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Construction of Toda flow via Sato–Segal–Wilson theory

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Abstract. A Toda flow is constructed on a space of bounded initial data through the Sato–Segal–Wilson theory. The flow is described in terms of the Weyl functions of the underlying Jacobi operators. This is a continuation of the previous work on the KdV flow.

1. Introduction

The Toda lattice introduced by M. Toda in 1967 is a simple model for a one-dimensional crystal in solid state physics, and it is known to be one of the earliest examples of a non-linear completely integrable system. Originally, it was an infinite-dimensional system of equations

$$\begin{aligned} \dot{p}_n(t) &= e^{-(q_n(t)-q_{n-1}(t))} - e^{-(q_{n+1}(t)-q_n(t))}, \\ \dot{q}_n(t) &= p_n(t), \end{aligned}$$

with $n \in \mathbb{Z}$, and later it was rewritten in an equivalent form

$$\begin{aligned} \dot{a}_n(t) &= a_n(t)(b_n(t) - b_{n-1}(t)), \\ \dot{b}_n(t) &= 2(a_{n+1}(t)^2 - a_n(t)^2), \end{aligned} \tag{1.1}$$

by Flaschka variables

$$a_n(t) = \frac{1}{2} e^{(q_n(t)-q_{n-1}(t))/2}, \quad b_n(t) = -\frac{1}{2} p_n(t).$$

This version is useful because it relates the Toda lattice with a Jacobi operator

$$(H_q u)_n = a_{n+1} u_{n+1} + a_n u_{n-1} + b_n u_n,$$

where $a_n > 0$, $b_n \in \mathbb{R}$, and $q \equiv \{a_n, b_n\}_{n\in\mathbb{Z}}$. The remarkable fact here is that if the coefficients q are replaced by a solution $q(t) \equiv \{a_n(t), b_n(t)\}_{n\in\mathbb{Z}}$ to (1.1), then the

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arising Jacobi operator $H_{q(t)}$ and the initial Jacobi operator $H_{q(0)}$ have the same spectrum. Flaschka discovered in 1973 that $H_{q(t)}$ satisfies the following operator equation:

$$\partial_t H_{q(t)} = [P(t), H_{q(t)}](\equiv P(t)H_{q(t)} - H_{q(t)}P(t)) \tag{1.2}$$

with a skew symmetric operator $P(t)$

$$(P(t)u)_n = a_{n+1}(t)u_{n+1} - a_n(t)u_{n-1},$$

which establishes the unitary equivalence of $H_{q(t)}$ and $H_{q(0)}$. If $P(t)$ is replaced by other suitable skew symmetric operators, we obtain infinitely many solutions to a hierarchy of non-linear equations by (1.2), which is called *Toda hierarchy.* The pair $\{H_{q(t)}, P(t)\}$ is called a *Lax pair* and the operator $H_{q(t)}$ is the underlying operator for this hierarchy of equations. There are many works treating Toda hierarchy and main results obtained in the last century can be found in the book by Teschl [7].

Recently, in 2018, another point of view for the Toda hierarchy was introduced by Remling [3] and Ong and Remling [2]. Remling considered the Lax equation in a form

$$\partial_t H_{q(t)} = [H_{q(t)}, p(H_{q(t)})_a]$$

with real polynomial p, where X_a for a self-adjoint bounded operator X on $\ell^2(\mathbb{Z})$ is defined by

$$(X_a)_{jk} = \begin{cases} X_{jk} & \text{if } j < k, \\ 0 & \text{if } j = k, \\ -X_{jk} & \text{if } j > k, \end{cases} \quad \text{with } X_{jk} = \langle \delta_j, X\delta_k \rangle. \tag{1.3}$$

He introduced a notion of cocycles or transfer matrices which map Weyl functions for Jacobi operators to Weyl functions. This notion played a crucial role in the study of spectral problems for ergodic Jacobi operators, so it is expected that the cocycle property of the Toda flow is also significant to investigate global behavior of solutions to the Toda lattice. Remling and Ong expanded the Toda hierarchy from polynomials to entire functions preserving the cocycle property.

The purpose of this article is to apply Sato's theory and provide another approach to the study of the Toda hierarchy. In 1980, Sato [4] obtained solutions to many integrable systems by constructing flows on an infinite-dimensional Grassmann manifold. Later, in 1985, Segal and Wilson [5] interpreted Sato's theory by Hardy space on the unit circle of the complex plane $\mathbb{C}$. They considered Grassmann manifolds on $L^2(|z| = 1)$ consisting of closed subspaces which are invariant under multiplication by z^n, and they showed the KdV hierarchy is obtained when $n = 2$. Segal and Wilson's approach was employed by Kotani [1] to construct general non-decaying solutions to the KdV equation including many almost-periodic initial data. In the present article, we apply Segal–Wilson's method to the Toda hierarchy by following Kotani's work.

For the KdV case, z^2 was employed as the spectral parameter for Schrödinger operators, which are the underlying operators for the KdV hierarchy. In the Toda case, we use $z + z^{-1}$ as its spectral parameter since $\{z^n\}_{n\in\mathbb{Z}}$ forms a system of generalised eigen-functions for the discrete Laplacian $u_{n+1} + u_{n-1}$. The bounded domain D_+ in $\mathbb{C}$ which is fundamental in the following argument is chosen so that it satisfies

$$\begin{aligned} &D_+ \ni z \to z^{-1}, \quad \bar{z} \in D_+ \ (\bar{z} \text{ denotes the complex conjugate of } z), \\ &D_+ \text{ contains the spectrum of Jacobi operators.} \end{aligned} \tag{1.4}$$

Throughout the paper, we assume the spectrum $\operatorname{sp} H_q$ of a Jacobi operator H_q is bounded, which is equivalent to the boundedness of the coefficients $q = \{a_n, b_n\}_{n\in\mathbb{Z}}$. Let

$$\phi(z) = z + z^{-1}.$$

If $\operatorname{sp} H_q \subset [-\lambda_0, \lambda_0]$ holds for $\lambda_0 > 2$ ($[-2, 2]$ is the spectrum for the discrete Laplacian), then

$$\Sigma_{\lambda_0} \equiv \phi^{-1}([-\lambda_0, \lambda_0]) = \{|z| = 1\} \cup [-\ell, -\ell^{-1}] \cup [\ell^{-1}, \ell]$$

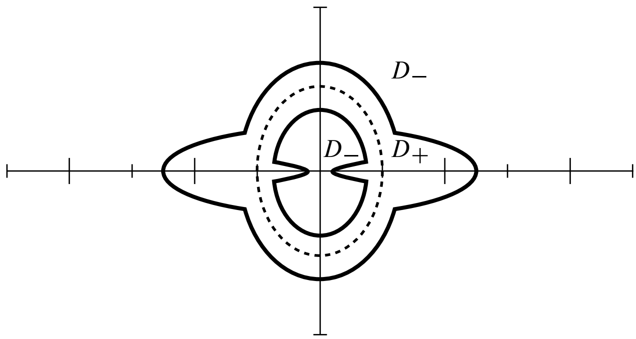
with $\ell = (\lambda_0 + \sqrt{\lambda_0^2 - 4})/2$. Therefore, D_+ should satisfy

$$\Sigma_{\lambda_0} \subset D_+.$$

Since D_+ is bounded, the origin 0 is not an element of D_+, otherwise the symmetry in (1.4) implies $\infty = 0^{-1} \in D_+$. We denote

$$\begin{aligned} &C = \partial D_+ \text{ (the boundary of } D_+), \quad D_- = \mathbb{C} \setminus (D_+ \cup C), \\ &C = C_1 \cup C_2 \ (C_1 \text{ is the outer curve and } C_2 \text{ is the inner curve of } C), \end{aligned}$$

and assume C_1, C_2 are closed smooth simple curves. The figure below is a typical example:



Denote

$$D_- = D_-^1 \cup D_-^2 \quad \text{(2 disjoint domains)},$$

where D_-^1 is unbounded and D_-^2 containing 0 is bounded.

The basic space is the Hilbert space $L^2(C)$, which is a direct sum of 2 closed subspaces $H_\pm$:

$$H_+ = L^2\text{-closure of } \{\text{all rational functions with no poles in } D_+\},$$

$$H_- = L^2\text{-closure of} \left\{ \begin{array}{l} \text{all functions } f \text{ on } D_- \text{ such that } f|_{D_-^j} \text{ are} \\ \text{rational functions with no poles on } D_-^j \\ \text{for } j = 1, 2 \text{ and } f(z) = o(1) \text{ as } z \to \infty \end{array} \right\}.$$

The Hardy space H_+ is generated by $\{z^n\}_{n\in\mathbb{Z}}$, and

$$r(z) = \begin{cases} z^m & \text{for } z \in D_-^1, \\ z^n & \text{for } z \in D_-^2, \end{cases} \quad \text{with } m \le -1 \text{ and } n \ge 0,$$

is an element of H_-. The projections $\mathfrak{p}_\pm$ from $L^2(C)$ onto $H_\pm$ are obtained by

$$(\mathfrak{p}_+ f)(z) = \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{\lambda - z} d\lambda \quad \text{for } z \in D_+,$$

$$(\mathfrak{p}_- f)(z) = \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{z - \lambda} d\lambda \quad \text{for } z \in D_-,$$

where the integrals on C are defined by

$$\int_C f(\lambda)d\lambda = \int_{C_1} f(\lambda)d\lambda + \int_{C_2} f(\lambda)d\lambda.$$

C_1 is oriented anti-clockwise and C_2 clockwise. It is known that $\mathfrak{p}_\pm$ are bounded on $L^2(C)$ (see [8]) and satisfy

$$\mathfrak{p}_\pm^2 = \mathfrak{p}_\pm, \quad \mathfrak{p}_+ + \mathfrak{p}_- = I \quad \text{(the identity operator on } L^2(C)).$$

In our Toda case, the Sato–Segal–Wilson theory is performed on a Grassmann manifold $\mathrm{Gr}^{(\mathrm{toda})}$ consisting of closed subspaces W of $L^2(C)$ satisfying

$$\phi(z)W \subset W. \tag{1.5}$$

Additionally, we assume also the comparability of W with H_+, that is

$$\mathfrak{p}_+ \colon W \to H_+ \text{ is bijective.} \tag{1.6}$$

Generally, a bounded vector function

$$\mathbf{a}(\lambda) = (a_1(\lambda), a_2(\lambda)) \text{ on } C$$

gives such W by

$$W_{\mathbf{a}} \equiv \mathbf{a}H_+ = \{\mathbf{a}u\,;\ u \in H_+\}, \tag{1.7}$$

where the product $\mathbf{a}u$ is defined by

$$(\mathbf{a}u)(\lambda) = a_1(\lambda)u(\lambda) + a_2(\lambda)\lambda^{-1}u(\lambda^{-1}).$$

The invariance $\phi(\lambda) = \phi(\lambda^{-1})$ implies

$$\phi(\lambda)(\mathbf{a}u)(\lambda) = a_1(\lambda)(\phi u)(\lambda) + a_2(\lambda)\lambda^{-1}(\phi u)(\lambda^{-1}) = (\mathbf{a}(\phi u))(\lambda),$$

hence $\phi W_{\mathbf{a}} \subset W_{\mathbf{a}}$ holds due to $\phi H_+ \subset H_+$, which shows (1.5). Moreover, defining an operator on H_+ by

$$T(\mathbf{a})u = \mathfrak{p}_+(\mathbf{a}u) \quad \text{for } u \in H_+,$$

one sees that the property (1.6) for $W_{\mathbf{a}}$ is equivalent to

$$T(\mathbf{a})\colon H_+ \to H_+ \text{ is bijective.}$$

The operator $T(\mathbf{a})$ is called a *Toeplitz operator with symbol* $\mathbf{a}$. Hereafter, we consider $W_{\mathbf{a}}$ instead of a general W.

Sato introduced the tau functions to represent his flows on a Grassmann manifold. In our context it is defined as follows. Set

$$\begin{cases} \mathbf{A}(C) = \{\mathbf{a} = (a_1, a_2)\,;\ \sup\limits_{\lambda\in C} \|\mathbf{a}(\lambda)\| < \infty\}, \\ \mathbf{A}^{\mathrm{inv}}(C) = \{\mathbf{a} \in \mathbf{A}(C)\,;\ T(\mathbf{a}) \text{ is invertible on } H_+\}, \end{cases}$$

with $\|\mathbf{a}\| = \sqrt{|a_1|^2 + |a_2|^2}$ and a group Γ acting on the space of symbols

$$\Gamma = \left\{\begin{array}{l} g = re^h\,;\ r \text{ is a rational function with no poles nor zeros in } \Sigma_{\lambda_0} \\ \text{and } h \text{ is analytic in a neighbourhood of } \Sigma_{\lambda_0} \end{array}\right\}.$$

Γ acts on $\mathbf{A}(C)$ by

$$(g\mathbf{a})(\lambda) = (g(\lambda)a_1(\lambda), g(\lambda)a_2(\lambda)) \in \mathbf{A}(C),$$

but not necessarily on $\mathbf{A}^{\mathrm{inv}}(C)$, since the invertibility of $T(g\mathbf{a})$ is not trivial for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$. The domain D_+ for each $g \in \Gamma$ is chosen so that g has no poles nor zeros in $\overline{D}_+$ (the closure). For $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$, $g \in \Gamma$ define

$$\tau_{\mathbf{a}}(g) = \det(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}).$$

Since $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I$ is a trace class operator on H_+ (refer to Lemma 7), one can define $\tau_{\mathbf{a}}(g)$. For $g =$ a rational function, this fact can be explained as follows. For $g \in \Gamma$ and $\mathbf{a} \in \mathbf{A}^{\text{inv}}(C)$, observe

$$T(g\mathbf{a})u = \mathfrak{p}_+(g\mathbf{a}u) = \mathfrak{p}_+(g\mathfrak{p}_+\mathbf{a}u) + \mathfrak{p}_+(g\mathfrak{p}_-\mathbf{a}u) = gT(g\mathbf{a})u + \mathfrak{p}_+(g\mathfrak{p}_-\mathbf{a}u)$$

for $u \in H_+$. If $g = q_\zeta$ (where $q_\zeta(z) = (1 - \zeta^{-1}z)^{-1}$) for $\zeta \in D_-$, the last term becomes

$$\begin{aligned}\mathfrak{p}_+(q_\zeta\mathfrak{p}_-\mathbf{a}u)(z) &= \frac{1}{2\pi i}\int_C \frac{(\mathfrak{p}_-\mathbf{a}u)(\lambda)}{(\lambda - z)(1 - \zeta^{-1}\lambda)}d\lambda \\ &= (\mathfrak{p}_-\mathbf{a}u)(\zeta)q_\zeta(z) \quad \text{for } z \in D_+,\end{aligned}$$

which says that the image of the operator $\mathfrak{p}_+(q_\zeta\mathfrak{p}_-\mathbf{a}\cdot)$ is spanned by a single $\{q_\zeta\}$. Similarly, for any rational $r = \sum_{1\le j\le n} r_j q_{\zeta_j} \in \Gamma$, the image of $\mathfrak{p}_+(r\mathfrak{p}_-\mathbf{a}\cdot)$ is spanned by $\{q_{\zeta_j}\}_{1\le j\le n}$. Thus, the operator $r^{-1}T(r\mathbf{a})T(\mathbf{a})^{-1} - I$ has an $n \times n$ matrix representation:

$$(r_j\mathfrak{p}_-(\mathbf{a}T(\mathbf{a})^{-1}r^{-1}q_{\zeta_i})(\zeta_j))_{1\le i,j\le n}.$$

The details can be found in Section 4.

Since we are trying to construct the Toda flow on a subclass of $\mathbf{A}^{\text{inv}}(C)$, the property $g\mathbf{a} \in \mathbf{A}^{\text{inv}}(C)$ is crucial and this is satisfied if and only if $\tau_{\mathbf{a}}(g) \neq 0$ (refer to Lemma 8). Therefore, it is one of the main issues in this paper to find a class of symbols $\mathbf{a}$ satisfying $\tau_{\mathbf{a}}(g) \neq 0$ for sufficiently enough $g \in \Gamma$. Since we are interested only in real valued solutions, we assume $\bar{\mathbf{a}} = \mathbf{a}$, where the conjugation $\bar{f}$ is defined by

$$\bar{f}(z) = \overline{f(\bar{z})}.$$

Then, it is easily seen that $\tau_{\mathbf{a}}(g) \in \mathbb{R}$ if $\bar{\mathbf{a}} = \mathbf{a}$ and $g \in \Gamma_{\text{real}}$ with

$$\Gamma_{\text{real}} = \{g \in \Gamma\,;\; g = \bar{g}\}.$$

Γ_{real} is endowed with the metric

$$\mathrm{d}_{C_0}(g_1, g_2) = \sup_{\lambda\in C_0} |g_1(\lambda) - g_2(\lambda)|,$$

where C_0 is a simple closed smooth curve surrounding Σ_{λ_0}. If $\tau_{\mathbf{a}}(z^n) > 0$ holds for any $n \in \mathbb{Z}$, then the coefficients $\{a_n(\mathbf{a}), b_n(\mathbf{a})\}_{n\in\mathbb{Z}}$ are obtained by

$$a_n(\mathbf{a}) = \sqrt{\frac{\tau_{\mathbf{a}}(z^n)\tau_{\mathbf{a}}(z^{n-2})}{\tau_{\mathbf{a}}(z^{n-1})^2}}, \quad b_n(\mathbf{a}) = \partial_\varepsilon \log \frac{\tau_{\mathbf{a}}(z^n e^{\varepsilon z})}{\tau_{\mathbf{a}}(z^{n-1}e^{\varepsilon z})}\Big|_{\varepsilon=0}. \tag{1.8}$$

This formula resembles with the conventional representation of $\{a_n, b_n\}_{n\in\mathbb{Z}}$ by the moment problem, but the present one is a representation on the whole $\mathbb{Z}$.

Set

$$Q_{\lambda_0} = \{q = \{a_n, b_n\}_{n\in\mathbb{Z}} \ ; \ a_n > 0, b_n \in \mathbb{R} \text{ and } \operatorname{sp} H_q \subset [-\lambda_0, \lambda_0]\}$$

and let $m_\pm$ be the Weyl functions of H_q (see Appendix A). Q_{λ_0} is endowed with the metric

$$\mathrm{d}(q_1, q_2) = \sum_{n\in\mathbb{Z}} 2^{-|n|}(|a_n^{(1)} - a_n^{(2)}| + |b_n^{(1)} - b_n^{(2)}|), \quad \text{where } q_j = \{a_n^{(j)}, b_n^{(j)}\}_{n\in\mathbb{Z}}.$$

The property $\operatorname{sp} H_q \subset [-\lambda_0, \lambda_0]$ implies

$$\lambda_0^2 \geq \|H_q\delta_n\|^2 = a_n^2 + a_{n+1}^2 + b_n^2,$$

which shows $a_n, |b_n| \leq \lambda_0$ uniformly. Define an analytic function m on $\mathbb{C} \setminus \Sigma_{\lambda_0}$ by

$$m(z) = \begin{cases} z + z^{-1} + a_1^2 m_+(z + z^{-1}) & \text{if } z \in \mathbb{C} \setminus \Sigma_{\lambda_0} \text{ and } |z| > 1, \\ -a_0^2 m_-(z + z^{-1}) + b_0 & \text{if } z \in \mathbb{C} \setminus \Sigma_{\lambda_0} \text{ and } |z| < 1. \end{cases}$$

It is known that q is completely recovered from m. If we define a symbol $\mathbf{m}$,

$$\mathbf{m}(z) = \Big(\frac{zm(z) - 1}{z^2 - 1}, z^2\frac{z - m(z)}{z^2 - 1}\Big),$$

we will prove that $\tau_{\mathbf{m}}(g) > 0$ for any $q \in Q_{\lambda_0}$ and $g \in \Gamma_{\text{real}}$ in Section 8. Then, one can define $q(g\mathbf{m}) = \{a_n(g\mathbf{m}), b_n(g\mathbf{m})\}_{n\in\mathbb{Z}}$ by (1.8). $q(g\mathbf{m}) \in Q_{\lambda_0}$ is also shown if $\mathbf{m}$ is generated by $\{m_\pm, a_0, a_1, b_0\}$ of $q \in Q_{\lambda_0}$, hence one can define

$$\mathrm{Toda}(g)q = q(g\mathbf{m}) \in Q_{\lambda_0} \quad \text{for } q \in Q_{\lambda_0}.$$

Our main theorems are as follows. Theorem 2 is a restatement of Proposition 3.

Theorem 1. $\{\mathrm{Toda}(g)\}_{g\in\Gamma_{\text{real}}}$ *defines a continuous flow on* Q_{λ_0}.

Theorem 2. *For a real polynomial p, the coefficients $q_t = \mathrm{Toda}(e^{-2t\hat{p}})q$ solve the Cauchy problem of the Toda hierarchy*

$$\partial_t H_{q_t} = [H_{q_t}, p(H_{q_t})_a] \quad \text{with } q_0 = q,$$

where $\hat{p}$ is the polynomial part of $p(z + z^{-1})$ and $(\cdot)_a$ denotes the anti-symmetrization of an operator $\cdot$. Especially, if $p(z) = z$, $\mathrm{Toda}(e^{-2tz})q$ provides a solution to the Toda lattice.

These theorems imply that one can construct the Toda flow on the space of bounded initial data $\{a_n, b_n\}_{n\in\mathbb{Z}}$, and the group Γ_{real} is more general than entire functions treated by Ong and Remling [2]. One can define the cocycles for the present flow.

One of the advantages for this construction is the possibility to extend the flow to a flow on an unbounded initial data $\{a_n, b_n\}_{n\in\mathbb{Z}}$. For this purpose, one has to replace the domain D_+ with a domain containing $\mathbb{R} \cup \{|z| = 1\}$ and trace the whole argument below. This extension is indeed possible and we have constructed solutions with initial data of power growth less than 1. This will be published in another paper.

2. Basic properties of $W_{\mathbf{a}}$

For a bounded vector symbol $\mathbf{a}$ on C, the space $W_{\mathbf{a}}$ is defined in (1.7). In this section we show basic properties of $W_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$.

Define

$$Rf(z) = z^{-1} f(z^{-1}).$$

Since C is invariant under the transformation $z \to z^{-1}$, R acts on functions on C. We have the following result.

Lemma 1. *The identities*

$$\mathfrak{p}_\pm R = R\mathfrak{p}_\pm \text{ on } L^2(C), \quad T(\mathbf{a})R = RT(\tilde{\mathbf{a}}) \text{ on } H_+$$

are satisfied, where $\tilde{\mathbf{a}}(\lambda) = (a_1(\lambda^{-1}), a_2(\lambda^{-1}))$. *In particular, this implies that* $\tilde{\mathbf{a}} \in \mathbf{A}^{\mathrm{inv}}(C)$ *if* $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$.

Proof. For $f \in L^2(C)$, note

$$\begin{aligned}
\int_C f(\lambda^{-1})d\lambda &= \int_{C_1} f(\lambda^{-1})d\lambda + \int_{C_2} f(\lambda^{-1})d\lambda \\
&= -\int_{C_1} f(\lambda)\lambda^{-2}d\lambda - \int_{C_2} f(\lambda)\lambda^{-2}d\lambda \\
&= -\int_C f(\lambda)\lambda^{-2}d\lambda.
\end{aligned}$$

Then one has

$$\mathfrak{p}_+(Rf)(z) = \frac{1}{2\pi i}\int_C \frac{\lambda^{-1}f(\lambda^{-1})}{\lambda - z}d\lambda = z^{-1}\frac{1}{2\pi i}\int_C \frac{f(\lambda)}{\lambda - z^{-1}}d\lambda = R(\mathfrak{p}_+ f)(z).$$

The second identity follows from $\mathbf{a}R = R\tilde{\mathbf{a}}$ and the first identity. ∎

For $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ and $n \in \mathbb{Z}$, define a sequence of functions of H_- by

$$\varphi_{\mathbf{a}}^{(n)}(z) = \mathfrak{p}_-(\mathbf{a}T(\mathbf{a})^{-1}z^n) \in H_-.$$

Lemma 2. $\varphi_{\mathbf{a}}^{(n)}$ *for* $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ *satisfies the following identities:*

(i)
$$\varphi_{\tilde{\mathbf{a}}}^{(n)}(z) = z^{-1}\varphi_{\mathbf{a}}^{(-n-1)}(z^{-1}),$$

(ii)
$$\begin{aligned}(z+z^{-1})(z^n+\varphi_{\mathbf{a}}^{(n)}) = {}& z^{n+1}+\varphi_{\mathbf{a}}^{(n+1)}+z^{n-1}+\varphi_{\mathbf{a}}^{(n-1)} \\ &+\varphi_{\tilde{\mathbf{a}}}^{(-n-1)}(0)(1+\varphi_{\mathbf{a}}^{(0)})+\varphi_{\mathbf{a}}^{(n)}(0)(z^{-1}+\varphi_{\mathbf{a}}^{(-1)}).\end{aligned}$$

(ii) *shows that linear combinations of* $\{1+\varphi_{\mathbf{a}}^{(0)}, z^{-1}+\varphi_{\mathbf{a}}^{(-1)}\}$ *generate* $W_{\mathbf{a}}$.

Proof. Lemma 1 implies

$$\begin{aligned}\varphi_{\tilde{\mathbf{a}}}^{(n)} &= \mathfrak{p}_-(\tilde{\mathbf{a}}T(\tilde{\mathbf{a}})^{-1}z^n) = \mathfrak{p}_-(\tilde{\mathbf{a}}RT(\mathbf{a})^{-1}R^{-1}z^n) = \mathfrak{p}_-(R\mathbf{a}T(\mathbf{a})^{-1}z^{-n-1}) \\ &= R\mathfrak{p}_-(\mathbf{a}T(\mathbf{a})^{-1}z^{-n-1}) = R\varphi_{\mathbf{a}}^{(-n-1)},\end{aligned}$$

which is (i).

To show (ii), first note $z^n+\varphi_{\mathbf{a}}^{(n)} \in W_{\mathbf{a}}$, hence $(z+z^{-1})(z^n+\varphi_{\mathbf{a}}^{(n)}) \in W_{\mathbf{a}}$ is satisfied. Now, decompose this quantity into elements of $H_\pm$, namely

$$(z+z^{-1})(z^n+\varphi_{\mathbf{a}}^{(n)}) = u+v$$

with

$$\begin{cases} u = z^{n+1}+z^{n-1}+\lim_{z\to\infty} z\varphi_{\mathbf{a}}^{(n)}(z)+\varphi_{\mathbf{a}}^{(n)}(0)z^{-1} \in H_+, \\ v = z\varphi_{\mathbf{a}}^{(n)}(z) - \lim_{z\to\infty} z\varphi_{\mathbf{a}}^{(n)}(z) + z^{-1}(\varphi_{\mathbf{a}}^{(n)}(z)-\varphi_{\mathbf{a}}^{(n)}(0)) \in H_-. \end{cases}$$

Here note due to (i),

$$\lim_{z\to\infty} z\varphi_{\mathbf{a}}^{(n)}(z) = \varphi_{\tilde{\mathbf{a}}}^{(-n-1)}(0).$$

The above $u \in H_+$ has the origin in $W_{\mathbf{a}}$ of

$$z^{n+1}+\varphi_{\mathbf{a}}^{(n+1)}+z^{n-1}+\varphi_{\mathbf{a}}^{(n-1)}+\varphi_{\tilde{\mathbf{a}}}^{(-n-1)}(0)(1+\varphi_{\mathbf{a}}^{(0)})+\varphi_{\mathbf{a}}^{(n)}(0)(z^{-1}+\varphi_{\mathbf{a}}^{(-1)}),$$

hence the bijectivity of $\mathfrak{p}_+ \colon W_{\mathbf{a}} \to H_+$ implies identity (ii). ■

Set

$$\Delta_{\mathbf{a}}(\zeta) = \frac{(1+\varphi_{\mathbf{a}}^{(0)}(\zeta))(\zeta+\varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1})) - (1+\varphi_{\mathbf{a}}^{(0)}(\zeta^{-1}))(\zeta^{-1}+\varphi_{\mathbf{a}}^{(-1)}(\zeta))}{\zeta-\zeta^{-1}}.$$

Later, we will need the non-vanishing of $\Delta_{\mathbf{a}}(\zeta)$.

Lemma 3. $\Delta_{\mathbf{a}}(\zeta) \neq 0$ *holds for* $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ *and* $\zeta \in D_-$.

Proof. Set $u_0 = T(\mathbf{a})^{-1}1$, $u_{-1} = T(\mathbf{a})^{-1}z^{-1}$. Since

$$\varphi_{\mathbf{a}}^{(0)} = \mathfrak{p}_-(\mathbf{a}u_0), \quad \varphi_{\mathbf{a}}^{(-1)} = \mathfrak{p}_-(\mathbf{a}u_{-1})$$

holds, one has

$$\mathbf{a}u_0(z) = 1 + \varphi_{\mathbf{a}}^{(0)}(z), \quad \mathbf{a}u_{-1}(z) = z^{-1} + \varphi_{\mathbf{a}}^{(-1)}(z).$$

Hence, observing that $r(z) = (\zeta - z)^{-1}(\zeta - z^{-1})^{-1}$ satisfies $r(z) = r(z^{-1})$, we have a decomposition into $H_\pm$:

$$\begin{aligned}\mathbf{a}ru_0(z) = r\mathbf{a}u_0(z) &= r(z)(1 + \varphi_{\mathbf{a}}^{(0)}(z)) \\ &= \Big(\frac{\varphi_{\mathbf{a}}^{(0)}(z) - \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - z^{-1})(\zeta - z)} - \frac{\varphi_{\mathbf{a}}^{(0)}(\zeta) - \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - \zeta^{-1})(\zeta - z)}\Big) \\ &\quad + \Big(\frac{\varphi_{\mathbf{a}}^{(0)}(\zeta) - \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - \zeta^{-1})(\zeta - z)} + \frac{1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - z)(\zeta - z^{-1})}\Big),\end{aligned}$$

which yields

$$\begin{aligned}(T(\mathbf{a})ru_0)(z) &= \frac{\varphi_{\mathbf{a}}^{(0)}(\zeta) - \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - \zeta^{-1})(\zeta - z)} + \frac{1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{(\zeta - z)(\zeta - z^{-1})} \\ &= \Big(z^{-1}\frac{1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1})}{\zeta - z^{-1}} + \zeta\frac{1 + \varphi_{\mathbf{a}}^{(0)}(\zeta)}{\zeta - z}\Big)\frac{1}{\zeta^2 - 1}.\end{aligned}$$

Similarly, we have

$$(T(\mathbf{a})ru_{-1})(z) = \Big(z^{-1}\frac{\zeta + \varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1})}{\zeta - z^{-1}} + \zeta\frac{\zeta^{-1} + \varphi_{\mathbf{a}}^{(-1)}(\zeta)}{\zeta - z}\Big)\frac{1}{\zeta^2 - 1}.$$

If we regard $-\zeta(\zeta - z)^{-1}$ as the unknown in these two identities, one obtains

$$T(\mathbf{a})((\zeta + \varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1}))ru_0 - (1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1}))ru_{-1}) = \frac{\Delta_{\mathbf{a}}(\zeta)}{\zeta - z}.$$

If $\Delta_{\mathbf{a}}(\zeta) = 0$ holds for some $\zeta \in D_-$, then

$$(\zeta + \varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1}))u_0 - (1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1}))u_{-1} = 0$$

follows. Applying $T(\mathbf{a})$ yields

$$(\zeta + \varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1})) - (1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1}))z^{-1} = 0,$$

which implies

$$\zeta + \varphi_{\mathbf{a}}^{(-1)}(\zeta^{-1}) = 1 + \varphi_{\mathbf{a}}^{(0)}(\zeta^{-1}) = 0.$$

Similarly, one has $\zeta^{-1} + \varphi_{\mathbf{a}}^{(-1)}(\zeta) = 1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) = 0$. Hence, $T(\mathbf{a})ru_0 = 0$ and $u_0 = 0$ follow. But this implies $1 = T(\mathbf{a})u_0 = 0$. ∎

Assuming $z^n \mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ for any $n \in \mathbb{Z}$, we define

$$f_n = \mathbf{a} T(z^n \mathbf{a})^{-1} 1 \in W_{\mathbf{a}}.$$

Then, one has the following result.

Lemma 4. $1 + \varphi^{(0)}_{z^n \mathbf{a}}(0) \neq 0$ *is satisfied for any* $n \in \mathbb{Z}$ *and* $\{f_n\}_{n\in\mathbb{Z}}$ *satisfies a recurrence relation*

$$\frac{1 + \varphi^{(0)}_{z^n \mathbf{a}}(0)}{1 + \varphi^{(0)}_{z^{n+1} \mathbf{a}}(0)} f_{n+1} + (\varphi^{(-1)}_{\widetilde{z^n \mathbf{a}}}(0) - \varphi^{(-1)}_{\widetilde{z^{n-1} \mathbf{a}}}(0)) f_n + f_{n-1} = (z + z^{-1}) f_n.$$

Proof. Observe

$$z^n f_n = z^n \mathbf{a} T(z^n \mathbf{a})^{-1} 1 = 1 + \varphi^{(0)}_{z^n \mathbf{a}}.$$

Moreover, $z^n f_{n+1}, z^n f_{n-1} \in W_{z^n \mathbf{a}}$ have decompositions into $H_\pm$:

$$\begin{aligned} z^n f_{n+1} &= z^{-1}(1 + \varphi^{(0)}_{z^{n+1}\mathbf{a}}(0)) + z^{-1}(\varphi^{(0)}_{z^{n+1}\mathbf{a}} - \varphi^{(0)}_{z^{n+1}\mathbf{a}}(0)) \\ z^n f_{n-1} &= (z + \lim_{z\to\infty} z\varphi^{(0)}_{z^{n-1}\mathbf{a}}(z)) + (z\varphi^{(0)}_{z^{n-1}\mathbf{a}} - \lim_{z\to\infty} z\varphi^{(0)}_{z^{n-1}\mathbf{a}}(z)) \\ &= (z + \varphi^{(-1)}_{\widetilde{z^{n-1}\mathbf{a}}}(0)) + (z\varphi^{(0)}_{z^{n-1}\mathbf{a}} - \varphi^{(-1)}_{\widetilde{z^{n-1}\mathbf{a}}}(0)). \end{aligned}$$

Therefore, we have the identities

$$\begin{cases} z^n f_n = 1 + \varphi^{(0)}_{z^n\mathbf{a}}, \\ z^n f_{n+1} = (1 + \varphi^{(0)}_{z^{n+1}\mathbf{a}}(0))(z^{-1} + \varphi^{(-1)}_{z^n\mathbf{a}}), \\ z^n f_{n-1} = z + \varphi^{(1)}_{z^n\mathbf{a}} + \varphi^{(-1)}_{\widetilde{z^{n-1}\mathbf{a}}}(0)(1 + \varphi^{(0)}_{z^n\mathbf{a}}). \end{cases} \tag{2.1}$$

If $1 + \varphi^{(0)}_{z^{n+1}\mathbf{a}}(0) = 0$ for some $n \in \mathbb{Z}$, then the second identity implies

$$1 + \varphi^{(0)}_{z^{n+1}\mathbf{a}} = f_{n+1} = 0,$$

which shows $\varphi^{(0)}_{z^{n+1}\mathbf{a}} = -1$. This is impossible since $\varphi^{(0)}_{z^{n+1}\mathbf{a}} \in H_-$ and $-1 \in H_+$, hence $1 + \varphi^{(0)}_{z^{n+1}\mathbf{a}}(0) \neq 0$ holds for any $n \in \mathbb{Z}$. Applying Lemma 2 (ii) for $n = 0$ and $z^n \mathbf{a}$, one has

$$z + \varphi^{(1)}_{z^n\mathbf{a}} = (z + z^{-1} - \varphi^{(-1)}_{\widetilde{z^n\mathbf{a}}}(0))(1 + \varphi^{(0)}_{z^n\mathbf{a}}) - (1 + \varphi^{(0)}_{z^n\mathbf{a}}(0))(z^{-1} + \varphi^{(-1)}_{z^n\mathbf{a}}).$$

This, combined with (2.1), completes the proof. ∎

3. A subclass $\mathbf{M}(C)$ of $\mathbf{A}^{\text{inv}}(C)$

The invertibility of $T(\mathbf{a})$ is crucial since it is equivalent to the non-existence of singularities of the flow. In this section we introduce a subclass $\mathbf{M}(C)$ of $\mathbf{A}^{\text{inv}}(C)$.

Let $\mathbf{M}(C)$ be the set of all symbols $\mathbf{m} = (m_1, m_2)$ satisfying

$$m_j \text{ is analytic in a neighbourhood of } \bar{D}_-, \text{ and } m_j = \bar{m}_j \text{ for } j = 1, 2, \tag{3.1a}$$

$$m_1(0) = \tilde{m}_1(0) = 1, \ m_2(0) = \tilde{m}_2(0) = 0, \tag{3.1b}$$

$$m_1(z)\tilde{m}_1(z) - m_2(z)\tilde{m}_2(z) \neq 0 \text{ on } \bar{D}_-, \tag{3.1c}$$

where $\tilde{f}(z) = f(z^{-1})$. For $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2) \in \mathbf{M}(C)$, define

$$\mathbf{m} \cdot \mathbf{n} = (m_1 n_1 + m_2 \tilde{n}_2, m_1 n_2 + m_2 \tilde{n}_1).$$

Since we have

$$\begin{aligned}(\mathbf{m}\cdot\mathbf{n})_1(z)(\widetilde{\mathbf{m}\cdot\mathbf{n}})_1(z) - (\mathbf{m}\cdot\mathbf{n})_2(z)(\widetilde{\mathbf{m}\cdot\mathbf{n}})_2(z)\\ = (m_1(z)\tilde{m}_1 - m_2\tilde{m}_2)(n_1\tilde{n}_1 - n_2\tilde{n}_2) \neq 0,\end{aligned}$$

clearly $\mathbf{m} \cdot \mathbf{n} \in \mathbf{M}(C)$ is satisfied. One has the following result.

Lemma 5. $\mathbf{M}(C)$ *is a group by the operation* $\mathbf{m} \cdot \mathbf{n}$*, whose identity* $\mathbf{1}$ *is* $(1, 0)$ *and inverse is*

$$\mathbf{m}^{-1} = \Big(\frac{\tilde{m}_1}{m_1\tilde{m}_1 - m_2\tilde{m}_2}, \frac{-m_2}{m_1\tilde{m}_1 - m_2\tilde{m}_2}\Big).$$

For $\mathbf{m}$*,* $\mathbf{n} \in \mathbf{M}(C)$*, it holds that*

$$T(\mathbf{m}\cdot\mathbf{n}) = T(\mathbf{m})T(\mathbf{n}), \tag{3.2}$$

and hence $\mathbf{M}(C) \subset \mathbf{A}^{\text{inv}}(C)$.

Proof. The associative law and the form of the inverse are clear. Identity (3.2) is verified as follows. Let $\mathbf{m} = (m_1, m_2)$ and $\mathbf{n} = (n_1, n_2) \in \mathbf{M}(C)$. Then, $n_1 H_-$ and $n_2 H_- \subset H_-$ are satisfied, hence, for $u \in H_+$,

$$\mathfrak{p}_+(m_i n_j u) = \mathfrak{p}_+(m_i \mathfrak{p}_+ n_j u) + \mathfrak{p}_+(m_i \mathfrak{p}_- n_j u) = \mathfrak{p}_+(m_i \mathfrak{p}_+ n_j u)$$

holds for i, $j = 1, 2$, which, together with $\mathfrak{p}_+, R = R\mathfrak{p}_+$ implies

$$\begin{aligned}T(\mathbf{m})T(\mathbf{n})u &= \mathfrak{p}_+(m_1\mathfrak{p}_+(n_1 u + n_2 Ru) + m_2 R\mathfrak{p}_+(n_1 u + n_2 Ru))\\ &= \mathfrak{p}_+(m_1(n_1 u + n_2 Ru)) + \mathfrak{p}_+(m_2 R(n_1 u + n_2 Ru))\\ &= \mathfrak{p}_+((m_1 n_1 + m_2\tilde{n}_2)u + (m_1 n_2 + m_2\tilde{n}_1)Ru) = T(\mathbf{m}\cdot\mathbf{n})u,\end{aligned}$$

which shows (3.2). ∎

$\{\varphi_{\mathbf{m}}^{(-1)}, \varphi_{\mathbf{m}}^{(0)}\}$ is computable for $\mathbf{m} \in \mathbf{M}(C)$.

Lemma 6. *For* $\mathbf{m} = (m_1, m_2) \in \mathbf{M}(C)$*, it holds that*

$$z^{-1} + \varphi_{\mathbf{m}}^{(-1)} = z^{-1}m_1 + m_2, \quad 1 + \varphi_{\mathbf{m}}^{(0)} = m_1 + m_2 z^{-1}.$$

Proof. Generally, for $\mathbf{m} = (m_1, m_2) \in \mathbf{M}(C)$, one has

$$\begin{cases} T(\mathbf{m})1 = \mathfrak{p}_+(m_1 + m_2 z^{-1}) = 1 + m_2(0)z^{-1} = 1, \\ T(\mathbf{m})z^{-1} = \mathfrak{p}_+(m_1 z^{-1} + m_2) = m_1(0)z^{-1} = z^{-1}, \end{cases}$$

which implies

$$T(\mathbf{m})^{-1}1 = 1, \quad T(\mathbf{m})^{-1}z^{-1} = z^{-1}.$$

Multiplying $\mathbf{m}$ on both sides gives the conclusion. ∎

4. Tau functions

The tau function is a key notion in Sato theory. In this article this quantity is used to examine $g\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ for $g \in \Gamma$ and to express the Toda flow.

Recall that the domain D_+ is chosen suitably so that D_+ contains neither poles nor zeros of $g \in \Gamma$. We have a new symbol $g\mathbf{a}$ for $g \in \Gamma$, $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$. However, we do not know if $g\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ holds or not; namely, the invertibility of $T(g\mathbf{a})$ does not always hold, which is certified by a Fredholm determinant $\det(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1})$ if it is well defined. For this purpose, we define

$$\begin{cases} S_{\mathbf{a}}u = \mathfrak{p}_-(\mathbf{a}u) & \text{for } u \in H_+, \\ H_g v = \mathfrak{p}_+(gv)F & \text{for } v \in H_-. \end{cases}$$

Note that for $v \in H_-$ and $z \in D_+$,

$$H_g v(z) = \frac{1}{2\pi i}\int_C \frac{g(\lambda)v(\lambda)}{\lambda - z}d\lambda = \frac{1}{2\pi i}\int_C \frac{(g(\lambda) - g(z))v(\lambda)}{\lambda - z}d\lambda, \tag{4.1}$$

which means H_g has a smooth kernel.

Lemma 7. *For any bounded symbol and* $g \in \Gamma$*, one has*

$$T(g\mathbf{a}) = gT(\mathbf{a}) + H_g S_{\mathbf{a}},$$

and $H_g S_{\mathbf{a}}$ *is of trace class.*

Proof. Observe $gu \in H_+$ for $u \in H_+$. Then, for $u \in H_+$, one has

$$\begin{aligned} T(g\mathbf{a})u &= \mathfrak{p}_+(g\mathbf{a}u) = \mathfrak{p}_+(g\mathfrak{p}_+\mathbf{a}u) + \mathfrak{p}_+(g\mathfrak{p}_-\mathbf{a}u) \\ &= g\mathfrak{p}_+\mathbf{a}u + \mathfrak{p}_+(g\mathfrak{p}_-\mathbf{a}u) = gT(\mathbf{a})u + H_g S_{\mathbf{a}} u, \end{aligned}$$

which shows the identity. Let Δ be the Laplacian C. In the identity

$$H_g = (I - \Delta)^{-1}(I - \Delta)H_g,$$

$(I - \Delta)^{-1}$ is a trace class operator on $L^2(C)$ and $(I - \Delta)H_g$ is a bounded operator from H_- to H_+ due to (4.1), the operator H_g turns to be of trace class. Since $S_{\mathbf{a}}$ is a bounded operator from H_+ to H_-, we have the trace class property of $H_g S_{\mathbf{a}}$. ■

Lemma 7 implies that for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$,

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} = I + g^{-1}H_g S_{\mathbf{a}} T(\mathbf{a})^{-1},$$

and the second term is of trace class. Then, the tau function can be defined by

$$\tau_{\mathbf{a}}(g) = \det(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}).$$

Tau functions satisfy several properties.

Lemma 8. *For $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ and $g, g_1, g_2 \in \Gamma$, it holds that*

(i) $g\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ *holds if and only if* $\tau_{\mathbf{a}}(g) \neq 0$;

(ii) $\tau_{\mathbf{a}}(g) = \tau_{\tilde{\mathbf{a}}}(\tilde{g})$ *(note that $\tilde{\mathbf{a}} \in \mathbf{A}^{\mathrm{inv}}(C)$, due to Lemma 1);*

(iii) *if* $\tau_{\mathbf{a}}(g_1) \neq 0$, *then*

$$\tau_{\mathbf{a}}(g_1 g_2) = \tau_{\mathbf{a}}(g_1)\tau_{g_1\mathbf{a}}(g_2) \quad \textit{(cocycle property);}$$

(iv) *if g_1 satisfies $g_1(z) = g_1(z^{-1})$, then one has $\tau_{\mathbf{a}}(g_1) \neq 0$ and*

$$\tau_{\mathbf{a}}(g_1 g_2) = \tau_{\mathbf{a}}(g_1)\tau_{\mathbf{a}}(g_2).$$

Proof. (i) follows from the property of Fredholm determinant (see [6]). Lemma 1 immediately implies (ii). To show (iii), assume $\tau_{\mathbf{a}}(g_1) \neq 0$. Then, (i) implies that $T(g_1\mathbf{a})$ is invertible. Then, the properties of determinant show

$$\begin{aligned} \tau_{\mathbf{a}}(g_1 g_2) &= \det(g_2^{-1}g_1^{-1}T(g_1g_2\mathbf{a})T(\mathbf{a})^{-1}) \\ &= \det(g_1^{-1}g_2^{-1}T(g_2g_1\mathbf{a})T(g_1\mathbf{a})^{-1}g_1(g_1^{-1}T(g_1\mathbf{a})T(\mathbf{a})^{-1})) \\ &= \det(g_2^{-1}T(g_2g_1\mathbf{a})T(g_1\mathbf{a})^{-1})\tau_{\mathbf{a}}(g_1) = \tau_{\mathbf{a}}(g_1)\tau_{g_1\mathbf{a}}(g_2), \end{aligned}$$

which is (iii). Suppose that $g_1(z) = g_1(z^{-1})$ holds. Then, the identity $g_1\mathbf{a}u = \mathbf{a}g_1u$ implies

$$T(g_1\mathbf{a}) = T(\mathbf{a})g_1, \quad T(g_2g_1\mathbf{a}) = T(g_2\mathbf{a})g_1,$$

which shows $g_1\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ and

$$\tau_{g_1\mathbf{a}}(g_2) = \det(g_2^{-1}T(g_2g_1\mathbf{a})T(g_1\mathbf{a})^{-1}) = \det(g_2^{-1}T(g_2\mathbf{a})T(\mathbf{a})^{-1}) = \tau_{\mathbf{a}}(g_2).$$

This, together with (iii), implies (iv). ∎

$\tau_{\mathbf{a}}(r)$ for rational functions $r \in \Gamma$ can be expressed by $\{\varphi_{\mathbf{a}}^{(0)}, \varphi_{\mathbf{a}}^{(-1)}\}$. Here, we compute $\tau_{\mathbf{a}}(r)$ in two simple cases. For $\zeta \in D_-$ set

$$q_\zeta(z) = (1 - \zeta^{-1}z)^{-1} \in \Gamma.$$

Then, for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ one has the following result.

Lemma 9. *The following statements hold:*

(i) $$\tau_{\mathbf{a}}(q_\zeta) = 1 + \varphi_{\mathbf{a}}^{(0)}(\zeta)$$

and

(ii) $$\tau_{\mathbf{a}}(q_{\zeta_1}q_{\zeta_2}) = \frac{(\zeta_1 + \varphi_{\mathbf{a}}^{(1)}(\zeta_1))(1 + \varphi_{\mathbf{a}}^{(0)}(\zeta_2)) - (1 + \varphi_{\mathbf{a}}^{(0)}(\zeta_1))(\zeta_2 + \varphi_{\mathbf{a}}^{(1)}(\zeta_2))}{\zeta_1 - \zeta_2}.$$

Proof. Recall the identity

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I = g^{-1}H_g\mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}$$

of Lemma 7. For $g = q_\zeta$ and $v \in H_-$, one has

$$\begin{aligned} q_\zeta^{-1}(H_{q_\zeta}v)(z) &= q_\zeta(z)^{-1}\frac{1}{2\pi i}\int_C \frac{\zeta}{(\lambda - z)(\zeta - \lambda)}v(\lambda)d\lambda \\ &= q_\zeta(z)^{-1}\frac{\zeta}{\zeta - z}v(\zeta) = v(\zeta), \end{aligned}$$

hence

$$q_\zeta^{-1}T(q_\zeta\mathbf{a})T(\mathbf{a})^{-1}u - u = q_\zeta^{-1}(H_{q_\zeta}\mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}u) = (\mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}u)(\zeta)$$

holds for any $u \in H_+$, which shows $q_\zeta^{-1}T(q_\zeta\mathbf{a})T(\mathbf{a})^{-1} - I$ is a rank 1 operator with image constant times 1. Since $\mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}1 = \varphi_{\mathbf{a}}^{(0)}$, one gets (i).

For $g = q_{\zeta_1}q_{\zeta_2}$, note that

$$(q_{\zeta_1}q_{\zeta_2})(z) = q_{\zeta_2}(\zeta_1)q_{\zeta_1}(z) + q_{\zeta_1}(\zeta_2)q_{\zeta_2}(z).$$

Then, one has

$$H_{q_{\zeta_1}q_{\zeta_2}}v = q_{\zeta_2}(\zeta_1)H_{q_{\zeta_1}}v + q_{\zeta_1}(\zeta_2)H_{q_{\zeta_2}}v = q_{\zeta_2}(\zeta_1)v(\zeta_1)q_{\zeta_1} + q_{\zeta_1}(\zeta_2)v(\zeta_2)q_{\zeta_2},$$

and $(q_{\zeta_1}q_{\zeta_2})^{-1}T(q_{\zeta_1}q_{\zeta_2}\mathbf{a})T(\mathbf{a})^{-1} - I$ is a rank 2 operator with image spanned by $\{q_{\zeta_1}^{-1}, q_{\zeta_2}^{-1}\}$. Since

$$\begin{cases} v_1 \equiv \mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}q_{\zeta_1}^{-1} = \varphi_{\mathbf{a}}^{(0)} - \zeta_1^{-1}\varphi_{\mathbf{a}}^{(1)}, \\ v_2 \equiv \mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}q_{\zeta_2}^{-1} = \varphi_{\mathbf{a}}^{(0)} - \zeta_2^{-1}\varphi_{\mathbf{a}}^{(1)}, \end{cases}$$

setting $a_j = 1 + \varphi_{\mathbf{a}}^{(0)}(\zeta_j)$, $b_j = \zeta_j + \varphi_{\mathbf{a}}^{(1)}(\zeta_j)$ for $j = 1, 2$, one has

$$\begin{aligned} \tau_{\mathbf{a}}(q_{\zeta_1}q_{\zeta_2}) &= \det\begin{pmatrix} 1 + q_{\zeta_1}(\zeta_2)v_1(\zeta_2) & q_{\zeta_1}(\zeta_2)v_2(\zeta_2) \\ q_{\zeta_2}(\zeta_1)v_1(\zeta_1) & 1 + q_{\zeta_2}(\zeta_1)v_2(\zeta_1) \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\zeta_1 a_2 - b_2}{\zeta_1 - \zeta_2} & \dfrac{\zeta_1(a_2 - \zeta_2^{-1}b_2)}{\zeta_1 - \zeta_2} \\ \dfrac{\zeta_2(a_1 - \zeta_1^{-1}b_1)}{\zeta_2 - \zeta_1} & \dfrac{\zeta_2 a_1 - b_1}{\zeta_2 - \zeta_1} \end{pmatrix} \\ &= \frac{a_2 b_1 - a_1 b_2}{\zeta_1 - \zeta_2}, \end{aligned}$$

which is (ii). ■

Later, we will need the continuity of $\tau_{\mathbf{a}}(g)$ with respect to $g \in \Gamma$ since we approximate g by rational functions.

Lemma 10. *Assume g_n, $g \in \Gamma$ are analytic and have no zeros on a fixed neighbourhood U of $\overline{D}_+$. If g_n converges to g uniformly on U, then $\tau_{\mathbf{a}}(g_n)$ converges to $\tau_{\mathbf{a}}(g)$ for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$.*

Proof. Suppose g_1, $g_2 \in \Gamma$ are analytic and have no zeros on U. Then trace class operators $A_j = g_j^{-1}H_{g_j}S_{\mathbf{a}}T(\mathbf{a})^{-1}$ $(j = 1, 2)$ satisfy (see [6])

$$|\det(I + A_1) - \det(I + A_2)| \le \|A_1 - A_2\|_1 \exp(1 + \|A_1\|_1 + \|A_2\|_1)$$

with trace norm $\|\cdot\|_1$. Observe that

$$\begin{aligned} \|A_1 - A_2\|_1 &\le \|g_1^{-1}H_{g_1} - g_2^{-1}H_{g_2}\|_1\|S_{\mathbf{a}}T(\mathbf{a})^{-1}\| \\ &\le c_1(\|g_1^{-1} - g_2^{-1}\|\|H_{g_1}\|_1 + \|g_2^{-1}\|\|H_{g_1} - H_{g_2}\|_1) \end{aligned}$$

holds with the operator norm $\|\cdot\|$ and $c_1 = \|S_{\mathbf{a}}T(\mathbf{a})^{-1}\|$. Since we have the estimates

$$\begin{cases} \|g_1^{-1} - g_2^{-1}\| \le \|g_1^{-1}g_2^{-1}\|_U\|g_1 - g_2\|_U, \\ \|H_{g_1} - H_{g_2}\|_1 \le \|(I - \Delta)^{-1}\|_1\|(I - \Delta)(H_{g_1} - H_{g_2})\| \le c_2\|g_1 - g_2\|_U, \\ \|H_{g_1}\|_1 \le \|(I - \Delta)^{-1}\|_1\|(I - \Delta)H_{g_1}\| \le c_2\|g_1\|_U. \end{cases}$$

with $\|g\|_U = \sup_{z\in U}|g(z)|$ and some constant c_2 depending only on U, the convergence of $\tau_{\mathbf{a}}(g_n) \to \tau_{\mathbf{a}}(g)$ is clear due to $\tau_{\mathbf{a}}(g_n) = \det(I + g_n^{-1}H_{g_n}S_{\mathbf{a}}T(\mathbf{a})^{-1})$. ■

5. Derivation of Toda hierarchy

In this section we show that the symbols $\mathbf{A}^{\mathrm{inv}}(C)$ and the group Γ generate Jacobi operators and Toda lattice under some conditions on symbols.

In view of Lemma 4, assuming $z^n\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ for $n \in \mathbb{Z}$, we define

$$a_n = \frac{\sqrt{1+\varphi^{(0)}_{z^{n-1}\mathbf{a}}(0)}}{\sqrt{1+\varphi^{(0)}_{z^n\mathbf{a}}(0)}}, \tag{5.1a}$$

$$b_n = \varphi^{(-1)}_{\widetilde{z^n\mathbf{a}}}(0) - \varphi^{(-1)}_{\widetilde{z^{n-1}\mathbf{a}}}(0), \tag{5.1b}$$

$$g_n = \frac{\mathbf{a}T(z^n\mathbf{a})^{-1}1}{\sqrt{1+\varphi^{(0)}_{z^n\mathbf{a}}(0)}}, \tag{5.1c}$$

where $\sqrt{\cdot}$ is taken arbitrary.

Lemma 11. *The coefficients in (5.1) can be expressed by tau functions as*

$$a_n^2 = \frac{\tau_{\mathbf{a}}(z^{n-2})\tau_{\mathbf{a}}(z^n)}{\tau_{\mathbf{a}}(z^{n-1})^2}, \quad b_n = \partial_\varepsilon \log \frac{\tau_{\mathbf{a}}(z^n q_{\varepsilon^{-1}})}{\tau_{\mathbf{a}}(z^{n-1}q_{\varepsilon^{-1}})}\bigg|_{\varepsilon=0}, \tag{5.2}$$

and g_n satisfies

$$a_{n+1}g_{n+1} + a_n g_{n-1} + b_n g_n = (z+z^{-1})g_n. \tag{5.3}$$

Proof. Equation (5.3) follows from Lemma 4 without difficulty. Lemma 9 (i) implies (setting $\zeta = 0$)

$$a_n^2 = \frac{1+\varphi^{(0)}_{z^{n-1}\mathbf{a}}(0)}{1+\varphi^{(0)}_{z^n\mathbf{a}}(0)} = \frac{\tau_{z^{n-1}\mathbf{a}}(z^{-1})}{\tau_{z^n\mathbf{a}}(z^{-1})} = \frac{\tau_{\mathbf{a}}(z^{n-2})\tau_{\mathbf{a}}(z^n)}{\tau_{\mathbf{a}}(z^{n-1})^2}.$$

In the last equality, Lemma 8 (iii) was used.

(i) of Lemma 2 shows

$$\begin{aligned}\varphi^{(-1)}_{\widetilde{z^n\mathbf{a}}}(0) &= \lim_{\varepsilon\to 0}\varphi^{(-1)}_{\widetilde{z^n\mathbf{a}}}(\varepsilon) = \lim_{\varepsilon\to 0}\varepsilon^{-1}\varphi^{(0)}_{z^n\mathbf{a}}(\varepsilon^{-1}) = \lim_{\varepsilon\to 0}\frac{\tau_{z^n\mathbf{a}}(q_{\varepsilon^{-1}})-1}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\tau_{\mathbf{a}}(z^n q_{\varepsilon^{-1}}) - \tau_{\mathbf{a}}(z^n)}{\varepsilon\tau_{\mathbf{a}}(z^n)} = \partial_\varepsilon \log \tau_{\mathbf{a}}(z^n q_{\varepsilon^{-1}})|_{\varepsilon=0},\end{aligned}$$

which completes the proof. ∎

The last derivative can be replaced by $\partial_\varepsilon \log \tau_{\mathbf{a}}(z^n e^{\varepsilon z})|_{\varepsilon=0}$, since $q_{\varepsilon^{-1}}(z) - e^{\varepsilon z} = O(\varepsilon^2)$. In this way, one can derive a Jacobi equation (5.3) from $z^n \in \Gamma$ and $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ under the condition $z^n\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ for any $n \in \mathbb{Z}$. Usually, Jacobi equations are considered for real coefficients a_n, b_n, and later we restrict ourselves only in the real case.

For a polynomial p, let $\hat{p}$ be the polynomial part of $p(z+z^{-1})$. Assume $z^n e^{t\hat{p}}\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Define a second order difference operator $L_{\mathbf{a}}$ by

$$(L_{\mathbf{a}} v)_n = a_{n+1}^2 v_{n+1} + b_n v_n + v_{n-1}$$

with a_n, b_n in (5.2), and set

$$f_n(t,z) = \mathbf{a}T(z^n e^{t\hat{p}}\mathbf{a})^{-1}1.$$

Then, Lemma 4 implies

$$(L_{e^{t\hat{p}}\mathbf{a}} f) = (z+z^{-1})f,$$

hence

$$p(L_{e^{t\hat{p}}\mathbf{a}})f = p(z+z^{-1})f.$$

We decompose $p(L_{\mathbf{a}})$ into 2 parts $p(L_{\mathbf{a}})_\pm$: Define coefficients α_{in} by

$$(p(L_{\mathbf{a}})v)_n = \sum_{-k\le i\le k} \alpha_{in} v_{i+n} \quad \text{with } k = \text{degree of } p,$$

and set

$$(p(L_{\mathbf{a}})_+ v)_n = \sum_{1\le i\le k} \alpha_{in} v_{i+n}, \quad (p(L_{\mathbf{a}})_- v)_n = \sum_{-k\le i\le 0} \alpha_{in} v_{i+n}.$$

Lemma 12. *$L_{e^{t\hat{p}}\mathbf{a}}$ satisfies an operator equation*

$$\partial_t L_{e^{t\hat{p}}\mathbf{a}} = [L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-]. \tag{5.4}$$

Proof. Since H_+ is generated by $\{z^n\}_{n\in\mathbb{Z}}$, one has $H_+ = H_+^1 \oplus H_+^2$ (direct sum) with

$$H_+^1 = \text{linear span of } \{z^n\}_{n\ge 0}, \ \ H_+^2 = \text{linear span of } \{z^n\}_{n<0}.$$

Observe that, for any integer $k \ge 0$, $f \in H_-$ and $z \in D_+$,

$$(\mathfrak{p}_+(z^k f))(z) = \frac{1}{2\pi i}\int_C \frac{\lambda^k f(\lambda)}{\lambda - z} d\lambda = \frac{1}{2\pi i}\int_{C_1} \frac{\lambda^k f(\lambda)}{\lambda - z} d\lambda \in H_+^1,$$

holds, since $\lambda^k f(\lambda)(\lambda - z)^{-1}$ is analytic in D_-^2 (inside of C_2). Then, we see that $\mathfrak{p}_+(z^k f) \in H_+^1$. Since

$$z^n e^{t\hat{p}} f_n(t,z) = 1 + \varphi^{(0)}_{z^n e^{t\hat{p}}\mathbf{a}}(z) \quad \text{with } \varphi^{(0)}_{z^n e^{t\hat{p}}\mathbf{a}} \in H_-,$$

one has

$$\mathfrak{p}_+(\hat{p} z^n e^{t\hat{p}} f_n(t,z)) \in H_+^1.$$

Similarly, $\mathfrak{p}_+((p(z+z^{-1}) - \hat{p}) z^n e^{t\hat{p}} f_n(t,z)) \in H_+^2$ holds. On the other hand, if $k \leq n$, then

$$z^n e^{t\hat{p}} f_k(t,z) = z^{n-k} z^k e^{t\hat{p}} f_k(t,z) = z^{n-k}(1 + \varphi^{(0)}_{z^k e^{t\hat{p}}\mathbf{a}}(z)),$$

hence

$$\mathfrak{p}_+ z^n e^{t\hat{p}} (p(L_{e^{t\hat{p}}\mathbf{a}})_- f)_n(t,z) \in H_+^1,$$

and similarly, $\mathfrak{p}_+ z^n e^{t\hat{p}} (p(L_{e^{t\hat{p}}\mathbf{a}})_+ f)_n(t,z) \in H_+^2$ hold. Since

$$p(L_{e^{t\hat{p}}\mathbf{a}}) f = \hat{p} f + (p(z+z^{-1}) - \hat{p}) f = p(L_{e^{t\hat{p}}\mathbf{a}})_- f + p(L_{e^{t\hat{p}}\mathbf{a}})_+ f,$$

one sees

$$\mathfrak{p}_+ z^n \hat{p} e^{t\hat{p}} f_n = \mathfrak{p}_+ z^n e^{t\hat{p}} (p(L_{e^{t\hat{p}}\mathbf{a}})_- f)_n.$$

Noting that $\mathfrak{p}_+ z^n e^{t\hat{p}} f_n(t,z) = 1$ implies

$$0 = \partial_t \mathfrak{p}_+ z^n e^{t\hat{p}} f_n(t,z) = \mathfrak{p}_+ z^n \hat{p} e^{t\hat{p}} f_n(t,z) + \mathfrak{p}_+ z^n e^{t\hat{p}} \partial_t f_n(t,z),$$

one has

$$\mathfrak{p}_+ z^n e^{t\hat{p}} (\partial_t f_n(t,z) + p(L_{e^{t\hat{p}}\mathbf{a}})_- f_n(t,z)) = 0.$$

Since $z^n e^{t\hat{p}} (\partial_t f_n(t,z) + p(L_{e^{t\hat{p}}\mathbf{a}})_- f_n(t,z)) \in W_{z^n e^{t\hat{p}}\mathbf{a}}$ and $\mathfrak{p}_+ \colon W_{z^n e^{t\hat{p}}\mathbf{a}} \to H_+$ is one-to-one, this implies

$$\partial_t f_n(t,z) = -p(L_{e^{t\hat{p}}\mathbf{a}})_- f_n(t,z).$$

Since

$$\partial_t (L_{e^{t\hat{p}}\mathbf{a}} f) = (\partial_t L_{e^{t\hat{p}}\mathbf{a}}) f + L_{e^{t\hat{p}}\mathbf{a}} \partial_t f$$

and $(L_{e^{t\hat{p}}\mathbf{a}} f) = (z + z^{-1}) f$ hold, one has

$$\begin{aligned}(\partial_t L_{e^{t\hat{p}}\mathbf{a}}) f &= (z+z^{-1}) \partial_t f - L_{e^{t\hat{p}}\mathbf{a}} \partial_t f \\ &= -p(L_{e^{t\hat{p}}\mathbf{a}})_- (z+z^{-1}) f + L_{e^{t\hat{p}}\mathbf{a}} p(L_{e^{t\hat{p}}\mathbf{a}})_- f \\ &= -p(L_{e^{t\hat{p}}\mathbf{a}})_- L_{e^{t\hat{p}}\mathbf{a}} f + L_{e^{t\hat{p}}\mathbf{a}} p(L_{e^{t\hat{p}}\mathbf{a}})_- f \\ &= [L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-] f.\end{aligned}$$

If we denote the coefficient of the deference operator $\partial_t L_{e^{t\hat{p}}\mathbf{a}} - [L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-]$ by α_{jk}, then we have

$$\sum_k \alpha_{jk} f_k(t,z) = 0 \quad \text{in } W_{\mathbf{a}} \text{ for each fixed } t,$$

where the summation is finite for each j. Since Lemma 13 below implies that $\{f_k(t,\cdot)\}_k$ is linearly independent in $W_{\mathbf{a}}$, one has $\alpha_{jk} = 0$, which yields (5.4). ∎

Lemma 13. *For a symbol* $\mathbf{a}$*, assume* $z^n\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ *for any* $n \in \mathbb{Z}$ *and set*

$$u_n = T(z^n\mathbf{a})^{-1}1, \quad u^{(n)} = T(\mathbf{a})^{-1}z^n \in H_+ \quad \text{for } n \in \mathbb{Z}.$$

Then, there exist constants $\{c^+_{kj}\}_{1\le j\le k}$, $\{c^-_{kj}\}_{-k\le j\le 0}$ *such that*

$$u_k = \begin{cases} \sum_{1\le j\le k} c^+_{kj} u^{(-j)} & \text{if } k \ge 1, \\ \sum_{k\le j\le 0} c^-_{kj} u^{(j)} & \text{if } k \le 0, \end{cases} \quad \text{with } c^+_{kk} \ne 0, c^-_{kk} = 1, \tag{5.5}$$

which implies $\{u_n\}_{n\in\mathbb{Z}}$ *and* $\{f_n = \mathbf{a}u_n\}_{n\in\mathbb{Z}}$ *are linearly independent.*

Proof. For $k \ge 1$, observe

$$\begin{aligned} 1 = T(z^k\mathbf{a})u_k &= \frac{1}{2\pi i}\int_C \frac{\lambda^k - z^k}{\lambda - z}\mathbf{a}u_k(\lambda)d\lambda + z^k T(\mathbf{a})u_k \\ &= \sum_{0\le j\le k-1} z^{k-1-j}\frac{1}{2\pi i}\int_C \lambda^j \mathbf{a}u_k(\lambda)d\lambda + z^k T(\mathbf{a})u_k. \end{aligned}$$

Multiplying by $T(\mathbf{a})^{-1}z^{-k}$, one has

$$T(\mathbf{a})^{-1}z^{-k} = \sum_{0\le j\le k-1}\left(\frac{1}{2\pi i}\int_C \lambda^j \mathbf{a}u_k(\lambda)d\lambda\right)T(\mathbf{a})^{-1}z^{-1-j} + u_k,$$

hence

$$\begin{aligned} u_k &= \left(1 - \frac{1}{2\pi i}\int_C \lambda^{k-1}\mathbf{a}u_k(\lambda)d\lambda\right) - \sum_{1\le j\le k-1}\left(\frac{1}{2\pi i}\int_C \lambda^j \mathbf{a}u_k(\lambda)d\lambda\right)u^{(-j)} \\ &= \sum_{1\le j\le k} c^+_{kj}u^{(-j)} \quad \text{with } c^+_{kj} = \begin{cases} 1 - \frac{1}{2\pi i}\int_C \lambda^{k-1}\mathbf{a}u_k(\lambda)d\lambda & \text{if } j = k, \\ -\frac{1}{2\pi i}\int_C \lambda^{j-1}\mathbf{a}u_k(\lambda)d\lambda & \text{if } 1 \le j < k. \end{cases} \end{aligned}$$

Here,

$$\frac{1}{2\pi i}\int_C \lambda^{k-1}\mathbf{a}u_k(\lambda)d\lambda = \frac{1}{2\pi i}\int_C \lambda^{-1}(1 + \varphi^{(0)}_{z^k\mathbf{a}}(\lambda))d\lambda = -\varphi^{(0)}_{z^k\mathbf{a}}(0)$$

holds, hence

$$c^+_{kk} = 1 + \varphi^{(0)}_{z^k\mathbf{a}}(0) = \tau_{z^k\mathbf{a}}(z^{-1}) = \frac{\tau_{\mathbf{a}}(z^{k-1})}{\tau_{\mathbf{a}}(z^k)} \ne 0.$$

Similarly, for $k \le 0$,

$$1 = T(z^{-k}\mathbf{a})u_{-k} = \frac{1}{2\pi i}\int_C \frac{\lambda^{-k} - z^{-k}}{\lambda - z}\mathbf{a}u_{-k}(\lambda)d\lambda + z^{-k}T(\mathbf{a})u_{-k}$$
$$= -\sum_{0\le j\le k-1}\left(\frac{1}{2\pi i}\int_C \lambda^{j-k}\mathbf{a}u_{-k}(\lambda)d\lambda\right)z^{-1-j} + z^{-k}T(\mathbf{a})u_{-k}$$

holds, hence one has

$$u_{-k} = u^{(k)} + \sum_{0\le j\le k-1}\left(\frac{1}{2\pi i}\int_C \lambda^{j-k}\mathbf{a}u_{-k}(\lambda)d\lambda\right)u^{(k-j-1)},$$

which shows the second identity of (5.5) by setting

$$c^-_{kk} = 1 \quad \text{and} \quad c^-_{kj} = \frac{1}{2\pi i}\int_C \lambda^{-j-1}\mathbf{a}u_{-k}(\lambda)d\lambda \quad \text{for } k+1 \le j \le 0.$$

The linear independence of $\{u^{(k)}\}_{k\in\mathbb{Z}}$ follows from that of $\{z^k\}_{k\in\mathbb{Z}}$, hence $\{u_k\}_{k\in\mathbb{Z}}$ is also linearly independent, since the triangular matrices $(c^+_{kj})_{1\le j\le k}$ and $(c^-_{kj})_{k\le j\le 0}$ are invertible. ∎

Our Jacobi operator H_q is related with $L_{\mathbf{a}}$ by

$$H_q = \Lambda_{\mathbf{a}}^{-1}L_{\mathbf{a}}\Lambda_{\mathbf{a}} \quad \text{with a diagonal matrix } \Lambda_{\mathbf{a}} = (d_n^{1/2})_n,$$

where $d_n = 1 + \varphi^{(0)}_{z^n\mathbf{a}}(0)$. Hence, for $q = q(t)$ with a_n, b_n replaced by those of $e^{t\hat{p}}\mathbf{a}$ one has from (5.4),

$$\begin{aligned}\partial_t H_{q(t)} &= -(\partial_t\Lambda_{e^{t\hat{p}}\mathbf{a}})\Lambda^{-2}_{e^{t\hat{p}}\mathbf{a}}L_{e^{t\hat{p}}\mathbf{a}}\Lambda_{e^{t\hat{p}}\mathbf{a}} + \Lambda^{-1}_{e^{t\hat{p}}\mathbf{a}}(\partial_t L_{e^{t\hat{p}}\mathbf{a}})\Lambda_{e^{t\hat{p}}\mathbf{a}}\\ &\quad + \Lambda^{-1}_{e^{t\hat{p}}\mathbf{a}}L_{e^{t\hat{p}}\mathbf{a}}\partial_t\Lambda_{e^{t\hat{p}}\mathbf{a}}\\ &= \Lambda^{-1}_{e^{t\hat{p}}\mathbf{a}}[L_{e^{t\hat{p}}\mathbf{a}}, \Lambda^{-1}_{e^{t\hat{p}}\mathbf{a}}\partial_t\Lambda_{e^{t\hat{p}}\mathbf{a}} + p(L_{e^{t\hat{p}}\mathbf{a}})_-]\Lambda_{e^{t\hat{p}}\mathbf{a}}\\ &= [H_{q(t)}, \Lambda^{-1}_{e^{t\hat{p}}\mathbf{a}}\partial_t\Lambda_{e^{t\hat{p}}\mathbf{a}} + p(H_{q(t)})_-].\end{aligned}$$

Let λ_n be the diagonal element of $p(L_{e^{t\hat{p}}\mathbf{a}})$. Then, identity (5.4) implies that $[L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-]$ takes a form

$$([L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-]f)_n = p_n f_{n+1} + q_n f_n + r_n f_{n-1},$$

hence

$$\begin{aligned}([L_{e^{t\hat{p}}\mathbf{a}}, p(L_{e^{t\hat{p}}\mathbf{a}})_-]f)_n &= (L_{e^{t\hat{p}}\mathbf{a}}p(L_{e^{t\hat{p}}\mathbf{a}})_- f)_n - (p(L_{e^{t\hat{p}}\mathbf{a}})_- L_{e^{t\hat{p}}\mathbf{a}}f)_n\\ &= a^2_{n+1}(\lambda_{n+1} - \lambda_n)f_{n+1} + q_n f_n + r_n f_{n-1},\end{aligned}$$

which yields

$$\partial_t a_{n+1}^2 = a_{n+1}^2(\lambda_{n+1} - \lambda_n).$$

Since $(\Lambda_{e^{t\hat{p}}\mathbf{a}}^{-1} \partial_t \Lambda_{e^{t\hat{p}}\mathbf{a}})_n = (\partial_t \log d_n)/2$, it holds that for $n \geq 1$,

$$\begin{aligned}
\partial_t \log d_n &= \partial_t \log d_0 - \sum_{k=1}^{n} \partial_t \log a_k^2 \quad (\text{due to } a_k^2 = d_{k-1}/d_k) \\
&= \partial_t \log d_0 - \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \\
&= \partial_t \log d_0 + \lambda_0 - \lambda_n.
\end{aligned}$$

This identity holds also for $n \leq 0$, hence we have

$$\begin{aligned}
\partial_t H_{q(t)} &= \left[H_{q(t)}, p(H_{q(t)})_- - \frac{1}{2}\lambda_n\right] \\
&= \left[H_{q(t)}, p(H_{q(t)})_- - \frac{1}{2}p(H_{q(t)}) - \frac{1}{2}\lambda_n\right] \\
&= \left[H_{q(t)}, -\frac{1}{2}p(H_{q(t)})_a\right],
\end{aligned}$$

where X_a is defined in (1.3). Therefore, replacing p by $-2p$, we have the following result.

Proposition 3. *Let p be a polynomial and assume $z^n e^{t\hat{p}}\mathbf{a} \in \mathbf{A}^{\text{inv}}(C)$ for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Let $q(t)$ be the coefficients consisting of $\{a_n, b_n\}$ of (5.2) for the symbol $e^{-2t\hat{p}}\mathbf{a}$. Then, $H_{q(t)}$ satisfies an operator equation*

$$\partial_t H_{q(t)} = [H_{q(t)}, p(H_{q(t)})_a].$$

Especially, if $p(z) = z$, this provides a Toda lattice:

$$\begin{cases} \partial_t a_n = a_n(b_n - b_{n-1}), \\ \partial_t b_n = 2(a_{n+1}^2 - a_n^2). \end{cases} \tag{5.6}$$

Proof. We only have to verify (5.6). If $p(z) = z$, then

$$(p(H_{q(t)})_a f)_n = a_n f_{n-1} - a_{n+1} f_{n+1},$$

hence

$$\begin{aligned}
([H_{q(t)}, p(H_{q(t)})_a] f)_n = a_{n+1}(a_{n+1}u_n - a_{n+2}u_{n+2}) \\
+ b_n(a_n u_{n-1} - a_{n+1}u_{n+1}) \\
+ a_n(a_{n-1}u_{n-2} - a_n u_n)
\end{aligned}$$

$$
\begin{aligned}
&+ a_{n+1}(a_{n+2}u_{n+2} + b_{n+1}u_{n+1} + a_{n+1}u_n) \\
&- a_n(a_n u_n + b_{n-1}u_{n-1} + a_{n-1}u_{n-2}) \\
= {}& a_{n+1}(b_{n+1} - b_n)u_{n+1} + 2(a_{n+1}^2 - a_n^2)u_n \\
&+ a_n(b_n - b_{n-1})u_{n-1},
\end{aligned}
$$

which leads us to (5.6). ■

6. *m*-function and non-vanishing of tau functions

In the previous sections, we have shown that symbols $\mathbf{a}$ generate the Toda hierarchy under a certain non-degenerate condition on $g\mathbf{a}$. However, as we will see later, the correspondence between $\mathbf{a}$ and $q = \{a_n, b_n\}_{n\in\mathbb{Z}}$ is not one-to-one. Different symbols $\mathbf{a}$ may create the same q. An essential quantity giving one-to one correspondence is called an *m-function,* which is defined by

$$
m_{\mathbf{a}}(z) = \frac{z + \varphi_{\mathbf{a}}^{(1)}(z)}{1 + \varphi_{\mathbf{a}}^{(0)}(z)} + \lim_{\zeta\to\infty} \zeta\varphi_{\mathbf{a}}^{(0)}(\zeta) = \frac{z + \varphi_{\mathbf{a}}^{(1)}(z)}{1 + \varphi_{\mathbf{a}}^{(0)}(z)} + \varphi_{\tilde{\mathbf{a}}}^{(-1)}(0)
$$

for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$. The constant term is added so that $m_{\mathbf{a}}$ satisfies

$$
m_{\mathbf{a}}(z) = z + O(z^{-1}) \quad \text{as } z \to \infty.
$$

m-functions also play a key role to show the non-vanishing of tau functions if we assume some non-negativity condition on tau functions.

Since $\varphi_{\mathbf{a}}^{(0)}(z) = O(z^{-1})$ as $z \to \infty$, we see that $1 + \varphi_{\mathbf{a}}^{(0)}$ is not identically 0 on $D_-^1 (= D_- \cap \{|z| > 1\})$. However, there is a possibility that $1 + \varphi_{\mathbf{a}}^{(0)}$ is identically 0 on $D_-^2 (= D_- \cap \{|z| < 1\})$, which makes it impossible to define $m_{\mathbf{a}}$ on D_-^2. In this section we investigate conditions under which the properties $1 + \varphi_{\mathbf{a}}^{(0)}(z) \neq 0$ on D_-^2 and $\tau_{\mathbf{a}}(g) \neq 0$ hold.

For simplicity, of notations we introduce an auxiliary quantity

$$
n_{\mathbf{a}}(z) = \frac{z^{-1} + \varphi_{\mathbf{a}}^{(-1)}(z)}{1 + \varphi_{\mathbf{a}}^{(0)}(z)} = \frac{z + z^{-1} - m_{\mathbf{a}}(z)}{1 + \varphi_{\mathbf{a}}^{(0)}(0)} \quad \text{(Lemma 2)}, \tag{6.1}
$$

which has a tau function expression

$$
n_{\mathbf{a}}(\zeta) = \frac{1 + \varphi_{\tilde{\mathbf{a}}}^{(0)}(\zeta^{-1})}{\zeta(1 + \varphi_{\mathbf{a}}^{(0)}(\zeta))} = \frac{\tau_{\tilde{\mathbf{a}}}(q_{\zeta^{-1}})}{\zeta\tau_{\mathbf{a}}(q_\zeta)} = \frac{\tau_{\mathbf{a}}(zq_\zeta)}{\zeta\tau_{\mathbf{a}}(q_\zeta)} \quad \text{(Lemmas 2 and 9)}. \tag{6.2}
$$

Lemma 14. *For* $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$, $\zeta \in D_-$ *assume* $\tau_{\mathbf{a}}(q_\zeta)$, $\tau_{\mathbf{a}}(z^{-1}) \neq 0$. *Then, we have*

$$m_{q_\zeta \mathbf{a}}(z) = (m_{\mathbf{a}}(0) - m_{\mathbf{a}}(\zeta))\Big(1 - \frac{\phi(z) - \phi(\zeta)}{m_{\mathbf{a}}(z) - m_{\mathbf{a}}(\zeta)}\Big) + \phi(z) \quad \text{on } D_- \tag{6.3}$$

with $\phi(z) = z + z^{-1}$. *Especially, setting* $\zeta = 0$, *we have*

$$m_{z^{-1}\mathbf{a}}(z) = \frac{m'_{\mathbf{a}}(0)}{m_{\mathbf{a}}(0) - m_{\mathbf{a}}(z)} + \phi(z) \quad \text{on } D_-. \tag{6.4}$$

Proof. First we compute $n_{q_\zeta \mathbf{a}}(\eta)$. From Lemma 9, we have

$$1 + \varphi^{(0)}_{q_\zeta \mathbf{a}}(\eta) = \tau_{q_\zeta \mathbf{a}}(q_\eta) = \frac{\tau_{\mathbf{a}}(q_\zeta q_\eta)}{\tau_{\mathbf{a}}(q_\zeta)} = (1 + \varphi^{(0)}_{\mathbf{a}}(\eta))\frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta - \eta}. \tag{6.5}$$

We compute $\eta^{-1} + \varphi^{(-1)}_{q_\zeta \mathbf{a}}(\eta)$ without using tau functions. Set $c_1 = \lim_{z\to\infty} z\varphi^{(-1)}_{q_\zeta \mathbf{a}}(z)$. Since

$$q_\zeta^{-1}(\eta^{-1} + \varphi^{(-1)}_{q_\zeta \mathbf{a}}) = q_\zeta^{-1} q_\zeta \mathbf{a} T(q_\zeta \mathbf{a})^{-1} z^{-1} = \mathbf{a} T(q_\zeta \mathbf{a})^{-1} z^{-1} \in W_{\mathbf{a}}$$

and

$$\mathfrak{p}_+(q_\zeta^{-1}(\eta^{-1} + \varphi^{(-1)}_{q_\zeta \mathbf{a}})) = (1 - \zeta^{-1}\eta)\eta^{-1} - c_1 \zeta^{-1}$$

hold, we have

$$q_\zeta^{-1}(\eta^{-1} + \varphi^{(-1)}_{q_\zeta \mathbf{a}}) = \eta^{-1} + \varphi^{(-1)}_{\mathbf{a}} - (1 + c_1)\zeta^{-1}(1 + \varphi^{(0)}_{\mathbf{a}}). \tag{6.6}$$

Substituting $\eta = \zeta$ yields

$$(1 + c_1)\zeta^{-1} = \frac{\zeta^{-1} + \varphi^{(-1)}_{\mathbf{a}}(\zeta)}{1 + \varphi^{(0)}_{\mathbf{a}}(\zeta)} = n_{\mathbf{a}}(\zeta),$$

hence (6.6) shows

$$\frac{\eta^{-1} + \varphi^{(-1)}_{q_\zeta \mathbf{a}}}{1 + \varphi^{(0)}_{\mathbf{a}}} = (n_{\mathbf{a}} - n_{\mathbf{a}}(\zeta))q_\zeta,$$

which together with (6.5) leads us to

$$\frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta - \eta} n_{q_\zeta \mathbf{a}}(\eta) = \zeta \frac{n_{\mathbf{a}}(\eta) - n_{\mathbf{a}}(\zeta)}{\zeta - \eta} \implies n_{q_\zeta \mathbf{a}}(\eta) = \zeta \frac{n_{\mathbf{a}}(\eta) - n_{\mathbf{a}}(\zeta)}{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}.$$

Then, (6.1) yields the proof of (6.3). ∎

Set

$$\begin{cases} \Gamma_{\mathrm{real}}(C) = \{g \in \Gamma;\ g = \bar{g},\ g \text{ has no zeros nor poles on } \overline{D}_+\}, \\ \mathbf{A}^{\mathrm{inv}}_+(C) = \{\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)\ ;\ \mathbf{a} = \bar{\mathbf{a}}, \tau_{\mathbf{a}}(r) \geq 0 \text{ for any rational } r \in \Gamma_{\mathrm{real}}(C)\}, \end{cases}$$

where we use the notation

$$\bar{g}(z) = \overline{g(\bar{z})} \quad \bar{\mathbf{a}}(\lambda) = \overline{\mathbf{a}(\bar{\lambda})}.$$

The continuity of $\tau_{\mathbf{a}}$ implies

$$\mathbf{A}_{+}^{\mathrm{inv}}(C) = \{\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)\,;\ \mathbf{a} = \bar{\mathbf{a}}, \tau_{\mathbf{a}}(g) \geq 0 \text{ for any } g \in \Gamma_{\mathrm{real}}(C)\}. \tag{6.7}$$

The condition $\mathbf{a} = \bar{\mathbf{a}}$ is necessary to have real a_n, b_n. Lemma 8 (ii) implies

$$\tilde{\mathbf{a}} \in \mathbf{A}_{+}^{\mathrm{inv}}(C) \text{ holds for } \mathbf{a} \in \mathbf{A}_{+}^{\mathrm{inv}}(C).$$

Lemma 15. *For* $\mathbf{a} \in \mathbf{A}_{+}^{\mathrm{inv}}(C)$*,*

(i) $1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) \neq 0$ *on* D_{-}^{1}*,* $\operatorname{Im} m_{\mathbf{a}}(\zeta) > 0$ *on* $D_{-}^{1} \cap \mathbb{C}_{+}$ *hold.*

Assume further $1 + \varphi_{\mathbf{a}}^{(0)}(0)\ (= \tau_{\mathbf{a}}(z^{-1})) > 0$ *and* $\tau_{\mathbf{a}}(z^{-2}) > 0$*. Then,*

(iii) $1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) \neq 0$ *on* D_{-}*, and* $m_{\mathbf{a}}'(0) > 0$*,* $\operatorname{Im} m_{\mathbf{a}}(\zeta) > 0$ *on* $D_{-} \cap \mathbb{C}_{+}$*,* $\operatorname{Im} m_{z^{-1}\mathbf{a}}(\zeta) > \operatorname{Im}(\zeta + \zeta^{-1})$ *on* $D_{-}^{1} \cap \mathbb{C}_{+}$*.*

Proof. Set

$$\mathcal{Z} = \{\zeta \in D_{-}\,;\ 1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) = 0\}.$$

$1 + \varphi_{\mathbf{a}}^{(0)}(\zeta)$ is not identically 0 on D_{-}^{1}, since $1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) \to 1$ as $\zeta \to \infty$. Hence, $m_{\mathbf{a}}(\zeta)$ is meromorphic on D_{-}^{1}. Lemma 9 yields an identity

$$0 \leq \tau_{\mathbf{a}}(q_z q_{\bar{z}}) = |1 + \varphi_{\mathbf{a}}^{(0)}(z)|^2 \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z} \quad \text{for } z \in D_{-}^{1} \setminus \mathcal{Z}, \tag{6.8}$$

hence $\operatorname{Im} m_{\mathbf{a}}(\zeta) \geq 0$ on $D_{-}^{1} \cap \mathbb{C}_{+}$. Let $\zeta_0 \in \mathcal{Z} \cap D_{-}^{1} \cap \mathbb{C}_{+}$. Then, $\operatorname{Im}(-m_{\mathbf{a}}(\zeta)^{-1}) \geq 0$ on a neighbourhood of ζ_0 and $\operatorname{Im}(-m_{\mathbf{a}}(\zeta_0)^{-1}) = 0$ hold, hence the minimum principle for harmonic functions implies $\operatorname{Im}(-m_{\mathbf{a}}(\zeta)^{-1}) = 0$ identically on $D_{-}^{1} \cap \mathbb{C}_{+}$, which shows $m_{\mathbf{a}}$ is a constant on $D_{-}^{1} \cap \mathbb{C}_{+}$. This contradicts $m_{\mathbf{a}}(\zeta) - \zeta \to 0$ as $\zeta \to \infty$, hence we have $\mathcal{Z} \cap D_{-}^{1} \cap \mathbb{C}_{+} = \varnothing$. Since $\varphi_{\mathbf{a}}^{(0)} = \bar{\varphi}_{\mathbf{a}}^{(0)}$ and $m_{\mathbf{a}} = \bar{m}_{\mathbf{a}}$ hold, $\mathcal{Z} \cap (D_{-}^{1} \setminus \mathbb{R}) = \varnothing$ is satisfied.

For $x \in D_{-}^{1} \cap \mathbb{R}$, one has from Lemma 9 (i),

$$0 \leq \tau_{\mathbf{a}}(q_x) = 1 + \varphi_{\mathbf{a}}^{(0)}(x) \quad (q_{\zeta}(z) = (1 - \zeta^{-1}z)^{-1}).$$

Suppose $1 + \varphi_{\mathbf{a}}^{(0)}(x_0) = 0$ holds for some $x_0 \in D_{-}^{1} \cap \mathbb{R}$. Choose an interval $(a, b) \subset D_{-}^{1} \cap \mathbb{R}$ such that $x_0 \in (a, b)$ and $1 + \varphi_{\mathbf{a}}^{(0)}(x) > 0$ for any $x \in (a, b) \setminus \{x_0\}$. Equation (6.8) implies $m_{\mathbf{a}}'(x) \geq 0$ for $x \in (a, b) \setminus \{x_0\}$. Hence, we have

$$m_{\mathbf{a}}(x_0 - 0) = +\infty \implies x_0 + \varphi_{\mathbf{a}}^{(1)}(x_0) > 0 \implies m_{\mathbf{a}}(x_0 + 0) = +\infty,$$

which contradicts $m'_{\mathbf{a}}(x) \geq 0$. Therefore, $\mathcal{Z} \cap D^1_- = \varnothing$ holds. The property $\operatorname{Im} m_{\mathbf{a}}(\zeta) > 0$ on $D^1_- \cap \mathbb{C}_+$ follows from $\operatorname{Im} m_{\mathbf{a}}(\zeta) \geq 0$ and $m_{\mathbf{a}}(\zeta) - \zeta \to 0$ as $\zeta \to \infty$, which shows (i).

To show (ii), observe the identity

$$\frac{\tau_{\mathbf{a}}(z^{-1} q_\zeta)}{\tau_{\mathbf{a}}(q_\zeta)\tau_{\mathbf{a}}(z^{-1})} = \frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(0)}{\zeta}$$

deduced from Lemma 9, which shows

$$m'_{\mathbf{a}}(0) = \frac{\tau_{\mathbf{a}}(z^{-2})}{\tau_{\mathbf{a}}(z^{-1})^2}.$$

If $\tau_{\mathbf{a}}(z^{-2}) > 0$, then $m_{\mathbf{a}}$ is not a constant on D^2_-, and by arguments similar to those used in (i), we have $1 + \varphi^{(0)}_{\mathbf{a}}(\zeta) \neq 0$ on D_- and $\operatorname{Im} m_{\mathbf{a}}(\zeta) > 0$ on $D^2_- \cap \mathbb{C}_+$. The inequality $\operatorname{Im} m_{z^{-1}\mathbf{a}}(\zeta) > \operatorname{Im}(\zeta + \zeta^{-1})$ on $D^1_- \cap \mathbb{C}_+$ follows from (6.4). ■

For $\mathbf{a} \in \mathbf{A}^{\text{inv}}_+(C)$, one can show partially the property $\tau_{\mathbf{a}}(g) > 0$. For $\zeta \in D_-$, set

$$r_\zeta(z) = q_\zeta(z) q_{\bar{\zeta}}(z) \in \Gamma_{\text{real}}(C).$$

Lemma 16. *If* $\mathbf{a} \in \mathbf{A}^{\text{inv}}_+(C)$*, then* $\tau_{\mathbf{a}}(re^h) > 0$ *for any rational function* $r \in \Gamma_{\text{real}}(C)$ *with no poles nor zeros on* D^2_- *and analytic function* h *on* $\overline{D}_+$ *satisfying* $h = \bar{h}$.

Proof. Note that any rational function $r \in \Gamma_{\text{real}}(C)$ with no poles nor zeros on D^2_- can be expressed by $r = r_1 r_2^{-1}$

$$\begin{cases} r_1 = q_{x_1} q_{x_2} \cdots q_{x_m} r_{\zeta_1} r_{\zeta_2} \cdots r_{\zeta_n} & \text{with } x_j \in D^1_- \cap \mathbb{R}, \zeta_j \in D^1_- \setminus \mathbb{R}, \\ r_2 = q_{y_1} q_{y_2} \cdots q_{y_{m'}} r_{\eta_1} r_{\eta_2} \cdots r_{\eta_{n'}} & \text{with } y_j \in D^1_- \cap \mathbb{R}, \eta_j \in D^1_- \setminus \mathbb{R}. \end{cases} \tag{6.9}$$

First, we show $\tau_{\mathbf{a}}(r_1) > 0$. For any $x \in D^1_- \cap \mathbb{R}$, $\zeta \in D^1_- \setminus \mathbb{R}$ we have

$$\begin{cases} \tau_{\mathbf{a}}(q_x) = 1 + \varphi^{(0)}_{\mathbf{a}}(x) > 0, \\ \tau_{\mathbf{a}}(r_\zeta) = \dfrac{|1 + \varphi^{(0)}_{\mathbf{a}}(\zeta)|^2 \operatorname{Im} m_{\mathbf{a}}(\zeta)}{\operatorname{Im} \zeta} > 0, \end{cases}$$

from Lemma 15. Then, $q_x \mathbf{a}, r_\zeta \mathbf{a} \in \mathbf{A}^{\text{inv}}_+(C)$, and the property $\tau_{\mathbf{a}}(r_1) > 0$ can be shown inductively. For general $r = r_1 r_2^{-1}$, Lemma 8 shows

$$\begin{aligned} \tau_{\mathbf{a}}(r) &= \tau_{\mathbf{a}}(r_1 r_2^{-1}) = \tau_{\mathbf{a}}(r_1 r_2^{-1} \tilde{r}_2^{-1} \tilde{r}_2) = \tau_{\mathbf{a}}((r_2 \tilde{r}_2)^{-1}) \tau_{\mathbf{a}}(r_1 \tilde{r}_2) \\ &= \tau_{\mathbf{a}}((r_2 \tilde{r}_2)^{-1}) \tau_{\mathbf{a}}(r_1) \tau_{r_1 \mathbf{a}}(\tilde{r}_2) = \tau_{\mathbf{a}}((r_2 \tilde{r}_2)^{-1}) \tau_{\mathbf{a}}(r_1) \tau_{\widetilde{r_1 \mathbf{a}}}(r_2), \end{aligned}$$

where $\tau_{\mathbf{a}}((r_2 \tilde{r}_2)^{-1}) > 0$ due to $(r_2 \tilde{r}_2)(z) = (r_2 \tilde{r}_2)(z^{-1})$. Then, $\tau_{\mathbf{a}}(r) > 0$ is satisfied, since $\widetilde{r_1 \mathbf{a}} \in \mathbf{A}^{\text{inv}}_+(C)$.

On the other hand, note that for $z \in D_+$,

$$h(z) = \frac{1}{2\pi i}\int_C \frac{h(\lambda)}{\lambda - z}d\lambda = \frac{1}{2\pi i}\int_{C_1} \frac{h(\lambda)}{\lambda - z}d\lambda + \frac{1}{2\pi i}\int_{C_2} \frac{h(\lambda)}{\lambda - z}d\lambda$$
$$\equiv h_1(z) + h_2(z),$$

where h_1 is analytic on $D_+ \cup D_-^2$ and h_2 is analytic on $D_+ \cup D_-^1$ (if it is necessary, we have only to move C_1 and C_2 in the inside of a neighbourhood of $\overline{D}_+$). We first consider h_1. Since $\overline{D_+ \cup D_-^2}$ is a simply connected domain containing 0, one sees that h_1 can be approximated by a real polynomial p. Observe

$$e^p = \lim_{n\to\infty}\left(1 - \frac{p}{n}\right)^{-n} \quad \text{uniformly on } \overline{D_+ \cup D_-^2},$$

and

$$r_1 \equiv \left(1 - \frac{p}{n}\right)^{-n} = q_{x_1}\cdots q_{x_{m_1}} r_{\zeta_1}\cdots r_{\zeta_{m_2}} \quad \text{for } x_j \in D_-^1 \cap \mathbb{R}, \zeta_j \in D_-^1 \setminus \mathbb{R},$$

since the zeros of $1 - p/n$ are located on D_-^1 for sufficiently large n. Then, the continuity of the tau function implies $\tau_{\mathbf{a}}(r_1^{-1}e^{h_1}) > 0$, hence

$$\tau_{\mathbf{a}}(e^{h_1}) = \tau_{\mathbf{a}}(r_1^{-1}e^{h_1}r_1) = \tau_{\mathbf{a}}(r_1^{-1}e^{h_1})\tau_{r_1^{-1}e^{h_1}\mathbf{a}}(r_1)$$

is satisfied. Since $r_1^{-1}e^{h_1}\mathbf{a} \in \mathbf{A}_+^{\mathrm{inv}}(C)$ due to (6.7), applying the above argument to $r_1^{-1}e^{h_1}\mathbf{a}$, one has $\tau_{r_1^{-1}e^{h_1}\mathbf{a}}(r_1) > 0$, which shows $\tau_{\mathbf{a}}(e^{h_1}) > 0$. Observe

$$\tau_{\mathbf{a}}(e^h) = \tau_{\mathbf{a}}(e^{h_1})\tau_{e^{h_1}\mathbf{a}}(e^{h_2}) = \tau_{\mathbf{a}}(e^{h_1})\tau_{\widetilde{e^{h_1}\mathbf{a}}}(e^{\tilde{h}_2}).$$

The property $e^{h_1}\mathbf{a} \in \mathbf{A}_+^{\mathrm{inv}}(C)$ implies $\widetilde{e^{h_1}\mathbf{a}} \in \mathbf{A}_+^{\mathrm{inv}}(C)$, hence $\tau_{\widetilde{e^{h_1}\mathbf{a}}}(e^{\tilde{h}_2}) > 0$ holds, which shows $\tau_{\mathbf{a}}(e^h) > 0$. Now, $\tau_{\mathbf{a}}(re^h) > 0$ clearly follows from $r\mathbf{a} \in \mathbf{A}_+^{\mathrm{inv}}(C)$. ■

$\mathbf{A}_+^{\mathrm{inv}}(C)$ is not sufficient to have the property $\tau_{\mathbf{a}}(g) > 0$ for any $g \in \Gamma_{\mathrm{real}}(C)$, since it is possible that $1 + \varphi_{\mathbf{a}}^{(0)}(z) = 0$ identically on D_-^2. We define

$$\mathbf{A}_{++}^{\mathrm{inv}}(C) = \{\mathbf{a} \in \mathbf{A}_+^{\mathrm{inv}}(C)\,;\ \tau_{\mathbf{a}}(z^n) > 0 \text{ for any } n \in \mathbb{Z}\}.$$

Proposition 4. *Suppose* $\mathbf{a} \in \mathbf{A}_{++}^{\mathrm{inv}}(C)$. *Then, for any* $g \in \Gamma_{\mathrm{real}}(C)$, *one has* $\tau_{\mathbf{a}}(g) > 0$ *and* $g\mathbf{a} \in \mathbf{A}_{++}^{\mathrm{inv}}(C)$.

Proof. If $\mathbf{a} \in \mathbf{A}_{++}^{\mathrm{inv}}(C)$, then $z^k\mathbf{a} \in \mathbf{A}_{++}^{\mathrm{inv}}(C)$ for $k \in \mathbb{Z}$ is satisfied. For $x \in D_- \cap \mathbb{R}$, Lemma 15 implies $\tau_{\mathbf{a}}(q_x) = 1 + \varphi_{\mathbf{a}}^{(0)}(x) > 0$, and the cocycle property of the tau functions shows

$$\tau_{q_x\mathbf{a}}(z^n) = \frac{\tau_{\mathbf{a}}(q_x z^n)}{\tau_{\mathbf{a}}(q_x)} = \frac{\tau_{\mathbf{a}}(z^n)\tau_{z^n\mathbf{a}}(q_x)}{\tau_{\mathbf{a}}(q_x)} > 0$$

due to $z^n\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$. Since $q_x\mathbf{a} \in \mathbf{A}^{\text{inv}}_{+}(C)$ clearly holds, one has $q_x\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$. Similarly, one can show $r_\zeta\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$ for $\zeta \in D_- \setminus \mathbb{R}$. Then, similarly to the proof of Lemma 16, we have $\tau_{\mathbf{a}}(r) > 0$ for any $r \in \Gamma_{\text{real}}(C)$, which implies $r\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$. Then, applying Lemma 16 to $r\mathbf{a}$, we have $\tau_{r\mathbf{a}}(e^h) > 0$, and hence $\tau_{\mathbf{a}}(re^h) > 0$, which shows $re^h\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$. ∎

The next issue is to find a sufficient condition for $\mathbf{a} \in \mathbf{A}^{\text{inv}}(C)$ to be an element of $\mathbf{A}^{\text{inv}}_{++}(C)$, which will be clarified in the next section.

7. *m*-function and Weyl function

In this section we relate m-functions to Weyl functions (see Appendix A for the definition).

First, we present a lemma which connects the existence of a positive solution to a Jacobi equation with the estimate of the spectrum of the Jacobi operator. For $a_n > 0$, $b_n \in \mathbb{R}$ let $q = \{a_n, b_n\}_{n\in\mathbb{Z}}$ and

$$(H_q u)_n = a_{n+1}u_{n+1} + a_n u_{n-1} + b_n u_n.$$

Set

$$\ell_0(\mathbb{Z}) = \{f \in \ell^2(\mathbb{Z})\ ;\ f \text{ has a finite support}\}.$$

Lemma 17. *Suppose that there exists $\lambda_0 < \lambda_1$ and two solutions u_n, v_n to*

$$\begin{cases} H_q u = \lambda_1 u,\ u_n > 0 & \text{for any } n \in \mathbb{Z}, \\ H_q v = \lambda_0 v,\ (-1)^n v_n > 0 & \text{for any } n \in \mathbb{Z}. \end{cases}$$

Then it holds that

$$\lambda_0(f, f) \le (H_q f, f) \le \lambda_1(f, f) \quad \text{for any } f \in \ell_0(\mathbb{Z})$$

Consequently, H_q can be uniquely extended to $\ell^2(\mathbb{Z})$ as a bounded self-adjoint operator, and the boundaries $\pm\infty$ are of limit point type.

Proof. Define

$$\phi_{n-1} = \frac{u_n}{u_{n-1}} > 0$$

and

$$\rho_o(n) = -\sqrt{\frac{a_{n+1}}{\phi_n}} < 0, \quad \rho_e(n) = \sqrt{a_{n+1}\phi_n} > 0,$$

and the operators A, A^* on $\ell_0^2(\mathbb{Z})$ by

$$(Af)_n = \rho_o(n) f_{n+1} + \rho_e(n) f_n,$$
$$(A^* f)_n = \rho_o(n-1) f_{n-1} + \rho_e(n) f_n.$$

Note that

$$(Af, g) = (f, A^* g) \quad \text{for any } f, g \in \ell_0^2(\mathbb{Z})$$

is satisfied. Then it holds that

$$\begin{cases} \rho_e(n)\rho_o(n) = -a_{n+1}, \\ \rho_o(n-1)^2 + \rho_e(n)^2 = \dfrac{a_n}{\phi_{n-1}} + a_{n+1}\phi_n = \lambda_1 - b_n, \end{cases}$$

and

$$\begin{aligned}(A^* A f)_n &= \rho_o(n-1)(\rho_o(n-1) f_n + \rho_e(n-1) f_{n-1}) \\ &\quad + \rho_e(n)(\rho_o(n) f_{n+1} + \rho_e(n) f_n) \\ &= \rho_e(n)\rho_o(n) f_{n+1} + (\rho_o(n-1)^2 + \rho_e(n)^2) f_n \\ &\quad + \rho_o(n-1)\rho_e(n-1) f_{n-1} \\ &= -a_n f_{n+1} - a_{n-1} f_{n-1} + (\lambda_1 - b_n) f_n \\ &= ((\lambda_1 - H_q) f)_n,\end{aligned}$$

hence $\lambda_1 - H_q = A^* A$ is satisfied on $\ell_0^2(\mathbb{Z})$, which shows for any $f \in \ell_0^2(\mathbb{Z})$

$$((\lambda_1 - H_q) f, f) = (A^* A f, f) = (Af, Af) \geq 0.$$

To show the converse inequality, set

$$\phi_{n-1} = -\frac{v_n}{v_{n-1}} > 0$$

and

$$\rho_o(n) = -\sqrt{\frac{a_{n+1}}{\phi_n}} < 0, \quad \rho_e(n) = \sqrt{a_{n+1}\phi_n} > 0,$$

and operators A, A^* on $\ell_0^2(\mathbb{Z})$ by

$$(Af)_n = \rho_o(n) f_{n+1} - \rho_e(n) f_n,$$
$$(A^* f)_n = \rho_o(n-1) f_{n-1} - \rho_e(n) f_n.$$

Then, $A^* A = H_q - \lambda_0$ holds, hence the converse inequality is satisfied. The rest of the statement is clear. ∎

Let $\lambda_\pm$ be

$$\begin{cases} \lambda_+ = \inf\{x + x^{-1} \ ; \ x \in D_- \cap \mathbb{R}_+\} > 0, \\ \lambda_- = \sup\{x + x^{-1} \ ; \ x \in D_- \cap \mathbb{R}_-\} < 0. \end{cases}$$

Now, we have the following result.

Proposition 5. *Suppose* $\mathbf{a} \in \mathbf{A}_{++}^{\mathrm{inv}}(C)$*. Then, the* $q = \{a_n, b_n\}_{n\in\mathbb{Z}}$ *associated with* $\mathbf{a}$ *by* (5.2) *provides a bounded Jacobi operator* H_q *satisfying* $\mathrm{sp}\, H_q \subset [\lambda_-, \lambda_+]$*. Moreover, its m-function* $m_{\mathbf{a}}$ *is given by the Weyl functions* $m_\pm$ *of* H_q *as*

$$m_{\mathbf{a}}(z) = \begin{cases} z + z^{-1} + a_1^2 m_+(z + z^{-1}) & \text{if } z \in D_-^1, \\ -a_0^2 m_-(z + z^{-1}) + b_0 & \text{if } z \in D_-^2. \end{cases}$$

Proof. Lemma 11 says that

$$g_n(\zeta) = \frac{\mathbf{a}T(z^n\mathbf{a})^{-1}1(\zeta)}{\sqrt{1 + \varphi^{(0)}_{z^n\mathbf{a}}(0)}} = \zeta^{-n} \frac{1 + \varphi^{(0)}_{z^n\mathbf{a}}(\zeta)}{\sqrt{1 + \varphi^{(0)}_{z^n\mathbf{a}}(0)}}$$

satisfies $H_q g = (\zeta + \zeta^{-1})g$ for $\zeta \in D_-$. Since $g_n(\zeta) > 0$ for any $n \in \mathbb{Z}$ if $\zeta \in D_- \cap \mathbb{R}_+$, and $(-1)^n g_n(\zeta) > 0$ for any $n \in \mathbb{Z}$ if $\zeta \in D_- \cap \mathbb{R}_-$, then from Lemma 17, one has that $\mathrm{sp}\, H_q \subset [\lambda_-, \lambda_+]$. The boundedness of H_q implies that the boundaries $\pm\infty$ are of limit point type.

We show $g_n(\zeta) \in \ell^2(\mathbb{Z}_+)$ for $\zeta \in D_-^1 \cap \mathbb{C}_+$. Observe

$$\frac{g_{k+1}(\zeta)}{g_k(\zeta)} = \frac{\sqrt{\tau_{z^k\mathbf{a}}(z^{-1})}}{\sqrt{\tau_{z^{k+1}\mathbf{a}}(z^{-1})}} \frac{\tau_{z^{k+1}\mathbf{a}}(q_\zeta)}{\zeta\tau_{z^k\mathbf{a}}(q_\zeta)} = \frac{\sqrt{\tau_{z^k\mathbf{a}}(z^{-1})}}{\sqrt{\tau_{z^{k+1}\mathbf{a}}(z^{-1})}} \frac{\tau_{z^{k+1}\mathbf{a}}(q_\zeta)}{\zeta\tau_{z^k\mathbf{a}}(q_\zeta)}.$$

Since

$$\tau_{z^{k+1}\mathbf{a}}(q_\zeta) = \tau_{z^{-1}\widetilde{z^k\mathbf{a}}}(\tilde{q}_\zeta) = \frac{\tau_{\widetilde{z^k\mathbf{a}}}(z^{-1}\tilde{q}_\zeta)}{\tau_{\widetilde{z^k\mathbf{a}}}(z^{-1})} = \frac{\tau_{\widetilde{z^k\mathbf{a}}}(q_{\zeta^{-1}})}{\tau_{z^k\mathbf{a}}(z)} \quad (z^{-1}\tilde{q}_\zeta = -\zeta q_{\zeta^{-1}})$$

holds, one has

$$\frac{g_{k+1}(\zeta)}{g_k(\zeta)} = \frac{\sqrt{\tau_{z^k\mathbf{a}}(z^{-1})}}{\sqrt{\tau_{z^{k+1}\mathbf{a}}(z^{-1})}} \frac{1}{\tau_{z^k\mathbf{a}}(z)} \frac{\tau_{\widetilde{z^k\mathbf{a}}}(q_{\zeta^{-1}})}{\zeta\tau_{z^k\mathbf{a}}(q_\zeta)} = c_k n_{z^k\mathbf{a}}(\zeta) \tag{7.1}$$

with $c_k = \sqrt{\tau_{\mathbf{a}}(z^{k-1})}/\sqrt{\tau_{\mathbf{a}}(z^{k+1})}$. Since $g_k(\zeta)$ satisfies (5.3), equation (7.1) implies

$$\begin{aligned} c_k n_{z^k\mathbf{a}}(\zeta) &= a_{k+1}^{-1} \frac{(\zeta + \zeta^{-1})g_k(\zeta) - a_k g_{k-1}(\zeta) - b_k g_k(\zeta)}{g_k(\zeta)} \\ &= a_{k+1}^{-1}(\zeta + \zeta^{-1} - b_k) - a_k a_{k+1}^{-1}(c_{k-1} n_{z^{k-1}\mathbf{a}}(\zeta))^{-1}. \end{aligned}$$

Taking the imaginary parts, one has

$$\operatorname{Im} c_k n_{z^k\mathbf{a}}(\zeta) = a_{k+1}^{-1} \operatorname{Im}(\zeta + \zeta^{-1}) + a_k a_{k+1}^{-1} \frac{\operatorname{Im} c_{k-1} n_{z^{k-1}\mathbf{a}}(\zeta)}{|c_{k-1} n_{z^{k-1}\mathbf{a}}(\zeta)|^2},$$

hence

$$1 - \frac{a_{k+1}^{-1} \operatorname{Im}(\zeta + \zeta^{-1})}{\operatorname{Im} c_k n_{z^k\mathbf{a}}(\zeta)} = a_k a_{k+1}^{-1} \frac{\operatorname{Im} c_{k-1} n_{z^{k-1}\mathbf{a}}(\zeta)}{\operatorname{Im} c_k n_{z^k\mathbf{a}}(\zeta)} |c_{k-1} n_{z^{k-1}\mathbf{a}}(\zeta)|^{-2}$$

follows. Note here $\operatorname{Im} c_k n_{z^k\mathbf{a}}(\zeta) < 0$ on $D_-^1 \cap \mathbb{C}_+$ due Lemma 15 (ii). Since

$$\frac{|g_{n+1}(\zeta)|^2}{|g_0(\zeta)|^2} = \frac{a_1 \operatorname{Im} c_0 n_{\mathbf{a}}}{a_{n+2} \operatorname{Im} c_{n+1} n_{z^{n+1}\mathbf{a}}} \prod_{1 \le k \le n+1} \Big(1 - \frac{\operatorname{Im}(\zeta + \zeta^{-1})}{a_{k+1} \operatorname{Im} c_k n_{z^k\mathbf{a}}}\Big)^{-1},$$

setting

$$A_k = \operatorname{Im}(\zeta + \zeta^{-1})(-a_{k+1} \operatorname{Im} c_k n_{z^k\mathbf{a}})^{-1} 2 \quad B = \frac{-a_1 \operatorname{Im} c_0 n_{\mathbf{a}}}{\operatorname{Im}(\zeta + \zeta^{-1})},$$

we have

$$\begin{aligned}\sum_{0 \le n \le N} \frac{|g_{n+1}(\zeta)|^2}{|g_0(\zeta)|^2} &= B \sum_{0 \le n \le N} A_{n+1} \prod_{1 \le k \le n+1} (1 + A_k)^{-1} \\ &= B\Big(1 - \prod_{1 \le k \le N+1} (1 + A_k)^{-1}\Big) \le B,\end{aligned}$$

which implies $g_n(\zeta) \in \ell^2(\mathbb{Z}_+)$ for $\zeta \in D_-^1 \cap \mathbb{C}_+$. Similarly, we have $g_n(\zeta) \in \ell^2(\mathbb{Z}_-)$ for $\zeta \in D_-^2 \cap \mathbb{C}_+$. Then, the definition of Weyl functions yields

$$m_+(\zeta + \zeta^{-1}) = -\frac{g_1(\zeta)}{a_1 g_0(\zeta)} = -\frac{c_0}{a_1} n_{\mathbf{a}}(\zeta) = -a_1^{-2}(\zeta + \zeta^{-1} - m_{\mathbf{a}}(\zeta)).$$

For m_-, one has

$$m_-(\zeta + \zeta^{-1}) = -\frac{g_{-1}(\zeta)}{a_0 g_0(\zeta)} = \frac{-1}{a_0 c_{-1} n_{z^{-1}\mathbf{a}}(\zeta)} = \frac{-1}{a_0 \sqrt{\tau_{\mathbf{a}}(z^{-2})}} \frac{\zeta \tau_{\mathbf{a}}(z^{-1} q_\zeta)}{\tau_{\mathbf{a}}(q_\zeta)}.$$

Since Lemma 9 implies

$$\zeta \tau_{\mathbf{a}}(z^{-1} q_\zeta) = \tau_{\mathbf{a}}(z^{-1}) \tau_{\mathbf{a}}(q_\zeta)(m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(0)),$$

one has

$$m_{\mathbf{a}}(z) = -a_0^2 m_-(z + z^{-1}) + m_{\mathbf{a}}(0).$$

To identify $m_{\mathbf{a}}(0)$ with b_0, we start from the definition (5.1) of b_0:

$$b_0 = \varphi_{\tilde{\mathbf{a}}}^{(-1)}(0) - \varphi_{\widetilde{z^{-1}\mathbf{a}}}^{(-1)}(0).$$

To change the symbol $z^{-1}\mathbf{a}$ to $\mathbf{a}$, we use tau functions. We have

$$\begin{aligned}\varphi^{(-1)}_{\widetilde{z^{-1}\mathbf{a}}}(\zeta) &= \zeta^{-1}\varphi^{(0)}_{z^{-1}\mathbf{a}}(\zeta^{-1}) = \zeta^{-1}(\tau_{z^{-1}\mathbf{a}}(q_{\zeta^{-1}})-1) = \frac{\tau_{\mathbf{a}}(z^{-1}q_{\zeta^{-1}})}{\zeta\tau_{\mathbf{a}}(z^{-1})} - \zeta^{-1}\\ &= (1+\varphi^{(0)}_{\mathbf{a}}(\zeta^{-1}))(m_{\mathbf{a}}(\zeta^{-1}) - m_{\mathbf{a}}(0)) - \zeta^{-1}\\ &= -m_{\mathbf{a}}(0) + \varphi^{(-1)}_{\tilde{\mathbf{a}}}(0) + O(\zeta) \quad \text{as } \zeta \to 0,\end{aligned}$$

which completes the proof. ■

This proposition implies that for $\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$, the associated m-function $m_{\mathbf{a}}$ is automatically extended analytically to $\mathbb{C} \setminus \Sigma_{\lambda_0}$ with $\lambda_0 = \max\{\lambda_+, -\lambda_-\}$.

8. Proof of the theorem by identifying $\mathbf{A}^{\text{inv}}_{++}$ with Q_{λ_0}

Although we have constructed a Toda flow on $\mathbf{A}^{\text{inv}}_{++}$ by Propositions 3 and 4, we have not been able to specify $\mathbf{A}^{\text{inv}}_{++}(C)$ in a concrete terminology. In this section we identify $\mathbf{A}^{\text{inv}}_{++}(C)$ with Q_{λ_0} ($\lambda_0 > 2$), which provides automatically a proof of the theorem.

Proposition 5 shows $\mathbf{A}^{\text{inv}}_{++}(C) \subset Q_{\lambda_0}$ in a certain sense. Our next task here is to verify the converse statement. For $q = \{a_n, b_n\}_{n\in\mathbb{Z}} \in Q_{\lambda_0}$, let $m_\pm$ be the Weyl functions for the Jacobi operator H_q and set

$$m(z) = \begin{cases} z + z^{-1} + a_1^2 m_+(z+z^{-1}) & \text{if } z \in \Sigma_{\lambda_0} \text{ and } |z| > 1,\\ -a_0^2 m_-(z+z^{-1}) + b_0 & \text{if } z \in \Sigma_{\lambda_0} \text{ and } |z| < 1,\end{cases}$$

with

$$\Sigma_{\lambda_0} = \{|z| = 1\} \cup [-\ell, -\ell^{-1}] \cup [\ell^{-1}, \ell], \quad \ell = \frac{\lambda_0 + \sqrt{\lambda_0^2 - 4}}{2}.$$

Then, m is an element of

$$M_{\lambda_0} = \left\{\begin{array}{l} m\,;\ m \text{ is analytic on } \mathbb{C}\setminus\Sigma_{\lambda_0} \text{ satisfying } m = \overline{m} \text{ and}\\ \text{(i)}\ \ m(z) = z + O(z^{-1}) \ \text{ as } z \to \infty\\ \text{(ii)}\ \ \operatorname{Im} m(z) > 0 \ \text{ on } \mathbb{C}_+ \setminus \Sigma_{\lambda_0}\\ \text{(iii)}\ m(z) - m(z^{-1}) \neq 0 \ \text{ on } \mathbb{C}\setminus\Sigma_{\lambda_0}\\ \text{(iv)}\ m(z) \text{ is not a rational function of } z + z^{-1}\end{array}\right\}, \tag{8.1}$$

which obeys from Lemma 21 in Appendix A.

$m \in M_{\lambda_0}$ has the following expression.

Lemma 18. *For $m \in M_{\lambda_0}$, there exist $a_0^2 > 0$, $a_1^2 > 0$, $b_0 \in \mathbb{R}$, and probability measures $\sigma_\pm$ on $[-\lambda_0, \lambda_0]$ such that*

$$m(z) = \begin{cases} z + z^{-1} + a_1^2 \int_{-\lambda_0}^{\lambda_0} \dfrac{\sigma_+(d\lambda)}{\lambda - (z + z^{-1})} & \text{if } |z| > 1, \\ -a_0^2 \int_{-\lambda_0}^{\lambda_0} \dfrac{\sigma_-(d\lambda)}{\lambda - (z + z^{-1})} + b_0 & \text{if } |z| < 1. \end{cases}$$

Proof. $\phi(z) = z + z^{-1}$ yields a conformal map

$$\phi \colon \begin{cases} \mathbb{C}_+ \cap \{|z| > 1\} \to \mathbb{C}_+, \\ \mathbb{C}_+ \cap \{|z| < 1\} \to \mathbb{C}_-. \end{cases}$$

The analytic functions $n_\pm(z) = \pm m(\phi^{-1}(z))$ on $\mathbb{C}_\pm$ satisfy $\pm \operatorname{Im} n_\pm(z) > 0$, hence they have Herglotz representations, namely there exist $\alpha_\pm \in \mathbb{R}$, $\beta_\pm \geq 0$ and measures $\sigma_\pm$ on $\mathbb{R}$ such that

$$n_\pm(z) = \alpha_\pm + \beta_\pm z + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1}\Big)\sigma_\pm(d\lambda),$$

hence

$$m(z) = \begin{cases} \alpha_+ + \beta_+\phi(z) + \int_{-\infty}^{\infty} \Big(\dfrac{1}{\lambda - \phi(z)} - \dfrac{\lambda}{\lambda^2 + 1}\Big)\sigma_+(d\lambda) & \text{if } |z| > 1, \\ -\alpha_- - \beta_-\phi(z) - \int_{-\infty}^{\infty} \Big(\dfrac{1}{\lambda - \phi(z)} - \dfrac{\lambda}{\lambda^2 + 1}\Big)\sigma_-(d\lambda) & \text{if } |z| < 1. \end{cases}$$

The property $m = \bar{m}$ of (8.1) imply the measures $\sigma_\pm$ should be supported on $[-\lambda_0, \lambda_0]$. Moreover, property (i) shows $\beta_+ = 1$ and

$$m(z) = \phi(z) + \int_{-\lambda_0}^{\lambda_0} \frac{1}{\lambda - \phi(z)}\sigma_+(d\lambda) \quad \text{if } |z| > 1.$$

Property (iv) implies $m(z) = \phi(z)$ not identically, hence setting

$$a_1^2 \equiv \sigma_+([-\lambda_0, \lambda_0]) > 0$$

and normalizing σ_+, one has the first expression of the lemma. Since we know $m(0)$ is finite, $m(z)$ on $\{|z| < 1\}$ takes a form

$$m(z) = -\int_{-\lambda_0}^{\lambda_0} \frac{1}{\lambda - \phi(z)} \sigma_-(d\lambda) + b_0$$

with $b_0 \in \mathbb{R}$. The strict positivity of $\operatorname{Im} m(z) > 0$ on $\mathbb{C}_+ \cap \{|z| < 1\}$ implies non-vanishing of σ_-. Then, setting $a_0^2 = \sigma_-([-\lambda_0, \lambda_0]) > 0$, one has the second expression of the lemma. ∎

For $m \in M_{\lambda_0}$, define a symbol

$$\mathbf{m}(z) = \Big(\frac{zm(z) - 1}{z^2 - 1}, z^2 \frac{z - m(z)}{z^2 - 1}\Big). \tag{8.2}$$

We fix any bounded domain D_+ containing Σ_{λ_0} and satisfying $D_+ \ni z \to z^{-1}$, $\bar{z} \in D_+$ and having smooth boundaries. Then, this $\mathbf{m}$ satisfies conditions (i) and (ii) of $\mathbf{M}(C)$ (see (3.1)). Condition (iii) of $\mathbf{M}(C)$ is verified by computing

$$m_1(z)\widetilde{m}_1(z) - m_2(z)\widetilde{m}_2(z) = \frac{m(z) - m(z^{-1})}{z - z^{-1}} \neq 0,$$

hence $\mathbf{m} \in \mathbf{M}(C)$ and $\mathbf{m} \in \mathbf{A}^{\mathrm{inv}}(C)$. Moreover, Lemma 6 yields

$$\begin{cases} 1 + \varphi_{\mathbf{m}}^{(0)}(z) = 1, \\ z^{-1} + \varphi_{\mathbf{m}}^{(-1)}(z) = \phi(z) - m(z) \end{cases} \Longrightarrow m_{\mathbf{m}}(z) = m(z).$$

Recall $D_-^1 = D_- \cap \{|z| > 1\}$ and $D_-^2 = D_- \cap \{|z| < 1\}$.

Lemma 19. *If $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$ satisfies $\mathbf{a} = \bar{\mathbf{a}}$ and*

$$1 + \varphi_{\mathbf{a}}^{(0)}(0) > 0, \quad m_{\mathbf{a}} \in M_{\lambda_0},$$

then it holds that

(i) $\tau_{\mathbf{a}}(z)\tau_{\mathbf{a}}(z^{-1}) = \lim_{\zeta \to \infty} \zeta(\phi(\zeta) - m_{\mathbf{a}}(\zeta)) > 0$,

(ii) $\tau_{\mathbf{a}}(q_x) > 0$ *on* $D_- \cap \mathbb{R}$ *and* $\tau_{\mathbf{a}}(r_\zeta) > 0$ *on* D_-,

(iii) *for any* $x \in D_- \cap \mathbb{R}$ *and* $\zeta \in D_- \setminus \mathbb{R}$,

$$1 + \varphi_{\tilde{\mathbf{a}}}^{(0)}(0) > 0, \quad 1 + \varphi_{q_x\mathbf{a}}^{(0)}(0) > 0, \quad 1 + \varphi_{r_\zeta\mathbf{a}}^{(0)}(0) > 0,$$

and

$$m_{\tilde{\mathbf{a}}}, m_{q_x\mathbf{a}}, m_{r_\zeta\mathbf{a}} \in M_{\lambda_0}.$$

Proof. Note

$$\tau_{\mathbf{a}}(z) = \tau_{\tilde{\mathbf{a}}}(z^{-1}) = 1 + \varphi_{\tilde{\mathbf{a}}}^{(0)}(0) = \lim_{\varepsilon \to 0}(1 + \varepsilon^{-1}\varphi_{\mathbf{a}}^{(-1)}(\varepsilon^{-1})).$$

Equations (6.1) and (6.2) imply

$$\tau_{\mathbf{a}}(z^{-1})(1 + \zeta\varphi_{\mathbf{a}}^{(-1)}(\zeta)) = \zeta(\phi(\zeta) - m_{\mathbf{a}}(\zeta))(1 + \varphi_{\mathbf{a}}^{(0)}(\zeta)),$$

hence

$$\tau_{\mathbf{a}}(z^{-1}) \lim_{\varepsilon \to 0}(1 + \varepsilon^{-1}\varphi_{\mathbf{a}}^{(-1)}(\varepsilon^{-1})) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(\phi(\varepsilon^{-1}) - m_{\mathbf{a}}(\varepsilon^{-1})). \tag{8.3}$$

And the expression of Lemma 18 shows (i).

If $1 + \varphi_{\mathbf{a}}^{(0)}(0) \neq 0$, then $1 + \varphi_{\mathbf{a}}^{(0)}$ and $z + \varphi_{\mathbf{a}}^{(1)}$ cannot vanish simultaneously on D_-, which together with the analyticity of $m_{\mathbf{a}}$, on $\mathbb{C} \setminus \Sigma_{\lambda_0}$ implies $1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) \neq 0$ on D_-. On $\mathbb{R}$, we have

$$1 + \varphi_{\mathbf{a}}^{(0)}(x) \to 1 \quad \text{as } x \to \infty$$

and

$$1 + \varphi_{\mathbf{a}}^{(0)}(0) > 0,$$

hence $\tau_{\mathbf{a}}(q_x) = 1 + \varphi_{\mathbf{a}}^{(0)}(x) > 0$ on $D_- \cap \mathbb{R}$. The property $\tau_{\mathbf{a}}(r_\zeta) > 0$ on $D_- \setminus \mathbb{R}$ follows from Lemma 9 (ii) and $1 + \varphi_{\mathbf{a}}^{(0)}(\zeta) \neq 0$ on D_-.

The equality $1 + \varphi_{\tilde{\mathbf{a}}}^{(0)}(0) = \tau_{\tilde{\mathbf{a}}}(z^{-1}) = \tau_{\mathbf{a}}(z) > 0$ holds, and (6.1) and (6.2) imply

$$m_{\tilde{\mathbf{a}}}(\zeta) = \phi(\zeta) - \frac{\tau_{\mathbf{a}}(z)}{n_{\mathbf{a}}(\zeta^{-1})} = \phi(\zeta) - \frac{\tau_{\mathbf{a}}(z)\tau_{\mathbf{a}}(z^{-1})}{\phi(\zeta) - m_{\mathbf{a}}(\zeta^{-1})}.$$

$\operatorname{Im} m_{\tilde{\mathbf{a}}}(\zeta) > \operatorname{Im} \phi(\zeta) > 0$ on D_-^1 obeys from $\tau_{\mathbf{a}}(z)\tau_{\mathbf{a}}(z^{-1}) > 0$ and (8.1). To see $\operatorname{Im} m_{\tilde{\mathbf{a}}}(\zeta) > 0$ on D_-^2 set $f(\zeta) = \tau_{\mathbf{a}}(z)\tau_{\mathbf{a}}(z^{-1})(\phi(\zeta) - m_{\mathbf{a}}(\zeta^{-1}))^{-1}$. Since ϕ maps $\mathbb{C}_+ \cap \{|z| < 1\}$ onto $\mathbb{C}_-$ conformaly, $f(\phi^{-1}(z))$ satisfies $\operatorname{Im} f(\phi^{-1}(z)) < 0$ on $\mathbb{C}_-$ and

$$\lim_{z \to \infty} z^{-1} f(\phi^{-1}(z)) = -1 \quad \text{due to (8.3)}.$$

Then, $f(\phi^{-1}(z))$ has an representation

$$f(\phi^{-1}(z)) = \alpha + z + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1}\Big)\sigma(d\lambda) \quad \text{on } \mathbb{C}_-$$

with $\alpha \in \mathbb{R}$ and a non-negative measure σ, hence we have $\operatorname{Im} f(\zeta) \leq \operatorname{Im} \phi(\zeta)$ if $\zeta \in D_-^2$. If we have an equality here for some ζ_0, then $f(\zeta) = \alpha + \phi(\zeta)$ identically should hold, which means $m_{\mathbf{a}}$ is a rational function of ϕ on D_-^1. This contradicts (iv) of (8.1), hence one has $m_{\tilde{\mathbf{a}}} \in M_{\lambda_0}$.

Note $m'_{\mathbf{a}}(x) > 0$ on $\mathbb{R} \setminus \Sigma_{\lambda_0}$ due to Lemma 18 (or Hopf lemma), which implies

$$1 + \varphi^{(0)}_{q_x \mathbf{a}}(0) = 1 + \varphi^{(0)}_{\mathbf{a}}(0) \frac{m_{\mathbf{a}}(x) - m_{\mathbf{a}}(0)}{x} > 0.$$

To verify $m_{q_x \mathbf{a}} \in M_{\lambda_0}$ we use (6.3):

$$\begin{cases} m_{q_x \mathbf{a}}(z) = (m_{\mathbf{a}}(0) - m_{\mathbf{a}}(x))\Big(1 - \dfrac{\phi(z) - \phi(x)}{m_{\mathbf{a}}(z) - m_{\mathbf{a}}(x)}\Big) + \phi(z), \\ m_{q_0 \mathbf{a}}(z) = m_{z^{-1}\mathbf{a}}(z) = \dfrac{m'_{\mathbf{a}}(0)}{m_{\mathbf{a}}(0) - m_{\mathbf{a}}(z)} + \phi(z). \end{cases}$$

The property $m_{q_x \mathbf{a}} = \overline{m}_{q_x \mathbf{a}}$ is easy to see, and $\operatorname{Im} m_{\mathbf{a}}(z) > 0$ on $\mathbb{C}_+ \setminus \Sigma_{\lambda_0}$ implies $m_{\mathbf{a}}(z) - m_{\mathbf{a}}(x) \neq 0$, hence $m_{q_x \mathbf{a}}$ is analytic on $\mathbb{C}_\pm \setminus \Sigma_{\lambda_0}$. Note that $m'_{\mathbf{a}}(x) > 0$ on $\mathbb{R} \setminus \Sigma_{\lambda_0}$, which yields $m_{\mathbf{a}}(z) - m_{\mathbf{a}}(x) \neq 0$ for any $z (\neq x) \in \mathbb{R} \setminus \Sigma_{\lambda_0}$. Then, we have the analyticity of $m_{\mathbf{a}}$ on $\mathbb{C} \setminus \Sigma_{\lambda_0}$. Properties (i), (iii), and (iv) of (8.1) can be easily verified. Property (ii) follows from Lemma 22 in Appendix A, hence we have $m_{q_x \mathbf{a}} \in M_{\lambda_0}$.

Note that

$$1 + \varphi^{(0)}_{r_\zeta \mathbf{a}}(x) = \frac{\tau_{\mathbf{a}}(r_\zeta q_x)}{\tau_{\mathbf{a}}(r_\zeta)} = \frac{\tau_{\mathbf{a}}(q_x) \tau_{q_x \mathbf{a}}(r_\zeta)}{\tau_{\mathbf{a}}(r_\zeta)} = \frac{\tau_{\mathbf{a}}(q_x) |\tau_{q_x \mathbf{a}}(q_\zeta)|^2 \operatorname{Im} m_{q_x \mathbf{a}}(\zeta)}{\operatorname{Im} m_{\mathbf{a}}(\zeta)},$$

which is positive, since $\tau_{\mathbf{a}}(q_x) |\tau_{q_x \mathbf{a}}(q_\zeta)|^2 > 0$ and $m_{\mathbf{a}}, m_{q_x \mathbf{a}} \in M_{\lambda_0}$.

Then $1 + \varphi^{(0)}_{r_\zeta \mathbf{a}}(0) > 0$ holds and $m_{r_\zeta \mathbf{a}}$ is analytic on $\mathbb{R} \setminus \Sigma_{\lambda_0}$. The analyticity of $m_{r_\zeta \mathbf{a}}$ on $\mathbb{C}_+ \setminus \Sigma_{\lambda_0}$ follows from $\operatorname{Im} m_{\mathbf{a}}(z) > 0$ by (ii). Properties (i), (iii), and (iv) are easily verified and property (ii) follows from Lemma 22. Then we complete the proof. ■

As in the introduction, for $\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$, we define

$$a_n(\mathbf{a}) = \sqrt{\frac{\tau_{\mathbf{a}}(z^n) \tau_{\mathbf{a}}(z^{n-2})}{\tau_{\mathbf{a}}(z^{n-1})^2}}, \quad b_n(\mathbf{a}) = \partial_\varepsilon \log \frac{\tau_{\mathbf{a}}(z^n e^{\varepsilon z})}{\tau_{\mathbf{a}}(z^{n-1} e^{\varepsilon z})}\Big|_{\varepsilon=0}.$$

Lemma 20. *For $\mathbf{a} \in \mathbf{A}^{\text{inv}}_{++}(C)$ and $\mathbf{a}' \in \mathbf{A}^{\text{inv}}_{++}(C')$, assume $m_{\mathbf{a}} = m_{\mathbf{a}'}$ on $D_- \cap D'_-$. Then one has $m_{g\mathbf{a}} = m_{g\mathbf{a}'}$ for any $g \in \Gamma_{\text{real}}(C) \cap \Gamma_{\text{real}}(C')$ and $a_n(\mathbf{a}) = a_n(\mathbf{a}')$, $b_n(\mathbf{a}) = b_n(\mathbf{a}')$ for any $n \in \mathbb{Z}$.*

Proof. The identity $m_{g\mathbf{a}} = m_{g\mathbf{a}'}$ follows from

$$\begin{aligned} m_{q_\zeta \mathbf{a}} &= (m_{\mathbf{a}}(0) - m_{\mathbf{a}}(\zeta))\Big(1 - \frac{\phi - \phi(\zeta)}{m_{\mathbf{a}} - m_{\mathbf{a}}(\zeta)}\Big) + \phi \\ &= (m_{\mathbf{a}'}(0) - m_{\mathbf{a}'}(\zeta))\Big(1 - \frac{\phi - \phi(\zeta)}{m_{\mathbf{a}'} - m_{\mathbf{a}'}(\zeta)}\Big) + \phi = m_{q_\zeta \mathbf{a}'} \end{aligned}$$

on $D_- \cap D'_-$. The identities for a_n, b_n are obtained by the uniqueness of the correspondence

$$q = \{a_n, b_n\}_{n\in\mathbb{Z}} \iff \{m_\pm, a_0, a_1, b_0\}. \qquad \blacksquare$$

Lemma 19 provides a proof of Theorem 1 as follows.

Proof of Theorem 1. We show $\mathbf{m} \in \mathbf{A}^{\mathrm{inv}}_{++}(C)$. A rational function $r \in \Gamma_{\mathrm{real}}(C)$ can be expressed by $r = r_1 r_2^{-1}$, where r_j are defined by (6.9). Applying Lemma 19 inductively to r_1, we see $\tau_{\mathbf{m}}(r_1) > 0$. Then, one has

$$\tau_{\mathbf{m}}(r_1 r_2^{-1}) = \tau_{\mathbf{m}}(r_1 r_2^{-1}\tilde{r}_2^{-1}\tilde{r}_2) = \tau_{\mathbf{m}}(r_2^{-1}\tilde{r}_2^{-1})\tau_{\mathbf{m}}(r_1\tilde{r}_2) = \tau_{\mathbf{m}}(r_2^{-1}\tilde{r}_2^{-1})\tau_{\widetilde{r_1\mathbf{m}}}(r_2).$$

Since we know $m_{\widetilde{r_1\mathbf{m}}} \in M_{\lambda_0}$, we see $\tau_{\widetilde{r_1\mathbf{m}}}(r_2) > 0$, which shows $\tau_{\mathbf{m}}(r) > 0$. The positivity $\tau_{\mathbf{m}}(z^n)$ is included in the statement $\tau_{\mathbf{m}}(r) > 0$, since $\tau_{\mathbf{a}}(z^{-1}) = \tau_{\mathbf{a}}(q_0)$, $\tau_{\mathbf{a}}(z) = \tau_{\tilde{\mathbf{a}}}(q_0)$ for any $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}(C)$, which proves $\mathbf{m} \in \mathbf{A}^{\mathrm{inv}}_{++}(C)$.

For $q = \{a_n, b_n\}_{n\in\mathbb{Z}} \in Q_{\lambda_0}$, define $\mathbf{m} \in M_{\lambda_0}$ by (8.2), and for $g \in \Gamma_{\mathrm{real}}$ set

$$\mathrm{Toda}(g)q = \{a_n(g\mathbf{m}), b_n(g\mathbf{m})\}_{n\in\mathbb{Z}}.$$

Then, Proposition 5 implies $\mathrm{Toda}(g)q \in Q_{\lambda'_0}$ with $\lambda'_0 = \max\{\pm\lambda_\pm\}$. Since $g\mathbf{m} \in \mathbf{A}^{\mathrm{inv}}_{++}(C)$ for arbitrary C containing Σ_{λ_0} and $a_n(g\mathbf{m}), b_n(g\mathbf{m})$ do not depend on C due to Lemma 20, one has $\mathrm{Toda}(g)q \in Q_{\lambda_0}$. The flow property $\mathrm{Toda}(g_1g_2)q = \mathrm{Toda}(g_1)\mathrm{Toda}(g_2)q$ is deduced by the following fact:

- for $\mathbf{a} \in \mathbf{A}^{\mathrm{inv}}_{++}(C)$,

$$g \in \Gamma_{\mathrm{real}}(C) \implies m_{g\mathbf{a}} = m_{g\mathbf{m_a}}$$

 with

$$\mathbf{m_a}(z) = \left(\frac{zm_{\mathbf{a}}(z) - 1}{z^2 - 1}, z^2\frac{z - m_{\mathbf{a}}(z)}{z^2 - 1}\right) \in \mathbf{A}^{\mathrm{inv}}_{++}(C).$$

This is a conclusion of Lemma 20, since $\mathbf{a}, \mathbf{m_a} \in \mathbf{A}^{\mathrm{inv}}_{++}(C)$ have the same m-function $m_{\mathbf{a}}$. Then, setting $\mathbf{a} = g_2\mathbf{m}$, we have $m_{g_1\mathbf{a}} = m_{g_1\mathbf{m_a}}$, which leads us to

$$\mathrm{Toda}(g_1g_2)q = \mathrm{Toda}(g_1)\mathrm{Toda}(g_2)q.$$

The continuity of the flow is straightforward from Lemma 10. This completes the proof of Theorem 1. $\blacksquare$

A. Appendix

A.1. Weyl functions of Jacobi operators

In this section, we provide necessary information on spectral theory for Jacobi operators. For $q \equiv \{a_n, b_n\}_{n\in\mathbb{Z}}$ with $a_n > 0$, $b_n \in \mathbb{R}$ the associated Jacobi operator is defined by

$$(H_q u)_n = a_{n+1}u_{n+1} + a_n u_{n-1} + b_n u_n$$

on $\ell_0(\mathbb{Z}) = \{f \in \ell^2(\mathbb{Z})\ ;\ f$ has a finite support$\}$ as a symmetric operator in $\ell^2(\mathbb{Z})$. It is possible to define a similar operator on $\ell^2(\mathbb{Z}_{\geq 1})$ by

$$(H_q^+ u)_n = \begin{cases} a_{n+1}u_{n+1} + a_n u_{n-1} + b_n u_n & \text{for } n \geq 2, \\ a_2 u_2 + b_1 u_1 & \text{for } n = 1. \end{cases}$$

There are two cases concerning the self-adjoint extension of H_q^+: the unique case and the non-unique case. The boundary $+\infty$ is called *limit point type* if the extension is unique and limit circle type if the extension is not unique. It is known (see [7]), that the boundary $+\infty$ is limit point type if and only if for any $z \in \mathbb{C} \setminus \mathbb{R}$ (or equivalently for some $z \in \mathbb{C} \setminus \mathbb{R}$) the space $\{u \in \ell^2(\mathbb{Z}_+)\ ;\ H_q^+ u = zu\}$ is one-dimensional, and in this case the Weyl function m_+ for H_q^+ is defined by

$$m_+(z) = -\frac{g_1^+(z)}{a_1 g_0^+(z)} \quad \text{for } g_n^+(z) \in \{u \in \ell^2(\mathbb{Z}_+)\ ;\ H_q^+ u = zu\} \setminus \{0\}.$$

The Weyl function m_- for H_q^- can be defined similarly by

$$m_-(z) = -\frac{g_{-1}^-(z)}{a_0 g_0^-(z)} \text{ for } g_n^-(z) \in \{u \in \ell^2(\mathbb{Z}_-)\ ;\ H_q^- u = zu\} \setminus \{0\},$$

if the boundary $-\infty$ is limit point type. The Weyl functions $m_\pm$ are known to be analytic on $\mathbb{C} \setminus \mathbb{R}$ and satisfy $\operatorname{Im} m_\pm(z)/\operatorname{Im} z > 0$ on $\mathbb{C} \setminus \mathbb{R}$. It is known that if coefficients $q \equiv \{a_n, b_n\}_{n\in\mathbb{Z}}$ are bounded, which is equivalent to the boundedness of H_q, then the boundaries $\pm\infty$ are limit point type. In this paper, we treat only bounded operators H_q. In this case, the Green functions for $H_q^\pm$, H_q are given by

$$(H_q^+ - z)^{-1}(j,k) = \frac{s_j^+(z) g_k^+(z)}{-a_1 g_0^+(z)} \quad \text{if } 1 \leq j \leq k \tag{A.1}$$

with s_j^+ the solutions to $H_q s^+ = z s^+$ satisfying $s_0^+ = 0$, $s_1^+ = 1$. Similarly, one can describe $(H_q^- - z)^{-1}(j,k)$ in terms of s_j^-, g_j^-, which yields

$$m_+(z) = (H_q^+ - z)^{-1}(1,1), \quad m_-(z) = (H_q^- - z)^{-1}(-1,-1).$$

The Green function for H_q is given by

$$(H_q - z)^{-1}(j,k) = \frac{g_j^-(z) g_k^+(z)}{a_1(g_0^-(z) g_1^+(z) - g_1^-(z) g_0^+(z))},$$

and one has

$$\begin{cases}(H_q - z)^{-1}(0,0) = \dfrac{-1}{a_1^2 m_+(z) + a_0^2 m_-(z) + z - b_0},\\ (H_q - z)^{-1}(1,1) = \dfrac{m_+(z)(z - b_0 + a_0^2 m_-(z))}{a_1^2 m_+(z) + a_0^2 m_-(z) + z - b_0}.\end{cases}$$

Generally, the spectrum sp A of a self-adjoint operator coincides with the singular set of the Green operator of A. For H_q, one has

$$\mathrm{sp} H_q = \text{the union of the singular sets of } (H_q - z)^{-1}(0,0), (H_q - z)^{-1}(1,1).$$

If sp $H_q \subset [-\lambda_0, \lambda_0]$ holds, then sp $H_q^{\pm} \subset [-\lambda_0, \lambda_0]$ are also satisfied, since the boundary 0 is absorbing. Hence, in this case, $m_{\pm}(z)$ are analytic on $\mathbb{C} \setminus [-\lambda_0, \lambda_0]$. Moreover, it should hold

$$a_1^2 m_+(z) + a_0^2 m_-(z) + z - b_0 \neq 0 \quad \text{on } \mathbb{C} \setminus [-\lambda_0, \lambda_0].$$

On the other hand, for $a_n > 0$ to hold for any $n \in \mathbb{Z}$, the set sp $H_q^{\pm}$ must be infinite. This can be shown as follows. Let $s_n = s_n(z)$ be a unique solution to

$$a_{n+1}s_{n+1} + a_n s_{n-1} + b_n s_n = z s_n \quad \text{with } s_0 = 0, s_1 = 1.$$

This is well defined, since $a_n \neq 0$ implies

$$s_1(z) = 1, \quad s_2(z) = a_2^{-1}(z - b_1), \quad s_3(z) = a_3^{-1}(a_2^{-1}(z - b_2)(z - b_1) - a_2), \quad \ldots.$$

The Herglotz function $m_+(z)$ has a representation

$$m_+(z) = \int_{-\lambda_0}^{\lambda_0} \frac{\sigma_+(d\lambda)}{\lambda - z},$$

and (A.1) yields

$$\int_{-\lambda_0}^{\lambda_0} s_m(\lambda) s_n(\lambda) \sigma_+(d\lambda) = (\delta_m, \delta_n),$$

namely $\{s_n(z)\}_{n\geq 1}$ forms orthogonal polynomials with respect to σ_+. Suppose that supp σ_+ is a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [-\lambda_0, \lambda_0]$. Then, there exists a polynomial $p(z)$ of degree $n+1$ such that

$$(p,p) = \int_{-\lambda_0}^{\lambda_0} |p(\lambda)|^2 \sigma_+(d\lambda) = 0.$$

However, the polynomials $\{s_n(z)\}_{n\geq 1}$ are linearly independent (due to deg $s_n = n-1$), and hence

$$p(z) = c_1 s_1(z) + c_2 s_2(z) + \cdots + c_n s_n(z) + c_{n+1} s_{n+1}(z)$$

holds with $c_j \in \mathbb{C}$. Then, the orthogonality of $\{s_n(z)\}_{n\geq 1}$ implies

$$c_j = (p, s_j) = 0 \quad \text{for all } j = 1, 2, \ldots, n+1,$$

which is a contradiction. Consequently, supp σ_+ should be infinite.

Summing up the above argument, we have the following result.

Lemma 21. *Suppose a Jacobi operator H_q satisfies* sp $H_q \subset [-\lambda_0, \lambda_0]$. *Then, the Weyl functions $m_\pm$ must satisfy the following conditions:*

- *$m_\pm(z)$ are analytic on $\mathbb{C} \setminus \Sigma_{\lambda_0}$ and*

$$a_1^2 m_+(z) + a_0^2 m_-(z) + z - b_0 \neq 0 \quad \text{on } \mathbb{C} \setminus [-\lambda_0, \lambda_0];$$

- *$m_\pm(z)$ are not rational functions.*

A.2. Transformation of Herglotz functions

Set $\phi(z) = z + z^{-1}$ and, in view of Lemma 14, define

$$d_\zeta m(z) = \phi(z) - (m(\zeta) - m(0))\Big(1 - \frac{\phi(z) - \phi(\zeta)}{m(z) - m(\zeta)}\Big).$$

Lemma 22. *Let $m \in M_{\lambda_0}$. Then, one has for $x \in \mathbb{R} \setminus \Sigma_{\lambda_0}$ and $\zeta \in \mathbb{C}_+ \setminus \Sigma_{\lambda_0}$,*

$$\operatorname{Im} d_x m(z) > 0, \quad \operatorname{Im} d_{\bar\zeta} d_\zeta m(z) > 0 \quad \text{on } \mathbb{C}_+ \setminus \Sigma_{\lambda_0}.$$

Proof. Note

$$d_{\bar\zeta} d_\zeta m(z) = (d_\zeta m(0) - d_\zeta m(\bar\zeta))\Big(1 - \frac{\phi(z) - \phi(\bar\zeta)}{d_\zeta m(z) - d_\zeta m(\bar\zeta)}\Big) + \phi(z). \tag{A.2}$$

Observe

$$d_\zeta m(z) = (m(0) - m(\zeta))\Big(1 + \frac{\phi(\zeta)}{m(z) - m(\zeta)}\Big) + z\phi(z)\frac{z^{-1}(m(z) - m(0))}{m(z) - m(\zeta)}, \tag{A.3}$$

and

$$(d_\zeta m)(0) = m(0) - m(\zeta) + \phi(\zeta) + \frac{m'(0)}{m(0) - m(\zeta)}.$$

Hence

$$d_\zeta m(0) - d_\zeta m(\bar\zeta) = \frac{\phi(\bar\zeta) - \phi(\zeta)}{m(\bar\zeta) - m(\zeta)}(m(0) - m(\bar\zeta)) + \frac{m'(0)}{m(0) - m(\zeta)} \tag{A.4}$$

holds. For notational simplicity, set

$$X = \frac{m(z) - m(\bar\zeta)}{\phi(z) - \phi(\bar\zeta)}, \quad Y = \frac{m(\bar\zeta) - m(\zeta)}{\phi(\bar\zeta) - \phi(\zeta)}.$$

Then, one has

$$\frac{\phi(z) - \phi(\zeta)}{m(z) - m(\zeta)} = \frac{\phi(z) - \phi(\zeta)}{(\phi(z) - \phi(\bar\zeta))X + m(\bar\zeta) - m(\zeta)},$$

and

$$\begin{aligned} d_\zeta m(z) - d_\zeta m(\bar\zeta) =& (m(0) - m(\zeta))\Big(1 - \frac{\phi(z) - \phi(\zeta)}{m(z) - m(\zeta)}\Big) + \phi(z) \\ & - (m(0) - m(\zeta))\Big(1 - \frac{\phi(\bar\zeta) - \phi(\zeta)}{m(\bar\zeta) - m(\zeta)}\Big) - \phi(\bar\zeta) \\ =& \Big(\frac{(m(0) - m(\zeta))(X - Y)}{Y((\phi(z) - \phi(\bar\zeta))X + (\phi(\bar\zeta) - \phi(\zeta))Y)} + 1\Big)(\phi(z) - \phi(\bar\zeta)). \end{aligned}$$

From (A.2) and (A.4),

$$\begin{aligned} & d_{\bar\zeta} d_\zeta m(z) - \phi(z) \\ &= \Big(Y^{-1}(m(0) - m(\bar\zeta)) + \frac{m'(0)}{m(0) - m(\zeta)}\Big)\Big(1 - \frac{\phi(z) - \phi(\bar\zeta)}{d_\zeta m(z) - d_\zeta m(\bar\zeta)}\Big) \\ &= \frac{Y^{-1}|m(0) - m(\zeta)|^2 + m'(0)}{m(0) - m(\zeta) + Y\dfrac{(\phi(z) - \phi(\bar\zeta))X + (\phi(\bar\zeta) - \phi(\zeta))Y}{X - Y}} \end{aligned}$$

follows. Now, we compute the imaginary part of

$$\begin{aligned} A &\equiv -m(\zeta) + Y\frac{(\phi(z) - \phi(\bar\zeta))X + (\phi(\bar\zeta) - \phi(\zeta))Y}{X - Y} \\ &= -m(\zeta) - Y\overline{\phi(\zeta)} + Y\frac{\phi(z)X - \phi(\zeta)Y}{X - Y}. \end{aligned}$$

Note that $Y \in \mathbb{R}$ and $\operatorname{Im} m(\zeta) = Y \operatorname{Im}\phi(\zeta)$. Then,

$$\begin{aligned} \operatorname{Im} A &= Y \operatorname{Im}\frac{\phi(z)X - \phi(\zeta)Y}{X - Y} \\ &= Y\frac{\operatorname{Im}(\phi(z)X - \phi(\zeta)Y)(\bar X - Y)}{|X - Y|^2} \\ &= Y\frac{|X|^2 \operatorname{Im}\phi(z) - Y \operatorname{Im} m(z)}{|X - Y|^2} \end{aligned}$$

holds. Therefore, we have

$$\operatorname{Im}(d_{\bar{\zeta}}d_{\zeta}m(z)-\phi(z)) = \operatorname{Im}\frac{Y^{-1}|m(0)-m(\zeta)|^2+m'(0)}{m(0)+A} = \frac{|m(0)-m(\zeta)|^2+m'(0)Y}{|m(0)+A|^2}\frac{Y\operatorname{Im}m(z)-|X|^2\operatorname{Im}\phi(z)}{|X-Y|^2}. \tag{A.5}$$

Without loss of generality, one can assume $\operatorname{Im}\zeta > 0$. Hence, $\zeta, z \in \mathbb{C}_+ \setminus \{|z| = 1\}$, and this is divided by two connected domains:

$$\mathbb{C}_+ \setminus \{|z|=1\} = (\mathbb{C}_+ \cap \{|z|>1\}) \cup (\mathbb{C}_+ \cap \{|z|<1\}).$$

The situation is different depending on the domain where ζ, z belong. If they are elements of the same domain, they can be expressed by a single m_+ or m_-, but if they belong to distinct domains, they have expressions by distinct $m_\pm$.

Case 1. *$\zeta, z \in \mathbb{C}_+ \cap \{|z| > 1\}$ or $\zeta, z \in \mathbb{C}_+ \cap \{|z| < 1\}$.* We consider the case $\zeta, z \in \mathbb{C}_+ \cap \{|z| > 1\}$. The other case can be treated similarly. Setting $\phi(z) = w$, $\phi(\zeta) = b$, one has from Lemma 18,

$$X = \frac{m(\phi^{-1}(w)) - \overline{m(\phi^{-1}(b))}}{w-\bar{b}} = 1 + a_1^2\int_{-\lambda_0}^{\lambda_0}\frac{\sigma_+(d\lambda)}{(\lambda-w)(\lambda-\bar{b})}.$$

Then, the Cauchy–Schwarz inequality shows

$$|X|^2 = \left|1 + a_1^2\int_{-\lambda_0}^{\lambda_0}\frac{\sigma_+(d\lambda)}{(\lambda-w)(\lambda-\bar{b})}\right|^2 \le \left(1+\int_{-\lambda_0}^{\lambda_0}\frac{a_1^2\sigma_+(d\lambda)}{|\lambda-w|^2}\right)\left(1+\int_{-\lambda_0}^{\lambda_0}\frac{a_1^2\sigma_+(d\lambda)}{|\lambda-b|^2}\right) = Y\frac{\operatorname{Im}m(z)}{\operatorname{Im}\phi(z)},$$

hence, noting

$$m'(0) = a_0^2\int_{-\lambda_0}^{\lambda_0}\sigma_-(d\lambda) > 0,$$

one has for $z \in \mathbb{C}_+ \cap \{|z| > 1\}$,

$$\operatorname{Im} d_{\bar{\zeta}}d_{\zeta}m(z) \ge \operatorname{Im}\phi(z) > 0.$$

Case 2. $\zeta \in \mathbb{C}_+ \cap \{|z| < 1\},\ z \in \mathbb{C}_+ \cap \{|z| > 1\}$. In this case, $\operatorname{Im}\phi(\zeta) < 0$, $\operatorname{Im} m(\zeta)$, $\operatorname{Im}\phi(z)$, and $\operatorname{Im} m(z) > 0$ hold. Therefore, one sees

$$|m(\zeta) - m(0)|^2 = \left| \int_{-\lambda_0}^{\lambda_0} \frac{a_0^2 \sigma_-(d\lambda)}{\lambda - \phi(\zeta)} \right|^2 \le m'(0) \int_{-\lambda_0}^{\lambda_0} \frac{a_0^2 \sigma_-(d\lambda)}{|\lambda - \phi(\zeta)|^2} = m'(0) \frac{\operatorname{Im} m(\zeta)}{-\operatorname{Im}\phi(\zeta)},$$

which implies

$$|m(0) - m(\zeta)|^2 + m'(0) Y \le 0,$$

hence (A.5) shows

$$\operatorname{Im}(d_{\bar\zeta} d_\zeta m(z) - \phi(z)) \ge 0.$$

Now

$$\operatorname{Im}(d_{\bar\zeta} d_\zeta m(z)) = \operatorname{Im}(d_{\bar\zeta} d_\zeta m(z) - \phi(z)) + \operatorname{Im}\phi(z) > 0$$

holds, if $z \in \mathbb{C}_+ \cap \{|z| > 1\}$.

Case 3. $\zeta \in \mathbb{C}_+ \cap \{|z| > 1\},\ z \in \mathbb{C}_+ \cap \{|z| < 1\}$. In this case $\operatorname{Im}\phi(z) < 0$, $\operatorname{Im}\phi(\zeta)$, $\operatorname{Im} m(\zeta)$, $\operatorname{Im} m(z) > 0$ hold. Therefore, $Y > 0$ and (A.5) show

$$\operatorname{Im}(d_{\bar\zeta} d_\zeta m(z) - \phi(z)) \ge 0,$$

which implies

$$-((d_{\bar\zeta} d_\zeta m)(\phi^{-1}(w)) - w) = \alpha + \beta w + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - w} - \frac{\lambda}{\lambda^2 + 1} \Big) \sigma(d\lambda)$$

with $w = \phi(z) \in \mathbb{C}_-$. Since (A.3) implies $d_\zeta m$ is analytic in a neighbourhood of 0, so does $d_{\bar\zeta} d_\zeta m$, which means $d_{\bar\zeta} d_\zeta m(0)$ exists finitely. Therefore, one has

$$(d_{\bar\zeta} d_\zeta m)(\phi^{-1}(w)) = O(1) \quad \text{as } w \to \infty \text{ in } \mathbb{C}_-,$$

and hence $\beta = 1$, which shows

$$-(d_{\bar\zeta} d_\zeta m)(z) = \alpha + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - \phi(z)} - \frac{\lambda}{\lambda^2 + 1} \Big) \sigma(d\lambda).$$

This proves $\operatorname{Im} d_{\bar\zeta} d_\zeta m(z) > 0$ in any case.

The proof of $\operatorname{Im} d_x m(z) > 0$ proceeds similarly to the above. Here we present the proof only for two cases. Suppose that $x \in \mathbb{R} \cap \{|z| > 1\}$ and $z \in \mathbb{C}_+ \cap \{|z| > 1\}$. The definition of $d_x m$ implies

$$\operatorname{Im}(d_x m(z) - \phi(z)) = (m(x) - m(0)) \operatorname{Im}\Big(\frac{m(z) - m(x)}{\phi(z) - \phi(x)} \Big)^{-1}, \tag{A.6}$$

hence the imaginary part is

$$\begin{aligned}\operatorname{Im}\frac{m(z)-m(x)}{\phi(z)-\phi(x)} &= \operatorname{Im}\Big(1+a_1^2\int_{-\lambda_0}^{\lambda_0}\frac{\sigma_+(d\lambda)}{(\lambda-\phi(z))(\lambda-\phi(x))}\Big)\\ &= a_1^2\operatorname{Im}\phi(z)\int_{-\lambda_0}^{\lambda_0}\frac{\sigma_+(d\lambda)}{|\lambda-\phi(z)|^2(\lambda-\phi(x))}.\end{aligned}$$

If $x>0$, then $\lambda-\phi(x)<0$ and

$$m(x)-m(0)=m(x)-m(x^{-1})+m(x^{-1})-m(0)>0,$$

due to $m(x^{-1})-m(0)>0$ ($m'(x)>0$), $m(x)-m(x^{-1})>0$. Therefore, we have $\operatorname{Im}(d_x m(z)-\phi(z))>0$, which yields $\operatorname{Im} d_x m(z)>0$, since $\operatorname{Im}\phi(z)>0$. The case $x<0$ can be treated similarly.

Suppose that $x\in\mathbb{R}\cap\{|z|<1\}$ and $z\in\mathbb{C}_+\cap\{|z|>1\}$. Then, in the expression (A.6), we decompose

$$\begin{aligned}\frac{m(z)-m(x)}{\phi(z)-\phi(x)} &= \frac{m(z)-m(x^{-1})}{\phi(z)-\phi(x^{-1})}+\frac{m(x^{-1})-m(x)}{\phi(z)-\phi(x)}\\ &= 1+a_1^2\int_{-\lambda_0}^{\lambda_0}\frac{\sigma_+(d\lambda)}{(\lambda-\phi(z))(\lambda-\phi(x^{-1}))}+\frac{m(x^{-1})-m(x)}{\phi(z)-\phi(x)},\end{aligned}$$

which yields

$$\operatorname{Im}\frac{m(z)-m(x)}{\phi(z)-\phi(x)}<0 \quad \text{if } x>0,$$

and $\operatorname{Im} d_x m(z)>0$. The other cases can be computed similarly and one can show $\operatorname{Im} d_x m(z)>0$ in any case. ■

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