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Asymptotic expansion for groupoids and Roe-type algebras

Xulong Lu, Qin Wang, and Jiawen Zhang

Abstract. In this paper, we introduce a notion of expansion for groupoids, which recovers the classical notion of expander graphs by a family of pair groupoids and expanding actions in measure by transformation groupoids. We also consider an asymptotic version for expansion and establish structural theorems, showing that asymptotic expansion can be approximated by domains of expansions. On the other hand, we introduce dynamical propagation and quasi-locality for operators on groupoids and the associated Roe-type algebras. Our main results characterise when these algebras possess block-rank-one projections by means of asymptotic expansion, which generalises the crucial ingredients in previous works to provide counterexamples to the coarse Baum–Connes conjecture.

1. Introduction

Over the last few decades, the phenomenon of expansion has been discovered and extensively studied across various branches of mathematics. For instance, in graph theory, the expansion phenomenon leads to the notion of expander graphs, which plays an important role not only in pure and applied mathematics but also in theoretical computer science (see, e.g., [24]). In dynamics, the expansion phenomenon leads to the notion of expanding actions in measure, which turns out to be equivalent to the classic notion of spectral gap for measure-preserving actions (see [32]), and numerous examples have been discovered (e.g., [5, 6, 13]).

Recently, an asymptotic version of expansion was introduced in different areas of mathematics [20, 23], which is more stable under small perturbations and hence leads to important applications in operator algebras and higher index theory [17, 21, 22]. A crucial step therein is structural type theorems, showing that objects with asymptotic expansion can be approximated by those with expansion. In dynamics, asymptotic expansion also provides a new quantitative viewpoint on the classic notion of strong ergodicity, introduced in [8, 29, 30] in relation to the Ruziewicz problem, Kazhdan's property (T) and amenability.

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In this paper, we aim to generalise and unify the theory of asymptotic expansion from different areas (including graph theory and dynamical systems) in the language of groupoids. Groupoids provide a framework encompassing both groups and spaces. They arise naturally in a variety of research areas such as dynamical systems, topology and geometry, geometric group theory and operator algebras, building bridges between all these areas of mathematics (see, e.g., [7, 14]).

To achieve this, we introduce the notion of expansion and asymptotic expansion for groupoids, generalising both (asymptotic) expander graphs and (asymptotically) expanding actions in measure, and establish structural type theorems in the groupoid setting. Furthermore, we introduce two classes of operator algebras associated to the dynamics of groupoids, generalising the classical Roe and quasi-local algebras from higher index theory. Our main results show that the existence of certain projection operators in these operator algebras characterises asymptotic expansion of groupoids, which provide a unified approach to results in [17, 21, 22] and allow a boarder range of examples and applications.

We will now give a more detailed overview of this work.

1.1. Expansion and asymptotic expansion

To motivate our notion of asymptotic expansion, let us first recall the notion of asymptotically expanding actions in measure from [23].

Let $\rho : \Gamma \curvearrowright X$ be a measure-class-preserving action of a countable group Γ acting on a probability space (X, μ) and ℓ_Γ be a proper length function on Γ (i.e., for each $L > 0$, the ball $B_L := \{\gamma \in \Gamma \mid \ell_\Gamma(\gamma) \leq L\}$ is finite). The action is called *asymptotically expanding in measure* μ if for any $\alpha \in \left(0, \frac{1}{2}\right]$ there exists $C_\alpha, L_\alpha > 0$ such that for any measurable $A \subseteq X$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, we have $\mu(B_{L_\alpha} \cdot A) > (1 + C_\alpha)\mu(A)$. When the functions $\alpha \mapsto L_\alpha$ and $\alpha \mapsto C_\alpha$ can be taken to be constant functions, then the action is called *expanding in measure* μ.

To generalise the above to groupoids (see Section 2.1 for definitions), we consider the associated *transformation groupoid* $X \rtimes \Gamma$ with source $\mathrm{s}(x, \gamma) = \gamma^{-1}x$, range $\mathrm{r}(x, \gamma) = x$ and inverse $(x, \gamma)^{-1} = (\gamma^{-1}x, \gamma^{-1})$. The length function ℓ_Γ naturally gives rise to a length function ℓ on $X \rtimes \Gamma$ by $\ell(x, \gamma) := \ell_\Gamma(\gamma)$ for $x \in X$ and $\gamma \in \Gamma$.

This leads to the following setting of our paper: Let $\mathcal{G}$ be a groupoid with a length function ℓ (see Definition 2.1) and the unit space $\mathcal{G}^{(0)}$ be equipped with a measure μ on some σ-algebra $\mathcal{R}$. To abstract measure-class-preserving transformations, we introduce the following: A bisection $K \subseteq \mathcal{G}$ is called *admissible* if its source $\mathrm{s}(K)$ and range $\mathrm{r}(K)$ are measurable, the induced bijection

$$\tau_K := \mathrm{r}|_K \circ (\mathrm{s}|_K)^{-1} : \mathrm{s}(K) \to \mathrm{r}(K) \tag{1.1}$$

is a measure-class-preserving measurable isomorphism and the length $\ell(K) := \sup\{\ell(x) \mid x \in K\}$ is finite. Moreover, a subset $K \subseteq \mathcal{G}$ is called *decomposable* if $K = \bigcup_{i=1}^{N} K_i$ for

$N \in \mathbb{N}$ and admissible bisections K_i. Note that for transformation groupoids, any measurable subset (with respect to the product structure) with finite length is decomposable, while unfortunately, this does not hold in general.

Now we are in the position to introduce our notion of asymptotic expansion.

Definition A (Definitions 3.1 and 3.2). Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. We say that $\mathcal{G}$ is *asymptotically expanding (in measure μ)* if for any $\alpha \in \left(0, \frac{1}{2}\right]$, there exists a decomposable $K_\alpha \subseteq \mathcal{G}$ with $\ell(K_\alpha) < \infty$ such that for any $A \in \mathcal{R}$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, then $\mu(\mathrm{r}(K_\alpha \cdot A) \setminus A) > C_\alpha \mu(A)$. If $K_\alpha \equiv K$, then we say that $\mathcal{G}$ is *expanding (in measure μ)*.

It is clear that in the case of transformation groupoids, this recovers the notion of (asymptotically) expanding actions (see Section 6.1). On the other hand, when considering pair groupoids, a family version of Definition A recovers the notion of (asymptotic) expander graphs (see Section 6.2 for details).

We establish the following structure theorem for asymptotic expansion, showing that it can be approximated by domains of expansion.

Theorem B (Theorems 3.9 and 3.18). *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Then the following are equivalent:*

(1) *$\mathcal{G}$ is asymptotically expanding in measure.*

(2) *$\mathcal{G}^{(0)}$ admits an exhaustion by domains of expansion with bounded ratio.*

(3) *$\mathcal{G}^{(0)}$ admits an exhaustion by domains of Markov expansion with bounded ratio.*

Roughly speaking, here the "domain of expansion" is a measurable subset Y of $\mathcal{G}^{(0)}$ such that the reduction $\mathcal{G}_Y^Y$ is expanding (see Definition 3.3), and "exhaustion" means a sequence of domains Y_n such that $\mu(\mathcal{G}^{(0)} \setminus Y_n) \to 0$. We also construct a Markov kernel for each domain (see Definition 3.14) and consider a Markovian version of expansion, which will play a key role later to produce certain projection operators. Details can be found in Sections 3.2 and 3.3.

Theorem B provides a unified approach for the structure results in [17,21,22] and also allows new examples (see Section 6). Even for asymptotic expander graphs and asymptotically expanding actions, our proof simplifies the original ones in a systematic way.

1.2. Dynamical propagation and quasi-locality

Now we introduce two classes of operator algebras associated to groupoids, which encode dynamical information and have their roots in higher index theory.

Recall that for a metric space, the Roe and the quasi-local algebra were introduced by Roe in his pioneering work on higher index theory [27], where he discovered that higher indices of elliptic differential operators on open manifolds belong to K-theories of the Roe algebra (see also Engel's work [10, 11]).

To compute their K-theories, a practical approach is to consult the coarse Baum–Connes conjecture [4], a central conjecture in higher index theory and closely related to other conjectures like the Novikov conjecture and the Gromov–Lawson conjecture. Unfortunately, counterexamples were discovered in [15,17,22,28] using (asymptotic) expanders and (asymptotically) expanding actions. Due to their importance, Roe and quasi-local algebras have been extensively studied, and several variants have also been introduced (see [3, 9, 16, 22, 25, 31, 34, 35]).

Inspired by these works, we introduce the following notions of dynamical propagation and quasi-locality for operators on groupoids. Note that in the context of transformation groupoids and pair groupoids, the following recovers the original (dynamical) Roe and quasi-local algebras (see Sections 6.1 and 6.2).

Definition C (Definitions 4.1 and 4.2). Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. For an operator $T \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$, we say:

1. T has finite *dynamical propagation* if there exists a unital decomposable $K \subseteq \mathcal{G}$ with $\ell(K) < \infty$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, then $\chi_A T \chi_B = 0$.
2. T is *dynamically quasi-local* if for any $\varepsilon > 0$, there exists a unital decomposable $K_\varepsilon \subseteq \mathcal{G}$ with $\ell(K_\varepsilon) < \infty$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K_\varepsilon \cdot A) \cap B) = 0$, then $\|\chi_A T \chi_B\| < \varepsilon$.

The *dynamical Roe algebra of* $\mathcal{G}$ is the norm closure of all operators with finite dynamical propagation, denoted by $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$. The *dynamical quasi-local algebra of* $\mathcal{G}$ is the set of all dynamically quasi-local operators, denoted by $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$.

Recall that in graph theory and dynamics, a key consequence of asymptotic expansion is that the Roe-type algebras possess block-rank-one projections (called the ghost projections), which are crucial to provide counterexamples to the coarse Baum–Connes conjecture (see [17, 21, 22]). In the context of groupoids, we study dynamical propagation and quasi-locality of block-rank-one projections and establish a more general result, which unifies and simplifies the previous ones.

More precisely, we first consider a rank-one projection $P \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ and a unit vector $\xi \in L^2(\mathcal{G}^{(0)}, \mu)$ in the range of P. This induces a probability measure ν on $\mathcal{G}^{(0)}$ given by $\mathrm{d}\nu(x) := |\xi(x)|^2 \mathrm{d}\mu(x)$ for $x \in \mathcal{G}^{(0)}$, called the *associated measure*.

The following is our main result.

Theorem D (Theorem 5.3). *Suppose $\mathcal{G}$ is a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ is a measure space. Let $P \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ be a rank-one projection, and ν the associated probability measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

1. $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$.
2. $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$.
3. *$\mathcal{G}$ is asymptotically expanding in measure ν.*

The proof of Theorem D is quite involved. Firstly, we reduce to averaging projections by changing measures. Applying Theorem B, we obtain an exhaustion by domain of Markov expansion, and on each domain, we obtain a rank-one projection in the dynamical Roe algebra by functional calculus. Finally, these projections converge in norm to the averaging projection, and we finish the proof.

After establishing Theorem D, we further consider a family version by chasing parameters. We obtain a family version of Theorem D (Theorem 5.3′), which characterises when the dynamical Roe and quasi-local algebras possess block-rank-one projections. This recovers the most technical results in [17, 21, 22].

Finally, we apply our results to several examples, including the known results on asymptotic expanders and asymptotically expanding actions. Moreover, we study new examples including groupoid actions on fibred spaces, the HLS groupoid and graph groupoids, and establish corresponding results therein.

1.3. Organisation

In Section 2, we recall background knowledge for groupoids and Markov kernels. In Section 3, we introduce the notion of (asymptotic) expansion for groupoids and establish the structure result (Theorem B). In Section 4, we introduce dynamical propagation and quasi-locality and relate asymptotic expansion to the quasi-locality of the averaging projection (Proposition 4.7). Then in Section 5, we accomplish the proof of Theorem D, making use of Theorem B. Finally, in Section 6, we explain in detail how our results recover the original ones for asymptotic expanders (by a family version of pair groupoids) and asymptotically expanding actions (by transformation groupoids) and provide new examples.

2. Preliminaries

In this section, we recall the notions of groupoids and Markov kernels.

2.1. Basic notions for groupoids

Recall that a *groupoid* $\mathcal{G}$ is a small category which consists of a set $\mathcal{G}$, a subset $\mathcal{G}^{(0)}$ called the *unit space*, two maps $\mathrm{s}, \mathrm{r} : \mathcal{G} \to \mathcal{G}^{(0)}$ called the *source* and the *range* maps, respectively, a composition law $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$ for

$$(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid \mathrm{s}(\gamma_1) = \mathrm{r}(\gamma_2)\}$$

and an inverse map $\gamma \mapsto \gamma^{-1}$. These operations satisfy a couple of axioms, including the associativity law and the fact that $\mathcal{G}^{(0)}$ acts as units. For $A, B \subseteq \mathcal{G}$, we denote $A^{-1} := \{\gamma^{-1} \mid \gamma \in A\}$ and

$$A \cdot B := \{\gamma \in \mathcal{G} \mid \gamma = \gamma_1\gamma_2, \text{ where } \gamma_1 \in A,\ \gamma_2 \in B \text{ with } \mathrm{s}(\gamma_1) = \mathrm{r}(\gamma_2)\}.$$

We say that A is *symmetric* if $A = A^{-1}$ and *unital* if $\mathcal{G}^{(0)} \subseteq A$. A subset $A \subseteq \mathcal{G}$ is called a *bisection* if the restrictions of s, r to A are injective, and recall from (1.1) that we have the induced bijection $\tau_A := \mathrm{r}|_A \circ (\mathrm{s}|_A)^{-1} : \mathrm{s}(A) \to \mathrm{r}(A)$.

Definition 2.1. Let $\mathcal{G}$ be a groupoid. A *length function* on $\mathcal{G}$ is a map $\ell : \mathcal{G} \to \mathbb{R}^+$ such that $\ell(\mathcal{G}^{(0)}) = \{0\}$, $\ell(\gamma^{-1}) = \ell(\gamma)$ for all $\gamma \in \mathcal{G}$ and $\ell(\gamma_1\gamma_2) \leq \ell(\gamma_1) + \ell(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \mathcal{G}$ with $\mathrm{s}(\gamma_1) = \mathrm{r}(\gamma_2)$.

Throughout the paper, let $\mathcal{G}$ be a groupoid with length function ℓ and $\mathcal{G}^{(0)}$ equipped with a (not necessarily finite) measure μ on some σ-algebra $\mathcal{R}$.

Definition 2.2. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. A bisection $K \subseteq \mathcal{G}$ is called *admissible* if $\mathrm{s}(K), \mathrm{r}(K) \in \mathcal{R}$, $\tau_K : \mathrm{s}(K) \to \mathrm{r}(K)$ is a measure-class-preserving measurable isomorphism and $\ell(K) := \sup\{\ell(x) \mid x \in K\}$ is finite.

We record the following, whose proof is straightforward and hence omitted.

Lemma 2.3. *If $K, K_1, K_2 \subseteq \mathcal{G}$ are admissible bisections, then K^{-1} and $K_1 \cdot K_2$ are also admissible bisections.*

Motivated by the case of group actions explained in Section 1, we introduce the following.

Definition 2.4. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. A subset $K \subseteq \mathcal{G}$ is called *decomposable* if $K = \bigcup_{i=1}^N K_i$ for $N \in \mathbb{N}$ and admissible bisections K_i.

Given a decomposable $K \subseteq \mathcal{G}$, a decomposition $K = \bigcup_{i=1}^N K_i$ is called *unital* if $K_i = \mathcal{G}^{(0)}$ for some i and *symmetric* if there exists a bijection $\sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ such that $K_i^{-1} = K_{\sigma(i)}$. We say that K is *unital symmetric N-decomposable* if there exists a unital symmetric decomposition $K = \bigcup_{i=1}^N K_i$ for admissible K_i.

Applying Lemma 2.3, we obtain the following.

Lemma 2.5. *If K is unital symmetric N-decomposable with $\ell(K) \leq L$, then K^m is unital symmetric N^m-decomposable with $\ell(K^m) \leq mL$ for any $m \in \mathbb{N}$.*

The following observation is useful to define expansion for groupoids later.

Lemma 2.6. *For decomposable $K \subseteq \mathcal{G}$ and measurable $A \subseteq \mathcal{G}^{(0)}$, $\mathrm{r}(K \cdot A)$ is measurable.*

Proof. By definition, we can write $K = \bigcup_{i=1}^N K_i$ for admissible bisection K_i. Then

$$\mathrm{r}(K \cdot A) = \mathrm{r}\left(\bigcup_{i=1}^N K_i \cdot A\right) = \bigcup_{i=1}^N \tau_{K_i}(A \cap \mathrm{s}(K_i)),$$

which is measurable since each τ_{K_i} is a measurable isomorphism on $\mathrm{s}(K_i)$. ∎

2.2. Markov kernels

Here, we recall a few elementary properties of reversible Markov kernels. We refer to the first chapters of [26] for more details.

Definition 2.7. Let $\mathcal{S}$ be a σ-algebra on a set X. A *Markov kernel* on the measurable space $(X, \mathcal{S})$ is a function $\Pi\colon X \times \mathcal{S} \to [0, 1]$ such that:

(1) For every $x \in X$, the function $\Pi(x, \text{-})\colon \mathcal{S} \to [0, 1]$ is a probability measure.

(2) For every $A \in \mathcal{S}$, the function $\Pi(\text{-}, A)\colon X \to [0, 1]$ is $\mathcal{S}$-measurable.

The associated *Markov operator* $\mathfrak{P}$ is a linear operator on the space of bounded $\mathcal{S}$-measurable functions, defined by

$$\mathfrak{P} f(x) := \int_X f(y)\Pi(x, \mathrm{d}y). \tag{2.1}$$

Definition 2.8. Given a measure μ on $(X, \mathcal{S})$ and $A \in \mathcal{S}$, the *(μ-)size of the boundary* of A (with respect to Π) is defined as

$$|\partial_\Pi(A)|_\mu := \int_A \Pi(x, X \setminus A)\mathrm{d}\mu(x).$$

We are only interested in the special case of reversible Markov kernels.

Definition 2.9. A Markov kernel Π is called *reversible* if there exists a measure m on $(X, \mathcal{S})$ such that

$$\int_X f(x)\mathfrak{P}g(x)\mathrm{d}m(x) = \int_X \mathfrak{P} f(x)g(x)\mathrm{d}m(x)$$

for every pair of measurable bounded functions $f, g : X \to \mathbb{R}$. The measure m is called a *reversing measure* for Π (note that m need not be unique in general). In this case, we also say that Π is a reversible Markov kernel *on* (X, m).

Given a reversing measure m on X, the Markov operator $\mathfrak{P}$ can be regarded as a bounded self-adjoint operator on $L^2(X, m)$ with $\|\mathfrak{P}\| \leq 1$. Define the *Laplacian* of Π as $\Delta := 1 - \mathfrak{P}$, which is positive self-adjoint with spectrum contained in $[0, 2]$.

Let Π be a reversible Markov kernel on (X, m), where m is a finite measure. Then all constant functions on X belong to $L^2(X, m)$ and are fixed by $\mathfrak{P}$. It follows that $\|\mathfrak{P}\| = 1$ and 1 belongs to the spectrum of $\mathfrak{P}$. Denote the orthogonal complement of the constant functions in $L^2(X, m)$ by $L^2_0(X, m)$, that is,

$$L^2_0(X, m) := \left\{ f \in L^2(X, m) \;\middle|\; \int_X f(x)\mathrm{d}m(x) = 0 \right\}.$$

Note that $L^2_0(X, m)$ is $\mathfrak{P}$-invariant and that the spectrum of the restriction of $\mathfrak{P}$ on $L^2_0(X, m)$ is contained in $[-1, 1]$. We denote the supremum of this spectrum by $\lambda \in \mathbb{R}$. We make the following definition.

Definition 2.10. A reversible Markov kernel on a finite measure space (X, m) is said to have a *spectral gap* if $\lambda < 1$.

On the other hand, we recall the notion of the Cheeger constant as follows.

Definition 2.11. The *Cheeger constant* for a reversible Markov kernel Π on a finite measure space (X, m) is defined to be

$$\kappa := \inf\left\{ \frac{|\partial_\Pi(A)|_m}{m(A)} \;\middle|\; A \in \mathcal{S},\ 0 < m(A) \le \frac{1}{2} m(X) \right\}.$$

Consequently, we have the following significant result relating the spectral gap to the Cheeger constant from [19] (see also [22]).

Theorem 2.12 ([19, Theorem 2.1]). *Let Π be a reversible Markov kernel on (X, m), where m is finite. Then*

$$\frac{\kappa^2}{2} \le 1 - \lambda \le 2\kappa.$$

3. Asymptotic expansion in measure and structure theorems

In this section, we introduce the notion of expansion and asymptotic expansion for groupoids and then establish their structure theory. This generalises both (measured) asymptotic expanders in [17,20,21] and asymptotic expansion in measure for group actions in [22,23].

Again, we always assume that $\mathcal{G}$ is a groupoid with a length function ℓ and $\mathcal{G}^{(0)}$ is equipped with a measure μ on some σ-algebra $\mathcal{R}$. In this section, we require μ to be finite. For simplicity, further assume that μ is a *probability measure*. Note that all the results in this section are also available for finite measures by rescaling.

3.1. Expansion and asymptotic expansion

We first introduce the following.

Definition 3.1. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. We say that $\mathcal{G}$ is *expanding (in measure μ)* if there exist $C, N, L > 0$ and a unital symmetric[1] N-decomposable $K \subseteq \mathcal{G}$ with $\ell(K) \le L$ such that for any $A \in \mathcal{R}$ with $0 < \mu(A) \le \frac{1}{2}$, then $\mu(\mathrm{r}(K \cdot A) \setminus A) > C\mu(A)$. In this case, we also say that $\mathcal{G}$ is (C, N, L)*-expanding*.

We also consider the following asymptotic version of Definition 3.1.

[1]Since we are only interested in the existence of K, we can always make it unital and symmetric.

Definition 3.2. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. We say that $\mathcal{G}$ is *asymptotically expanding (in measure μ)* if for any $\alpha \in \left(0, \frac{1}{2}\right]$, there exist $C_\alpha, N_\alpha, L_\alpha > 0$ and a unital symmetric N_α-decomposable $K_\alpha \subseteq \mathcal{G}$ with $\ell(K_\alpha) \leq L_\alpha$ such that for any $A \in \mathcal{R}$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, then $\mu(\mathrm{r}(K_\alpha \cdot A) \setminus A) > C_\alpha \mu(A)$. The functions $\alpha \mapsto C_\alpha$, $\alpha \mapsto N_\alpha$ and $\alpha \mapsto L_\alpha$ are called *expansion parameters* of $\mathcal{G}$ (which are not unique).

To establish the structure theorem, we also need the following notion.

Definition 3.3. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. A measurable subset $Y \subseteq \mathcal{G}^{(0)}$ is called a *domain of asymptotic expansion*[2] if for any $\alpha \in \left(0, \frac{1}{2}\right]$, there exist $C_\alpha, N_\alpha, L_\alpha > 0$ and a unital symmetric N_α-decomposable $K_\alpha \subseteq \mathcal{G}$ with $\ell(K_\alpha) \leq L_\alpha$ such that for any measurable $A \subseteq Y$ with $\alpha\mu(Y) \leq \mu(A) \leq \frac{1}{2}\mu(Y)$, we have

$$\mu((\mathrm{r}(K_\alpha \cdot A) \setminus A) \cap Y) > C_\alpha \mu(A).$$

The functions $\alpha \mapsto C_\alpha$, $\alpha \mapsto N_\alpha$ and $\alpha \mapsto L_\alpha$ are called *expansion parameters* for Y.

If $C_\alpha \equiv C$, $N_\alpha \equiv N$, $L_\alpha \equiv L$ and $K_\alpha \equiv K$, then we also say that Y is a *domain of (C, N, L)-expansion* (or simply a *domain of (N, L)-expansion* or a *domain of expansion*).

Here, we collect several useful facts, generalising those in [21, 23].

Lemma 3.4. *Let $Y \subseteq \mathcal{G}^{(0)}$ be a domain of asymptotic expansion with expansion parameters $\alpha \mapsto C_\alpha, \alpha \mapsto N_\alpha, \alpha \mapsto L_\alpha$. Then for any $\alpha \in \left(0, \frac{1}{2}\right]$ and $\beta \in \left[\frac{1}{2}, 1\right)$, there exist $C, N, L > 0$ and a unital symmetric N-decomposable $K \subseteq \mathcal{G}$ with $\ell(K) \leq L$ such that for any measurable $A \subseteq Y$ with $\alpha\mu(Y) \leq \mu(A) \leq \beta\mu(Y)$, we have*

$$\mu((\mathrm{r}(K \cdot A) \setminus A) \cap Y) > C\mu(A).$$

Here, C, N, L only depend on α, β and expansion parameters for Y with $N = N_\alpha + N_{\frac{1-\beta}{2\beta}}$, $L = \max\left\{L_\alpha, L_{\frac{1-\beta}{2\beta}}\right\}$ and $C = \min\left\{C_\alpha, \frac{1-\beta}{2\beta} C_{\frac{1-\beta}{2}}, \frac{1-\beta}{2\beta}\right\}$.

The proof of Lemma 3.4 is similar to that of [23, Lemma 3.8] and hence omitted.

Lemma 3.5. *Assume $\mathcal{G}$ is asymptotically expanding in measure. Then for any $\beta \in (0, 1], \alpha \in \left(0, \frac{1}{2}\right]$ and $C \in (0, 1)$, there exists a unital symmetric N-decomposable $K \subseteq \mathcal{G}$ with $\ell(K) \leq L$ such that for any measurable $Y \subseteq \mathcal{G}^{(0)}$ with $\mu(Y) \geq \beta$ and measurable $A \subseteq Y$ with $\alpha\mu(Y) \leq \mu(A) \leq \frac{1}{2}\mu(Y)$, we have $\mu((\mathrm{r}(K \cdot A) \setminus A) \cap Y) > C\mu(A)$. Here, N, L only depend on α, β, C and expansion parameters of $\mathcal{G}$.*

[2]This is equivalent to saying that the reduction $\mathcal{G}_Y^Y := \mathrm{s}^{-1}(Y) \cap \mathrm{r}^{-1}(Y)$ is asymptotically expanding.

Proof. Fix $\alpha \in (0, \frac{1}{2}]$, $\beta \in (0, 1]$, $C \in (0, 1)$ and measurable $Y \subseteq \mathcal{G}^{(0)}$ with $\mu(Y) \geq \beta$. Set $\varepsilon := \frac{1+C}{2}\mu(Y)$ and $\gamma := \mu(\mathcal{G}^{(0)} \setminus Y) + \varepsilon = 1 - \frac{1-C}{2}\mu(Y)$. It suffices to find a unital symmetric decomposable $K \subseteq \mathcal{G}$ such that for any measurable $A \subseteq Y$ with $\alpha\mu(Y) \leq \mu(A) \leq \frac{1}{2}\mu(Y)$, then $\mu(\mathrm{r}(K \cdot A)) > \gamma$. If this holds, then the proof would be completed by the following estimate:

$$\mu(\mathrm{r}(K \cdot A) \cap Y) > \gamma - \mu(\mathcal{G}^{(0)} \setminus Y) = \varepsilon \geq \frac{2\varepsilon}{\mu(Y)}\mu(A) = (1 + C)\mu(A).$$

Now we aim to find such a K. Since $\mathcal{G}$ is asymptotically expanding, Lemma 3.4 provides $C', L', N' > 0$ and a unital symmetric N'-decomposable $K' \subseteq \mathcal{G}$ with $\ell(K') \leq L'$ such that for any measurable $A' \subseteq \mathcal{G}^{(0)}$ with $\alpha\beta \leq \mu(A') \leq 1 - \frac{1-C}{2}\beta$, then

$$\mu(\mathrm{r}(K' \cdot A')) > (1 + C')\mu(A') \geq (1 + C')\alpha\beta. \tag{3.1}$$

Set m to be the minimal integer satisfying $(1 + C')^m \geq \frac{1}{\alpha\beta}$, and take $K := (K')^m$. Then Lemma 2.5 shows that K is unital symmetric $(N')^m$-decomposable and $\ell(K) \leq mL'$.

Now for any measurable $A \subseteq Y$ with $\alpha\mu(Y) \leq \mu(A) \leq \frac{1}{2}\mu(Y)$, we need to show that $\mu(\mathrm{r}(K \cdot A)) > \gamma$. If not, then $\mu(\mathrm{r}((K')^m \cdot A)) \leq \gamma$. This shows that

$$\alpha\beta \leq \mu(\mathrm{r}((K')^i \cdot A)) \leq \gamma \leq 1 - \frac{1-C}{2}\beta \quad \text{for any } i = 0, 1, \ldots, m.$$

Applying inequality (3.1) inductively, we obtain

$$\mu(\mathrm{r}((K')^m \cdot A)) > (1 + C')^m\mu(A) \geq (1 + C')^m\alpha\beta \geq 1 > \gamma,$$

which leads to a contradiction. ∎

3.2. The structure theorem

Here, we introduce the structure theorem for asymptotic expansion on groupoids. The main idea is to approximate by domains of expansions in the following sense.

Definition 3.6. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. We say that a sequence of measurable subsets $\{Y_n\}_{n\in\mathbb{N}}$ in $\mathcal{G}^{(0)}$ *forms a (measured) exhaustion of* $\mathcal{G}^{(0)}$ if $\lim_{n\to\infty} \mu(Y_n) = 1$.

For technical reasons, we also need to consider the following.

Definition 3.7. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Given an admissible bisection $K \subseteq \mathcal{G}$, we set

$$\mathfrak{R}(K, x) := \frac{\mathrm{d}(\tau_K^{-1})_*\mu|_{\mathrm{r}(K)}}{\mathrm{d}\mu|_{\mathrm{s}(K)}}(x)$$

the Radon–Nikodym derivative at $x \in \mathrm{s}(K)$, where τ_K comes from (1.1). Here, $\mu|_{\mathrm{s}(K)}$ and $\mu|_{\mathrm{r}(K)}$ are the restrictions of μ, and $(\tau_K^{-1})_*\mu|_K$ is the pushforward measure.

To simplify the statement of our main result, let us package the derivative information into the notion of a domain of expansion.

Definition 3.8. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. For measurable $Y \subseteq \mathcal{G}^{(0)}$, we say that Y is a *domain of expansion with bounded ratio* if there exist $C, N, L > 0$ and a unital symmetric N-decomposable $K \subseteq \mathcal{G}$ with $\ell(K) \leq L$, together with a unital symmetric decomposition $K = \bigcup_{i=1}^{N} K_i$, satisfying the following:

(1) For measurable $A \subseteq Y$ with $0 < \mu(A) \leq \frac{1}{2}\mu(Y)$, then we have

$$\mu((\mathrm{r}(K_\alpha \cdot A) \setminus A) \cap Y) > C_\alpha \mu(A).$$

(2) There exists $\theta \geq 1$ such that $\frac{1}{\theta} \leq \mathfrak{R}(K_i, x) \leq \theta$ for any $x \in Y$ and $i \in \{1, 2, \ldots, N\}$ with $\tau_{K_i}(x) \in Y$. (As a priori to $\tau_{K_i}(x) \in Y$, we have $x \in \mathrm{s}(K_i)$ and hence $\mathfrak{R}(K_i, x)$ makes sense.)

In this case, we say that Y is a *domain of* (C, N, L)*-expansion with ratio bounded by* θ.

The following is our structure theorem.

Theorem 3.9. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Then the following are equivalent:*

(1) *The groupoid $\mathcal{G}$ is asymptotically expanding in measure.*

(2) *The unit space $\mathcal{G}^{(0)}$ admits an exhaustion by domains Y_n of (C_n, N_n, L_n)-expansion of ratio bounded*[3] *by θ_n for $C_n, N_n, L_n > 0$ and $\theta_n \geq 1$.*

(3) *The unit space $\mathcal{G}^{(0)}$ admits an exhaustion by domains of asymptotic expansion.*

A key tool to prove Theorem 3.9 is to consider maximal Følner sets as follows.

Definition 3.10. Given measurable $Y \subseteq \mathcal{G}^{(0)}$, decomposable $K \subseteq \mathcal{G}$ and $\varepsilon > 0$, we say that a measurable subset $F \subseteq Y$ is (ε, K)*-Følner in* Y if $\mu(F) \leq \frac{1}{2}\mu(Y)$ and

$$\mu((\mathrm{r}(K \cdot F) \setminus F) \cap Y) \leq \varepsilon \mu(F).$$

Now fix measurable $Y \subseteq \mathcal{G}^{(0)}$, decomposable $K \subseteq \mathcal{G}$ and $\varepsilon > 0$. Denote the set of all (ε, K)-Følner sets in Y by $\mathcal{F}_{\varepsilon,K}$. Consider the equivalence relation on $\mathcal{F}_{\varepsilon,K}$ by setting $F \sim F'$ in $\mathcal{F}_{\varepsilon,K}$ if and only if they differ by a null set. Define a partial order on $\mathcal{F}_{\varepsilon,K}/\sim$ by setting $[F] \sqsubseteq [F']$ if $F \subseteq F'$ up to null sets.

The following lemmas are similar to [23, Lemmas 4.2 and 4.4], and hence we omit their proofs.

[3] The extra requirement on the Radon–Nikodym derivatives will play an important role in proving our main results in Section 5.

Lemma 3.11. *The partial ordered set* $(\mathcal{F}_{\varepsilon,K}/\sim, \sqsubseteq)$ *has maximal elements.*

Lemma 3.12. *Given measurable* $Y \subseteq \mathcal{G}^{(0)}$*, decomposable* $K \subseteq \mathcal{G}$ *and* $\varepsilon > 0$*, let* $F_{\varepsilon,K} \subseteq Y$ *be a maximal* (ε, K)*-Følner set in* Y*. Then for any measurable* $A \subseteq Y \setminus F_{\varepsilon,K}$ *with* $0 < \mu(A) \leq \frac{1}{2}\mu(Y) - \mu(F_{\varepsilon,K})$*, we have*

$$\mu(((\mathrm{r}(K \cdot A) \setminus A) \cap Y) \setminus F_{\varepsilon,K}) > \varepsilon\mu(A).$$

Now we are ready to prove Theorem 3.9.

Proof of Theorem 3.9. "(1) $\Rightarrow$ (2)": Fix $C \in \left(0, \frac{1}{2}\right)$, and let $\alpha_n := \frac{C}{(4+2C)(n+1)}$ for each $n \in \mathbb{N}$. Since $\mathcal{G}$ is asymptotically expanding, Lemma 3.5 provides a unital symmetric N_n-decomposable K_n with $\ell(K_n) \leq L_n$ satisfying the condition therein for $\beta = \frac{1}{2}$, $\alpha = \alpha_n$ and C. Take a unital symmetric decomposition $K_n = \bigcup_{i=1}^{N_n} K_{n,i}$ for admissible $K_{n,i}$ and set $\theta_n := N_n \cdot (n+1)$. Denote

$$Z_n := \left\{\tau_{K_{n,i}}(x) \;\middle|\; x \in \mathrm{s}(K_{n,i}) \text{ and } i = 1, \ldots, N_n \text{ such that } \Re(K_{n,i}, x) < \frac{1}{\theta_n}\right\}.$$

Then $\mu(Z_n) < \frac{N_n}{\theta_n} \cdot \mu(\mathcal{G}^{(0)}) = \frac{1}{n+1}$, which implies that $\mu(\mathcal{G}^{(0)} \setminus Z_n) \geq \frac{n}{n+1} \geq \frac{1}{2}$.

Let $X_n := \mathcal{G}^{(0)} \setminus Z_n$, and take F_n to be a maximal (C, K_n)-Følner sets in X_n, ensured by Lemma 3.11. Then we have $\mu(F_n) < \alpha_n \cdot \mu(X_n)$. Setting $Y_n := X_n \setminus F_n$, then $\mu(Y_n) > (1 - \alpha_n)\mu(X_n) \geq (1 - \alpha_n) \cdot \frac{n}{n+1}$.

Now we claim that Y_n is a domain of $\left(\frac{C}{2}, N_n, L_n\right)$-expansion. In fact, we take an arbitrary measurable $A \subseteq Y_n$ with $0 < \mu(A) \leq \frac{1}{2}\mu(Y_n)$ and divide into two cases.

If $\mu(A) \leq \frac{1}{2}\mu(X_n) - \mu(F_n)$, then $\mu((\mathrm{r}(K_n \cdot A) \setminus A) \cap Y_n) > C\mu(A)$ by Lemma 3.12.

If $\mu(A) > \frac{1}{2}\mu(X_n) - \mu(F_n)$, then $\frac{1}{2}\mu(X_n) \geq \mu(A) > \left(\frac{1}{2} - \alpha_n\right)\mu(X_n) \geq \alpha_n\mu(X_n)$. It follows from the requirement on K_n that $\mu((\mathrm{r}(K_n \cdot A) \setminus A) \cap X_n) > C\mu(A)$. Since

$$\mu((\mathrm{r}(K_n \cdot A) \setminus A) \cap Y_n) \geq \mu((\mathrm{r}(K_n \cdot A) \setminus A) \cap X_n) - \mu(F_n) > C\mu(A) - \mu(F_n)$$

and

$$\mu(A) > \left(\frac{1}{2} - \alpha_n\right)\mu(X_n) \geq \left(\frac{1}{2} - \frac{C}{4+2C}\right)\mu(X_n) = \frac{1}{C+2}\mu(X_n),$$

then we have

$$C\mu(A) - \mu(F_n) > C\mu(A) - \alpha_n\mu(X_n) > C\mu(A) - \alpha_n(C+2)\mu(A) \geq \frac{C}{2}\mu(A).$$

In conclusion, we showed that Y_n is a domain of $\left(\frac{C}{2}, N_n, L_n\right)$-expansion with ratio bounded by θ_n and $\mu(Y_n) > (1 - \alpha_n) \cdot \frac{n}{n+1}$.

"(2) $\Rightarrow$ (3)" is trivial.

"(3) $\Rightarrow$ (1)": Take an exhaustion of $\mathcal{G}^{(0)}$ by domains Y_n of asymptotic expansion. Assume that $\mathcal{G}$ were not asymptotically expanding. Then there exists $\alpha_0 \in \left(0, \frac{1}{2}\right)$ such that for any $C > 0$ and any unital symmetric decomposable $K \subseteq \mathcal{G}$, there exists $A_{C,K} \in \mathcal{R}$ with $\alpha_0 \leq \mu(A_{C,K}) \leq \frac{1}{2}$ such that $\mu(\mathrm{r}(K \cdot A_{C,K}) \setminus A_{C,K}) \leq C\mu(A_{C,K})$.

Take $n \in \mathbb{N}$ such that $\mu(Y_n) \geq 1 - \frac{\alpha_0}{2}$. Then for any measurable $A \subseteq \mathcal{G}^{(0)}$ with $\alpha_0 \leq \mu(A) \leq \frac{1}{2}$, direct calculations show that $\frac{\alpha_0}{2}\mu(Y_n) \leq \mu(A \cap Y_n) \leq \frac{1}{2-\alpha_0}\mu(Y_n)$. Then Lemma 3.4 provides $\varepsilon > 0$ and unital symmetric decomposable $K \subseteq \mathcal{G}$ such that

$$\mu((\mathrm{r}(K \cdot (A \cap Y_n)) \setminus (A \cap Y_n)) \cap Y_n) > \varepsilon\mu(A \cap Y_n) \geq \varepsilon\left(\mu(A) - \frac{\alpha_0}{2}\right) \geq \frac{\varepsilon}{2}\mu(A),$$

which implies that

$$\mu(\mathrm{r}(K \cdot A) \setminus A) \geq \mu((\mathrm{r}(K \cdot (A \cap Y_n)) \setminus A) \cap Y_n) > \frac{\varepsilon}{2}\mu(A).$$

This leads to a contradiction. ∎

Remark 3.13. From the proof above, if $\mathcal{G}$ is asymptotically expanding, then in condition (2) we can take $C_n \equiv \frac{C}{2}$ for any chosen $C \in \left(0, \frac{1}{2}\right)$, domains Y_n satisfying $\mu(Y_n) > \left(1 - \frac{C}{(4+2C)(n+1)}\right) \cdot \frac{n}{n+1}$, N_n, L_n and θ_n only depend on the expansion parameters.

3.3. Markov kernels on groupoids

Here, we construct reversible Markov kernels on groupoids and study the relation between their Cheeger constants and the expansion of groupoids.

Firstly, we construct a Markov kernel for decomposable subsets. Let us fix a unital symmetric decomposable subset K together with a unital symmetric decomposition $K = \bigcup_{i=1}^{N} K_i$.

Definition 3.14. For measurable $Y \subseteq \mathcal{G}^{(0)}$ and $x \in Y$, denote

$$K_{Y,x} := \{i \mid x \in \mathrm{s}(K_i) \text{ and } \tau_{K_i}(x) \in Y\} \quad \text{and} \quad \sigma_{Y,K}(x) = \sum_{i \in K_{Y,x}} \mathfrak{R}(K_i, x)^{\frac{1}{2}}. \quad (3.2)$$

The *normalised local Markov kernel* associated to Y and K is the Markov kernel on Y defined as follows:

$$\Pi_{Y,K}(x, -) = \frac{1}{\sigma_{Y,K}(x)} \sum_{i \in K_{Y,x}} \mathfrak{R}(K_i, x)^{\frac{1}{2}} \delta_{\tau_{K_i}(x)}(-) \quad \text{for } x \in Y. \quad (3.3)$$

Here, δ_y is the Dirac delta measure on y.

Since the decomposition is unital, it is clear that $\sigma_{Y,K} > 0$ on Y and (3.3) makes sense. It is also routine to check that (3.3) is indeed a Markov kernel on Y. To see that $\Pi_{Y,K}$ is reversible, we consider the measure on Y defined by

$$\mathrm{d}\tilde{\mu}_{Y,K} := \sigma_{Y,K} \cdot \mathrm{d}(\mu|_Y). \quad (3.4)$$

Note that both $\Pi_{Y,K}$ and $\tilde{\mu}_{Y,K}$ depend on the decomposition of K.

We collect several useful properties of $\Pi_{Y,K}$ in the following. The proof is similar to that of [22, Proposition 3.10], and hence we omit the details.

Proposition 3.15. *In the setting above, we have:*

(1) *The measure $\widetilde{\mu}_{Y,K}$ is reversing for the Markov kernel $\Pi_{Y,K}$.*

(2) *For measurable $A \subseteq Y$, we have $\mu(A) \leq \widetilde{\mu}_{Y,K}(A) \leq N\sqrt{\mu(A)\mu(Y)}$. Hence $\widetilde{\mu}_{Y,K}$ is equivalent to the restriction $\mu|_Y$.*

Definition 3.16. For a measurable $Y \subseteq \mathcal{G}^{(0)}$ and $C, N, L > 0$, we call Y a *domain of Markov (C, N, L)-expansion* (or simply, *domain of Markov expansion*) if there exists a unital symmetric N-decomposable K together with a unital symmetric decomposition $K = \bigcup_{i=1}^{N} K_i$ such that $\ell(K) \leq L$ and the associated normalised local Markov kernel $\Pi_{Y,K}$ on $(Y, \widetilde{\mu}_{Y,K})$ has the Cheeger constant greater than C.

Moreover, if the decomposition for K above has ratio bounded by $\theta \geq 1$, then we say that Y is a *domain of Markov (C, N, L)-expansion with ratio bounded by θ.*

The following lemma relates the domain of Markov expansion to the ordinary expansion. The proof is similar to that of [22, Lemma 3.14] and hence omitted.

Lemma 3.17. *For a measurable $Y \subseteq \mathcal{G}^{(0)}$, we have:*

(1) *If Y is a domain of (C, N, L)-expansion with ratio bounded by θ, then Y is a domain of $\big(\frac{C}{N\theta}, N, L\big)$-Markov expansion.*

(2) *If Y is a domain of (κ, N, L)-Markov expansion with ratio bounded by θ, then Y is a domain of $\big(\frac{\kappa}{N\sqrt{\theta}+\kappa}, N, L\big)$-expansion.*

Combining Theorem 3.9 with Lemma 3.17, we obtain the following Markovian version of the structure theorem.

Theorem 3.18. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Then the following are equivalent:*

(1) *The groupoid $\mathcal{G}$ is asymptotically expanding in measure.*

(2) *The unit space $\mathcal{G}^{(0)}$ admits an exhaustion by domains Y_n of (C_n, N_n, L_n)-Markov expansion with ratio bounded by θ_n for $C_n, N_n, L_n > 0$ and $\theta_n \geq 1$.*

Remark 3.19. Combining Remark 3.13 with the dependence of parameters established in Lemma 3.17, we know that if $\mathcal{G}$ is asymptotically expanding, then in Theorem 3.18 (2) we can take domains Y_n to satisfy $\mu(Y_n) > t_n$ for some universal t_n independent of $\mathcal{G}$, while C_n, N_n, L_n and θ_n only depend on expansion parameters of $\mathcal{G}$.

4. Dynamical propagation and quasi-locality

In this section, we introduce the notion of dynamical propagation and dynamical quasi-locality for operators on groupoids. Again we always assume that $\mathcal{G}$ is a groupoid with a length function ℓ and $\mathcal{G}^{(0)}$ is equipped with a measure μ on some σ-algebra $\mathcal{R}$. To include more examples, here we do *not* require μ to be finite.

4.1. Basic notions

We start with the following key notions.

Definition 4.1. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. An operator $T \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ is said to have *finite dynamical propagation* if there is a unital symmetric N-decomposable subset $K \subseteq \mathcal{G}$ with $\ell(K) \le L$ for some $N, L > 0$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, then $\chi_A T \chi_B = 0$. In this case, we also say that T has *(N, L)-dynamical propagation*.

Denote the set of all operators with finite dynamical propagation by $\mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$ and its norm completion by $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$, called the *dynamical Roe algebra of $\mathcal{G}$*.

Similarly, we consider its quasi-local counterpart.

Definition 4.2. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. An operator $T \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ is called *dynamically quasi-local* if for any $\varepsilon > 0$, there is a unital symmetric N_ε-decomposable subset $K_\varepsilon \subseteq \mathcal{G}$ with $\ell(K_\varepsilon) \le L_\varepsilon$ for some $N_\varepsilon, L_\varepsilon > 0$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K_\varepsilon \cdot A) \cap B) = 0$, then $\|\chi_A T \chi_B\| < \varepsilon$. The maps $\varepsilon \mapsto N_\varepsilon, \varepsilon \mapsto L_\varepsilon$ are called *quasi-local parameters* for T.

Let us denote the set of all dynamically quasi-local operators in $\mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ by $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$, called the *dynamical quasi-local algebra of $\mathcal{G}$*.

The following shows that $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$ and $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$ are indeed C^*-algebras. The proof is straightforward from definitions and hence omitted.

Lemma 4.3. *Given $\mathcal{G}$, ℓ and μ as above, we have:*

(1) *The set $\mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$ is a $*$-algebra, and hence $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$ is a C^*-algebra.*

(2) *The set $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$ is a C^*-algebra.*

For a sequence of groupoids, we combine them into a single groupoid and translate the notion of dynamical propagation and quasi-locality thereon in a uniform version. More precisely, for each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a groupoid with a length function ℓ_n and $(\mathcal{G}_n^{(0)}, \mathcal{R}_n, \mu_n)$ be a measure space. Form a groupoid $\mathcal{G}$ to be their disjoint union $\bigsqcup_{n\in\mathbb{N}} \mathcal{G}_n$ and equip $\mathcal{G}^{(0)} = \bigsqcup_{n\in\mathbb{N}} \mathcal{G}_n^{(0)}$ with a measure μ on some σ-algebra $\mathcal{R}$ generated by $\bigsqcup_{n\in\mathbb{N}} \mathcal{R}_n$, determined by $\mu|_{\mathcal{G}_n^{(0)}} := \mu_n$ for each n. Also take a length function ℓ on $\mathcal{G}$ to be the disjoint union of ℓ_n. Then we have

$$L^2(\mathcal{G}^{(0)}, \mu) = \bigoplus_{n\in\mathbb{N}} L^2(\mathcal{G}_n^{(0)}, \mu_n).$$

The following shows that operators in $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$ are always diagonal and can be described in a uniform version.

Lemma 4.4. *Given $T \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$, we have:*

(1) $T \in \mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$ if and only if *there exists* $T_n \in \mathfrak{B}(L^2(\mathcal{G}_n^{(0)}, \mu_n))$ *for each* $n \in \mathbb{N}$ *with* $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ *such that* $T = (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} T_n$ *and there exist* $N, L > 0$ *satisfying that for any* $n \in \mathbb{N}$, *there exists a unital symmetric* N*-decomposable* $K_n \subseteq \mathcal{G}_n$ *with* $\ell_n(K_n) \leq L$ *such that for any* $A, B \in \mathcal{R}_n$ *with* $\mu(\mathrm{r}(K_n \cdot A) \cap B) = 0$, *we have* $\chi_A T_n \chi_B = 0$.

(2) $T \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$ if and only if *there exists* $T_n \in \mathfrak{B}(L^2(\mathcal{G}_n^{(0)}, \mu_n))$ *for each* $n \in \mathbb{N}$ *with* $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ *such that* $T = (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} T_n$ *and for any* $\varepsilon > 0$, *there exist* $N, L > 0$ *satisfying that for any* $n \in \mathbb{N}$, *there exists a unital symmetric* N*-decomposable* $K_n \subseteq \mathcal{G}_n$ *with* $\ell_n(K_n) \leq L$ *such that for any* $A, B \in \mathcal{R}_n$ *with* $\mu(\mathrm{r}(K_n \cdot A) \cap B) = 0$, *we have* $\|\chi_A T_n \chi_B\| < \varepsilon$.

Proof. (1): We divide the proof into two directions.

Necessity. Firstly, we show that T is diagonal. Note that for any $n \neq m$ and $A \in \mathcal{R}_n$, $B \in \mathcal{R}_m$, we have $(K \cdot A) \cap B = \emptyset$ for any decomposable $K \subseteq \mathcal{G}$. Then by definition, $\chi_A T \chi_B = 0$. Hence $T = (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} T_n$ for $T_n = \chi_{\mathcal{G}_n^{(0)}} T \chi_{\mathcal{G}_n^{(0)}}$.

Furthermore, there exists a unital symmetric decomposable $K \subseteq \mathcal{G}$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, then $\chi_A T \chi_B = 0$. For each $n \in \mathbb{N}$, set $K_n := K \cap \mathcal{G}_n$. It is easy to check that K_n satisfies the requirement.

Sufficiency. Assume K_n satisfies the requirement, and write $K_n = \bigcup_{i=1}^N K_n^{(i)}$ for admissible $K_n^{(i)} \subseteq \mathcal{G}_n$ with $\ell_n(K_n^{(i)}) \leq L$. Define $K := \bigsqcup_{n\in\mathbb{N}} K_n \subseteq \mathcal{G}$ and for $i = 1, \ldots, N$, define $K^{(i)} := \bigsqcup_{n\in\mathbb{N}} K_n^{(i)}$. Since $K_n^{(i)}$ is admissible, it is easy to see that $K^{(i)} \subseteq \mathcal{G}$ is admissible for each i. Hence K is N-decomposable and $\ell(K) \leq L$.

Now for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, set $A_n := A \cap \mathcal{G}_n^{(0)}$ and $B_n := B \cap \mathcal{G}_n^{(0)}$, and then we have $\mu_n(\mathrm{r}(K_n \cdot A_n) \cap B_n) = 0$ for each $n \in \mathbb{N}$. By assumption, we have $\chi_{A_n} T_n \chi_{B_n} = 0$. Finally, note that

$$\chi_A T \chi_B = \sum_{n\in\mathbb{N}} \chi_{A_n} T_n \chi_{B_n} = 0,$$

which concludes (1). Since (2) is similar to (1), the details are omitted. ∎

4.2. Quasi-locality of the averaging projection

Now we focus on a special projection operator in $\mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ when μ is a probability measure, whose dynamical quasi-locality is closely related to the asymptotic expansion of the groupoid.

Definition 4.5. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. For any measurable $Y \subseteq \mathcal{G}^{(0)}$ with $0 < \mu(Y) < \infty$, denote by $P_Y \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ the *averaging projection on* Y, which is the orthogonal projection onto the one-dimensional subspace in $L^2(\mathcal{G}^{(0)}, \mu)$ spanned by χ_Y. In other words,

$$P_Y(f) := \frac{\langle f, \chi_Y \rangle}{\mu(Y)} \chi_Y \quad \text{for } f \in L^2(\mathcal{G}^{(0)}, \mu).$$

If $\mu(\mathcal{G}^{(0)}) < \infty$, we simply write $P_{\mathcal{G}}$ for $P_{\mathcal{G}^{(0)}}$.

Similar to [20, Lemma 3.8], we have the following.

Lemma 4.6. *Assume that $\mu(\mathcal{G}^{(0)}) = 1$. Then for any measurable $A, B \subseteq \mathcal{G}^{(0)}$, we have*

$$\|\chi_A P_{\mathcal{G}} \chi_B\| = \sqrt{\mu(A)\mu(B)}.$$

The following relates the quasi-locality of $P_{\mathcal{G}}$ to asymptotic expansion of $\mathcal{G}$.

Proposition 4.7. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Then the averaging projection $P_{\mathcal{G}}$ is dynamically quasi-local* if and only if *$\mathcal{G}$ is asymptotically expanding in measure μ.*

Proof. We divide the proof into two directions.

Necessity. Assume $\varepsilon \mapsto N_\varepsilon$, $\varepsilon \mapsto L_\varepsilon$ are quasi-local parameters for $P_{\mathcal{G}}$. Given $\alpha \in \left(0, \frac{1}{2}\right]$, set $N := N_{\sqrt{\alpha}/2}$ and $L := L_{\sqrt{\alpha}/2}$. Then there exists a unital symmetric N-decomposable K with $\ell(K) \leq L$ such that for any $A, B \in \mathcal{R}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, we have $\|\chi_A T \chi_B\| < \frac{\sqrt{\alpha}}{2}$. Then for any $A \in \mathcal{R}$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, Lemma 4.6 shows

$$\sqrt{\mu(A) \cdot \mu(\mathcal{G}^{(0)} \setminus \mathrm{r}(K \cdot A))} = \|\chi_A P_{\mathcal{G}} \chi_{\mathcal{G}^{(0)} \setminus \mathrm{r}(K \cdot A)}\| < \frac{\sqrt{\alpha}}{2}.$$

Hence we obtain $\mu(A) \cdot \mu(\mathcal{G}^{(0)} \setminus \mathrm{r}(K \cdot A)) < \frac{\alpha}{4}$, which implies that $\mu(\mathcal{G}^{(0)} \setminus \mathrm{r}(K \cdot A)) < \frac{1}{4}$. Therefore, we obtain $\mu(\mathrm{r}(K \cdot A)) > \frac{3}{4} \geq \left(1 + \frac{1}{2}\right) \cdot \mu(A)$. This concludes that $\mathcal{G}$ is asymptotically expanding in measure.

Sufficiency. Assuming $\alpha \mapsto C_\alpha$, $\alpha \mapsto N_\alpha$ and $\alpha \mapsto L_\alpha$ are expansion parameters for $\mathcal{G}$, we take K_α as in Definition 3.2. Given $0 < \varepsilon \leq \frac{1}{2}$, take $n \in \mathbb{N}$ to be the smallest number such that $(1 + C_\varepsilon)^n \cdot \varepsilon > \frac{1}{2}$. Set $K := K_\varepsilon^{2n}$, which is unital symmetric N_ε^{2n}-decomposable with length at most $2nL_\varepsilon$ by Lemma 2.5.

For measurable $A, B \subseteq \mathcal{G}^{(0)}$ with $\mu(\mathrm{r}(K \cdot A) \cap B) = 0$, we have

$$\mu(\mathrm{r}(K_\varepsilon^n \cdot A) \cap \mathrm{r}(K_\varepsilon^n \cdot B)) = 0.$$

Hence we can assume $\mu(\mathrm{r}(K_\varepsilon^n \cdot A)) \leq \frac{1}{2}$. If $\mu(A) < \varepsilon$, it follows from Lemma 4.6 that

$$\|\chi_A P_{\mathcal{G}} \chi_B\| = \sqrt{\mu(A)\mu(B)} < \sqrt{\varepsilon}.$$

If $\mu(A) \geq \varepsilon$, then using asymptotic expansion inductively, we have

$$\frac{1}{2} \geq \mu(\mathrm{r}(K_\varepsilon^n \cdot A)) \geq (1 + C_\varepsilon)\mu(\mathrm{r}(K_\varepsilon^{n-1} \cdot A)) \geq \cdots \geq (1 + C_\varepsilon)^n \mu(A) \geq (1 + C_\varepsilon)^n \cdot \varepsilon.$$

This leads to a contradiction to the choice of n. ∎

Remark 4.8. From the proof above, it is clear that if $P_{\mathcal{G}}$ is dynamically quasi-local, then we can choose expansion parameters for $\mathcal{G}$ only depending on quasi-local parameters for $P_{\mathcal{G}}$, and vice versa.

5. Main results

Now we are ready to prove the following fundamental case of the main result.

Theorem 5.1. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a probability measure space. Then the following are equivalent:*

(1) *The averaging projection $P_{\mathcal{G}} \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$.*

(2) *The averaging projection $P_{\mathcal{G}} \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$.*

(3) *The groupoid $\mathcal{G}$ is asymptotically expanding in measure.*

Proof. "(1) $\Rightarrow$ (2)" is trivial, and "(2) $\Leftrightarrow$ (3)" comes from Proposition 4.7. Hence we only focus on "(3) $\Rightarrow$ (1)".

Fix $C \in \left(0, \frac{1}{2}\right)$. By Theorem 3.18 and Remarks 3.13 and 3.19, there exists a sequence of measurable subsets $Y_n \subseteq \mathcal{G}^{(0)}$ with $\mu(Y_n) > \left(1 - \frac{C}{(4+2C)(n+1)}\right) \cdot \frac{n}{n+1}$ such that each Y_n is a domain of (C_n, N_n, L_n)-Markov expansion with ratio bounded by θ_n for $C_n, N_n, L_n > 0$ and $\theta_n \geq 1$ only depending on expansion parameters of $\mathcal{G}$. From Definition 3.16, we can choose a unital symmetric N_n-decomposable K_n together with a unital symmetric decomposition $K_n = \bigcup_{i=1}^{N_n} K_n^{(i)}$ such that $\ell(K_n) \leq L_n$, and the associated normalised local Markov kernel $\Pi_n := \Pi_{Y_n, K_n}$ from (3.3) on $(Y_n, \widetilde{\mu}_n)$ has the Cheeger constant greater than C_n, where $\widetilde{\mu}_n := \widetilde{\mu}_{Y_n, K_n}$ is the reversing measure defined in (3.4). We have a function $\sigma_n := \sigma_{Y_n, K_n}$ from (3.2) such that $\sigma_n \geq 1$ on Y_n. Denote the associated Markov operator by $\mathfrak{P}_n$ from (2.1) with spectral gap λ_n.

For each $n \in \mathbb{N}$, consider the embedding $I_n : L^2(Y_n, \widetilde{\mu}_n) \to L^2(\mathcal{G}^{(0)}, \mu)$ simply by extending each function in $L^2(Y_n, \widetilde{\mu}_n)$ by zero on $\mathcal{G}^{(0)} \setminus Y_n$. Thanks to Proposition 3.15, this is well defined and $\|I_n\| \leq 1$. Direct calculations show that $I_n^*(g) = \frac{1}{\sigma_n} \cdot g|_{Y_n}$ for $g \in L^2(\mathcal{G}^{(0)}, \mu)$. Denote the adjoint map

$$\mathrm{Ad}_n : \mathfrak{B}(L^2(Y_n, \widetilde{\mu}_n)) \to \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu)), \quad T \mapsto I_n \circ T \circ I_n^* \quad \text{for } T \in \mathfrak{B}(L^2(Y_n, \widetilde{\mu}_n)).$$

Recall from Definition 4.5 that we have the averaging projection $P_n := P_{Y_n}$ in $\mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$. Denote the orthogonal projection $\widetilde{P}_n \in \mathfrak{B}(L^2(Y_n, \widetilde{\mu}_n))$ onto constant functions on Y_n in $L^2(Y_n, \widetilde{\mu}_n)$. Direct calculations show that

$$\mathrm{Ad}_n(\widetilde{P}_n) = \frac{\mu(Y_n)}{\widetilde{\mu}_n(Y_n)} \cdot P_n. \tag{5.1}$$

On the other hand, Theorem 2.12 shows that $\frac{C_n^2}{2} \leq 1 - \lambda_n \leq 2C_n$. Hence $\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n$ has spectrum contained in $\left[-\frac{3}{4}, 1 - \frac{C_n^2}{4}\right] \cup \{1\}$. Therefore, for any $m \in \mathbb{N}$, we have

$$\left\| \left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^m - \widetilde{P}_n \right\| \leq \left(1 - \frac{C_n^2}{4} \right)^m.$$

Applying Ad_n and using (5.1), we obtain

$$\left\| \frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \mathrm{Ad}_n\left[\left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^m \right] - P_n \right\| \leq \frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \left(1 - \frac{C_n^2}{4} \right)^m.$$

Note that

$$\frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \leq \|\sigma_n\|_\infty \leq \left\| \sum_{i=1}^{N_n} \mathfrak{R}(K_n^{(i)}, x)^{\frac{1}{2}} \right\|_\infty \leq N_n \sqrt{\theta_n}.$$

Combining the above together, we obtain

$$\left\| \frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \mathrm{Ad}_n \left[\left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^m \right] - P_n \right\| \leq N_n \sqrt{\theta_n} \cdot \left(1 - \frac{C_n^2}{4}\right)^m. \tag{5.2}$$

Hence given $\varepsilon > 0$, we can choose $m_n \in \mathbb{N}$ such that

$$N_n \sqrt{\theta_n} \cdot \left(1 - \frac{C_n^2}{4}\right)^{m_n} < \frac{\varepsilon}{2}. \tag{5.3}$$

Moreover, direct calculations show that for each $n \in \mathbb{N}$, we have

$$\|P_n - P_{\mathcal{G}}\| \leq \sqrt{\mu(\mathcal{G}^{(0)} \setminus Y_n)} = \sqrt{\frac{1}{n+1} + \frac{nC}{(4+2C)(n+1)^2}}.$$

For ε above, we can further choose $\widetilde{N} \in \mathbb{N}$ such that for any $n > \widetilde{N}$, we have

$$\sqrt{\frac{1}{n+1} + \frac{nC}{(4+2C)(n+1)^2}} < \frac{\varepsilon}{2}. \tag{5.4}$$

Combining with (5.2), (5.3) and (5.4), we obtain that for any $n > \widetilde{N}$, we have

$$\begin{aligned}
&\left\| \frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \mathrm{Ad}_n \left[\left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^{m_n} \right] - P_{\mathcal{G}} \right\| \\
&\quad < \left\| \frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \mathrm{Ad}_n \left[\left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^{m_n} \right] - P_n \right\| + \|P_n - P_{\mathcal{G}}\| \\
&\quad < N_n \sqrt{\theta_n} \cdot \left(1 - \frac{C_n^2}{4}\right)^{m_n} + \sqrt{\frac{1}{n+1} + \frac{nC}{(4+2C)(n+1)^2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{5.5}$$

Finally, for $n \in \mathbb{N}$ and measurable $A, B \subseteq Y_n$ with $\mu((K_n \cdot A) \cap B) = 0$, we have $\widetilde{\mu}_n((K_n \cdot A) \cap B) = 0$ by Proposition 3.15 (2). For $x \in Y_n$ and $\xi \in L^2(Y_n, \widetilde{\mu}_n)$, we have

$$\begin{aligned}
(\chi_A \mathfrak{P}_n \chi_B \xi)(x) &= \chi_A(x) \cdot \int_B \xi(y) \Pi_n(x, \mathrm{d}y) \\
&= \chi_A(x) \cdot \frac{1}{\sigma_n(x)} \sum_{i : \tau_{K_n^{(i)}}(x) \in B} \mathfrak{R}(K_n^{(i)}, x)^{\frac{1}{2}} \xi(\tau_{K_n^{(i)}}(x)).
\end{aligned}$$

It follows that $\chi_A \mathfrak{P}_n \chi_B = 0$. Note that the operator I_n does not change the propagation, and hence the following operator

$$\frac{\widetilde{\mu}_n(Y_n)}{\mu(Y_n)} \cdot \mathrm{Ad}_n \left[\left(\frac{1}{2}\chi_{Y_n} + \frac{1}{2}\mathfrak{P}_n \right)^{m_n} \right] \quad \text{in } \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$$

has $(N_n^{m_n}, m_n \cdot L_n)$-propagation. Combining with (5.5), we conclude the proof. ∎

Remark 5.2. From the proof above together with Remarks 3.13 and 3.19, we know that if $\mathcal{G}$ is asymptotically expanding in measure, then for any $\varepsilon > 0$, we can choose $T_\varepsilon \in \mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$ with $(N_\varepsilon, L_\varepsilon)$-dynamical propagation such that $\|T_\varepsilon - P_{\mathcal{G}}\| < \varepsilon$ and the functions $\varepsilon \mapsto N_\varepsilon$ and $\varepsilon \mapsto L_\varepsilon$ only depend on expansion parameters of $\mathcal{G}$. Conversely, it follows from Remark 4.8 that parameters of $\mathcal{G}$ only depend on functions $\varepsilon \mapsto N_\varepsilon$ and $\varepsilon \mapsto L_\varepsilon$ satisfying the conditions above.

In the following, we consider general rank-one projections on $L^2(\mathcal{G}^{(0)}, \mu)$ for general μ. Let $P \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ be a rank-one projection and $\xi \in L^2(\mathcal{G}^{(0)}, \mu)$ be a unit vector in the range of P. Then $P(\eta) = \langle \eta, \xi \rangle \xi$ for any $\eta \in L^2(\mathcal{G}^{(0)}, \mu)$. This induces a probability measure ν on $(\mathcal{G}^{(0)}, \mathcal{R})$ defined by

$$\mathrm{d}\nu(x) := |\xi(x)|^2 \mathrm{d}\mu(x) \quad \text{for } x \in \mathcal{G}^{(0)}.$$

It is clear that the measure ν only depends on P, called the *associated measure to P*.

Then we have the following.

Theorem 5.3. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a (not necessarily finite) measure space. Let $P \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ be a rank-one projection and ν the associated probability measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

1. $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$.
2. $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$.
3. *$\mathcal{G}$ is asymptotically expanding in measure ν.*

Our idea is to apply Theorem 5.1 to the probability measure ν. Firstly, note that ν might *not* be equivalent to μ, and hence consider $Z := \{x \in \mathcal{G}^{(0)} \mid \xi(x) = 0\}$. Here, Z is well defined up to μ-null sets. We set $Y := \mathcal{G}^{(0)} \setminus Z$ and then the restrictions $\mu|_Y$ and $\nu|_Y$ are equivalent. Now we consider the reduction $\mathcal{G}_Y^Y = \mathrm{s}^{-1}(Y) \cap \mathrm{r}^{-1}(Y)$, equipped with the restriction ℓ_Y of the length function ℓ. Denote

$$Q : L^2(Y, \mu|_Y) \to L^2(\mathcal{G}^{(0)}, \mu)$$

the embedding by extending functions in $L^2(Y, \mu|_Y)$ to 0 on Z. Then we have the following, whose proof is similar to [21, Lemmas 4.5 and 4.6] and hence omitted.

Lemma 5.4. *With the notation above, we have:*

1. $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$ if and only if $Q^*PQ \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}_Y^Y)$.
2. $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$ if and only if $Q^*PQ \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}_Y^Y)$.

Here, we equip $\mathcal{G}_Y^Y$ with ℓ_Y and Y with $\mu|_Y$ to define $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}_Y^Y)$ and $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}_Y^Y)$.

Note that for $T \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$, Q^*TQ and T have the same quasi-local parameters, and the same holds for $T \in \mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$. Hence to prove Theorem 5.3, it suffices to consider that μ

and ν are equivalent, that is, $\mu(Z) = 0$. Therefore, in the following, we assume $Z = \emptyset$. To tell the difference, denote $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G};\mu)$ and $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G};\mu)$ the dynamical Roe and quasi-local algebras defined using μ, while $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G};\nu)$ and $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G};\nu)$ for those using ν. We have the following.

Lemma 5.5. *With the notation above and assuming $Z = \emptyset$, we have:*

(1) $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G};\mu)$ if and only if $P_{\mathcal{G}} \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G};\nu)$.

(2) $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G};\mu)$ if and only if $P_{\mathcal{G}} \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G};\nu)$.

Here, $P_{\mathcal{G}}$ is the averaging projection in $\mathfrak{B}^2(L^2(\mathcal{G}^{(0)},\nu))$ from Definition 4.5.

Proof. We only prove (1), since (2) is similar. Denote

$$U : L^2(\mathcal{G}^{(0)},\nu) \to L^2(\mathcal{G}^{(0)},\mu), \quad f \mapsto f\cdot\xi \quad \text{for } f \in L^2(\mathcal{G}^{(0)},\nu).$$

By the construction and the assumption $Z = \emptyset$ (which implies that ξ is non-zero everywhere), it is clear that U is unitary with $U^*\eta = \frac{1}{\xi}\cdot\eta$ for $\eta \in L^2(\mathcal{G}^{(0)},\mu)$. Denote

$$\mathrm{Ad}_U : \mathfrak{B}(L^2(\mathcal{G}^{(0)},\nu)) \to \mathfrak{B}(L^2(\mathcal{G}^{(0)},\mu)), \quad T \mapsto UTU^* \quad \text{for } T \in \mathfrak{B}(L^2(\mathcal{G}^{(0)},\nu)).$$

We claim that $\mathrm{Ad}_U(P_{\mathcal{G}}) = P$. In fact, given $\eta \in L^2(\mathcal{G}^{(0)},\mu)$, we have

$$\begin{aligned}
\mathrm{Ad}_U(P_{\mathcal{G}})\eta &= UP_{\mathcal{G}}\left(\frac{1}{\xi}\cdot\eta\right) = U\left(\left\langle\frac{1}{\xi}\cdot\eta, \chi_{\mathcal{G}^{(0)}}\right\rangle_{L^2(\mathcal{G}^{(0)},\nu)}\chi_{\mathcal{G}^{(0)}}\right)\\
&= \left\langle\frac{1}{\xi}\cdot\eta, \chi_{\mathcal{G}^{(0)}}\right\rangle_{L^2(\mathcal{G}^{(0)},\nu)}\xi\\
&= \left(\int_{\mathcal{G}^{(0)}}\frac{1}{\xi}(x)\eta(x)\mathrm{d}\nu(x)\right)\cdot\xi = \left(\int_{\mathcal{G}^{(0)}}\eta(x)\xi(x)\mathrm{d}\mu(x)\right)\cdot\xi\\
&= \langle\eta,\xi\rangle_{L^2(\mathcal{G}^{(0)},\mu)}\xi = P\eta.
\end{aligned}$$

Hence we prove the claim. Therefore, for any measurable $A, B \subseteq \mathcal{G}^{(0)}$, we have

$$\mathrm{Ad}_U(\chi_A P_{\mathcal{G}}\chi_B) = \mathrm{Ad}_U(\chi_A)\circ\mathrm{Ad}_U(P_{\mathcal{G}})\circ\mathrm{Ad}_U(\chi_B) = \chi_A P\chi_B.$$

Finally, since μ and ν are equivalent, we have $K \subseteq \mathcal{G}$ is decomposable with respect to μ if and only if it is decomposable with respect to ν. Moreover, for any measurable $A, B \subseteq \mathcal{G}^{(0)}$ and unital symmetric decomposable $K \subseteq \mathcal{G}$, we have $\mu(\mathrm{r}(K\cdot A)\cap B) = 0$ if and only if $\nu(\mathrm{r}(K\cdot A)\cap B) = 0$. So we finish the proof. ■

Combining Theorem 5.1 with Lemmas 5.4 and 5.5, we obtain Theorem 5.3.

Remark 5.6. From the proof above together with Remark 5.2, we know that if the rank-one projection P belongs to $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$, then for any $\varepsilon > 0$, we can choose $T_\varepsilon \in \mathbb{C}_{\mathrm{dyn}}(\mathcal{G})$ with $(N_\varepsilon, L_\varepsilon)$-dynamical propagation such that $\|T_\varepsilon - P\| < \varepsilon$ and the functions $\varepsilon \mapsto N_\varepsilon$ and $\varepsilon \mapsto L_\varepsilon$ only depend on quasi-local parameters of P. Conversely, quasi-local parameters of P only depend on functions $\varepsilon \mapsto N_\varepsilon$ and $\varepsilon \mapsto L_\varepsilon$ satisfying the conditions above.

Finally, we introduce two variants of Theorem 5.3. The first one deals with a family version, following the discussion in Section 4.1. Combining Lemma 4.4 and Theorem 5.3 together with Remark 5.6, we obtain the following.

Theorem 5.3′. *For each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a groupoid with a length function ℓ_n and $(\mathcal{G}_n^{(0)}, \mathcal{R}_n, \mu_n)$ be a measure space. Form the groupoid $\mathcal{G} = \bigsqcup_{n\in\mathbb{N}} \mathcal{G}_n$ together with a length function ℓ and a measure μ on $\mathcal{G}^{(0)}$ as in Section 4.1. Moreover, let $P_n \in \mathfrak{B}(L^2(\mathcal{G}_n^{(0)}, \mu_n))$ be a rank-one orthogonal projection for each $n \in \mathbb{N}$ and consider their direct sum*

$$P := (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} P_n \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu)).$$

Then the following are equivalent:

(1) $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$.

(2) $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$.

(3) *$\mathcal{G}_n$ is asymptotically expanding in the associated measure ν_n to P_n uniformly in the sense that they have the same expansion parameters.*

The second focuses on a specific family of admissible and decomposable subsets allowed to build the notion of (asymptotic) expansion, dynamical propagation and quasi-locality. We introduce the following refined version of Definition 2.4.

Definition 5.7. Let $\mathcal{G}$ be a groupoid with a length function ℓ and $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space. Let $\mathcal{K}$ be a family of admissible bisections which is closed under taking compositions and inverses and $\mathcal{G}^{(0)} \in \mathcal{K}$. A subset $K \subseteq \mathcal{G}$ is called *$\mathcal{K}$-decomposable* if $K = \bigcup_{i=1}^N K_i$ for $N \in \mathbb{N}$ and admissible bisection $K_i \in \mathcal{K}$.

Given such $\mathcal{K}$, we replace the word "decomposable" by "$\mathcal{K}$-decomposable" in Definitions 3.1, 3.2, 4.1 and 4.2 to define the notion of $\mathcal{K}$-expansion (in measure), $\mathcal{K}$-asymptotic expansion (in measure), $\mathcal{K}$-dynamical propagation and $\mathcal{K}$-dynamical quasi-local, together with C^*-algebras $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}, \mathcal{K})$ and $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}, \mathcal{K})$.

Remark 5.8. Note that if $\mathcal{K}$ is *cofinal* in the sense that any admissible bisection is contained in the union of finitely many elements in $\mathcal{K}$, then these notions coincide with the original ones of *possibly different* parameters.

Applying exactly the same proofs, we obtain the following.

Theorem 5.3″. *Let $\mathcal{G}$ be a groupoid with a length function ℓ, $(\mathcal{G}^{(0)}, \mathcal{R}, \mu)$ be a measure space and $\mathcal{K}$ be a family of admissible bisections which is closed under taking compositions and inverses and $\mathcal{G}^{(0)} \in \mathcal{K}$. Let $P \in \mathfrak{B}(L^2(\mathcal{G}^{(0)}, \mu))$ be a rank-one projection and ν the associated probability measure on $\mathcal{G}^{(0)}$. Then the following are equivalent:*

(1) $P \in \mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}, \mathcal{K})$.

(2) $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}, \mathcal{K})$.

(3) *$\mathcal{G}$ is $\mathcal{K}$-asymptotically expanding in measure ν.*

6. Examples

In this section, we apply our theory to several classes of groupoids.

6.1. Transformation groupoids

Our first example comes from group actions, which is one of our main motivations for this work.

Let $\rho : \Gamma \curvearrowright X$ be a countable group Γ acting on a set X and ℓ_Γ be a proper word length function on Γ in the sense that for any $L > 0$, the closed ball denoted by

$$B_L := \{\gamma \in \Gamma \mid \ell_\Gamma(\gamma) \le L\}$$

is finite. Consider the *transformation groupoid* $X \rtimes \Gamma$ as in Section 1 with the length function ℓ by $\ell(x, \gamma) := \ell_\Gamma(\gamma)$ for $x \in X$ and $\gamma \in \Gamma$.

Moreover, let μ be a measure on $(X, \mathcal{R})$ for some σ-algebra $\mathcal{R}$ and assume that the action ρ is measure class preserving. Then for $\gamma \in \Gamma$, the map $\tau_{X\times\{\gamma\}} : X \to X$ coincides with $\rho(\gamma)$ (simply denoted by γ) and $X \times \{\gamma\}$ is admissible.

Concerning decomposable subsets in $X \rtimes \Gamma$, we have the following. The proof is straightforward and hence omitted.

Lemma 6.1. *For any $L > 0$, the subset $X \times B_L$ is unital symmetric $|B_L|$-decomposable with $\ell(X \times B_L) \le L$, and it admits a unital symmetric decomposition $X \times B_L = \bigcup_{\gamma\in B_L} X \times \{\gamma\}$, where each $X \times \{\gamma\}$ is admissible. Here, we use $|\cdot|$ to denote the cardinality. Conversely, for any decomposable $K \subseteq X \rtimes \Gamma$, we have $K \subseteq X \times B_{\ell(K)}$.*

Hence it suffices to consider decomposable subsets of the form $X \times B_L$, which is determined by its length. Therefore, the notion of (asymptotic) expansion in measure can be translated as follows.

Proposition 6.2. *In the above setting, the transformation groupoid $X \rtimes \Gamma$ is asymptotically expanding in measure in the sense of Definition 3.2* if and only if *the action is asymptotically expanding in measure. A similar result holds for the notion of expansion and the domain of (asymptotic) expansion.*

Consequently, Theorem 3.9 and Theorem 3.18 recover [23, Theorem 4.6] and [22, Theorem 3.20], respectively.

On the other hand, applying Lemma 6.1 again, the notion of dynamical propagation and quasi-locality can be translated as follows.

Proposition 6.3. *In the above setting, an operator $T \in L^2(X,\mu)$ is dynamically quasi-local in the sense of Definition 4.2* if and only if *T is ρ-quasi-local in the sense of* [22, Definition 4.3]*, that is, for any $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any measurable $A, B \subseteq X$ with $\mu((B_{L_\varepsilon} \cdot A) \cap B) = 0$, then $\|\chi_A T \chi_B\| < \varepsilon$. A similar result holds for the notion of finite dynamical propagation.*

Consequently, Theorem 5.1 recovers [22, Proposition 4.6 and Theorem 4.16] for the averaging projection when μ is finite.

Corollary 6.4. *Let $\rho : \Gamma \curvearrowright X$ be a countable group Γ acting on a probability measure space (X,μ) and assume that ρ is measure class preserving. For the averaging projection $P_X \in \mathfrak{B}(L^2(X,\mu))$, the following are equivalent:*

(1) *P is ρ-quasi-local.*

(2) *P is a norm limit of operators in $B(L^2(X,\mu))$ with finite ρ-propagations.*

(3) *The action is asymptotically expanding in measure μ.*

Furthermore, we have the following generalised version from Theorem 5.3, which deals with an arbitrary rank-one projection.

Corollary 6.5. *Let $\rho : \Gamma \curvearrowright X$ be a countable group Γ acting on a (not necessarily finite) measure space (X,μ) and assume that ρ is measure class preserving. Then for any rank-one projection $P \in \mathfrak{B}(L^2(X,\mu))$, we have that P is ρ-quasi-local* if and only if *P is a norm limit of operators in $B(L^2(X,\mu))$ with finite ρ-propagations.*

6.2. Pair groupoids: A family version

Now we consider pair groupoids and aim to recover the results for uniform Roe and quasi-local algebras of metric spaces.

Let (X,d) be a discrete metric space, $x \in X$ and $R > 0$. Denote $B(x,R)$ the *closed ball* with radius R and centre x. We say that (X,d) has *bounded geometry* if $\sup_{x \in X} |B(x,R)|$ is finite for each $R > 0$. A sequence of metric spaces $\{(X_n,d_n)\}_{n\in\mathbb{N}}$ has *uniformly bounded geometry* if for all $R > 0$, $\sup_{n\in\mathbb{N}} \sup_{x\in X_n} |B(x,R)|$ is finite.

Definition 6.6. Let $\{(X_n,d_n)\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces. A *coarse disjoint union* of $\{(X_n,d_n)\}_{n\in\mathbb{N}}$ is a metric space (X,d), where $X = \bigsqcup_{n\in\mathbb{N}} X_n$ is the disjoint union of $\{X_n\}$ as a set and d is a metric on X such that:

- The restriction of d on each X_n coincides with d_n.
- $d(X_n,X_m) \to \infty$ as $n+m \to \infty$ and $n \neq m$.

Definition 6.7. Let (X,d) be a discrete metric space with bounded geometry. For an operator $T \in \mathfrak{B}(\ell^2(X))$, we say that:

(1) T has *finite propagation* if there exists $R > 0$ such that for any $A, B \subseteq X$ with $d(A,B) > R$, we have that $\chi_A T \chi_B = 0$.

(2) T is *quasi-local* if for any $\varepsilon > 0$, there exists $R > 0$ such that for any $A, B \subseteq X$ with $d(A, B) > R$, we have $\|\chi_A T \chi_B\| < \varepsilon$.

The set of all finite propagation operators in $\mathfrak{B}(\ell^2(X))$ forms a $*$-algebra, called the *algebraic uniform Roe algebra* and denoted by $\mathbb{C}_u[X]$. The *uniform Roe algebra of* X is defined to be the operator norm closure of $\mathbb{C}_u[X]$ in $\mathfrak{B}(\ell^2(X))$, which is a C^*-algebra and denoted by $\mathbf{C}^*_u(X)$. The set of all quasi-local operators in $\mathfrak{B}(\ell^2(X))$ forms a C^*-algebra, called the *uniform quasi-local algebra of* X and denoted by $\mathbf{C}^*_{uq}(X)$.

In the following, we focus on a coarse disjoint union (X, d) of a sequence of finite metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ with uniformly bounded geometry.

For each $n \in \mathbb{N}$, we consider the pair groupoid $X_n \times X_n$ with $(x, y)\cdot(y, z) = (x, z)$, $(x, y)^{-1} = (y, x)$ for $x, y, z \in X_n$. Its unit space can be identified with X_n with source and range maps given by $\mathrm{s}(x, y) = y$ and $\mathrm{r}(x, y) = x$ for $x, y \in X_n$. Moreover, let μ_n be a finite measure on $(X_n, \mathcal{R}_n)$ for $\mathcal{R}_n = \mathcal{P}(X_n)$ and consider the length function ℓ_n on $\mathcal{G}_n$ defined by $\ell_n(x, y) := d_n(x, y)$ for $x, y \in X_n$. For $L > 0$, we denote

$$E_L^{(n)} := \{(x, y) \in X_n \times X_n \mid \ell_n(x, y) \le L\}.$$

It is clear that any bisection in $\mathcal{G}_n$ is admissible, and moreover, we have the following. The proof follows from a well-known fact in coarse geometry (see, e.g., [34, Lemma 12.2.3]) and hence omitted.

Lemma 6.8. *For any $L > 0$, there exists $N_L \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, the set $E_L^{(n)}$ is unital symmetric and N_L-decomposable. Conversely, for any $n \in \mathbb{N}$ and decomposable $K \subseteq \mathcal{G}_n$, we have $K \subseteq E^{(n)}_{\ell(K)}$.*

Hence it suffices to consider decomposable subsets of the form $E_L^{(n)}$. For $A \subseteq X_n$, $\mathrm{r}(E_L^{(n)} \cdot A) \setminus A = \partial_L A$, where $\partial_L A := \{x \in X_n \setminus A \mid d_n(x, A) \le L\}$. Therefore, we recover[4] the notion of measured asymptotic expanders introduced in [23].

Proposition 6.9. *In the above setting, the following are equivalent:*

(1) *There exist functions $\underline{C}, \underline{N}, \underline{L} : \left(0, \frac{1}{2}\right] \to (0, \infty)$ such that for all $n \in \mathbb{N}$, $\mathcal{G}_n$ is asymptotically expanding with parameters $\underline{C}, \underline{N}, \underline{L}$.*

(2) *$\{(X_n, d_n, \mu_n)\}_{n\in\mathbb{N}}$ forms a sequence of measured asymptotic expanders in the sense of* [23, Definition 6.1].

Consequently, Theorem 3.18 and Remark 3.19 recover [21, Theorem 4.15].

Now we consider finite propagation and quasi-local operators. Choose μ_n to be the counting measure on X_n. From the discussions in Section 4.1, $\mathcal{G}_n$ gives rise to a single

[4]Note that although we define the notion of (asymptotic) expansion in the setting of probability measure in Definitions 3.1 and 3.2, these can be naturally extended to all finite measure spaces as mentioned at the beginning of Section 3.

groupoid $\mathcal{G} = \bigsqcup_{n\in\mathbb{N}} \mathcal{G}_n$ together with a length function ℓ and a measure μ on $\mathcal{G}^{(0)} = X$. Combining Lemmas 4.4 and 6.8, it is easy to see the following.

Corollary 6.10. *Given $T \in \mathfrak{B}(\ell^2(X))$, we have $T \in \mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G})$* if and only if *there exists $T_n \in \mathfrak{B}(\ell^2(X_n))$ for each $n \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ such that $T = (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} T_n$ and for any $\varepsilon > 0$, there exists $L > 0$ satisfying that for any $n \in \mathbb{N}$ and $A_n, B_n \subseteq X_n$ with $d_n(A_n, B_n) > L$, we have $\|\chi_{A_n} T_n \chi_{B_n}\| < \varepsilon$. A similar result holds for $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G})$.*

Combining Lemma 4.4 and Proposition 6.9 with [3, Lemmas 3.13 and 3.14], we obtain the following.

Proposition 6.11. *With the same notation as above, we have:*

(1) $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}) = \mathbf{C}^*_{\mathrm{u}}(X) \cap \prod_{n\in\mathbb{N}} \mathfrak{B}(\ell^2(X_n))$.

(2) $\mathbf{C}^*_{\mathrm{dyn}}(\mathcal{G}) + \mathfrak{K}(\ell^2(X)) = \mathbf{C}^*_{\mathrm{u}}(X)$.

(3) $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}) = \mathbf{C}^*_{\mathrm{uq}}(X) \cap \prod_{n\in\mathbb{N}} \mathfrak{B}(\ell^2(X_n))$.

(4) $\mathbf{C}^*_{\mathrm{dyn,q}}(\mathcal{G}) + \mathfrak{K}(\ell^2(X)) = \mathbf{C}^*_{\mathrm{uq}}(X)$.

Finally, combining with Theorem 5.3′ and Proposition 6.9, we recover both [17, Theorem C] and [21, Theorem B].

Corollary 6.12. *Let $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces with uniformly bounded geometry and (X, d) be their coarse disjoint union. For each $n \in \mathbb{N}$, let $P_n \in \mathfrak{B}(\ell^2(X_n))$ be a rank-one projection and ν_n the associated probability measure on X_n. For $P = (\mathrm{SOT}) - \sum_{n\in\mathbb{N}} P_n \in \mathfrak{B}(\ell^2(X))$, the following are equivalent:*

(1) $P \in \mathbf{C}^*_{\mathrm{u}}(X)$.

(2) $P \in \mathbf{C}^*_{\mathrm{uq}}(X)$.

(3) $\{(X_n, d_n, \nu_n)\}_{n\in\mathbb{N}}$ *forms a sequence of measured asymptotic expanders.*

6.3. Semi-direct product groupoids

Here, we consider groupoid actions on fibre spaces, generalising those in Section 6.1. Our idea is to package the information of groupoid actions into semi-direct product groupoids.

Let us start with some preliminaries for groupoid actions. A *fibre space* over a set X is a pair (Y, p), where Y is a set and $p\colon Y \to X$ is a surjective map. For two fibre spaces (Y_1, p_1) and (Y_2, p_2) over X, we form their *fibred product* to be

$$Y_1 \;{}_{p_1}\!*_{p_2}\; Y_2 := \{(y_1, y_2) \in Y_1 \times Y_2 \mid p_1(y_1) = p_2(y_2)\}.$$

Definition 6.13. Let $\mathcal{G}$ be a groupoid. A *left $\mathcal{G}$-space* is a fibre space (Y, p) over $\mathcal{G}^{(0)}$, equipped with a map $(\gamma, y) \mapsto \gamma y$ from $\mathcal{G} \;{}_{\mathrm{s}}\!*_p\; Y$ to Y (called a *$\mathcal{G}$-action on* (Y, p)) which satisfies the following:

- $p(\gamma y) = \mathrm{r}(\gamma)$ for $(\gamma, y) \in \mathcal{G} \;{}_{\mathrm{s}}\!*_p\; Y$, and $p(y)y = y$.

- $\gamma_2(\gamma_1 y) = (\gamma_2\gamma_1)y$ for $(\gamma_1, y) \in \mathcal{G}\ {}_{\mathrm{s}}{*}_p\ Y$ and $\mathrm{s}(\gamma_2) = \mathrm{r}(\gamma_1)$.

Given a $\mathcal{G}$-space (Y, p), the associated *semi-direct product groupoid* $Y \rtimes \mathcal{G}$ is defined (as a set) to be $Y\ {}_p{*}_{\mathrm{r}}\ \mathcal{G}$. For $(y, \gamma) \in Y \rtimes \mathcal{G}$, define its range to be $(y, \mathrm{r}(\gamma))$ and its source to be $(\gamma^{-1}y, \mathrm{s}(\gamma))$. The product and inverse are given by

$$(y,\gamma)(\gamma^{-1}y,\gamma') = (y,\gamma\gamma') \quad \text{and} \quad (y,\gamma)^{-1} = (\gamma^{-1}y,\gamma^{-1}).$$

Clearly, $(y, p(y)) \mapsto y$ is a bijection from the unit space of $Y \rtimes \mathcal{G}$ onto Y.

Given a bisection $K \subseteq \mathcal{G}$, we define the following map:

$$\beta_K : p^{-1}(\mathrm{s}(K)) \to p^{-1}(\mathrm{r}(K)), \quad y \mapsto \gamma_y \cdot y \quad \text{for } y \in p^{-1}(\mathrm{s}(K)), \tag{6.1}$$

where γ_y is the unique point in K such that $\mathrm{s}(\gamma_y) = p(y)$. The following is easy.

Lemma 6.14. *For a groupoid $\mathcal{G}$ acting on a fibre space (Y, p) and a bisection $K \subseteq \mathcal{G}$, then $Y\ {}_p{*}_{\mathrm{r}}\ K$ is a bisection in $Y \rtimes \mathcal{G}$. Moreover, we have $\tau_{Y\ {}_p{*}_{\mathrm{r}} K} = \beta_K$.*

Analogous to Definitions 2.2 and 2.4, we introduce the following.

Definition 6.15. Let $\mathcal{G}$ be a groupoid with a length function ℓ and (Y, p) be a $\mathcal{G}$-space equipped with a probability measure structure $(Y, \mathcal{R}, \mu)$.

- A bisection $K \subseteq \mathcal{G}$ is called *dynamically admissible* if $p^{-1}(\mathrm{s}(K))$, $p^{-1}(\mathrm{r}(K))$ are measurable, the bijection β_K from (6.1) is a measure-class-preserving measurable isomorphism and $\ell(K)$ is finite.
- A subset $K \subseteq \mathcal{G}$ is called *dynamically decomposable* if $K = \bigcup_{i=1}^N K_i$ for $N \in \mathbb{N}$ and dynamically admissible bisection $K_i \subseteq \mathcal{G}$.

To translate the notions above in terms of $Y \rtimes \mathcal{G}$, we first notice that a length function ℓ on $\mathcal{G}$ gives rise to a length function $\widehat{\ell}$ on $Y \rtimes \mathcal{G}$ by

$$\widehat{\ell}(y,\gamma) := \ell(\gamma) \quad \text{for } (y,\gamma) \in Y \rtimes \mathcal{G}. \tag{6.2}$$

Then Lemma 6.14 implies the following.

Lemma 6.16. *A bisection $K \subseteq \mathcal{G}$ is dynamically admissible* if and only if *$Y\ {}_p{*}_{\mathrm{r}}\ K$ is admissible in $Y \rtimes \mathcal{G}$.*

Hence we consider the following specific family of admissible bisections in $Y \rtimes \mathcal{G}$:

$$\mathcal{K}_{\mathcal{G},Y} = \{Y\ {}_p{*}_{\mathrm{r}}\ K \subseteq Y \rtimes \mathcal{G} \mid K \subseteq \mathcal{G} \text{ dynamically admissible}\}.$$

It is clear that $\mathcal{K}_{\mathcal{G},Y}$ is closed under taking composition and inverse and contains Y. Moreover, Lemma 6.16 directly implies the following.

Corollary 6.17. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and (Y, p) be a $\mathcal{G}$-space equipped with a probability measure structure $(Y, \mathcal{R}, \mu)$. Equip $Y \rtimes \mathcal{G}$ with the length function $\widehat{\ell}$ from (6.2). Then for $K \subseteq \mathcal{G}$, K is dynamically decomposable* if and only if $Y\ {}_p{*}_{\mathrm{r}}\ K \subseteq Y \rtimes \mathcal{G}$ *is $\mathcal{K}_{\mathcal{G},Y}$-decomposable (see the end of Section 5).*

Similar to Lemma 2.6, it is clear that for dynamically decomposable $K \subseteq \mathcal{G}$ and measurable $A \subseteq Y$, $K \cdot A$ is also measurable. Moreover, direct calculations show

$$K \cdot A = \mathrm{r}((Y\ {}_p{*}_{\mathrm{r}}\ K) \cdot A).$$

Hence the $\mathcal{K}_{\mathcal{G},Y}$-asymptotic expansion of $\mathcal{G} \rtimes Y$ is equivalent to the following.

Definition 6.18. Let $\mathcal{G}$ be a groupoid with a length function ℓ and (Y, p) be a $\mathcal{G}$-space equipped with a probability measure structure $(Y, \mathcal{R}, \mu)$. We say that the $\mathcal{G}$-action on (Y, p) is *asymptotically expanding* if for any $\alpha \in \left(0, \frac{1}{2}\right]$, there exist $C_\alpha, N_\alpha, L_\alpha > 0$ and a unital symmetric dynamically N_α-decomposable subset $K_\alpha \subseteq \mathcal{G}$ with $\ell(K_\alpha) \leq L_\alpha$ such that for any $A \in \mathcal{R}$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, then $\mu((K_\alpha \cdot A) \setminus A) > C_\alpha \mu(A)$.

Combining Theorem 5.3$''$ with the discussions above, we obtain the following.

Corollary 6.19. *Let $\mathcal{G}$ be a groupoid with a length function ℓ and (Y, p) be a $\mathcal{G}$-space equipped with a probability measure structure $(Y, \mathcal{R}, \mu)$. Equip $Y \rtimes \mathcal{G}$ with the length function $\widehat{\ell}$ from (6.2). Let P be a rank-one projection in $\mathfrak{B}(L^2(Y, \mu))$ and ν the associated measure to P constructed in Section 5, then the following are equivalent:*

(1) $P \in \mathbf{C}^*_{\mathrm{dyn}}(Y \rtimes \mathcal{G}, \mathcal{K}_{\mathcal{G},Y})$.

(2) $P \in \mathbf{C}^*_{\mathrm{dyn,q}}(Y \rtimes \mathcal{G}, \mathcal{K}_{\mathcal{G},Y})$.

(3) *$Y \rtimes \mathcal{G}$ is $\mathcal{K}_{\mathcal{G},Y}$-asymptotically expanding in measure ν.*

(4) *The $\mathcal{G}$-action on (Y, p) is asymptotically expanding in measure ν.*

Remark 6.20. In the case of a group G acting on a space Y, Lemma 6.1 shows that when the action is measure class preserving, then the family $\mathcal{K}_{G,Y}$ is cofinal in the sense of Remark 5.8. Hence combining Remark 5.8 with Proposition 6.3, Corollary 6.19 generalises Corollary 6.5.

6.4. The HLS groupoid and its variant

Throughout this subsection, let Γ be a finitely generated group and $\{N_i\}_{i\in\mathbb{N}}$ be a family of nested, finite index normal subgroups of Γ with trivial intersection. For each $i \in \mathbb{N}$, denote the quotient map $\pi_i : \Gamma \to \Gamma/N_i$ and $\pi_\infty : \Gamma \to \Gamma$ the identity map.

The associated *HLS groupoid* (after Higson, Lafforgue and Skandalis from [15], see also [33]) $\mathcal{G}$ is defined to be

$$\mathcal{G} := \bigsqcup_{i\in\mathbb{N}\cup\{\infty\}} \{i\} \times X_i, \quad \text{where } X_i = \begin{cases} \Gamma/N_i & \text{if } i \in \mathbb{N}; \\ \Gamma & \text{if } i = \infty. \end{cases}$$

Since $\mathcal{G}$ is a bundle of groups, it cannot be asymptotically expanding for any given measure and length function. Regarding $\mathcal{G}$ as a family of groups, it is uniformly asymptotically expanding since each unit space consists of a single point.

Now we consider a variant of the HLS groupoid from [2]. To recall their construction, we use the same notation as above and denote $\widehat{\Gamma} := \varprojlim \Gamma/N_i$ the profinite completion of Γ with respect to the family $\{\Gamma/N_i\}_{i\in\mathbb{N}}$, that is, the inverse limit of $\{\Gamma/N_i\}_{i\in\mathbb{N}}$.

For each $i \in \mathbb{N}$, let $\mathcal{G}_i^{\mathrm{AFS}}$ be the transformation groupoid $X_i \rtimes (\Gamma/N_i)$ with the action by left multiplication. For $i = \infty$, let $\mathcal{G}_\infty^{\mathrm{AFS}}$ be the transformation groupoid $\widehat{\Gamma} \rtimes \Gamma$ with the natural free action. Then the groupoid constructed in [2] is defined to be the disjoint union (thanks to [2, Lemma 2.1])

$$\mathcal{G}^{\mathrm{AFS}} := \bigsqcup_{i\in\mathbb{N}\cup\{\infty\}} \mathcal{G}_i^{\mathrm{AFS}}.$$

For $i \in \mathbb{N}$, we take the normalised counting measure μ_i on Γ/N_i. For $i = \infty$, we take the induced probability Γ-invariant measure μ_∞ on $\widehat{\Gamma}$. On the other hand, fix a length function ℓ on Γ, which induces a quotient length function ℓ_i on Γ/N_i for each $i \in \mathbb{N}$. Following Section 6.1, we obtain a length function on $\mathcal{G}_i^{\mathrm{AFS}}$ for each $i \in \mathbb{N}\cup\{\infty\}$. By the discussions in Section 4.1, these can be combined to provide a measure and a length function for $\mathcal{G}^{\mathrm{AFS}}$.

We have the following characterisation.

Proposition 6.21. *With the same notation as above, the following are equivalent:*

(1) *$\mathcal{G}_i^{\mathrm{AFS}}$ is asymptotically expanding in measure uniformly for $i \in \mathbb{N}\cup\{\infty\}$ (i.e., having the same expansion parameters).*

(2) *The natural action of Γ on $\widehat{\Gamma}$ is asymptotically expanding in measure.*

(3) *The natural action of Γ on $\widehat{\Gamma}$ is expanding in measure.*

(4) *The sequence $\{\Gamma/N_i\}_{i\in\mathbb{N}}$ forms a sequence of asymptotic expander graphs.*

(5) *The sequence $\{\Gamma/N_i\}_{i\in\mathbb{N}}$ forms a sequence of expander graphs.*

(6) *$\mathcal{G}_i^{\mathrm{AFS}}$ is expanding in measure uniformly for $i \in \mathbb{N}\cup\{\infty\}$.*

Proof. "(1) $\Rightarrow$ (2)": Condition (1) implies that $\mathcal{G}_\infty^{\mathrm{AFS}}$ is asymptotically expanding in measure. Then by Proposition 6.2, this implies (2).

"(2) $\Leftrightarrow$ (3)" is due to [1, Theorem 4] and [23, Proposition 3.5].

"(3) $\Leftrightarrow$ (5)" is fairly well known (see, e.g., [1, Lemma 2.2]).

"(2) $\Leftrightarrow$ (4)" is due to [23, Theorem 6.16, Corollary 6.17].

"(5) $\Rightarrow$ (6)": For each $i \in \mathbb{N}$, note that $\mathcal{G}_i^{\mathrm{AFS}}$ is isomorphic to the pair groupoid $(\Gamma/N_i)\times(\Gamma/N_i)$, where the length function corresponds to the one given by $(x, y) \mapsto d_i(x, y)$ for the left-invariant metric d_i on Γ/N_i induced by ℓ_i. According to Proposition 6.9, (4) is equivalent to that $\mathcal{G}_i^{\mathrm{AFS}}$ is expanding uniformly for $i \in \mathbb{N}$. This concludes (5) since we already showed "(4) $\Rightarrow$ (3)".

"(6) $\Rightarrow$ (1)" is trivial. ∎

Finally, combining with Theorem 5.3′, we obtain the following.

Corollary 6.22. *For the groupoid $\mathcal{G}^{\mathrm{AFS}}$, take $P_i \in \mathfrak{B}(L^2(X_i, \mu_i))$ to be the averaging projection for $i \in \mathbb{N} \cup \{\infty\}$. For $P = (\mathrm{SOT}) - \sum_{i \in \mathbb{N} \cup \{\infty\}} P_i$, the following are equivalent:*

(1) $P \in \mathbf{C}^*_{\mathrm{u}}(\mathcal{G}^{\mathrm{AFS}})$.

(2) $P \in \mathbf{C}^*_{\mathrm{uq}}(\mathcal{G}^{\mathrm{AFS}})$.

(3) *The natural action of Γ on $\widehat{\Gamma}$ is asymptotically expanding in measure.*

(4) *The sequence $\{\Gamma/N_i\}_{i \in \mathbb{N}}$ forms a sequence of expander graphs.*

6.5. Graph groupoids

We first recall background knowledge from [18].

Throughout this subsection, let $G = (V, E, r, s)$ be a directed uniformly locally finite connected graph, where V is the vertex set, E is the edge set and $r, s : E \to V$ describe the range and source of edges. Here, G is called *uniformly locally finite* if $\sup_{v \in V} |s^{-1}(v)|$ and $\sup_{v \in V} |r^{-1}(v)|$ are finite. Given $v \in V$, denote $s\text{-}\deg(v) := |s^{-1}(v)|$.

A *finite path* in G is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of edges in E with $s(\alpha_{j+1}) = r(\alpha_j)$ for $1 \le j \le k-1$. Write $s(\alpha) = s(\alpha_1)$ and $r(\alpha) = \mathrm{r}(\alpha_k)$. The *length* of α is $|\alpha| := k$. Denote by v the path of length 0 with $s(v) = r(v) = v$. Denote $F(G)$ (resp. $P(G)$) the set of all finite (resp. infinite) paths in G and $F(G, v)$ (resp. $P(G, v)$) those starting at $v \in V$. For $\alpha, \mu \in F(G)$ satisfying $r(\alpha) = s(\mu)$, we define a path $\alpha\mu \in F(G)$ of length $|\alpha| + |\mu|$ by $\alpha\mu = (\alpha_1, \dots, \alpha_{|\alpha|}, \mu_1, \dots, \mu_{|\mu|})$. We can similarly define $\alpha x \in P(G)$ for $\alpha \in F(G)$ and $x \in P(G)$ satisfying $s(x) = r(\alpha)$.

Equip $P(G)$ with the induced topology from the product topology on the infinite product ΠE, which is locally compact, σ-compact, totally disconnected and Hausdorff. Moreover, it has a basis consisting of the following subsets:

$$Z(\alpha) = \{x \in P(G) \mid x_1 = \alpha_1, \dots, x_{|\alpha|} = \alpha_{|\alpha|}\}, \quad \text{where } \alpha \in F(G).$$

By [18, Lemma 2.1], $Z(\alpha) \cap Z(\beta) \neq \emptyset$ if and only if either α is a *prefix* of β (i.e., $\beta = \alpha\beta'$ for some $\beta' \in F(G)$) or β is a prefix of α.

To construct the groupoid associated to the graph G, we consider the following equivalence relation on $P(G)$: two paths $x, y \in P(G)$ are *shift equivalent* with lag $k \in \mathbb{Z}$ (written $x \sim_k y$) if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i \ge N$.

Definition 6.23. For a directed uniformly locally finite connected graph $G = (V, E, r, s)$, the associated *graph groupoid* $\mathcal{G}$ is defined to be (as a set)

$$\mathcal{G} = \{(x, k, y) \in P(G) \times \mathbb{Z} \times P(G) \mid x \sim_k y\}$$

with $(x, k, y) \cdot (y, l, z) := (x, k + l, z)$ and $(x, k, y)^{-1} := (y, -k, x)$. The range and source maps are given by $\mathrm{r}(x, k, y) = x$ and $\mathrm{s}(x, k, y) = y$, respectively, with $\mathcal{G}^{(0)} = P(G)$.

For $\alpha, \beta \in F(G)$ with $r(\alpha) = r(\beta)$, denote

$$Z(\alpha, \beta) := \{(x, k, y) \mid x \in Z(\alpha),\ y \in Z(\beta),\ k = |\beta| - |\alpha|,\ x_i = y_{i+k} \text{ for } i \geq |\alpha|\},$$

which is a bisection in $\mathcal{G}$ with $\mathrm{r}(Z(\alpha, \beta)) = Z(\alpha)$ and $\mathrm{s}(Z(\alpha, \beta)) = Z(\beta)$. The sets $Z(\alpha, \beta)$ form a basis for a locally compact Hausdorff topology on $\mathcal{G}$, which makes it a second countable étale groupoid where each $Z(\alpha, \beta)$ is compact open. The induced topology on $\mathcal{G}^{(0)} = P(G)$ coincides with the product topology above.

To apply our results to the graph groupoid $\mathcal{G}$, we first endow $\mathcal{G}$ with the following length function ℓ. For $(x, k, y) \in \mathcal{G}$, choose the smallest $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i \geq N$, denoted by $N(x, k, y)$, and set

$$\ell(x, k, y) = 2N(x, k, y) + k. \tag{6.3}$$

Note that $N(x, k, y) + k \geq 0$ for any $(x, k, y) \in \mathcal{G}$. Intuitively, $\ell(x, k, y)$ is the total length of the shortest starting segments in x and y, deleting which x and y are the same with lag k. It is routine to check that ℓ is indeed a length function on $\mathcal{G}$ taking values in $\mathbb{N}$. Moreover, given $n \in \mathbb{N}$, the ball in $\mathcal{G}$ defined by $B_n := \{(x, k, y) \in \mathcal{G} \mid \ell(x, k, y) \leq n\}$ can be reconstructed as follows:

$$B_n = \bigcup\{Z(\alpha, \beta) \mid \alpha, \beta \in F(G) \text{ with } r(\alpha) = r(\beta) \text{ and } |\alpha| + |\beta| \leq n\}. \tag{6.4}$$

Moreover, we have

$$B_n = (B_1)^n \quad \text{for any } n \in \mathbb{N}. \tag{6.5}$$

These can be verified by direct calculations, and we omit the details.

Next, we will define a Borel measure μ on $\mathcal{G}^{(0)} = P(G)$. Given $v \in V$ and $b_v > 0$, we define a measure μ_v on $P(G, v)$ by first setting

$$\mu_v(Z(\alpha)) := b_v \cdot \prod_{i=1}^{k} \frac{1}{s\text{-}\deg(s(\alpha_i))} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in F(G, v). \tag{6.6}$$

Since the sets $Z(\alpha)$ for $\alpha \in F(G, v)$ form a basis for the topology on $P(G, v)$, then the above gives rise to a Borel measure μ_v on $P(G, v)$.

Combine μ_v into a single probability Borel measure μ on $P(G)$. Since G is connected and uniformly locally finite, V is countable. For each $v \in V$, choose $b_v > 0$ such that $\sum_{v \in V} b_v = 1$. Then the measures μ_v defined above give rise to a Borel measure μ on $P(G)$. In the sequel, we fix such a measure μ.

By the construction of ℓ and μ above, it is clear that for any $\alpha, \beta \in F(G)$ with $r(\alpha) = r(\beta)$, the subset $Z(\alpha, \beta)$ is an admissible bisection. Moreover, we have the following.

Lemma 6.24. *Each ball $B_n = \{(x, k, y) \in \mathcal{G} \mid \ell(x, k, y) \leq n\}$ is decomposable.*

Proof. Fix $n \in \mathbb{N}$. We aim to decompose the ball B_n into finitely many admissible bisections. To achieve this, we construct another graph $\widehat{G} = (\widehat{V}, \widehat{E})$ with

$$\widehat{V} := \{(\alpha, \beta) \mid \alpha, \beta \in F(G) \text{ with } r(\alpha) = r(\beta) \text{ and } |\alpha| + |\beta| \leq n\},$$

and two vertices (α_1, β_1) and (α_2, β_2) in $\widehat{V}$ are connected by an edge if and only if at least one of the following four cases holds: (i) α_1 is a prefix of α_2; (ii) α_2 is a prefix of α_1; (iii) β_1 is a prefix of β_2; (iv) β_2 is a prefix of β_1. Since G is uniformly locally finite and connected, then $\widehat{G}$ is uniformly locally finite as well. It is a well-known fact (see, e.g., [31, Lemma 4.2]) that $\widehat{V}$ can be decomposed into $\widehat{V} = \bigsqcup_{p=1}^{N} \widehat{V}_p$ such that for each p, any two distinct vertices in $\widehat{V}_p$ are *not* connected by an edge.

For each $p = 1, \ldots, N$, set

$$K_p := \bigsqcup_{(\alpha,\beta)\in\widehat{V}_p} Z(\alpha, \beta).$$

By the construction of $\widehat{E}$, it is routine to check that K_p is an admissible bisection. Finally, (6.4) implies $B_n = \bigcup_{p=1}^{N} K_p$, and hence we conclude the proof. ■

Lemma 6.24 shows that the set $\{B_n \mid n \in \mathbb{N}\}$ is cofinal in the set of all decomposable subsets of $\mathcal{G}$ in the sense of Remark 5.8. Hence we can merely use these balls to define asymptotic expansion, dynamical propagation and quasi-locality for graph groupoids by the discussion at the end of Section 5.

Thanks to Lemma 6.24 and (6.5), we have the following.

Lemma 6.25. *The graph groupoid $\mathcal{G}$ is asymptotically expanding in measure* if and only if *for any $\alpha \in \left(0, \frac{1}{2}\right]$, there exists $C_\alpha > 0$ such that for any measurable $A \subseteq \mathcal{G}^{(0)}$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$, then $\mu(\mathrm{r}(B_1 \cdot A)) > (1 + C_\alpha)\mu(A)$.*

Lemma 6.25 can be further refined as follows under an extra assumption.

Proposition 6.26. *Let G be a uniformly locally finite connected graph with $\mathcal{G}, \ell, \mu$ as above. Assume there exists $C \geq 1$ such that $\frac{b_w}{C} \leq b_v \leq C b_w$ for any $v, w \in V$ connected by an edge, where $(b_v)_v$ are the weights used to define μ. Then $\mathcal{G}$ is asymptotically expanding in measure* if and only if *for any $0 < \alpha \leq \frac{1}{2}$, there exists $C_\alpha > 0$ such that for any measurable $A \subseteq \mathcal{G}^{(0)}$ of the form $A = \bigcup_{i=1}^{N} Z(\alpha_i)$ with $\alpha \leq \mu(A) \leq \frac{1}{2}$ for $N \in \mathbb{N}$ and $\alpha_i \in F(G)$, we have $\mu(\mathrm{r}(B_1 \cdot A)) > (1 + C_\alpha)\mu(A)$.*

Proof. By Lemma 6.25, we only need to prove sufficiency. Since $\mathcal{G}^{(0)} = P(G)$ is a locally compact Hausdorff space in which every open subset is σ-compact and μ is finite, it follows from [12, Theorem 7.8] that μ is regular. Take C as in the assumption and set $D = \max\{\sup_{v\in V}|s^{-1}(v)|, \sup_{v\in V}|r^{-1}(v)|\}$.

We claim that for any measurable $A' \subseteq P(G)$, we have

$$\mu(\mathrm{r}(B_1 \cdot A')) \leq CD(D + 2)\mu(A'). \tag{6.7}$$

Indeed, for any $\alpha = (\alpha_1, \dots, \alpha_k) \in F(G)$ with $k \geq 1$, set

$$F(G)_\alpha := \{\alpha\} \cup \{(\alpha_2, \dots, \alpha_k)\} \cup \{(\alpha_0, \alpha_1, \dots, \alpha_k) \mid \alpha_0 \in E \text{ with } r(\alpha_0) = s(\alpha_1)\}.$$

Here, we take the convention that if $k = 1$, then $(\alpha_2, \dots, \alpha_k) = r(\alpha_1)$. By the choice of D, we have $|F(G)_\alpha| \leq D + 2$. Moreover, it follows from the definition of μ in (6.6) that $\mu(Z(\beta)) \leq CD\mu(Z(\alpha))$ for any $\beta \in F(G)_\alpha$. Hence we obtain

$$\mu(\mathrm{r}(B_1 \cdot Z(\alpha))) \leq \sum_{\beta \in F(G)_\alpha} \mu(Z(\beta)) \leq CD(D+2)\mu(Z_\alpha).$$

Denote $\mathcal{M} := \{\text{measurable } A' \subseteq P(G) \mid (6.7) \text{ holds for } A'\}$. Note that the set $\mathcal{P} := \{Z(\alpha) \mid \alpha \in F(G)\}$ is a semi-ring, and hence the analysis above implies that $\mathcal{M}$ contains the ring $R(\mathcal{P})$ generated by $\mathcal{P}$. Due to the monotone class theorem, $\mathcal{M}$ contains the σ-algebra generated by $\mathcal{P}$, which coincides with the Borel σ-algebra on $P(G)$. Hence we conclude the claim.

Given $0 < \alpha \leq \frac{1}{2}$, we have a constant $C_{\alpha/2} > 0$ from the assumption for $\frac{\alpha}{2}$. Also given a measurable $A \subseteq P(G)$ with $\alpha \leq \mu(A) < \frac{1}{2}$, take

$$\varepsilon_0 := \min\left\{\frac{\alpha}{2}, \frac{1}{2} - \mu(A), \frac{\alpha C_{\alpha/2}}{2 + 2C_{\alpha/2} + 2CD(D+2)}\right\} > 0.$$

Due to regularity, there is an open subset $\widetilde{A} = \bigcup_{i=1}^N Z(\alpha_i)$ for $\alpha_i \in F(G)$ such that $\mu(A \triangle \widetilde{A}) < \varepsilon_0$, which implies that $\frac{\alpha}{2} \leq \mu(\widetilde{A}) \leq \frac{1}{2}$. By the claim above, we have

$$\begin{aligned}
\mu(\mathrm{r}(B_1 \cdot A)) &\geq \mu(\mathrm{r}(B_1 \cdot \widetilde{A})) - \mu(\mathrm{r}(B_1 \cdot (A \triangle \widetilde{A}))) \\
&> (1 + C_{\alpha/2})\mu(\widetilde{A}) - CD(D+2)\mu(A \triangle \widetilde{A}) \\
&\geq (1 + C_{\alpha/2})(\mu(A) - \varepsilon_0) - CD(D+2)\varepsilon_0 \\
&\geq (1 + C_{\alpha/2})\mu(A) - \frac{\alpha C_{\alpha/2}}{2} \geq \left(1 + \frac{C_{\alpha/2}}{2}\right)\mu(A).
\end{aligned} \tag{6.8}$$

Finally, we consider measurable $A \subseteq P(G)$ with $\mu(A) = \frac{1}{2}$. Take

$$\varepsilon_1 := \min\left\{\frac{3}{8}, \frac{C_{1/8}}{32 + 4C_{1/8} + 32CD(D+2)}\right\} > 0.$$

Without loss of generality, assume $C_{1/8} \leq 1$. Again due to regularity, we choose an open subset $\widetilde{A} = \bigcup_{i=1}^N Z(\alpha_i)$ for $\alpha_i \in F(G)$ such that $\mu(A \triangle \widetilde{A}) < \varepsilon_1$. This implies that $\frac{1}{2} - \varepsilon_1 < \mu(\widetilde{A}) < \frac{1}{2} + \varepsilon_1$. We need to control the expansion of $\widetilde{A}$.

If $\mu(\widetilde{A}) \leq \frac{1}{2}$, then $\mu(\mathrm{r}(B_1 \cdot \widetilde{A}) \setminus \widetilde{A}) \geq C_{1/8}\mu(A)$. Otherwise, set

$$\widetilde{B} := P(G) \setminus (\mathrm{r}(B_1 \cdot \widetilde{A}))$$

and then $\mu(\widetilde{B}) < \frac{1}{2}$. If $\mu(\widetilde{B}) < \frac{1}{4}$, then $\mu(\mathrm{r}(B_1 \cdot \widetilde{A})) = 1 - \mu(\widetilde{B}) > 1 - \frac{1}{4} \geq \frac{3}{2}\mu(A)$. If $\mu(\widetilde{B}) \geq \frac{1}{4}$, then by applying (6.8) to $\widetilde{B}$, we obtain

$$\mu(\mathrm{r}(B_1 \cdot \widetilde{B})) > \left(1 + \frac{C_{1/8}}{2}\right)\mu(\widetilde{B}).$$

Hence

$$\mu(\mathrm{r}(B_1 \cdot \widetilde{A}) \setminus \widetilde{A}) \geq \mu(\mathrm{r}(B_1 \cdot \widetilde{B}) \setminus \widetilde{B}) > \frac{C_{1/8}}{2}\mu(\widetilde{B}) \geq \frac{C_{1/8}}{8} \cdot \mu(\widetilde{A}).$$

In conclusion, we obtain $\mu(\mathrm{r}(B_1 \cdot \widetilde{A})) > \big(1 + \frac{C_{1/8}}{8}\big)\mu(\widetilde{A})$. Therefore,

$$\begin{aligned}\mu(\mathrm{r}(B_1 \cdot A)) &\geq \mu(\mathrm{r}(B_1 \cdot \widetilde{A})) - \mu(\mathrm{r}(B_1 \cdot (A \triangle \widetilde{A}))) \\ &> \left(1 + \frac{C_{1/8}}{8}\right)\mu(\widetilde{A}) - CD(D+2)\mu(A \triangle \widetilde{A}) \\ &\geq \left(1 + \frac{C_{1/8}}{8}\right)(\mu(A) - \varepsilon_1) - CD(D+2)\varepsilon_1 \geq \left(1 + \frac{C_{1/8}}{16}\right)\mu(A),\end{aligned}$$

which concludes the proof. ∎

Finally, we provide a concrete example.

Example 6.27. Fix $k \in \mathbb{N}$. Consider the graph $G = (V, E)$, where $V = \mathbb{N}$ and $E = \{(n, n+i) \mid n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, k\}$. Let $\mathcal{G}$ be the associated graph groupoid with the length function from (6.3), and equip $P(G)$ with the Borel probability measure from (6.6) by setting $b_n = \mu(Z(n)) := \frac{1}{2^{n+1}}$ for $n \in \mathbb{N}$. More precisely, for $\alpha = (\alpha_1, \ldots, \alpha_m) \in F(G)$ with $\mathrm{s}(\alpha_1) = n$, set $\mu(Z(\alpha)) = \frac{1}{2^{n+1}k^m}$. We have the following:

(1) If $k = 1$, then $\mathcal{G}$ is expanding in measure.

(2) If $k \geq 2$, then $\mathcal{G}$ is not asymptotically expanding in measure.

For (1). For any $\alpha \in F(G)$, the set $Z(\alpha)$ consists of a single point, which is the infinite path starting at $s(\alpha)$. Hence $Z(\alpha) = Z(s(\alpha))$. Now for any measurable $A \subseteq P(G)$ with $0 < \mu(A) \leq \frac{1}{2}$, take $n_A := \min\{s(\alpha) \mid \alpha \in A\}$.

If $n_A = 0$, then $\mu(A) \leq \frac{1}{2}$ implies that $A = Z(0)$. Hence $Z(1) \subseteq \mathrm{r}(B_1 \cdot A)$, which implies that $\mu(\mathrm{r}(B_1 \cdot A)) \geq \frac{3}{2}\mu(A)$. If $n_A \geq 1$, then by the choice of μ, we have $\mu(Z(n_A)) \geq \frac{1}{2}\mu(A)$. Since $Z(n_A - 1) \subseteq \mathrm{r}(B_1 \cdot A)$, then

$$\mu(\mathrm{r}(B_1 \cdot A)) \geq \mu(A) + 2\mu(Z(n_A)) \geq 2\mu(A).$$

Hence $\mathcal{G}$ is expanding in measure, and we conclude (1).

For (2). Fix $k \geq 2$. For $n_1 \leq n_2 \in \mathbb{N}$, denote

$$F(G)_{n_1,n_2} := \{\alpha \in F(G) \mid s(\alpha) = n_1,\ r(\alpha) = n_2\}$$

and

$$Z_{n_1,n_2} := \bigcup \{Z(\alpha) \mid \alpha \in F(G)_{n_1,n_2}\}.$$

Direct calculations show that for $n \geq k$, we have

$$\mu(Z_{0,n}) + \sum_{j=1}^{k-1} \frac{k-j}{k}\mu(Z_{0,n-j}) = \frac{1}{2}.$$

Note that $1 + \sum_{j=1}^{k-1} \frac{k-j}{k} = \frac{k+1}{2}$. Hence for any $p \in \mathbb{N}$, there exists an integer $n_p \in \{pk+1, pk+2, \dots, (p+1)k\}$ such that $\mu(Z_{0,n_p}) \geq \frac{1}{k+1}$.

Now for positive integer $p \in \mathbb{N}$, define

$$F(G)_{n_p} := \bigsqcup_{m=0}^{n_p} F(G)_{m,n_p} \quad \text{and} \quad A_p = \bigcup \{Z(\alpha\alpha_p) \mid \alpha \in F(G)_{n_p}\},$$

where $\alpha_p := (n_p, n_p + 1)$. Then we have

$$\mu(A_p) > \frac{1}{k}\mu(Z_{0,n_p}) \geq \frac{1}{k(k+1)} \quad \text{and} \quad \mu(A_p) \leq \frac{1}{k}\mu(P(G)) = \frac{1}{k}.$$

By the construction of $F(G)_{n_p}$, it is clear that for any $\alpha \in F(G)_{n_p}$ with $|\alpha| \geq 1$, we have $\mathrm{r}(B_1 \cdot Z(\alpha\alpha_p)) \subseteq A_p$. Hence

$$\mu(\mathrm{r}(B_1 \cdot A_p)) = \mu(A_p \sqcup Z(n_p + 1)) = \mu(A_p) + \frac{1}{2^{n_p+2}},$$

which shows that $\mu(\mathrm{r}((B_1 \cdot A_p) \setminus A_p)) = \frac{1}{2^{n_p+2}} \to 0$ as $p \to \infty$. This concludes (2).

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Xulong Lu
Research Center for Operator Algebras, School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, 200241 Shanghai, P. R. China; 52290155009@stu.ecnu.edu.cn
Author IDs: zbMATH lu.xulong ORCID 0009-0004-3346-5469

Qin Wang
Research Center for Operator Algebras, School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, 200241 Shanghai, P. R. China; qwang@math.ecnu.edu.cn
Author IDs: zbMATH wang.qin.1 MR 618295 ORCID 0000-0002-1347-506X

Jiawen Zhang (corresponding author)
School of Mathematical Sciences, Fudan University, 220 Handan Road, 200433 Shanghai, P. R. China; jiawenzhang@fudan.edu.cn
Author IDs: zbMATH zhang.jiawen MR 1108765 ORCID 0000-0002-6383-6732