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Optimal Runge approximation for damped nonlocal wave equations and simultaneous determination results

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Abstract. The main purpose of this article is to establish new uniqueness results for Calderón-type inverse problems related to damped nonlocal wave equations. To achieve this goal, we extend the theory of very weak solutions to our setting, which allows to deduce an optimal Runge approximation theorem. With this result at our disposal, we can prove simultaneous determination results in the linear and semilinear regime.

1. Introduction

In recent years, inverse problems for nonlocal partial differential equations (PDEs) of elliptic, parabolic and hyperbolic type have been studied. This line of research was initiated by Ghosh, Salo, and Uhlmann [11], in which they have considered the (partial data) Calderón problem related to the *fractional Schrödinger equation*

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \Omega_e, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$, $0 < s < 1$, q is a suitable potential and $(-\Delta)^s$ is the *fractional Laplacian* which is the operator with Fourier symbol $|\xi|^{2s}$. In this problem, one asks whether the knowledge of the *(partial) Dirichlet to Neumann (DN) map*

$$\Lambda_q \varphi = (-\Delta)^s u_\varphi|_{W_2}, \quad \varphi \in C_c^\infty(W_1),$$

where $W_1, W_2 \subset \Omega_e$ are given measurement sets (i.e., nonempty open sets) and u_φ denotes the unique solution to (1.1), uniquely determines the potential q. The overall strategy to establish unique determination results for the above Calderón problem is as follows (see [11, 23, 25]).

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(i) *Integral identity.* Assume that the potentials q_j are suitably regular, then one can write

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})\varphi_1, \varphi_2 \rangle = \int_\Omega (q_1 - q_2)(u_{\varphi_1} - \varphi_1)(u_{\varphi_2} - \varphi_2)\, dx,$$

when the right-hand side is interpreted accordingly.

(ii) Establish one of the following *Runge approximation theorems*:

(I) $\mathcal{R}_W = \{u_f|_\Omega\,;\, f \in C_c^\infty(W)\}$ is dense in $L^2(\Omega)$ (see [11] for $q \in L^\infty(\Omega)$);

(II) $\mathcal{R}_W = \{u_f - f\,;\, f \in C_c^\infty(W)\}$ is dense in $\widetilde{H}^s(\Omega)$ (see [25] for Sobolev multipliers q or [23] for local, bounded bilinear forms).

(iii) If the potentials q_j for $j = 1, 2$ have suitable continuity properties, then $\Lambda_{q_1} = \Lambda_{q_2}$ together with (ii) ensure that there holds $q_1 = q_2$ in Ω.

In (II), the space $\widetilde{H}^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the energy space

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \|u\|_{H^s(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)} < \infty\}.$$

Here, $\langle D\rangle^s$ stands for the *Bessel potential operator* of order s, that is the Fourier multiplier with symbol $\langle\xi\rangle^s$, where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. Observe the similarity of the above strategy to the one of [26] for showing unique determination for the classical Calderón problem, where instead of the Runge approximation theorem suitable geometric optics solutions are used. Moreover, the Runge approximation (ii) relies on a Hahn–Banach argument and the *unique continuation property* (UCP) of the fractional Laplacian $(-\Delta)^s$. For more results on Calderón problems for elliptic nonlocal PDEs, we refer the interested reader to [1–4, 6–8, 10, 12–15, 15–19, 21–24] and the references therein.

1.1. Mathematical model and main results

Recently, the above approach for solving elliptic nonlocal inverse problems has also been adapted to deduce uniqueness results for the Calderón problem of nonlocal hyperbolic equations. Let us next describe some of these results in more detail and for this purpose consider the problem

$$\begin{cases} \partial_t^2 u + \lambda(-\Delta)^s \partial_t u + (-\Delta)^s u + f(u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = 0,\ \partial_t u(0) = 0 & \text{on } \Omega, \end{cases} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a possibly nonlinear (Carathéodory) function and $f(u)$ denotes the corresponding Nemytskii operator, which is defined by $f(u)(x,t) =$

$f(x,u(x,t))$ for measurable functions $u\colon \Omega_T \to \mathbb{R}$. If the problem (1.2) is well posed in the energy class $H^s(\mathbb{R}^n)$, then for any two given measurement sets $W_1, W_2 \subset \Omega_e$, we may introduce the DN map Λ_f^λ via

$$\Lambda_f^\lambda \varphi = (\lambda(-\Delta)^s \partial_t u_\varphi + (-\Delta)^s u_\varphi)|_{(W_2)_T},$$

whenever φ is supported in $(W_1)_T$ and u_φ is the solution to (1.2). The *Calderón problem* for (1.2) reads as follows.

Question 1. *Does the DN map* Λ_f^λ *uniquely determine the function* f*?*

A suitable class of nonlinearities are the so-called *weak nonlinearities,* which are defined next.

Definition 1.1. We call a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *weak nonlinearity*, if it satisfies the following conditions.

(i) f has partial derivative $\partial_\tau f$, which is a Carathéodory function, and there exists $a \in L^p(\Omega)$ such that

$$|\partial_\tau f(x,\tau)| \lesssim a(x) + |\tau|^r$$

for all $\tau \in \mathbb{R}$ and a.e. $x \in \Omega$. Here the exponents p and r satisfy the restrictions

$$\begin{cases} n/s \leq p \leq \infty, & \text{if } 2s < n, \\ 2 < p \leq \infty, & \text{if } 2s = n, \\ 2 \leq p \leq \infty, & \text{if } 2s \geq n, \end{cases} \tag{1.3}$$

and

$$\begin{cases} 0 \leq r < \infty, & \text{if } 2s \geq n, \\ 0 \leq r \leq \frac{2s}{n-2s}, & \text{if } 2s < n, \end{cases} \tag{1.4}$$

respectively. Moreover, f fulfills the integrability condition $f(\cdot,0) \in L^2(\Omega)$.

(ii) The function $\mathfrak{F}\colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined via

$$\mathfrak{F}(x,\tau) = \int_0^\tau f(x,\rho)\,d\rho,$$

satisfies $\mathfrak{F}(x,\tau) \geq 0$ for all $\tau \in \mathbb{R}$ and a.e. $x \in \Omega$.

Remark 1.2. We note that fractional power-type nonlinearities of the form

$$f(x,\tau) = q(x)|\tau|^r \tau,$$

with $q \in L^\infty(\Omega)$, $q \geq 0$, and $r \geq 0$ satisfying (1.4), comply with the assumptions in Definition 1.1; see also [19, Remark 3.5] for further discussion.

Using the above notions, we can now discuss some of the existing results.

(a) The article [13] gives a positive answer for $\lambda = 0$, $f(x,\tau) = q(x)\tau$ with $q \in L^\infty(\Omega)$. Their proof relied on the observation that in this case (1.2) satisfies an $L^2(\Omega_T)$ Runge approximation theorem.

(b) The work [27] deals on the one hand with the linear case $\lambda = 1$, $f(x,\tau) = q(x)\tau$ with $q \in L^\infty(0,T;L^p(\Omega))$, where p satisfies the restrictions (1.3), and q is weakly continuous in t and, on the other hand, with the nonlinear case $\lambda = 1$ and f is a $r+1$ homogeneous, weak nonlinearity. The uniqueness proofs use substantially that due to presence of the viscosity term $(-\Delta)^s\partial_t$ solutions u to (1.2) satisfy $\partial_t u \in L^2(0,T;H^s(\mathbb{R}^n))$ and as a consequence the linearized equations have the Runge approximation property in $L^2(0,T;\widetilde{H}^s(\Omega))$.

(c) In [19], uniqueness is proved in the case $\lambda = 0$ and f satisfies the same properties as in (b), but with the additional restriction $r \leq 1$. This article only uses an $L^2(\Omega_T)$ Runge approximation result for the linearized nonlocal wave equation.

(d) By establishing a theory for very weak solutions to linear nonlocal wave equations with $\lambda = 0$, the authors of [20] could deduce a $L^2(0,T;\widetilde{H}^s(\Omega))$ Runge approximation theorem for these wave equations. This allowed to extend the results in (c) to the cases $r > 1$ and additionally showed that one can recover any linear perturbation $q \in L^p(\Omega)$ with p satisfying the restrictions (1.3). Furthermore, one can also recover serially or asymptotically polyhomogeneous nonlinearities ([20, Theorem 1.5]).

In this context, let us also mention the recent article [9] which deals with the Calderón problem for a third order semilinear, nonlocal, viscous wave equation.

The goal of this paper is to present an extension of the models described in (b) and (c), which we discuss next. Let $0 < s < 1$ and suppose that we have given coefficients $(\gamma, q) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$, where $0 < s < \alpha \leq 1$ and $1 \leq p \leq \infty$ satisfies the restrictions in (1.3). Then we define the following *damped, nonlocal wave operator*

$$L_\gamma := \partial_t^2 + \gamma\partial_t + (-\Delta)^s \tag{1.5}$$

and consider the problem

$$\begin{cases} L_\gamma u + f(u) = F & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{1.6}$$

where f is a weak nonlinearity or $f(u) = q(x)u$. In fact, this is a possibly nonlinear generalization of the model (G2) with $s_1 = 0$ in [27, Section 1.1]. By [19, Proposition 3.7] (see Section 2.1 for the linear case), we know that the problem (1.6) is well posed, whenever the source F, exterior condition φ and initial conditions u_0, u_1 are sufficiently regular. Thus, we can introduce the related (partial) DN map via

$$\Lambda_{\gamma,f}\varphi := (-\Delta)^s u_\varphi|_{(W_2)_T},$$

where $W_1, W_2 \subset \Omega_e$ are some measurement sets, φ is supported in $(W_1)_T$ and u_φ is the unique solution to (1.6) with $u_0 = u_1 = F = 0$. Then we ask the following question.

Question 2. *Does the partial DN map $\Lambda_{\gamma,f}$ uniquely determine the damping coefficient γ and the function f?*

In this work we establish the following affirmative answers to this question.

Theorem 1.3 (Uniqueness for linear perturbations). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$ and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that for $j = 1, 2$ we have given coefficients $(\gamma_j, q_j) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$ and let Λ_{γ_j,q_j} be the DN map associated to the problem*

$$\begin{cases} (L_{\gamma_j} + q_j)u = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = 0,\ \partial_t u(0) = 0 & \text{on } \Omega, \end{cases}$$

for $j = 1, 2$. If $W_1, W_2 \subset \Omega_e$ are two measurement sets such that

$$\Lambda_{\gamma_1,q_1}\varphi|_{(W_2)_T} = \Lambda_{\gamma_2,q_2}\varphi|_{(W_2)_T} \tag{1.7}$$

for all $\varphi \in C_c^\infty((W_1)_T)$, then there holds

$$\gamma_1 = \gamma_2 \quad \text{and} \quad q_1 = q_2 \quad \text{in } \Omega.$$

Theorem 1.4 (Uniqueness for semilinear perturbations). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $0 < s < \alpha \leq 1$. Assume that for $j = 1, 2$ we have given coefficients $\gamma_j \in C^{0,\alpha}(\mathbb{R}^n)$ and $r + 1$ homogeneous, weak nonlinearities f_j, where $r > 0$ satisfies (1.4). Let Λ_{γ_j,f_j} be the DN map associated to the problem*

$$\begin{cases} L_{\gamma_j}u + f_j(u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = 0,\ \partial_t u(0) = 0 & \text{on } \Omega, \end{cases}$$

for $j = 1, 2$. If $W_1, W_2 \subset \Omega_e$ are two measurement sets such that

$$\Lambda_{\gamma_1,f_1}\varphi|_{(W_2)_T} = \Lambda_{\gamma_2,f_2}\varphi|_{(W_2)_T} \tag{1.8}$$

for all $\varphi \in C_c^\infty((W_1)_T)$*, then there holds*

$$\gamma_1 = \gamma_2 \quad \text{in } \Omega \quad \text{and} \quad f_1 = f_2 \quad \text{in } \Omega \times \mathbb{R}.$$

Remark 1.5. For simplicity, we restrict our attention to homogeneous nonlinearities f, but the unique determination remains valid in some polyhomogeneous cases as described in [20] for $\gamma = 0$. Additionally, [9] discusses the Calderón problem for third order nonlocal wave equations with polyhomogeneous nonlinearities of finite order, special subclasses of nonlinearities $f(x,t,u)$ without a homogeneity structure and nonlinearities of Kuznetsov type.

2. Weak and very weak solutions to damped, nonlocal wave equations

The main purpose of this section is to show existence of unique weak and very weak solutions to *damped, nonlocal wave equations* (DNWEQ)

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{2.1}$$

where L_γ is given by (1.5) and only the case $\varphi = 0$ is considered for very weak solutions.

2.1. Weak solutions

This section deals with the well-posedness of (2.1) for regular sources, exterior conditions and initial data. We also prove well-posedness for the case when instead of initial values the values at $t = T$ are specified, which will be needed for the development of the theory of very weak solutions.

Theorem 2.1 (Weak solutions to homogeneous DNWEQ). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain,* $T > 0$*,* $0 < s < 1$*, and suppose that* $1 \leq p \leq \infty$ *satisfies* (1.3)*. Assume that we have given coefficients* $(\gamma, q) \in L^\infty(\Omega) \times L^p(\Omega)$*. Then for any source*[1] $F \in L^2(0,T;\widetilde{L}^2(\Omega))$ *and initial conditions* $(u_0,u_1) \in \widetilde{H}^s(\Omega) \times \widetilde{L}^2(\Omega)$*, there exists a unique weak solution* $u \in C([0,T];\widetilde{H}^s(\Omega)) \cap C^1([0,T];\widetilde{L}^2(\Omega))$ *of*

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{2.2}$$

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[1] Here and below we set $\widetilde{L}^2(\Omega) := \widetilde{H}^0(\Omega)$.

which means that $(u(0), \partial_t u(0)) = (u_0, u_1)$ *in* $\widetilde{H}^s(\Omega) \times \widetilde{L}^2(\Omega)$ *and there holds*

$$\begin{aligned}
&\frac{d}{dt}\langle \partial_t u, v\rangle_{L^2(\Omega)} + \langle \gamma \partial_t u, v\rangle_{L^2(\Omega)} \\
&+ \langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v\rangle_{L^2(\mathbb{R}^n)} + \langle qu, v\rangle_{L^2(\Omega)} \\
&\quad = \langle F, v\rangle_{L^2(\Omega)}
\end{aligned} \tag{2.3}$$

for all $v \in \widetilde{H}^s(\Omega)$ *in the sense of distributions on* $(0, T)$. *Moreover, the unique solution* u *obeys the energy identity*

$$\begin{aligned}
&\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2} u(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \langle \gamma \partial_t u(\tau) + qu(\tau), \partial_t u(\tau)\rangle_{L^2(\Omega)}\, d\tau \\
&= \|(-\Delta)^{s/2} u_0\|^2_{L^2(\mathbb{R}^n)} + \|u_1\|^2_{L^2(\Omega)} + 2\int_0^t \langle F(\tau), \partial_t u(\tau)\rangle_{L^2(\Omega)}\, d\tau
\end{aligned} \tag{2.4}$$

which implies

$$\begin{aligned}
&\|\partial_t u(t)\|_{L^2(\Omega)} + \|(-\Delta)^{s/2} u(t)\|_{L^2(\mathbb{R}^n)} \\
&\quad \lesssim \|u_1\|_{L^2(\Omega)} + \|(-\Delta)^{s/2} u_0\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^2(0,t;L^2(\Omega))}
\end{aligned} \tag{2.5}$$

for all $0 \leq t \leq T$.

Proof. Throughout the proof, we endow $\widetilde{H}^s(\Omega)$ with the equivalent norm

$$\|u\|_{\widetilde{H}^s(\Omega)} = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}$$

(see [19, Lemma 2.3]) and we introduce the following continuous sesquilinear forms

$$a_0(u, v) = \langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v\rangle_{L^2(\mathbb{R}^n)}, \quad a_1(u, v) = \langle qu, v\rangle_{L^2(\Omega)}$$

for $u, v \in \widetilde{H}^s(\Omega)$ and $b(u, v) = \langle \gamma u, v\rangle_{L^2(\Omega)}$ for $u, v \in \widetilde{L}^2(\Omega)$. Next, recall that by [19, eq. (3.7)] one has

$$\|qu\|_{L^2(\Omega)} \lesssim \|q\|_{L^p(\Omega)} \|u\|_{\widetilde{H}^s(\Omega)} \tag{2.6}$$

for all $u \in \widetilde{H}^s(\Omega)$. It is not hard to see that we can invoke the existence and uniqueness results [5, Chapter XVIII, Section 5, Theorem 3 & 4] (see [5, p. 571]), which ensure the existence of a unique, real-valued solution $u \in C([0, T]; \widetilde{H}^s(\Omega)) \cap C^1([0, T]; \widetilde{L}^2(\Omega))$ to (2.2). Furthermore, by [5, Chapter XVIII, Section 5, Lemma 7] the solution

u satisfies the following energy identity:

$$\begin{aligned}&\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}u(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \langle \gamma\partial_t u(\tau) + qu(\tau), \partial_t u(\tau)\rangle_{L^2(\Omega)}\,d\tau \\ &\quad = \|(-\Delta)^{s/2}u_0\|^2_{L^2(\mathbb{R}^n)} + \|u_1\|^2_{L^2(\Omega)} + 2\int_0^t \langle F(\tau), \partial_t u(\tau)\rangle_{L^2(\Omega)}\,d\tau \qquad (2.7)\end{aligned}$$

for $0 \leq t \leq T$. Hence, we have shown the identity (2.4). If we set

$$\Psi(t) := \|\partial_t u(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}u(t)\|^2_{L^2(\mathbb{R}^n)} \in C([0,T]),$$

then using (2.6), (2.7), and $\gamma \in L^\infty(\Omega)$, we get

$$\Psi(t) \leq \Psi(0) + \int_0^t \|F(\tau)\|^2_{L^2(\Omega)}\,d\tau + C\int_0^t (1 + \|\gamma\|_{L^\infty(\Omega)} + \|q\|^2_{L^p(\Omega)})\Psi(\tau)\,d\tau$$

and via Gronwall's inequality we deduce the energy estimate

$$\Psi(t) \lesssim \Psi(0) + \|F\|^2_{L^2(0,t;L^2(\Omega))}$$

for all $0 \leq t \leq T$, which establishes the estimate (2.5). ■

As a consequence we have the following result.

Proposition 2.2 (Weak solutions to inhomogeneous DNWEQ). *Let $\Omega \subset \mathbb{R}^n$ be a bound-ed Lipschitz domain, $T > 0$, $0 < s < 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma, q) \in L^\infty(\Omega) \times L^p(\Omega)$. Then for any source $F \in L^2(0,T;\widetilde{L}^2(\Omega))$, exterior condition $\varphi \in C^2([0,T];H^{2s}(\mathbb{R}^n))$ and initial conditions $(u_0,u_1) \in H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying the compatibility conditions $u_0 - \varphi(0) \in \widetilde{H}^s(\Omega)$ and $u_1 - \partial_t\varphi(0) \in \widetilde{L}^2(\Omega)$, there exists a unique weak solution $u \in C([0,T];H^s(\mathbb{R}^n)) \cap C^1([0,T];L^2(\mathbb{R}^n))$ of*

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \qquad (2.8)$$

which means that u satisfies (2.3), the $(u(0),\partial_t u(0)) = (u_0,u_1)$ in $H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $u = \varphi$ in $(\Omega_e)_T$ means that $u(t) = \varphi(t)$ a.e. in Ω_e for any $0 < t < T$. Furthermore, the following energy estimate holds:

$$\begin{aligned}&\|\partial_t u(t)\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2}u(t)\|_{L^2(\mathbb{R}^n)} \\ &\quad \lesssim \|u_1\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2}u_0\|_{L^2(\mathbb{R}^n)} + \|\varphi\|_{C^2([0,t];H^{2s}(\mathbb{R}^n))} + \|F\|_{L^2(0,t;L^2(\Omega))}\end{aligned}$$

for any $0 \leq t \leq T$.

Proof. Observe, under the current regularity assumptions and compatibility conditions, that u solves (2.8) if and only if $w := u - \varphi$ solves

$$\begin{cases} (L_\gamma + q)w = F - (L_\gamma + q)\varphi & \text{in } \Omega_T, \\ w = 0 & \text{on } (\Omega_e)_T, \\ w(0) = u_0 - \varphi(0),\ \partial_t w(0) = u_1 - \partial_t\varphi(0) & \text{on } \Omega. \end{cases} \tag{2.9}$$

The only fact to keep in mind is that if $u \in C([0,T]; H^s(\mathbb{R}^n))$, then the condition $u(t) = \varphi(t)$ a.e. in Ω_e is equivalent to $u(t) - \varphi(t) \in \widetilde{H}^s(\Omega)$ as $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. So, the assertions of Propsition 2.2 follow immediately from Theorem 2.1. ■

Denoting by $g^\star(x,t) = g(x, T-t)$ the *time-reversal* of $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n_T)$, we have the following.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma, q) \in L^\infty(\Omega) \times L^p(\Omega)$. Let $F \in L^2(0,T; \widetilde{L}^2(\Omega))$, $\varphi \in C^2([0,T]; H^{2s}(\mathbb{R}^n))$ and $(u_0, u_1) \in H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying the compatibility conditions $u_0 - \varphi(0) \in \widetilde{H}^s(\Omega)$ and $u_1 - \partial_t\varphi(0) \in \widetilde{L}^2(\Omega)$. Then u solves*

$$\begin{cases} (L_\gamma + q)u = F & \textit{in } \Omega_T, \\ u = \varphi & \textit{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \textit{on } \Omega, \end{cases}$$

if and only if $u^\star$ solves

$$\begin{cases} (L_{-\gamma} + q)v = F^\star & \textit{in } \Omega_T, \\ v = \varphi^\star & \textit{on } (\Omega_e)_T, \\ v(T) = u_0,\ \partial_t v(T) = u_1 & \textit{on } \Omega. \end{cases}$$

In particular, for any $F \in L^2(\Omega_T)$, $\varphi \in C^2([0,T]; H^{2s}(\mathbb{R}^n))$ and any $(u_0, u_1) \in H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying the compatibility conditions $u_0 - \varphi(T) \in \widetilde{H}^s(\Omega)$ and $u_1 - \partial_t\varphi(T) \in \widetilde{L}^2(\Omega)$, there exists a unique solution $u^\star$ of

$$\begin{cases} (L_{-\gamma} + q)v = F & \textit{in } \Omega_T, \\ v = \varphi & \textit{on } (\Omega_e)_T, \\ v(T) = u_0,\ \partial_t v(T) = u_1 & \textit{on } \Omega. \end{cases} \tag{2.10}$$

Proof. First, note that by the proof of Proposition 2.2, we can assume without loss of generality, that $\varphi = 0$. Secondly, one easily sees that $\partial_t u^\star = -(\partial_t u)^\star$ and thus a change of variables in (2.3) gives the asserted equivalence. The unique solvability of (2.10) follows from the equivalence and Theorem 2.1. ■

2.2. Very weak solutions

Let us start by making some simple observations. Suppose that u and v are smooth solutions to the problems

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{2.11}$$

and

$$\begin{cases} (L_{-\gamma} + q)v = G & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(T) = 0,\ \partial_t v(T) = 0 & \text{on } \Omega, \end{cases}$$

respectively. If we multiply the PDE (2.11) by v and integrate over Ω_T, then we get

$$\begin{aligned} \int_{\Omega_T} Fv\,dx\,dt &= \int_{\Omega_T} [(L_\gamma + q)u]v\,dx\,dt \\ &= \int_\Omega u_0 \partial_t v(0)\,dx - \int_\Omega u_1 v(0)\,dx - \int_\Omega \gamma u_0 v(0)\,dx + \int_{\Omega_T} u(L_{-\gamma} + q)v\,dx\,dt \\ &= \int_\Omega u_0 \partial_t v(0)\,dx - \int_\Omega u_1 v(0)\,dx - \int_\Omega \gamma u_0 v(0)\,dx + \int_{\Omega_T} Gu\,dx\,dt. \end{aligned} \tag{2.12}$$

Notice that if one has $G \in L^2(0,T;\widetilde{L}^2(\Omega))$, $v \in L^2(0,T;\widetilde{H}^s(\Omega))$ and $(v(0), \partial_t v(0)) \in \widetilde{H}^s(\Omega) \times \widetilde{L}^2(\Omega)$, then one can make sense of the first integral and the last line in (2.12), even in the case $F \in L^2(0,T;H^{-s}(\Omega))$, $(u_0,u_1) \in \widetilde{L}^2(\Omega) \times H^{-s}(\Omega)$ and $u \in L^2(0,T;\widetilde{L}^2(\Omega))$. Here, $H^{-s}(\Omega) \subset \mathcal{D}'(\Omega)$ is defined by

$$H^{-s}(\Omega) = \{u|_\Omega\,;\, u \in H^{-s}(\mathbb{R}^n)\},$$

which can be identified with $(\widetilde{H}^s(\Omega))'$ for $\partial\Omega \in C^{0,1}$. This motivates the following.

Definition 2.4 (Very weak solutions). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma, q) \in L^\infty(\Omega) \times L^p(\Omega)$, source $F \in L^2(0,T;H^{-s}(\Omega))$ and initial conditions $(u_0,u_1) \in \widetilde{L}^2(\Omega) \times H^{-s}(\Omega)$. Then we say that $u \in C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega))$ is a *very weak solution* of

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases}$$

whenever there holds[2]

$$\begin{aligned}\int_0^T \langle G, u\rangle_{L^2(\Omega)}\,dt \\ &= \int_0^T \langle F, v\rangle\,dt + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2(\Omega)} + \langle \gamma u_0, v(0)\rangle \end{aligned} \tag{2.13}$$

for all $G \in L^2(0,T;\widetilde{L}^2(\Omega))$, where $v \in C([0,T];\widetilde{H}^s(\Omega)) \cap C^1([0,T];\widetilde{L}^2(\Omega))$ is the unique weak solution to the adjoint equation

$$\begin{cases} (L_{-\gamma} + q)v = G & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(T) = 0,\ \partial_t v(T) = 0 & \text{on } \Omega, \end{cases} \tag{2.14}$$

(see Theorem 2.1).

Next, let us recall the following well-posedness result of very weak solutions.

Theorem 2.5 (Very weak solutions for $\gamma = q = 0$, [20, Theorem 3.6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $0 < s < 1$. Then for any given source $F \in L^2(0,T;H^{-s}(\Omega))$ and initial conditions $(u_0, u_1) \in \widetilde{L}^2(\Omega) \times H^{-s}(\Omega)$, there exists a unique solution to*

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u = F & \textit{in } \Omega_T, \\ u = 0 & \textit{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \textit{on } \Omega \end{cases} \tag{2.15}$$

and it satisfies the following energy estimate:

$$\begin{aligned} &\|u(t)\|_{L^2(\Omega)} + \|\partial_t u(t)\|_{H^{-s}(\Omega)} \\ &\quad \lesssim \|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-s}(\Omega)} + \|F\|_{L^2(0,t;H^{-s}(\Omega))} \end{aligned} \tag{2.16}$$

for all $0 \le t \le T$.

Hence, we have a well-defined solution map.

---

[2] Here and below we sometimes write $\langle \cdot, \cdot \rangle$ to denote the duality pairing between $H^{-s}(\Omega) \times \widetilde{H}^s(\Omega)$.

Proposition 2.6 (Solution map). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain,* $T > 0$, $0 < s < 1$ *and let* $X_s := \widetilde{L}^2(\Omega) \times H^{-s}(\Omega)$ *be endowed with the usual product norm*

$$\|(u, w)\|_{X_s} := (\|u\|^2_{L^2(\Omega)} + \|w\|^2_{H^{-s}(\Omega)})^{1/2}.$$

Then the solution map $S \colon L^2(0, T; H^{-s}(\Omega)) \to C([0, T]; X_s)$ *defined by*

$$S(F) := (u, \partial_t u),$$

where $u \in C([0, T]; \widetilde{L}^2(\Omega)) \cap C^1([0, T]; H^{-s}(\Omega))$ *is the unique solution to* (2.15) *with* $(u_0, u_1) = 0$. *Moreover, the solution map is continuous and satisfies the estimate*

$$\|S(F)(t)\|_{X_s} \leq C\|F\|_{L^2(0,t;H^{-s}(\Omega))} \tag{2.17}$$

for any $0 \leq t \leq T$.

Proof. First of all note that the solution map S is well defined by Theorem 2.5. The estimate (2.17) follows from (2.16), which together with the linearity of S gives the continuity of S. Observe that the linearity of S is a direct consequence of the unique solvability of (2.15) and the fact that the PDE is linear. ■

Theorem 2.7. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain,* $T > 0$, $0 < s < 1$, *and suppose that* $\mathcal{F} \colon C([0, T]; X_s) \to L^2(0, T; H^{-s}(\Omega))$ *satisfies the Lipschitz estimate*

$$\|\mathcal{F}(U)(t) - \mathcal{F}(V)(t)\|_{H^{-s}(\Omega)} \leq C\|U(t) - V(t)\|_{X_s} \tag{2.18}$$

for a.e. $0 \leq t \leq T$ *and* $U, V \in C([0, T]; X_s)$. *Then for all* $(u_0, u_1) \in X_s$, *there exists a unique solution* u *of*

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u = \mathcal{F}(u, \partial_t u) & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{2.19}$$

that is the formula (2.13) *holds with* F *replaced by* $\mathcal{F}(u, \partial_t u)$ *in which we test against every weak solution* v *of the adjoint equation*

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)v = G & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(T) = 0,\ \partial_t v(T) = 0 & \text{on } \Omega \end{cases} \tag{2.20}$$

with $G \in L^2(0, T; \widetilde{L}^2(\Omega))$.

Proof of Theorem 2.7. Let $u_h \in C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega))$ be the unique solution to

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega \end{cases}$$

and let us set $U_h := (u_h, \partial_t u_h) \in C([0,T];X_s)$. Furthermore, we define the operator $\mathcal{T}\colon C([0,T];X_s) \to C([0,T];X_s)$ as

$$\mathcal{T}(U) := U_h + S(\mathcal{F}(U)),$$

which is well defined by (2.6) and the properties of $\mathcal{F}$. Next, we show that $\mathcal{T}$ has a unique fixed point $U = (U_1, U_2)$.

Step 1. *Existence.* Let $U, V \in C([0,T];X_s)$, then by linearity of S, (2.17) and (2.18) we get

$$\begin{aligned} \|\mathcal{T}(U)(t) - \mathcal{T}(V)(t)\|_{X_s} &= \|S(\mathcal{F}(U))(t) - S(\mathcal{F}(V))(t)\|_{X_s} \\ &= \|S(\mathcal{F}(U) - \mathcal{F}(V))(t)\|_{X_s} \\ &\leq C\|\mathcal{F}(U) - \mathcal{F}(V)\|_{L^2(0,t;H^{-s}(\Omega))} \\ &\leq C\|U - V\|_{L^2(0,t;X_s)}. \end{aligned}$$

Next, let us define the following norm on X_s:

$$\|U\|_\theta := \sup_{0\leq t\leq T} (e^{-\theta t}\|U(t)\|_{X_s})$$

for $\theta > 0$, which will be fixed in a moment. Then we have the estimate

$$\|\mathcal{T}(U)(t) - \mathcal{T}(V)(t)\|_{X_s} \leq C\left(\int_0^t e^{2\theta\tau}\,d\tau\right)^{1/2} \|U - V\|_\theta \leq \frac{C}{(2\theta)^{1/2}} e^{\theta t}\|U - V\|_\theta$$

and hence there holds

$$\|\mathcal{T}(U)(t) - \mathcal{T}(V)(t)\|_\theta \leq \frac{C}{(2\theta)^{1/2}}\|U - V\|_\theta.$$

Therefore, we deduce that $\mathcal{T}$ is a strict contraction from the complete metric space $(C([0,T];X_s), \|\cdot\|_\theta)$ to itself, when $\theta > 0$ is chosen such that $C/(2\theta)^{1/2} < 1$. Now, we may invoke Banach's fixed point theorem to obtain a unique fixed point $U = (u, w)$ of $\mathcal{T}$. Next, observe that the definition of the solution map S and $U = \mathcal{T}(U) = U_h + S(\mathcal{F}(U))$ imply

$$u = u_h + u_n \quad \text{and} \quad w = \partial_t u,$$

where u_n solves

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)v = \mathscr{F}(U) & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(0) = 0, \ \partial_t v(0) = 0 & \text{on } \Omega. \end{cases}$$

Going back to the definition of very weak solutions, we see this implies that u solves (2.19).

Step 2. *Uniqueness.* Suppose $\tilde{u} \in C([0,T]; \widetilde{L}^2(\Omega)) \cap C^1([0,T]; H^{-s}(\Omega))$ is any other solution to (2.19), then $\bar{u} := u - \tilde{u}$ solves

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)v = \mathscr{F}(u, \partial_t u) - \mathscr{F}(\tilde{u}, \partial_t \tilde{u}) & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(0) = 0, \ \partial_t v(0) = 0 & \text{on } \Omega. \end{cases}$$

Thus, applying the energy estimate (2.16) together with the Lipschitz assumption on $\mathscr{F}$, we see that

$$\begin{aligned} \|U(t) - \widetilde{U}(t)\|_{X_s}^2 &\lesssim \int_0^t \|\mathscr{F}(U)(\tau) - \mathscr{F}(\widetilde{U})(\tau)\|_{H^{-s}(\Omega)}^2 \, d\tau \\ &\lesssim \int_0^t \|U(t) - \widetilde{U}(t)\|_{X_s}^2 \, d\tau, \end{aligned}$$

where $U = (u, \partial_t u)$ and $\widetilde{U} = (\tilde{u}, \partial_t \tilde{u})$. So, Gronwall's inequality shows that $u = \tilde{u}$. This establishes the uniqueness assertion and we can conclude the proof. ■

As an application of Theorem 2.7, we can show the unique solvability of (2.1) for rough source and initial data.

Theorem 2.8 (Very weak solutions to DNWEQ). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, and suppose that $1 \leq p \leq \infty$ satisfies* (1.3). *Assume that we have given coefficients $(\gamma, q) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$. Then for any $F \in L^2(0,T; H^{-s}(\Omega))$ and $(u_0, u_1) \in \widetilde{L}^2(\Omega) \times H^{-s}(\Omega)$, there exists a unique solution to*

$$\begin{cases} (L_\gamma + q)u = F & \text{in } \Omega_T, \\ u = 0 & \text{on } (\Omega_e)_T, \\ u(0) = u_0, \ \partial_t u(0) = u_1 & \text{on } \Omega. \end{cases} \tag{2.21}$$

Proof. Let us define the mapping $\mathscr{F} \colon C([0,T]; X_s) \to L^2(0,T; H^{-s}(\Omega))$ by

$$\mathscr{F}(U)(t) := F - \gamma w(t) - q u(t),$$

where $U=(u,w)\in C([0,T];X_s)$. On the one hand, using the estimate (2.6), we see that for any $u\in\widetilde{L}^2(\Omega)$ one has $qu\in H^{-s}(\Omega)$ and there holds

$$\|qu\|_{H^{-s}(\Omega)}=\sup_{\|v\|_{\widetilde{H}^s(\Omega)}\leq 1}|\langle u,qv\rangle_{L^2(\Omega)}|\lesssim\|q\|_{L^p(\Omega)}\|u\|_{L^2(\Omega)}. \tag{2.22}$$

On the other hand, by applying [4, Lemma 3.1] and $\partial\Omega\in C^0$, we deduce that for any $v\in\widetilde{H}^s(\Omega)$ one has $\gamma v\in\widetilde{H}^s(\Omega)$ and it obeys the estimate

$$\|\gamma v\|_{\widetilde{H}^s(\Omega)}\leq C\|\gamma\|_{C^{0,\alpha}(\mathbb{R}^n)}\|v\|_{\widetilde{H}^s(\Omega)}. \tag{2.23}$$

Thus, we can again infer from a duality argument that $H^{-s}(\Omega)\ni w\mapsto\gamma w\in H^{-s}(\Omega)$ is a continuous map satisfying

$$\|\gamma w\|_{H^{-s}(\Omega)}=\sup_{\|v\|_{\widetilde{H}^s(\Omega)}\leq 1}|\langle w,\gamma v\rangle|\lesssim\|\gamma\|_{C^{0,\alpha}(\mathbb{R}^n)}\|w\|_{H^{-s}(\Omega)}. \tag{2.24}$$

From the estimates (2.22) and (2.24), we easily deduce that $\mathcal{F}$ is well defined and satisfies the Lipschitz estimate

$$\|\mathcal{F}(U)(t)-\mathcal{F}(V)(t)\|_{H^{-s}(\Omega)}\lesssim(\|\gamma\|_{C^{0,\alpha}(\mathbb{R}^n)}+\|q\|_{L^p(\Omega)})\|U(t)-V(t)\|_{X_s}$$

for all $U,V\in C([0,T];X_s)$. Thus, we can apply Theorem 2.7 to get the existence of a unique solution to (2.21) in the sense that for any $G\in L^2(0,T;\widetilde{L}^2(\Omega))$ and corresponding solution v of (2.20), there holds

$$\int_0^T\langle G,u\rangle_{L^2(\Omega)}\,dt=\int_0^T\langle(F-\gamma\partial_t u-qu),v\rangle\,dt+\langle u_1,v(0)\rangle-\langle u_0,\partial_t v(0)\rangle_{L^2(\Omega)}. \tag{2.25}$$

It remains to verify that u is indeed a solution to (2.21) in the sense of Definition 2.4. For this purpose, let $G\in L^2(0,T;\widetilde{L}^2(\Omega))$ and suppose that v is the unique solution to (2.14). Hence, v solves

$$\begin{cases}(\partial_t^2+(-\Delta)^s)v=\widetilde{G} & \text{in } \Omega_T,\\ v=0 & \text{on } (\Omega_e)_T,\\ v(T)=0,\ \partial_t v(T)=0 & \text{on } \Omega\end{cases}$$

with $\widetilde{G}=G+\gamma\partial_t v-qv\in L^2(0,T;\widetilde{L}^2(\Omega))$ (see (2.6)). Next, we claim that there holds

$$\int_0^T\langle\gamma\partial_t u,v\rangle\,dt=-\int_0^T\langle\gamma\partial_t v,u\rangle_{L^2(\Omega)}\,dt-\langle\gamma u_0,v(0)\rangle. \tag{2.26}$$

For this purpose, let us consider for $\varepsilon > 0$ the unique solution $v_\varepsilon \in H^1(0,T;\widetilde{H}^s(\Omega))$ with $\partial_t^2 v_\varepsilon \in L^2(0,T;H^{-s}(\Omega))$ to the following parabolically regularized problem:

$$\begin{cases} (\partial_t^2 - \varepsilon(-\Delta)^s\partial_t + (-\Delta)^s)v = \widetilde{G} & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(T) = \partial_t v(T) = 0 & \text{in } \Omega \end{cases}$$

(see [5, Chapter XVIII, Section 5.3.1]). By [5, Chapter XVIII, Section 5.3.4], we know that there holds

$$\begin{aligned} v_\varepsilon &\overset{*}{\rightharpoonup} v && \text{in } L^\infty(0,T;\widetilde{H}^s(\Omega)), \\ \partial_t v_\varepsilon &\overset{*}{\rightharpoonup} \partial_t v && \text{in } L^\infty(0,T;\widetilde{L}^2(\Omega)), \\ v_\varepsilon(t) &\to v(t) && \text{in } \widetilde{H}^s(\Omega) \text{ for all } 0 \leq t \leq T. \end{aligned} \tag{2.27}$$

First, note that the conditions $u \in C^1([0,T];H^{-s}(\Omega))$ and $v_\varepsilon \in C^1([0,T];\widetilde{L}^2(\Omega))$, where the latter follows from the Sobolev embedding, guarantee that one we have

$$\langle \gamma u, v_\varepsilon\rangle \in C^1([0,T])$$

and

$$\partial_t\langle \gamma u, v_\varepsilon\rangle = \langle \partial_t u, \gamma v_\varepsilon\rangle + \langle u, \gamma\partial_t v_\varepsilon\rangle_{L^2(\Omega)}.$$

Thus, by the fundamental theorem of calculus we deduce that there holds

$$\langle \gamma u(T), v_\varepsilon(T)\rangle - \langle \gamma u_0, v_\varepsilon(0)\rangle = \int_0^T \langle \partial_t u, \gamma v_\varepsilon\rangle + \langle u, \gamma\partial_t v_\varepsilon\rangle_{L^2(\Omega)}\,dt.$$

By the convergence assertions (2.27) and $v_\varepsilon(T) = 0$, we get

$$-\langle \gamma u_0, v(0)\rangle = \int_0^T \langle \partial_t u, \gamma v\rangle + \langle u, \gamma\partial_t v\rangle_{L^2(\Omega)}\,dt.$$

This proves (2.26). Hence, inserting this into (2.25), we obtain

$$\begin{aligned} \int_0^T \langle \widetilde{G}, u\rangle_{L^2(\Omega)}\,dt &= \int_0^T \langle (F - \gamma\partial_t u - qu), v\rangle\,dt + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2(\Omega)} \\ &= \int_0^T \langle F, v\rangle\,dt + \int_0^T \langle u, \gamma\partial_t v\rangle_{L^2(\Omega)}\,dt - \int_0^T \langle u, qv\rangle\,dt \\ &\quad + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2(\Omega)} + \langle \gamma u_0, v(0)\rangle. \end{aligned}$$

As $\widetilde{G} = G + \gamma\partial_t v - qv$, this gives

$$\int_0^T \langle G, u\rangle_{L^2(\Omega)}\,dt = \int_0^T \langle F, v\rangle\,dt + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2(\Omega)} + \langle \gamma u_0, v(0)\rangle.$$

Hence, we observe that u is indeed a solution to (2.21) in the sense of Definition 2.4. By reversing the above arguments, one can also observe that if u is a solution in the sense of Definition 2.4, then by (2.26), it is a solution in the sense of (2.25) and thus the solution in the sense of Definition 2.4 is unique. ■

3. The inverse problem

After establishing the theory of very weak solutions to damped, nonlocal wave equations, we now turn our attention to the inverse problem part. First, in Section 3.1 we prove the optimal Runge approximation theorem (Theorem 3.1) and in Section 3.2 a suitable integral identity. Using these results, we then show in Section 3.3 our first main result dealing with linear perturbations (Theorem 1.3). Finally, in Section 3.4 we prove Theorem 1.4 showing that the damping coefficient and the nonlinearity can be determined simultaneously.

3.1. Runge approximation

With the material from Section 2 at our disposal, we can now show the following Runge approximation theorem, whose proof is very similar to the one of [20, Theorem 1.2].

Theorem 3.1 (Runge approximation). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma, q) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$. Then for any measurement set $W \subset \Omega_e$ and initial conditions $(u_0, u_1) \in \widetilde{H}^s(\Omega) \times \widetilde{L}^2(\Omega)$, the* Runge set

$$\mathcal{R}_W^{u_0,u_1} := \{u_\varphi - \varphi\,;\, \varphi \in C_c^\infty(W_T)\}$$

is dense in $L^2(0, T; \widetilde{H}^s(\Omega))$, where u_φ is the unique solution to

$$\begin{cases} (L_\gamma + q)u = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{on } \Omega, \end{cases} \tag{3.1}$$

(see Proposition 2.2).

Proof. First of all note that it is enough to consider the case $(u_0, u_1) = 0$. To see this, assume that the density holds for $\mathcal{R}_W := \mathcal{R}_W^{0,0}$ and let $f \in L^2(0, T; \widetilde{H}^s(\Omega))$. Let v_0 be the unique solution to

$$\begin{cases} (L_\gamma + q)v = 0 & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(0) = u_0,\ \partial_t v(0) = u_1 & \text{on } \Omega \end{cases}$$

and define $\tilde{f} := f - v_0 \in L^2(0, T; \widetilde{H}^s(\Omega))$. By assumption, there exists $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty(W_T)$ such that $u_k - \varphi_k \to \tilde{f}$ in $L^2(0, T; \widetilde{H}^s(\Omega))$ as $k \to \infty$, where u_k is the unique solution to

$$\begin{cases} (L_\gamma + q)u = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = 0,\ \partial_t u(0) = 0 & \text{on } \Omega \end{cases} \tag{3.2}$$

with $\varphi = \varphi_k$. Then $v_k := u_k + v_0$ is the unique solution to (3.1) with $\varphi = \varphi_k$. The above convergence now implies $v_k - \varphi_k \to f$ in $L^2(0, T; \widetilde{H}^s(\Omega))$ as $k \to \infty$ and we get that $\mathcal{R}_W^{u_0,u_1}$ is dense in $L^2(0, T; \widetilde{H}^s(\Omega))$.

Therefore, it remains to show that $\mathcal{R}_W$ is dense in $L^2(0, T; \widetilde{H}^s(\Omega))$. As usual, we show this by a Hahn–Banach argument. Thus, suppose that $F \in L^2(0, T; H^{-s}(\Omega))$ vanishes on $\mathcal{R}_W$. Let us recall that if $\varphi \in C_c^\infty(W_T)$ and u solves (3.2), then, by (2.9) and Lemma 2.3, the function $v = (u - \varphi)^\star$ satisfies

$$\begin{cases} (L_{-\gamma} + q)v = -(-\Delta)^s \varphi^\star & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(T) = \partial_t v(T) = 0 & \text{in } \Omega. \end{cases}$$

Next, let w be the unique solution to

$$\begin{cases} (L_\gamma + q)w = F^\star & \text{in } \Omega_T, \\ w = 0 & \text{in } (\Omega_e)_T, \\ w(0) = \partial_t w(0) = 0 & \text{in } \Omega \end{cases}$$

(see Theorem 2.8). By testing the equation for w by v, we get

$$\begin{aligned} -\int_0^T \langle (-\Delta)^s \varphi^\star, w\rangle_{L^2(\Omega)}\, dt &= \int_0^T \langle F^\star, v\rangle\, dt \\ &= \int_0^T \langle F, u_\varphi - \varphi\rangle\, dt = 0 \end{aligned}$$

for any $\varphi \in C_c^\infty(W_T)$. This ensures that there holds

$$(-\Delta)^s w = 0 \quad \text{in } W_T.$$

Furthermore, by construction w vanishes in $(\Omega_e)_T$ and hence the unique continuation principle for the fractional Laplacian guarantees $w = 0$ in $\mathbb{R}^n_T$ (see [11]). As very weak solutions are distributional solutions, we get

$$\int_0^T \langle F^\star, \Phi\rangle \, dt = \int_0^T \langle (L_{-\gamma} + q)\Phi, w\rangle_{L^2(\Omega)} \, dt = 0$$

for all $\Phi \in C_c^\infty(\Omega_T)$. To see that very weak solutions are distributional solutions, one can simply take $G = \chi_\Omega (L_\gamma + q)\Phi$ with $\Phi \in C_c^\infty(\Omega \times [0,T))$ in Definition 2.4, where χ_Ω denotes the characteristic function of Ω (see also [20, Proposition 3.8]). By density of $C_c^\infty(\Omega_T)$ in $L^2(0,T;\widetilde{H}^s(\Omega))$, we deduce that $F = 0$ and so can conclude the proof. ■

As a consequence, we have the following lemma.

Lemma 3.2 (Convergence of time derivative). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma, q) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$. Let $\Phi, \Psi \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap H^1(0,T;H^{-s}(\Omega))$ and suppose $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty((\Omega_e)_T)$ is such that*

$$u_k - \varphi_k \to \Phi \quad \text{in } L^2(0,T;\widetilde{H}^s(\Omega)) \text{ as } k \to \infty, \tag{3.3}$$

where u_k solves

$$\begin{cases} (L_\gamma + q)u = 0 & \text{in } \Omega_T, \\ u = \varphi_k & \text{on } (\Omega_e)_T, \\ u(0) = 0, \ \partial_t u(0) = 0 & \text{on } \Omega, \end{cases} \tag{3.4}$$

for $k \in \mathbb{N}$. If Φ, Ψ satisfy one of the conditions

(a) $\Psi(T) = \Phi(0) = 0$

(b) *or* $\Psi(T) = \Psi(0) = 0$,

then we have

$$\lim_{k\to\infty} \int_0^T \langle \partial_t (u_k - \varphi_k), \Psi\rangle \, dt = \int_0^T \langle \partial_t \Phi, \Psi\rangle \, dt. \tag{3.5}$$

Remark 3.3. Let us note that the same formula (3.5) holds for second order time derivatives under appropriate conditions.

Proof. Using the integration by parts formula, we may compute

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^T \langle \partial_t(u_k-\varphi_k),\Psi\rangle\,dt\\
&\quad=\lim_{k\to\infty}\Bigg(\langle (u_k-\varphi_k)(T),\Psi(T)\rangle_{L^2(\Omega)}-\langle (u_k-\varphi_k)(0),\Psi(0)\rangle_{L^2(\Omega)}\\
&\qquad\qquad-\int_0^T\langle \partial_t\Psi,u_k-\varphi_k\rangle\,dt\Bigg)\\
&\quad=-\lim_{k\to\infty}\int_0^T\langle \partial_t\Psi,u_k-\varphi_k\rangle\,dt\\
&\quad=-\int_0^T\langle \partial_t\Psi,\Phi\rangle\,dt\\
&\quad=\langle \Phi(0),\Psi(0)\rangle_{L^2(\Omega)}-\langle \Phi(T),\Psi(T)\rangle_{L^2(\Omega)}+\int_0^T\langle \partial_t\Phi,\Psi\rangle\,dt\\
&\quad=\int_0^T\langle \partial_t\Phi,\Psi\rangle\,dt.
\end{aligned}$$

In the first equality sign we used an integration by parts, in the second equality we used (3.4), $\Psi(T)=0$ and (3.4), in the third equality the convergence (3.3), in the fourth equality again an integration by parts and finally in the last equality the conditions (a) or (b). ■

3.2. DN map and integral identities

Next, we define the *Dirichlet to Neumann (DN) map* $\Lambda_{\gamma,q}$ related to

$$\begin{cases}(L_\gamma+q)u=0 & \text{in } \Omega_T,\\ u=\varphi & \text{on } (\Omega_e)_T,\\ u(0)=0,\ \partial_t u(0)=0 & \text{on } \Omega,\end{cases}\tag{3.6}$$

via

$$\langle \Lambda_{\gamma,q}\varphi,\psi\rangle=\int_{\mathbb{R}^n_T}(-\Delta)^{s/2}u_\varphi(-\Delta)^{s/2}\psi\,dx$$

for all $\varphi, \psi \in C_c^\infty((\Omega_e)_T)$, where u_φ is the unique solution to (3.6) with exterior condition φ. Using the above preparation, we now establish the following integral identity.

Proposition 3.4 (Integral identity for linear perturbations). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, and suppose that $1 \leq p \leq \infty$ satisfies (1.3). Assume that we have given coefficients $(\gamma_j, q_j) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$ for $j = 1, 2$. Let $\varphi_j \in C_c^\infty((\Omega_e)_T)$ and denote by u_j the corresponding solution to (3.6) with $(\gamma, q) = (\gamma_j, q_j)$. Then there holds*

$$\begin{aligned}
&\langle (\Lambda_{\gamma_1,q_1} - \Lambda_{\gamma_2,q_2})\varphi_1, \varphi_2^\star \rangle \\
&\quad = \int_{\Omega_T} \{[(\gamma_1 - \gamma_2)\partial_t + q_1 - q_2](u_1 - \varphi_1)\}(u_2 - \varphi_2)^\star \, dx \, dt.
\end{aligned} \tag{3.7}$$

Proof. Let $(\Gamma_j, Q_j) \in C^{0,\alpha}(\mathbb{R}^n) \times L^p(\Omega)$, $j = 1, 2$, and suppose U_j is the unique solutions to (3.6) with $(\gamma, q) = (\Gamma_j, Q_j)$ and exterior condition $\varphi = \psi_j$. Then we may compute

$$\begin{aligned}
&\int_{\Omega_T} \{[(\Gamma_1 - \Gamma_2)\partial_t + Q_1 - Q_2](U_1 - \psi_1)\}(U_2 - \psi_2)^\star \, dx \, dt \\
&\quad = \int_{\Omega_T} \{[\Gamma_1 \partial_t + Q_1](U_1 - \psi_1)\}(U_2 - \psi_2)^\star \, dx \, dt \\
&\qquad - \int_{\Omega_T} (U_1 - \psi_1)[-\Gamma_2 \partial_t + Q_2](U_2 - \psi_2)^\star \, dx \, dt \\
&\quad = \int_0^T \langle L_{\Gamma_1,Q_1}(U_1 - \psi_1), (U_2 - \psi_2)^\star \rangle \, dt \\
&\qquad - \int_0^T \langle (\partial_t^2 + (-\Delta)^s)(U_1 - \psi_1), (U_2 - \psi_2)^\star \rangle \, dt \\
&\qquad - \int_0^T \langle L_{-\Gamma_2,Q_2}(U_2 - \psi_2)^\star, (U_1 - \psi_1) \rangle \, dt \\
&\qquad + \int_0^T \langle (\partial_t^2 + (-\Delta)^s)(U_2 - \psi_2)^\star, (U_1 - \psi_1) \rangle \, dt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T \langle (-\Delta)^s \psi_1, (U_2 - \psi_2)^\star \rangle_{L^2(\Omega)}\, dt \\
&\quad + \int_0^T \langle (-\Delta)^s \psi_2^\star, (U_1 - \psi_1) \rangle_{L^2(\Omega)}\, dt \\
&= \int_{\mathbb{R}^n_T} ((-\Delta)^s \psi_2^\star) U_1 \, dx\, dt - \int_{\mathbb{R}^n_T} ((-\Delta)^s \psi_1) U_2^\star \, dx\, dt \\
&= \langle \Lambda_{\Gamma_1, Q_1} \psi_1, \psi_2^\star \rangle - \langle \Lambda_{\Gamma_2, Q_2} \psi_2, \psi_1^\star \rangle. \qquad (3.8)
\end{aligned}
$$

In the first equality we used that $U_1 - \psi_1$ has vanishing initial conditions, $(U_2 - \psi_2)^\star$ has vanishing terminal conditions and an integration by parts. In the third equality we used that the PDEs for $U_1 - \psi_1$ and $(U_2 - \psi_2)^\star$ hold in the sense of

$$L^2(0,T; H^{-s}(\Omega)) = (L^2(0,T; \widetilde{H}^s(\Omega))'$$

(see Lemma 2.3). In the fourth equality, we used the PDEs for U_1 and U_2, Lemma 2.3 and that there holds

$$
\begin{aligned}
&\int_0^T \langle (\partial_t^2 + (-\Delta)^s)(U_2 - \psi_2)^\star, (U_1 - \psi_1) \rangle \, dt \\
&\quad = \int_0^T \langle (\partial_t^2 + (-\Delta)^s)(U_1 - \psi_1), (U_2 - \psi_2)^\star \rangle \, dt,
\end{aligned}
$$

which can be established along the lines of [19, Claim 4.2] (see also the proof of Theorem 2.8). In the last equality, we have made the change of variables $\tau = T - t$ for the second integral. On the one hand, using (3.8) with $\Gamma_1 = \Gamma_2 = \gamma_j$ and $Q_1 = Q_2 = q_j$, we observe that

$$\langle \Lambda_{\gamma_j, q_j} \psi_1, \psi_2^\star \rangle = \langle \Lambda_{\gamma_j, q_j} \psi_2, \psi_1^\star \rangle \qquad (3.9)$$

for all $\psi_j \in C_c^\infty((\Omega_e)_T)$, $j = 1, 2$. On the other hand, choosing $\Gamma_j = \gamma_j$, $Q_j = q_j$ and $\psi_j = \varphi_j$ in (3.8) and taking into account the self-adjointness (3.9), we get (3.7).

■

3.3. Simultaneous determination of damping coefficient and linear perturbations

Proof of Theorem 1.3. First note that by the integral identity in Proposition 3.4, we may deduce from the condition (1.7) that there holds

$$\int_{\Omega_T} \{[(\gamma_1-\gamma_2)\partial_t+q_1-q_2](u_1-\varphi_1)\}(u_2-\varphi_2)^\star\,dx\,dt=0 \tag{3.10}$$

for all $\varphi_j\in C_c^\infty((W_j)_T)$, where u_j denotes the unique solution to

$$\begin{cases}(L_{\gamma_j}+q_j)u=0 & \text{in } \Omega_T,\\ u=\varphi_j & \text{on } (\Omega_e)_T,\\ u(0)=0,\ \partial_t u(0)=0 & \text{on } \Omega.\end{cases} \tag{3.11}$$

Let $\omega\Subset\Omega$ and choose a cutoff function $\Phi_1\in C_c^\infty(\Omega)$ satisfying $\Phi_1=1$ on ω. Moreover, let $\Phi_2\in C_c^\infty(\omega_T)$. By the Runge approximation (Theorem 3.1), there exist sequences $(\varphi_j^k)_{k\in\mathbb{N}}\subset C_c^\infty((W_j)_T)$ with corresponding solutions u_j^k of (3.11) with $\varphi_j=\varphi_j^k$ such that $u_j^k-\varphi_j^k\to\Phi_j$ in $L^2(0,T;\widetilde{H}^s(\Omega))$. Taking $\varphi_1=\varphi_1^k$ and $\varphi_2=\varphi_2^\ell$ in (3.10) gives

$$\int_{\Omega_T} \{[(\gamma_1-\gamma_2)\partial_t+q_1-q_2](u_1^k-\varphi_1^k)\}(u_2^\ell-\varphi_2^\ell)^\star\,dx\,dt=0$$

for all $k,\ell\in\mathbb{N}$. First, we let $\ell\to\infty$ to deduce

$$\int_{\Omega_T} \{[(\gamma_1-\gamma_2)\partial_t+q_1-q_2](u_1^k-\varphi_1^k)\}\Phi_2^\star\,dx\,dt=0 \tag{3.12}$$

for all $k\in\mathbb{N}$. As $\gamma_1-\gamma_2\in C^{0,\alpha}(\mathbb{R}^n)$, the estimate (2.23) ensures that we can apply Lemma 3.2 under the condition (b) and so $\partial_t\Phi_1=0$ shows that the first term in (3.12) goes to zero. So in the limit $k\to\infty$ what remains is

$$\int_{\Omega_T}(q_1-q_2)\Phi_2^\star\,dx\,dt=0,$$

where we used $\Phi_1=1$ on ω. This ensures that $q_1=q_2$ on ω. As the set ω is arbitrary, we get $q_1=q_2$ in Ω. Now, the identity (3.10) reduces to

$$\int_{\Omega_T}\{[(\gamma_1-\gamma_2)\partial_t](u_1-\varphi_1)\}(u_2-\varphi_2)^\star\,dx\,dt=0$$

for all $\varphi_j \in C_c^\infty((W_j)_T)$. We choose $\eta \in C_c^\infty(\Omega_T)$, define

$$\Phi_1(x,t) = \int_0^t \eta(x,\tau)\,d\tau \in C_c^\infty(\Omega \times (0,T])$$

and take $\Phi_2 \in C_c^\infty(\Omega_T)$. Then using $\partial_t \Phi_1 = \eta$ and arguing as above via a Runge approximation and Lemma 3.2, we get from (3.10) the identity

$$\int_{\Omega_T} (\gamma_1 - \gamma_2)\eta\Phi_2^\star \,dx\,dt = 0.$$

This again implies $\gamma_1 = \gamma_2$ in Ω. ■

3.4. Simultaneous determination of damping coefficient and nonlinearity

Before turning to the proof of our second main result, let us recall that the *DN map* related to the problem

$$\begin{cases} L_\gamma u + f(u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } (\Omega_e)_T, \\ u(0) = 0,\ \partial_t u(0) = 0 & \text{on } \Omega \end{cases} \tag{3.13}$$

is defined by

$$\langle \Lambda_{\gamma,f}\varphi, \psi \rangle := \int_{\mathbb{R}^n_T} (-\Delta)^{s/2} u_\varphi (-\Delta)^{s/2}\psi \,dx\,dt,$$

where $\varphi, \psi \in C_c^\infty((\Omega_e)_T)$ and u_φ is the unique solution to (3.13). We note that the well-posedness of (3.13) follows from [19, Proposition 3.7] with $g(x,\partial u) := \gamma(x)\partial_t u$.

Proof of Theorem 1.4. Let $\varepsilon > 0$ and denote by $u_\varepsilon^{(j)}$ the unique solutions to (3.13) with $f = f_j, \gamma = \gamma_j$ and $\varphi = \varepsilon\eta$ for some fixed $\eta \in C_c^\infty((W_1)_T)$. Let us observe that the UCP for the fractional Laplacian and the condition (1.8) imply that $u_\varepsilon := u_\varepsilon^{(1)} = u_\varepsilon^{(2)}$. Next, let us note that we can write

$$u_\varepsilon = \varepsilon v_j + R_\varepsilon^{(j)} \tag{3.14}$$

for $j = 1,2$, where v_j and $R_\varepsilon^{(j)}$ are the unique solutions to

$$\begin{cases} L_{\gamma_j} v = 0 & \text{in } \Omega_T, \\ v = \eta & \text{on } (\Omega_e)_T, \\ v(0) = 0,\ \partial_t v(0) = 0 & \text{on } \Omega, \end{cases}$$

and

$$\begin{cases} L_{\gamma_j} R = -f_j(u_\varepsilon) & \text{in } \Omega_T, \\ R = 0 & \text{on } (\Omega_e)_T, \\ R(0) = 0,\ \partial_t R(0) = 0 & \text{on } \Omega, \end{cases}$$

respectively. This simply follows from the unique solvability of (3.13) and both functions u_ε and $\varepsilon v_j + R_\varepsilon^{(j)}$ are solutions. Furthermore, we notice that the energy estimate of [19, Theorem 3.1], [19, eq. (3.18)], and the $r+1$ homogeneity of f_j ensure that $R_\varepsilon^{(j)}$ satisfies

$$\begin{aligned} \|\partial_t R_\varepsilon^{(j)}\|_{L^\infty(0,T;L^2(\Omega))} + \|R_\varepsilon^{(j)}(t)\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} &\lesssim \|f_j(u_\varepsilon)\|_{L^2(\Omega_T)} \\ &\lesssim \|u_\varepsilon\|^{r+1}_{L^\infty(0,T;H^s(\mathbb{R}^n))}. \end{aligned} \tag{3.15}$$

Moreover, we may estimate

$$\begin{aligned} &\|\partial_t u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|u_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \\ &\quad\lesssim \|\partial_t(u_\varepsilon - \varepsilon\eta)\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon \\ &\qquad - \varepsilon\eta\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \varepsilon\|\eta\|_{W^{1,\infty}(0,T;H^{2s}(\mathbb{R}^n))} \\ &\quad\lesssim \varepsilon\|\eta\|_{W^{1,\infty}(0,T;H^{2s}(\mathbb{R}^n))}. \end{aligned} \tag{3.16}$$

This follows from the following observations. If u solves (3.13) for a damping coefficient $\gamma \in C^{0,\alpha}(\mathbb{R}^n)$, a weak nonlinearity f, and $\varphi \in C_c^\infty((\Omega_e)_T)$, then $v = u - \varphi$ solves

$$\begin{cases} L_\gamma v + f(v) = -(-\Delta)^s\varphi & \text{in } \Omega_T, \\ v = 0 & \text{on } (\Omega_e)_T, \\ v(0) = 0,\ \partial_t v(0) = 0 & \text{on } \Omega. \end{cases}$$

Now, we may invoke [19, eq. (3.15)] to find that there holds

$$\begin{aligned} &\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{H^s(\mathbb{R}^n)} \\ &\quad\lesssim \int_0^t |\langle \gamma\partial_t v, \partial_t v\rangle_{L^2(\Omega)}|\,d\tau + \int_0^t |\langle (-\Delta)^s\varphi, \partial_t v\rangle_{L^2(\Omega)}|\,d\tau \\ &\quad\lesssim \|(-\Delta)^s\varphi\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \|\partial_t v\|^2_{L^2(\Omega)}\,d\tau. \end{aligned}$$

Thus, Gronwall's inequality gives

$$\|\partial_t v(t)\|_{L^2(\Omega)} + \|v(t)\|_{H^s(\mathbb{R}^n)} \lesssim \|(-\Delta)^s\varphi\|_{L^2(0,t;L^2(\Omega))}.$$

This ensures the validity of the second estimate in (3.16). Next, observe that by subtracting the PDEs for $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$, we deduce that

$$(\gamma_1-\gamma_2)\partial_t u_\varepsilon = f_2(u_\varepsilon)-f_1(u_\varepsilon) \quad \text{in } \Omega_T. \tag{3.17}$$

By (3.14), we may write

$$(\gamma_1-\gamma_2)(\varepsilon\partial_t v_1+\partial_t R_\varepsilon^{(1)}) = f_2(u_\varepsilon)-f_1(u_\varepsilon) \quad \text{in } \Omega_T. \tag{3.18}$$

Combining (3.15) and (3.16), we see that

$$\|\partial_t R_\varepsilon^{(j)}\|_{L^\infty(0,T;L^2(\Omega))}+\|R_\varepsilon^{(j)}(t)\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \lesssim \varepsilon^{r+1}. \tag{3.19}$$

Multiplying 3.18 by ε^{-1} gives

$$(\gamma_1-\gamma_2)(\partial_t v_1+\varepsilon^{-1}\partial_t R_\varepsilon^{(1)}) = f_2(\varepsilon^{-1/(r+1)}u_\varepsilon)-f_1(\varepsilon^{-1/(r+1)}u_\varepsilon) \quad \text{in } \Omega_T. \tag{3.20}$$

Next, let us focus one the case $2s<n$ as the other one can be treated similarly. As $r>0$ we deduce from (3.16) that $\varepsilon^{-1/(1+r)}u_\varepsilon \to 0$ in $L^\infty(0,T;H^s(\mathbb{R}^n))$ and so by Sobolev's embedding in $L^q(0,T;L^{2_s^*}(\Omega))$ for all $1\leq q\leq\infty$ and $2_s^*=\frac{2n}{n-2s}$. Hence, by our assumptions on f_j and [27, Lemma 3.6], we get

$$f_j(\varepsilon^{-1/(r+1)}u_\varepsilon)\to 0 \quad \text{in } L^{q/(r+1)}(0,T;L^{2_s^*/(r+1)}(\Omega)) \tag{3.21}$$

for all $q\geq r+1$ as $\varepsilon\to 0$. Additionally, using (3.19) we know that

$$\varepsilon^{-1}\partial_t R_\varepsilon^{(j)}\to 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{3.22}$$

Therefore, from (3.20)–(3.22), we infer

$$(\gamma_1-\gamma_2)\partial_t v_1 = 0 \quad \text{in } \Omega_T.$$

In particular, this ensures that there holds

$$\int_{\Omega_T}(\gamma_1-\gamma_2)\partial_t(v_1-\eta)(w_2-\psi)^\star\,dx\,dt = 0$$

for any $\psi\in C_c^\infty((W_2)_T)$, where w_2 is the unique solution to

$$\begin{cases} L_{\gamma_2}w=0 & \text{in } \Omega_T,\\ w=\psi & \text{on } (\Omega_e)_T,\\ w(0)=0,\ \partial_t w(0)=0 & \text{on } \Omega. \end{cases}$$

Now, arguing as in the previous section, we get $\gamma_1 = \gamma_2$ in Ω. Hence, (3.17) reduces to

$$f_1(u_\varepsilon) = f_2(u_\varepsilon) \quad \text{in } \Omega_T.$$

Multiplying this identity by $\varepsilon^{-(r+1)}$ and arguing as before, we deduce that

$$f_1(v) = f_2(v) \quad \text{in } \Omega_T,$$

where $v := v_1 = v_2$ as $\gamma_1 = \gamma_2$. One can now show $f_1(x,\tau) = f_2(x,\tau)$ for all $x \in \Omega$ and $\tau \in \mathbb{R}$ exactly as described in [20, p. 29]. Hence, we can conclude the proof. ■

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References

[1] X. Cao, Y.-H. Lin, and H. Liu, Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators. *Inverse Probl. Imaging* **13** (2019), no. 1, 197–210 Zbl 1407.35225 MR 3917858

[2] M. Cekić, Y.-H. Lin, and A. Rüland, The Calderón problem for the fractional Schrödinger equation with drift. *Calc. Var. Partial Differential Equations* **59** (2020), no. 3, article no. 91 Zbl 1439.35563 MR 4092686

[3] G. Covi, T. Ghosh, A. Rüland, and G. Uhlmann, A reduction of the fractional Calderón problem to the local Calderón problem by means of the Caffarelli–Silvestre extension. 2023, arXiv:2305.04227v2

[4] G. Covi, J. Railo, T. Tyni, and P. Zimmermann, Stability estimates for the inverse fractional conductivity problem. *SIAM J. Math. Anal.* **56** (2024), no. 2, 2456–2487 Zbl 1537.35409 MR 4719389

[5] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 5: Evolution problems* I. Springer, Berlin, 1992 Zbl 0755.35001 MR 1156075

[6] A. Feizmohammadi, Fractional Calderón problem on a closed Riemannian manifold. *Trans. Amer. Math. Soc.* **377** (2024), no. 4, 2991–3013 Zbl 1543.35266 MR 4745418

[7] A. Feizmohammadi, T. Ghosh, K. Krupchyk, and G. Uhlmann, Fractional anisotropic Calderón problem on closed Riemannian manifolds. 2021, arXiv:2112.03480v1

[8] A. Feizmohammadi, K. Krupchyk, and G. Uhlmann, Calderón problem for fractional Schrödinger operators on closed Riemannian manifolds. 2024, arXiv:2407.16866v1

[9] S.-R. Fu, Y. Yu, and P. Zimmermann, The Calderón problem for third order nonlocal wave equations with time-dependent nonlinearities and potentials. *J. Differential Equations* **463** (2026), article no. 114164 MR 5026190

[10] T. Ghosh, Y.-H. Lin, and J. Xiao, The Calderón problem for variable coefficients nonlocal elliptic operators. *Comm. Partial Differential Equations* **42** (2017), no. 12, 1923–1961 Zbl 1387.35619 MR 3764930

[11] T. Ghosh, M. Salo, and G. Uhlmann, The Calderón problem for the fractional Schrödinger equation. *Anal. PDE* **13** (2020), no. 2, 455–475 Zbl 1439.35530 MR 4078233

[12] M. Kar, Y.-H. Lin, and P. Zimmermann, Determining coefficients for a fractional p-Laplace equation from exterior measurements. *J. Differential Equations* **406** (2024), 338–365 Zbl 1551.35485 MR 4770286

[13] P.-Z. Kow, Y.-H. Lin, and J.-N. Wang, The Calderón problem for the fractional wave equation: uniqueness and optimal stability. *SIAM J. Math. Anal.* **54** (2022), no. 3, 3379–3419 Zbl 1492.35427 MR 4434352

[14] R.-Y. Lai and Y.-H. Lin, Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.* **216** (2022), article no. 112699 Zbl 1481.35212 MR 4348315

[15] C.-L. Lin, Y.-H. Lin, and G. Uhlmann, The Calderón problem for nonlocal parabolic operators: A new reduction from the nonlocal to the local. 2023, arXiv:2308.09654v1

[16] Y.-H. Lin and H. Liu, Inverse problems for fractional equations with a minimal number of measurements. *Commun. Anal. Comput.* **1** (2023), no. 1, 72–93 Zbl 1554.35408 MR 4807750

[17] Y.-H. Lin, G. Nakamura, and P. Zimmermann, The Calderón problem for the Schrödinger equation in transversally anisotropic geometries with partial data. 2024, arXiv:2408.08298v1

[18] Y.-H. Lin, J. Railo, and P. Zimmermann, The Calderón problem for a nonlocal diffusion equation with time-dependent coefficients. *Rev. Mat. Iberoam.* **41** (2025), no. 3, 1129–1172 Zbl 1564.35299 MR 4892768

[19] Y.-H. Lin, T. Tyni, and P. Zimmermann, Well-posedness and inverse problems for semilinear nonlocal wave equations. *Nonlinear Anal.* **247** (2024), article no. 113601 Zbl 1547.35767 MR 4767473

[20] Y.-H. Lin, T. Tyni, and P. Zimmermann, Optimal Runge approximation for nonlocal wave equations and unique determination of polyhomogeneous nonlinearities. *Calc. Var. Partial Differential Equations* **64** (2025), no. 9, article no. 280 Zbl 08105628 MR 4970252

[21] Y.-H. Lin and P. Zimmermann, Unique determination of coefficients and kernel in nonlocal porous medium equations with absorption term. 2023, arXiv:2305.16282v1

[22] Y.-H. Lin and P. Zimmermann, Approximation and uniqueness results for the nonlocal diffuse optical tomography problem. 2024, arXiv:2406.06226v2 Zbl arXiv:2406.06226

[23] J. Railo and P. Zimmermann, Fractional Calderón problems and Poincaré inequalities on unbounded domains. *J. Spectr. Theory* **13** (2023), no. 1, 63–131 Zbl 1526.35323 MR 4620353

[24] J. Railo and P. Zimmermann, Low regularity theory for the inverse fractional conductivity problem. *Nonlinear Anal.* **239** (2024), article no. 113418 Zbl 1530.35368 MR 4658532

[25] A. Rüland and M. Salo, The fractional Calderón problem: low regularity and stability. *Nonlinear Anal.* **193** (2020), article no. 111529 Zbl 1448.35581 MR 4062981

[26] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)* **125** (1987), no. 1, 153–169 Zbl 0625.35078 MR 0873380

[27] P. Zimmermann, Calderón problem for nonlocal viscous wave equations: Unique determination of linear and nonlinear perturbations. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **120** (2026), no. 2,, article no. 31 MR 5013439

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