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Localization on the quantum graph connecting the points of the lattice $\mathbb{Z}^d$

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Abstract. We consider the graph whose edges connect all nearest neighbours of the lattice $\mathbb{Z}^d$. We prove several theorems establishing localization of eigenfunctions of the Schrödinger operator on this graph. The energies for which we establish the localization belong to specific intervals whose union is an unbounded subset of the real line.

1. Introduction

This paper contains several results on the localised states for a random Hamiltonian (Schrödinger-type operator) on the graph $\Gamma = \Gamma_d$, $d \geqslant 2$. This graph, whose vertices are points of $\mathbb{Z}^d$, and the edges are the intervals connecting the nearest neighbours, was introduced by Pavel Exner [6, 7, 9] before the general quantum graphs theory has been thoroughly developed (see [3]).

In particular, P. Exner and his collaborators considered the "Laplace operator" on Γ_d and the Hamiltonian with some kind of periodic potential. One of the central observations made by the authors of [6, 9] is that the spectrum of such operators can have infinitely many gaps. This means that the corresponding system has "semiconducting" properties even for waves with arbitrarily high frequencies. The latter fact has substantial applications.

For instance, we may consider the optical network of thin channels (of small diameter ε) that have been "drilled" by a laser inside an optical material (like silicon). The propagation of electromagnetic waves with high frequencies through this network is described by Maxwell's equations. An important result of [15] says that in the limit as $\varepsilon \to 0$, the system of Maxwell's equations reduces to the Schrödinger equation on the graph Γ_3 with Kirchhoff-type gluing conditions at the vertices. The statistical errors appear in such systems mainly in the junctions (intersections of the channels). This is exactly Exner's model with a random δ-like potential, and we have

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a physically justified problem of Anderson's localization for the random Hamiltonian on Γ_3.

The paper [15] is not the only work addressing graphs as limits of net-work of tubes. Another appropriate reference here is the one to the book of O. Post [18] in which the author considers tubes with Dirichlet and Neumann boundary conditions.

The model studied in the present paper is similar to, but not the same as the one studied by Pavel Exner, Mario Helm, and Peter Stollmann in [10]. Loosely speaking, we deal with the operator

$$H = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}^d} V(n)\delta_n(x)$$

defined on the graph whose edges connect the points of $\mathbb{Z}^d$. The symbol δ_n denotes the delta finction "sitting" at the point n. The precise definition of the operator H is given below.

For each pair $m, n \in \mathbb{Z}^d$ of neighbouring (i.e., such that $|n-m| = 1$) points of the lattice $\mathbb{Z}^d$, define $I_{n,m}$ to be the interval connecting the two points. We will study the graph $\Gamma_d \subset \mathbb{R}^d$ defined as the union of all intervals of the form $I_{n,m}$.

$$\Gamma_d = \bigcup_{n,m:|m-n|=1} I_{n,m}.$$

There is a natural measure on Γ_d whose restriction to $I_{n,m}$ coincides with the Lebesgue measure on this interval. In the Hilbert space $L^2(\Gamma_d)$, we consider the self-adjoint operator H generated by the quadratic form

$$h[u] = \int_{\Gamma_d} |u'(x)|^2\,dx + \sum_{n\in\mathbb{Z}^d} V(n)|u(n)|^2,$$

where $V\colon \mathbb{Z}^d \to \mathbb{R}$ is a bounded real potential on the lattice $\mathbb{Z}^d$:

$$V = \overline{V} \in \ell^\infty(\mathbb{Z}^d).$$

The domain of the quadratic form h is the collection of all absolutely continuous square-integrable functions on Γ_d having square integrable derivatives. It would be natural to call this collection the "Sobolev space" $W^{1,2}(\Gamma_d)$. Such spaces are called the "maximal (or decoupled) Sobolev spaces" in the book [18] by Olaf Post. A standard analysis shows that

$$[Hu](x) = -u''(x), \quad \text{for a.e. } x \in \Gamma_d, \text{ for all } u \in D(H).$$

The domain $D(H)$ of H is the set of all functions $u \in W^{1,2}(\Gamma_d)$ such that u' are absolutely continuous on each $I_{n,m}$, the second derivatives obey the condition $u'' \in$

$L^2(\Gamma_d)$, and

$$\sum_{|n-m|=1} \frac{\partial u}{\partial(m-n)}(n) = V(n)u(n) \quad \text{for each } n \in \mathbb{Z}^d. \tag{1.1}$$

By the symbol $\frac{\partial u}{\partial(m-n)}(n)$, we denote the value of the derivative of u at the initial point n of the interval $I_{n,m}$. This time, we distinguish between $I_{n,m}$ and $I_{m,n}$.

Note that if $V = 0$, then (1.1) turns into the Kirchhoff condition. The spectrum of the operator with $V = 0$ is the positive half-line $\mathbb{R}_+ = [0, \infty)$. In this case, $-H$ is the generator of a certain "diffusion" process on Γ. The transition probability of this process is the integral kernel $p(t, x_1, x_2)$ of the operator $\exp(-tH)$. The corresponding random motion could be viewed as a combination of two types of Brownian motion. Namely, a particle moves along one of edges of Γ until it reaches a vertex. After that, it starts moving along the next edge which is randomly selected with the probability $1/(2d)$ similar to the way the next point is selected for the Brownian particle moving on $\mathbb{Z}^d$.

While the spectrum of the operator with $V = 0$ coincides with the spectrum of the operator on $-\Delta$ on $L^2(\mathbb{R}^d)$, it is not absolutely continuous. This is very well illustrated in the paper [8] by Pavel Exner. If $d \geqslant 2$, the points $(\pi n)^2$ with $n \in \mathbb{N}$ are eigenvalues of infinite multiplicity. The corresponding eigenfunctions can be selected so that they are compactly supported.

Another interesting fact is that, if $V \neq 0$ is constant on the lattice Γ, then the spectrum of the operator H has gaps situated either to the left or to the right of the points $(\pi n)^2$. This phenomenon was discovered by Pavel Exner and is also described in [8]. The latter fact is rather surprising because the spectrum of a usual periodic Schrödinger operator on $L^2(\mathbb{R}^d)$ with $d \geqslant 2$ has only finitely many gaps (see [17, 20–22]).

The main results of this paper are Theorems 1.1 and 1.2 described below. Let $\rho(H)$ and $\Sigma(H)$ be the resolvent set and the spectrum of the operator H, respectively. Let $v_0 > 0$ be a positive number. In Theorems 1.1 and 1.2, we consider the case where V is a potential of the form

$$V(n) = \pm(v_0 + \sigma\omega_n),$$

where $\sigma > 0$ is a coupling constant and ω_n are independent random variables identically distributed on $[0, 1]$. We will assume that there is a constant $C > 0$ and $\varkappa > 0$ for which the density of distribution $\nu(\omega_n)$ of the random variable ω_n obeys the condition

$$0 < \nu(\omega_n) \leqslant C\omega_n^{\varkappa}, \quad \text{for all } \omega_n \in (0, 1].$$

Theorem 1.1. *Suppose that* $V(n) = v_0 + \sigma\omega_n$ *with* $v_0 > 0$*. Then the set* $\rho(H) \cap \mathbb{R} = \mathbb{R} \setminus \Sigma(H)$ *contains infinitely many intervals of the form*

$$((\pi n)^2, \lambda_n), \quad n = 1, 2, \dots,$$

where $\lambda_n > 0$ *satisfy the equation*

$$2d(\cos(\sqrt{\lambda_n}) + (-1)^{n+1}) + \frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} v_0 = 0.$$

For each $n \in \mathbb{N}$*, there is a* $\tilde{\lambda}_n > \lambda_n$ *for which the operator* H *has dense pure point spectrum in* $[\lambda_n, \tilde{\lambda}_n]$ *with probability one.*

Theorem 1.2. *Suppose that* $V(n) = -(v_0 + \sigma\omega_n)$ *with* $0 < v_0 < 2d$*. Then the set* $\rho(H) \cap \mathbb{R} = \mathbb{R} \setminus \Sigma(H)$ *contains infinitely many intervals of the form*

$$(\lambda_n, (\pi n)^2), \quad n = 1, 2, \dots,$$

where $\lambda_n > 0$ *satisfy the equation*

$$2d(\cos(\sqrt{\lambda_n}) + (-1)^{n+1}) - \frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} v_0 = 0.$$

For each $n \in \mathbb{N}$*, there is a* $\tilde{\lambda}_n < \lambda_n$ *for which the operator* H *has dense pure point spectrum in* $[\tilde{\lambda}_n, \lambda_n]$ *with probability one.*

Remark. Theorems 1.1 and 1.2 remain valid for $v_0 = 0$ with $\lambda_n = (\pi n)^2$.

These results are essentially different from the theorems in [10] where the authors consider potentials defined on the edges of the graph rather than on the vertices. They are also different from the results of the paper [14] whose assumptions are not fulfilled in the models treated by Theorems 1.1 and 1.2. In fact, the corresponding statement in [14] establishes localization for energies near points E that do not belong to the spectrum of the operator with $V = 0$. Since the spectrum of this operator is $[0, \infty)$, the intervals $[\lambda_n, \tilde{\lambda}_n]$ or $[\tilde{\lambda}_n, \lambda_n]$ considered in Theorems 1.1 and 1.2 are excluded from the discussion in [14] simply because $\lambda_n \to \infty$.

The paper [15] by S. Molchanov and B. Vainberg explains the physics related to the considered model. The graph Γ_d could be interpreted as the union of infinitely many conducting channels that meet at the points of the lattice $\mathbb{Z}^d$. These points are called "junctions." No matter how hard one tries to make the junctions perfect, they still have different properties changing randomly as one passes from one point of junction to another. In reality, such channels are made in silicon by a laser with a very high accuracy, while the junctions are always imperfect. This imperfection is described by the potential V whose values at the points of $\mathbb{Z}^d$ are selected randomly.

Clearly, one could study more general graphs whose theory is given in the book by G. Berkolaiko and P. Kuchment [3]. Our goal is to discuss the most natural model that could serve as a basis for further investigations.

The paper is structured so that it consists of six sections. Theorems 1.1 and 1.2 will be proved in Section 3. Section 4 contains a proof of the exponential decay of eigenfunctions of the operators studied in Section 3. Anderson's localization in the problem with a big coupling constant is discussed in Section 5. Finally, Section 6 is devoted to the theory unbounded random potentials.

2. Preliminaries

The following lemma reduces the questions of our interest to the study of a discrete Schrödinger operator. This reduction is known at least since the publication of the paper by von Below [23]. One can also find it in the articles by Cattaneo [4], Exner [8], and Pankrashkin [16].

Lemma 2.1. *Let $k \in \mathbb{C}$ satisfy the condition $k \neq \pi n$ for all $n \in \mathbb{Z}$. Let ψ be a solution of the equation*

$$-\psi''(x) = k^2\psi(x), \quad x \in \Gamma_d \setminus \mathbb{Z}^d,$$

that is continuous on Γ_d. Then

$$-\sum_{|m-n|=1} \psi(m) + \frac{\sin k}{k} V(n)\psi(n) = -(2d\cos k)\psi(n), \quad \textit{for all } n \in \mathbb{Z}^d.$$

Proof. The statement of the lemma is a simple consequence of the fact that if

$$-\varphi''(t) = k^2\varphi(t), \quad \text{for } t \in [0,1],$$

then

$$\varphi(t) = -\frac{\sin(k(t-1))}{\sin k}\varphi(0) + \frac{\sin(kt)}{\sin k}\varphi(1)$$

Therefore, the derivative of φ at 0 equals

$$\varphi'(0) = -\frac{k\cos(k)}{\sin k}\varphi(0) + \frac{k}{\sin k}\varphi(1)$$

This implies the relation

$$\sum_{|n-m|=1} \frac{\partial\psi}{\partial(m-n)}(n) = -\frac{2dk\cos(k)}{\sin k}\psi(n) + \frac{k}{\sin k}\sum_{|n-m|=1}\psi(m). \qquad \blacksquare$$

Consider two different cases $V > 0$ and $V < 0$. Throughout the paper, $\Sigma(H)$ denotes the spectrum of H, and $\rho(H) = \mathbb{C} \setminus \Sigma(H)$ is its resolvent set.

Corollary 2.2. *Suppose there is a constant $c_0 > 0$ for which*

$$V \geqslant c_0 \quad \textit{on } \mathbb{Z}^d.$$

Then the set $\rho(H) \cap \mathbb{R} = \mathbb{R} \setminus \Sigma(H)$ contains infinitely many intervals of the form

$$((\pi n)^2, \lambda_n), \quad n = 1, 2, \dots,$$

where $\lambda_n > 0$ satisfy the equation

$$2d(\cos(\sqrt{\lambda_n}) + (-1)^{n+1}) + \frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} c_0 = 0. \tag{2.1}$$

The bottom of the spectrum $m_H = \inf\{\lambda \in \Sigma(H)\}$ is not smaller than the least positive solution $\lambda_0 > 0$ of the equation (2.1) for $n = 0$. If $V = c_0$ is constant, then $\lambda_n \in \Sigma(H)$.

In the case $V < 0$, we have to consider two different situations.

Corollary 2.3. *Suppose there is a constant $c_0 > 0$ for which*

$$V \leqslant -c_0 \quad \textit{on } \mathbb{Z}^d.$$

Then the set $\rho(H) \cap \mathbb{R} = \mathbb{R} \setminus \Sigma(H)$ contains infinitely many intervals of the form

$$(\lambda_n, (\pi n)^2), \quad n = 1, 2, \dots,$$

where $\lambda_n > 0$ satisfy the equation

$$2d(\cos(\sqrt{\lambda_n}) + (-1)^{n+1}) - \frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} c_0 = 0,$$

except for the case $c_0 \geqslant 2d$ where the equation for $\lambda_1 \leqslant 0$ has to be replaced by

$$2d(\cosh(\sqrt{-\lambda_1}) + 1) - \frac{\sinh\sqrt{-\lambda_1}}{\sqrt{-\lambda_1}} c_0 = 0.$$

The bottom of the spectrum $\Sigma(H)$ is not higher than the solution $\lambda_0 < 0$ of the equation

$$2d(\cosh(\sqrt{-\lambda_0}) - 1) - \frac{\sinh\sqrt{-\lambda_0}}{\sqrt{-\lambda_0}} c_0 = 0.$$

If $V = -c_0$ is constant, then $\lambda_n \in \Sigma(H)$.

Let us also mention the following interesting fact (see [3, Section 3.4]).

Proposition 2.4. *Let $d \geqslant 2$. The points $(\pi n)^2$ with $n \in \mathbb{N}$ are eigenvalues of H no matter what V is. The multiplicity of these eigenvalues is infinite.*

Proof. If n is odd, then let γ be a closed path in Γ_d containing an even number of intervals $I_{n,m}$ and having no self-intersections. On one of the edges of the path define the function $\psi(t) = \sin(\pi n t)$ for $t \in [0, 1]$. After that, we extend this function to a function on γ so that ψ changes its sign each time one passes through a vertex. Finally, we extend ψ to a function on Γ_d by setting $\psi = 0$ on $\Gamma_d \setminus \gamma$. It is easy to see that $H\psi = (\pi n)^2\psi$.

If n is even, then let γ be any closed path in Γ_d having no self-intersections. On one of the edges of the path, define the function $\psi(t) = \sin(\pi n t)$ for $t \in [0, 1]$. After that, we extend this function to a periodic function on γ. Finally, we extend ψ to a function on Γ_d by setting $\psi = 0$ on $\Gamma_d \setminus \gamma$. It is also easy to see that $H\psi = (\pi n)^2\psi$. ■

Define $\mathfrak{H}$ to be the orthogonal sum of the eigenspaces of H corresponding to the eigenvalues $(\pi m)^2$ with $m \in \mathbb{N}$. The next result describes the orthogonal complement $L^2(\Gamma_d) \ominus \mathfrak{H}$.

Proposition 2.5. *Let $\gamma > 0$ be a fixed positive number. Assume that $V : \mathbb{Z}^d \to \mathbb{R}$ is non-negative and bounded. For each $n \in \mathbb{Z}^d$, set $\varphi_n = \varphi(\cdot, n)$, where $\varphi(x, y)$ denotes the value of the integral kernel of the operator $(H + \gamma I)^{-1}$ at $x, y \in \Gamma_d$. Then the span of all vectors $(H + \gamma I)^{-l}\varphi_n$, corresponding to different $n \in \mathbb{Z}^d$ and $l \in \mathbb{N} \cup \{0\}$, is dense in $L^2(\Gamma_d) \ominus \mathfrak{H}$.*

Proof. Indeed, if $f \in L^2(\Gamma)$ is a function obeying the condition

$$(f, (H + \gamma I)^{-l}\varphi_n) = 0, \quad \text{for all } n \in \mathbb{Z}^d,\ l \in \mathbb{N} \cup \{0\}, \tag{2.2}$$

then the function $u = (H + \gamma I)^{-1} f$ is the solution of the equation

$$(H + \gamma I)u = f$$

that vanishes at each point $n \in \mathbb{Z}^d$. Denote now the subspace of all vectors f satisfying (2.2) by the symbol $\mathfrak{H}_0$. Then $\mathfrak{H}_0$ is an invariant subspace of the operator H. In particular, if $u \in D(H) \cap \mathfrak{H}_0$, then $(H + \gamma I)u \in \mathfrak{H}_0$, which implies that u vanishes at each point $n \in \mathbb{Z}^d$ of the lattice. Thus, the part of the operator H in the subspace $\mathfrak{H}_0$ is also a restriction of the orthogonal sum of operators $-d^2/dx^2$ with the Dirichlet boundary conditions at the endpoints of the intervals $I_{n,m}$. Therefore, $\mathfrak{H}_0 \subset \mathfrak{H}$. ■

Throughout the paper, the symbol $-\Delta$ denotes the operator defined on $\ell^2(\mathbb{Z}^d)$ by

$$[-\Delta u](n) = -\sum_{|m-n|=1} u(m) + 2du(n), \quad \text{for all } n \in \mathbb{Z}^d. \tag{2.3}$$

The following very-well-known result describes the dependence of the matrix elements $R(n,m)$ of the operator $(-\Delta - z + V)^{-1}$ on the values of V. A slightly different proof of this statement can be found in [5].

Lemma 2.6. *Let $n_0 \in \mathbb{Z}^d$ be a fixed point of the lattice. Then for any $z \in \mathbb{C} \setminus \mathbb{R}$, the Green's function $R(n,m) = ((-\Delta - z + V)^{-1}\delta_n, \delta_m)$ of the operator*

$$-\Delta - z + V$$

is a linear fractional function of $V(n_0)$ the form

$$R(n,m) = \frac{\alpha V(n_0) + \beta}{\tilde{\alpha} V(n_0) + \tilde{\beta}}, \tag{2.4}$$

where α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ do not depend on $V(n_0)$.

Proof. Set $\widetilde{V} = V - V(n_0)\delta_{n_0}$ where δ_{n_0} is the operator of multiplication by the characteristic function of the one point set $\{n_0\}$. Then

$$(-\Delta - z + V)^{-1} = (-\Delta - z + \widetilde{V})^{-1} - V(n_0)(-\Delta - z + \widetilde{V})^{-1}\delta_{n_0}(-\Delta - z + V)^{-1}. \tag{2.5}$$

Denote now the integral kernel of the operator $(-\Delta - z + \widetilde{V})^{-1}$ by $\widetilde{R}(n_0, m)$. It follows from (2.5) that

$$R(n_0, m) = \widetilde{R}(n_0, m) - V(n_0)\widetilde{R}(n_0, n_0)R(n_0, m).$$

Thus, $R(n_0, m)$ is a linear fractional function of $V(n_0)$. It remains to note that

$$R(n,m) = \widetilde{R}(n,m) - V(n_0)\widetilde{R}(n, n_0)R(n_0, m). \qquad \blacksquare$$

The following statement gives further information about the role of the points $(\pi n)^2$ in the spectrum of H.

Proposition 2.7. *Suppose*

$$\lim_{|n|\to\infty} V(n) = \infty$$

Then the spectrum of H in the complement

$$\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} \{(\pi n)^2\}$$

is discrete.

Proof. Let $-\Delta$ be the operator defined on $\ell^2(\mathbb{Z}^d)$ by (2.3). Then the spectrum of this operator is the interval $[0, 4d]$. For $M > 0$, define the potential V_M by

$$V_M(n) = \begin{cases} V(n) & \text{if } V(n) \geqslant M, \\ M & \text{if } V(n) < M. \end{cases}$$

Then for any $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an $M > 0$ such that the operator

$$T_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin(k)}{k} V_M$$

has a bounded inverse for all $k \in [\pi n + \varepsilon, \pi(n+1) - \varepsilon]$. Since $V - V_M$ is a finite rank operator, applying the analytic Fredholm alternative, we conclude that the operator

$$I + \frac{\sin(k)}{k}(V - V_M)T_k^{-1} \tag{2.6}$$

has a bounded inverse for all $k \in [(\pi n + \varepsilon, \pi(n+1) - \varepsilon)$ except for a discrete sequence of points. It remains to note that the operator

$$A_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin(k)}{k} V$$

has a bounded inverse whenever the operator (2.6) is invertible. Moreover,

$$A_k^{-1} = \Big(I + \frac{\sin(k)}{k}(V - V_M)T_k^{-1}\Big)^{-1} T_k^{-1}.$$

We leave it as an exercise for the reader to prove that $H - k^2 I$ has a bounded inverse (defined on all of ℓ^2) if and only if A_k has a bounded inverse (defined on all of ℓ^2). ■

Let us mention a similarity of this proposition to the main result of the paper [11] by A. Gordon, S. Molchanov, and B. Tsagani where the authors show that the spectrum of a one-dimensional Schrödinger operator with a special unbounded potential is discrete outside of the union of the points $(\pi n)^2$.

We also need the following result on monotonicity of the Green's function of the discrete Laplace operator.

Proposition 2.8. *Let $V_1 \geqslant 0$ and $V_2 \geqslant 0$ be two bounded non-negative potentials on $\mathbb{Z}^d$ satisfying the inequality*

$$V_1 \leqslant V_2 \quad \text{on } \mathbb{Z}^d.$$

Let $\Gamma_{V_1,\lambda}$ and $\Gamma_{V_2,\lambda}$ be the integral kernels of the operators $(-\Delta + V_1 - \lambda)^{-1}$ and $(-\Delta + V_2 - \lambda)^{-1}$. Then

$$\Gamma_{V_1,\lambda}(n,m) \geqslant \Gamma_{V_2,\lambda}(n,m), \quad \text{for all } n, m \in \mathbb{Z}^d \text{ and } \lambda < 0.$$

Proof. Consider the case where

$$V_1 = V \geqslant 0, \quad V_2 = V + \alpha W \quad \text{for some } W \geqslant 0 \text{ and } \alpha > 0.$$

Then

$$\frac{d}{d\alpha}\Gamma_{V,\lambda}(n,m) = -\sum_{l\in\mathbb{Z}^d}\Gamma_{V,\lambda}(n,l)W(l)\Gamma_{V,\lambda}(l,m) \leqslant 0,$$

because $\Gamma_{V,\lambda} \geqslant 0$. Indeed, for large values of $|\lambda|$ and $n \neq m$

$$\Gamma_{V,\lambda}(n,m) = \sum_{\gamma}\prod_{\tilde{n}\in\gamma}\frac{1}{2d-\lambda+V(\tilde{n})}\Gamma_{V,\lambda}(m,m)$$

where the sum is taken over all paths γ connecting the point m with the point n. Since $\Gamma_{V,\lambda}(m,m) = -1/\lambda + O(1/|\lambda|^2)$ as $\lambda \to -\infty$, we obtain that $\Gamma_{V,\lambda} > 0$ for large values of $|\lambda|$. On the other hand, it is easy to show that

$$\frac{d}{d\lambda}\Gamma_{V,\lambda}(n,m) = \sum_{l\in\mathbb{Z}^d}\Gamma_{V,\lambda}(n,l)\Gamma_{V,\lambda}(l,m) \geqslant 0.$$

Thus,

$$\Gamma_{V,\lambda_1} \leqslant \Gamma_{V,\lambda_2} \quad \text{for } \lambda_1 < \lambda_2 < 0. \qquad \blacksquare$$

To prove localization near the edges of the gaps, we need to know the rate of the decay of the integral kernel $g_\lambda(n-m)$ of the free resolvent $(-\Delta-\lambda)^{-1}$, where $-\Delta$ is the operator (2.3).

Proposition 2.9. *Let* $\lambda \in (-1,0)$*, let* $r \geqslant 0$ *satisfy the inequality* $r+2 > d$ *and let* $g_\lambda(n-m)$ *be the integral kernel of the operator* $(-\Delta-\lambda)^{-1}$*. Then there is a constant* $C_r > 0$ *for which*

$$|n|^r|g_\lambda(n)| \leqslant \frac{C_r}{|\lambda|^{r/2+1-d/2}}, \quad \textit{for all } n \in \mathbb{Z}^d,\ \lambda \in (-1,0). \tag{2.7}$$

The constant C_r *depends only on* r *and* d*.*

Proof. Consider first the case where r is integer. Observe that

$$g_\lambda(n) = (2\pi)^{-d}\int_{[-\pi,\pi)^d}\frac{e^{-i\xi n}\,d\xi}{2d-2\sum_{j=1}^d\cos(\xi_j)-\lambda}.$$

Denote

$$\varphi_\lambda(\xi) = \frac{1}{2d-2\sum_{j=1}^d\cos(\xi_j)-\lambda}.$$

Then there is a constant $C_r > 0$ for which

$$|\nabla^r \varphi_\lambda(\xi)| \leqslant C_r(\varphi_\lambda(\xi))^{r/2+1}, \quad \text{for all } \lambda \in (-1,0). \tag{2.8}$$

This inequality follows from the fact that the r-th order derivatives of $\varphi_\lambda(\xi)$ are functions of the form

$$Q(\xi) = \sum_{m=1}^{r+1} \frac{P_m(\xi)}{(2d - 2\sum_{j=1}^d \cos(\xi_j) - \lambda)^m}, \tag{2.9}$$

where P_m are linear combinations of products of $\sin(\xi_{j_1})$ and $\cos(\xi_{j_2})$ such that $|P_m(\xi)| \leqslant C|\xi|^{2m-r-2}$ for $m > r/2+1$. The representation (2.9) is in its turn established by induction in r. Indeed,

$$\frac{\partial Q}{\partial \xi_i} = \sum_{m=1}^{r} \frac{2P_m(\xi)\sin(\xi_i)}{(2d - 2\sum_{j=1}^d \cos(\xi_j) - \lambda)^{m+1}} + \sum_{m=1}^{r} \frac{\partial P_m/\partial \xi_i}{(2d - 2\sum_{j=1}^d \cos(\xi_j) - \lambda)^m}.$$

The estimate (2.8) implies that

$$|n|^r|g_\lambda(n)| \leqslant \widetilde{C}_r \int\limits_{[-\pi,\pi)^d} (\varphi_\lambda(\xi))^{r/2+1}\, d\xi. \tag{2.10}$$

Obviously, this inequality is also valid for $r = 0$. In the case r is not integer, (2.10) is established by interpolation.

On the other hand, there is constant $c_0 > 0$ such that

$$c_0|\xi|^2 \leqslant 2d - 2\sum_{j=1}^d \cos(\xi_j), \quad \text{for all } \xi \in [-\pi,\pi)^d. \tag{2.11}$$

Combining (2.10) with (2.11), we obtain that

$$|n|^r|g_\lambda(n)| \leqslant \widetilde{C}_r \int\limits_{\mathbb{R}^d} (c_0|\xi|^2 - \lambda)^{-(r/2+1)}\, d\xi = \frac{C_r}{|\lambda|^{r/2+1-d/2}}. \qquad \blacksquare$$

The next statement plays a crucial role in the proof of Theorem 1.1.

Proposition 2.10. *Let* $-1 < \lambda < 0$ *and let* $g_\lambda(n-m)$ *be the integral kernel of the operator* $(-\Delta - \lambda)^{-1}$. *Let also* $1/2 < s < 1$. *Then there is a constant* C_s, *depending only on* s *and* d, *for which*

$$\sum_{n\in\mathbb{Z}^d} |g_\lambda(n)|^s \leqslant C_s|\lambda|^{-s-\frac{(1-s)d}{2}}, \quad \textit{for all } \lambda < (-1,0). \tag{2.12}$$

Proof. Note first that there is a constant $C_0 > 0$ depending only on d for which

$$|g_\lambda(n)| \leqslant f(\lambda) = \begin{cases} C_0|\lambda|^{-1/2} & \text{if } d = 1, \\ C_0 \ln|\lambda|^{-1} & \text{if } d \geqslant 2, \\ C_0 & \text{if } d \geqslant 3. \end{cases}$$

Let us choose parameters r and $r_1 > d - 2$ so that $rs > d$, but $sr_1 < d$. In view of (2.7), the left-hand side of (2.12) can be estimated as follows:

$$\begin{aligned} \sum_{n\in\mathbb{Z}^d} |g_\lambda(n)|^s &\leqslant 5\sqrt{d}\, f(\lambda) \;+ \sum_{2\sqrt{d}\leqslant|n|\leqslant 2\sqrt{d/|\lambda|}} |g_\lambda(n)|^s \;+ \sum_{|n|\geqslant 2\sqrt{d/|\lambda|}} |g_\lambda(n)|^s \\ &\leqslant 5\sqrt{d}\, f(\lambda) + \frac{|C_{r_1}|^s}{d^{sr_1/2}|\lambda|^{s(r_1/2+1-d/2)}} \int\limits_{|x|<2\sqrt{d/|\lambda|}} |x|^{-sr_1}\, dx \\ &\quad + \frac{|C_r|^s}{d^{sr/2}|\lambda|^{s(r/2+1-d/2)}} \int\limits_{|x|>2\sqrt{d/|\lambda|}} |x|^{-sr}\, dx \\ &\leqslant C_s|\lambda|^{-s-\frac{(1-s)d}{2}}. \end{aligned}$$

■

3. Proof of Theorems 1.1 and 1.2

Recall that we consider the case where V is a potential of the form

$$V(n) = \pm(v_0 + \sigma\omega_n),$$

where $v_0 > 0$, $\sigma > 0$ is a coupling constant and ω_n are independent random variables identically distributed on $[0, 1]$. We will assume that there is a constant $C > 0$ and $\varkappa > 0$ for which the density of distribution $\nu(\omega_n)$ of the random variable ω_n obeys the condition

$$0 < \nu(\omega_n) \leqslant C\omega_n^{\varkappa}, \quad \text{for all } \omega_n \in (0, 1].$$

We will prove only Theorem 1.1 in which V is positive. The proof of Theorem 1.2 follows the same pattern. Let λ_* be the solution of the equation

$$2d(\cos(\sqrt{\lambda_*}) - 1) + \frac{\sin\sqrt{\lambda_*}}{\sqrt{\lambda_*}} v_0 = 0$$

such that $(\pi n)^2 < \lambda_* < (\pi(n+1))^2$ where with n is an even number and

$$2d(\cos(\sqrt{\lambda}) - 1) + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} v_0 > 0, \quad \text{for all } \lambda \in ((\pi n)^2, \lambda_*).$$

We will show that there is a $\tilde{\lambda} > \lambda_*$ such that H has dense pure point spectrum in $[\lambda_*, \tilde{\lambda}]$.

For a fixed $0 < \varepsilon < 1$, we introduce the random variables ξ_n by

$$\xi_n = \begin{cases} \omega_n - \varepsilon & \text{if } \omega_n < \varepsilon, \\ 0 & \text{if } \omega_n \geqslant \varepsilon, \end{cases}$$

and set

$$\eta_n = \omega_n - \xi_n.$$

Then $\eta_n \geqslant \varepsilon$ and the potential V can be decomposed into the sum

$$V = V_\varepsilon + \widetilde{V}_\varepsilon,$$

where $V_\varepsilon(n) = \sigma\xi_n$ and $\widetilde{V}_\varepsilon(n) = v_0 + \sigma\eta_n$. Therefore, the operator

$$A_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin k}{k} V$$

can be written in the form

$$A_k = \widetilde{A}_k + \frac{\sin k}{k} V_\varepsilon, \quad \text{where } \widetilde{A}_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin k}{k} \widetilde{V}_\varepsilon.$$

We see now that A_k^{-1} may be considered as a solution of the equation

$$A_k^{-1} - \widetilde{A}_k^{-1} = -\frac{\sin k}{k} \widetilde{A}_k V_\varepsilon A_k,$$

which is convenient to write in terms of integral kernels $G_k(n,m)$ and $\widetilde{G}_k(n,m)$ of the operators A_k^{-1} and $\widetilde{A}_k^{-1}$:

$$G_k(n,m) - \widetilde{G}_k(n,m) = -\frac{\sin k}{k} \sum_l \widetilde{G}_k(n,l) V_\varepsilon(l) G_k(l,m),$$

Using the obvious inequality $|\sum_j c_j|^s \leqslant \sum_j |c_j|^s$ for $0 < s < 1$, we obtain

$$|G_k(n,m)|^s \leqslant |\widetilde{G}_k(n,m)|^s + \left|\frac{\sin k}{k}\right|^s \sum_l |\widetilde{G}_k(n,l)|^s |V_\varepsilon(l)|^s |G_k(l,m)|^s, \quad (3.1)$$

Finally, using monotonicity of the Green's function with respect to the potential, we estimate the positive function $\widetilde{G}_k(n,m)$ by the kernel $\Gamma_k(n,m)$ of the operator

$$\Big(-\Delta + 2d(\cos(k) - 1) + \frac{\sin k}{k}(v_0 + \sigma\varepsilon)\Big)^{-1}.$$

Namely, we have the inequality

$$\widetilde{G}_k(n,m) \leqslant \Gamma_k(n,m), \quad \text{for all } n,m \in \mathbb{Z}^d. \tag{3.2}$$

Now, let $k_\varepsilon \in (\sqrt{\lambda_*}, \pi(n+1))$ be the solution of the equation

$$2d(\cos(k_\varepsilon)-1) + \frac{\sin k_\varepsilon}{k_\varepsilon}(v_0 + \sigma\varepsilon/2) = 0$$

obeying the condition $k_\varepsilon \to \sqrt{\lambda_*}$ as $\varepsilon \to 0$. Then for any $k \in (\sqrt{\lambda_*}, k_\varepsilon)$,

$$2d(\cos(k)-1) + \frac{\sin k}{k}(v_0 + \sigma\varepsilon/2) > 0.$$

Consequently,

$$2d(\cos(k)-1) + \frac{\sin k}{k}(v_0 + \sigma\varepsilon) > \frac{\sin k}{2k}\sigma\varepsilon.$$

Therefore,

$$\Gamma_k(n,m) \leqslant g(n-m), \quad \text{for all } n,m \in \mathbb{Z}^d, \tag{3.3}$$

where g is the integral kernel of the operator

$$(-\Delta + \tau\sigma\varepsilon)^{-1} \quad \text{and} \quad \tau = \inf_{k\in[\sqrt{\lambda_*},k_\varepsilon]} \left(\frac{\sin k}{2k}\right) > 0$$

Combining (3.1), (3.2), and (3.3), we obtain that $|G_k(n,m)|^s$ satisfies the inequality

$$|G_k(n,m)|^s \leqslant |\widetilde{G}_k(n,m)|^s + \left|\frac{1}{\sqrt{\lambda_*}}\right|^s \sum_l |g(n-l)|^s |V_\varepsilon(l)|^s |G_k(l,m)|^s, \tag{3.4}$$

Now, we will use the following lemma.

Lemma 3.1. *There is a constant* $C(s,\varkappa) > 0$ *depending only on* $s \in (0,1)$ *and* $\varkappa > 0$ *such that for all* $0 < \varepsilon < 1$,

$$\int_0^\varepsilon |v-\varepsilon|^s \left|\frac{v-a}{v-b}\right|^s v^\varkappa \, dv \leqslant C(s,\varkappa)\varepsilon^{\varkappa+1} \int_0^1 \left|\frac{v-a}{v-b}\right|^s v^\varkappa \, dv \tag{3.5}$$

for all a, b *in* $\mathbb{C}$.

Proof. Note that

$$\int_0^1 |v-1|^s \left|\frac{v-a}{v-b}\right|^s v^\varkappa \, dv \leqslant \int_0^1 \left|\frac{v-a}{v-b}\right|^s v^\varkappa \, dv \tag{3.6}$$

Also note that one can find two constants $c_{s,\varkappa} > 0$ and $C_{s,\varkappa} > 0$ for which

$$c_{s,\varkappa}\frac{1+|a|^s}{1+|b|^s} \leqslant \int_0^1 \left|\frac{v-a}{v-b}\right|^s v^\varkappa\, dv \leqslant C_{s,\varkappa}\frac{1+|a|^s}{1+|b|^s} \tag{3.7}$$

To prove (3.7), we consider the function

$$f(a,b) = \left(\frac{1+|a|^s}{1+|b|^s}\right)^{-1} \int_0^1 \left|\frac{v-a}{v-b}\right|^s v^\varkappa\, dv. \tag{3.8}$$

If $|a| \leqslant 2$ and $|b| \leqslant 2$, then $f(a,b)$ is a continuous positive function of a and b defined on the square. Therefore, it is bounded from above and separated from zero by a positive constant.

On the other hand, if $|a| > 2$ or $|b| > 2$ then $|v-a|^s$ and $|v-b|^s$ are equivalent to the constant functions $1+|a|^s$ and $1+|b|^s$ correspondingly.

To prove (3.5), we make a substitution

$$\int_0^\varepsilon |v-\varepsilon|^s \left|\frac{v-a}{v-b}\right|^s v^\varkappa\, dv = \varepsilon^{s+\varkappa+1} \int_0^1 |v-1|^s \left|\frac{v-a/\varepsilon}{v-b/\varepsilon}\right|^s v^\varkappa\, dv$$
$$\leqslant \varepsilon^{s+\varkappa+1} C_{s,\varkappa}\frac{\varepsilon^s+|a|^s}{\varepsilon^s+|b|^s}.$$

Consequently,

$$\int_0^\varepsilon |v-\varepsilon|^s \left|\frac{v-a}{v-b}\right|^s v^\varkappa\, dv \leqslant \varepsilon^{\varkappa+1} C_{s,\varkappa}\frac{1+|a|^s}{1+|b|^s}.$$

Combining this estimate with the left inequality in (3.7), we obtain (3.5). ■

Denote by $\langle f \rangle$ the mean value of a random variable f defined on Ω. That is,

$$\langle f \rangle = \int_\Omega f(\omega)\, d\omega$$

where $d\omega$ is the probability measure on Ω.

Clearly, if $|\operatorname{Im} k| > 0$, then

$$\langle |G_k(n,m)|^s \rangle < \infty, \tag{3.9}$$

for any $s \in (0,1)$. Our nearest goal is to prove (3.9) for some real k's with $k^2 \in \Sigma(H)$.

Corollary 3.2. *There is a constant* $C > 0$ *that depends only on* s, $\varkappa$ *and* σ *for which the mean values of* $|G_k(n,m)|^s$ *satisfy the inequality*

$$\langle |G_k(n,m)|^s\rangle \leqslant |\widetilde{G}_k(n,m)|^s + \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s}\sum_l |g(n-l)|^s \langle |G_k(l,m)|^s\rangle,$$

Proof. This is a consequence of (3.4), Lemma 2.4 and Lemma 3.1. ■

Consider now the "integral" operator T from $\ell^1(\mathbb{Z}^d)$ to $\ell^1(\mathbb{Z}^d)$ defined by

$$[Tu](n) = \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s}\sum_l |g(n-l)|^s u(l)$$

Obviously,

$$\|T\| \leqslant \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s}\sum_n |g(n)|^s.$$

Proposition 3.3. *Let*

$$(1-s)(d-2) < 2\varkappa.$$

Then there is an $\varepsilon > 0$ *for which*

$$\|T\| < 1. \tag{3.10}$$

Proof. According to Proposition 2.10, for $1/2 < s < 1$,

$$\sum_{n\in\mathbb{Z}^d} |g(n)|^s \leqslant \frac{C_s}{|\sigma\tau\varepsilon|^{s+\frac{(1-s)d}{2}}}, \quad \text{for all } \varepsilon > 0.$$

Consequently, if $(1-s)(d-2) < 2\varkappa$, then

$$\|T\| \to 0, \quad \text{as } \varepsilon \to 0.$$ ■

Corollary 3.4. *Let* (3.10) *be fulfilled. Then there is a finite positive constant* $C > 0$ *such that*

$$\sum_n \langle |G_k(n,m)|^s\rangle < C, \tag{3.11}$$

for each $k \in [\sqrt{\lambda_*}, k_\varepsilon]$ *and for each* $m \in \mathbb{Z}^d$.

To finish the proof of Theorem 1.1 we will use the Simon–Wolff Theorem (see [12, 19]) described below.

Let A be a cyclic self-adjoint operator on a separable Hilbert space, let φ be its normalised cyclic vector, and P be the orthogonal projection onto the span of φ. Denote by μ the spectral measure of the operator A corresponding to the vector φ.

Theorem 3.5. *Let $\mathfrak{U}$ be a Borel subset of $\mathbb{R}$. The operator $A - \alpha P$ has pure point spectrum in $\mathfrak{U}$ for almost every value of $\alpha \in \mathbb{R}$ if and only if*

$$\int_0^\infty \frac{d\mu(t)}{(t-\lambda)^2} < \infty \quad \textit{for Lebesgue a.e. } \lambda \in \mathfrak{U}. \tag{3.12}$$

The condition (3.12) literally means that

$$\|(A - \lambda I)^{-1}\varphi\|^2 < \infty.$$

Let us now write the Hilbert identity for the operator $H\colon D(H) \to L^2(\Gamma_d)$ considered in Theorem 1.1. Changing the value of V by α at $x = n$ can be formally viewed as adding $\alpha\delta_n$ to H. Here, we understand δ_n as a distribution on the graph Γ_d (not as an element of $\ell^2(\mathbb{Z}^d)$). For any $\gamma > 0$, and any fixed point $n \in \mathbb{Z}^d$,

$$(H + \alpha\delta_n + \gamma I)^{-1} = (H + \gamma I)^{-1} - \beta(\alpha)P,$$

where P is the projection onto the span of the vector $\varphi = (H + \gamma I)^{-1}\delta_n$ and $\beta(\alpha)$ is a meromorphic function of α such that $\frac{d\beta}{d\alpha}(0) = \|\varphi\|^2$. Applying the Simon–Wolff theorem, we obtain that the operator $(H + \gamma I)^{-1}$ almost surely has pure point spectrum in the Borel set $\mathfrak{U}$ if and only if

$$((H + \gamma I)^{-1} - \lambda I)^{-1}(H + \gamma I)^{-1}\delta_n \in L^2(\Gamma_d), \quad \text{for all } n \in \mathbb{Z}^d \tag{3.13}$$

for almost every $\lambda \in \mathfrak{U}$. Note that condition (3.13) can be rewritten in the form

$$(H + \gamma I - \lambda^{-1} I)^{-1}\delta_n \in L^2(\Gamma_d), \quad \text{for all } n \in \mathbb{Z}^d,$$

which is, in its turn, equivalent to

$$\sum_m |G_k(m,n)|^2 < \infty \quad \text{for } k^2 = \lambda^{-1} - \gamma. \tag{3.14}$$

Since (3.11) implies (3.14) for almost every $k \in [\sqrt{\lambda_*}, k_\varepsilon]$, it also implies Theorem 1.1. One could argue that $\varphi = (H + \gamma I)^{-1}\delta_n$ is not a cyclic vector, but this condition is not needed. Instead, one has to define $\mathfrak{H}$ to be the orthogonal sum of the eigenspaces of H corresponding to the eigenvalues $(\pi m)^2$ with $m \in \mathbb{N}$ and realise that, according to Proposition 2.5, the span of all vectors $(H + \gamma I)^{-l}\delta_n$, corresponding to different $n \in \mathbb{Z}^d$ and $l \in \mathbb{N}$, is dense in $L^2(\Gamma_d) \ominus \mathfrak{H}$.

The proof of Theorem 1.2 is analogous to the one of Theorem 1.1.

4. Exponential decay of eigenfunctions

Here we establish an exponential decay of the eigenfunctions of the operators considered in Section 3. Let us recall that one of the assumptions made in Section 3 was that there is a constant $C > 0$ and $\varkappa > 0$ for which the density of distribution $\nu(\omega_n)$ of the random variable ω_n obeys the condition

$$0 < \nu(\omega_n) \leqslant C\omega_n^{\varkappa}, \quad \text{for all } \omega_n \in (0,1].$$

Theorem 4.1. *Let the function V and the operator H be the same as in Theorem 1.1 Assume that $\varkappa > d-1$. Let $\lambda_n > 0$ be the points described in Theorem 1.1. Then for each $n \in \mathbb{N}$, there is a $\tilde\lambda_n > \lambda_n$ for which the eigenfunctions corresponding to the eigenvalues in $[\lambda_n, \tilde\lambda_n]$ decay exponentially fast almost surely.*

Theorem 4.2. *Let the function V and the operator H be the same as in Theorem 1.2 Assume that $\varkappa > d-1$. Let $\lambda_n \in \mathbb{R}$ be the points described in Theorem 1.2. Then for each $n \in \mathbb{N}$, there is a $\tilde\lambda_n < \lambda_n$ for which the eigenfunctions corresponding to the eigenvalues in $[\tilde\lambda_n, \lambda_n]$ decay exponentially fast almost surely.*

Remark. Theorems 4.1 and 4.2 remain valid for $v_0 = 0$ with $\lambda_n = (\pi n)^2$.

The remaining part of this section consists of the proofs of these theorems. Let us first establish the exponential decay of the Green's function $g_\lambda(n-m)$ of the integral kernel of the operator $(-\Delta - \lambda)^{-1}$ for $\lambda < 0$.

Theorem 4.3. *Let $\lambda < 0$. Then*

$$0 \leqslant g_\lambda(n-n_0) \leqslant \left|\frac{2d}{2d-\lambda}\right|^{|n-n_0|}\Big(\frac{2d-\lambda}{|\lambda|^2}\Big).$$

Proof. We will use the equation

$$2d\, g_\lambda(n-n_0) - \sum_{m:|m-n|=1} g_\lambda(m-n_0) = \lambda g_\lambda(n-n_0) + \delta_{n_0}(n)$$

that is satisfied by the function g_λ. This equation can be written in the form

$$g_\lambda(n-n_0) = \frac{1}{2d-\lambda}\sum_{m:|m-n|=1} g_\lambda(m-n_0) + \frac{1}{2d-\lambda}\delta_{n_0}(n) \tag{4.1}$$

Repeating the arguments of in [5, Lemma 2.3], we obtain

$$0 \leqslant g_\lambda(n-n_0) \leqslant \left|\frac{2d}{2d-\lambda}\right|^{|n-n_0|}\Big(\frac{2d-\lambda}{|\lambda|}\Big) g_\lambda(0). \tag{4.2}$$

Indeed, equation (4.1) implies that for $n \neq n_0$,

$$g_\lambda(n-n_0) = \sum_\gamma \Big(\frac{1}{2d-\lambda}\Big)^{|\gamma|} g_\lambda(0),$$

where the sum is taken over all paths connecting the points n and n_0. Since the number of the paths γ of length k that start at n is $(2d)^k$, and the paths that we consider possess the property $|\gamma| \geqslant |n-n_0|$, we conclude that

$$0 \leqslant g_\lambda(n-n_0) \leqslant \sum_{k \geqslant |n-n_0|} \Big(\frac{2d}{2d-\lambda}\Big)^k g_\lambda(0),$$

which implies (4.2). ■

Corollary 4.4. *Let* $c_0 = (6d)^{-1}$ *and* $C_1 = 2d+1$. *Then*

$$0 \leqslant g_\lambda(n-n_0) \leqslant \frac{C_1 e^{-c_0|\lambda||n-n_0|}}{|\lambda|^2}, \quad \textit{for all } n, n_0 \in \mathbb{Z}^d \textit{ and } \lambda \in (-1,0). \tag{4.3}$$

Corollary 4.5. *Let* $s \in (0,1)$ *and let* $\mu = \frac{s|\lambda|}{12d}$. *Then there is a constant* $C_s > 0$ *depending only on* s *and* d, *for which*

$$\sum_{n\in\mathbb{Z}^d} e^{\mu|n|} |g_\lambda(n)|^s \leqslant \frac{C_s}{|\lambda|^{2s+d}}, \quad \textit{for all } \lambda \in (-1,0). \tag{4.4}$$

Proof. It follows from (4.3) that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} e^{\mu|n|}|g_\lambda(n)|^s &\leqslant \sum_{|n|\leqslant 2\sqrt{d}} \frac{C_1^s e^{-2^{-1}c_0 s|\lambda||n|}}{|\lambda|^{2s}} + \sum_{|n|>2\sqrt{d}} \frac{C_1^s e^{-2^{-1}c_0 s|\lambda||n|}}{|\lambda|^{2s}} \\ &\leqslant \sum_{|n|\leqslant 2\sqrt{d}} \frac{C_1^s}{|\lambda|^{2s}} + \int_{\mathbb{R}^d} \frac{\tilde{C} e^{-2^{-1}c_0 s|\lambda||x|}}{|\lambda|^{2s}}\, dx.\end{aligned}$$

It remains to make a substitution $x = |\lambda|^{-1}\tilde{x}$ in the last integral. ■

Let us now prove Theorem 4.1. As in Section 3, consider the case where $v_0 > 0$, and λ_* is the solution of the equation

$$2d(\cos(\sqrt{\lambda_*})-1) + \frac{\sin\sqrt{\lambda_*}}{\sqrt{\lambda_*}} v_0 = 0.$$

such that $(\pi n)^2 < \lambda_* < (\pi(n+1))^2$ where with n is an even number and

$$2d(\cos(\sqrt{\lambda})-1) + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} v_0 > 0, \quad \text{for all } \lambda \in ((\pi n)^2, \lambda_*).$$

For a fixed $0 < \varepsilon < 1$, we introduce the random variables ξ_n by

$$\xi_n = \begin{cases} \omega_n - \varepsilon & \text{if } \omega_n < \varepsilon, \\ 0 & \text{if } \omega_n \geqslant \varepsilon, \end{cases}$$

and set

$$\eta_n = \omega_n - \xi_n.$$

Then $\eta_n \geqslant \varepsilon$ and the potential V can be decomposed into the sum

$$V = V_\varepsilon + \widetilde{V}_\varepsilon,$$

where $V_\varepsilon(n) = \sigma \xi_n$ and $\widetilde{V}_\varepsilon(n) = v_0 + \sigma \eta_n$.

Our goal is to obtain an estimate for the integral kernel $G_k(n, m)$ of the operator A_k^{-1}, where

$$A_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin k}{k} V.$$

For that purpose, we define $k_\varepsilon \in (\sqrt{\lambda_*}, \pi(n+1))$ to be the solution of the equation

$$2d(\cos(k_\varepsilon) - 1) + \frac{\sin k_\varepsilon}{k_\varepsilon}(v_0 + \sigma\varepsilon/2) = 0$$

obeying the condition $k_\varepsilon \to \sqrt{\lambda_*}$ as $\varepsilon \to 0$. After that, we set

$$\tau = \inf_{k \in [\sqrt{\lambda_*}, k_\varepsilon]} \left(\frac{\sin k}{2k} \right) > 0,$$

and define g to be the integral kernel of the operator

$$(-\Delta + \tau\sigma\varepsilon)^{-1}.$$

Combining (3.1), (3.2), and (3.3), we obtain that $|G_k(n, m)|^s$ satisfies the inequality

$$|G_k(n, m)|^s \leqslant |g(n-m)|^s + \left| \frac{1}{\sqrt{\lambda_*}} \right|^s \sum_l |g(n-l)|^s |V_\varepsilon(l)|^s |G_k(l, m)|^s, \quad (4.5)$$

Combining (3.5) and (4.7) we obtain the following statement.

Corollary 4.6. *There is a constant $C > 0$ that depends only on s, $\varkappa$ and σ for which the mean values of $|G_k(n, m)|^s$ satisfy the inequality*

$$\langle |G_k(n, m)|^s \rangle \leqslant |g(n-m)|^s + \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s} \sum_l |g(n-l)|^s \langle |G_k(l, m)|^s \rangle, \quad (4.6)$$

Since we intend to establish an exponential decay of the Green's function, it makes sense to work with the quantities of the form

$$\Theta_k(n,m) = \exp(\mu_*|n-m|)\langle |G_k(n,m)|^s\rangle \quad \text{where } \mu_* = \frac{s|\tau\varepsilon\sigma|}{12d}.$$

Using the triangle inequality $|n-m| \leqslant |n-l| + |l-m|$, we infer from (4.6) that

$$\Theta_k(n,m) \leqslant |g(n-m)|^s e^{\mu_*|n-m|} + \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s} \sum_l |g(n-l)|^s e^{\mu_*|n-l|}\Theta_k(l,m). \tag{4.7}$$

Consider now the "integral" operator T from $\ell^1(\mathbb{Z}^d)$ to $\ell^1(\mathbb{Z}^d)$ defined by

$$[Tu](n) = \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s} \sum_l |g(n-l)|^s e^{\mu_*|n-l|} u(l).$$

Obviously,

$$\|T\| \leqslant \frac{C\varepsilon^{\varkappa+1}}{|\sqrt{\lambda_*}|^s} \sum_n |g(n)|^s e^{\mu_*|n|}.$$

Proposition 4.7. *Let*

$$\varkappa > 2s - 1 + d.$$

Then there is an $\varepsilon > 0$ *for which*

$$\|T\| < 1. \tag{4.8}$$

Proof. According to (4.4), for $0 < s < 1$,

$$\sum_{n\in\mathbb{Z}^d} |g(n)|^s e^{\mu_*|n|} \leqslant \frac{C_s}{|\sigma\tau\varepsilon|^{2s+d}}, \quad \text{for all } \varepsilon > 0.$$

Consequently, if $\varkappa > 2s - 1 + d$, then

$$\|T\| \to 0, \quad \text{as } \varepsilon \to 0.$$

■

Corollary 4.8. *Let* $\varepsilon > 0$ *be a number for which* (4.8) *is fulfilled for all* $k \in [\sqrt{\lambda_*}, k_\varepsilon]$. *Then there is a finite positive constant* $C > 0$ *such that*

$$\sum_n \langle |G_k(n,m)|^s\rangle \exp(\mu_*|n-m|) < C,$$

for each $k \in [\sqrt{\lambda_*}, k_\varepsilon]$ *and for each* $m \in \mathbb{Z}^d$.

We see that the Green's function $G_k(n,m)$ decays exponentially fast as $|n-m| \to \infty$ for a.e. $k \in [\sqrt{\lambda_*}, k_\varepsilon]$. Therefore, the integral kernel of $(H - \lambda I)^{-1}$ decays exponentially as well for a.e. $\lambda \in [\lambda_n, \tilde{\lambda}_n]$. This implies Theorem 4.1 (see [19]). The proof in [19] is based on the following observation. For a.e. $\omega = \{\omega_n\}$, the Green's function $G_k(n,m)$ satisfies

$$\sum_n |G_k(n,m)|^s \exp(\mu_*|n-m|) < \infty, \tag{4.9}$$

for each $m \in \mathbb{Z}^d$ and almost every $k \in [\sqrt{\lambda_*}, k_\varepsilon]$. Thus, (4.9) holds almost surely for all $m \in \mathbb{Z}^d$ on a subset $K \subset [\sqrt{\lambda_*}, k_\varepsilon]$ of full measure. The fact that the Lebesgue measure of $K_0 = \{k^2 : k \in [\sqrt{\lambda_*}, k_\varepsilon] \setminus K\}$ equals zero implies that the spectral measure of this set is also zero almost surely. To show this, we use the Hilbert identity for the operator H. For any $\gamma > 1$, and any fixed point $n \in \mathbb{Z}^d$,

$$(H + \alpha\delta_n + \gamma I)^{-1} = (H + \gamma I)^{-1} - \beta(\alpha)P,$$

where P is the projection onto the span of the vector $\varphi = (H + \gamma I)^{-1}\delta_n$ and $\beta(\alpha)$ is an meromorphic function of α such that $\frac{d\beta}{d\alpha}(0) = \|\varphi\|^2$. Denote by μ_β the spectral measure of the operator $(H + \gamma I)^{-1} - \beta P$ corresponding to the vector φ. Then according to Atkinson's averaging formula (see [2]), the Lebesgue measure of K_0 equals

$$|K_0| = \int_{\mathbb{R}} \mu_\beta(K_0)\, d\beta.$$

Therefore, $\mu_\beta(K_0) = 0$ for almost every β, and hence, $\mu_{\beta(\alpha)}(K_0) = 0$ for almost every $\alpha \in \mathbb{R}$. By Fubini's theorem, the spectral measure of the operator H corresponding to any vector $(H + \gamma I)^{-1}\delta_n$ vanishes on the set K_0 almost surely. This is enough to claim that K_0 is free of eigenvalues of H for almost every ω.

Note now that, for $\alpha \neq 0$, the sequence $\{\psi(n)\}$ of the values of an eigenfunction of $H + \alpha\delta_{n_0}$ corresponding to the eigenvalue $k^2 \notin K_0$, satisfies the equation $A_k\psi + \psi(n_0)\frac{\sin(k)}{k}\alpha\delta_{n_0} = 0$. Consequently, the values $\psi(n)$ are given by the formula

$$\psi(n) = -\frac{\sin(k)}{k}\alpha G_k(n, n_0)\psi(n_0), \quad \text{for all } n \in \mathbb{Z}^d.$$

Thus, $\psi(n)$ decays exponentially.

This proves Theorem 4.1. The proof of Theorem 4.2 is analogous to the one of Theorem 4.1.

5. Localization in the problem with the large coupling constant

Here we consider potentials of the form

$$V(n) = V_0(n) + \sigma\omega_n, \quad n \in \mathbb{Z}^d,$$

where V_0 is a fixed bounded function and ω_n are independent random variables uniformly distributed on the interval [0, 1]. To establish localization, we repeat some of the steps of the proof of the main result of [1].

Since no specific condition was imposed on the potential V in Lemma 2.4, we may state that for any fixed point $n_0 \in \mathbb{Z}^d$, the Green's function $G_k(n,m)$ of the operator

$$A_k = -\Delta + 2d(\cos(k) - 1) + \frac{\sin(k)}{k} V$$

is a linear fractional function of V_{n_0} the form

$$G_k(n,m) = \frac{\alpha V(n_0) + \beta}{\tilde{\alpha} V(n_0) + \tilde{\beta}}, \tag{5.1}$$

where α, β, $\tilde{\alpha}$, and $\tilde{\beta}$ do not depend on $V(n_0)$.

The following lemma is one of the key statements in the theory of random operators on the lattice $\mathbb{Z}^d$ (the proofs can be found in [1, 5]).

Lemma 5.1. *For each $0 < s < 1/2$, there is a constant $C_0(s) > 0$ such that*

$$\int_0^1 \frac{1}{|v-c|^s} \left|\frac{v-a}{v-b}\right|^s dv \leqslant C_0(s) \int_0^1 \left|\frac{v-a}{v-b}\right|^s dv$$

for all a, b, and c in $\mathbb{C}$. The constant $C_0(s)$ depends only on s.

Without loss of generality, we may assume that

$$\int_0^1 \frac{1}{|v-c|^s} dv \leqslant C_0(s)$$

for all c in $\mathbb{C}$.

Theorem 5.2. *Let $\operatorname{Im} k^2 > 0$, $0 < s < 1/2$ and let $G_k(n,m)$ be the integral kernel of the operator A_k^{-1} where $A_k \colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is defined by*

$$[A_k u](n) = -\sum_{|m-n|=1} u(m) + \frac{\sin k}{k} V(n) u(n) + (2d \cos k) u(n), \quad \text{for all } n \in \mathbb{Z}^d.$$

Suppose that V is a random potential function of the form

$$V(n) = V_0(n) + \sigma \omega_n,$$

where ω_n *are independent random variables uniformly distributed on* $[0,1]$. *Then for each* $n_0 \in \mathbb{Z}^d$,

$$\langle |G_k(n,n_0)|^s\rangle \leqslant \frac{C_0(s)|k|^s}{\sigma^s|\sin k|^s}\Big(\sum_{|m-n|=1}\langle |G_k(m,n_0)|^s\rangle + \delta_{n_0}\Big), \quad \textit{for all } n \in \mathbb{Z}^d. \tag{5.2}$$

Proof. Green's function satisfies the equation

$$-\sum_{|m-n|=1} G_k(m,n_0) + \frac{\sin k}{k}V(n)G_k(n,n_0) + (2d\cos k)G_k(n,n_0) = \delta_{n_0}.$$

Consequently,

$$G_k(n,n_0) = \Big(\frac{\sin k}{k}V(n) + (2d\cos k)\Big)^{-1}\Big(\sum_{|m-n|=1} G_k(m,n_0) + \delta_{n_0}\Big).$$

It remains to use (5.1) and apply Lemma 5.1. ■

Theorem 5.3. *Let* $[\lambda_1,\lambda_2] \subset [0,\infty)$ *be a bounded closed interval. Suppose*

$$\frac{C_0(s)|k|^s}{\sigma^s|\sin k|^s} < \frac{1}{2d} \quad \textit{for all } k^2 \in [\lambda_1,\lambda_2].$$

Then the operator H *has pure point spectrum in* $[\lambda_1,\lambda_2]$. *The eigenfunctions corresponding to eigenvalues in* $[\lambda_1,\lambda_2]$ *decay at infinity exponentially.*

Proof. Under the conditions of the theorem, the inequality (5.2) implies that there is an $\varepsilon > 0$ such that

$$\Big(1+\frac{\varepsilon}{2d}\Big)\langle |G_k(n,n_0)|^s\rangle \leqslant \frac{1}{2d}\Big(\sum_{|m-n|=1}\langle |G_k(m,n_0)|^s\rangle + \delta_{n_0}\Big), \quad \text{for all } n \in \mathbb{Z}^d.$$

Therefore,

$$(-\Delta + \varepsilon I)\langle |G_k(\cdot,n_0)|^s\rangle \leqslant \delta_{n_0}. \tag{5.3}$$

It follows from (5.3) that

$$\langle |G_k(n,n_0)|^s\rangle \leqslant g_{-\varepsilon}(n-n_0),$$

where $g_\lambda(n-m)$ is the integral kernel of the operator $(-\Delta-\lambda)^{-1}$. It remains to mention that $g_{-\varepsilon}(n-n_0)$ decays exponentially fast as $|n-n_0| \to \infty$. ■

In particular, this theorem implies that, if

$$\frac{C_0(s)}{\sigma^s} < \frac{1}{2d},$$

then there is a positive number $\lambda_* > 0$ such that the operator H has pure point spectrum in $[0,\lambda_*]$.

6. Localization in the problem with unbounded random potentials

Here we consider potentials of the form

$$V(n) = \sigma_n \omega_n, \quad n \in \mathbb{Z}^d,$$

where $\{\sigma_n\}$ is a fixed sequence of positive numbers and ω_n are independent random variables uniformly distributed on the interval $[0, 1]$. Assume that

$$\lim_{|n|\to\infty} \sigma_n = \infty.$$

Note that similar (but still different) models with unbounded random potentials were considered earlier by A. Gordon, S. Molchanov, V. Jaksic, and B. Simon in [13].

Let $C_0(s)$ be the constant from Lemma 5.1. We may always choose $C_0(s)$ so that

$$\int_0^1 \frac{1}{|v-c|^s}\, dv \leqslant C_0(s)$$

for all c in $\mathbb{C}$.

Theorem 6.1. *Let* $\operatorname{Im} k^2 > 0$, $0 < s < 1/2$ *and let* $G_k(n,m)$ *be the integral kernel of the operator* A_k^{-1} *where* $A_k\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ *is defined by*

$$[A_k u](n) = -\sum_{|m-n|=1} u(m) + \frac{\sin k}{k} V(n)u(n) + (2d\cos k)u(n), \quad \textit{for all } n \in \mathbb{Z}^d. \tag{6.1}$$

Suppose that V *is a random potential function of the form*

$$V(n) = \sigma_n \omega_n,$$

where ω_n *are independent random variables uniformly distributed on* $[0, 1]$. *Then for each* $n_0 \in \mathbb{Z}^d$,

$$\langle |G_k(n,n_0)|^s \rangle \leqslant \frac{C_0(s)|k|^s}{\sigma_n^s |\sin k|^s} \Big(\sum_{|m-n|=1} \langle |G_k(m,n_0)|^s \rangle + \delta_{n_0} \Big), \quad \textit{for all } n \in \mathbb{Z}^d. \tag{6.2}$$

Proof. Green's function satisfies the equation

$$-\sum_{|m-n|=1} G_k(m,n_0) + \frac{\sin k}{k} V(n) G_k(n,n_0) + (2d\cos k) G_k(n,n_0) = \delta_{n_0}.$$

Consequently,

$$G_k(n,n_0) = \Big(\frac{\sin k}{k} V(n) + (2d\cos k)\Big)^{-1} \Big(\sum_{|m-n|=1} G_k(m,n_0) + \delta_{n_0} \Big).$$

It remains to apply Lemma 5.1. ∎

Theorem 6.2. *Let the constants σ_n obey the condition*

$$\lim_{|n|\to\infty} \sigma_n = \infty.$$

Then the operator H has pure point spectrum.

Proof. Let us show that H has pure point spectrum in each interval of the form

$$J = [(\pi n + \varepsilon)^2, (\pi(n+1) - \varepsilon)^2].$$

For that purpose, we choose $\tilde{\sigma}_n$ so that

$$\frac{\tilde{\sigma}_n^s |\sin k|^s}{C_0(s)|k|^s} - 2d \geqslant 1, \quad \text{for all } k^2 \in J,\ n \in \mathbb{Z}^d.$$

and represent V in the form

$$V = \widetilde{V} + (V - \widetilde{V}), \quad \text{where } \widetilde{V}(n) = \tilde{\sigma}_n \omega_n \text{ for all } n \in \mathbb{Z}^d.$$

Since $\sigma_n \to \infty$ as $|n| \to \infty$, we may choose $\tilde{\sigma}_n$ so that there is an $R > 0$ for which

$$\tilde{\sigma}_n = \sigma_n, \quad \text{for all } |n| > R.$$

In this case, $V - \widetilde{V}$ is a finite rank operator on $\ell^2(\mathbb{Z}^d)$. Let $\widetilde{A}_k$ be the operator

$$[\widetilde{A}_k u](n) = -\sum_{|m-n|=1} u(m) + \frac{\sin k}{k} \widetilde{V}(n) u(n) + (2d \cos k) u(n), \quad \text{for all } n \in \mathbb{Z}^d.$$

Then the inverse of the operator (6.1) satisfies the condition

$$A_k^{-1} - \widetilde{A}_k^{-1} = \frac{\sin k}{k} \widetilde{A}_k^{-1} (\widetilde{V} - V) A_k^{-1}$$

which could be viewed as an equation for A_k^{-1}

$$A_k^{-1} + \frac{\sin k}{k} \widetilde{A}_k^{-1} (V - \widetilde{V}) A_k^{-1} = \widetilde{A}_k^{-1}. \tag{6.3}$$

We will show that the operator

$$\widetilde{A}_k^{-1}(V - \widetilde{V})$$

is compact almost surely. Therefore, (6.3) is solvable if and only if 1 is not an eigenvalue of the operator

$$\frac{\sin k}{k} \widetilde{A}_k^{-1} (\widetilde{V} - V).$$

On the other hand, if 1 is an eigenvalue of this operator, then there is a non-zero $\psi \in \ell^2(\mathbb{Z}^d)$ for which

$$\psi = \frac{\sin k}{k} \widetilde{A}_k^{-1} (\widetilde{V} - V) \psi.$$

Multiplying both sides of this equality by $\widetilde{A}_k$, we obtain that

$$A_k \psi = 0.$$

Thus, k^2 is an eigenvalue of H. The collection of eigenvalues is a countable set of measure zero. Consequently, equation (6.3) is solvable for almost every $k^2 \in J$.

It remains to prove that the integral kernel $\widetilde{G}_k$ of $\widetilde{A}_k$ almost surely satisfies the condition

$$\sum_{n \in \mathbb{Z}^d} |\widetilde{G}_k(n, n_0)|^2 < \infty \quad \text{for all } n_0 \in \mathbb{Z}^d,$$

for almost every $k^2 \in J$. The inequality (6.2), with G_k and σ_n replaced by $\widetilde{G}_k$ and $\tilde{\sigma}_n$, implies that

$$\frac{\tilde{\sigma}_n^s |\sin k|^s}{C_0(s)|k|^s} \langle |\widetilde{G}_k(n, n_0)|^s \rangle \leqslant \Big(\sum_{|m-n|=1} \langle |\widetilde{G}_k(m, n_0)|^s \rangle + \delta_{n_0} \Big), \quad \text{for all } n \in \mathbb{Z}^d.$$

Therefore,

$$(-\Delta + W)\langle |\widetilde{G}_k(\cdot, n_0)|^s \rangle \leqslant \delta_{n_0},$$

where the potential W is defined by

$$W(n) = \frac{\tilde{\sigma}_n^s |\sin k|^s}{C_0(s)|k|^s} - 2d.$$

Since $W \geqslant 1$,

$$(-\Delta + 1)\langle |\widetilde{G}_k(\cdot, n_0)|^s \rangle \leqslant \delta_{n_0}. \tag{6.4}$$

It follows from (6.4) that

$$\langle |\widetilde{G}_k(\cdot, n_0)|^s \rangle \leqslant (-\Delta + I)^{-1} \delta_{n_0},$$

Put differently,

$$\langle |\widetilde{G}_k(n, n_0)|^s \rangle \leqslant g_{-1}(n - n_0)$$

where $g_\lambda(n - m)$ is the integral kernel of the operator $(-\Delta - \lambda)^{-1}$. Taking into account the fact that $g_{-1}(n - n_0)$ decays exponentially fast as $|n - n_0| \to \infty$, we conclude that

$$\int_{k^2 \in J} \sum_{n \in \mathbb{Z}^d} \langle |\widetilde{G}_k(n, n_0)|^s \rangle \, dk < \infty.$$

By Fubini theorem,

$$\sum_{n \in \mathbb{Z}^d} |\widetilde{G}_k(n, n_0)|^s < \infty$$

almost surely for almost every $k^2 \in J$. ∎

We would like to mention that the additional assumption

$$\sum_{n\in\mathbb{Z}^d}\frac{1}{\sigma_n}<\infty \tag{6.5}$$

implies that $V(n)\to\infty$ as $|n|\to\infty$ almost surely. Indeed, for any $M>0$ the probability P_n of the event $\{V(n)<M\}$ equals M/σ_n. The condition (6.5) tells us that

$$\sum_{n\in\mathbb{Z}^d}P_n<\infty.$$

Consequently, $V(n)\geqslant M$ for all sufficiently large $|n|>R_0$. According to Proposition 2.7, the essential spectrum of H in this case is the union of the points $(\pi n)^2$ with $n\in\mathbb{N}$, which means that the spectrum is discrete between $(\pi n)^2$ and $(\pi(n+1))^2$.

On the other hand, if $\sigma_n\leqslant\ln(1+|n|)$, then using the second Borel–Cantelli lemma one can easily show that H has a dense pure point spectrum in $[0,\infty)$.

Theorem 6.3. *Let $\sigma_n\to\infty$ as $|n|\to\infty$ so slowly that*

$$\sigma_n\leqslant\ln(1+|n|),\quad \textit{for all } n\in\mathbb{Z}^d.$$

Then H has a dense pure point spectrum in $[0,\infty)$.

Proof. It is sufficient to show that for any small $\delta>0$ and for any large natural number $m\in\mathbb{N}$, there is a point $n_0\in\mathbb{Z}^d$ such that $V(n)<\delta$ for all $n\in n_0+[0,m)^d$. The probability $P(n_0,m)$ of the event

$$\{V(n)<\delta \text{ for all } n\in n_0+[0,m)^d\}.$$

is not smaller than $(\delta/(\ln(1+|n_0|+m\sqrt{d})))^{(m+1)^d}$. Consequently,

$$\sum_{n_0\in m\mathbb{Z}^d}P(n_0,m)=\infty.$$

By the second Borel–Cantelli lemma, there are infinitely many $n_0\in\mathbb{Z}^d$ for which

$$V(n)<\delta\quad \text{for all } n\in n_0+[0,m)^d. \tag{6.6}$$

Now, for any $\varepsilon>0$ and $n_0\in\mathbb{Z}^d$, one can find a large number $m\in\mathbb{N}$ for which there exists a function ψ supported on $n_0+[0,m)^d$ satisfying the inequality $\|(-\Delta+2d(\cos k-1))\psi\|\leqslant\varepsilon\|\psi\|$. If (6.6) holds, then

$$\|A_k\psi\|\leqslant\Big(\varepsilon+\frac{\delta}{k}\Big)\|\psi\|,\quad \text{for all } k>0.$$

This implies that either A_k is not invertible, or $\|A_k^{-1}\|\geqslant\big(\varepsilon+\frac{\delta}{k}\big)^{-1}$. Since δ and ε are arbitrary, we conclude that A_k almost surely does not have a bounded inverse. Thus, $k^2\in\Sigma(H)$. ∎

References

[1] M. Aizenman and S. Molchanov, Localization at large disorder and at extreme energies: an elementary derivation. *Comm. Math. Phys.* **157** (1993), no. 2, 245–278 Zbl 0782.60044 MR 1244867

[2] F. V. Atkinson, *Discrete and continuous boundary problems*. Mathematics in Science and Engineering 8, Academic Press, New York and London, 1964 Zbl 0117.05806 MR 0176141

[3] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*. Math. Surveys Monogr. 186, American Mathematical Society, Providence, RI, 2013 Zbl 1318.81005 MR 3013208

[4] C. Cattaneo, The spectrum of the continuous Laplacian on a graph. *Monatsh. Math.* **124** (1997), no. 3, 215–235 Zbl 0892.47001 MR 1476363

[5] F. den Hollander, S. A. Molchanov, and O. Zeitouni, *Random media at Saint-Flour*. Probability at Saint-Flour, Springer, Berlin etc., 2012 Zbl 1260.60008 MR 3059554

[6] P. Exner, Weakly coupled states on branching graphs. *Lett. Math. Phys.* **38** (1996), no. 3, 313–320 Zbl 0861.35068 MR 1416707

[7] P. Exner, A duality between Schrödinger operators on graphs and certain Jacobi matrices. *Ann. Inst. H. Poincaré Phys. Théor.* **66** (1997), no. 4, 359–371 Zbl 0949.34073 MR 1459512

[8] P. Exner, Topologically induced spectral behavior: the example of quantum graphs. 2020, arXiv:2003.06189v1

[9] P. Exner, and R. Gawlista, Band spectra of rectangular graph superlattices. *Phys. Rev. B* **53** (1996), 7275–7286

[10] P. Exner, M. Helm, and P. Stollmann, Localization on a quantum graph with a random potential on the edges. *Rev. Math. Phys.* **19** (2007), no. 9, 923–939 Zbl 1147.82018 MR 2355567

[11] A. Y. Gordon, S. A. Molchanov, and B. Tsagani, Spectral theory of one-dimensional Schrödinger operators with strongly fluctuating potentials. *Funktsional. Anal. i Prilozhen.* **25** (1991), no. 3, 89–92. English translation: *Funct. Anal. Appl.* **25** (1991), no. 3, 236–238 Zbl 0741.34054 MR 1139884

[12] A. Y. Gordon and O. Safronov, A real-analytic proof of the Simon–Wolff theorem. *Rep. Math. Phys.* **85** (2020), no. 3, 427–442 Zbl 1441.81094 MR 4118435

[13] Y. A. Gordon, V. Jakšić, S. Molčanov, and B. Simon, Spectral properties of random Schrödinger operators with unbounded potentials. *Comm. Math. Phys.* **157** (1993), no. 1, 23–50 Zbl 0788.60077 MR 1244857

[14] F. Klopp and K. Pankrashkin, Localization on quantum graphs with random vertex couplings. *J. Stat. Phys.* **131** (2008), no. 4, 651–673 Zbl 1144.82061 MR 2398947

[15] S. Molchanov and B. Vainberg, Scattering solutions in networks of thin fibers: small diameter asymptotics. *Comm. Math. Phys.* **273** (2007), no. 2, 533–559 Zbl 1210.35036 MR 2318317

[16] K. Pankrashkin, Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures. *J. Math. Anal. Appl.* **396** (2012), no. 2, 640–655 Zbl 1260.47010 MR 2961258

[17] L. Parnovski and A. V. Sobolev, Bethe-Sommerfeld conjecture for periodic operators with strong perturbations. *Invent. Math.* **181** (2010), no. 3, 467–540 Zbl 1200.47067 MR 2660451

[18] O. Post, *Spectral analysis on graph-like spaces*. Lecture Notes in Math. 2039, Springer, Berlin etc., 2012 Zbl 1247.58001 MR 2934267

[19] B. Simon and T. Wolff, Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math.* **39** (1986), no. 1, 75–90 Zbl 0609.47001 MR 0820340

[20] M. M. Skriganov, Proof of the Bethe–Sommerfeld conjecture in dimension 2 (in Russian). *Dokl. Akad. Nauk SSSR* **248** (1979), no. 1, 39–42. English translation: *Sov. Math., Dokl.* **20** (1979), 956–959 Zbl 0435.35028 MR 0549367

[21] M. M. Skriganov, The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential. *Invent. Math.* **80** (1985), no. 1, 107–121 Zbl 0578.47003 MR 0784531

[22] O. A. Veliev, Perturbation theory for the periodic multidimensional Schrödinger operator and the Bethe-Sommerfeld conjecture. *Int. J. Contemp. Math. Sci.* **2** (2007), no. 1-4, 19–87 Zbl 1118.35019 MR 2292470

[23] J. von Below, A characteristic equation associated to an eigenvalue problem on c^2-networks. *Linear Algebra Appl.* **71** (1985), 309–325 Zbl 0617.34010 MR 0813056

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