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Sharp L^p estimates for generalized Steklov eigenfunctions with an application to nodal sets

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Abstract. We study a generalized Steklov problem involving a rough potential on the boundary. We establish sharp L^p estimates for the Steklov eigenfunctions on compact manifolds with boundary, controlled by their L^2 norms on the boundary. We first establish sharp boundary estimates by heat kernel bounds and resolvent estimates for the Dirichlet-to-Neumann operator with a rough potential. And then we combine harmonic extension with the Littlewood–Paley decomposition to obtain sharp interior estimates. These results are new even when there is no potential. As an application, we prove the eigenfunctions are C^1 if the potential is Lipschitz and refine the previous results by Wang and Zhu (2015) on the lower bound of the size of the boundary nodal sets. A key tool is the commutator estimate for first-order pseudo-differential operators by Calderón (1965), and Coifman and Meyer (1978).

1. Introduction

Eigenfunction estimates have been recently considered in the case of Schrödinger operators with singular potentials (see e.g., [3,4,29,38–41,53]). In the present paper, we investigate a generalization of the well-known Steklov problem with rough potentials. For surveys on the Steklov problem, see e.g., [13,34].

Let (Ω, h) be a smooth manifold with boundary (M, g), where $\dim \Omega = n+1 \geq 2$ and $h|_M = g$. The Steklov eigenvalue problem with potential V is

$$\begin{cases} \Delta_h e_\lambda(x) = 0, & x \in \Omega, \\ \partial_\nu e_\lambda(x) + V(x) e_\lambda(x) = \lambda e_\lambda(x), & x \in \partial\Omega = M. \end{cases}$$

Here ν is a unit outer normal vector on M. Then the restriction of the eigenfunction $e_\lambda(x)$ (denoted also by e_λ to simplify notations) to the boundary M is an eigenfunction of $\mathcal{D} + V$:

$$(\mathcal{D} + V) e_\lambda(x) = \lambda e_\lambda(x), \ x \in M.$$

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Here $\mathscr{D}$ is the Dirichlet-to-Neumann operator $\mathscr{D}\colon H^{1/2}(M) \to H^{-1/2}(M)$,

$$\mathscr{D} f = \partial_\nu u|_M,$$

where u is the harmonic extension of f:

$$\begin{cases} \Delta_h u(x) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega = M. \end{cases} \tag{1.1}$$

Such a type of Steklov problem with potential has been considered in [15] from the point of view of conformal geometry, where the potential V is the mean curvature on the boundary $\partial\Omega$. See e.g., [21–23, 47] for related works on Yamabe problem on compact manifolds with boundary. In the current paper, we derive estimates whenever the potential is merely bounded or Lipschitz.

For $m \in \mathbb{R}$, we denote OPS^m the class of pseudo-differential operators of order m. It is known that $\mathscr{D} \in \mathrm{OPS}^1$ and one can write (see e.g., [66, Proposition C.1])

$$\mathscr{D} = \sqrt{-\Delta_g} + P_0,$$

for some $P_0 \in \mathrm{OPS}^0$. Therefore, up to a classical pseudo-differential operator of order zero, the problem of eigenfunction bounds (among other results) on the boundary M has been treated in our previous paper [38]. In this setting, the model is related to relativistic matter (see e.g., [11, 16, 27, 28, 42, 43]).

In our first result below, we provide a control of the L^p norms of the Steklov eigenfunctions in the domain by their L^2 norms on the boundary.

Theorem 1. *Let $V \in L^\infty(M)$. Then for $\lambda \geq 1$, we have*

$$\|e_\lambda\|_{L^p(\Omega)} \lesssim \lambda^{-1/p+\sigma(p)} \|e_\lambda\|_{L^2(M)}, \quad 2 \leq p \leq \infty, \tag{1.2}$$

where

$$\sigma(p) = \begin{cases} \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{p}\right), & 2 \leq p < \dfrac{2(n+1)}{n-1}, \\ \dfrac{n-1}{2} - \dfrac{n}{p}, & \dfrac{2(n+1)}{n-1} \leq p \leq \infty. \end{cases}$$

This result is new, even for $V \equiv 0$. Note that the estimate is sharp when $V \equiv 0$ and Ω is the unit ball $B(0,1) \subset \mathbb{R}^{n+1}$ with boundary $M = S^n$. In this case, the Steklov eigenfunction $e_\lambda(x) = r^k e_k(\omega)$ in the polar coordinate $r \in [0,1]$, $\omega \in S^n$, where $\lambda^2 = k(k+n-1)$, $k \in \mathbb{N}$ and $e_k(\omega)$ is a spherical harmonic of degree k, that is, the restriction to S^n of homogeneous harmonic polynomials of degree k. It is straightforward to see that

$$\|e_\lambda\|_{L^p(B(0,1))} \approx \lambda^{-1/p} \|e_\lambda\|_{L^p(S^n)}. \tag{1.3}$$

The L^p estimates of the Laplacian eigenfunctions on compact manifolds were proved by Sogge [55], and they are sharp on S^n

$$\|e_\lambda\|_{L^p(S^n)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(S^n)}, \tag{1.4}$$

and they are saturated by zonal spherical harmonic for $p \geq \frac{2(n+1)}{n-1}$ and highest weight spherical harmonic for $p \leq \frac{2(n+1)}{n-1}$ (see e.g., [56]). Thus, combining (1.3) with (1.4), we see that (1.2) is sharp.

The motivation for this result is to investigate the feature that Steklov eigenfunctions concentrate near the boundary, and rapidly decay away from the boundary (see e.g., [19, 30, 31, 37, 50]). Motivated by the elliptic inverse boundary value problems such as Calderón problem (see e.g., [10, 46]), Hislop and Lutzer [37] proved that for any compact set $K \subset \Omega$,

$$\|e_\lambda\|_{L^2(K)} \leq C_N \lambda^{-N} \|e_\lambda\|_{L^2(M)} \quad \text{for all } N.$$

This bound reflects the fact that the Steklov eigenfunctions become highly oscillatory as the eigenvalue increases, hence they decay rapidly away from the boundary. Hislop and Lutzer [37] conjectured that the decay is actually of order $e^{-\lambda d_h(K,\partial\Omega)}$. One may see by examining the case of unit ball $B(0,1) \subset \mathbb{R}^{n+1}$ that the exponential decay is optimal. For real-analytic surfaces ($n = 1$), Polterovich, Sher, and Toth [50] obtained a pointwise bound and the eigenfunction decay is a key feature in their main results on nodal length. They proved that for any real-analytic compact surface Ω with boundary $M = \partial\Omega$, there exist constants $C, c > 0$ depending only on Ω, such that

$$|e_\lambda(x)| \leq C e^{-c\lambda d_g(x,\partial\Omega)} \|e_\lambda\|_{L^2(M)}.$$

Their methods are specific to the case of real-analytic surfaces. A different method of proving this bound was communicated to them by M. Taylor. See also Galkowski and Toth [30] for recent results in higher-dimensional real-analytic manifolds. Moreover, this interesting concentration feature is also related to the restriction estimates of eigenfunctions to submanifolds (see e.g., [7, 8, 57, 64]).

To prove Theorem 1, we will need to establish the following boundary L^p estimates. These types of estimates are important in their own right (see surveys e.g., [59, 72]).

Lemma 1. *If $V \in L^\infty(M)$, then the following two eigenfunction estimates hold:*

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)}, \quad 2 \leq p \leq \infty. \tag{1.5}$$

$$\|e_\lambda\|_{L^1(M)} \gtrsim \lambda^{-(n-1)/4} \|e_\lambda\|_{L^2(M)}. \tag{1.6}$$

Both (1.5) and (1.6) are sharp on S^n. Indeed, they can be saturated by zonal spherical harmonic or highest weight spherical harmonic (see e.g., [56,59]). For smooth V, (1.5) was proved by Seeger and Sogge [51]. They obtained the eigenfunction estimates for self-adjoint elliptic pseudo-differential operators satisfying a convexity assumption on the principal symbol. However, to obtain Lemma 1 for rough V, one has to use a different approach. We prove it by establishing new heat kernel estimates and resolvent estimates for $\mathcal{D} + V$.

Remark 1. As noticed in [38, Remark 1], the resolvent method cannot efficiently handle singular potentials when the order α of the leading term $(-\Delta_g)^{\alpha/2}$ is too small, say $\alpha < \frac{2n}{n+1}$. Here we encounter the same difficulty and can only handle bounded potentials, since Dirichlet-to-Neumann operator has order $\alpha = 1$.

Lemma 2. *Let* $2 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* $V \in L^{np'}(M)$, *then*

$$\|e_\lambda\|_{L^p(\Omega)} \lesssim \lambda^{-1/p} \|e_\lambda\|_{L^p(M)}. \tag{1.7}$$

From (1.3), we see that the estimate (1.7) is sharp for $\Omega = B(0,1)$. The endpoint $p = \infty$ follows from the maximum principle, since e_λ is harmonic in Ω. The other endpoint $p = 2$ can be obtained from the trace theorem and standard regularity estimates. Then (1.7) is proved by an interpolation argument involving the harmonic extension and the Littlewood–Paley decomposition.

Remark 2. It is worth mentioning that it is possible to extend Lemma 2 to more singular potentials (e.g., Kato class) by an interpolation theorem between Besov spaces [2, Theorem 6.4.5 (6)]. By Remark 1 and for simplicity, we shall give a direct proof for Lemma 2 that already suffices for our purpose.

Next, in the second theorem we exploit the boundary eigenfunction estimates in Lemma 1 to estimate the Hausdorff measure of the boundary nodal sets for Lipschitz potentials.

Theorem 2. *If* $V \in \mathrm{Lip}^1(M)$, *then we have the following lower bound for the Hausdorff measure of the boundary nodal set* $N_\lambda = \{x \in M : e_\lambda(x) = 0\}$,

$$\mathcal{H}^{n-1}(N_\lambda) \gtrsim \lambda^{(3-n)/2}. \tag{1.8}$$

This refines the previous results of Wang and Zhu [70]. Inspired by Yau's conjecture for nodal sets, one may expect that the "correct" bound should be $\sim \lambda$. To our knowledge, (1.8) is still the best lower bound estimate for the boundary nodal sets of the Steklov eigenfunctions up to now. The difficulty lies in the nonlocal property of the Dirichlet-to-Neumann operator. Unlike the smooth case, the rough potential requires us to investigate the regularity of eigenfunctions before estimating the nodal sets. We first prove that the eigenfunctions are C^1 if the potential V is Lipschitz (Lemma 8), and then estimate the size of nodal sets by applying the Gauss–Green theorem on nodal domains. A key tool is the commutator estimate (Lemma 7) for first-order pseudo-differential operators by Calderón [9] and Coifman and Meyer [12]. We also establish a useful result on the equivalence between two kinds of Sobolev norms on compact manifolds (Proposition 2), which is interesting in its own right. We cannot find this basic result in the literature, so we give a detailed proof for completeness.

The paper is structured as follows. In Section 2, we prove Lemma 1. In Section 3, we prove Lemma 2. In Section 4, we apply these estimates to the study of boundary nodal sets. In Appendix A, we establish heat kernel bounds. In Appendix B, we establish the kernel estimates of pseudo-differential operators.

Throughout this paper, $X \lesssim Y$ (or $X \gtrsim Y$) means $X \le CY$ (or $X \ge CY$) for some positive constant C independent of λ. This constant may depend on V and the domain Ω. $X \approx Y$ means $X \lesssim Y$ and $X \gtrsim Y$.

2. Boundary eigenfunction estimates: proof of Lemma 1

To prove Lemma 1, we begin with the following resolvent estimate.

Proposition 1. *For $\lambda \ge 1$, we have*

$$\|(\sqrt{-\Delta_g} - (\lambda + i))^{-1}\|_{L^2 \to L^p} \lesssim \lambda^{\sigma(p)}, \; 2 < p \le \frac{2(n+1)}{n-1}, \tag{2.1}$$

where

$$\sigma(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \le p < \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \le p \le \infty. \end{cases}$$

Proof. For $k \in \mathbb{N}$, let $\chi_{[k,k+1)}$ be the spectral projection operator for $\sqrt{-\Delta_g}$ corresponding to the spectral interval $[k, k+1)$, and let $\chi_{[2[\lambda],\infty)}$ be spectral projection operator onto the interval $[2[\lambda], \infty)$, where $[\lambda]$ denotes the largest integer that is smaller than λ. Then for any function f, by Cauchy–Schwarz inequality,

$$\begin{aligned} &(\sqrt{-\Delta_g} - (\lambda + i))^{-1} f \\ &\quad = \sum_{k<2[\lambda]} \frac{1}{k - (\lambda + i)} (k - (\lambda + i))(\sqrt{-\Delta_g} - (\lambda + i))^{-1} \chi_{[k,k+1)} f \\ &\qquad\qquad + \chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g} - (\lambda + i))^{-1} f \\ &\quad \lesssim \Big(\sum_{k<2[\lambda]} |(k - (\lambda + i))(\sqrt{-\Delta_g} - (\lambda + i))^{-1} \chi_{[k,k+1)} f|^2\Big)^{1/2} \\ &\qquad\qquad + |\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g} - (\lambda + i))^{-1} f|. \end{aligned}$$

Thus, by Minkowski's inequality,

$$\begin{aligned}
&\|(\sqrt{-\Delta_g}-(\lambda+i))^{-1}f\|_{L^p}\\
&\quad\le\Big(\sum_{k<2[\lambda]}\|(k-(\lambda+i))(\sqrt{-\Delta_g}-(\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^p}^2\Big)^{1/2}\\
&\qquad+\|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g}-(\lambda+i))^{-1}f\|_{L^p}.
\end{aligned}$$

To handle the first term on the right, note that $\chi_{[k,k+1)}=\chi_{[k,k+1)}\circ\chi_{[k,k+1)}$, and by the classical results in [55],

$$\|\chi_{[k,k+1)}f\|_{L^p}\lesssim(1+k)^{\sigma(p)}\|f\|_{L^2}\lesssim\lambda^{\sigma(p)}\|f\|_{L^2},\quad \text{if } k<2[\lambda].$$

Thus,

$$\begin{aligned}
&\Big(\sum_{k<2[\lambda]}\|(k-(\lambda+i))(\sqrt{-\Delta_g}-(\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^p}^2\Big)^{1/2}\\
&\quad\lesssim\lambda^{\sigma(p)}\Big(\sum_{k<2[\lambda]}\|(k-(\lambda+i))(\sqrt{-\Delta_g}-(\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^2}^2\Big)^{1/2}\\
&\quad\lesssim\lambda^{\sigma(p)}\Big(\sum_{k<2[\lambda]}\|\chi_{[k,k+1)}f\|_{L^2}^2\Big)^{1/2}ls\lambda^{\sigma(p)}\|f\|_{L^2},
\end{aligned}$$

where in the second inequality we used the fact that by spectral theorem,

$$\|(k-(\lambda+i))(\sqrt{-\Delta_g}-(\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^2}\lesssim\|\chi_{[k,k+1)}f\|_{L^2}\quad\text{for all } k\in\mathbb{N}.$$

To handle the second term, we use Sobolev estimates to see that

$$\begin{aligned}
&\|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g}-(\lambda+i))^{-1}f\|_{L^p}\\
&\quad\lesssim\|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g})^{n(\frac12-\frac1p)}(\sqrt{-\Delta_g}-(\lambda+i))^{-1}f\|_{L^2}.
\end{aligned}$$

When $2<p\le\frac{2(n+1)}{n-1}$, it is straightforward to check that $n\big(\frac12-\frac1p\big)<1$, thus by spectral theorem,

$$\|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g})^{n(\frac12-\frac1p)}(\sqrt{-\Delta_g}-(\lambda+i))^{-1}f\|_{L^2}\lesssim\|f\|_{L^2},$$

which is better than the desired bound in (2.1). ■

2.1. Proof of Lemma 1

It follows from similar strategies as in [3]. Recall that $\mathcal{D}=\sqrt{-\Delta_g}+P_0$; by using the second resolvent formula, we have

$$\begin{aligned}
&(\mathcal{D}+V-(\lambda+i))^{-1}\\
&\quad=(\sqrt{-\Delta_g}-(\lambda+i))^{-1}-(\sqrt{-\Delta_g}-(\lambda+i))^{-1}(P_0+V)(\mathcal{D}+V-(\lambda+i))^{-1}.
\end{aligned}\tag{2.2}$$

Since $P_0 \in \mathrm{OPS}^0$ and the eigenvalues of $\mathcal{D} + V$ are real, by spectral theorem, we have

$$\|P_0(\mathcal{D} + V - (\lambda + i))^{-1}\|_{L^2 \to L^2} \lesssim \|(\mathcal{D} + V - (\lambda + i))^{-1}\|_{L^2 \to L^2} \lesssim 1. \tag{2.3}$$

Similarly, since $V \in L^\infty(M)$, we have

$$\|V(\mathcal{D} + V - (\lambda + i))^{-1}\|_{L^2 \to L^2} \lesssim 1. \tag{2.4}$$

Thus, (2.2), (2.3), (2.4), and (2.1) yield that

$$\|(\mathcal{D} + V - (\lambda + i))^{-1}\|_{L^2 \to L^p} \lesssim \lambda^{\sigma(p)}, \ 2 < p \le \frac{2(n+1)}{n-1}. \tag{2.5}$$

If we let $\chi^V_{[\lambda,\lambda+1)}$ denote the spectral projection operator associated with

$$\sqrt{-\Delta_g} + P_0 + V$$

for the interval $[\lambda, \lambda + 1)$, then (2.5) implies

$$\|\chi^V_{[\lambda,\lambda+1)} f\|_{L^p} \lesssim \lambda^{\sigma(p)} \|f\|_{L^2}, \quad 2 < p \le \infty. \tag{2.6}$$

Note that if we take $f = e_\lambda$ in (2.6), and use the fact that $\chi^V_{[\lambda,\lambda+1)} e_\lambda = e_\lambda$, we obtain (1.5). We postpone the proof of (2.6). By using the arguments from Sogge and Zelditch [62], we note that (1.6) can be obtained from Hölder's inequality and (1.5)

$$\begin{aligned}\|e_\lambda\|^{1/\theta}_{L^2(M)} &\le \|e_\lambda\|_{L^1(M)} \|e_\lambda\|^{1/\theta - 1}_{L^p(M)} \\ &\lesssim \|e_\lambda\|_{L^1(M)} (\lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)})^{1/\theta - 1} = \|e_\lambda\|_{L^1(M)} \lambda^{(n-1)/4} \|e_\lambda\|^{1/\theta-1}_{L^2(M)}.\end{aligned}$$

Here $2 < p < \frac{2(n+1)}{n-1}$, and $\theta = \frac{p}{p-1}\left(\frac{1}{2} - \frac{1}{p}\right)$.

Proof of (2.6). If $2 < p \le \frac{2(n+1)}{n-1}$, this follows from (2.5) by letting $f = \chi^V_{[\lambda,\lambda+1)} f$ there along with the fact that

$$\|(\mathcal{D} + V - (\lambda + i))\chi^V_{[\lambda,\lambda+1)}\|_{L^2 \to L^2} \lesssim 1.$$

If $p > \frac{2(n+1)}{n-1}$, we shall use the heat kernel bounds in Proposition 3. Indeed, the operator $H_V = \mathcal{D} + V$ with $V \in L^\infty(M)$ satisfies the heat kernel bound (A.2). Then Young's inequality implies

$$\|e^{-tH_V}\|_{L^q(M) \to L^p(M)} \lesssim t^{-n(1/q - 1/p)}, \quad \text{if } 0 < t \le 1 \text{ and } 1 \le q \le p \le \infty.$$

If we fix $t = \lambda^{-1}$ and $p_c = \frac{2(n+1)}{n-1}$, and apply the above bound, we have for $p > \frac{2(n+1)}{n-1}$,

$$\begin{aligned}
\|\chi^V_{[\lambda,\lambda+1)} f\|_{L^p} &\lesssim \lambda^{n(1/p_c - 1/p)} \|e^{\lambda^{-1} H_V} \chi^V_{[\lambda,\lambda+1)} f\|_{L^{p_c}} \\
&= \lambda^{n(1/p_c - 1/p)} \|\chi^V_{[\lambda,\lambda+1)} e^{\lambda^{-1} H_V} \chi^V_{[\lambda,\lambda+1)} f\|_{L^{p_c}} \\
&\lesssim \lambda^{n(1/p_c - 1/p)} \lambda^{(n-1)/2 - n/p_c} \|e^{\lambda^{-1} H_V} \chi^V_{[\lambda,\lambda+1)} f\|_{L^2} \\
&\lesssim \lambda^{(n-1)/2 - n/p} \|f\|_{L^2},
\end{aligned}$$

where in the third line we applied (2.6) at $p = p_c$ and in the last line we applied spectral theorem. Since $\frac{n-1}{2} - \frac{n}{p} = \sigma(p)$ when $p \geq p_c$, we complete the proof. ■

3. Interior eigenfunction estimates: Proof of Lemma 2

In this section, we prove Lemma 2, and then Theorem 1 follows from the L^p bounds in Lemma 1. To proceed, we shall use the following lemma.

Lemma 3. *For any $f \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the weak solution to the Dirichlet boundary value problem (1.1). Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C\|f\|_{H^{-1/2}(\partial\Omega)}.$$

This lemma was proved in [19, Proposition 2.17]. It follows from the trace theorem and standard regularity estimates (see e.g., [35, Theorem 1.5.1.2, Theorem 1.5.1.3, Corollary 2.2.2.4, and Corollary 2.2.2.6]).

Lemma 4. *Let $Q \in \mathrm{OPS}^0$. Then Q is bounded on L^p for $1 < p < \infty$, i.e.,*

$$\|Qf\|_{L^p} \leq C\|f\|_{L^p}.$$

Here the L^p norm can be taken on $\mathbb{R}^n$ and compact manifolds. See e.g., [60, Theorem 3.1.6 and Theorem 4.3.1] for the proofs.

We also need the kernel estimates of the pseudo-differential operators on compact manifolds.

Lemma 5. *Let $\mu \in \mathbb{R}$, and $m \in C^\infty(\mathbb{R})$ belong to the symbol class S^μ, that is, assume that*

$$|\partial_t^\alpha m(t)| \leq C_\alpha (1 + |t|)^{\mu - \alpha} \quad \text{for all } \alpha. \tag{3.1}$$

If $P = \sqrt{-\Delta_g}$, *then* $m(P)$ *is a pseudo-differential operator of order* μ. *Moreover, if* $R \geq 1$, *then the kernel of the operator* $m(P/R)$ *satisfies for all* $N \in \mathbb{N}$,

$$\left|m\Big(\frac{P}{R}\Big)(x,y)\right| \lesssim \begin{cases} R^n(Rd_g(x,y))^{-n-\mu}(1+Rd_g(x,y))^{-N}, & n+\mu>0, \\ R^n\log(2+(Rd_g(x,y))^{-1})(1+Rd_g(x,y))^{-N}, & n+\mu=0, \\ R^n(1+Rd_g(x,y))^{-N}, & n+\mu<0. \end{cases} \tag{3.2}$$

We shall give a proof of this lemma in Appendix B.

3.1. Proof of Lemma 2

It suffices to consider two cases, $p=\infty$ and $p<\infty$.

Case 1. $p=\infty$. From the maximum principle (see e.g., [32, Theorem 8.1]), since e_λ is harmonic in Ω. We get

$$\|e_\lambda\|_{L^\infty(\Omega)} \lesssim \|e_\lambda\|_{L^\infty(\partial\Omega)},$$

and since $V \in L^\infty(M)$, by Lemma 1, we have

$$\|e_\lambda\|_{L^\infty(\partial\Omega)} \lesssim \lambda^{(n-1)/2}\|e_\lambda\|_{L^2(\partial\Omega)},$$

which yields (1.2) for the case $p=\infty$.

Case 2. $p<\infty$. Let us fix a Littlewood–Paley bump function $\beta \in C_0^\infty\big((\frac{1}{2},2)\big)$ satisfying

$$\sum_{\ell=-\infty}^{\infty}\beta(2^{-\ell}s)=1, \quad s>0,$$

and define

$$\beta_0(s) = 1-\sum_{\ell>0}\beta(2^{-\ell}|s|),$$
$$\beta_\ell(s) = \beta(2^{-\ell}|s|), \quad \text{for } \ell>0.$$

Let $P=\sqrt{-\Delta_g}$. Then we have for $\ell \geq 0$,

$$\|\beta_\ell(P)f\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}, \quad 1 \leq p \leq \infty. \tag{3.3}$$

The implicit constant is independent of ℓ. Indeed, by Lemma 5 with $\mu=-n-1$ and $R=2^\ell$, we have

$$|\beta_\ell(P)(x,y)| \lesssim 2^{n\ell}(1+2^\ell d_g(x,y))^{-N} \quad \text{for all } N. \tag{3.4}$$

Then, (3.3) follows from Young's inequality.

Let T_H be the harmonic extension operator from $\partial\Omega$ to Ω. Then by Lemma 3, we have

$$\|T_H(\beta_\ell(P)f)\|_{L^2(\Omega)} \lesssim \|\beta_\ell(P)f\|_{H^{-1/2}(\partial\Omega)} \lesssim 2^{-\ell/2}\|f\|_{L^2(\partial\Omega)}. \tag{3.5}$$

And from the maximal principle and (3.3), we have

$$\|T_H(\beta_\ell(P)f)\|_{L^\infty(\Omega)} \lesssim \|\beta_\ell(P)f\|_{L^\infty(\partial\Omega)} \lesssim \|f\|_{L^\infty(\partial\Omega)}. \tag{3.6}$$

By (3.5), (3.6), and interpolation, we have the following L^p estimate of the frequency-localized harmonic extension operator:

$$\|T_H(\beta_\ell(P)f)\|_{L^p(\Omega)} \lesssim 2^{-\ell/p}\|f\|_{L^p(\partial\Omega)},\ 2 \le p \le \infty. \tag{3.7}$$

Thus, if $2^\ell \gtrsim \lambda$, we have

$$\Big\|T_H\Big(\sum_{2^\ell\gtrsim\lambda}\beta_\ell(P)e_\lambda\Big)\Big\|_{L^p(\Omega)} \lesssim \sum_{2^\ell\gtrsim\lambda} 2^{-\ell/p}\|e_\lambda\|_{L^p(\partial\Omega)} \lesssim \lambda^{-1/p}\|e_\lambda\|_{L^p(\partial\Omega)}.$$

So it remains to consider $2^\ell \lesssim \lambda$. Let $\tilde\beta \in C_0^\infty$ with $\tilde\beta \equiv 1$ in a neighborhood of $\left(\frac{1}{2}, 2\right)$ and define $\tilde\beta_\ell(s) = \tilde\beta(2^{-\ell}|s|)$. Then by (3.7),

$$\begin{aligned}\|T_H(\beta_\ell(P)e_\lambda)\|_{L^p(\Omega)} &= \|T_H(\beta_\ell(P)\tilde\beta_\ell(P)e_\lambda)\|_{L^p(\Omega)}\\ &\lesssim 2^{-\ell/p}\|\tilde\beta_\ell(P)e_\lambda\|_{L^p(\partial\Omega)}.\end{aligned} \tag{3.8}$$

Moreover, for $2 \le p < \infty$,

$$\begin{aligned}&\|\tilde\beta_\ell(P)e_\lambda\|_{L^p(\partial\Omega)}\\ &\quad= (1+\lambda)^{-1}\|\tilde\beta_\ell(P)(1+\sqrt{-\Delta_g}+P_0+V)e_\lambda\|_{L^p(\partial\Omega)}\\ &\quad\lesssim (1+\lambda)^{-1}\|\tilde\beta_\ell(P)(1+\sqrt{-\Delta_g})e_\lambda\|_{L^p(\partial\Omega)}\\ &\qquad+(1+\lambda)^{-1}\|\tilde\beta_\ell(P)(P_0+V)e_\lambda\|_{L^p(\partial\Omega)}\\ &\quad\lesssim (1+\lambda)^{-1}2^\ell\|e_\lambda\|_{L^p(\partial\Omega)}+(1+\lambda)^{-1}\|e_\lambda\|_{L^p(\partial\Omega)}\\ &\qquad+(1+\lambda)^{-1}\|\tilde\beta_\ell(P)(Ve_\lambda)\|_{L^p(\partial\Omega)},\end{aligned}$$

where we use (3.3) and Lemma 4. For the third term, by using (3.4) with $2^\ell \lesssim \lambda$ and Young's inequality, we obtain

$$\begin{aligned}(1+\lambda)^{-1}\|\tilde\beta_\ell(P)(Ve_\lambda)\|_{L^p(\partial\Omega)} &\lesssim (1+\lambda)^{-1+1/p'}\|Ve_\lambda\|_{L^q(\partial\Omega)}\\ &\lesssim (1+\lambda)^{-1/p}\|V\|_{L^{np'}(\partial\Omega)}\|e_\lambda\|_{L^p(\partial\Omega)}.\end{aligned}$$

Here $\frac{1}{q} = \frac{1}{np'} + \frac{1}{p}$.

Combining these with (3.8) , we obtain

$$\Big\| T_H\Big(\sum_{2^\ell \lesssim \lambda} \beta_\ell(P)e_\lambda\Big)\Big\|_{L^p(\Omega)} \lesssim \sum_{2^\ell \lesssim \lambda} 2^{-\ell/p}\|\tilde{\beta}_\ell(P)e_\lambda\|_{L^p(\partial\Omega)} \lesssim \lambda^{-1/p}\|e_\lambda\|_{L^p(\partial\Omega)}.$$

So we obtain (1.7) in Lemma 2.

4. Applications to nodal sets

Inspired by Yau's conjecture on the Hausdorff measure of nodal sets of Laplace eigenfunctions ([20, 36, 44, 45, 71]), an analogous question has been asked for nodal sets of Steklov eigenfunctions (see e.g., [13, 34]). The study of the boundary nodal set

$$N_\lambda = \{x \in M : e_\lambda(x) = 0\}$$

was largely initiated by Bellova and Lin [1]. They conjectured that the $(n-1)$-dimensional Hausdorff measure

$$\mathcal{H}^{n-1}(N_\lambda) \approx \lambda.$$

The optimal upper bound $\mathcal{H}^{n-1}(N_\lambda) \lesssim \lambda$ was proved by Zelditch [73] for real analytic manifolds. See also the results by Sogge, Wang, and Zhu [61], Zhu [74], and Decio [17, 18] for the references on interior nodal sets. In this section, we will apply the eigenfunction estimates from the previous sections to study the measure of the boundary nodal set of generalized Steklov eigenfunctions in Theorem 2.

When $V \equiv 0$, Wang and Zhu [70] proved (1.8) for smooth manifolds under the assumption that zero is a regular value. Their proof follows from the idea in Sogge and Zelditch [62] (see also Colding and Minicozzi [14]), and the assumption that zero is a regular value is used to ensure the validity of the Gauss–Green theorem. Recently, this assumption has been proved to be "generic" by Wang [69], and the proof is based on the transversality theorems of Uhlenbeck [67]. In this paper, we remove this assumption by following Sogge and Zelditch [62, Proof of Proposition 1], where they used the Gauss–Green theorem for domains with rough boundaries. We refer to Federer [26, Section 2.10.6, p. 173, Theorem 4.5.11, p. 506], Evans and Gariepy [25, Theorem 1 on p. 209], and Pfeffer [48, Theorem 5.19].

Unlike the smooth case, the low regularity of potential requires us to investigate the regularity of eigenfunctions before estimating the nodal set. The Lipschitz assumption is used to ensure that the eigenfunctions are C^1 (see Lemma 8), so that the restriction of ∇e_λ to the nodal set makes sense and that the nodal set N_λ is locally C^1 near the non-critical points of e_λ. These allow us to apply the Gauss–Green theorem.

Before proceeding to the proof, it is important to establish some general results for Sobolev spaces on compact manifolds. These results are interesting in their own right but we cannot find a direct reference, so detailed proofs are provided for completeness.

Let $s > 0$ and $1 < p < \infty$. We can define the Sobolev norm on M by local coordinates

$$\|f\|_{W^{s,p}(M)} = \sum_{\nu} \|(I-\Delta)^{s/2} f_\nu\|_{L^p(\mathbb{R}^n)}, \tag{4.1}$$

where $f_\nu = (\phi_\nu f) \circ \kappa_\nu^{-1}$, and $\{\phi_\nu\}$ is a partition of unity subordinate to a finite covering $M = \bigcup \Omega_\nu$, and $\kappa_\nu \colon \Omega_\nu \to \widetilde{\Omega}_\nu \subset \mathbb{R}^n$ is the coordinate map. For simplicity, we sometimes do not distinguish between Ω_ν and $\widetilde{\Omega}_\nu$, f_ν and $\phi_\nu f$, since they are identical up to the coordinate map.

Moreover, we can also define another Sobolev norm by pseudo-differential operators

$$\|f\|_{H^{s,p}(M)} = \|(I-\Delta_g)^{s/2} f\|_{L^p(M)}. \tag{4.2}$$

By [60, Theorem 4.3.1], we see that $(I-\Delta_g)^{s/2}$ is an invertible pseudo-differential operator of order s with elliptic principal symbol $(\sum g^{jk}(x)\xi_j\xi_k)^{s/2}$. Moreover, if we replace $(I-\Delta_g)^{s/2}$ in (4.2) by any invertible pseudo-differential operator of order s, then it still gives a comparable norm, by Lemma 4.

We prove that these two Sobolev norms are equivalent.

Proposition 2. *For $s > 0$ and $1 < p < \infty$, we have*

$$\|f\|_{W^{s,p}(M)} \approx \|f\|_{H^{s,p}(M)}.$$

The implicit constants are independent of f.

As a corollary, different partitions of unity and such coordinate atlases in the definition (4.1) give comparable norms. When $p = 2$, Proposition 2 follows from Plancherel theorem and the L^2-boundedness of zero order pseudo-differential operators, see e.g., [58, Section 4.2]. The case $p \neq 2$ is more complicated, and it is difficult to find good references. To prove this on our own, we start with the following key lemma. Roughly speaking, this lemma establishes a “linear relation” between any two pseudo-differential operators of the same order.

Lemma 6. *Let $s > 0$. Let V_1, V, Ω be open sets such that $\overline{V}_1 \subset V \subset \Omega$. Let $P_1, P \in \mathrm{OPS}^s$ with symbols supported in V_1, V respectively. If the principal symbol $\bar{p}(x,\xi)$ of P is elliptic on $\overline{V}_1$, i.e., for any $x \in \overline{V}_1$,*

$$\bar{p}(x,\xi) \neq 0 \quad \textit{for all } \xi \neq 0,$$

then there is a $Q \in \mathrm{OPS}^0$ with symbol supported in V_1 such that

$$P_1 - QP \in \mathrm{OPS}^0.$$

Proof. Let $p_1(x,\xi)$ be the symbols of P_1 on Ω. Since $\bar{p}(x,\xi)$ is elliptic on the support of $p_1(x,\xi)$, we have

$$\frac{\varphi(\xi)p_1(x,\xi)}{\bar{p}(x,\xi)} \in S^0$$

where $\varphi \in C^\infty$ vanishes near the origin but equals one near infinity. Denote the associated zero order pseudo-differential operator by Q_0. Let $R_{-1} = P_1 - Q_0 P$. Then by the Kohn–Nirenberg theorem (see e.g., [60, Theorem 3.1.1]), we have $R_{-1} \in \mathrm{OPS}^{s-1}$. The symbol of R_{-1} is supported in V_1. If $s \le 1$, then we are done by setting $Q = Q_0$, since $P_1 - Q_0 P \in \mathrm{OPS}^{s-1} \subset \mathrm{OPS}^0$.

Next, it remains to consider $s > 1$. Let $k = \lceil s \rceil \ge 2$. We need to construct $Q_{-i} \in \mathrm{OPS}^{-i}$, $R_{-i-1} \in \mathrm{OPS}^{s-i-1}$ recursively for $1 \le i \le k-1$. If $r_i(x,\xi)$ is the symbol of R_{-i}, and Q_{-i} has the symbol

$$\frac{\varphi(\xi)r_i(x,\xi)}{\bar{p}(x,\xi)} \in S^{-i},$$

then using the Kohn–Nirenberg theorem, we have $R_{-i-1} = R_{-i} - Q_{-i}P \in \mathrm{OPS}^{s-i-1}$. The symbol of R_{-i-1} is supported in V_1. Let

$$Q = \sum_{i=1}^{k-1} Q_{-i}.$$

The symbol of Q is supported in V_1. Then $P_1 - QP = R_{-k} \in \mathrm{OPS}^{s-k} \subset \mathrm{OPS}^0$. ∎

Proof of Proposition 2. The basic idea is to verify these two equivalences

$$\begin{aligned}\|(I-\Delta_g)^{s/2} f\|_{L^p(M)} &\approx \sum_\nu \|(I-\Delta_g)^{s/2} f_\nu\|_{L^p(M)} \\ &\approx \sum_\nu \|(I-\Delta)^{s/2} f_\nu\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{4.3}$$

The first equivalence is straightforward. Indeed, The relation $\lesssim$ follows from the Minkowski inequality. And for the other direction, we use Lemma 4 to see that

$$\begin{aligned}\|(I-\Delta_g)^{s/2} f_\nu\|_{L^p(M)} &= \|(I-\Delta_g)^{s/2} M_{\phi_\nu}(I-\Delta_g)^{-s/2}((I-\Delta_g)^{s/2} f)\|_{L^p(M)} \\ &\lesssim \|(I-\Delta_g)^{s/2} f\|_{L^p(M)}, \end{aligned} \tag{4.4}$$

where M_{ϕ_ν} stands for the operator of multiplying by $\phi_\nu(x)$. Summing up of (4.4) over ν, we obtain the first equivalence in (4.3).

To prove the second equivalence in (4.3), it suffices to show that for each ν

$$\|(I-\Delta_g)^{s/2} f_\nu\|_{L^p(M)} \approx \|(I-\Delta)^{s/2} f_\nu\|_{L^p(\mathbb{R}^n)}. \tag{4.5}$$

For each Ω_ν, $\phi_\nu \in C_0^\infty(\Omega_\nu)$ in (4.1), we can find open subsets V_ν, U_ν, W_ν of Ω_ν, and cutoff functions $\psi_\nu \in C_0^\infty(V_\nu)$, $\psi_{\nu 1} \in C_0^\infty(U_\nu)$, $\psi_{\nu 2} \in C_0^\infty(W_\nu)$, $\eta_\nu \in C_0^\infty(\Omega_\nu)$ such that

$$\operatorname{supp}\phi_\nu \subset\subset U_\nu \subset V_\nu \subset\subset W_\nu$$

and $\psi_\nu \equiv 1$ on $\bar{U}_\nu$, $\psi_{\nu 2} \equiv 1$ on $\bar{V}_\nu$, $\eta_\nu \equiv 1$ on $\bar{W}_\nu$.

Let $P_\nu = \psi_\nu(I-\Delta)^{s/2}$, $P_{\nu 1} = \psi_{\nu 1}(I-\Delta_g)^{s/2}M_{\eta_\nu}$. We see that $M_{\eta_\nu} \in \mathrm{OPS}^0$, and $P_\nu, P_{\nu 1} \in \mathrm{OPS}^s$. Note that the principal symbol of P_ν is $\psi_\nu(x)|\xi|^s$, which is elliptic on $\bar{U}_\nu$. By Lemma 6, we can find $Q_{\nu 1} \in \mathrm{OPS}^0$ supported in U_ν such that

$$P_{\nu 1} - Q_{\nu 1}P_\nu \in \mathrm{OPS}^0 .$$

Then by Lemma 4, we obtain the local estimate

$$\begin{aligned} \|P_{\nu 1}(f_\nu)\|_{L^p(\Omega_\nu)} &= \|(P_{\nu 1} - Q_{\nu 1}P_\nu)(f_\nu) + Q_{\nu 1}P_\nu(f_\nu)\|_{L^p(\Omega_\nu)} \\ &\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_\nu f_\nu\|_{L^p(\Omega_\nu)}. \end{aligned} \tag{4.6}$$

Moreover, if $P_{\nu 2} = \psi_{\nu 2}(I-\Delta_g)^{s/2}M_{\eta_\nu}$, then $P_{\nu 2}$ has the principal symbol $\psi_{\nu 2}(x)(\sum g^{jk}(x)\xi_j\xi_k)^{s/2}$, which is elliptic on $\bar{V}_\nu$. Similarly, by applying Lemma 6 to P_ν and $P_{\nu 2}$, we obtain the local estimate

$$\|P_\nu(f_\nu)\|_{L^p(\Omega_\nu)} \lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_{\nu 2}f_\nu\|_{L^p(\Omega_\nu)}. \tag{4.7}$$

Next, we handle the nonlocal part. We write

$$\begin{aligned} (1-\psi_\nu)(I-\Delta)^{s/2}f_\nu &= (1-\psi_\nu)(I-\Delta)^{s/2}(\phi_\nu\eta_\nu f) \\ &= (1-\psi_\nu)(I-\Delta)^{s/2}M_{\phi_\nu}(\eta_\nu f). \end{aligned}$$

Since $\operatorname{dist}(\operatorname{supp}(1-\psi_\nu), \operatorname{supp}\phi_\nu) = \delta_\nu > 0$, using integration by parts, we see that the kernel of $(1-\psi_\nu)(I-\Delta)^{s/2}M_{\phi_\nu}$ satisfies

$$\left| \int_{\mathbb{R}^n} (1-\psi_\nu(x))e^{i(x-y)\cdot\xi}\phi_\nu(y)(1+|\xi|^2)^{s/2}d\xi \right| \lesssim (1+|x-y|)^{-N} \quad \text{for all } N.$$

By Young's inequality, we get

$$\|(1-\psi_\nu)(I-\Delta)^{s/2}(f_\nu)\|_{L^p(\mathbb{R}^n)} \lesssim \|\eta_\nu f\|_{L^p(\mathbb{R}^n)} = \|f_\nu\|_{L^p(\Omega_\nu)}. \tag{4.8}$$

Similarly, using the fact that the kernel of pseudo-differential operators on compact manifolds is smooth away from diagonal, we have

$$\begin{aligned} \|(1-\psi_{\nu 1})(I-\Delta_g)^{s/2}(f_\nu)\|_{L^p(M)} &= \|(1-\psi_{\nu 1})(I-\Delta_g)^{s/2}(\phi_\nu\eta_\nu f)\|_{L^p(M)} \\ &\lesssim \|\eta_\nu f\|_{L^p(M)} = \|f_\nu\|_{L^p(\Omega_\nu)} \end{aligned} \tag{4.9}$$

and

$$\|(1-\psi_{\nu 2})(I-\Delta_g)^{s/2}(f_\nu)\|_{L^p(M)} \lesssim \|f_\nu\|_{L^p(\Omega_\nu)}. \tag{4.10}$$

Combining (4.6) with the nonlocal estimates (4.8) and (4.9), we obtain

$$\begin{aligned}
&\|(I-\Delta_g)^{s/2}f_\nu\|_{L^p(M)}\\
&\quad\lesssim \|(1-\psi_{\nu 1})(I-\Delta_g)^{s/2}f_\nu\|_{L^p(M)} + \|\psi_{\nu 1}(I-\Delta_g)^{s/2}f_\nu\|_{L^p(M)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|\psi_{\nu 1}(I-\Delta_g)^{s/2}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad= \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_{\nu 1}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_\nu f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad= \|f_\nu\|_{L^p(\Omega_\nu)} + \|(I-\Delta)^{s/2}f_\nu - (1-\psi_\nu)(I-\Delta)^{s/2}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|(I-\Delta)^{s/2}f_\nu\|_{L^p(\mathbb{R}^n)}\\
&\quad\lesssim \|(I-\Delta)^{s/2}f_\nu\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Here in the last step we apply Lemma 4 to $(I-\Delta)^{-s/2} \in \mathrm{OPS}^0$.

Similarly, combining (4.7) with the nonlocal estimates (4.8) and (4.10), we have

$$\begin{aligned}
&\|(I-\Delta)^{s/2}f_\nu\|_{L^p(\mathbb{R}^n)}\\
&\quad\lesssim \|(1-\psi_\nu)(I-\Delta)^{s/2}f_\nu\|_{L^p(\mathbb{R}^n)} + \|\psi_\nu(I-\Delta)^{s/2}f_\nu\|_{L^p(\mathbb{R}^n)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|\psi_\nu(I-\Delta)^{s/2}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad= \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_\nu f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|P_{\nu 2}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad= \|f_\nu\|_{L^p(\Omega_\nu)} + \|(I-\Delta_g)^{s/2}f_\nu - (1-\psi_{\nu 2})(I-\Delta_g)^{s/2}f_\nu\|_{L^p(\Omega_\nu)}\\
&\quad\lesssim \|f_\nu\|_{L^p(\Omega_\nu)} + \|(I-\Delta_g)^{s/2}f_\nu\|_{L^p(M)}\\
&\quad\lesssim \|(I-\Delta_g)^{s/2}f_\nu\|_{L^p(M)}.
\end{aligned}$$

In the last step we used Lemma 4 for $(I-\Delta_g)^{-s/2} \in \mathrm{OPS}^0$. So we finish the proof of (4.5). Thus, the proof of Proposition 2 is complete. ■

Let $[\mathcal{D}, V] = \mathcal{D}V - V\mathcal{D}$. We need to following commutator estimate.

Lemma 7. *Let $1 < p < \infty$. Given $P \in \mathrm{OPS}^1$,*

$$\|[P, f]u\|_{L^p} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{L^p}.$$

Here $\|f\|_{\mathrm{Lip}^1}$ is the Lipschitz norm of f.

Here the L^p norm can be taken on $\mathbb{R}^n$ and compact manifolds. See Taylor [65, Proposition 1.3]. The result was proven in Calderón [9] for classical first-order pseudodifferential operators and by Coifman and Meyer [12] for OPS^1.

Lemma 8. *If* $V \in \mathrm{Lip}^1(M)$*, then* $e_\lambda \in C^{1,\alpha}(M)$*, for any* $0 < \alpha < 1$.

Proof. By Sobolev embedding (see e.g., [24]), we only need to show $\|e_\lambda\|_{W^{2,p}(M)} < \infty$ for any $1 < p < \infty$. Indeed, using the commutator estimate in Lemma 7 and the equation $(\mathcal{D} + V)e_\lambda = \lambda e_\lambda$, we have

$$\begin{aligned}
&\|\mathcal{D}(Ve_\lambda)\|_{L^p(M)} \\
&\quad \le \|V(\mathcal{D}+V)e_\lambda\|_{L^p(M)} + \|V^2 e_\lambda\|_{L^p(M)} + \|[\mathcal{D},V]e_\lambda\|_{L^p(M)} \\
&\quad \lesssim \lambda\|V\|_{L^\infty}\|e_\lambda\|_{L^p(M)} + \|V\|_{L^\infty}^2\|e_\lambda\|_{L^p(M)} + \|V\|_{\mathrm{Lip}^1}\|e_\lambda\|_{L^p(M)} \\
&\quad \lesssim (1+\lambda)\|e_\lambda\|_{L^p(M)}.
\end{aligned}$$

So, by Proposition 2, we obtain

$$\begin{aligned}
&\|e_\lambda\|_{W^{2,p}(M)} \\
&\quad \approx \|(1+\mathcal{D})^2 e_\lambda\|_{L^p(M)} \\
&\quad \lesssim \|(1+\mathcal{D})(1+\mathcal{D}+V)e_\lambda\|_{L^p(M)} + \|\mathcal{D}(Ve_\lambda)\|_{L^p(M)} + \|V\|_{L^\infty}\|e_\lambda\|_{L^p(M)} \\
&\quad \lesssim (1+\lambda)(\|(1+\mathcal{D})e_\lambda\|_{L^p(M)} + \|e_\lambda\|_{L^p(M)}) \\
&\quad \le (1+\lambda)(\|(1+\mathcal{D}+V)e_\lambda\|_{L^p(M)} + \|V\|_{L^\infty}\|e_\lambda\|_{L^p(M)} + \|e_\lambda\|_{L^p(M)}) \\
&\quad \lesssim (1+\lambda)^2\|e_\lambda\|_{L^p(M)}. \qquad \blacksquare
\end{aligned}$$

Now, we are ready to prove the nodal set estimates. Let

$$\begin{aligned}
N_\lambda &= \{x \in M : e_\lambda(x) = 0\}, \\
D_+ &= \{x \in M : e_\lambda(x) > 0\}, \\
D_- &= \{x \in M : e_\lambda(x) < 0\}.
\end{aligned}$$

We first express the manifold M as a (essentially) disjoint union

$$M = \bigcup_{j\ge 1} D_{j,+} \cup \bigcup_{j \ge 1} D_{j,-} \cup N_\lambda$$

where $D_{j,+}$ and $D_{j,-}$ are the positive and negative nodal domains of e_λ, i.e., the connected components of the sets D_+ and D_-. To simplify the notation, without loss of generality, we may assume that there are only two nodal domains D_+ and D_-. Since ∇e_λ is continuous by Lemma 8, the restriction of ∇e_λ to N_λ is well defined. Moreover, if the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(N_\lambda) = \infty$, then the lower bound (1.8) trivially holds. So we may assume that $\mathcal{H}^{n-1}(N_\lambda) < \infty$ in the following.

We mainly follow the exposition in Evans and Gariepy [25]. Since the eigenfunctions are continuous, the "measure theoretic boundaries" (see [25, definition on p. 208]) of the nodal domains must be subsets of N_λ, so they also have finite

$(n-1)$-dimensional Hausdorff measure. Then we can apply the Gauss–Green theorem (see [25, Theorem 1 on p. 209]) on nodal domains $D_\pm$ with "measure theoretic boundaries" $\partial D_\pm$. Let $\nu_\pm(x)$ be the "measure theoretic unit outer normal" at $x \in \partial D_\mp$ (see [25, Definition on p. 203 and Theorem 2 on p. 205]). Since e_λ is C^1, the implicit function theorem implies that the nodal set is C^1 near $x_0 \in \partial D_\pm$ whenever $\nabla e_\lambda(x_0) \neq 0$. Thus, we have $\nu_\pm(x) = \pm\frac{\nabla e_\lambda(x)}{|e_\lambda(x)|}$ whenever $\nabla e_\lambda(x) \neq 0$. So we have

$$\int\limits_{D_+} \operatorname{div}(f\nabla e_\lambda)dV_g = \int\limits_{\partial D_+} \langle f\nabla e_\lambda, \nu_-\rangle dS = -\int\limits_{\partial D_+} f|\nabla e_\lambda|dS,$$
$$\int\limits_{D_-} \operatorname{div}(f\nabla e_\lambda)dV_g = \int\limits_{\partial D_-} \langle f\nabla e_\lambda, \nu_+\rangle dS = \int\limits_{\partial D_-} f|\nabla e_\lambda|dS,$$

Then

$$\sum_\pm \int\limits_{\partial D_\pm} f|\nabla e_\lambda| = \int\limits_{D_-} \operatorname{div}(f\nabla e_\lambda) - \int\limits_{D_+} \operatorname{div}(f\nabla e_\lambda). \tag{4.11}$$

Note that by Cauchy–Schwarz and the fact that $\partial D_\pm \subset N_\lambda$

$$\begin{aligned}\int\limits_{\partial D_\pm} |\nabla e_\lambda| &\lesssim \Big(\int\limits_{\partial D_\pm} |\nabla e_\lambda|^2\Big)^{1/2} \mathcal{H}^{n-1}(\partial D_\pm)^{1/2}\\ &\lesssim \Big(\int\limits_{\partial D_\pm} |\nabla e_\lambda|^2\Big)^{1/2} \mathcal{H}^{n-1}(N_\lambda)^{1/2}.\end{aligned} \tag{4.12}$$

Therefore, to estimate the lower bound of $\mathcal{H}^{n-1}(N_\lambda)$, it suffices to estimate both $\int_{\partial D_\pm} |\nabla e_\lambda|$ and $\int_{\partial D_\pm} |\nabla e_\lambda|^2$. This can be done by applying the eigenfunction estimates from previous sections.

Lemma 9. *If* $V \in \operatorname{Lip}^1(M)$, *then*

$$\sum_\pm \int\limits_{\partial D_\pm} |\nabla e_\lambda| \geq \frac{\lambda^2}{4}\|e_\lambda\|_{L^1(M)}.$$

Proof. We set $f = 1$ in (4.11). We have

$$\sum_\pm \int\limits_{\partial D_\pm} |\nabla e_\lambda| \leq \Big|\int\limits_{D_-} \Delta_g e_\lambda - \int\limits_{D_+} \Delta_g e_\lambda\Big|.$$

Since $\sqrt{-\Delta_g} = \mathcal{D} - P_0$, we have

$$\begin{aligned}-\Delta_g = &(\mathcal{D}+V)^2 - (\mathcal{D}V - V\mathcal{D}) - 2V(\mathcal{D}+V) + V^2\\ &- 2P_0(\mathcal{D}+V) + 2P_0V + Q_0,\end{aligned}$$

where $Q_0 = P_0\mathcal{D} - \mathcal{D}P_0 + P_0^2 \in \mathrm{OPS}^0$. Thus,

$$\begin{aligned}
&\sum_{\pm}\int_{\partial D_\pm} |\nabla e_\lambda| \\
&\quad= \int_{D_+} - \int_{D_-} (\lambda^2 e_\lambda - [\mathcal{D}, V]e_\lambda - 2\lambda V e_\lambda + V^2 e_\lambda - 2\lambda P_0 e_\lambda + 2P_0 V e_\lambda + Q_0 e_\lambda) \\
&\quad\geq \lambda^2 \|e_\lambda\|_{L^1(M)} - \|[\mathcal{D}, V]e_\lambda\|_{L^1(M)} - 2\lambda\|Ve_\lambda\|_{L^1(M)} - \|V^2 e_\lambda\|_{L^1(M)} \\
&\qquad - 2\lambda\|P_0 e_\lambda\|_{L^1(M)} - 2\|P_0 V e_\lambda\|_{L^1(M)} - \|Q_0 e_\lambda\|_{L^1(M)}.
\end{aligned}$$

By Hölder's inequality and (1.6), we have

$$\|e_\lambda\|_{L^{1+\varepsilon}(M)} \lesssim \lambda^{((n-1)\varepsilon)/(2(1+\varepsilon))}\|e_\lambda\|_{L^1(M)}, \ 0 < \varepsilon < 1.$$

Combining this estimate with Lemma 7, we have

$$\begin{aligned}
\|[\mathcal{D}, V]e_\lambda\|_{L^1(M)} &\lesssim \|[\mathcal{D}, V]e_\lambda\|_{L^{1+\varepsilon}(M)} \\
&\lesssim \|V\|_{\mathrm{Lip}^1}\|e_\lambda\|_{L^{1+\varepsilon}(M)} \lesssim \lambda\|V\|_{\mathrm{Lip}^1}\|e_\lambda\|_{L^1(M)},
\end{aligned}$$

if $\varepsilon > 0$ is small enough. Moreover, if $\varepsilon > 0$ is small enough, then by Lemma 4 we have

$$\begin{aligned}
\lambda\|Ve_\lambda\|_{L^1(M)} &\lesssim \lambda\|V\|_{L^\infty}\|e_\lambda\|_{L^1(M)}, \\
\|V^2 e_\lambda\|_{L^1(M)} &\lesssim \|V\|_{L^\infty}^2\|e_\lambda\|_{L^1(M)}, \\
\lambda\|P_0 e_\lambda\|_{L^1(M)} \lesssim \lambda\|P_0 e_\lambda\|_{L^{1+\varepsilon}(M)} &\lesssim \lambda\|e_\lambda\|_{L^{1+\varepsilon}(M)} \lesssim \lambda^{3/2}\|e_\lambda\|_{L^1(M)} \\
\|P_0 V e_\lambda\|_{L^1(M)} \lesssim \|P_0 V e_\lambda\|_{L^{1+\varepsilon}(M)} &\lesssim \|Ve_\lambda\|_{L^{1+\varepsilon}(M)} \lesssim \lambda\|V\|_{L^\infty}\|e_\lambda\|_{L^1(M)} \\
\|Q_0 e_\lambda\|_{L^1(M)} \lesssim \|Q_0 e_\lambda\|_{L^{1+\varepsilon}(M)} &\lesssim \|e_\lambda\|_{L^{1+\varepsilon}(M)} \lesssim \lambda\|e_\lambda\|_{L^1(M)}.
\end{aligned}$$

So we finish the proof Lemma 9. ∎

Lemma 10. *If* $V \in \mathrm{Lip}^1(M)$, *then*

$$\sum_{\pm}\int_{\partial D_\pm} |\nabla e_\lambda|^2 \lesssim \lambda^3\|e_\lambda\|_{L^2(M)}.$$

Proof. We set $f = \sqrt{1 + |\nabla e_\lambda|^2}$ in (4.11). And then

$$\begin{aligned}
&\sum_{\pm}\int_{\partial D_\pm} \sqrt{1 + |\nabla e_\lambda|^2}\,|\nabla e_\lambda| \\
&\quad= \int_{D_-} \mathrm{div}(\sqrt{1 + |\nabla e_\lambda|^2}\nabla e_\lambda) - \int_{D_+} \mathrm{div}(\sqrt{1 + |\nabla e_\lambda|^2}\nabla e_\lambda)
\end{aligned}$$

$$\begin{aligned}&\lesssim \int_M |\operatorname{div}(\sqrt{1+|\nabla e_\lambda|^2}\nabla e_\lambda)| \lesssim \int_M \sqrt{1+|\nabla e_\lambda|^2}\,|\nabla^2 e_\lambda|\\ &\lesssim (\|e_\lambda\|_{L^2(M)}+\|\nabla e_\lambda\|_{L^2(M)})\|\nabla^2 e_\lambda\|_{L^2(M)} \lesssim \lambda^3\|e_\lambda\|_{L^2(M)}.\end{aligned}$$

Here we use the Sobolev estimates of eigenfunctions in the last step. Indeed, we have the following Sobolev estimates:

$$\begin{aligned}\|\nabla e_\lambda\|_{L^2(M)} &\lesssim \|\mathcal{D}e_\lambda\|_{L^2(M)}+\|e_\lambda\|_{L^2(M)}\\ &\le \|(\mathcal{D}+V)e_\lambda\|_{L^2(M)}+\|Ve_\lambda\|_{L^2(M)}+\|e_\lambda\|_{L^2(M)}\\ &\lesssim \lambda\|e_\lambda\|_{L^2(M)}+\|V\|_{L^\infty}\|e_\lambda\|_{L^2(M)}\\ &\lesssim \lambda\|e_\lambda\|_{L^2(M)};\end{aligned}$$

similarly, we may exploit Lemma 7 to obtain

$$\begin{aligned}\|\nabla^2 e_\lambda\|_{L^2(M)} &\lesssim \|\mathcal{D}^2 e_\lambda\|_{L^2(M)}+\|\mathcal{D}e_\lambda\|_{L^2(M)}+\|e_\lambda\|_{L^2(M)}\\ &\lesssim \|(\mathcal{D}+V)^2 e_\lambda\|_{L^2(M)}+\|[\mathcal{D},V]e_\lambda\|_{L^2(M)}+\|V(\mathcal{D}+V)e_\lambda\|_{L^2(M)}\\ &\quad+\|V^2 e_\lambda\|_{L^2(M)}+\lambda\|e_\lambda\|_{L^2(M)}\\ &\lesssim \lambda^2\|e_\lambda\|_{L^2(M)}+\lambda\|V\|_{\mathrm{Lip}^1}\|e_\lambda\|_{L^2(M)}+\|V\|_{L^\infty}^2\|e_\lambda\|_{L^2(M)}\\ &\lesssim \lambda^2\|e_\lambda\|_{L^2(M)}.\end{aligned}$$

So Lemma 10 is proved. ■

Finally, we finish the proof of (1.8) by inserting Lemma 9 and Lemma 10 into (4.12) and applying (1.6).

A. Heat kernel bounds

In this section, we prove the heat kernel estimates for the operators

$$H_V = (-\Delta_g)^{\alpha/2} + P_{\alpha-1} + V,$$

where $P_{\alpha-1}$ is a classical pseudo-differential operator of order $\alpha-1$, and the real-valued potential V belongs to the Kato class on the closed manifold (M,g). These results generalize those of Gimperlein and Grubb [33, Theorem 4.3] for

$$H^0 = (-\Delta_g)^{\alpha/2} + P_{\alpha-1},$$

under the minimal assumption on V. They are more general estimates than we need in this paper ($\alpha = 1$), but they may be useful for future research.

Definition 1. For $n \geq 2$ and $0 < \alpha < 2$, the potential V is said to be in the Kato class $\mathcal{K}_\alpha(M)$ if

$$\lim_{r \downarrow 0} \sup_{x \in M} \int_{B_r(x)} d_g(x,y)^{\alpha-n} |V(y)| dy = 0 \tag{A.1}$$

where $d_g(\cdot,\cdot)$ denotes geodesic distance and $B_r(x)$ is the geodesic ball of radius r about x and dy denotes the volume element on (M, g). To define the Kato class for $n = 1$ and $0 < \alpha < 2$, we replace the function $d_g(x,y)^{\alpha-n}$ in (A.1) by

$$w(x,y) = \begin{cases} d_g(x,y)^{\alpha-1}, & \alpha < 1, \\ \log(2 + d_g(x,y)^{-1}), & \alpha = 1, \\ 1, & \alpha > 1. \end{cases}$$

Since M is compact we have $\mathcal{K}_\alpha(M) \subset L^1(M)$, and for any $p > \frac{n}{\alpha}$, we have $L^p(M) \subset \mathcal{K}_\alpha(M)$ by Hölder's inequality. We recall that the assumption $V \in \mathcal{K}_\alpha(M)$ implies that the operators $H_V = (-\Delta_g)^{\alpha/2} + V$ are self-adjoint and bounded from below. See [38, Proof of Proposition 2]. The same argument is still valid to prove that $H_V = (-\Delta_g)^{\alpha/2} + P_{\alpha-1} + V$ is self-adjoint and bounded from below, whenever $P_{\alpha-1}$ is self-adjoint.

Proposition 3. *Let $n \geq 1$, $0 < \alpha < 2$ and $V \in \mathcal{K}_\alpha(M)$. Let $p_V(t,x,y)$ be the heat kernel of $H_V = (-\Delta_g)^{\alpha/2} + P_{\alpha-1} + V$. Then we have*

$$|p_V(t,x,y)| \lesssim q_\alpha(t,x,y), \quad 0 < t \leq 1,\ x, y \in M, \tag{A.2}$$

where $q_\alpha(t,x,y) = \min\{t^{-n/\alpha}, t d_g(x,y)^{-n-\alpha}\}$.

A.1. Proof of Proposition 3

Let $p_V = p_V(t,x,y)$ denote the heat kernel $e^{-tH_V}(x,y)$. Let $p_0 = p_0(t,x,y)$ denote the heat kernel $e^{-tH^0}(x,y)$. We employ the following fact derived from Duhamel's principle.

Lemma 11. *Suppose $V \in L^\infty(M)$. Then,*

$$p_V = \sum_{k \geq 0} S_k$$

with $S_0 = p_0$ and

$$S_k(t,x,y) = \int_0^t \int_M p_0(t-s,x,z) V(z) S_{k-1}(s,z,y) dz ds.$$

Proof. Since V acts as a bounded multiplication operator on L^2, we apply [49, Theorem 6.2] with $A = -H^0$ and $B(s) = V$ to complete the proof. ■

Lemma 12. *For* $0 < \alpha < 2$*, let*

$$N_t^\alpha(V) = \sup_{y\in M} \int_0^t \int_M q_\alpha(s, y, z)|V(z)|dzds.$$

Then we have

$$\int_0^t \int_M q_\alpha(s, x, z)q_\alpha(t-s, z, y)|V(z)|dzds \le CN_t^\alpha(V)q_\alpha(t, x, y),$$

where $C > 0$ *is a constant.*

This lemma follows from the 3P-inequality in [5, Theorem 4] and [68, Proposition 2.4]. We remark that such 3P-inequality holds for all $\alpha \in (0, 2)$ but fails to hold for the Gaussian kernel ($\alpha = 2$).

Lemma 13 (3P-inequality). *We have for any* $s, t > 0$ *and* $x, y, z \in M$

$$q_\alpha(t, x, z)q_\alpha(s, z, y) \le C_1 q_\alpha(s+t, x, y)(q_\alpha(t, x, z) + q_\alpha(s, z, y)),$$

where $C_1 > 0$ *is a constant.*

Proof. Note that for $A, B > 0$,

$$\min\{A, B\} \approx \frac{AB}{A+B}, \quad (A+B)^{n/\alpha} \approx A^{n/\alpha} + B^{n/\alpha},$$

and the triangle inequality $d_g(x, y) \le d_g(x, z) + d_g(z, y)$. The implicit constants may depend on n and α.

We have

$$\begin{aligned}
\frac{q_\alpha(t, x, z) + q_\alpha(s, z, y)}{q_\alpha(t, x, z)q_\alpha(s, z, y)} &= \frac{1}{q_\alpha(t, x, z)} + \frac{1}{q_\alpha(s, z, y)} \\
&\approx t^{n/\alpha} + t^{-1}d_g(x, z)^{n+\alpha} + s^{n/\alpha} + s^{-1}d_g(z, y)^{n+\alpha} \\
&\approx (t+s)^{n/\alpha} + t^{-1}d_g(x, z)^{n+\alpha} + s^{-1}d_g(z, y)^{n+\alpha} \\
&\ge (t+s)^{n/\alpha} + (s+t)^{-1}(d_g(x, z)^{n+\alpha} + d_g(z, y)^{n+\alpha}) \\
&\approx (t+s)^{n/\alpha} + (s+t)^{-1}(d_g(x, z) + d_g(z, y))^{n+\alpha} \\
&\ge (t+s)^{n/\alpha} + (s+t)^{-1}d_g(x, y)^{n+\alpha} \\
&\approx \frac{1}{q_\alpha(s+t, x, y)}.
\end{aligned}$$

■

Now, we give a proof of Proposition 3. We use an approximation argument similar to the one for the Feynman-Kac formula as presented in [52, Theorem 6.2]. By Gimperlein and Grubb [33, Theorem 4.3],

$$|p_0(t,x,y)| \le C_0 q_\alpha(t,x,y), \ \ 0 < t \le 1. \tag{A.3}$$

For $m, l \in \mathbb{N}_+$, let

$$V_m^l(x) = \begin{cases} V(x), & -m \le V(x) \le l, \\ -m, & V(x) < -m, \\ l, & V(x) > l. \end{cases}$$

Then Lemma 11 yields

$$p_{V_m^l} = \sum_{k\ge 0} S_k^{m,l},$$

where

$$S_0^{m,l}(t,x,y) = p_0(t,x,y)$$

and for $k \ge 1$,

$$S_k^{m,l}(t,x,y) = \int_0^t \int_M p_0(t-s,x,z)(V_m^l)(z) S_{k-1}^{m,l}(s,z,y)dz\, ds. \tag{A.4}$$

We claim that, for the constant $C > 0$ in Lemma 12, we have

$$|S_k^{m,l}(t,x,y)| \le C_0^{k+1}(CN_t^\alpha(V))^k q_\alpha(t,x,y), \quad k \ge 0,\ 0 < t \le 1.$$

We prove the claim by induction. The base case $k = 0$ follows immediately from (A.3). Suppose the claim holds for $k-1$. Then (A.3), (A.4), and Lemma 12 imply

$$\begin{aligned} |S_k^{m,l}(t,x,y)| &\le C_0^k(CN_t^\alpha(V))^{k-1} \int_0^t \int_M C_0 q_\alpha(t-s,x,z) q_\alpha(s,z,y)|V(z)|dz\, ds \\ &\le C_0^k(CN_t^\alpha(V))^{k-1} \cdot C_0 C N_t^\alpha(V) q_\alpha(t,x,y) \\ &= C_0^{k+1}(CN_t^\alpha(V))^k q_\alpha(t,x,y). \end{aligned}$$

This establishes the claim.

Moreover, the Kato class property of V implies that for any $0 < \alpha < 2$,

$$\lim_{t\downarrow 0} N_t^\alpha(V) = 0. \tag{A.5}$$

Indeed, for $n \geq 2$,

$$\int_0^t \int_M q_\alpha(r, y, z)|V(z)|dz\,dr \lesssim \int_{d_g(z,y)<t^{1/(2\alpha)}} d_g(z, y)^{\alpha-n}|V(z)|dz + \int_M t\, d_g(z, y)^{\alpha-n}|V(z)|dz,$$

which implies (A.5) by the definition of Kato class. The case $n = 1$ is similar.

Therefore, there exists $0 < t_1 \leq 1$ such that $C_0 C N_t^\alpha(V) \leq \frac{1}{3}$ for all $t \in (0, t_1]$. Thus, we have the uniform upper bound

$$|p_{V_m^l}(t, x, y)| \leq \widetilde{C} q_\alpha(t, x, y), \quad 0 < t \leq t_1.$$

By the dominated convergence theorem, the functions $p_{V_m^l}$ converge pointwise to a function $\tilde{p}_V(t, x, y)$ as $m, l \to \infty$, with

$$|\tilde{p}_V(t, x, y)| \leq \widetilde{C} q_\alpha(t, x, y), \quad 0 < t \leq t_1. \tag{A.6}$$

Let $\widetilde{E}(t)$ be the operator associated with the integral kernel $\tilde{p}_V(t, x, y)$. For any fixed $t > 0$, by the dominated convergence theorem, with the uniform upper bound obtained above, we have

$$\lim_{l\to\infty} \lim_{m\to\infty} \|(e^{-tH_{V_m^l}} - \widetilde{E}(t))f\|_{L^2} = 0 \quad \text{for all } f \in L^2(M).$$

To verify that $\tilde{p}_V(t, x, y)$ is indeed the kernel of e^{-tH_V}, it suffices to show

$$\lim_{l\to\infty} \lim_{m\to\infty} \|(e^{-tH_{V_m^l}} - e^{-tH_V})f\|_{L^2} = 0 \quad \text{for all } f \in L^2(M).$$

Indeed, by the monotone convergence theorem for forms [54, Theorem 7.5.18 (b)], we obtain

$$H_{V_m^l} \to H_{V_\infty^l} \text{ in strong resolvent sense.}$$

Since $H_{V_\infty^l}$ is closed by the KLMN Theorem [54, Theorem 7.5.7], and [54, Theorem 7.2.10] gives

$$\lim_{m\to\infty} e^{-tH_{V_m^l}} f = e^{-tH_{V_\infty^l}} f \quad \text{in } L^2(M).$$

Similarly, [54, Theorem 7.5.18 (a)] yields

$$\lim_{l\to\infty} e^{-tH_{V_\infty^l}} f = e^{-tH_V} f \quad \text{in } L^2(M).$$

Finally, we extend the upper bound (A.6) in the full range $0 < t \leq 1$ by using the semigroup property.

A.2. Two-sided estimates

For $H^0 = (-\Delta_g)^{\alpha/2}$ $(0 < \alpha < 2)$ or $H^0 = \mathscr{D}$ (Dirichlet-to-Neumann operator, $\alpha = 1$), the heat kernel $p_0 = e^{-tH^0}(x, y)$ satisfies the two-sided estimates

$$C^{-1} q_\alpha(t, x, y) \le p_0(t, x, y) \le C q_\alpha(t, x, y), \quad 0 < t \le 1, x, y \in M.$$

See e.g., [33, Theorems 4.2 an 4.4] and [6, Theorem 3.1]. For $H_V = H^0 + V$ with $V \in K_\alpha(M)$, by modifying the argument above, we can obtain the two-sided heat kernel estimates

$$p_V(t, x, y) \approx q_\alpha(t, x, y), \quad 0 < t \le 1, x, y \in M.$$

B. Kernel bounds of pseudo-differential operators

In this section, we prove the kernel bounds of pseudo-differential operators in Lemma 5. See [60, Theorem 4.3.1] for the proof of the fact that $m(P)$ is a pseudo-differential operator of order μ. The kernel bounds (3.2) can be viewed as the rescaled version on compact manifolds compared to the Euclidean estimates in [63, Proposition 1 on p. 241]. We mean that the bounds hold near the diagonal (so that $d_g(x, y)$ is smaller than the injectivity radius of M) and that outside the neighborhood of the diagonal they are $O(R^{-N})$. Roughly speaking, modulo lower order terms, $m(P/R)(x, y)$ equals

$$(2\pi)^{-n} \int_{\mathbb{R}^n} m\Big(\frac{|\xi|}{R}\Big) e^{i d_g(x,y)\xi_1} d\xi$$

near the diagonal, which satisfies the bounds in (3.2), while outside of a fixed neighborhood of the diagonal $m(P/R)(x, y)$ is $O(R^{-N})$. For completeness, we give a detailed proof by using the Hadamard parametrix.

B.1. Proof of Lemma 5

Since the spectrum of $P = \sqrt{-\Delta_g}$ is nonnegative, we may assume that $m(t)$ is an even function on $\mathbb{R}$. Let $\delta > 0$ be smaller than the injectivity radius of (M, g). Let $\rho \in C_0^\infty(-1, 1)$ be even and satisfy $\rho \equiv 1$ on $\big(-\frac{\delta}{2}, \frac{\delta}{2}\big)$. So we can write

$$\begin{aligned} m(P/R) &= \frac{R}{2\pi} \int_{\mathbb{R}} \hat{m}(tR) \cos(tP) dt \\ &= \frac{R}{2\pi} \int \rho(t) \hat{m}(tR) \cos(tP) dt + \frac{R}{2\pi} \int (1 - \rho(t)) \hat{m}(tR) \cos(tP) dt. \end{aligned} \tag{B.1}$$

To handle the first term in (B.1), we need to use the Hadamard parametrix (see e.g., [58, Section 1.2 and Theorem 3.1.5]). For $0 < t < \delta$ and $N_0 > n+3$, we have

$$\cos tP(x,y) = \sum_{\nu=0}^{N_0} \omega_\nu(x,y)\partial_t E_\nu(t, d_g(x,y)) + R_{N_0}(t,x,y) \tag{B.2}$$

where the leading term

$$\partial_t E_0 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{id_g(x,y)\xi_1} \cos(t|\xi|) d\xi \tag{B.3}$$

and E_ν satisfies $2\partial_t E_\nu = tE_{\nu-1}$, and $\partial_t E_\nu(t/R, r) = R^{n-2\nu}\partial_t E_\nu(t, Rr)$ for any $R > 0$. Here $\omega_\nu \in C^\infty(M \times M)$, and $\omega_0(x,x) = 1$ for all $x \in M$. For $\nu \geq 1$, we have the following explicit formula (see e.g., [58, Section 1.2]):

$$E_\nu = \nu!(2\pi)^{-n} \int_{0\leq s_1 \leq \cdots \leq s_\nu \leq t} \int_{\mathbb{R}^n} e^{id_g(x,y)\xi_1} \frac{\sin(t-s_\nu)|\xi|}{|\xi|} \frac{\sin(s_\nu - s_{\nu-1})|\xi|}{|\xi|} \cdots \cdot \frac{\sin(s_2-s_1)|\xi|}{|\xi|} \frac{\sin s_1|\xi|}{|\xi|} d\xi ds_1 \dots ds_\nu.$$

So for $\nu \geq 1$, we obtain (see e.g., [58, Section 1.2])

$$\begin{aligned} \partial_t E_\nu = \frac{1}{2} t E_{\nu-1} &= \int e^{id_g(x,y)\xi_1} a_\nu(t, |\xi|) d\xi \\ &= \sum_{\pm} \sum_{j=0}^{\nu-1} a_{j\nu}^{\pm} \int e^{id_g(x,y)\xi_1 \pm it|\xi|} t^{j+1} |\xi|^{-2\nu+1+j} d\xi, \end{aligned} \tag{B.4}$$

where $a_{j\nu}^{\pm}$ are constants, and $a_\nu \in C^\infty$. The remainder kernel $R_{N_0} \in C^{N_0-n-3}$ satisfies

$$|\partial_{t,x,y}^\alpha R_{N_0}(t,x,y)| \lesssim |t|^{2N_0+2-n-|\alpha|}, \quad |\alpha| \leq N_0 - n - 2. \tag{B.5}$$

Then we plug (B.2) into the first term of (B.1). We first handle the contribution of the leading term in (B.2). By (B.3), we can write

$$\begin{aligned} &\frac{R}{2\pi} \iint \rho(t)\hat{m}(tR)\cos(t|\xi|)e^{id_g(x,y)\xi_1} dt\, d\xi \\ &= \int m(|\xi|/R) e^{id_g(x,y)\xi_1} d\xi \\ &\qquad + \frac{R}{2\pi} \iint (1-\rho(t))\hat{m}(tR)\cos(t|\xi|)e^{id_g(x,y)\xi_1} dt\, d\xi \\ &:= I_1 + I_2. \end{aligned}$$

Using the property (3.1) and integration by parts, we see that for any $N \in \mathbb{N}$,

$$|I_1| \lesssim \begin{cases} R^n (Rd_g(x,y))^{-n-\mu}(1+Rd_g(x,y))^{-N}, & n+\mu>0, \\ R^n \log(2+(Rd_g(x,y))^{-1})(1+Rd_g(x,y))^{-N}, & n+\mu=0, \\ R^n(1+Rd_g(x,y))^{-N}, & n+\mu<0, \end{cases} \tag{B.6}$$

and

$$\begin{aligned} |I_2| &\lesssim \left| R \iiint (1-\rho(t))(tR)^{-N} m^{(N)}(s) e^{-itRs}\cos(t|\xi|)e^{id_g(x,y)\xi_1}\,ds\,dt\,d\xi \right| \\ &\lesssim R^{-N+1}\iint (1+||\xi|-R|s||)^{-N_1}(1+|s|)^{-N+\mu}\,ds\,d\xi \\ &\lesssim R^{-N}\int (1+|\xi|/R)^{-N+\mu}\,d\xi \\ &\lesssim R^{-N+n}. \end{aligned} \tag{B.7}$$

Here we choose $N_1 > N > n+\mu$.

Similarly, we can handle the contributions of the remaining terms in (B.2). For each $\nu \geq 1$, we can write

$$\frac{R}{2\pi}\int \rho(t)\hat{m}(tR)\partial_t E_\nu(t,d_g(x,y))dt = \frac{R}{2\pi}\int \hat{m}(tR)\partial_t E_\nu(t,d_g(x,y))dt - \frac{R}{2\pi}\int (1-\rho(t))\hat{m}(tR)\partial_t E_\nu(t,d_g(x,y))dt := I_3 + I_4.$$

Using the scaling property $\partial_t E_\nu(t/R,r) = R^{n-2\nu}\partial_t E_\nu(t,Rr)$ and formula (B.4), we can integrate by parts to see that

$$\begin{aligned} |I_3| &= (2\pi)^{-1}R^{n-2\nu}\left|\int \hat{m}(t)\partial_t E_\nu(t,Rd_g(x,y))dt\right| \\ &= (2\pi)^{-1}R^{n-2\nu}\left|\sum_{\pm}\sum_{j=0}^{\nu-1} a_{j\nu}^{\pm}\iint e^{iRd_g(x,y)\xi_1 \pm it|\xi|}\hat{m}(t)t^{j+1}|\xi|^{-2\nu+1+j}\,dt\,d\xi\right| \\ &= R^{n-2\nu}\left|\sum_{\pm}\sum_{j=0}^{\nu-1} i^{-j-1}a_{j\nu}^{\pm}\int e^{iRd_g(x,y)\xi_1} m^{(j+1)}(\pm|\xi|)|\xi|^{-2\nu+1+j}\,d\xi\right| \\ &\lesssim R^{n-2\nu}(1+Rd_g(x,y))^{-N} \\ &\quad + \sum_{j=0}^{\nu-1}\left|\int e^{iRd_g(x,y)\xi_1} m^{(j+1)}(|\xi|)|\xi|^{-2\nu+1+j}\varphi(|\xi|)\,d\xi\right| \end{aligned} \tag{B.8}$$

$$\lesssim \begin{cases} R^{n-2\nu}(Rd_g(x,y))^{-n-\mu}(1+Rd_g(x,y))^{-N}, & n+\mu>0, \\ R^{n-2\nu}\log(2+(Rd_g(x,y))^{-1})(1+Rd_g(x,y))^{-N}, & n+\mu=0, \\ R^{n-2\nu}(1+Rd_g(x,y))^{-N}, & n+\mu<0, \end{cases} \tag{B.9}$$

where $\varphi \in C^\infty$ vanishes near the origin but equals one near infinity. The first term in (B.8) follows from the smoothness of a_ν in (B.4) near $\xi = 0$ and integration by parts. Moreover,

$$\begin{aligned}|I_4| &\lesssim \sum_{\pm}\sum_{j=0}^{\nu-1}\Big| R\iiint (1-\rho(t))(tR)^{-N} m^{(N+j+1)}(s) \\ &\qquad\qquad \cdot e^{-itRs} e^{id_g(x,y)\xi_1 \pm it|\xi|}\phi_{j\nu}(|\xi|)d\xi\, ds\, dt\Big| \\ &\lesssim R^{-N+1}\iint (1+||\xi|-R|s||)^{-N_1}(1+|s|)^{-N+\mu} ds d\xi \\ &\lesssim R^{-N}\int (1+|\xi|/R)^{-N+\mu} d\xi \lesssim R^{-N+n}. \end{aligned} \tag{B.10}$$

The remainder term R_{N_0} in (B.2) is easy to handle. Indeed, for $n+\mu < N \le N_0 - n - 2$, using (B.5) we integrate by parts to obtain

$$\begin{aligned}&\left|\frac{R}{2\pi}\int \rho(t)\hat{m}(tR)R_{N_0}(t,x,y)dt\right| \\ &\quad\lesssim R^{-N+1}\left|\iint \rho(t)t^{-N}R_{N_0}(t,x,y)m^{(N)}(s)e^{-itRs}ds\, dt\right| \\ &\quad\lesssim R^{-N+1}\int (1+R|s|)^{-N}(1+|s|)^{\mu-N}ds \lesssim R^{-N+1}. \end{aligned} \tag{B.11}$$

To handle the second term in (B.1), we notice that for $\lambda \ge 0$

$$\begin{aligned}&\left|\frac{R}{2\pi}\int (1-\rho(t))\hat{m}(tR)\cos(t\lambda)dt\right| \\ &\quad\lesssim \left| R\iint (1-\rho(t))(tR)^{-N}m^{(N)}(s)e^{-itRs}\cos(t\lambda)dtds\right| \\ &\quad\lesssim R^{-N+1}\int (1+|\lambda-R|s||)^{-N_1}(1+|s|)^{-N+\mu}ds \lesssim R^{-N}\Big(1+\frac{\lambda}{R}\Big)^{-N+\mu}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\left|\frac{R}{2\pi}\int (1-\rho(t))\hat{m}(tR)\cos(tP)(x,y)dt\right| \\ &\quad\lesssim R^{-N}\sum_j (1+\lambda_j/R)^{-N+\mu}|e_j(x)e_j(y)| \\ &\quad\lesssim R^{-N}\sum_k (1+k/R)^{-N+\mu}\sum_{\lambda_j\in[k,k+1)}|e_j(x)e_j(y)| \\ &\quad\lesssim R^{-N}\sum_k (1+k/R)^{-N+\mu}(1+k)^{n-1} \lesssim R^{-N+n}. \end{aligned} \tag{B.12}$$

Here we used the L^∞ bound of Laplace eigenfunctions (see e.g., [60, Lemma 4.2.4])

$$\sum_{\lambda_j \in [k,k+1)} |e_j(x)e_j(y)| \lesssim \sup_{x \in M} \sum_{\lambda_j \in [k,k+1)} |e_j(x)|^2 \lesssim (1+k)^{n-1}.$$

Combining the bounds (B.6), (B.7), (B.9), (B.10), (B.11), and (B.12), we complete the proof.

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