closed subschemes. We know that  $((p \times \mathrm{id})^* K)|_{U_i}$  is concentrated in a single degree say  $n_i$ . Let  $x \in X(k)$  be such that there exists  $u_0 \in U_0(k)$  and  $u_1 \in U_1(k)$  which maps to  $x$ . Such a point exists since  $X$  is connected. Thus we get the equality  $n_0 = n_1$ . This finishes proof of assertion (1).

Now we prove the second assertion. From part (1) we get that  $K$  is flat over  $X$ . Thus for any  $F \in \mathrm{QCoh}(X)$ ,

$$\mathrm{pr}_X^* F \otimes K \in \mathrm{QCoh}(X \times Y).$$

Let  $Z \xrightarrow{iz} X \times Y$  denote the schematic support of  $K$ , then  $(p \times \mathrm{id})^{-1}(Z)$  is the schematic support of  $(p \times \mathrm{id})^*(K)$  since  $p$  is étale. Suppose we know that  $(p \times \mathrm{id})^{-1}(Z)$  is quasi-finite over  $Y$ , then  $\mathrm{pr}_Y \circ iz =: a: Z \rightarrow Y$  is a quasi-finite map. (The fibers of the map

<sup>2</sup>We remark that Calabrese states this lemma with the hypothesis that the restriction of  $K$  to the fiber of  $\mathrm{pr}_U$  over all points (and not just closed points) of  $U$  is concentrated in a single degree, but the proof uses the hypothesis for only closed points. Note that the assumption that  $K$  has bounded cohomology is satisfied, since  $K$  is perfect and  $X$  is proper and hence can be covered by finitely many smooth affine schemes.

$a = \text{pr}_Y \circ i_Z$  are discrete since fibers of  $(p \times \text{id})^{-1}(Z) \rightarrow Y$  is discrete and  $(p \times \text{id})$  is an open map.) Since  $K$  is supported on  $Z$ , hence  $\text{pr}_X^* F \otimes K$  is supported on  $Z$  as well. Let  $G \in \text{QCoh}(Z)$  be such that  $\text{pr}_X^* F \otimes K = i_{Z*} G$ . Thus we get  $\Phi_K(F) = a_* G$ . Since  $a: Z \rightarrow Y$  is quasi-finite and proper, higher derived functors  $R^i a_* \equiv 0$  vanish (see [23, Tag 0A4T]). Now, we may use [23, Tag 0FZH] or rather a more general version from [21, Proposition A.4] to conclude. All that remains to show is that  $(p \times \text{id})^{-1}(Z) \rightarrow Y$  is a quasi-finite map.

Let us denote  $(p \times \text{id})^{-1}(Z)$  by  $Z'$ . Let  $q: V \rightarrow Y$  be an étale cover of  $Y$  with  $V$  a finite type  $k$ -scheme and  $p: U \rightarrow X$  as before. Define  $Z''$  as the pullback of  $Z'$  along  $q \times \text{id}_U$ . We have the following diagram in which squares are cartesian.

$$\begin{array}{ccccc}
 & & s & & \\
 & \curvearrowright & & \curvearrowleft & \\
 Z'' & \longrightarrow & U \times V & \longrightarrow & V \\
 \downarrow & & \downarrow & & \downarrow q \\
 Z' & \longrightarrow & U \times Y & \longrightarrow & Y
 \end{array}$$

Since quasi-finite maps satisfy faithfully flat descent, it is enough to show that  $s: Z'' \rightarrow V$  is quasi-finite. Let  $(u, v), (u_1, v) \in Z''(k)$ . Since the schematic support of  $K$  is  $Z$  and  $(u, v), (u_1, v) \in Z''(k)$  it is an easy check that  $\Phi_K(k(p(u)))$  and  $\Phi_K(k(p(u_1)))$  are two sheaves supported on a zero-dimensional set containing  $q(v)$ . Also,  $\Phi_K$  is an equivalence of categories, hence  $\Phi_K(k(p(u)))$  and  $\Phi_K(k(p(u_1)))$  satisfy the assumptions of Lemma 3.3. Thus  $\Phi_K(k(u)) \cong \Phi_K(k(u_1)) \cong k(q(v))$ . Since  $\Phi_K$  is assumed to be fully faithful, we get that  $k(p(u)) \cong k(p(u_1))$ . Since  $p$  is étale and of finite type, its fibers are discrete finite sets. Hence the subset  $\{u_1 \in U(k) \mid (u_1, v) \in Z''(k)\} \subset p^{-1}(v)$  is a discrete finite set. This implies that the map  $Z'' \rightarrow V$  is quasi-finite. This finishes the proof of assertion (2). ■

**Theorem 3.5.** *Let  $X$  and  $Y$  be smooth proper connected algebraic spaces over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $\Phi: \text{D}_{\text{qcoh}}(X) \rightarrow \text{D}_{\text{qcoh}}(Y)$  be a Fourier–Mukai equivalence which satisfies*

$$\Phi \circ F_{X*} \cong F_{Y*} \circ \Phi.$$

*Then there exists an isomorphism  $f: X \xrightarrow{\sim} Y$ , a line bundle  $L$  on  $Y$  and an integer  $n$  such that*

$$\Phi \simeq f_* \circ (- \otimes L)[n].$$

*Proof.* Observe that  $\Phi$  induces an equivalence among the full subcategory of compact objects since  $\Phi$  is an equivalence. Since both  $X$  and  $Y$  are algebraic spaces and hence concentrated [10, Theorem C], that is the category of perfect complexes coincides with the full subcategory of compact objects in  $\text{D}(X)$ . By [2, Theorem 1.2(2)]<sup>3</sup> we get that

<sup>3</sup>As noted in [11, §Perfect stacks (i), p. 2321] quasi-compact separated algebraic spaces are perfect stacks.

the Fourier–Mukai kernel inducing  $\Phi$  is a perfect complex. Now by hypothesis, for any  $x \in X(k)$ ,

$$F_{X*}\Phi(k(x)) \cong \Phi(F_{X*}k(x)) \cong \Phi(k(x)).$$

By Lemma 3.2, the support of  $\Phi(k(x))$  is a zero-dimensional. Since  $\Phi$  is fully faithful, Lemma 3.3 implies that  $\Phi(k(x))$  must be isomorphic to  $k(y)[m]$  for some  $y \in Y(k)$  and an integer  $m$ . Proposition 3.4 implies that  $\Phi[-m]$  restricts to a  $k$ -linear exact fully faithful functor  $\mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ . Arguing similarly with  $\Phi^{-1}[m]$  we get that  $\Phi[-m]$  induces a  $k$ -linear equivalence  $\mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$  of abelian categories. Finally, an application of [5, Theorems 2.10 and 4.5] finishes the proof.  $\blacksquare$

## 4. Proof of Theorem 1.3

In this section, we will complete the proof of Theorem 1.3. The proof proceeds in a manner analogous to that of Theorem 1.2 with appropriate adjustments at a few places to account for the fact that twisted sheaves are objects on a gerbe on the algebraic space. This extra subtlety requires some care at a certain stages of the proof which we explain below.

We recall the setup.  $X$  and  $Y$  are quasi-compact quasi-separated algebraic spaces over an algebraically closed field  $k$  of characteristic  $p > 0$ ,  $\mathcal{X}_\alpha$  (respectively  $\mathcal{Y}_\beta$ ) denote the gerbes associated to  $\alpha \in \mathrm{Br}(X)$  (respectively  $\beta \in \mathrm{Br}(Y)$ ). We assume that  $n\alpha = n\beta = 0$  for an integer  $n$  that is coprime to  $p$ . Let  $r > 0$  be an integer such that  $p^r \equiv 1 \pmod{n}$ . Symbols for stacks and sheaves on stacks are written in cursive letter while the algebraic spaces and sheaves on algebraic spaces are written in usual font.

**Definition 4.1.** For an  $\alpha$ -twisted sheaf  $\mathcal{F} \in \mathrm{Coh}(X, \alpha)$ , the support of  $\mathcal{F}$  is defined to be the Zariski closed subspace  $|p_\alpha|(\mathrm{Supp}(\mathcal{F})) \subset |X|$ , where  $p_\alpha: \mathcal{X}_\alpha \rightarrow X$  is the gerbe associated to  $\alpha \in \mathrm{Br}(X)$ . Define  $\mathrm{Supph}(\mathcal{F})$ , the cohomological support of  $\mathcal{F} \in D_{\mathrm{coh}}^b(X, \alpha)$  to be the finite union of Zariski closed subspaces

$$\mathrm{Supph}(\mathcal{F}) := \bigcup_i p_\alpha(\mathrm{Supp}(\mathcal{H}^i(\mathcal{F}))).$$

**Lemma 4.2.** *Let  $X$  be of finite type over  $k$  and  $\alpha$  be as before. Suppose  $(F_{X*})^r \mathcal{F} \cong \mathcal{F}$  for some  $r$  such that  $p^r \alpha = \alpha$  in the Brauer group. Then  $\mathrm{Supph}(\mathcal{F})$  is zero-dimensional.*

*Proof.* Since  $F_{X*}$  is an exact functor ( $F_X$  is finite, see for instance [13, Proposition 3.9]), it suffices to prove the lemma for  $\mathcal{F} \in \mathrm{Coh}(X, \alpha)$ . The equality  $(F_{X*})^r \mathcal{F} \cong \mathcal{F}$  from the hypothesis implies the equality  $(F_{X*})^r \mathcal{F} \otimes \mathcal{E}_\alpha^\vee \cong \mathcal{F} \otimes \mathcal{E}_\alpha^\vee$ . Here  $\mathcal{E}_\alpha$  is the sheaf introduced in (2.2). Using projection formula, the above is equivalent to  $(F_{X*})^r (\mathcal{F} \otimes (F_X^*)^r \mathcal{E}_\alpha^\vee) \cong \mathcal{F} \otimes \mathcal{E}_\alpha^\vee$ . It can be checked, for instance using the description of twisted sheaves in terms of 2-cocycles on algebraic space  $X$  that  $F_X^* \mathcal{E}_\alpha = \mathcal{E}_{p\alpha}$ , hence  $(F_X^*)^r \mathcal{E}_\alpha \cong \mathcal{E}_{p^r \alpha} \cong \mathcal{E}_\alpha$ . Thus we get the equality  $(F_{X*})^r (\mathcal{F} \otimes \mathcal{E}_\alpha^\vee) \cong \mathcal{F} \otimes \mathcal{E}_\alpha^\vee$ . Recall from Section 2.2 that  $\mathcal{F} \otimes \mathcal{E}_\alpha^\vee$  is now  $p_\alpha^* G$  for a quasi-coherent  $\mathcal{O}_X$ -sheaf  $G$ . Applying Lemma 3.2 to  $G$  implies that the support of  $G$  is zero dimensional. We may write

$$\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(\mathcal{F} \otimes \mathcal{E}_\alpha^\vee) = \mathrm{Supp}(p_\alpha^* G) = |p|_\alpha^{-1}(\mathrm{Supp}(G)).$$

The first equality follows since  $\mathcal{E}_\alpha^\vee$  is a locally free sheaf of finite rank and the last equality follows from the definition of support and the fact that  $p_\alpha$  is a faithfully flat map. Hence,  $|p|_\alpha(\text{Supp}(\mathcal{F}))$  is zero dimensional. Note that  $|p_\alpha|$  is a homeomorphism by [23, Tag 06R9] and hence  $\text{Supp}(\mathcal{F})$  is zero dimensional as well. The proof is complete. ■

In this paragraph, we review the definition of twisted skyscraper sheaves. Recall that  $p_\alpha: \mathcal{X}_\alpha \rightarrow X$  denotes the gerbe associated to  $\alpha \in \text{Br}(X)$ . It is also clear from [23, Tag 06QD] that  $p_\alpha$  is a coarse moduli space map. Since  $\mathcal{X}_\alpha$  has finite inertia stack, the coarse moduli space map  $p_\alpha$  is a universal homeomorphism. Let  $x \in |\mathcal{X}_\alpha|$  be the canonical lift of a point  $x \in X(k)$ . Then the morphism  $x \rightarrow \mathcal{X}_\alpha$  admits the following factorization

$$x \rightarrow BG_x \xrightarrow{\iota_x} \mathcal{X}_\alpha,$$

where  $G_x = \text{Aut}_{\mathcal{X}_\alpha}(x)$ . Since  $\alpha \in H_{\text{ét}}^2(X, \mu_n)$  by assumption, the group  $G_x$  is isomorphic to  $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ . The latter isomorphism holds since  $(n, p) = 1$  and  $k = \bar{k}$ . Define  $k(x, \alpha) \in \text{Coh}(BG_x, \iota_x^* \alpha)$  to be the unique simple coherent sheaf. Since  $\iota_x$  is a closed immersion,  $\iota_{x*}(k(x, \alpha))$  is again a coherent sheaf and is an  $\alpha$ -twisted sheaf by construction. We denote it by  $k(x, \alpha)$ . Next, we need the lemma characterizing twisted skyscraper sheaves.

**Lemma 4.3.** *Let  $X$  be smooth and proper over  $k$  and  $\alpha \in \text{Br}(X)$ . Let  $\mathcal{F} \in \text{D}_{\text{coh}}^b(X, \alpha)$  be such that*

- (1)  $\text{Supp}(\mathcal{F})$  is zero-dimensional.
- (2)  $\text{Hom}_{\text{D}_{\text{qcoh}}(X, \alpha)}(\mathcal{F}, \mathcal{F}[i]) = 0$  for  $i < 0$ .
- (3)  $\text{Hom}_{\text{D}_{\text{qcoh}}(X, \alpha)}(\mathcal{F}, \mathcal{F}) = k$ .

*Then  $\mathcal{F} \cong k(x, \alpha)[m]$  for some  $x \in X(k)$  and an integer  $m$ .*

*Proof.* Arguing as in [12, Lemma 4.5] we get that  $\mathcal{F} = \mathcal{G}[m]$  for an integer  $m$  and  $\mathcal{G} \in \text{Coh}(X, \alpha)$  supported on a singleton  $x \in \mathcal{X}_\alpha(k)$ . Define  $\mathcal{J} := \ker(\mathcal{O}_{\mathcal{X}_\alpha} \rightarrow \underline{\text{End}}(\mathcal{G}))$ , then we know that  $\mathcal{O}_{\mathcal{X}_\alpha}/\mathcal{J}$  is a coherent sheaf supported at one point  $x \in \mathcal{X}_\alpha(k)$ . Let  $\sqrt{\mathcal{J}}$  be radical ideal, then  $\mathcal{O}_{\mathcal{X}_\alpha}/\sqrt{\mathcal{J}}$  is a reduced closed substack  $\mathcal{Z} \subset \mathcal{X}_\alpha$ . Now, [23, Tag 0H25] implies that  $\mathcal{Z} \subset BG_x$  is a closed substack. Hence  $\mathcal{Z} = BG_x$ . Since  $X$  is assumed to be smooth and proper, so is  $\mathcal{X}_\alpha$ . Hence  $\mathcal{X}_\alpha$  is a Noetherian algebraic stack. So there exists  $N_0$  a non-negative integer such that  $\sqrt{\mathcal{J}}^{N_0} \not\subset \mathcal{J}$  and  $\sqrt{\mathcal{J}}^{N_0+1} \subset \mathcal{J}$ . Then we have the following series of maps

$$\mathcal{G} \twoheadrightarrow \mathcal{G}/\sqrt{\mathcal{J}} \twoheadrightarrow k(x, \alpha) \hookrightarrow \sqrt{\mathcal{J}}^{N_0} \mathcal{G} \hookrightarrow \mathcal{G}.$$

The composite map is nonzero, so it must be an isomorphism due to hypothesis 3. Hence all the surjective maps and injective maps must be isomorphisms. This completes the proof. ■

To any object  $K \in \text{D}_{\text{qcoh}}(X \times_k Y, \alpha^{-1} \boxtimes \beta)$ , we may associate the following functor

$$\Phi_K: \text{D}_{\text{qcoh}}(X, \alpha) \rightarrow \text{D}_{\text{qcoh}}(Y, \beta); \quad F \mapsto \text{pr}_{Y*}(K \otimes \text{pr}_X^* F). \quad (4.1)$$

The above functor is called the twisted Fourier–Mukai transform with kernel  $K$ . Note that  $\text{D}_{\text{qcoh}}(X \times_k Y, \alpha^{-1} \boxtimes \beta)$  is a full subcategory of  $\text{D}(\mathcal{X}_\alpha \times \mathcal{Y}_\beta)$  and thus the object  $K$  defines

a functor

$$\Phi_K: D_{\text{qcoh}}(\mathcal{X}_\alpha) \rightarrow D_{\text{qcoh}}(\mathcal{Y}_\beta).$$

It follows from Lemma 2.1 that  $\Phi_K(k(x, \alpha')) = 0$  for all  $\alpha' \neq \alpha$ . This fact will be used when adapting the proof of Proposition 3.4 to prove the following

**Proposition 4.4.** *Let  $X$  and  $Y$  be smooth proper connected algebraic spaces over  $k$ ,  $\alpha \in \text{Br}(X)$  and  $\beta \in \text{Br}(Y)$ , and  $K \in D_{\text{perf}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ . Assume that  $\Phi_K: D_{\text{qcoh}}(X, \alpha) \rightarrow D_{\text{qcoh}}(Y, \beta)$  is an equivalence of categories.*

- (1) *Assume that  $\Phi_K(k(x, \alpha))$  is concentrated in a single degree for all  $x \in X(k)$ . Then  $K$  is a (possibly shifted) quasi-coherent sheaf on  $\mathcal{X}_\alpha \times \mathcal{Y}_\beta$  which is flat over  $\mathcal{X}_\alpha$ .*
- (2) *Furthermore, if  $\Phi_K(k(x, \alpha))$  is a sheaf concentrated in degree zero whose cohomological support is zero dimensional, then  $\Phi_K$  restricts to a  $k$ -linear exact functor  $\Phi_K: \text{QCoh}(X, \alpha) \rightarrow \text{QCoh}(Y, \beta)$ .*

*Proof.* The projection map  $\text{pr}_{\mathcal{X}_\alpha}: \mathcal{X}_\alpha \times \mathcal{Y}_\beta \rightarrow \mathcal{X}_\alpha$  is flat. Let  $p: U \rightarrow \mathcal{X}_\alpha$  be an étale cover of  $\mathcal{X}_\alpha$  with  $U$  a finite type  $k$ -scheme. For any  $u \in U(k)$  we have the following commutative diagram where the squares are cartesian

$$\begin{array}{ccccccc}
 & & & r & & & \\
 & & & \curvearrowright & & & \\
 u \times \mathcal{Y}_\beta & \xrightarrow{i_u \times \text{id}} & U \times \mathcal{Y}_\beta & \xrightarrow{p \times \text{id}} & \mathcal{X}_\alpha \times \mathcal{Y}_\beta & \xrightarrow{\text{pr}_{\mathcal{Y}_\beta}} & \mathcal{Y}_\beta. \\
 \downarrow q & & \downarrow \text{pr}_U & & \downarrow \text{pr}_{\mathcal{X}_\alpha} & & \\
 u & \xrightarrow{i_u} & U & \xrightarrow{p} & \mathcal{X}_\alpha & & 
 \end{array}$$

It suffices to prove that  $(p \times \text{id})^* K$  is flat over  $U$ . Since  $U$  is a scheme, by [4, Lemma 4.1]<sup>4</sup> (see also [12, Lemma 3.31]) it suffices to prove that  $((p \times \text{id})^* K)|_{u \times \mathcal{Y}_\beta}$  is concentrated in a single degree. This is what we will show now. We have the following series of equations

$$\begin{aligned}
 & ((p \times \text{id})^* K)|_{u \times \mathcal{Y}_\beta} \\
 &= \text{pr}_{\mathcal{Y}_\beta *} r_* (((p \times \text{id})^* K)|_{u \times \mathcal{Y}_\beta}) && \text{use the isomorphism } \text{pr}_{\mathcal{Y}_\beta} \circ r: u \times \mathcal{Y}_\beta \rightarrow \mathcal{Y}_\beta \\
 &= \text{pr}_{\mathcal{Y}_\beta *} (r_* r^* K) && r = (p \times \text{id}) \circ (i_u \times \text{id}) \\
 &= \text{pr}_{\mathcal{Y}_\beta *} (K \otimes r_* (\mathcal{O}_{u \times \mathcal{Y}_\beta})) && \text{since } K \text{ is perfect, the projection formula holds} \\
 & && \text{for the map } r \\
 &= \text{pr}_{\mathcal{Y}_\beta *} (K \otimes r_* q^* k(u)) && \text{use that } q^* k(u) = \mathcal{O}_{u \times \mathcal{Y}_\beta} \\
 &= \text{pr}_{\mathcal{Y}_\beta *} (K \otimes \text{pr}_{\mathcal{X}_\alpha}^* (p \circ i_u)_* k(u)) && \text{use flat base change in the relevant diagram} \\
 &= \Phi_K((p \circ i_u)_* k(u)) && \text{use definition of Fourier–Mukai transforms.}
 \end{aligned}$$

<sup>4</sup>We remark that Calabrese states the Lemma where the hypothesis is imposed for all points, but for the proof the hypothesis is required for closed points only.

The map  $p \circ i_u: \text{Spec}(k) \rightarrow \mathcal{X}_\alpha$  factors as

$$\text{Spec}(k) \rightarrow BG_u \xrightarrow{t_x} \mathcal{X}_\alpha.$$

Computing the pushforward we see that  $(p \circ i_u)_*k(u) = \bigoplus_{i=0}^n k(u, i\alpha)$ . Recall from the discussion above,  $\Phi_K(k(x, \xi))$  either vanishes or is concentrated in a single degree from our hypothesis. Therefore  $((p \times \text{id})^*K)|_{u \times \mathcal{Y}_\beta}$  is concentrated in a single degree. The argument in the proof of Proposition 3.4 (1) needed to deal with the case of a disconnected  $U$  works verbatim in this case. This finishes proof of assertion (1).

Now we prove the second assertion. From part (1) we get that  $K$  is flat over  $\mathcal{X}_\alpha$  and is a sheaf placed in degree zero. Thus for any  $\mathcal{F} \in \text{Coh}(\mathcal{X}_\alpha)$

$$\text{pr}_{\mathcal{X}_\alpha}^* \mathcal{F} \otimes K \in \text{Coh}(\mathcal{X}_\alpha \times \mathcal{Y}_\beta).$$

Let  $\mathcal{Z} \xrightarrow{i_{\mathcal{Z}}} \mathcal{X}_\alpha \times \mathcal{Y}_\beta$  denote the schematic support of  $K$  (see [8, Lemma 2.6]), then  $(p \times \text{id})^{-1}(\mathcal{Z})$  is the schematic support of  $(p \times \text{id})^*(K)$  since  $p$  is étale. Suppose we know that  $(p \times \text{id})^{-1}(\mathcal{Z})$  is quasi-finite over  $\mathcal{Y}_\beta$ , then  $\text{pr}_{\mathcal{Y}_\beta} \circ i_{\mathcal{Z}} =: a: \mathcal{Z} \rightarrow \mathcal{Y}_\beta$  is quasi-finite<sup>5</sup> over  $\mathcal{Y}_\beta$ . Since  $K$  is supported on  $\mathcal{Z}$ ,  $\text{pr}_{\mathcal{X}_\alpha}^* \mathcal{F} \otimes K$  is supported on  $\mathcal{Z}$  as well. Let  $\mathcal{G} \in \text{Coh}(\mathcal{Z})$  be such that  $\text{pr}_{\mathcal{X}_\alpha}^* \mathcal{F} \otimes K = i_{\mathcal{Z}*} \mathcal{G}$ . Thus we get  $\Phi_K(\mathcal{F}) = a_* \mathcal{G}$ . Since  $a: \mathcal{Z} \rightarrow \mathcal{Y}_\beta$  is quasi-finite and proper, higher derived functors  $R^i a_* \equiv 0$  vanish (see Proposition B.1). Now, we may use [23, Tag 0FZH] or rather its generalization [21, Proposition A.4] to conclude. All that remains to show is that  $(p \times \text{id})^{-1}(\mathcal{Z}) \rightarrow \mathcal{Y}_\beta$  is a quasi-finite map.

Let us denote  $(p \times \text{id})^{-1}(\mathcal{Z})$  by  $\mathcal{Z}'$ . Let  $q: V \rightarrow \mathcal{Y}_\beta$  be an étale cover of  $Y$  with  $V$  a finite type  $k$ -scheme. Define  $\mathcal{Z}''$  as the pullback of  $\mathcal{Z}'$  along  $q \times \text{id}_U$ . We have the following cartesian diagram.

$$\begin{array}{ccccc} \mathcal{Z}'' & \hookrightarrow & U \times V & \longrightarrow & V \\ \downarrow & & \downarrow \text{id}_U \times q & & \downarrow q \\ \mathcal{Z}' & \hookrightarrow & U \times \mathcal{Y}_\beta & \longrightarrow & \mathcal{Y}_\beta \end{array}$$

Since quasi-finite maps satisfy faithfully flat descent, it is enough to show  $\mathcal{Z}'' \rightarrow V$  is quasi-finite. Let  $(u, v), (u_1, v) \in \mathcal{Z}''(k)$ . By hypothesis,  $\Phi_K(k(p(u), \alpha))$  and  $\Phi_K(k(p(u_1), \alpha))$  are two coherent sheaves supported on a zero-dimensional set containing  $q(v)$ . Also,  $\Phi_K$  is an equivalence of categories and hence  $\Phi_K(k(p(u), \alpha))$  and  $\Phi_K(k(p(u_1), \alpha))$  satisfy the hypothesis of Lemma 4.3. Thus  $\Phi_K(k(p(u), \alpha)) \cong \Phi_K(k(p(u_1), \alpha)) \cong k(q(v), \beta)$ . Since  $\Phi_K$  is assumed to be fully faithful, we get that  $k(p(u), \alpha) \cong k(p(u_1), \alpha)$ . Since  $p$  is étale and of finite type, its fibers are discrete finite sets. This shows that the set  $\{u_1 \in U(k) \mid (u_1, v) \in \mathcal{Z}''(k)\}$  is a finite discrete set. Hence the map  $\mathcal{Z}'' \rightarrow V$  is quasi-finite. This finishes the proof of assertion (2). ■

<sup>5</sup>The fibers of the map  $a$  are discrete since fibers of  $(p \times \text{id})^{-1}(\mathcal{Z}) \rightarrow Y$  is discrete and  $(p \times \text{id})$  is an open map. The map  $a$  is locally of finite type and quasi-compact since  $\mathcal{Z}$  and  $\mathcal{Y}_\beta$  are locally of finite presentation, quasi-compact and quasi-separated algebraic stacks. The map  $a$  is quasi-DM follows from [23, Tag 050M] and the fact that  $\mathcal{Z} \rightarrow \text{Spec}(k)$  is quasi-DM. The latter fact follows since [23, Tag 06R5] shows that  $I_{\mathcal{Z}} \rightarrow \mathcal{Z}$  is finite, and hence  $\mathcal{Z} \rightarrow \text{Spec}(k)$  is quasi-DM [23, Tag 0CL0].

**Theorem 4.5.** *Let  $X$  and  $Y$  be smooth proper algebraic space over an algebraically closed field of characteristic  $p > 0$ . Let  $\alpha \in \text{Br}(X)$  and  $\beta \in \text{Br}(Y)$  such that  $n\alpha = 0 = n\beta$  for a positive integer  $n$  coprime to  $p$ . Let  $r$  be a natural number such that  $p^r \equiv 1 \pmod{n}$ . Let  $\Phi$  be a Fourier–Mukai equivalence which satisfies*

$$\Phi \circ (F_{X*})^r = (F_{Y*})^r \circ \Phi.$$

*Then up to a shift and twists by a line bundle,  $\Phi$  is given by pullback via a morphism  $f: Y \rightarrow X$  such that  $f^*\alpha = \beta$ .*

*Proof.* The proof of the theorem is verbatim except we substitute the use of Lemma 3.2, Lemma 3.3, Proposition 3.4 and [5, Theorems 2.1 and 4.5] with the use of Lemma 4.2, Lemma 4.3, Proposition 4.4 and [5, Theorem 4.3]. ■

## A. Frobenius morphism for stacks

Recall that for a scheme  $S$  over a field of characteristic  $p > 0$ , we have the absolute Frobenius morphism  $F_S: S \rightarrow S$  which is defined to be identity on the underlying topological space of  $S$ , and is defined to be the  $p$ -power map  $s \rightarrow s^p$  on the structure sheaf  $\mathcal{O}_S$ . For an algebraic stack  $\mathcal{X}$  over a field  $\text{Spec}(k)$  of characteristic  $p > 0$ , define the absolute Frobenius morphism  $F_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X}$  as follows : Let  $S$  be a scheme over  $k$ , and let  $\xi \in \mathcal{X}(S)$ . Since  $\mathcal{X}$  is a fibered category over  $\text{Sch}/k$ , we have the pullback  $F_S^*\xi$  which is unique up to unique isomorphism. We define  $F_{\mathcal{X}}(\xi) := F_S^*\xi$ . For a morphism  $\tilde{f}: \xi \rightarrow \eta \in \mathcal{X}$  lying over  $f: T \rightarrow S \in \text{Sch}/k$ , we have the following commutative diagram

$$\begin{array}{ccccc}
 F_T^*\xi & \overset{\text{-----}}{\longrightarrow} & F_S^*\eta & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \xi & \xrightarrow{\tilde{f}} & \eta \\
 & & \downarrow & & \downarrow \\
 T & \xrightarrow{f} & S & & S \\
 \downarrow F_T & & \downarrow F_S & & \downarrow \\
 T & \xrightarrow{f} & S & & S
 \end{array}$$

It follows that there exists a unique map  $F_T^*\xi \rightarrow F_T^*\eta$  which makes the above diagram commute. We define  $F_T^*\tilde{f}$  to be this unique map.

## B. Vanishing of certain higher derived functors

**Proposition B.1.** *Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a quasi-finite morphism among proper tame Noetherian stacks. Then  $f_*$  is exact.*

*Proof.* The map  $f$  induces a morphism among associated coarse moduli spaces  $f: X \rightarrow Y$ . By standard arguments, the map  $f: X \rightarrow Y$  is proper and quasi-finite. Since  $\mathcal{X}$  and  $\mathcal{Y}$  are tame stacks, the coarse moduli space map is exact [1]. Now, we are done by [23, Tag 0A4T]. ■

More generally, we have the following.

**Proposition B.2.** *Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a proper morphism among quasi-compact separated Noetherian algebraic stacks such that the cohomological dimension of the fibers for any  $p \in |\mathcal{Y}|$  is bounded above by  $n$ . Then  $R^i f_* \equiv 0$  for any  $i > n$ .*

*Proof.* It suffices to prove the theorem for  $\mathcal{Y} = \text{Spec}(A)$ , where  $A$  is a Noetherian ring. Since  $f$  is a proper morphism,  $R^i f_* \mathcal{F}$  is a finite  $A$ -module for any  $\mathcal{F} \in \text{Coh}(\mathcal{X})$  [7]. To prove  $(R^i f_* \mathcal{F}) = 0$  it suffices to prove that  $(R^i f_* \mathcal{F})_{\mathfrak{m}} = 0$  for all maximal ideals  $\mathfrak{m} \subset A$ . Since completion  $A_{\mathfrak{m}} \rightarrow \hat{A}_{\mathfrak{m}}$  is faithfully flat, it suffices to prove that  $(R^i f_* \mathcal{F})_{\mathfrak{m}}^{\hat{}} = (R^i f_* \mathcal{F}) \otimes \hat{A}_{\mathfrak{m}} = 0$ . Let  $\mathcal{X}_n = \mathcal{X} \times_{\text{Spec}(A)} \text{Spec}(A/\mathfrak{m}^n)$ ,  $\mathcal{F}_n = \mathcal{F}|_{\mathcal{X}_n}$ , and  $f_n = f \times_{\text{Spec}(A)} A/\mathfrak{m}^n: \mathcal{X}_n \rightarrow \text{Spec}(A/\mathfrak{m}^n)$ . Consider the pro-system  $\{\ker(\varphi_n)\}_n$ , where  $\varphi_n$  is the map

$$\varphi_n: \frac{R^i f_* \mathcal{F}}{\mathfrak{m}^n R^i f_* \mathcal{F}} \rightarrow R^i f_{n*} \mathcal{F}_n$$

induced by the map  $\mathcal{F} \rightarrow \mathcal{F}_n$ . Note that the proof of [23, Tag 08AY (i)] holds since derived pushforward along proper morphisms of Noetherian stacks preserves the subcategory of coherent sheaves [7]. The aforementioned lemma implies that the pro-system  $\{\ker(\varphi_n)\}_n$  is null. Thus we get an injective map on taking the limit

$$(R^i f_* \mathcal{F})_{\mathfrak{m}}^{\hat{}} \hookrightarrow \varprojlim_n R^i f_{n*} \mathcal{F}_n.$$

Thus it suffices to show that  $R^i f_{n*} \mathcal{F}_n = 0$  for all  $i > n$ . We have the following sequence of  $A/\mathfrak{m}^n$ -modules

$$0 \rightarrow \mathfrak{m}^{n-1}/\mathfrak{m}^n \rightarrow A/\mathfrak{m}^n \rightarrow A/\mathfrak{m}^{n-1} \rightarrow 0.$$

Tensoring with  $\mathcal{O}_{\mathcal{X}}$ , we get the following exact sequence of coherent  $\mathcal{O}_{\mathcal{X}_n}$ -modules

$$\mathcal{G} \rightarrow \mathfrak{F}_n \rightarrow \mathcal{F}_{n-1} \rightarrow 0.$$

Note that  $\mathcal{G}$  is supported on  $\mathcal{X}_{n-1}$ . With  $\mathcal{G}' = \text{im}(\mathcal{G} \rightarrow \mathfrak{F}_n)$ , we get a short exact sequence in which the extreme terms are supported on  $\mathcal{X}_{n-1}$

$$0 \rightarrow \mathcal{G}' \rightarrow \mathcal{F}_n \rightarrow \mathcal{F}_{n-1} \rightarrow 0.$$

Thus we get a long exact sequence

$$\cdots \rightarrow R^i f_{n-1*} \mathcal{G}' \rightarrow R^i f_{n*} \mathcal{F}_n \rightarrow R^i f_{n-1*} \mathcal{F}_{n-1} \rightarrow R^{i+1} f_{n-1*} \mathcal{G}' \rightarrow \cdots .$$

Now we can complete the proof by induction on  $n$ . ■

**Funding.** The first named author was partially supported by the INSPIRE faculty fellowship(Reference number IFA21-MA 157) funded by the DST, Govt. of India. The second named author was supported by postdoctoral fellowships at Chennai Mathematical Institute, Chennai, India and University at Buffalo, US.

## References

- [1] D. Abramovich, M. Olsson, and A. Vistoli, [Tame stacks in positive characteristic](#). *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 4, 1057–1091 Zbl [1222.14004](#) MR [2427954](#)
- [2] D. Ben-Zvi, J. Francis, and D. Nadler, [Integral transforms and Drinfeld centers in derived algebraic geometry](#). *J. Amer. Math. Soc.* **23** (2010), no. 4, 909–966 Zbl [1202.14015](#) MR [2669705](#)
- [3] D. Bragg, [Fourier-Mukai transforms commuting with Frobenius](#). *Bull. Lond. Math. Soc.* **56** (2024), no. 11, 3477–3483 Zbl [1568.14036](#) MR [4828027](#)
- [4] J. Calabrese, [Relative singular twisted Bondal-Orlov](#). *Math. Res. Lett.* **25** (2018), no. 2, 393–414 Zbl [1423.14117](#) MR [3826827](#)
- [5] J. Calabrese and M. Groechenig, [Moduli problems in abelian categories and the reconstruction theorem](#). *Algebr. Geom.* **2** (2015), no. 1, 1–18 Zbl [1322.14038](#) MR [3322195](#)
- [6] A. Canonaco and P. Stellari, [Twisted Fourier-Mukai functors](#). *Adv. Math.* **212** (2007), no. 2, 484–503 Zbl [1116.14009](#) MR [2329310](#)
- [7] G. Faltings, [Finiteness of coherent cohomology for proper fppf stacks](#). *J. Algebraic Geom.* **12** (2003), no. 2, 357–366 Zbl [1086.14003](#) MR [1949648](#)
- [8] J. Hall, [Cohomology and base change for algebraic stacks](#). *Math. Z.* **278** (2014), no. 1–2, 401–429 Zbl [1346.14005](#) MR [3267585](#)
- [9] J. Hall, A. Neeman, and D. Rydh, [One positive and two negative results for derived categories of algebraic stacks](#). *J. Inst. Math. Jussieu* **18** (2019), no. 5, 1087–1111 Zbl [1440.14083](#) MR [3995721](#)
- [10] J. Hall and D. Rydh, [Algebraic groups and compact generation of their derived categories of representations](#). *Indiana Univ. Math. J.* **64** (2015), no. 6, 1903–1923 Zbl [1348.14045](#) MR [3436239](#)
- [11] J. Hall and D. Rydh, [Perfect complexes on algebraic stacks](#). *Compos. Math.* **153** (2017), no. 11, 2318–2367 Zbl [1390.14057](#) MR [3705292](#)
- [12] D. Huybrechts, [Fourier-Mukai transforms in algebraic geometry](#). Oxford Math. Monogr., The Clarendon Press, Oxford University Press, Oxford, 2006 Zbl [1095.14002](#) MR [2244106](#)
- [13] P. Lank and F. Peng, [Frobenius generation for algebraic stacks](#). 2025, arXiv:[2512.05026v2](#)
- [14] G. Laumon and L. Moret-Bailly, [Champs algébriques](#). *Ergeb. Math. Grenzgeb.* (3) 39, Springer, Berlin, 2000 Zbl [0945.14005](#) MR [1771927](#)
- [15] G. Laumon and B. C. Ngô, [Le lemme fondamental pour les groupes unitaires](#). *Ann. of Math.* (2) **168** (2008), no. 2, 477–573 Zbl [1179.22019](#) MR [2434884](#)
- [16] M. Lieblich, [Moduli of twisted sheaves and generalized Azumaya algebras](#). Ph.D. thesis, Massachusetts Institute of Technology, 2004. <https://dspace.mit.edu/handle/1721.1/30145> visited on 20 April 2026
- [17] M. Lieblich, [Twisted sheaves and the period-index problem](#). *Compos. Math.* **144** (2008), no. 1, 1–31 Zbl [1133.14018](#) MR [2388554](#)
- [18] M. Olsson, [Sheaves on Artin stacks](#). *J. Reine Angew. Math.* **603** (2007), 55–112 Zbl [1137.14004](#) MR [2312554](#)

- [19] M. Olsson, *Algebraic spaces and stacks*. Amer. Math. Soc. Colloq. Publ. 62, American Mathematical Society, Providence, RI, 2016 Zbl 1346.14001 MR 3495343
- [20] D. O. Orlov, *Equivalences of derived categories and K3 surfaces*. *J. Math. Sci. (New York)* **84** (1997), no. 5, 1361–1381 Zbl 0938.14019 MR 1465519
- [21] F. Peng, *Equivalences of derived categories of sheaves on tame stacks*. 2024, arXiv:2405.19676v1
- [22] A. Rizzardo, M. Van den Bergh, and A. Neeman, *An example of a non-Fourier-Mukai functor between derived categories of coherent sheaves*. *Invent. Math.* **216** (2019), no. 3, 927–1004 Zbl 1504.14037 MR 3955712
- [23] The Stacks project, <https://stacks.math.columbia.edu> visited on 20 April 2026
- [24] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*. *Invent. Math.* **167** (2007), no. 3, 615–667 Zbl 1118.18010 MR 2276263
- [25] V. Vologodsky, *Triangulated endofunctors of the derived category of coherent sheaves which do not admit DG liftings*. *Arnold Math. J.* **5** (2019), no. 2–3, 387–391 Zbl 1439.18019 MR 4031363

Communicated by Henning Krause

Received 14 January 2025; revised 19 October 2025.

### Chandranandan Gangopadhyay

Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, 201314, India; [chandranandan.g@snu.edu.in](mailto:chandranandan.g@snu.edu.in)

Author IDs: zbMATH [gangopadhyay.chandranandan](https://zbmath.org/authors/?q=gangopadhyay.chandranandan) MR 1292182 ORCID 0009-0006-2696-546x

### Ankit Rai

Department of Mathematics, University at Buffalo, Mathematics Building, North Campus, Buffalo, NY 14260-2900, USA; [arai22@buffalo.edu](mailto:arai22@buffalo.edu)

Author IDs: zbMATH [rai.ankit](https://zbmath.org/authors/?q=raiki) MR 1570544 ORCID 0009-0001-9197-8893