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Singularities of the magnetic spectral shift function for potentials of variable sign

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Abstract. We consider the 3D-Schrödinger operator H_0 with constant magnetic field B of scalar intensity $b > 0$, and its perturbations $H = H_0 + V$ by a potential V of not necessarily fixed sign. Then we study the singularities at the Landau levels of the Kreĭn spectral shift functions $\xi(\cdot; H, H_0)$ for the pair (H, H_0). We consider power-like decaying potentials and cylindrically supported potentials (with axis B).

1. Introduction

The spectral properties of quantum systems in the presence of magnetic fields have been a central focus in mathematical physics due to their profound connections to quantum mechanics, scattering theory, and operator theory (see e.g., [1, 11, 12, 27, 30] to name just a few). A fundamental tool in this study is the spectral shift function (SSF), which quantifies the change in the spectrum of a self-adjoint operator under perturbation. The SSF not only encapsulates scattering properties, but also provides crucial information about spectral and eigenvalue asymptotics [5, 8, 37, 38].

Let H_0 and H be two self-adjoint operators acting on the same Hilbert space, such that the resolvent difference $(H - z)^{-1} - (H_0 - z)^{-1}$ belongs to the trace class. Under this assumption, M. Kreĭn [17] showed there exists a unique (up to an additive constant) function $\xi(\cdot; H, H_0)$, which satisfies the Lifshits–Kreĭn trace formula

$$\operatorname{Tr}(\phi(H) - \phi(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0)\phi'(E)\, dE,$$

for sufficiently smooth test functions ϕ. Such a function $\xi(\cdot; H, H_0)$ is called the *spectral shift function* for the pair (H_0, H). Moreover, the SSF plays a dual role, characterizing the eigenvalue counting function of H below the essential spectrum and encoding scattering information through the Birman–Kreĭn formula (see [37]).

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Fernández and Raikov [8] conducted a detailed study of the singularities of the SSF for the three-dimensional magnetic Schrödinger operator

$$H_0 = (i\nabla + \mathbb{A})^2,$$

where the magnetic potential $\mathbb{A} = \frac{b}{2}(-x_2, x_1, 0)$ generates a constant magnetic field $B = (0, 0, b)$ of strength $b > 0$. This operator exhibits a purely absolutely continuous spectrum $\sigma(H_0) = [b, \infty)$, with thresholds at the Landau levels $\Lambda_q = b(2q + 1)$, $q \in \mathbb{Z}_+$. They considered perturbations $H_\pm = H_0 \pm V$ by scalar potentials $V \geq 0$ decaying sufficiently rapidly at infinity and demonstrated that the singular behavior of the SSF near the Landau levels is directly linked to the decay properties of V. By the Birman–Schwinger principle, they reduced the problem to the study of eigenvalue asymptotics for certain Toeplitz-type operators, and provided explicit asymptotic formulas describing the SSFs singularities near the Landau levels.

In many models of mathematical physics, however, we must consider perturbations of the operator H_0 that are not of definite sign. A primary example involves introducing an electric potential with variable sign. Electric potentials of variables sign have been shown to produce qualitatively new spectral behaviour. A classic illustration is the von Neumann–Wigner construction, where an oscillatory, sign-indefinite electric potential that decays like r^{-1} produces a bound state embedded in the continuous spectrum [31,36]. In 2009, Klopp and Raikov [16] showed that for the two-dimensional Landau Hamiltonian, all Landau levels cease to be eigenvalues when the added electric potential keeps one sign, whereas for a sign-changing perturbation a Landau level can be still an eigenvalue of infinite multiplicity. Another important case of sign indefinite perturbations are the perturbations of the magnetic field itself: even adding a positive perturbation of the magnetic field does not yield a perturbed operator larger than H_0. Obstacle perturbations with mixed boundary conditions (Dirichlet in one part of the boundary and Neumann on the other) also fall into this category.

Although representations of the SSF exist for sign-indefinite perturbations, their structure is more intricate, and a comprehensive analysis in the context of magnetic Schrödinger operators has been limited. In particular, the behavior of the SSF near Landau levels in such settings remains insufficiently understood.

In this paper, we address the first of the aforementioned cases. Specifically, we analyse the SSF for the perturbed operator $H = H_0 + V$, where H_0 is the three-dimensional magnetic Schrodinger operator as above and V is a scalar potential of variable sign, satisfying appropriate decay conditions. As a result, we generalise the work of Fernández and Raikov by removing their restriction to potentials V with fixed sign. To some extent, we also generalise previous works where the SSF is studied below the first Landau level Λ_0. In that case, the SSF coincides almost everywhere with the eigenvalue counting function and its asymptotic behaviour have been studied for different kind of potentials (see Remark 3.3).

Our main results extend the asymptotics of the SSF near the Landau levels $\Lambda_q = b(2q+1)$, $q \in \mathbb{Z}_+$, to a broader class of potentials V, with the case of sign-definite perturbations being a particular case of the results presented in this paper.

In Theorem 3.2 we show that as $E \to \Lambda_q$, the behaviour of the SSF $\xi(E; H, H_0)$ is captured by the distribution of eigenvalues of a Toeplitz operator on the q-th Landau eigenspace. Namely, let p_q be the orthogonal projection of the two-dimensional Landau Hamiltonian associated to the eigenvalue Λ_q. Set $E = \Lambda_q \mp \lambda$ with $\lambda \downarrow 0$, then

$$\xi(\Lambda_q - \lambda) \sim -\operatorname{Tr} \mathbb{1}_{(-\infty,-1)}\Big(\frac{p_q W p_q}{\sqrt{\lambda}}\Big), \quad \xi(\Lambda_q + \lambda) \sim \frac{1}{\pi} \operatorname{Tr}\Big[\arctan\Big(\frac{p_q W p_q}{\sqrt{\lambda}}\Big)\Big]$$

where W is the "transverse averaged" version of V:

$$W(x_\perp) = \frac{1}{2} \int_{\mathbb{R}} V(x_\perp, x_3)\, dx_3.$$

We then further specify the exact behaviour of the SSF $\xi(\cdot; H, H_0)$ for two classes of potentials. In the first of these results (see Theorem 4.1) we consider potentials with power-like decay, i.e., $W(x_\perp) \sim |x_\perp|^{-m+1}$ for $|x_\perp| \to \infty$. Then we show that for $\lambda \to 0$

$$\xi(\Lambda_q + \lambda) \sim C \lambda^{-1/(m-1)}$$

with some explicit constants C. Thus, the power-like decay in V produces a single power-like blow-up of the SSF $\xi(\cdot; H, H_0)$.

For the second class where the exact behaviour of the SSF $\xi(\cdot; H, H_0)$ is established, we consider potentials such that the support of W is compact, which means that V is supported in a cylinder with axis parallel to the magnetic field. Here, we consider two complementary scenarios for potentials V that vanish outside a cylinder in the $(x_\perp, x_3)$-space, but differ in how their positive and negative parts are arranged.

In Theorem 5.2 and Theorem 5.3 we assume there exists a bounded Lipschitz region $\Omega \subset \mathbb{R}^2$ and a compact subset $K \subset \Omega$ such that $\operatorname{supp} V \subset \Omega \times \mathbb{R}$ and the transverse average W of V has fixed sign on $\Omega \setminus K$. In this scenario, the associated Toeplitz eigenvalues decay factorially, leading to the asymptotic

$$\xi(\Lambda_q + \lambda) \sim -\frac{|\ln \lambda|}{(\ln |\ln \lambda|)} \quad \lambda \to 0, \tag{1.1}$$

in the case when W is negative on $\Omega \setminus K$. The result in Theorem 5.2 is actually a three term asymptotic formula where the third term depends on the logarithmic capacity of Ω. In Theorem 5.3 we obtain the correponding result for W positive on $\Omega \setminus K$.

In the second scenario (see Theorem 5.4) of a potential V supported in a cylinder, we suppose instead that the positive and negative parts of W are confined to two disjoint Lipschitz domains Ω_+ and Ω_- in $\mathbb{R}^2$, with

$$\operatorname{supp} V \subset (\Omega_+ \cup \Omega_-) \times \mathbb{R}, \quad W > 0 \text{ on } \Omega_+, \quad W < 0 \text{ on } \Omega_-.$$

In some cases, the singularity of the SSF still follows (1.1), but its amplitude is now bounded above and below by two constants $0 < \delta < \Delta < 1$, whose optimal values remain an open problem.

These three complementary theorems (Theorems 5.2, 5.3, and 5.4) demonstrate how the compact support of V and the geometry of its sign distribution together dictate the precise iterated–logarithm scaling of the singularities of the SSF $\xi(\cdot; H, H_0)$ at each Landau level.

We use advanced tools, such as index-theoretic representations of the SSF [14] and refined asymptotic analysis of Toeplitz operators, to show that the singularities obtained in [8] persist and to better understand the interplay between the variable-sign potential V and the magnetic field.

The remainder of the paper is organised as follows. In Section 2 we discuss some preliminaries, introduce the spectral shift function $\xi(\cdot; H, H_0)$, and describe its main properties. Section 3 introduces the necessary theoretical framework, including the representation of the SSF in terms of compact operators. In Section 4 we prove the asymptotic behaviour for the spectral shift function $\xi(\cdot; H, H_0)$ under the assumption that the potential V has power-like decay. Section 5 is specialised for cylindrically supported perturbations V.

2. Preliminaries

In this section we recall the representation formula of the SSF in terms of an integral of an index for a Fredholm pair of spectral projections (formula (2.10) below). We begin by recalling some basic properties of compact operators and the index of a pair of projections. For proofs and more properties on this notion of index, we refer to [5, 14, 22].

Throughout the paper, for a linear operator T in a Hilbert space we denote by $\sigma(T)$ (respectively, $\sigma_{\mathrm{ess}}(T)$ and $\sigma_{\mathrm{ac}}(T)$) the spectrum of T (respectively, the essential and absolutely continuous spectrum of T).

2.1. Counting function of eigenvalues

We denote by S_∞ the class of linear compact operators acting on a fixed Hilbert space. Let $T = T^* \in S_\infty$. Denote by $E_T(I)$ the spectral projection of T associated with the interval $I \subset \mathbb{R}$. For $s > 0$, set

$$n_\pm(s; T) := \operatorname{rank} E_{\pm T}(s, \infty).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$, we have

$$n_+(s; T^*T) = n_+(s; TT^*), \quad s > 0,$$

and let us put

$$n_*(s; T) := n_+(s^2; T^*T), \quad s > 0. \tag{2.1}$$

If $T = T^*$, then evidently

$$n_*(s; T) = n_+(s, T) + n_-(s; T), \quad s > 0. \tag{2.2}$$

Moreover, if $T_j = T_j^* \in S_\infty$, $j = 1, 2$, the Weyl's inequalities

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2) \tag{2.3}$$

hold for each $s_1, s_2 > 0$.
Further, we denote by S_p, $p \in (0, \infty)$, the Schatten–von Neumann class of compact operators for which the functional $\|T\|_{S_p} := \left(p \int_0^\infty s^{p-1} n_*(s; T)\, ds\right)^{1/p}$ is finite. If $T \in S_p$, $p \in (0, \infty)$, then the elementary inequality

$$n_*(s; T) \le s^{-p} \|T\|_{S_p}^p \tag{2.4}$$

holds for every $s > 0$. If $T = T^* \in S_p$, $p \in (0, \infty)$, then (2.1) and (2.2) imply

$$n_\pm(s; T) \le s^{-p} \|T\|_{S_p}^p, \quad s > 0. \tag{2.5}$$

Finally, for a bounded operator T, we define the self-adjoint operators

$$\operatorname{Re} T := \frac{1}{2}(T + T^*) \quad \text{and} \quad \operatorname{Im} T := \frac{1}{2i}(T - T^*).$$

Evidently,

$$n_\pm(s; \operatorname{Re} T) \le 2n_*(s; T), \quad n_\pm(s; \operatorname{Im} T) \le 2n_*(s; T).$$

2.2. Index of a pair of projections

A pair of orthogonal projections P, Q in a Hilbert space is called *Fredholm* if

$$\{1, -1\} \cap \sigma_{\mathrm{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q$ is compact, then the pair P, Q is Fredholm. The *index* of a Fredholm pair is given by the formula

$$\operatorname{index}(P, Q) = \dim \operatorname{Ker}(P - Q - I) - \dim \operatorname{Ker}(P - Q + I).$$

In particular, if $P - Q \in S_1$, then $\operatorname{index}(P, Q) = \operatorname{Tr}(P - Q)$. If both P, Q and Q, R are Fredholm pairs and either $P - Q$ or $Q - R$ is compact, then the pair P, R is also Fredholm and the following "chain rule" holds true:

$$\operatorname{index}(P, R) = \operatorname{index}(P, Q) + \operatorname{index}(Q, R).$$

See e.g., [2] for the proof of the last statement and the details.

If the spectral projections $E_M(\infty, 0)$, $E_{\widetilde{M}}(\infty, 0)$ (for bounded self-adjoint operators M and $\widetilde{M}$) are a Fredholm pair, we will use the shorthand notation

$$\operatorname{ind}(\widetilde{M}, M) := \operatorname{index}(E_{\widetilde{M}}(-\infty, 0), E_M(-\infty, 0)).$$

We will mostly use this notation in the case $\widetilde{M} = M + A$, where A is a compact self-adjoint operator and M is a bounded self-adjoint operator such that 0 is not in the essential spectrum of M. In this case, representing the spectral projections by Riesz integrals and using the resolvent identity, it is easy to see that the difference $E_{\widetilde{M}}(-\infty, 0) - E_M(-\infty, 0)$ is compact and therefore the above pair of spectral projections is Fredholm. This result is still true for unbounded operators (see for instance [20, Corollary 3.5]).

In the proposition below we list some of the properties of ind that we will need in the paper.

Proposition 2.1. *The following statements hold true.*

(1) (see e.g., [23, (4.4), (4.5)]) *If $G \in S_\infty$, and 0 is not in the spectrum of M, then the following Birman–Schwinger-type formula holds*

$$\begin{aligned} \operatorname{ind}(M + G^*G, M) &= -n_-(1; GM^{-1}G^*), \\ \operatorname{ind}(M - G^*G, M) &= n_+(1; GM^{-1}G^*). \end{aligned} \tag{2.6}$$

(2) (see e.g., [5, Section 3.2]) *If $\widetilde{M} \geq M$, then* $\operatorname{ind}(\widetilde{M}, M) \leq 0$.

(3) (see e.g., [5, Section 3.2] and [23, Corollary 3.3]) *Let M be a bounded self-adjoint operator such that $[-a, a] \subset \rho(M)$ for some $a > 0$ and let S_0 and S be compact self-adjoint operators. Then*

$$\begin{aligned} &\operatorname{ind}(M + S_0 + a, M + a) - n_+(a, S) \\ &\quad \leq \operatorname{ind}(M + S_0 + S, M) \\ &\quad \leq \operatorname{ind}(M + S_0 - a, M - a) + n_-(a, S). \end{aligned} \tag{2.7}$$

2.3. Magnetic Schrödinger operator and its perturbation

Next we introduce the magnetic Schrödinger operator and the class of potentials we will consider in the present paper. Let us consider the magnetic Laplacian

$$-\Delta_{\mathbb{A}} := \Big(-i\frac{\partial}{\partial x_1} + \frac{bx_2}{2}\Big)^2 + \Big(-i\frac{\partial}{\partial x_2} - \frac{bx_1}{2}\Big)^2 - \frac{\partial^2}{\partial x_3^2}$$

corresponding to the Hamiltonian of a particle in 3d in presence of a constant magnetic field $B = (0, 0, b)$, $b > 0$ generated by the magnetic potential

$$\mathbb{A}(x) := \frac{b}{2}(-x_2, x_1, 0), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

i.e., $\operatorname{curl} \mathbb{A} = B$.

Denote by $\mathrm{H}^s_{\mathbb{A}}$ the closure of the space $C_0^\infty(\mathbb{R}^3)$ of all compactly supported infinitely differentiable functions on $\mathbb{R}^3$ in the norm of $\mathrm{H}^s_{\mathbb{A}}$ defined by

$$\|u\|^2_{\mathrm{H}^s_{\mathbb{A}}} := \sum_{\alpha\in\mathbb{Z}^3_+ : 0\le|\alpha|\le s} \int_\Omega |(-i\nabla - \mathbb{A})^\alpha u|^2\, dx.$$

Then the operator $H_0 := -\Delta_{\mathbb{A}}$ with domain $\mathfrak{D}(H_0) := \mathrm{H}^2_{\mathbb{A}}$ is self-adjoint in $L^2(\mathbb{R}^3)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ (see e.g., [13, Appendix] and [1, Theorem 2.4]). It is well known that

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [b, \infty),$$

and the *Landau levels* (see e.g., [10, 18])

$$\Lambda_q := b(2q + 1), \quad q \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

play the role of *thresholds* in the spectrum $\sigma(H_0)$ of H_0, in the sense that the limiting absorption principle in its standard form does not hold at Λ_q (see (2.9) below).

We assume that V satisfies

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad |V(\mathbf{x})| \le C_0\langle\mathbf{x}\rangle^{-m}, \quad m > 3, \quad \mathbf{x} \in \mathbb{R}^3, \tag{2.8}$$

with $C_0 > 0$.

Set

$$H := H_0 + V, \quad \mathfrak{D}(H) = \mathfrak{D}(H_0).$$

It is clear that H is a self-adjoint operator and $\inf\sigma(H) \ge -C_0$.

By the diamagnetic inequality (see e.g., [1]), $|V|^{1/2}(H_0 - \lambda_0)^{-1}$ with $\lambda_0 < b$ is a Hilbert–Schmidt operator. In particular, V is a relatively compact perturbation of H_0, and so stability of essential spectrum implies that $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = [b, \infty)$.

In particular, for any potential V satisfying the above assumptions, we have that $\inf \sigma(H) \leq \inf \sigma(H_0)$.

Since $|V|^{1/2}(H_0 - \lambda_0)^{-1} \in S_2$ for any $\lambda_0 < b$, the resolvent identity implies

$$(H - \lambda_0)^{-1} - (H_0 - \lambda_0)^{-1} \in S_1,$$

for $\lambda_0 < \inf \sigma(H) \leq \inf \sigma(H_0)$. Referring to [37, Theorem 8.9.1], we conclude that there exists a function $\xi(\cdot; H, H_0)$ from the class $L^1((\lambda_0, \infty), (E - \lambda_0)^{-2} dE)$, which satisfies the Lifshits–Kreĭn trace formula

$$\operatorname{Tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE$$

for all smooth compactly supported function f on $\mathbb{R}$. This function is unique (see [37, Section 8.9]) if we choose the normalisation of the following form

$$\xi(\lambda; H, H_0) = 0, \quad \lambda < \inf \sigma(H) \leq \inf \sigma(H_0).$$

The function $\xi(\cdot; H, H_0)$ is referred to as the *spectral shift function* (SSF) $\xi(\cdot; H, H_0)$ for the pair (H_0, H).

By [5, Proposition 2.5], the SSF $\xi(\cdot; H, H_0)$ possesses the following features:

- $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$;
- $\xi(\cdot; H, H_0)$ is continuous on $\mathbb{R} \setminus (\bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\} \cup \sigma_{\mathrm{pp}}(H))$, where $\sigma_{\mathrm{pp}}(H)$ is the set of the eigenvalues of H.

2.4. Representation of the SSF

Now, we are ready to describe the required representation for the spectral shift function, which will serve as basis for our results.

For $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, set

$$T(z) := |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}.$$

For every $E \in \mathbb{R} \setminus \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$, we know (see [5, Lemma 4.2]) that the operator norm limit

$$T(E + i0) := \operatorname*{n-lim}_{\delta \downarrow 0} T(E + i\delta) \tag{2.9}$$

exists, and

$$0 \leq \operatorname{Im} T(E + i0) \in S_1.$$

Denote

$$A(E) = \operatorname{Re} T(E + i0), \quad B(E) = \operatorname{Im} T(E + i0), \quad J = \operatorname{sign} V.$$

Then for almost every $E \in \mathbb{R}$, we have

$$\xi(E; H, H_0) = \int_{-\infty}^{\infty} \operatorname{ind}(J + A(E) + tB(E), J) d\mu(t), \tag{2.10}$$

where $d\mu(t) = \pi^{-1}(1+t^2)^{-1}dt$ and the right-hand side being well defined for every $E \in \mathbb{R} \setminus \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\}$.

The above representation formula (2.10) was obtained in [14] for the case of trace class perturbation and generalised in [22] to the case of relatively trace class perturbations. The formula generalises earlier results of [25, 34], and can be regarded as a far going extension of the Birman–Schwinger principle (see [3]).

Let us comment on the convergence of the integral in (2.10). Below we mimick the proof of [25, Lemma 2.1]. Choose $0 < s < 1$. Then the interval $[-s, s]$ does not contain the spectrum of J. Using (2.7), we obtain

$$|\operatorname{ind}(J + A(E) + tB(E), J)| \le n_*(s, A(E) + tB(E)), \quad t \in \mathbb{R}.$$

Applying Weyl's inequality (2.3) and (2.5), we obtain for any $s_1, s_2 \in (0,1)$ such that $s_1 + s_2 = s$,

$$\begin{aligned}\int_{-\infty}^{\infty} n_*(s, A(E) + tB(E)) d\mu(t) &\le n_*(s_1, A(E)) + \int_0^{\infty} n_*(s_2, tB(E)) d\mu(t) \\ &\le n_*(s_1, A(E)) + \frac{1}{\pi s_2} \|B(E)\|_{S_1}. \end{aligned} \tag{2.11}$$

This proves absolute convergence of the integral in (2.10) and provides a bound which we will use in the sequel.

3. Asymptotics of spectral shift function via effective Hamiltonian

In this section we prove our first result of the present paper. Namely, we show that the asympototics of the spectral shift function $\xi(\cdot; H, H_0)$ can be expressed via eigenvalue counting function of some compact operator.

Let us begin by introducing some notations. For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote by $x_\perp = (x_1, x_2)$ the variables in the plane perpendicular to the magnetic field. Let $H_\perp$ be the so-called *Landau Hamiltonian* in $L^2(\mathbb{R}^2)$, i.e., the self-adjoint operator

$$H_\perp := \left(-i\frac{\partial}{\partial x_1} + \frac{bx_2}{2}\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \frac{bx_1}{2}\right)^2$$

having a pure point spectrum $\sigma(H_\perp) = \{\Lambda_q \mid q \in \mathbb{Z}_+\}$ with eigenvalues of infinite multiplicity.

We fix a Landau Level Λ_q, $q \in \mathbb{Z}_+$. The first step in our proof is the reduction to Birman–Schwinger operators corresponding to this Landau level.

In the following, if $F(\lambda)$ and $\mathcal{F}_\varepsilon^{\pm}(\lambda)$ are real functions depending on λ, we write

$$F(\lambda) \asymp \mathcal{F}_\varepsilon^{\pm}(\lambda), \quad \lambda \to \lambda_0,$$

if for each $\varepsilon \in (0,1)$ we have that

$$\mathcal{F}_\varepsilon^{-}(\lambda) + O_\varepsilon(1) \le F(\lambda) \le \mathcal{F}_\varepsilon^{+}(\lambda) + O_\varepsilon(1), \quad \lambda \to \lambda_0.$$

For p_q being the orthogonal projection onto $\mathrm{Ker}(H_\perp - \Lambda_q)$ and $I_\parallel$ the identity on $L^2(\mathbb{R})$ (i.e., in the variable x_3), let us introduce the operators

$$T_q(z) := |V|^{1/2}(p_q \otimes I_\parallel)(H_0 - z)^{-1}|V|^{1/2},$$

and

$$A_q(E) = \mathrm{Re}\, T_q(E + i0), \quad B_q(E) = \mathrm{Im}\, T_q(E + i0).$$

Then, we have the following result.

Lemma 3.1. *Let the Landau level Λ_q be fixed. For any V satisfying (2.8), with the above notations we have*

$$\xi(E; H, H_0) \asymp \int_{-\infty}^{\infty} \mathrm{ind}((J \mp \varepsilon) + A_q(E) + tB_q(E), (J \mp \varepsilon))d\mu(t), \quad E \to \Lambda_q. \tag{3.1}$$

Proof. Introduce the operator

$$\widetilde{T_q}(z) := |V|^{1/2}(H_0 - z)^{-1}(I - p_q \otimes I_\parallel)|V|^{1/2}.$$

Then, using the decomposition

$$(H_0 - z)^{-1} = p_q \otimes (D_3^2 + \Lambda_q - z)^{-1} + (H_0 - z)^{-1}(I - p_q \otimes I_\parallel),$$

we obtain that $T(z) = T_q(z) + \widetilde{T_q}(z)$. In [8, Corollary 4.3] it is proved that the operator $T(E + i\delta)$ admits an operator-norm limit $T(E + i0)$ as $\delta \downarrow 0$, for $E \in (\Lambda_{q-1}, \Lambda_q) \cup (\Lambda_q, \Lambda_{q+1})$. Moreover,

$$\mathrm{Re}\, T(E + i0) = \mathrm{Re}\, T_q(E + i0) + \mathrm{Re}\, \widetilde{T_q}(E + i0) = A_q(E) + \mathrm{Re}\, \widetilde{T_q}(E + i0),$$

$$\mathrm{Im}\, T(E + i0) = \mathrm{Im}\, T_q(E + i0) + \mathrm{Im}\, \widetilde{T_q}(E + i0) = B_q(E) + \mathrm{Im}\, \widetilde{T_q}(E + i0).$$

Next, in [8, equation 5.3], by exploiting that $\widetilde{T_q}(E+i0)$ is uniformly bounded near Λ_q, it is shown that for any $\varepsilon \in (0,1)$,

$$\int_{-\infty}^{\infty} n_{\pm}(\varepsilon; \operatorname{Re}\widetilde{T_q}(E+i0) + t\operatorname{Im}\widetilde{T_q}(E+i0))d\mu(t) = O_{\varepsilon}(1). \tag{3.2}$$

Therefore, referring to (2.7) (with $M = J$, $a = \varepsilon$, $A_0 = A_q(E) + tB_q(E)$ and $A = \operatorname{Re}\widetilde{T_q}(E+i0) + t\operatorname{Im}\widetilde{T_q}(E+i0)$), together with (2.10) and (3.2) we conclude that

$$\xi(E; H, H_0) \asymp \int_{-\infty}^{\infty} \operatorname{ind}((J \mp \varepsilon) + A_q(E) + tB_q(E), (J \mp \varepsilon))d\mu(t),$$

as required. ∎

With this initial lemma at hand, we are ready to prove the main result of this section. We first introduce some additional notations.

For a potential V and $\varepsilon \in (0,1)$, we define

$$V_{\varepsilon}^{\pm} := |V|(J \mp \varepsilon)^{-1} = V(I \mp \varepsilon J)^{-1} = V \pm \varepsilon |V|(I \mp \varepsilon J)^{-1}. \tag{3.3}$$

Here, as before, J stands for $\operatorname{sign}(V)$. It is clear that for any bounded potential V we have that $V_{\varepsilon}^{\pm} - V = O(\varepsilon)$ as $\varepsilon \to 0$.

For V satisfying (2.8) and for $x_{\perp} \in \mathbb{R}^2$, denote by W the function defined by

$$W(x_{\perp}) := \frac{1}{2}\int_{\mathbb{R}} V(x_{\perp}, x_3)dx_3. \tag{3.4}$$

In addition, for $\lambda \geq 0$ we introduce

$$\mathcal{W}_{\lambda} = \mathcal{W}_{\lambda}(x_{\perp}) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \tag{3.5}$$

where

$$w_{11} := \frac{1}{2}\int_{\mathbb{R}} V(x_{\perp}, x_3)\cos^2(\sqrt{\lambda}x_3)dx_3,$$

$$w_{22} := \frac{1}{2}\int_{\mathbb{R}} V(x_{\perp}, x_3)\sin^2(\sqrt{\lambda}x_3)dx_3,$$

$$w_{12} = w_{21} := \frac{1}{2}\int_{\mathbb{R}} V(x_{\perp}, x_3)\cos(\sqrt{\lambda}x_3)\sin(\sqrt{\lambda}x_3)dx_3.$$

For any $\varepsilon \in (0,1)$, we also introduce the notations

$$W_\varepsilon^\pm, \quad \mathcal{W}_{\lambda,\varepsilon}^\pm, \tag{3.6}$$

as in (3.4) and (3.5), respectively, with V replaced by $V_\varepsilon^\pm$ (given by (3.3)).

The following theorem is the main result of the present section. It extends [8, Theorems 3.1, 3.2] to potentials of non-definite sign.

Theorem 3.2. *Let V satisfy* (2.8)*, and $W_\varepsilon^\pm$ (resp. $\mathcal{W}_{\lambda,\varepsilon}^\pm$) be the functions defined by* (3.6)*. Then, below each Landau level Λ_q, $q \in \mathbb{Z}_+$, we have*

$$\xi(\Lambda_q - \lambda; H, H_0) \asymp -\operatorname{Tr} \mathbb{1}_{(-\infty,-\sqrt{\lambda})}(p_q W_\varepsilon^\pm p_q), \quad \lambda \downarrow 0. \tag{3.7}$$

Moreover, when approaching each Landau level Λ_q from above, we have

$$\xi(\Lambda_q + \lambda; H, H_0) \asymp \frac{1}{\pi} \operatorname{Tr} \arctan\Big(\frac{p_q \mathcal{W}_{\lambda,\varepsilon}^\pm p_q}{\sqrt{\lambda}}\Big), \quad \lambda \downarrow 0. \tag{3.8}$$

Proof. We follow the strategy for the proof of [8, Theorem 3.1 and Theorem 3.2], but start with the representation formula (2.10), valid in the non-sign definite case.

By Lemma 3.1, we have that

$$\xi(E; H, H_0) \asymp \int_{-\infty}^{\infty} \operatorname{ind}((J \mp \varepsilon) + A_q(E) + tB_q(E), (J \mp \varepsilon)) d\mu(t), \tag{3.9}$$

as $E \to \Lambda_q$.

Now, in order to analyse $A_q(E)$ and $B_q(E)$, let us use that for $\operatorname{Im} k > 0$, we have that $k^2 \in \mathbb{C} \setminus [0,\infty)$, and so the integral kernel of the operator $(D_3^2 - k^2)^{-1}$ (in $L_2(\mathbb{R})$) is $-\frac{r(ik)}{ik}$ with $r(z)$, the one-dimensional operator with the integral kernel $\frac{e^{z|x_3 - x_3'|}}{2}$.

First, we consider the case when E is below Λ_q, i.e., write $E = \Lambda_q - \lambda$ with $\lambda > 0$. Then $A_q(\Lambda_q - \lambda) = T_q(\Lambda_q - \lambda) = |V|^{1/2} p_q \otimes (D_3^2 + \lambda)^{-1} |V|^{1/2}$ is self-adjoint and $B_q(\Lambda_q - \lambda) = 0$. In consequence,

$$\begin{aligned} &\int_{-\infty}^{\infty} \operatorname{ind}((J \mp \varepsilon) + A_q(E) + tB_q(E), (J \mp \varepsilon)) d\mu(t) \\ &= \int_{-\infty}^{\infty} \operatorname{ind}((J \mp \varepsilon) + A_q(E), (J \mp \varepsilon)) d\mu(t) \\ &= \operatorname{ind}((J \mp \varepsilon) + A_q(E), (J \mp \varepsilon)), \end{aligned} \tag{3.10}$$

with

$$A_q(E) = |V|^{1/2} p_q \otimes \frac{r(-\sqrt{\lambda})}{\sqrt{\lambda}} |V|^{1/2}.$$

Define r_0 as the operator with constant integral kernel $\frac{1}{2}$ and $A_{q,0} := |V|^{1/2} p_q \otimes r_0 |V|^{1/2}$. By [8, equation 5.11], using that $A_q(\Lambda_q - \lambda) - \frac{A_{q,0}}{\sqrt{\lambda}}$ is uniformly bounded when λ goes to 0, we have that

$$n_\pm\Big(s; A_q(\Lambda_q - \lambda) - \frac{A_{q,0}}{\sqrt{\lambda}}\Big) = O_s(1).$$

Then, using again (2.7) (applied for $M = J \mp \varepsilon$, $a = s \in (0, 1-\varepsilon)$, $A_0 = A_{q,0}$ and $A = A_q(E) - \frac{A_{q,0}}{\sqrt{\lambda}}$) we deduce

$$\begin{aligned}
&\operatorname{ind}((J-\varepsilon) + A_q(E), J-\varepsilon) \\
&\quad \le \operatorname{ind}\Big((J-\varepsilon-s) + \frac{A_{q,0}}{\sqrt{\lambda}}, J-\varepsilon-s\Big) + O_s(1), \\
&\operatorname{ind}\Big((J+\varepsilon+s) + \frac{A_{q,0}}{\sqrt{\lambda}}, J+\varepsilon+s\Big) + O_s(1) \\
&\quad \le \operatorname{ind}((J+\varepsilon) + A_q(E), J+\varepsilon).
\end{aligned}$$

Since $s+\varepsilon \in (0,1)$, combining the latter estimates with (3.9) and (3.10), we conclude that

$$\xi(\Lambda_q - \lambda; H, H_0) \asymp \operatorname{ind}\Big((J \mp \varepsilon) + \frac{A_{q,0}}{\sqrt{\lambda}}, (J \mp \varepsilon)\Big). \tag{3.11}$$

Recall (see e.g., [27]) that p_q has an integral kernel given by

$$\mathcal{P}_q(x_\perp, x'_\perp) = \frac{b}{2\pi} L_q\Big(\frac{b|x_\perp - x'_\perp|}{2}\Big) \exp\Big(-\frac{b}{4}(|x_\perp - x'_\perp|^2 + 2i(x_1 x'_2 - x'_1 x_2))\Big),$$

where L_q are the Laguerre polynomials:

$$L_q(t) := \frac{1}{q!} e^t \frac{d^q(t^q e^{-t})}{dt^q}.$$

Let $K\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ be the integral operator defined, for $u \in L^2(\mathbb{R}^3)$, $x_\perp \in \mathbb{R}^2$, by

$$(Ku)(x_\perp) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_q(x_\perp, x'_\perp) \sqrt{|V(x'_\perp, x'_3)|}\, u(x'_\perp, x'_3) dx'_3 dx'_\perp.$$

It is easy to see that the adjoint $K^*\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^3)$ is given by

$$(K^* f)(x_\perp, x_3) = \frac{1}{\sqrt{2}} \sqrt{|V(x_\perp, x_3)|} \int_{\mathbb{R}^2} \mathcal{P}_q(x'_\perp, x_\perp) f(x'_\perp)\, dx'_\perp,$$

and that $A_{q,0}(E) = K^* K$. Moreover,

$$K(J \pm \varepsilon)^{-1} K^* = p_q W_\varepsilon^\mp p_q. \tag{3.12}$$

Then, combining (2.6), (3.11), and (3.12) we deduce that

$$\begin{aligned}\xi(\Lambda_q - \lambda; H, H_0) &\asymp -n_-\Big(1; \frac{K(J \mp \varepsilon)^{-1} K^*}{\sqrt{\lambda}}\Big) \\ &\asymp -n_-(\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q), \quad \varepsilon \in (0,1),\end{aligned}$$

which proves (3.7).

Now, consider the case E above Λ_q, i.e., $E = \Lambda_q + \lambda$ with $\lambda \downarrow 0$. Starting from (3.1), we need to analyse $T_q(E)$ which is given by

$$T_q(E) = -|V|^{1/2} p_q \otimes \frac{r(i\sqrt{\lambda})}{i\sqrt{\lambda}} |V|^{1/2}.$$

In [8, Proposition 5.4], it is shown that

$$n_\pm(s; \operatorname{Re} T_q(\Lambda_q + \lambda)) = O_s(1), \tag{3.13}$$

as $\lambda \downarrow 0$, for any $s > 0$. It exploits that the integral kernel of the real part of $T_q(E)$ involves $\frac{\sin\sqrt{\lambda}(x_3 - x_3')}{2\sqrt{\lambda}}$ which is uniformly bounded with respect to $\lambda \downarrow 0$.

Let $r_+(\lambda)$ be the operator with integral kernel $\frac{\cos\sqrt{\lambda}(x_3 - x_3')}{2\sqrt{\lambda}}$, $x_3, x_3' \in \mathbb{R}$ then $B_q(E) = |V|^{1/2}(p_q \otimes r_+(\lambda))|V|^{1/2}$. Thus, combining (2.7) with (3.13), we deduce the estimate

$$\xi(\Lambda_q + \lambda; H, H_0) \asymp \int_{-\infty}^{\infty} \operatorname{ind}(J \mp \varepsilon + t B_q(E), J \mp \varepsilon) d\mu(t). \tag{3.14}$$

Moreover, in [8, Proposition 5.5] it is shown that there exists an operator

$$\mathcal{K} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)^2$$

such that $B_q(E) = \mathcal{K}^* \mathcal{K}$. More precisely, for $u \in L^2(\mathbb{R}^3)$, $\mathcal{K} u = (v_1, v_2)$ is defined by

$$v_1(x_\perp) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_q(x_\perp, x_\perp') \cos(\sqrt{\lambda} x_3') \sqrt{|V(x_\perp', x_3')|} u(x_\perp', x_3') dx',$$

$$v_2(x_\perp) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_q(x_\perp, x_\perp') \sin(\sqrt{\lambda} x_3') \sqrt{|V(x_\perp', x_3')|} u(x_\perp', x_3') dx'.$$

A direct verification shows that the adjoint operator

$$\mathcal{K}^*\colon L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^3)$$

is given by

$$\begin{aligned}(\mathcal{K}^* v)(x_\perp, x_3) = {} & \cos(\sqrt{\lambda}x_3)\sqrt{|V(x_\perp, x_3)|}\int_{\mathbb{R}^2} \mathcal{P}_q(x_\perp, x'_\perp)v_1(x'_\perp)\,dx'_\perp \\ & + \sin(\sqrt{\lambda}x_3)\sqrt{|V(x_\perp, x_3)|}\int_{\mathbb{R}^2} \mathcal{P}_q(x_\perp, x'_\perp)v_2(x'_\perp)\,dx'_\perp,\end{aligned}$$

for $v = (v_1, v_2) \in L^2(\mathbb{R}^2)^2$ and $(x_\perp, x_3) \in \mathbb{R}^3$.

Since $J = \operatorname{sign}(V)$ acts by pointwise multiplication, we have that

$$\begin{aligned}((J \pm \varepsilon)^{-1}\mathcal{K}^* v)(x_\perp, x_3) = {} & (J(x_\perp, x_3) \pm \varepsilon)^{-1}\sqrt{|V(x_\perp, x_3)|} \\ & \times \Big\{\cos(\sqrt{\lambda}x_3)\int_{\mathbb{R}^2} \mathcal{P}_q(x_\perp, x'_\perp)v_1(x'_\perp)dx'_\perp \\ & + \sin(\sqrt{\lambda}x_3)\int_{\mathbb{R}^2} \mathcal{P}_q(x_\perp, x'_\perp)v_2(x'_\perp)dx'_\perp\Big\}.\end{aligned}$$

Applying the operator $\mathcal{K}$ we have

$$\begin{aligned}&(\mathcal{K}(J \pm \varepsilon)^{-1}\mathcal{K}^* v)(x_\perp) \\ &\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{P}_q(x_\perp, x'_\perp)\mathcal{W}^{\mp}_{\lambda,\varepsilon}(x'_\perp)\mathcal{P}_q(x'_\perp, x''_\perp)v(x''_\perp)dx''_\perp dx'_\perp \\ &\quad = (p_q \mathcal{W}^{\mp}_{\lambda,\varepsilon} p_q v)(x_\perp), \quad v \in L^2(\mathbb{R}^2)^2, x_\perp \in \mathbb{R}^2.\end{aligned}$$

Thus,

$$\mathcal{K}(J \pm \varepsilon)^{-1}\mathcal{K}^* = p_q \mathcal{W}^{\mp}_{\lambda,\varepsilon} p_q.$$

Thus, using again (2.6) we can see that for $t > 0$,

$$\begin{aligned}\operatorname{ind}(J \pm \varepsilon + tB_q(E), J \pm \varepsilon) &= -n_-(1; t\mathcal{K}(J \pm \varepsilon)^{-1}\mathcal{K}^*) \\ &= -n_-\Big(1; t\frac{p_q \mathcal{W}^{\mp}_{\lambda,\varepsilon} p_q}{\sqrt{\lambda}}\Big),\end{aligned}$$

and

$$\begin{aligned}\operatorname{ind}(J \pm \varepsilon - tB_q(E), J \pm \varepsilon) &= n_+(1; t\mathcal{K}(J \pm \varepsilon)^{-1}\mathcal{K}^*) \\ &= n_+\Big(1; t\frac{p_q \mathcal{W}^{\mp}_{\lambda,\varepsilon} p_q}{\sqrt{\lambda}}\Big).\end{aligned}$$

Finally, using that for any $T = T^*$, compact

$$\operatorname{Tr}(\arctan(T)) = \int_0^\infty (n_+(s;T) - n_-(s;T)) \frac{ds}{1+s^2} = \int_0^\infty (n_+(1;tT) - n_-(1;tT)) \frac{dt}{1+t^2}, \tag{3.15}$$

we get

$$\int_{-\infty}^{\infty} \operatorname{ind}(J \pm \varepsilon + tB_q(E), J \pm \varepsilon) d\mu(t) = \frac{1}{\pi} \operatorname{Tr} \arctan\Big(\frac{p_q \mathcal{W}^{\mp}_{\lambda,\varepsilon} p_q}{\sqrt{\lambda}} \Big). \tag{3.16}$$

Now, putting together (3.14) and (3.16) yields (3.8). ∎

Remark 3.3. Note that in the case $q = 0$, the asymptotic relation (3.7) concerns the distribution of the discrete eigenvalues of the operator H near the first Landau level Λ_0 which coincides with the infimum of its essential spectrum. Such results on the discrete spectrum have been known for a long time, and could be found in [15,26,32, 33,35] for the case of *a power-like decay of* V; in [28] for the case of *an exponential decay of* V; and in [21,28] for the case of *compactly supported potentials* V. However, in the last two cases, the perturbation V is of fixed sign.

4. Application to power like perturbations

In this section, we apply Theorem 3.2 to potentials V for which W (defined by (3.4)) is power like decaying with respect to $x_\perp$ by exploiting known results on Toeplitz operators $p_q W p_q$ (see [26]).

Theorem 4.1. *Suppose that* V *satisfies* (2.8) *and* $W \in C^1(\mathbb{R}^2)$ *satisfies*

$$|\nabla W(x_\perp)| \le C \langle x_\perp \rangle^{-m}.$$

Suppose moreover that

$$W(x_\perp) = \omega(x_\perp/|x_\perp|)|x_\perp|^{-m+1}(1 + o(1)), \quad |x_\perp| \to \infty,$$

where ω *is a continuous function on* $\mathbb{S}^1$ *that does not vanish identically. Let us define*

$$\mathcal{C}_\pm := \frac{b}{4\pi} \int_{\mathbb{S}^1} \omega_\pm^{\frac{2}{m-1}}; \quad \omega_\pm := \max(\pm\omega(\theta), 0).$$

Then, for any $q \in \mathbb{Z}_+$, the spectral shift function satisfies

$$\xi(\Lambda_q - \lambda; H, H_0) = -\lambda^{-\frac{1}{m-1}}(\mathscr{C}_- + o(1)), \quad \lambda \downarrow 0. \tag{4.1}$$

On the other side,

$$\xi(\Lambda_q + \lambda; H, H_0) = \lambda^{-\frac{1}{m-1}}\Big(\frac{\mathscr{C}_+ - \mathscr{C}_-}{\cos(\pi/(m-1))} + o(1)\Big), \quad \lambda \downarrow 0. \tag{4.2}$$

Proof. First, let us recall that for W satisfying the assumptions of the theorem, for any $q \in \mathbb{Z}_+$, the counting function of the eigenvalues of $p_q W p_q$ satisfies (see [26, Theorem 2.6]):

$$\begin{aligned} n_\pm(s; p_q W p_q) &= \frac{b}{2\pi}|\{x_\perp \in \mathbb{R}^2 : \pm W(x_\perp) > s\}|(1 + o(1)) \\ &= s^{-2/(m-1)}(\mathscr{C}_\pm + o(1)) \quad s \downarrow 0. \end{aligned} \tag{4.3}$$

From (3.7), the study of $\xi(\Lambda_q - \lambda; H, H_0)$ when $\lambda > 0$ tends to 0 is reduced to that of $n_-(\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q) = n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q)$. By writing

$$V_\varepsilon^\pm = V \pm \varepsilon|V|(I \mp \varepsilon J)^{-1},$$

we can use Weyl's inequalities (2.3) to obtain that for any $\delta \in (0, 1)$,

$$\begin{aligned} n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q) \le{}& n_+(\sqrt{\lambda}(1-\delta); p_q(-W)p_q) \\ &+ n_+(\sqrt{\lambda}\delta\varepsilon^{-1}; p_q(-\widetilde{W}_\varepsilon^\pm)p_q), \end{aligned} \tag{4.4}$$

where $\widetilde{W}_\varepsilon^\pm = \frac{1}{2}\int_{\mathbb{R}} |V|(I \mp \varepsilon J)^{-1} dx_3$. Hence, by (4.3), we obtain that

$$\begin{aligned} n_+(\sqrt{\lambda}(1-\delta); p_q(-W)p_q) &= (1-\delta)^{-2/(m-1)}(\sqrt{\lambda})^{-2/(m-1)}(\mathscr{C}_- + o(1)) \\ &= \lambda^{-1/(m-1)}(1-\delta)^{-2/(m-1)}(\mathscr{C}_- + o(1)). \end{aligned}$$

Next, by (2.8),

$$|\widetilde{W}_\varepsilon^\pm(x_\perp)| \le C\langle x_\perp\rangle^{-m+1},$$

then the min-max principle and (4.3) imply

$$\begin{aligned} n_+(\sqrt{\lambda}\delta\varepsilon^{-1}; p_q(-\widetilde{W}_\varepsilon^\pm)p_q) &\le n_+(\sqrt{\lambda}\delta(C\varepsilon)^{-1}; p_q(\langle x_\perp\rangle^{-m+1})p_q) \\ &\le C(\delta\lambda^{1/2}\varepsilon^{-1})^{-\frac{2}{m-1}}. \end{aligned}$$

Combining these two estimates with (4.4) gives

$$n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q) \le \lambda^{-1/(m-1)}[(1-\delta)^{-2/(m-1)}\mathscr{C}_- + O(\varepsilon^{2/(m-1)}) + o(1)].$$

Multiplying by $\lambda^{1/(m-1)}$, then successively letting $\lambda \to 0$, next $\varepsilon \to 0$, and finally $\delta \to 0$, we obtain that

$$\limsup_{\lambda \to 0} \lambda^{1/(m-1)} n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q) \le \mathcal{C}_-,$$

which gives the lower bound in (4.1). Similarly, one can prove that

$$\liminf_{\lambda \to 0} \lambda^{1/(m-1)} n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q) \ge \mathcal{C}_-.$$

Combining the latter two inequalities with (3.7), we infer (4.1).

For (4.2), from (3.8) we are reduced to study

$$\frac{1}{\pi} \operatorname{Tr} \arctan\Big(\frac{p_q \mathcal{W}_{\lambda,\varepsilon}^\pm p_q}{\sqrt{\lambda}}\Big),$$

where $\mathcal{W}_{\lambda,\varepsilon}^\pm$ is given by (3.6). Then, by defining

$$\widetilde{\mathcal{W}}_{\lambda,\varepsilon}^\pm(x_\perp) := \begin{pmatrix} W_\varepsilon^\pm & 0 \\ 0 & 0 \end{pmatrix},$$

we can use the proof of [8, Proposition 6.1] to show that if $m > 4$ in (2.8) then

$$\operatorname{Tr}\Big(\arctan\Big(p_q \frac{\mathcal{W}_{\lambda,\varepsilon}^\pm}{\sqrt{\lambda}} p_q\Big) - \arctan\Big(p_q \frac{\widetilde{\mathcal{W}}_{\lambda,\varepsilon}^\pm}{\sqrt{\lambda}} p_q\Big)\Big) = O(1), \quad \lambda \downarrow 0. \tag{4.5}$$

As in (3.13), it exploits that $\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$ is uniformly bounded with respect to λ as $\lambda \downarrow 0$.

Likewise, if $m \in (3,4]$ in (2.8), the difference in (4.5) is of order $O(\lambda^{-\delta})$ for any $\delta > \frac{4-m}{2}$. More precisely, we use that for

$$T = p_q \frac{\mathcal{W}_{\lambda,\varepsilon}^\pm}{\sqrt{\lambda}} p_q \quad \text{and} \quad \widetilde{T} = p_q \frac{\widetilde{\mathcal{W}}_{\lambda,\varepsilon}^\pm}{\sqrt{\lambda}} p_q,$$

$T - \widetilde{T}$ is trace class and using (2.11) with (2.4), for any $s > 0$, we obtain

$$|\operatorname{Tr}(\arctan(T) - \arctan(\widetilde{T}))| = \left| \int_{\mathbb{R}} \xi(E; T, \widetilde{T}) d\mu(E) \right| \le s^{-1} \|T - \widetilde{T}\|_1,$$

with

$$\begin{aligned} \|T - \widetilde{T}\|_1 \le & \frac{b}{2\pi\sqrt{\lambda}} \int_{\mathbb{R}^2} (w_{22,\varepsilon}^\pm(x_\perp)^2 + w_{12,\varepsilon}^\pm(x_\perp)^2)^{1/2} dx_\perp \\ \le & \frac{b}{\pi\sqrt{\lambda}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{1}{1-\varepsilon} |V(x_\perp, x_3)| |\sin(\sqrt{\lambda}x_3)| dx_3\, dx_\perp. \end{aligned} \tag{4.6}$$

Then the claimed estimates work exactly as in [8, Proposition 6.1].

Now, since

$$\operatorname{Tr}\Big(\arctan\Big(p_q \frac{\widetilde{W}^{\pm}_{\lambda,\varepsilon}}{\sqrt{\lambda}} p_q\Big)\Big) = \operatorname{Tr}\Big(\arctan\Big(p_q \frac{W^{\pm}_{\varepsilon}}{\sqrt{\lambda}} p_q\Big)\Big),$$

by (3.15), we can proceed as in (4.4) and using a dominated convergence argument we obtain that the main term in the asymptotic behavior is given by

$$\frac{1}{\pi}\int_0^\infty ((1-\delta)t\sqrt{\lambda})^{-\frac{2}{m-1}} \frac{dt}{1+t^2}(\mathcal{C}_+ - \mathcal{C}_-) = \lambda^{-\frac{1}{m-1}} \frac{(1-\delta)^{-\frac{2}{m-1}}}{\cos(\pi/(m-1))}(\mathcal{C}_+ - \mathcal{C}_-),$$

where we use known results for Beta functions. We conclude (4.2) by successively taking the limits $\limsup_{\lambda\downarrow 0}$, $\lim_{\varepsilon\to 0}$ and $\lim_{\delta\to 0}$. ■

Remark 4.2. One can verify that for $m > 3$, $\omega_0 \in C^1(\mathbb{S}^1)$ and $w \in C^1(\mathbb{R}^2)$ such that $w(x_\perp) = \omega_0(x_\perp/|x_\perp|)$ on $\{x_\perp \in \mathbb{R}^2 \mid |x_\perp| \geq r_0\}$, $r_0 > 0$, the function $V\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $V(\mathbf{x}) = w(x_\perp)\langle \mathbf{x}\rangle^{-m}$ satisfies the assumption of Theorem 4.1 with

$$W(x_\perp) = w(x_\perp)\langle x_\perp\rangle^{-m+1}\int_{\mathbb{R}} \langle u\rangle^{-m}\,du.$$

5. Application to cylindrically supported perturbations

Let us now consider potentials V which are supported in a cylinder with axis B, the magnetic field. Then W (defined by (3.4)) is compactly supported and we can apply results of [4,9] when W is supported on a domain $\Omega \subset \mathbb{R}^2$ or results of [24] when the sign of W is different on two "well-separated" domains Ω_+ and Ω_-. In this section, a domain will be a connected bounded subset of $\mathbb{R}^2$ (sometimes identified with $\mathbb{C}$), with Lipschitz boundary.

Some of the following asymptotics involve the *logarithmic capacity* introduced in the framework of the potential theory. For $\mathcal{E} \subset \mathbb{R}^2$ a Borel set, and $\mathfrak{M}(\mathcal{E})$ the set of compactly supported probability measures on $\mathcal{E}$, the *logarithmic capacity* of $\mathcal{E}$ is defined as $\operatorname{Cap}(\mathcal{E}) := e^{-\mathcal{I}(\mathcal{E})}$ where

$$\mathcal{I}(\mathcal{E}) := \inf_{\mu\in\mathfrak{M}(\mathcal{E})} \int_{\mathcal{E}}\int_{\mathcal{E}} \ln|x-y|^{-1}\,d\mu(x)\,d\mu(y).$$

A systematic exposition of the theory of the logarithmic capacity can be found, for example, in [29, Chapter 5] and [19, Chapter II, Section 4]. In this section, we mainly use that the logarithmic capacity of a Lipschitz domain Ω coincides with the logarithmic capacity of its outer boundary (the boundary of the unbounded component of $\mathbb{R}^2 \setminus \Omega$) and with $\operatorname{Cap}(\overline{\Omega})$ (see e.g., [7, Proposition 5.6] or [6, Section 5.5]).

For the first two results of this section, we say that a compact set K is *encircled* by an open set Ω if there exists a Jordan curve $\Gamma \subset \Omega$ such that K is contained in the interior part of Γ (see [4, 24]). It is clear that if Ω encircles a compact set K, then we have

$$\mathrm{Cap}(\bar{\Omega} \setminus K) = \mathrm{Cap}(\bar{\Omega}) = \mathrm{Cap}(\Omega). \tag{5.1}$$

Here and throughout the section, χ_K denotes the characteristic function of the set K and

$$\mathfrak{C}(\Omega) := 1 + \ln\left(b\,\mathrm{Cap}(\Omega)^2\right).$$

For the proof of the main results of this section, we shall need the following auxiliary lemma. For $\lambda > 0$ small enough, we introduce the notations

$$\ln_2(\lambda) := \ln|\ln\lambda|, \quad \ln_3(\lambda) := \ln\ln_2(\lambda) \tag{5.2}$$

Lemma 5.1 ([6, Corollaries 5.11 and 5.12]). *Let T be a self-adjoint compact operator having a finite number of negative eigenvalues and whose positive eigenvalues $(\nu_k)_k$ satisfy*

$$\lim_{k\to\infty} (k!\nu_k)^{1/k} = \mathfrak{C}_0/2, \quad \mathfrak{C}_0 > 0.$$

Then for $\mathfrak{C} := 1 + \ln\mathfrak{C}_0$ and for any constant $c > 0$, we have, as $\lambda \downarrow 0$,

$$n_+(c\sqrt{\lambda}; T) = \frac{1}{2}\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \mathfrak{C}\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big) + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big),$$

and

$$\frac{1}{\pi}\mathrm{Tr}\arctan\Big(\frac{T}{c\sqrt{\lambda}}\Big) = \frac{1}{4}\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \mathfrak{C}\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big) + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big). \tag{5.3}$$

With this auxiliary lemma at hand, we are ready to prove the first main result of this section.

Theorem 5.2. *Assume that V satisfies (2.8) and there exists a bounded domain $\Omega \subset \mathbb{R}^2$ such that $\mathrm{supp}\,V \subset \bar{\Omega} \times \mathbb{R}$. Assume also that there exists a compact set K encircled by Ω and $\alpha > 0$ such that $W(x) \le -\alpha$ for any $x \in \Omega \setminus K$.*

Then, for any $q \in \mathbb{Z}_+$, as $\lambda \downarrow 0$, the spectral shift function satisfies

$$\xi(\Lambda_q - \lambda; H, H_0) = -\frac{1}{2}\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \frac{|\ln\lambda|}{\ln_2(\lambda)^2}\mathfrak{C}(\Omega)\Big) + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big), \tag{5.4}$$

where $\ln_2$ and $\ln_3$ are defined in (5.2).

Moreover,

$$\xi(\Lambda_q + \lambda; H, H_0) = -\frac{1}{4}\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \frac{|\ln\lambda|}{\ln_2(\lambda)^2}\mathfrak{C}(\Omega)\Big) + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big). \tag{5.5}$$

Proof. As in Section 4, from (3.7), we know that the study of $\xi(\Lambda_q - \lambda; H, H_0)$, when $\lambda > 0$ tends to 0, is reduced to that of $n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q)$, for any $\varepsilon \in (0,1)$. Let us denote by $\{\nu_{k,q}^\pm\}_{k\in\mathbb{Z}_+}$ the non-increasing sequences of positive eigenvalues of $p_q(-W_\varepsilon^\pm)p_q$. Under the assumption on V and W, from the relation

$$W_\varepsilon^\pm(x_\perp) = W(x_\perp) \pm \frac{\varepsilon}{2}\int_{\mathbb{R}} (I \mp \varepsilon \,\mathrm{sign}\, V)^{-1}|V|(x_\perp, x_3)dx_3,$$

we deduce that for ε small enough, there exists $C > 0$ such that

$$C\chi_{\bar\Omega} \geq -W_\varepsilon^\pm(x_\perp) \geq \frac{\alpha}{2}\chi_{\bar\Omega\setminus K}.$$

By [24, Theorem 1.1] (see also [4, Proposition 3.4]), the operator $p_q(-W_\varepsilon^\pm)p_q$ has only finitely many negative eigenvalues. On the other hand, for the positive eigenvalues, the min-max principle combined with known asymptotic behaviors of the eigenvalues of $p_q\chi_{\bar\Omega}p_q$ and $p_q\chi_{\bar\Omega\setminus K}p_q$ (see [9] and [4, Proposition 3.7]) yields

$$\limsup_{k\to\infty}(k!\nu_{k,q}^\pm)^{1/k} \leq \frac{b\,\mathrm{Cap}(\bar\Omega)^2}{2}, \quad \liminf_{k\to\infty}(k!\nu_{k,q}^\pm)^{1/k} \geq \frac{b\,\mathrm{Cap}(\bar\Omega\setminus K)^2}{2}.$$

From (5.1), we deduce

$$\lim_{k\to\infty}(k!\nu_{k,q}^\pm)^{1/k} = \frac{b\,\mathrm{Cap}(\Omega)^2}{2},$$

and (5.4) follows from Lemma 5.1.

For the result in (5.5), by using again (4.6) and that $\mathrm{supp}\, V \subset \bar\Omega \times \mathbb{R}$, we are reduced to study

$$\frac{1}{\pi}\,\mathrm{Tr}\Big(\arctan\Big(p_q\frac{\widetilde{W}_{\lambda,\varepsilon}^\pm}{\sqrt{\lambda}}p_q\Big)\Big) = \frac{1}{\pi}\,\mathrm{Tr}\Big(\arctan\Big(p_q\frac{W_\varepsilon^\pm}{\sqrt{\lambda}}p_q\Big)\Big),$$

because

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} |V(x_\perp, x_3)||\sin(\sqrt{\lambda}x_3)|dx_3dx_\perp \leq \sqrt{\lambda}\int_{\bar\Omega}\int_{\mathbb{R}} \langle x_3\rangle^{-m+1}\,dx_3\,dx_\perp.$$

Since under the assumption on V, the compact operator $p_q(-W_\varepsilon^\pm)p_q$ satisfies the hypothesis of Lemma 5.1 (with $\mathfrak{C}_0 = b\,\mathrm{Cap}(\Omega)^2$), we deduce (5.5) from (5.3). ■

When the main contribution of W is positive, we have the following.

Theorem 5.3. *Assume that V satisfies* (2.8) *and there exists a bounded domain $\Omega \subset \mathbb{R}^2$ such that* $\operatorname{supp} V \subset \overline{\Omega} \times \mathbb{R}$. *Assume also that there exists a compact set K encircled by Ω and $\alpha > 0$ such $W(x) \geq \alpha$ for any $x \in \Omega \setminus K$.*

Then, for any $q \in \mathbb{Z}_+$, the spectral shift function satisfies

$$\xi(\Lambda_q - \lambda; H, H_0) = O(1), \quad \lambda \downarrow 0, \tag{5.6}$$

and

$$\begin{aligned} &\xi(\Lambda_q + \lambda; H, H_0) \\ &= \frac{1}{4}\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)} + \frac{|\ln\lambda|\ln_3(\lambda)}{\ln_2(\lambda)^2} + \frac{|\ln\lambda|}{\ln_2(\lambda)^2}\mathfrak{C}(\Omega)\Big) + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)^2}\Big), \quad \lambda \downarrow 0, \end{aligned} \tag{5.7}$$

where $\ln_2$ *and* $\ln_3$ *are defined in* (5.2).

Proof. As in the previous theorem, the study of $\xi(\Lambda_q - \lambda; H, H_0)$ when $\lambda > 0$ tends to 0 is still reduced to that of $n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q)$. By [24, Theorem 1.1] (see also [4, Proposition 3.4]), we have that the compact operator $p_q(-W_\varepsilon^\pm)p_q$ has a finite number of positive eigenvalues Then $n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q)$ is uniformly bounded when $\lambda \downarrow 0$ and we deduce (5.6).

The proof of (5.7) works as for (5.5). We just apply Lemma 5.1 with $T = p_q W_\varepsilon^\pm p_q$ instead of $p_q(-W_\varepsilon^\pm)p_q$ (which explains the difference in sign in the asymptotic). ∎

In the previous results, even if the perturbation is not of defined sign, W has a sign on the "external" part of its support. In the main contribution, governed by this "external" part, the "encircled" part of the support of W plays no role. Let us now give a result when the negative part and the positive part of the effective potential W contribute in the main term. It is for a compactly supported potential with "well-separated" supports of positive and negative contributions (a 3-D analogue of [24, Theorem 1.2]).

Theorem 5.4. *Assume that V satisfies* (2.8) *and there exist Ω_+, Ω_-, two compact domains in $\mathbb{C}$ (with Lipschitz boundaries) such that*

(1) *the closed convex hull of Ω_+ and Ω_- are disjoint;*

(2) $\operatorname{supp} V \subset (\Omega_+ \cup \Omega_-) \times \mathbb{R}$;

(3) *W defined by* (3.4) *satisfies*

$$W(x_\perp) = w_+(x_\perp)\chi_{\Omega_+}(x_\perp) - w_-(x_\perp)\chi_{\Omega_-}(x_\perp)$$

with $w_\pm \in L^\infty(\mathbb{R}^2)$, and on $\Omega_\pm$, $w_\pm \geq \alpha$ for some $\alpha > 0$.

Then, there exist constants $0 < \delta \leq \Delta < 1$*, such that for any* $q \in \mathbb{Z}_+$*, the spectral shift function satisfies*

$$\frac{1}{2}\frac{|\ln\lambda|}{\ln_2(\lambda)}(\delta + o(1)) \leq -\xi(\Lambda_q - \lambda; H, H_0) \leq \frac{1}{2}\frac{|\ln\lambda|}{\ln_2(\lambda)}(\Delta + o(1)), \quad \lambda \downarrow 0, \tag{5.8}$$

and

$$\frac{1}{2}\frac{|\ln\lambda|}{\ln_2(\lambda)}\Big(\frac{1}{2} - \Delta + o(1)\Big) \leq \xi(\Lambda_q + \lambda; H, H_0) \leq \frac{1}{2}\frac{|\ln\lambda|}{\ln_2(\lambda)}\Big(\frac{1}{2} - \delta + o(1)\Big), \quad \lambda \downarrow 0. \tag{5.9}$$

Proof. As in the previous theorems, the study of $\xi(\Lambda_q - \lambda; H, H_0)$ when $\lambda > 0$ tends to 0 is reduced to that of $n_-(\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q) = n_+(\sqrt{\lambda}; p_q(-W_\varepsilon^\pm)p_q)$ and for $\xi(\Lambda_q + \lambda; H, H_0)$ we must analyse

$$\frac{1}{\pi}\operatorname{Tr}\Big(\arctan\Big(p_q\frac{W_\varepsilon^\pm}{\sqrt{\lambda}}p_q\Big)\Big) = \frac{1}{\pi}\int_0^\infty n_+(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q) - n_-(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2},$$

with

$$W_\varepsilon^\pm(x_\perp) = W(x_\perp) \pm \frac{\varepsilon}{2}\int_{\mathbb{R}}(I \mp \varepsilon \operatorname{sign} V)^{-1}|V|(x_\perp, x_3)dx_3.$$

We easily check that for ε sufficiently small the assumptions on W are still true for $W_\varepsilon^\pm$ (possibly by changing α by $\alpha/2$). Then from [24, Theorem 1.2], we have the existence of constants $0 < \delta \leq \Delta < 1$ such that as $s \downarrow 0$,

$$\frac{|\ln s|}{\ln_2(s)}(\delta + o(1)) \leq n_-(s; p_q W_\varepsilon^\pm p_q) \leq \frac{|\ln s|}{\ln_2(s)}(\Delta + o(1)), \tag{5.10}$$

$$\frac{|\ln s|}{\ln_2(s)}(1 - \Delta + o(1)) \leq n_+(s; p_q W_\varepsilon^\pm p_q) \leq \frac{|\ln s|}{\ln_2(s)}(1 - \delta + o(1)). \tag{5.11}$$

We deduce (5.8), and for (5.9) we conclude by using the idea of the proof of Lemma 5.1. More precisely, by using that the counting functions $s \mapsto n_\pm(s, T)$ are non-increasing, $\lambda \downarrow 0$, we have

$$0 \leq \int_{|\ln\lambda|^{-1}}^\infty n_\pm(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2} \leq n_\pm(|\ln\lambda|^{-1}\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\int_{|\ln\lambda|^{-1}}^\infty \frac{dt}{1+t^2},$$

and

$$\int_0^\infty n_\pm(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2} = \int_0^{|\ln\lambda|^{-1}} n_\pm(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2} + o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)}\Big).$$

Then (5.10) and (5.11) imply

$$\begin{aligned}\frac{1}{4}\frac{|\ln\lambda|}{\ln_2(\lambda)}(\delta+o(1)) &\le \frac{1}{\pi}\int_0^{|\ln\lambda|^{-1}} n_-(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2}\\ &\le \frac{1}{4}\frac{|\ln\lambda|}{\ln_2(\lambda)}(\Delta+o(1)),\\ \frac{1}{4}\frac{|\ln\lambda|}{\ln_2(\lambda)}(1-\Delta+o(1)) &\le \frac{1}{\pi}\int_0^{|\ln\lambda|^{-1}} n_+(t\sqrt{\lambda}; p_q W_\varepsilon^\pm p_q)\frac{dt}{1+t^2}\\ &\le \frac{1}{4}\frac{|\ln\lambda|}{\ln_2(\lambda)}(1-\delta+o(1)),\end{aligned}$$

and we deduce (5.9). ■

Remark 5.5. From [24, Theorem 1.3], we know that if, in the previous result, Ω_+ and Ω_- can be interchanged by an euclidian motion in $\mathbb{C}$, then $\delta = \Delta = \frac{1}{2}$. In this case, the lower and upper bounds of (5.8) coincide while the main contribution of (5.9) vanishes, and we obtain

$$\begin{aligned}\xi(\Lambda_q - \lambda; H, H_0) &= -\frac{1}{4}\frac{|\ln\lambda|}{\ln_2(\lambda)}(1+o(1)), \quad \lambda\downarrow 0,\\ \xi(\Lambda_q + \lambda; H, H_0) &= o\Big(\frac{|\ln\lambda|}{\ln_2(\lambda)}\Big), \quad \lambda\downarrow 0.\end{aligned}$$

As shown in [24], the constants δ and Δ come from the spectral bounds on an auxiliary operator for two-dimensional Bargmann–Toeplitz operators. Their optimal values for more general domains $\Omega_\pm$ remain an open problem.

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