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Weyl calculus on graded groups

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Abstract. The aim of this paper is to establish a pseudo-differential Weyl calculus on graded nilpotent Lie groups G which extends the celebrated Weyl calculus on $\mathbb{R}^n$. To reach this goal, we develop a symbolic calculus for a very general class of quantization schemes, following the work by Măntoiu and Ruzhansky (2017), using the Hörmander symbol classes $S^m_{\rho,\delta}(G)$ introduced in the book by Fischer and Ruzhansky (2016). We particularly focus on the so-called symmetric calculi, for which quantizing and taking the adjoint commute, among them the Euclidean Weyl calculus, but we also recover the (non-symmetric) Kohn–Nirenberg calculus, on $\mathbb{R}^n$ and on general graded groups (Fischer and Ruzhansky (2016)). Several interesting applications follow directly from our calculus: expected mapping properties on Sobolev spaces, the existence of one-sided parametrices and the Gårding inequality for elliptic operators, and a generalization of the Poisson bracket for symmetric quantizations on stratified groups.

In the particular case of the Heisenberg group $\mathbb{H}_n$, we are able to answer the fundamental question of this paper: which, among all the admissible quantizations, is the natural Weyl quantization on $\mathbb{H}_n$?

Among other things, we discuss and investigate an analog of the symplectic invariance property of the Weyl quantization in the setting of graded groups, as well as a notion of noncommutative Poisson bracket for symbols in the setting of stratified groups.

The authors dedicate this work to the memory of Marius Măntoiu.

Contents

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1. Introduction

The Weyl quantization of pseudo-differential operators on $\mathbb{R}^n$,

$$\mathrm{Op}^{\mathrm{w}}(\sigma) f(x) = \frac{1}{(2\pi)^n} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\xi}\, \sigma\Big(\frac{x+y}{2}, \xi\Big)\, f(y)\, dy\, d\xi, \tag{1.1}$$

has many remarkable properties that are appreciated by mathematicians and physicists alike. Two particularly noteworthy properties are:

- Preservation of involution:
$$\mathrm{Op}^{\mathrm{w}}(\overline{\sigma}) = \mathrm{Op}^{\mathrm{w}}(\sigma)^*. \tag{1.2}$$
- Symplectic invariance: for each symplectic map $S \in \mathrm{Sp}(2n, \mathbb{R})$ there exists an up to a factor ± 1 uniquely determined unitary operator $U_S \in \mathcal{U}(L^2(\mathbb{R}^n))$ such that
$$\mathrm{Op}^{\mathrm{w}}(\sigma \circ S) = U_S^{-1} \mathrm{Op}^{\mathrm{w}}(\sigma) U_S. \tag{1.3}$$

These properties hold true for very general types of symbols σ on $\mathbb{R}^n \times \mathbb{R}^n$, in particular for the Hörmander symbol classes $S^m_{\rho,\delta}(\mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$, as was shown in Hörmander's seminal paper on the Weyl calculus of pseudo-differential operators [37]. Among the family of so-called τ-quantizations

$$\mathrm{Op}^{\tau}(\sigma) f(x) = \frac{1}{(2\pi)^n} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\xi}\, \sigma\big(x - \tau(x-y), \xi\big)\, f(y)\, dy\, d\xi, \quad \tau \in [0,1], \tag{1.4}$$

all of which give rise to a full-fledged calculus for the classes $S^m_{\rho,\delta}(\mathbb{R}^n)$, especially the Kohn–Nirenberg calculus for $\mathrm{Op}^0 \equiv \mathrm{Op} \equiv \mathrm{Op}^{\mathrm{KN}}$,

$$\mathrm{Op}(\sigma) f(x) = \frac{1}{(2\pi)^n} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\xi}\, \sigma(x, \xi)\, f(y)\, dy\, d\xi, \tag{1.5}$$

the properties (1.2) and (1.3) hold true for precisely one quantization scheme: the Weyl quantization $\mathrm{Op}^{\mathrm{w}} = \mathrm{Op}^{\frac{1}{2}}$.

Several notable papers have developed Weyl quantizations for nilpotent groups, especially the Heisenberg group $\mathbb{H}_n$, which by construction satisfy (1.2) (see, e.g., [4, 5, 14, 17, 18, 32, 38, 44–46]). Dynin's early works [17, 18] are based on the fact that for a given nilpotent group G one can find another nilpotent group $N_G \geq G$ whose generic unitary irreducible representations $\pi \in \widehat{N}_{G_{\mathrm{gen}}}$ act on $\mathcal{H}_\pi = L^2(G)$. For complex-valued symbols defined on the corresponding co-adjoint orbits $\mathcal{O}_\pi \subseteq \mathfrak{n}_G^*$, the representation then gives rise to a Weyl-type quantization of operators a priori defined on $\mathcal{S}(G) \subseteq L^2(G)$. The papers [17, 18] establish this procedure for $G = \mathbb{H}_n$ and a group $N_{\mathbb{H}_n}$ given as a semi-direct product $\mathbb{R}^{2n+2} \rtimes \mathbb{H}_n$. Folland [32] later called $\mathbb{R}^{2n+2} \rtimes \mathbb{H}_n$ the meta-Heisenberg group $H(\mathbb{H}_n)$ and studied it as a special case of meta-Heisenberg groups $H(G)$ of general 2-step nilpotent groups G. As Folland showed, one obtains invariance of the Weyl quantization for the automorphisms of $H(\mathbb{H}_n)$ and it was conjectured that this might hold true for all meta-Heisenberg groups $H(G)$ up to degenerate examples. For Dynin's Weyl quantization on $\mathbb{H}_n$, this can in fact be considered as the adequate analog of (1.3). The papers [17, 18], moreover, give an outlook on a symbolic calculus for $G = \mathbb{H}_n$, for which [32] establishes a connection with Beals and Greiner's calculus on Heisenberg manifolds [3]. A symbolic calculus in the sense of [37] for this setting is, however, only partially developed in [17, 18]. The composition of two operators with symbols in Dynin's Hörmander-type classes, for example, is only available when one of them is a differential operator. This restriction can essentially not be removed because the asymptotic series of the Dynin-Weyl product generally features infinitely many summands of growing order, as was shown in [48, §5.6.1]. Other notable works of this type also employ a quantization procedure via some representation $\pi \in \widehat{N}_{\text{gen}}$ of a nilpotent group N for symbols $\sigma\colon \mathcal{O}_\pi \to \mathbb{C}$, but restrict to classes of classical (e.g., [14]) or even non-differentiable symbols (e.g., [4, 5]) or do not make an attempt at a full-fledged symbolic calculus in the first place (see, e.g., [38, 44–46]).

A very different type of calculus was first mentioned in Taylor [54] in order to study pseudo-differential operators on $\mathbb{H}_n$ and contact manifolds. Its novel use of a generalized version of Kohn–Nirenberg quantization on Lie groups is based on the generally operator-valued group Fourier transform and appropriate operator-valued symbols. One of its main ingredients for the analysis on $\mathbb{H}_n$ is the fundamental connection between the operator-valued symbols on $\mathbb{H}_n \times \widehat{\mathbb{H}}_n$ and Weyl-quantized operators on $\mathbb{R}^n$, which consequently permits the use of scalar-valued symbols on an extended phase space. The purely kernel based pseudo-differential calculus on homogeneous groups [12] by Christ, Geller, Głowacki and Polin, which is essentially equivalent to the one in [54] in the case $G = \mathbb{H}_n$, provides explicit formulas and expansions for adjoints and compositions as well as one-sided parametrices, but refrains from using symbols and essentially focuses on classical pseudo-differential operators. Much later, however, the connection between the operator-valued symbols on $\mathbb{H}_n \times \widehat{\mathbb{H}}_n$ and the Hörmander-Weyl calculus on $\mathbb{R}^n$ was further explored by Bahouri, Fermanian-Kammerer and Gallagher [1] to develop a comprehensive symbolic pseudo-differential calculus on $\mathbb{H}_n$, which provides asymptotic expansions for the (scalar-valued) adjoint and composite symbols. The use of scalar-valued symbols on an extended phase space, a feature which is very specific of $\mathbb{H}_n$, was eventually dropped by Fischer and the third author, in [28], in order to develop a full-fledged Kohn–Nirenberg-type calculus on general graded nilpotent Lie groups G for appropriately generalized operator-valued Hörmander classes $S^m_{\rho,\delta}(G)$. The successful use of operator-valued symbols throughout the calculus is made possible by an extensive use of harmonic analysis on graded groups and novel tools and function spaces. A profound comprehension of the properties of the associated kernels, however, remains one of the key ingredients. The work [28] provided a parallel theory to the one developed earlier in the setting of compact Lie groups in [49, 50]. (For further developments see, e.g., [10, 15, 25, 26, 51, 52].)

The existence of a well-functioning Kohn–Nirenberg calculus on graded groups naturally opens up the following questions.

Questions. Is it possible to find a class of τ-quantizations on general graded groups G which

(Q1) give rise to substantial pseudo-differential calculi for the symbol classes $S^m_{\rho,\delta}(G)$,

(Q2) include the classical τ-calculi (1.4) on $G = \mathbb{R}^n$ as special cases,

(Q3) include the Kohn–Nirenberg calculus on graded groups [28] as a special case,

(Q4) include at least one viable Weyl-type calculus that satisfies appropriate versions of (1.2) and (1.3) on any graded G?

This paper aims at positively answering all these questions. In order to treat these questions in a rigorous way, we recall one of the main tools in [28], the operator-valued group Fourier transform. It is determined by the generally infinite-dimensional irreducible unitary representations $\pi \in \widehat{G}$, the noncommutative analogs of the exponential functions $x \mapsto e^{ix\xi}$ on $G = \mathbb{R}^n$, and accordingly defined by

$$\hat{f}(\pi) \equiv (\mathcal{F} f)(\pi) := \int_G f(x)\pi(x)^*\, dx = \int_G f(x)\pi(x^{-1})\, dx. \tag{1.6}$$

Despite being an operator-valued transform, its values $\hat{f}(\pi)$ are generally well-behaved operators on the representation space $\mathcal{H}_\pi \cong L^2(\mathbb{R}^k)$, for some $k = k(\pi) \in \mathbb{N}$, e.g., compact, Hilbert–Schmidt, trace-class, etc., if the function f lies in nice function spaces like $L^1(G)$, $L^2(G)$, $\mathcal{S}(G)$, etc, and in these specific cases actually the whole field of operators $\hat{f} = \{\hat{f}(\pi)\}_{\pi\in\widehat{G}}$ has desirable properties. In particular, for Schwartz functions $f \in \mathcal{S}(G)$, one gets the Fourier inversion formula

$$f(x) = \int_{\widehat{G}} \mathrm{Tr}\big(\pi(x)\hat{f}(\pi)\big)\, d\mu(\pi) = \int_{\widehat{G}} \mathrm{Tr}\Big(\int_G \pi(y^{-1}x) f(y)\, dy\Big)\, d\mu(\pi), \tag{1.7}$$

a generalization of

$$f(x) = \int_{\widehat{\mathbb{R}}^n} e^{ix\xi} \hat{f}(\xi)\, \frac{d\xi}{(2\pi)^n} = \int_{\widehat{\mathbb{R}}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} f(y)\, dy\, \frac{d\xi}{(2\pi)^n},$$

which, in [28], motivated the definition of the Kohn–Nirenberg-type quantization

$$\begin{aligned}\mathrm{Op}(\sigma) f(x) &= \int_{\widehat{G}} \mathrm{Tr}\big(\pi(x)\sigma(x,\pi)\hat{f}(\pi)\big)\, d\mu(\pi) \\ &= \int_{\widehat{G}} \mathrm{Tr}\Big(\int_G \pi(y^{-1}x)\sigma(x,\pi) f(y)\, dy\Big)\, d\mu(\pi)\end{aligned} \tag{1.8}$$

for fields of operator-valued symbols $\sigma = \{\sigma(x,\pi) \mid x \in G,\ \pi \in \widehat{G}\}$ with $\sigma(x,\pi)\colon \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty}$ for all $x \in G$ and almost every $\pi \in \widehat{G}$. For the special case $G = \mathbb{R}^n$ this quantization coincides with (1.5).

Starting from (1.8), Măntoiu and the third author [41] investigated the question whether interesting τ-quantizations would arise if one replaced the point $x \in G$ in $\sigma(x,\pi)$ by a suitable generalization of $x - \tau(x-y) \in \mathbb{R}^n$, say the points $x\tau(y^{-1}x)^{-1} \in G$ determined by a measurable function $\tau\colon G \to G$, called the *quantizing function*. The resulting quantizations

$$\mathrm{Op}^\tau(\sigma) f(x) = \int_{\widehat{G}} \mathrm{Tr}\Big(\int_G \pi(y^{-1}x)\sigma\big(x\tau(y^{-1}x)^{-1}, \pi\big) f(y)\, dy\Big)\, d\mu(\pi), \tag{1.9}$$

were subsequently studied for the much wider class of second countable unimodular type I locally compact groups by means of C^*-algebra theory, producing surprising noncommutative generalizations of well-known concepts like, e.g., the Wigner function and the short time Fourier transform.[1] Although the generality of the setting necessitated a restriction of $\mathrm{Op}^\tau(\sigma)$ to the class of Hilbert–Schmidt operators on $L^2(G)$, the approach pointed in the right direction: choosing the constant function $\tau\colon x \mapsto e_G$ for the unit element $e_G \in G$, one recovers the Kohn–Nirenberg quantization (1.8), while for the functions $\tau(x) := \tau x$, $\tau \in [0, 1]$, on $G = \mathbb{R}^n$ one recovers the quantizations (1.4).

A particularly valuable insight was gained from investigating those quantizing functions τ which satisfy

$$\mathrm{Op}^\tau(\sigma)^* = \mathrm{Op}^\tau(\sigma^*) \tag{1.10}$$

for all Hilbert–Schmidt operators $\mathrm{Op}^\tau(\sigma)$ on $L^2(G)$. The authors of [41] named them *symmetry functions* and every quantization (1.9) arising from a symmetry function is called a *symmetric quantization* on G. (Equivalently, we will often simply say that τ is symmetric.) Although (1.10) seems to be the appropriate generalization of (1.2), it may in practice be hard to check for operator-valued symbols. The authors could show, however, that a quantizing function τ is symmetric if and only if it satisfies

$$\tau(x) = \tau(x^{-1})x \tag{1.11}$$

for almost all $x \in G$ (cf. [41, Cor. 4.2]), a condition that can often be checked in a straightforward manner for a whole class of groups. It is, for example, obviously satisfied by the quantizing function $\tau(x) := \frac{1}{2}x$ on $G = \mathbb{R}^n$, which defines the Weyl quantization, but by no other $\tau(x) := \tau x$ with $\tau \in [0, 1]$.

Clearly, the class of Hilbert–Schmidt operators $\mathrm{Op}^\tau(\sigma)$ on $L^2(G)$ is a rather restricted class of pseudo-differential operators since, equivalently, its integral kernel Ker^τ_σ must lie in $L^2(G \times G)$. Moreover, the lack of smooth structure on generic second countable unimodular type I locally compact groups makes this setting too general to extend the quantizations $\sigma \mapsto \mathrm{Op}^\tau(\sigma)$ to full-fledged pseudo-differential calculi for differentiable symbol classes. Focusing on graded nilpotent groups, however, one can study the symmetry condition (1.10) for the Hörmander-type classes $S^m_{\rho,\delta}(G)$ introduced in [28]. The explicit description and asymptotic expansion of the symbol of the adjoint $\mathrm{Op}(\sigma)^*$ of a Kohn–Nirenberg-quantized symbol $\sigma \in S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, $0 \le \delta < \rho \le 1$, provided in [28, §5.5.3], indicates that this is also a good setting to study symmetric calculi on graded groups. It turns out that we can indeed extend the symmetry condition to the symbol classes $S^m_{\rho,\delta}(G)$, as shown in Theorem 4.14.

[1] For a subsequent extension including non-unimodular type I groups we refer to Măntoiu and Sandoval [43].

At this point the question arises whether symmetry functions are available for a given group or class of groups. For the setting of this paper, namely graded groups, this is always the case. In fact, there are at least two canonical examples of symmetry functions available on every exponential Lie group G. If we denote by $\log\colon G \to \mathfrak{g}$ the inverse of the exponential map $\exp\colon \mathfrak{g} \to G$, then the first example, due to [41, Prop. 4.3 (3)], is given by

$$\tau(x) = \int_0^1 \exp\big(s \log(x)\big)\, ds. \tag{1.12}$$

The second example, which seems to be new, is defined by

$$\tau(x) := \exp\Big(\frac{1}{2}\log(x)\Big). \tag{1.13}$$

By the Taylor–Campbell–Hausdorff formula and the anti-symmetry of the Lie bracket, this function is easily seen to satisfy (1.11). If we express the elements $x \in G$ in exponential coordinates with respect to a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, i.e., $x = \exp(x_1 X_1 + \cdots + x_n X_n) = (x_1, \ldots, x_n)$ for uniquely determined $x_1, \ldots, x_n \in \mathbb{R}$, then τ takes the form

$$\tau(x_1, \ldots, x_n) = \exp\Big(\frac{x_1}{2}X_1 + \cdots + \frac{x_n}{2}X_n\Big) = \Big(\frac{x_1}{2}, \ldots, \frac{x_n}{2}\Big).$$

Since on $G=\mathbb{R}^n$ the exponential function coincides with the identity function, one also gets

$$\int_0^1 \exp\big(s \log(x)\big)\, ds = \frac{x}{2},$$

so the two symmetry functions τ defined by (1.12) and (1.13) coincide.

On $G = \mathbb{H}_n$, however, these two functions are utterly distinct. Using the standard upper triangular matrix representation of $\mathbb{H}_n$, the integral (1.12) is easily seen to give[2]

$$\tau(x) = \tau(x_1, \ldots, x_{2n+1}) = \Big(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{2n+1}}{2} + \sum_{j=1}^{n} \frac{x_j x_{j+n}}{24}\Big). \tag{1.14}$$

The symmetry function (1.13), on the other hand, is given by

$$\tau(x) = \tau(x_1, \ldots, x_{2n+1}) = \big(\frac{x_1}{2}, \ldots, \frac{x_{2n+1}}{2}\big), \tag{1.15}$$

thus clearly differs from (1.14). Interestingly, both functions are members of the infinite family of symmetry functions

$$\tau(x) = \Big(\frac{x_1}{2}, \ldots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\Big) \tag{1.16}$$

with $c_{j,k} \in \mathbb{R}$, $j,k = 1, \ldots, 2n$ (cf. Example 4.5), and in fact one can easily come up with infinitely many symmetry functions with non-linear terms on any graded group.

In view of the abundant supply of (especially symmetric) quantizing functions, this paper sets out to develop a versatile pseudo-differential τ-calculus on general graded

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[2]The factor $\frac{1}{24}$ corrects the erroneous $\frac{1}{6}$ in [19, (A.4)].

groups G that gives positive answers to (Q1), (Q2), (Q3) and (Q4). To achieve this goal, however, we have to deal with a few serious difficulties due to the generally noncommutative nature of G:

- In contrast to $\mathbb{R}^n \times \widehat{\mathbb{R}}^n \cong T^*\mathbb{R}^n$, the set $G \times \widehat{G}$ is not a symplectic manifold in any obvious fashion. So, we cannot rely on a type of phase space analysis akin to the one $T^*\mathbb{R}^n$ but instead have to work with the integral kernels of $\mathrm{Op}^\tau(\sigma)$. However, the convolution-based kernel identities from the Kohn–Nirenberg calculus [28] are not available for general τ.[3] This has to be compensated by a more intricate kernel analysis.
- Moreover, the quantizing function $\tau\colon G \to G$ cannot be assumed to be a Lie group homomorphism since generally $\tau(xy) \neq \tau(x)\tau(y)$ for given $x, y \in G$. This complicates the whole calculus substantially.
- In order to control all the oscillatory integrals, one has to impose reasonably restrictive conditions on τ, which, however, should not rule out any of the relevant well-known or new examples of τ.
- Since there is no immediate generalization of the symplectomorphisms of $\mathbb{R}^n \times \widehat{\mathbb{R}}^n \cong T^*\mathbb{R}^n$ for $G \times \widehat{G}$, one has to find an appropriate version of condition (1.3).

In what follows we develop such a τ-calculus on graded groups for symbols in the Hörmander classes $S^m_{\rho,\delta}(G)$. Any given graded group G, whose topological dimension we denote by $n \in \mathbb{N}$, is understood to be equipped with a fixed family of automorphic dilations, characterized by a set of weights $1 = \upsilon_1 \leq \cdots \leq \upsilon_n \in \mathbb{N}$. The main results of our calculus are valid for all symbol classes $S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, which satisfy $0 \leq \delta < \min\{\rho, \frac{1}{\upsilon_n}\} \leq \rho \leq 1$. This peculiar restriction of the usual condition $0 \leq \delta < \rho \leq 1$ appears throughout the calculus to ensure the convergence of crucial oscillatory integrals. Essentially, it arises from left and right-invariant differentiation of the quantizing function τ and the fact that the latter generally *fails to be* a group homomorphism. It is therefore not surprising that the restriction disappears when $\tau = e_G$, the most trivial group homomorphism possible, and our calculus agrees with the Kohn–Nirenberg calculus [28]. Clearly, the restriction is also absent in the Euclidean case, where the canonical dilations are assumed to be isotropic, that is, when all weights $\upsilon_1, \ldots, \upsilon_n$ equal to 1. So the restriction, in a sense, reflects the noncommutative nature of the group and the fact that τ does not respect the group structure. We also note that the analysis of this paper covers completely the case of the symbol classes $S^m_{1,0}(G)$, $m \in \mathbb{R}$. This analysis will appear elsewhere.

However, in the case of the $G = \mathbb{H}_n$ we show in a forthcoming paper that the restriction on the gap between the optimal range $0 \leq \delta < \rho \leq 1$ and $0 \leq \delta < \min\{\rho, \frac{1}{\upsilon_n}\} \leq \rho \leq 1$ can be removed due to the particular structure of two-step nilpotent groups.

We conclude this introduction by describing the organization of the paper: Section 2 recalls some necessary basics from harmonic analysis on graded groups and the theory of Rockland operators. Section 3 gives a quick review of the Hörmander-type symbol classes

[3] Already the Weyl calculus on $\mathbb{R}^n$ has to make due without them.

$S^m_{\rho,\delta}(G)$ and the inhomogeneous Sobolev spaces $L^p_s(G)$ developed in [28], highlighting their well-known special cases on $\mathbb{R}^n$. Section 4, the main section of this paper, we give positive answers to the questions (Q1)–(Q3) by developing a symbolic τ-calculus for a very wide range of quantizing functions τ and for symbols in $S^m_{\rho,\delta}(G)$ classes with $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq \rho \leq 1$. This includes asymptotic expansions for the symbols of adjoint and composite operators as well as a novel notion of G-homogeneous Poisson bracket for symmetric quantizations on stratified groups, which coincides with the classical Poisson bracket on $\mathbb{R}^n$. All the corresponding results for the Weyl and Kohn–Nirenberg calculi on $\mathbb{R}^n$ and the Kohn–Nirenberg calculus on graded groups [28] are recovered as special cases. In Section 5 we give a sufficiently affirmative answer to (Q4) by providing an appropriate version of the symplectic invariance (1.3) for any general graded group G and explain why the special case $G = \mathbb{H}_n$ singles out the symmetric quantization defined by (1.15) as the most viable generalization of the Weyl quantization on $\mathbb{R}^n$.

2. Preliminaries

In this section, we recall the notions and tools from harmonic analysis on graded groups that we consider essential for a proper definition of τ-quantized operators on G with symbols in the Hörmander-type classes introduced in [28]. Although we focus on the most necessary material, we want to provide enough details and references in order not to lose among our readers the interested experts on pseudo-differential theory on $\mathbb{R}^n$. Readers who are familiar with harmonic analysis on nilpotent Lie groups may well skip this section.

2.1. Homogeneous groups

A Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ is called *homogeneous* if it admits a family of dilations

$$D_r = \exp\big(\log(r)A\big), \quad r > 0,$$

for some diagonalizable A on $\mathfrak{g}$ with positive eigenvalues $0 < v_1 \leq \cdots \leq v_n$, the so-called weights, such that each D_r is a vector space isomorphism that satisfies $D_r\big([X, Y]\big) = [D_r(X), D_r(Y)]$. The real number $Q := v_1 + \cdots + v_n$ denotes the so-called homogeneous dimension of G. By standard convention, the weights $v_1, \ldots, v_n$ are jointly rescaled so that the lowest weight $v_1 \geq 1$, hence $Q \geq n$. The existence of such a family of automorphic dilations implies that $\mathfrak{g}$ is nilpotent.

Families of automorphic dilations naturally arise if a nilpotent Lie algebra $\mathfrak{g}$ is equipped with a gradation, that is, a vector space decomposition $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$, for which all but finitely many of the $\mathfrak{g}_i$'s are $\{0\}$ and satisfying $[\mathfrak{g}_i, \mathfrak{g}_{i'}] \subseteq \mathfrak{g}_{i+i'}$ for all $i, i' \in \mathbb{N}$. Since the sequence of subspaces $\mathfrak{h}_k := \bigoplus_{i=k}^{\infty} \mathfrak{g}_i$ forms a (finite) nested sequence of ideals in $\mathfrak{g}$, any basis $\{X_1, \ldots, X_n\}$ given as the union of bases $\{X_{1_i}, \ldots, X_{n_i}\}$ is a strong Malcev basis of $\mathfrak{g}$, passing through the ideals $\mathfrak{h}_k$ (cf. [13, Thm. 1.1.13] and, e.g., [34, Lem. 4.16]), and defines a canonical family of dilations $\{D_r\}_{r>0}$ via $AX_j := iX_j$ if $X_j \in \mathfrak{g}_i$, that is, dilations that satisfy $D_r(X_j) = r^i X_j$. So, every graded nilpotent Lie algebra is homogeneous, but the converse is also true, that is, every homogeneous Lie algebra admits a gradation (cf. [42, Prop. 1.1]). Note, however, that the original dilations need not coincide with the dilations that arise from the gradation. Finally, let us recall that a graded Lie algebra is called *stratified* if the gradation satisfies $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for all $i \in \mathbb{N}$. Such a gradation is called a stratification of $\mathfrak{g}$ and the direct summand $\mathfrak{g}_i$, $i \in \mathbb{N}$, is referred to as the i-th stratum.

The connected, simply connected nilpotent Lie group $G = \exp(\mathfrak{g})$ is called homogeneous, graded or stratified if $\mathfrak{g}$ is homogeneous, graded or stratified, respectively.[4] Now suppose that G is homogeneous and that $\{X_1, \dots, X_n\}$ is eigenbasis of $A = \log(D_1)$ with weights $0 < \upsilon_1 \leq \cdots \leq \upsilon_n$. If G is equipped with the corresponding exponential coordinates, then, due to nilpotency, its group multiplication can be written as

$$\begin{aligned} xy &= \exp(x_1 X_1 + \cdots + x_n X_n) \exp(y_1 X_1 + \cdots + y_n X_n) \\ &= \exp(R_1(x,y) X_1 + \cdots + R_n(x,y) X_n) \end{aligned}$$

for some polynomials $R_j(x, y)$, $j = 1, \dots, n$. Since we chose $\{X_1, \dots, X_n\}$ to be an eigenbasis of A, these polynomials are homogeneous of degree υ_j with respect to the dilations $\{D_r\}_{r>0}$ and can be written as

$$\begin{aligned} R_j(x,y) = \sum_{[\alpha]+[\beta]=\upsilon_j} c_{j,\alpha,\beta}\, x^\alpha y^\beta &= x_j + y_j + \sum_{\substack{[\alpha]+[\beta]=\upsilon_j, \\ [\alpha]\neq 0 \neq [\beta]}} c_{j,\alpha,\beta}\, x^\alpha y^\beta \\ &=: x_j + y_j + \widetilde{R}_j(x_1, \dots, x_{j-1}, y_1, \dots, y_{j-1}), \end{aligned}$$

where $[\alpha] := \upsilon_1 \alpha_1 + \cdots + \upsilon_n \alpha_n \in \mathbb{R}$, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$. Moreover, the global chart map $\log := \exp^{-1} \colon G \to \mathfrak{g} \cong \mathbb{R}^n$ immediately gives rise to a family of automorphic dilations $G \to G : \exp \circ D_r \circ \log$, also denoted by $\{D_r\}_{r>0}$.

We will frequently denote an element $x \in G$ by the n-tuple of coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n$ computed with respect to the basis $\{X_1, \dots, X_n\}$. As a consequence, the dilations' action on group elements and the bi-invariant Haar measure can be conveniently expressed by $D_r(x) = (r^{\upsilon_1} x_1, \dots, r^{\upsilon_n} x_n)$ and $d(D_r(x)) = r^Q dx$. Note that all the Lebesgue spaces $L^p(G)$, $p \in [1, \infty]$, are defined with respect to the Haar measure dx, which can be identified as a pullback under log of the Lebesgue measure on $\mathfrak{g} \cong \mathbb{R}^n$. Similarly, the test function space $\mathcal{D}(G)$ is defined as $\log^*(\mathcal{D}(\mathbb{R}^n))$.

Homogeneous groups admit so-called *homogeneous quasi-norms* (cf. [33, §1 (A)]), that is, continuous functions $x \mapsto |x| \colon G \to [0, \infty)$ that are

- definite: $|x| = 0$ if and only if $x = e_G$,
- symmetric: $|x^{-1}| = |x|$,
- 1-homogeneous: $|D_r x| = r|x|$.

Every quasi-norm satisfies a quasi-triangle inequality $|xy| \leq C(|x| + |y|)$, $x, y \in G$, but every homogeneous group admits at least one with $C = 1$ (cf. [35] and [28, Thm. 3.1.39]).

[4] Although there is no algebraic distinction between homogeneous and graded Lie algebras, the attribute *graded* allows us to indicate the use of a given gradation.

In analogy to $\mathbb{R}^n$, any two homogeneous quasi-norms on a given G equivalent, and frequently used examples are

$$|x|_p := \left(|x_1|^{\frac{p}{\nu_1}} + \cdots + |x_n|^{\frac{p}{\nu_n}}\right)^{\frac{1}{p}}, \quad p \in [1,\infty),$$
$$|x|_\infty := \max_{j=1,\ldots,n} |x_j|^{\frac{1}{\nu_j}}. \tag{2.1}$$

Let us furthermore recall that the open balls $B_r(x) := \{y \in G : |y^{-1}x| < r\}$ satisfy $B_r(x) = D_r(B_1(r^{-1}x))$ and $|B_r(x)| = r^Q|B_1(x)| = r^Q|B_1(0)|$.

2.2. Invariant derivatives and homogeneous polynomials

Let G be a homogeneous group equipped with dilations $\{D_r\}_{r>0}$ and let $\{X_1,\ldots,X_n\}$ be an eigenbasis of the matrix A that determines the dilations via $D_r = \exp(\log(r)A)$. Since the space of left-invariant (respectively right-invariant) vector fields can be identified with $G \times \mathfrak{g}$, we will denote the left-invariant and right-invariant vector basis fields on G that are associated to X_j, $j = 1,\ldots,n$, by $X_j f = \frac{d}{dt}|_{t=0} f(\cdot \exp(tX_j))$ and $\widetilde{X}_j f = \frac{d}{dt}|_{t=0} f(\exp(tX_j)\cdot)$, respectively. Their higher order iterates $X^\alpha := X_1^{\alpha_1}\cdots X_n^{\alpha_n}$ and $\widetilde{X}^\alpha := \widetilde{X}_1^{\alpha_1}\cdots \widetilde{X}_n^{\alpha_n}$, $\alpha \in \mathbb{N}_0^n$, are the natural generalizations of $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ on $\mathbb{R}^n$. Like their Abelian counterparts, the operators X^α and $\widetilde{X}^\alpha$ are homogeneous of degree $[\alpha]$. Such higher order derivatives do not commute in general, but any iterate $X^\alpha X^\beta$, $\alpha,\beta \in \mathbb{N}_0^n$, can be written as a linear combination of X^γ with $\gamma \in \mathbb{N}_0^n$, $[\gamma] = [\alpha] + [\beta]$:

$$X^\alpha X^\beta = \sum_{\substack{\gamma\in\mathbb{N}_0^n,\, |\gamma|\le|\alpha|+|\beta| \\ [\gamma]=[\alpha]+[\beta]}} c'_{\alpha,\beta,\gamma} X^\gamma.$$

An analogous formula holds for $\widetilde{X}^\alpha \widetilde{X}^\beta$. Throughout this paper, we will frequently need to rewrite right-invariant derivatives in terms of left-invariant ones via

$$\widetilde{X}^\alpha = \sum_{\substack{|\beta|\le|\alpha|, \\ [\beta]\ge[\alpha]}} Q_{\alpha,\beta}\, X^\beta, \tag{2.2}$$

where the $Q_{\alpha,\beta}$ are uniquely determined polynomials of homogeneous degree $[\beta] - [\alpha]$.[5]

Given a homogeneous Lie group G, equipped with a quasi-norm $|\cdot|$, the left Taylor polynomial of homogeneous order $M \in \mathbb{N}$ of a function $f : G \to \mathbb{C}$ at a point $x \in G$ is

$$P^f_{x,M}(y) = \sum_{[\alpha]\le M} q_\alpha(y)(X^\alpha f)(x), \tag{2.3}$$

where q_α is the homogeneous polynomial of degree $[\alpha]$ uniquely determined by

$$\left(X^\beta q_\alpha\right)(e_G) = \delta_{\beta,\alpha} \quad \text{for all } \beta \in \mathbb{N}_0^n. \tag{2.4}$$

[5] Cf. [33, Prop. 1.29] for a proof of (2.2) and similar identities for left-invariant and standard partial derivatives.

The remainder of order M, given by $R^f_{x,M}(y) := f(xy) - P^{(f)}_{x,M}(y)$, is controlled uniformly if $f \in C^{\lceil M \rfloor+1}(G)$, with $\lceil M \rfloor := \max\{|\alpha| : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \leq M\}$: there exist constants $C_M > 0$[6] and $\eta > 1$[7] such that

$$\left|R^f_{x,M}(y)\right| \leq C_M \sum_{M<[\alpha]\leq M+\upsilon_n} |y|^{[\alpha]} \sup_{|z|\leq \eta^{\lceil M \rfloor+1}|yx^{-1}|} \left|(X^\alpha f)(xz)\right|. \tag{2.5}$$

Moreover, for any $M \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, and for $f \in C^\infty(G)$, one has

$$X_y^\alpha P^f_{x,M}(y) = P^{X^\alpha f}_{x,M-[\alpha]}(y) \quad \text{and} \quad X_y^\alpha R^f_{x,M}(y) = R^{X^\alpha f}_{x,M-[\alpha]}(y),$$
$$\widetilde{X}_y^\alpha P^f_{x,M}(y) = P^{X_x^\alpha f(xx_1)}_{x_1=0,M-[\alpha]}(y) \quad \text{and} \quad \widetilde{X}_y^\alpha R^f_{x,M}(y) = R^{X_x^\alpha f(xx_1)}_{x_1=0,M-[\alpha]}(y).$$

We recall that for every possible homogeneous order M in $\mathcal{W} := \{\upsilon_1\alpha_1 + \cdots + \upsilon_n\upsilon_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N}_0\}$, the polynomials q_α with $[\alpha] = M$ form a basis of the finite-dimensional subspace of polynomials of homogeneous degree equal to M, which allows us to rewrite

$$q_\alpha(xy) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\, q_{\alpha_1}(x)\, q_{\alpha_2}(y) \tag{2.6}$$

for any $\alpha \in \mathbb{N}_0^n$ and any $x, y \in G$, with coefficients $c_{\alpha_1,\alpha_2} \in \mathbb{R}$ which are independent of x, y. Since the collection of all $q_\alpha, \alpha \in \mathbb{N}_0^n$ more generally forms a basis of the vector space of all polynomials on G, one can equip the Schwartz class $\mathcal{S}(G) := \log^*(\mathcal{S}(\mathbb{R}^n))$ with the equivalent family of seminorms $\|f\|_{\mathcal{S},N} := \max_{[\alpha],[\beta]\leq N} \|q_\alpha X^\beta f\|_{L^\infty(G)}$, $N \in \mathbb{N}_0$.[8]

2.3. The group Fourier transform

In the following, we recall a few essential properties of the group Fourier transform (1.6) as well as some crucially related notions. For more details we refer to the monograph [13].

The unitary irreducible representations employed in (1.6) are group homomorphism $\pi: G \to \mathcal{U}(\mathcal{H}_\pi)$, for some complex Hilbert space $\mathcal{H}_\pi$, that are

- strongly continuous: $\pi(x_j)v \to \pi(x)v$ for each $v \in \mathcal{H}_\pi$ whenever $x_j \to x$ in G;
- irreducible: $\pi(G)V \subseteq V$ for any subspace $V \subseteq \mathcal{H}_\pi \Rightarrow V = \mathcal{H}_\pi$ or $V = \{0\}$.

The unitary dual $\widehat{G}$ of G is the set of equivalence classes $[\pi]$ of unitary irreducible representations of G, where any two representations $\pi: G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\pi': G \to \mathcal{U}(\mathcal{H}'_\pi)$ are called unitarily equivalent if there exists a $U \in \mathcal{U}(\mathcal{H}_\pi, \mathcal{H}'_\pi)$ such that $U^{-1}\pi'(x)U = \pi(x)$ holds for all $x \in G$. By a common abuse of notation, we will denote the equivalence classes by π instead of $[\pi]$ when there is no risk of confusion. A vector $v \in \mathcal{H}_\pi$ is called smooth if the Hilbert space-valued function $x \mapsto \pi(x)v: G \to \mathcal{H}_\pi$ is infinitely Fréchet-differentiable on G[9], and we denote the space of smooth vectors by $\mathcal{H}_\pi^\infty$. We recall that if G is nilpotent

[6] Cf. [33, Thm. 1.33].
[7] Cf. the proof of [28, Prop 3.1.46].
[8] Cf. [33, §1 (D)] and [28, Lem. 3.1.56].
[9] Equivalently, if $\pi(\log(\cdot))v$ is infinitely Fréchet-differentiable on $\mathfrak{g} \cong \mathbb{R}^n$.

and $\pi \in \widehat{G}$ is realized to act on the Hilbert space $\mathcal{H}_\pi = L^2(\mathbb{R}^d)$, then $\mathcal{H}_\pi^\infty$ can be identified with $\mathcal{S}(\mathbb{R}^d)$. For a given $\pi \in \widehat{G}$, its infinitesimal representation is the skew-adjoint representation of the Lie algebra $\mathfrak{g}$ defined by

$$d\pi(X)v := \frac{d}{dt}\Big|_{t=0}\big(\pi\big(\exp(tX)\big)v - v\big) \tag{2.7}$$

for any $X \in \mathfrak{g}$ and $v \in \mathrm{dom}(d\pi(X)) = \{v \in \mathcal{H}_\pi \mid (2.7) \text{ converges in } \mathcal{H}_\pi\} \supseteq \mathcal{H}_\pi^\infty$. Employing the Poincaré–Birkhoff–Witt theorem, this representation can be extended to the algebra $\mathrm{Diff}(G)$ of left-invariant (respectively right-invariant) differential operators with complex coefficients. This allows us to define the symbol $\pi(T) := d\pi(T)$ in the representation $\pi \in \widehat{G}$ of any $T \in \mathrm{Diff}(G)$, such as the derivatives X^α and $\widetilde{X}^\alpha$, $\alpha \in \mathbb{N}_0^n$. Since for $G = \mathbb{R}^n$ the unitary dual is given by $\widehat{\mathbb{R}}^n = \{x \mapsto e^{ix\xi} \mid \xi \in \mathbb{R}^n\} \cong \mathbb{R}^n$, we simply have $\pi_\xi(T) = \sum_{|\alpha|\leq k} c_\alpha i^{|\alpha|}\xi^\alpha = \sigma_T(x,\xi)$ for any $T = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha \in \mathrm{Diff}(\mathbb{R}^n)$.

For the Kohn–Nirenberg calculus on graded Lie groups established in [28] and the τ-calculus developed in this paper, two noncommutative L^p-spaces on the unitary dual $\widehat{G}$ are of particular importance: the Hilbert space $L^2(\widehat{G})$ and the von Neumann algebra $L^\infty(\widehat{G})$. To discuss these spaces, however, we need to equip $\widehat{G}$ with a suitable topology, namely one that generates the σ-algebra of Borel sets which is used to prove the Fourier inversion and Plancherel theorems, the so-called Mackey Borel structure. Following the mainstream convention, we always assume that $\widehat{G}$ is equipped with the so-called Fell topology. The resulting topological space $\widehat{G}$ is generally (T_0) but not Hausdorff (see, e.g., [16, Ch. 18]). Thus, viewing $\widehat{G}$ as a topological space, the space $L^2(\widehat{G})$ can be defined as a direct integral Hilbert space which arises from the Fourier inversion formula for Schwartz functions: for every $f \in \mathcal{S}(G)$ the Fourier transform $\hat{f}$ is a field of Hilbert–Schmidt operators $\hat{f}(\pi)$ on $\mathcal{H}_\pi$ which is measurable with respect to the Mackey-Borel structure. Moreover, there exists a measure μ on $\widehat{G}$, the so-called Plancherel measure, for which the Plancherel formula

$$\int_G |f(x)|^2\,dx = \int_{\widehat{G}} \|\hat{f}(\pi)\|^2_{\mathtt{HS}(\mathcal{H}_\pi)}\,d\mu(\pi), \tag{2.8}$$

holds for all $f \in \mathcal{S}(G)$ and for which the inversion formula (1.7) holds pointwise for all $x \in G$. The latter is clearly crucial to derive a rigorous version of the quantization formula (1.9). In addition, the restriction of $\mathcal{F}$ to $\mathcal{S}(G) \subseteq L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ onto the direct integral Hilbert space of measurable fields of Hilbert–Schmidt operators

$$L^2(\widehat{G}) := \int_{\widehat{G}}^\oplus \mathtt{HS}(\mathcal{H}_\pi)\,d\mu(\pi) \text{ with norm } \|\sigma\|_{L^2(\widehat{G})} = \left(\int_{\widehat{G}} \|\sigma_\pi\|^2_{\mathtt{HS}(\mathcal{H}_\pi)}\,d\mu(\pi)\right)^{1/2}.$$

Note that its elements are the equivalence classes of such fields determined up to measure zero and unitary equivalence of $\pi \in [\pi] \in \widehat{G}$. The unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$ is called the Plancherel transform and its existence implies that μ is uniquely determined.

The space $L^\infty(\widehat{G})$ is called the group von Neumann algebra (see, e.g. [28]) and is defined as the space of all measurable fields $\sigma = \{\sigma_\pi\}_{\pi\in\widehat{G}}$ of bounded operators $\sigma_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi$ which are essentially uniformly bounded with respect to the Plancherel measure μ, i.e.,

$$L^\infty(\widehat{G}) := \Big\{\sigma = \{\sigma_\pi\}_{\pi\in\widehat{G}} \text{ meas.} \mid \sigma_\pi \in \mathcal{L}(\mathcal{H}_\pi),\ \operatorname*{ess\,sup}_{\pi\in\widehat{G}} \|\sigma_\pi\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty\Big\}.$$

As in the case of $L^2(\widehat{G})$, the elements of $L^\infty(\widehat{G})$ are equivalence classes determined up to Plancherel measure zero and unitary equivalence of $\pi \in [\pi] \in \widehat{G}$, and pointwise multiplication of any two fields σ_1 and σ_2 is easily seen to turn $L^\infty(\widehat{G})$ into a C^*-algebra. For graded groups G it plays the same important role for defining symbol classes on $G \times \widehat{G}$ that the C^*-algebra $L^\infty(\widehat{\mathbb{R}}^n) \cong L^\infty(\mathbb{R}^n)$ plays for symbols defined on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \widehat{\mathbb{R}}^n$ in the classical case.

Note that because of the bound $\|\pi\|_{\mathcal{L}(\mathcal{H}_\pi)} = 1$ for all $\pi \in \widehat{G}$, the Fourier integral (1.6) converges for all $f \in L^1(G)$ and the Fourier transform $\mathcal{F}$ maps the space continuously into $L^\infty(\widehat{G})$. Since the elements of $L^1(G)$ act continuously on $L^2(G)$ by convolution from the right (cf. Young's convolution inequality), the Fourier transform defined on $L^1(G)$ extends, as a bounded linear map, to the space $\mathcal{L}_{\mathrm{L}}(L^2(G))$ of all bounded left-invariant operators on $L^2(G)$. By a special version of the Schwartz kernel theorem (see, e.g., [28, Cor. 3.2.1]), these operators can be viewed as convolution operators with kernels $\kappa \in \mathcal{S}'(G)$. If $\mathcal{K}(G)$ denotes the space of all $\kappa \in \mathcal{D}'(G)$ such that the linear map

$$f \mapsto f * \kappa : \mathcal{D}(G) \to L^2(G)$$

extends to a bounded operator on $L^2(G)$, then $\mathcal{K}(G)$ equipped with the convolution product and the usual $\mathcal{L}(L^2(G))$-operator norm is a von Neumann algebra, which is isomorphic to $\mathcal{L}_{\mathrm{L}}(L^2(G))$. As a consequence, the Fourier transform on $L^1(G)$ extends to a von Neumann algebra isomorphism from $\mathcal{K}(G) \cong \mathcal{L}_{\mathrm{L}}(L^2(G))$ to $L^\infty(\widehat{G})$, which justifies its name. The Plancherel formula helps express these isomorphisms in a very clear fashion: given $T \in \mathcal{L}_{\mathrm{L}}(L^2(G))$, its uniquely determined convolution kernel $\kappa \in \mathcal{K}(G)$ and the corresponding Fourier multiplier $\sigma = \widehat{\kappa} \in L^\infty(\widehat{G})$, one has

$$\langle T\phi, \psi\rangle_{L^2(G)} = \langle \phi * \kappa, \psi\rangle_{L^2(G)} = \int_{\widehat{G}} \mathrm{Tr}\big(\sigma_\pi \widehat{\phi}(\pi)\widehat{\psi}(\pi)^*\big)\, d\mu$$

for all $\phi, \psi \in L^2(G)$

In our case we need not distinguish the Fourier transforms on $L^1(G)$, $L^2(G)$ or $\mathcal{K}(G)$ if the context is unambiguous, in which case we will simply speak of the (group) Fourier transform $\mathcal{F}$ and denote the images of functions/distributions by $\mathcal{F}(f)$ or $\widehat{f}$.

2.4. The Heisenberg group

The simplest non-Abelian nilpotent Lie algebras are the *Heisenberg Lie algebras* $\mathfrak{h}_n$ with $n \in \mathbb{N}$, which are defined as the vector spaces $\mathbb{R}^{2n+1}$ equipped with the Lie bracket whose only non-trivial relation is

$$[X_j, X_{n+j}] = X_{2n+1}, \quad j = 1, \dots, n,$$

for the Euclidean standard basis $\{X_1, \dots, X_{2n+1}\}$. Any such Lie algebra admits a canonical stratification, $\mathfrak{h}_n = \mathbb{R}-\text{span}\{X_1, \dots, X_{2n}\} \oplus \mathbb{R}X_{2n+1}$, and hence a canonical family of homogeneous dilations: $D_r(X_j) = rX_j$ for $j = 1, \dots, 2n$ and $D_r(X_{2n+1}) = r^2 X_{2n+1}$. As a The group law of the *Heisenberg group* $\mathbb{H}_n := \exp(\mathfrak{h}_n)$ is given by

$$\begin{aligned} xy &= (x_1, \dots, x_{2n+1})(y_1, \dots, y_{2n+1}) \\ &= \Big(x_1 + y_1, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \sum_{j=1}^{n} \frac{x_j y_{n+j} - x_{n+j} y_j}{2} \Big). \end{aligned}$$

and its left-invariant basis vector fields by

$$X_j = \partial_j - \frac{1}{2} x_{n+j} \partial_{2n+1}, \; X_{n+j} = \partial_{n+j} + \frac{1}{2} x_j \partial_{2n+1}, \; X_{2n+1} = \partial_{2n+1}, \; j = 1, \dots, n.$$

Apart from the quasi-norms 2.1 and 2.1, which are defined for any family of dilations on $\mathbb{H}_n$, the canonical dilations admit a homogeneous norm[10] of the form

$$|x| := \Big(\big(|x_1|^2 + \dots + |x_{2n}|^2 \big)^2 + \frac{1}{16} |x_{2n+1}|^2 \Big)^{\frac{1}{4}}.$$

The unitary dual $\widehat{\mathbb{H}}_n$ is exhausted by[11] the so-called Schrödinger representations $\pi_\lambda \in \widehat{\mathbb{H}}_n$, $\lambda \in \mathbb{R} \setminus \{0\}$, which act on $\mathcal{H}_{\pi_\lambda} := L^2(\mathbb{R}^n)$ by

$$\big(\pi_\lambda(x)h\big)(u) = \big(\pi_\lambda(x', x'', x_{2n+1})h\big)(u) = e^{i\lambda(x_{2n+1} + \frac{1}{2}x'x'')} e^{i\sqrt{\lambda}x''u} h\big(u + \sqrt{|\lambda|}x'\big)$$

for $u \mapsto h(u) \in L^2(\mathbb{R}^n)$ and $x' := (x_1, \dots, x_n)$, $x'' := (x_{n+1}, \dots, x_{2n}) \in \mathbb{R}^n$, and the Plancherel identity (2.8) takes the simple form

$$\int_{\mathbb{H}_n} \big| f(x) \big|^2 \, dx = \int_{\mathbb{R}\setminus\{0\}} \| \hat{f}(\pi_\lambda) \|^2_{\mathrm{HS}(\mathcal{H}_{\pi_\lambda})} \, |\lambda|^n \, d\lambda.$$

Since the infinitesimal Schrödinger representations of the basis vector fields are given by $(\pi_\lambda(X_j)h)(u) = \sqrt{|\lambda|}\partial_{u_j} h(u)$, $(\pi_\lambda(X_{n+j})h)(u) = i\sqrt{\lambda}u_j h(u)$ and $(\pi_\lambda(X_{2n+1})h)(u) = i\lambda h(u)$, the symbol in $\pi_\lambda \in \widehat{\mathbb{H}}_n$ of the canonical sub-Laplacian $\mathcal{L}_{\mathbb{H}_n} = X_1^2 + \dots + X_{2n}^2$ equals $\pi_\lambda(\mathcal{L}_{\mathbb{H}_n}) = |\lambda| \sum_{j=1}^n (\partial_j^2 - u_j^2)$, a multiple of the harmonic oscillator on $\mathbb{R}^n$.

3. Sobolev spaces and symbol classes

To set the stage for the symbolic calculus for the τ-quantizations (1.9), we recall some crucial facts about the Hörmander-type symbol classes and the inhomogeneous Sobolev spaces on graded groups established in [28].

[10] The so-called Korányi–Cygan norm.

[11] Up to Plancherel measure zero and unitary equivalence.

3.1. Rockland operators, Bessel potentials and Sobolev spaces

The notion of regularity employed in the definition of the Hörmander-type symbol classes of [28] is based on a family of Bessel potentials adapted to setting of graded groups.[12] These Bessel potentials are defined in terms of the natural generalization of the Euclidean Laplace operator, so-called *Rockland operators*. On a given homogeneous group G, any positive, left-invariant, essentially self-adjoint operator $\mathcal{R}$ on $L^2(G)$ is Rockland if and only if[13] it is homogeneous of degree $\nu \in \mathbb{N}$ and hypoelliptic, that is, $\mathcal{R} f \in C^\infty(G) \Rightarrow f \in C^\infty(G)$. We recall that a family of homogeneous dilations admits a Rockland operator if and only if there exists some $r_o \in \mathbb{R}$ such that $r_o v_1, \dots, r_o v_n \in \mathbb{N}$, which in turn holds if and only if the dilations are the canonical dilations of some gradation.[14] Every fixed gradation admits infinitely many Rockland operators, such as

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{v_j}} c_j X_j^{\frac{2\nu_0}{v_j}} \quad \text{with } c_1, \dots, c_n > 0,$$

defined for any common multiple $\nu_o \in \mathbb{N}$ of $v_1, \dots, v_n$. Other special cases are the canonical sub-Laplacians on stratified groups and the Euclidean Laplacians.[15]

Given a positive Rockland operator $\mathcal{R}$ of homogeneous degree $\nu \in \mathbb{N}$, let us denote by $\mathcal{R}_2$ its self-adjoint extension on $L^2(G)$ and by $\varphi(\mathcal{R}_2)$ any spectral multiplier of $\mathcal{R}_2$ defined via the spectral calculus for unbounded self-adjoint operators in $L^2(G)$. Since $\mathcal{R}_2$ is left-invariant, the operators $\varphi(\mathcal{R}_2)$ are also left-invariant, which in combination with the Schwarz kernel theorem for $\mathcal{D}'(G)$ yields the identity $\varphi(\mathcal{R}_2) f = f * \kappa_{\varphi(\mathcal{R}_2)}$ for all $f \in \mathcal{D}(G)$ and some $\kappa_{\varphi(\mathcal{R}_2)} \in \mathcal{D}'(G)$. A particularly important special case are the generally unbounded self-adjoint operators $(I + \mathcal{R}_2)^{\frac{s}{\nu}}$, $s \in \mathbb{R}$, called Bessel potentials. For any integer $s = k\nu$, $k \in \mathbb{N}$, the spectral multiplier $(I + \mathcal{R}_2)^{\frac{s}{\nu}}$ equals the left-invariant differential operator $(I + \mathcal{R})^k$ on $\mathcal{D}(G)$. For general $s \in \mathbb{R}$ one can use properties of the heat kernel h_t on $L^2(G)$, defined by $f * h_t := e^{-t\mathcal{R}_2} f$, to show that

$$(I + \mathcal{R}_2)^{-\frac{s}{\nu}} f = f * \mathcal{B}_s \tag{3.1}$$

holds for all $f \in \mathcal{S}(G)$ and a kernel $\mathcal{B}_s \in \mathcal{S}'(G)$ and that in fact $(I + \mathcal{R}_2)^{\frac{s}{\nu}}(\mathcal{S}(G)) = \mathcal{S}(G)$. Since the group Fourier transform relates the spectral calculus of $\mathcal{R}_2$ on $L^2(G)$ with that of the self-adjoint operator $\pi(\mathcal{R})$ on $\mathcal{H}_\pi$ for every $\pi \in \widehat{G}$ (cf. [28, Ch. 4]), an application of the Fourier transform to (3.1) yields the identity

$$\pi(I + \mathcal{R})^{-\frac{s}{\nu}} \hat{f}(\pi) = \mathcal{F}(f * \mathcal{B}_s)(\pi) \quad \text{a.e.} \quad \pi \in \widehat{G}$$

for the symbol of the Bessel potential. This identity will prove crucial for the rest of the

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[12] We recall that these are homogeneous groups equipped with a fixed gradation.

[13] For Rockland's original definition, the Rockland conjecture and its proof, we refer to [2, 36, 47].

[14] Cf. [42, 55]. Counterexamples on R^n and $\mathbb{H}_n$ can easily be constructed for mixtures of rational and irrational weights.

[15] To satisfy the positivity criterion, one actually has to multiply these operators by (-1).

paper since by (1.9) we obtain, for now at least formally,

$$\mathrm{Op}^\tau\big(\pi(I+\mathcal{R})^{-\frac{s}{\nu}}\big) = (I+\mathcal{R}_2)^{-\frac{s}{\nu}}$$

for all quantizing functions τ.

The notion of Sobolev space on a graded group established in [28, 29] is based on the use of a positive Rockland operator $\mathcal{R}$ in a very canonical fashion. This makes them the ideal spaces to express and judge the mapping properties of the τ-calculus developed in this paper. This class of Sobolev spaces includes, in particular, the Sobolev spaces on stratified groups by [30] and the classical homogeneous and inhomogeneous Sobolev spaces $\dot{L}^p_s(\mathbb{R}^n)$ and $L^p_s(\mathbb{R}^n)$, respectively, with $p \in (1,\infty)$, $s \in \mathbb{R}$. The original definition of the inhomogeneous Sobolev spaces $L^p_s(G) = L^p_{s,\mathcal{R}}(G)$ in [28, 29] relies on the properties of the heat semi-group $f \mapsto f * h_t$, $t > 0$, on $L^p(G)$, and on the spectral theory of its infinitesimal generator $\mathcal{R}_p$. Since this is outside the scope of this paper, we choose a slightly different, nevertheless equivalent definition, reminiscent of [53, §5.2], on $\mathbb{R}^n$.[16] Namely, for $p \in (1,\infty)$, $s \in \mathbb{R}$, we define $L^p_s(G) = L^p_{s,\mathcal{R}}(G)$ as the space of all $f \in \mathcal{S}'(G)$ such that

$$\|f\|_{L^p_s(G)} := \big\|(I+\mathcal{R}_2)^{\frac{s}{\nu}} f\big\|_{L^p(G)} < \infty.$$

Notation 3.1. Throughout the rest of this paper we will no longer distinguish $\mathcal{R}$ and $\mathcal{R}_2$ notationally and simply write $\mathcal{R}$, since it will be clear from the context whether we mean the differential operator on its natural domain $\mathrm{dom}(\mathcal{R}) \supseteq \mathcal{S}(G)$ or its self-adjoint extension on $L^2(G)$.

3.2. Hörmander-type symbol classes

Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu \in \mathbb{N}$. A smooth symbol is a field of operators

$$\sigma = \big\{\sigma(x,\pi)\colon \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi \mid x \in G,\ \pi \in \widehat{G}\big\}$$

for which there exist $a, b \in \mathbb{R}$ such that for each $x \in G$ the field

$$\big\{\pi(I+\mathcal{R})^{\frac{a}{\nu}}\sigma(x,\pi)\pi(I+\mathcal{R})^{-\frac{b}{\nu}} \mid \pi \in \widehat{G}\big\}$$

is an element of $L^\infty(\widehat{G})$ and such that this field is smooth as a Banach space-valued function on G. By the properties of $L^\infty(\widehat{G})$ discussed in Section 2.3, the condition on $a, b \in \mathbb{R}$ is equivalent to L^2-continuity of the convolution operators $T_x f = f * \kappa_x$ with kernels $\kappa_x \in \mathcal{S}'(G)$, $x \in G$, defined by

$$\kappa_x(y) := \mathcal{F}^{-1}\Big(\big\{\pi(I+\mathcal{R})^{\frac{a}{\nu}}\sigma(x,\pi)\pi(I+\mathcal{R})^{-\frac{b}{\nu}}\big\}_{\pi\in\widehat{G}}\Big)(y) = \big(\mathcal{B}_b * \kappa_\sigma(x,\cdot) * \mathcal{B}_{-a}\big)(y).$$

[16] To see that this definition avoids any circular or otherwise problematic arguments, we refer the interested reader to [21].

The generalization of differentiability in the dual variable $\xi \in \mathbb{R}^n$ of symbols $\sigma : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \to \mathbb{C}$ to general graded groups G involves the so-called difference operators on $\widehat{G}$. For any $\alpha \in \mathbb{N}_0^n$, we write $\tilde{q}_\alpha(x) := q_\alpha(x^{-1})$, $x \in G$, for the homogeneous polynomial q_α determined by (2.4). Then the difference operator Δ^α is defined by

$$(\Delta^\alpha \hat{f})(\pi) := \widehat{\tilde{q}_\alpha f}(\pi), \quad \pi \in \widehat{G},$$

for any $f \in \mathcal{D}'(G)$ for which

$$\mathcal{B}_b * f * \mathcal{B}_{-a}, \quad \mathcal{B}_{b'} * (\tilde{q}_\alpha f) * \mathcal{B}_{-a'} \in \mathcal{K}(G)$$

for some $a, a', b, b' \in \mathbb{R}$, i.e., for which these kernels define continuous convolution operators on $L^2(G)$. While on $G = \mathbb{R}^n$ this recovers the classical notion, with

$$(\Delta^\alpha \hat{f})(\xi) = (-i)^{|\alpha|}(\partial_\xi^\alpha \hat{f})(\xi)$$

due to $q_\alpha(x) = (\alpha_1! \cdots \alpha_n!)^{-1} x^\alpha$, for the Heisenberg group $\mathbb{H}_n$, for example, one obtains a combination of Euclidean derivatives ∂_λ and commutators with the symbols $\pi_\lambda(X_j)$, $j = 1, \ldots, 2n$ [28, §6.3.3]. (See also [11] for the difference operators on the Engel and Cartan groups.)

Now let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. Following [28], we will denote by $S^m_{\rho,\delta}(G)$ the class of smooth symbols which for all $a, b \in \mathbb{N}_0$ satisfy

$$\begin{aligned} \|\sigma\|_{S^m_{\rho,\delta},a,b} &:= \sup_{\substack{(x,\pi)\in G\times\widehat{G}\\ [\alpha]\leq a,\,[\beta]\leq b}} \left\| (\Delta^\alpha X_x^\beta \sigma)(x,\pi)\pi(I+\mathcal{R})^{-\frac{m-\rho[\alpha]+\delta[\beta]}{\nu}} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &= \sup_{\substack{x\in G\\ [\alpha]\leq a,\,[\beta]\leq b}} \left\| \left\{ (\Delta^\alpha X_x^\beta \sigma)(x,\pi)\pi(I+\mathcal{R})^{-\frac{m-\rho[\alpha]+\delta[\beta]}{\nu}} \mid \pi\in\widehat{G} \right\} \right\|_{L^\infty(\widehat{G})} < \infty. \end{aligned} \tag{3.2}$$

These symbols are called smooth symbols of order m and type (ρ, δ).

Note that the seminorms defined by (3.2) are precisely the seminorms $\|\cdot\|_{S^{m,R}_{\rho,\delta},a,b}$ in [28], which for technical reasons prove to be more convenient in this paper. However, due to [28, Thm. 5.5.20], they define the same topology on $S^m_{\rho,\delta}(G)$ as the original seminorms in Definition [28, Def. 5.2.11]. Several equivalent characterizing families of seminorms are provided by [28, Thm. 5.5.20] and [10, Thm. 13.16], and in each case the topology is independent of the choice of Rockland operator $\mathcal{R}$ on G.

For $G = \mathbb{R}^n$ and $\mathcal{R} = -\sum_{j=1}^n \partial_{x_j}^2$, one recovers the seminorms

$$\|\sigma\|_{S^m_{\rho,\delta},a,b} = \sup_{\substack{(x,\xi)\in\mathbb{R}\times\widehat{\mathbb{R}}^n\\ [\alpha]\leq a,\,[\beta]\leq b}} \left| (\partial_\xi^\alpha \partial_x^\beta \sigma)(x,\xi)\big(I + |\xi|^2\big)^{-\frac{m-\rho[\alpha]+\delta[\beta]}{2}} \right|,$$

which define the Hörmander symbol classes $S^m_{\rho,\delta}(\mathbb{R}^n)$.

Remark 3.2. It was shown in [28, Thm. 5.2.22 (ii)] that if $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$, for $m_1, m_2 \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, then the smooth symbol defined by

$$\sigma(x,\pi) := \sigma_1(x,\pi)\sigma_2(x,\pi)$$

lies in $S^{m_1+m_2}_{\rho,\delta}(G)$.

3.3. Associated kernels

By [28, Thm. 5.2.15], any pseudo-differential operator

$$\mathrm{Op}(\sigma)f(x) = \iint\limits_{\widehat{G}\times G} \mathrm{Tr}\big(\pi(y^{-1}x)\sigma(x,\pi)f(y)\big)\,dy\,d\mu(\pi)$$

with $\sigma \in S^m_{\rho,\delta}(G)$ for some $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, is continuous on the Schwartz space $\mathcal{S}(G)$. It follows from the Schwartz kernel theorem that $\mathrm{Op}(\sigma)$ has a distributional kernel Ker_σ in $\mathcal{S}'(G \times G)$. Abusing the notation of integral kernels, we will frequently express this fact by writing

$$\mathrm{Op}(\sigma)f(x) = \int_G \mathsf{Ker}_\sigma(x,y)f(y)\,dy$$

for $f \in \mathcal{S}(G)$. The so-called *associated kernel* κ_σ is formally defined by

$$\mathrm{Op}(\sigma)f(x) = \int_G \kappa_\sigma(x, y^{-1}x)f(y)\,dy, \tag{3.3}$$

so that, again formally, it is related to the symbol by

$$\sigma(x,\pi) = \mathcal{F}\big(\kappa_\sigma(x,\cdot)\big)(\pi) \tag{3.4}$$

for $x \in G$, $\pi \in \widehat{G}$. The change of variables

$$\mathrm{cv}\colon G \times G \to G \times G\colon (x,y) \mapsto (x, y^{-1}x), \tag{3.5}$$

which formally relates Ker_σ and κ_σ to each other, is a Lie group automorphism of the direct product group $G \times G$, and its pullback CV maps $\mathcal{S}(G \times G)$ continuously onto itself. By the open mapping theorem for Fréchet spaces (cf. [56, Thm. 17.1]), it is even a Fréchet space isomorphism on $\mathcal{S}(G \times G)$. So, under the assumption that $\mathsf{Ker}_\sigma \in \mathcal{S}'(G \times G)$, by duality, we also have $\kappa_\sigma \in \mathcal{S}'(G \times G)$.

Maintaining the formal relation (3.3) for a given τ-quantization (1.9), we may rewrite the quantization as

$$\mathrm{Op}^\tau(\sigma)f(x) = \int_G \kappa_\sigma(x\tau(y^{-1}x)^{-1}, y^{-1}x)f(y)\,dy = \int_G \mathsf{Ker}^\tau_\sigma(x,y)f(y)\,dy, \tag{3.6}$$

for its distributional kernel Ker^τ_σ. Under the assumption that the quantizing function $\tau\colon G \to G$ is smooth and that its pullback maps the Schwartz space $\mathcal{S}(G \times G)$ continuously into itself, the change of coordinates

$$\mathrm{cv}^\tau\colon G \times G \to G \times G\colon (x,y) \mapsto \big(x\tau(y^{-1}x)^{-1}, y^{-1}x\big), \tag{3.7}$$

which formally relates Ker^τ_σ to κ_σ, is easily seen to be a Lie group automorphism of $G \times G$ with inverse $(\mathrm{cv}^\tau)^{-1}(x,y) = (x\tau(y), x\tau(y)y^{-1})$. So, the associated pullbacks $(\mathrm{CV}^\tau \circ \kappa_\sigma)(x,y) := \kappa(\mathrm{cv}^\tau(x,y))$ and $((\mathrm{CV}^\tau)^{-1} \circ \kappa_\sigma)(x,y) := \kappa((\mathrm{cv}^\tau)^{-1}(x,y))$ are in fact Fréchet space isomorphisms on $\mathcal{S}(G \times G)$ and $\mathsf{Ker}^\tau_\sigma \in \mathcal{S}'(G \times G)$. For any reasonable pseudo-differential calculus on graded groups this is an indispensable condition, hence throughout this paper we will assume

(CV) τ is smooth and the pullback of cv^τ is a Fréchet space isomorphism on the Schwartz space $\mathcal{S}(G \times G)$.

Let us finally point out that the formal relation between the distributional Ker^τ_σ kernel and the symbol σ, given by

$$\sigma(x,\pi) = \big((\mathrm{id} \otimes \mathcal{F}_G) \circ \kappa\big)(x,\pi) = \mathcal{F}_{y\to\pi}\big((\mathrm{CV}^\tau)^{-1} \circ \mathsf{Ker}^\tau_\sigma\big)(x,y), \tag{3.8}$$

which, for symbols $\sigma \in S^{-\infty}(G)$, is given by an absolutely convergent integral.

3.4. Kernel estimates

The following estimates from [28] for the kernels κ_σ associated to symbols $\sigma \in S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, will be used frequently throughout the paper. The quasi-norm in question will always be the canonical supremum quasi-norm defined by (2.1).[17]

The first result gives a general L^2-decay condition, which for large negative m gives square integrability.

Proposition 3.3 ([28, Prop. 5.2.16]). *Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and let $\sigma \in S^m_{\rho,\delta}(G)$. If $m < -\frac{Q}{2}$, then for any $x \in G$, the tempered distribution $y \mapsto \kappa_\sigma(x,y)$ is square-integrable and*

$$\begin{aligned} \big\|\kappa(x,\cdot)\big\|_{L^2(G)} &\le C \sup_{\pi\in\widehat{G}} \big\|\pi(I+\mathcal{R})^{-\frac{m}{\nu}}\sigma(x,\pi)\big\|_{\mathcal{L}(\mathcal{H}_\pi)}, \\ \big\|\kappa(x,\cdot)\big\|_{L^2(G)} &\le C \sup_{\pi\in\widehat{G}} \big\|\sigma(x,\pi)\pi(I+\mathcal{R})^{-\frac{m}{\nu}}\big\|_{\mathcal{L}(\mathcal{H}_\pi)}, \end{aligned} \tag{3.9}$$

with $C = C(m) \in (0,\infty)$ independent of σ and x.

The second estimate ensures Schwartz decay away from the origin.

Proposition 3.4 ([28, Prop. 5.4.4]). *Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\rho > 0$. Let $\sigma \in S^m_{\rho,\delta}(G)$ and let $|\cdot|$ be a quasi-norm on G. Then for any $M \in \mathbb{R}$ and any $\alpha, \beta_1, \beta_2, \beta_o \in \mathbb{N}_0^n$ there exist some $C > 0$ and $a, b \in \mathbb{N}$ independent of σ such that for all $x \in G$ and all $z \in G$ with $|z| \ge 1$, we have*

$$\big|X^{\beta_1}_{z_1=z} X^{\beta_2}_{z_2=z_1} X^{\beta_o}_{x_1=x} \tilde{q}_\alpha(z)\kappa_\sigma(x_1,z_2)\big| \le C\|\sigma\|_{S^m_{\rho,\delta}}|z|^{-M}.$$

[17] This is a matter of convenience since any two quasi-norms on a given G are equivalent and therefore give the same kernel estimates.

The third result, a direct consequence of Lemma 5.5.6 and Propositions 5.4.4 and 5.4.6 in [28], provides some integrability conditions, dictated by the behavior of the kernel at the origin. In addition, the result shows that such oscillatory integrals can be bounded by suitable seminorms of the corresponding symbol, a fact that will turn out to be crucial to perform the approximation argument we use in Section 4.

Lemma 3.5. *Let $\sigma \in S^m_{\rho,\delta}(G)$, with $0 \le \delta \le \rho \le 1$ and $\rho \neq 0$, and let κ_x be the associated kernel. Then for every $\gamma \in \mathbb{R}$, with $\gamma + Q > \max\left\{\frac{Q+m+\delta[\beta_0]-\rho[\alpha]+[\beta_1]+[\beta_2]}{\rho}, 0\right\}$, there exists $C > 0$ and a seminorm $\|\cdot\|_{S^m_{\rho,\delta}a,b}$ such that*

$$\int_G |z|^\gamma \sup_{x\in G} \left| X_z^{\beta_1} \tilde{X}_z^{\beta_2} \left(X_x^{\beta_0} \tilde{q}_\alpha(z) \kappa_x(z) \right) \right| dz \le C \|\sigma\|_{S^m_{\rho,\delta}a,b}.$$

4. Symbolic calculus

In this section, we develop a symbolic calculus for a wide range of τ-quantizations on graded groups and the Hörmander symbol classes $S^m_{\rho,\delta}(G)$ introduced in [28]. For technical reasons that will become clear soon, we will mostly have to require that ρ and δ satisfy $0 \le \delta < \min\{\rho, \frac{1}{v_n}\} \le \rho \le 1$. Note that the fundamental case $(\rho, \delta) = (1, 0)$ always satisfies this condition.

4.1. Admissible quantizing functions

In order to make the proposed τ-calculus work for the symbol classes $S^m_{\rho,\delta}(G)$, we need to control oscillatory integrals such as (3.6). We will therefore require that the quantizing functions $\tau\colon G \to G$ satisfy certain properties that are trivial for the τ-quantizations (1.4) on $G = \mathbb{R}^n$. While on $\mathbb{R}^n$ this is owed to the trivial nature of its Lie algebra, on non-trivially graded groups it hinges on the compatibility with the given homogeneous structure. Thus, for any graded G under consideration we fix, once and for all, a family of dilations that admits a Rockland operator and a strong Malcev $\{X_1, \dots, X_n\}$ basis which passes through the gradation and forms an eigenbasis of the dilations (cf. Section 2). Without loss of generality, we can choose such a basis to be a strong Malcev basis that passes through the gradation, that is, a basis whose pairwise Lie brackets generate the gradation (cf. [13]). We then define the following primary condition on quantizing functions τ on G:

(HP) The exponential coordinates of $\tau(x) = (c_1^\tau(x), \dots, c_n^\tau(x)) \in G$ as functions of $x \in G$ either vanish identically, i.e., $c_j^\tau(x) = 0$ for all $x \in G$, or are *homogeneous polynomials* of the form

$$c_j^\tau(x) = c_j^\tau(x_1, \dots, x_j) = C_j^\tau x_j + d_j^\tau(x_1, \dots, x_{j-1})$$

for some $C_j^\tau \neq 0$ and some homogeneous polynomial d_j^τ of degree v_j which only depends on x_k with $k = 1, \dots, j-1$.

Whenever τ satisfies the symmetry condition (1.10) for the classes $S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, $0 \le \delta < \min\{\rho, \frac{1}{v_n}\} \le \rho \le 1$, we will call τ a *symmetry function* and the corresponding τ-calculus a *symmetric calculus*.

Since for such τ the change of coordinates cv^τ defined by (3.7) is a Fréchet space isomorphism on the Schwartz space $\mathcal{S}(G)$ (see, e.g., [13, Lem. A.2.1]), the condition (HP) supersedes the condition (CV) from Section 3.3, which we have considered to be the minimal assumption on τ so far. To prove the main theorems of the τ-calculus presented in this paper, we will always assume (HP). This covers all the classical τ-quantizations (1.4) on $\mathbb{R}^n$ and the Kohn–Nirenberg quantization (1.8) as well as the symmetric quantizations arising from (1.12) and (1.13) on general graded groups.

Remark 4.1. The explicit dependence of (HP) on the choice of Malcev basis may seem a bit stark at first but turns out to be useful to prove important auxiliary lemmas.[18] Admittedly, a more natural, larger class of admissible quantizing functions may emerge from relaxing the conditions on (HP) to merely depend on subspaces $\mathfrak{g}_i$ of the gradation rather than individual coordinates.[19] While this is a very interesting aspect worth exploring in the future, we have tried not to involve even more algebraic properties of the Lie groups under consideration in our endeavor to develop a symbolic calculus for interesting families of quantizations.

Remark 4.2. One could also dispense with the homogeneity of the coefficients of τ and admit quantizing functions with more general polynomial coordinates, including quadratic maps $\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ as in [6] and the non-linear quantizing functions τ studied in [19]. This would come at the cost of carefully keeping track of products and sums of homogeneous and isotropic orders of polynomials and vector fields throughout the already quite lengthy estimates of many oscillatory integrals, while it would not generate much additional insight. Implementing this in our framework would also require that if $k_o \in \mathbb{N}$ denoted the highest isotropic order of any polynomial coefficient of τ, then the parameters $\rho, \delta \in [0, 1]$ would generally have to satisfy a stricter condition than the one required currently.

Let us provide a few examples of very distinct quantizing functions to illustrate that neither condition is overly restrictive but, on the contrary, rather natural.

Example 4.3. The quantizing function $\tau\colon x \mapsto e_G = (0, \dots, 0)$ and the corresponding change of variables $\mathrm{cv}^\tau(x, y) = (x, y^{-1}x)$ clearly characterizes the Kohn–Nirenberg quantization from [28].

Example 4.4. The quantizing function $\tau\colon x \mapsto x$ trivially satisfy (HP) and the corresponding change of variables $\mathrm{cv}^\tau(x, y) = (y, y^{-1}x)$ corresponds to the quantization which on $G = \mathbb{R}^n$ is sometimes called the right quantization.

[18] This becomes more visible in the unabridged preprint version of this paper [21].

[19] This very reasonable assumption was suggested to us by the referee.

Example 4.5. On the Heisenberg group $\mathbb{H}_n$, equipped with the canonical homogeneous dilations, we consider the quantizing functions

$$\tau(x) = \left(\frac{x_1}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\right) \tag{4.1}$$

for any choice of $c_{j,k} \in \mathbb{R}$, $j, k = 1, \dots, 2n$.

Since the canonical dilation weights are given by $v_1 = \cdots = v_{2n} = 1$ and $v_{2n+1} = 2$, each function τ clearly satisfies (HP). To see that it is also symmetric, by Theorem 4.14 below, it suffices to show that $\tau(x) = \tau(x^{-1})x$ holds for all $x \in \mathbb{H}_n$. Explicitly, this means that

$$\begin{aligned} &\left(\frac{x_1}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\right) \\ &= \Bigg(-\frac{x_1}{2} + x_1, \dots, -\frac{x_{2n}}{2} + x_{2n}, -\frac{x_{2n+1}}{2} \\ &\qquad + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k + x_{2n+1} - \frac{1}{2} \sum_{l=1}^{n} \underbrace{(x_l x_{n+l} - x_{n+l} x_l)}_{=0}\Bigg) \end{aligned} \tag{4.2}$$

has to hold for all $x \in \mathbb{H}_n$, which is clearly the case.

In fact, these are the only symmetric quantizing functions $\tau \colon \mathbb{H}_n \to \mathbb{H}_n$ which satisfy (HP): because of the homogeneous structure of $\mathbb{H}_n$ the function τ has to be of the form

$$\tau(x) = \left(a_1 x_1 + L_1(x), \dots, a_{2n} x_{2n} + L_{2n}(x), a_{2n+1} x_{2n+1} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\right)$$

for some linear forms $L_l(x) = b_{l,1} x_1 + \cdots + b_{l,2n} x_{2n}$ with $b_{l,1}, \dots, b_{l,2n} \in \mathbb{R}$ and some coefficients $c_{j,k} \in \mathbb{R}$, $j, k, l = 1, \dots, n$. Now, by a comparison of coefficients as in (4.2), the condition $\tau(x) = \tau(x^{-1})x$ for all $x \in \mathbb{H}_n$ requires that

$$\begin{cases} a_j x_j + L_j(x) = (1 - a_j) x_j - L_j(x) & \Rightarrow a_j = \frac{1}{2},\ L_j(x) = 0, \\ a_{2n+1} x_{2n+1} = (1 - a_{2n+1}) x_{2n+1} & \Rightarrow a_{2n+1} = \frac{1}{2} \end{cases}$$

for all $j = 1, \dots, 2n$. Hence, any such τ is necessarily of the form (4.1).

Example 4.6. The quantizing function (1.13), which for $G = \mathbb{R}^n$ coincides with $\tau(x) = (\frac{x_1}{2}, \dots, \frac{x_n}{2})$ and thus yields the Weyl quantization and which for $G = \mathbb{H}_n$ is an instance of (4.1), trivially satisfying (HP) on general graded G.

Example 4.7. The quantizing function (1.12) by [41], which for $G = \mathbb{R}^n$ also coincides with $\tau(x) = (\frac{x_1}{2}, \dots, \frac{x_n}{2})$ and which for $G = \mathbb{H}_n$ is another instance of (4.1), can also easily be seen to satisfy (HP) if one, for example, employs an upper diagonal matrix representation of the nilpotent group G in question.

The following lemma, which characterizes the chain rule of differentiation for compositions involving the quantizing function τ, will be crucial to prove the convergence of the main oscillatory integrals in this section.

Lemma 4.8. *Let E be a Banach space and let $f: G \to E$. Moreover, let $p: G \times G \to G$ be a smooth function defined by*

$$p(y,z) = \big(p_1(y,z), \dots, p_n(y,z)\big),$$

where, for every $j = 1, \dots, n$, the j-th coefficient $p_j(y,z)$ is homogeneous of degree v_j in y and z, that is,

$$p_j(y,z) = \sum_{[\eta_1]+[\eta_2]=v_j} C_{\eta_1,\eta_2} y^{\eta_1} z^{\eta_2}.$$

Then, for every $\alpha \in \mathbb{N}_0^n$ there exist coefficients $c_{\alpha_1,\alpha_2}, c'_{\alpha_1,\alpha_2} \in \mathbb{R}$ such that

$$X^{\alpha}_{y_1=y} f\big(p(y_1,z)\big) = \sum_{\substack{|\alpha_3|\le|\alpha|,\\ [\alpha_3]\ge[\alpha]}} \sum_{[\alpha_1]+[\alpha_2]=[\alpha_3]-[\alpha]} c_{\alpha_1,\alpha_2} q_{\alpha_1}(y) q_{\alpha_2}(z) (X^{\alpha_3} f)\big(p(y,z)\big), \quad (4.3)$$

$$\widetilde{X}^{\alpha}_{y_2=y} f\big(p(y_2,z)\big) = \sum_{\substack{|\alpha_3|\le|\alpha|,\\ [\alpha_3]\ge[\alpha]}} \sum_{[\alpha_1]+[\alpha_2]=[\alpha_3]-[\alpha]} c'_{\alpha_1,\alpha_2} q_{\alpha_1}(y) q_{\alpha_2}(z) (\widetilde{X}^{\alpha_3} f)\big(p(y,z)\big), \quad (4.4)$$

where the polynomials q_{α_2} are determined by (2.4). Moreover, the same conclusion holds true if we replace $X^{\alpha}_{y_1=y}$ by $X^{\alpha}_{z_1=z}$ in (4.3), and $\widetilde{X}^{\alpha}_{y_2=y}$ by $\widetilde{X}^{\alpha}_{z_2=z}$ in (4.4).

Proof. We omit the easy but lengthy proof and refer the reader to [21, Lem. 4.9]. ∎

4.2. Change of quantization

In this subsection, we show that one can switch between any τ-quantization and the Kohn–Nirenberg quantization, and therefore between any two τ-quantizations, without losing the essential properties of the Kohn–Nirenberg calculus for the range of symbol classes under consideration.

First, however, we need an auxiliary proposition on the continuity on the Schwartz space $\mathcal{S}(G)$ of τ-quantized operators. This result extends [28, Thm. 5.2.15] to τ-quantized operators with symbols in Hörmander classes for $0 < \rho, \delta < \frac{1}{v_n}$. Note that these restrictions can be omitted for $\tau = e_G$, i.e., in the setting of [28]. While at least $\delta < 1$ is necessary[20] whenever $\tau \neq e_G$, the condition $0 < \rho$ serves us to control the derivatives of τ at infinity (cf. [28, Thm. 5.4.1]). Explicit computations in the case of $G = \mathbb{H}_n$ equipped with the canonical dilations show that $\delta < \frac{1}{v_n} = \frac{1}{2}$ can be relaxed to $\delta < 1$. Whether this can be improved for general graded groups remains an open question.

Proposition 4.9. *Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $0 < \rho$, $\delta < \frac{1}{v_n}$, and let $\sigma \in S^m_{\rho,\delta}(G)$. Then for every $f \in \mathcal{S}(G)$ the quantization formula (1.9) defines an element in $\mathcal{S}(G)$ given by*

[20] Confer, e.g., [31, Thm. 2.21] for $\delta < 1$ for the Weyl quantization on $\mathbb{R}^n$, in which case $v_1 = \cdots = v_n = 1$.

the absolutely convergent integral

$$\mathrm{Op}^\tau(\sigma) f(x) = \int_G \kappa_\sigma\big(x\tau(y^{-1}x)^{-1}, y^{-1}x\big) f(y)\, dy. \tag{4.5}$$

Moreover, the linear operator $\mathrm{Op}^\tau(\sigma)\colon \mathcal{S}(G) \to \mathcal{S}(G)$ *is continuous: for each seminorm* $\|\cdot\|_{\mathcal{S},N_1}$, $N_1 \in \mathbb{N}$, *there exist a constant* $C > 0$, *a seminorm* $\|\cdot\|_{\mathcal{S},N_2}$, $N_2 \in \mathbb{N}$, *and a seminorm* $\|\cdot\|_{S^m_{\rho,\delta}a,b}$, $a, b \in \mathbb{N}$, *such that* $\|\mathrm{Op}^\tau(\sigma) f\|_{\mathcal{S},N_1} \le C\|\sigma\|_{S^m_{\rho,\delta}a,b}\|f\|_{\mathcal{S},N_2}$.

Proof. Thus, let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $\delta < \frac{1}{v_n}$, and let $\sigma \in S^m_{\rho,\delta}(G)$. By Proposition 3.3, we have that

$$y \mapsto \widetilde{\kappa}(x, y) := \big((I + \widetilde{\mathcal{R}})^{-N}\kappa_\sigma(x, \cdot)\big)(y) = \mathcal{F}^{-1}\Big(\big\{\sigma(x, \pi)\pi(I + \mathcal{R})^{-N}\big\}_{\pi\in\widehat{G}}\Big)(y),$$

is in $L^2(G)$ for all $x \in G$ whenever $N \in \mathbb{N}$ satisfies $m + Q/2 < N\nu$, and $\|\widetilde{\kappa}(x, \cdot)\|_{L^2(G)}$ is uniformly bounded in x due to (3.9) with

$$\sup_{x\in G}\big\|\widetilde{\kappa}(x, \cdot)\big\|_{L^2(G)} \le C_1 \sup_{(x,\pi)\in G\times\widehat{G}}\big\|\sigma(x, \pi)\pi(I + \mathcal{R})^{-N}\big\|_{\mathcal{H}_\pi} \le C_2\|\sigma\|_{S^m_{\rho,\delta},a,b} \tag{4.6}$$

for some $a, b \in \mathbb{N}_0$ and constants $C_1, C_2 > 0$ independent of σ and x. Since $\kappa_\sigma, \mathrm{Ker}^\tau_\sigma \in \mathcal{S}'(G \times G)$ whenever $\sigma \in S^m_{\rho,\delta}(G)$ (cf. Section 3.3), for any $f \in \mathcal{S}(G)$ the integral (4.5) defines a tempered distribution $g := \mathrm{Op}^\tau(\sigma) f$. To see that $g \in \mathcal{S}(G)$, we fix some $N_0 \in \mathbb{N}$ and use integration by parts (in the distributional sense) to rewrite (4.5) as

$$\begin{aligned} g(x) &= \int_G f(y)\big[(I + \widetilde{\mathcal{R}})^{N_0}(I + \widetilde{\mathcal{R}})^{-N_0}\big]_{y_0=y}\kappa_\sigma\big(x\tau(y^{-1}x)^{-1}, y_0^{-1}x\big)\, dy \\ &= \int_G \overline{\sum_{\substack{[\alpha]\le\nu N_0,\\ [\beta]\ge[\alpha],\\ |\beta|\le|\alpha|}}} Q_{\alpha,\beta}(y) \overline{\sum_{[\beta_1]+[\beta_2]=[\beta]}} X^{\beta_1}_{y_1=y} f(y_1) \\ &\qquad\qquad \times X^{\beta_2}_{y_2=y}(I + \widetilde{\mathcal{R}})^{-N_0}_{y_0=y}\kappa_\sigma\big(x\tau(y_2^{-1}x)^{-1}, y_0^{-1}x\big)\, dy, \end{aligned}$$

where the homogeneous polynomials $Q_{\alpha,\beta}$ are determined by (2.2). Before we estimate $g(x)$ by an application of the Cauchy–Schwarz inequality for $L^2(G, dy)$, we observe that, due to (HP), we can use Lemma 4.8 and write

$$\begin{aligned} &X^{\beta_2}_{y_2=y}(I + \widetilde{\mathcal{R}})^{-N_0}_{y_0=y}\kappa_\sigma\big(x\tau(y_2^{-1}x)^{-1}, y_0^{-1}x\big) \\ &\quad= \overline{\sum_{\substack{[\beta_{2,3}]\ge[\beta_2]\\ |\beta_{2,3}|\le|\beta_2|\\ [\beta_{2,1}]+[\beta_{2,2}]=[\beta_{2,3}]-[\beta_2]}}} q_{\beta_{2,1}}(y) q_{\beta_{2,2}}(x) X^{\beta_{2,3}}_{x_1=x\tau(y^{-1}x)^{-1}}(I + \widetilde{\mathcal{R}})^{-N_0}_{y_0=y}\kappa_\sigma(x_1, y_0^{-1}x) \\ &\quad= \overline{\sum_{\substack{[\beta_{2,3}]\ge[\beta_2]\\ |\beta_{2,3}|\le|\beta_2|\\ [\beta_{2,1}]+[\beta_{2,2}]=[\beta_{2,3}]-[\beta_2]}}} \overline{\sum_{[\beta'_{2,2}]+[\beta''_{2,2}]=[\beta_{2,2}]}} q_{\beta_{2,1}}(y) q_{\beta'_{2,2}}(y) q_{\beta''_{2,2}}(y^{-1}x) \\ &\qquad\qquad \times X^{\beta_{2,3}}_{x_1=x\tau(y^{-1}x)^{-1}}(I + \widetilde{\mathcal{R}})^{-N_0}_{y_0=y}\kappa_\sigma(x_1, y_0^{-1}x), \end{aligned}$$

so that the polynomials $q_{\beta_{2,1}+\beta'_{2,2}}(y)$ in these summands can be grouped with the Schwartz function $X^{\beta_1} f$ and the polynomials $q_{\beta''_{2,2}}(y^{-1}x)$ with the associated kernel. Since $\rho > 0$, the kernel κ_σ decays like a Schwartz function away from the origin, due to Proposition 3.4, so the factors $q_{\beta''_{2,2}}(y^{-1}x)$ do not affect our eventual estimate, while close to the origin they at most improve the L^2-convergence of

$$y \mapsto X^{\beta_{2,3}}_{x_1 = x\tau(y_2^{-1}x)^{-1}} (I + \widetilde{\mathcal{R}})^{-N_0}_{y_0 = y} \kappa_\sigma(x_1, y_0^{-1}x). \tag{4.7}$$

So we can ignore these polynomial factors without loss of generality and focus on the fact that, by (4.6), the L^2-norm of (4.7) is uniformly bounded in $x \in G$ whenever we choose $N_0 \in N$ such that $m + \delta[\beta_{2,3}] + Q/2 \leq m + \delta\nu N_0 v_n + Q/2 < N_0\nu$. Such a choice is in fact possible due to $\delta < \frac{1}{v_n}$. We can now estimate

$$\begin{aligned} \left|g(x)\right| \leq & \overline{\sum_{\substack{[\alpha] \leq \nu N_0, \\ [\beta] \geq [\alpha], \\ |\beta| \leq |\alpha|}}} \overline{\sum_{\substack{[\beta_{2,3}] \geq [\beta_2] \\ |\beta_{2,3}| \leq |\beta_2| \\ [\beta_{2,1}] + [\beta_{2,2}] = [\beta_{2,3}] - [\beta_2]}}} \overline{\sum_{[\beta'_{2,2}] + [\beta''_{2,2}] = [\beta_{2,2}]}} \int_G \left|q_{\beta_{2,1} + \beta'_{2,2}}(y)\right| \left|Q_{\alpha,\beta}(y)\right| \\ & \times \left|(X^{\beta_1} f)(y)\right| \left|q_{\beta''_{2,2}}(y^{-1}x)\right| \sup_{x_1 \in G} \left|X^{\beta_{2,3}}_{x_1} (I + \widetilde{\mathcal{R}})^{-N_0}_{y_0 = y} \kappa_\sigma(x_1, y_0^{-1}x)\right| dy \\ \leq & \; C \|f\|_{\mathcal{S}, N_0} \|\sigma\|_{S^m_{\rho,\delta} a, b}, \end{aligned}$$

for some sufficiently large $N_0 \in \mathbb{N}$, where for each of the integrals in the sum we have used (4.6) to estimate $\sup_{x_1 \in G} |X^{\beta_{2,3}}_{x_1} \widetilde{\kappa}(x_1, y)|$ and the Cauchy–Schwarz inequality for $L^2(G, dy)$. This in turn gives the desired result for $N_1 = 0$.

To show that $x^{\alpha_o}(X^{\beta_o} g)(x)$ is defined for all $x \in G$ and that $x \mapsto x^{\alpha_o}(X^{\beta_o} g)(x) \in L^2(G)$ for all $\alpha_o, \beta_o \in \mathbb{N}_0^n$, we use dominated convergence to differentiate under the integral sign. Next, we split x^{α_o} as

$$x^{\alpha_o} = \overline{\sum_{[\alpha_{o,1}] + [\alpha_{o,2}] = [\alpha_o]}} q_{\alpha_{o,1}}(y) q_{\alpha_{o,2}}(y^{-1}x)$$

in order to apply essentially the same argument as above, with $N_0 \in N$ chosen sufficiently large in order to compensate additional polynomial factors in the integral when α_o or β_o are different from zero. This completes the proof. ■

Assumptions 4.10. In what follows we will often assume $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq \frac{1}{v_n}$, which in turn implies the conditions $0 < \rho$ and $\delta < \frac{1}{v_n}$ in Proposition 4.9. With the exception of $G = \mathbb{R}^n$ equipped with isotropic dilations, for which $v_n = 1$, this condition is stricter than $0 \leq \delta < \rho \leq 1$. The restriction $\delta < \rho$ is always assumed because of the authors' personal interest in asymptotic extensions for the symbols of adjoints, composite operators, etc, and it cannot be relaxed even in the case of $G = \mathbb{R}^n$, while $\delta < \frac{1}{v_n}$ and, in particular, sometimes $\delta < \frac{\rho}{v_n}$ are needed to make the proofs work for general graded G. Note that for the Kohn–Nirenberg quantization it suffices to assume $0 \leq \delta < \rho \leq 1$ as in [28] since the interaction of the quantizing function $\tau = e_G$ with the calculus developed here becomes trivial. The proofs given in this paper simplify in this case and recover the well-known results due to [28].

Theorem 4.11. *Let T be a continuous linear operator from $\mathcal{S}(G)$ to $\mathcal{S}(G)$ and let $m \in \mathbb{R}$ and $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq 1$. Let $\tau: G \to G$ be a quantizing function different from the constant function $\tau = e_G$ which satisfies (HP). Then, if σ and σ_τ are two symbols such that $T = \mathrm{Op}^\tau(\sigma_\tau) = \mathrm{Op}(\sigma)$, $\sigma \in S^m_{\rho,\delta}(G)$ if and only if $\sigma_\tau \in S^m_{\rho,\delta}(G)$.*

The map $\sigma \mapsto \sigma_\tau$ is a Fréchet space isomorphism from $S^m_{\rho,\delta}(G)$ onto itself, and the symbols are related to one another by the asymptotic expansions

$$\sigma_\tau \sim \sum_{j=0}^{\infty}\Big(\sum_{[\alpha]=j}\ \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \Delta^{\alpha'} X^{\alpha} \sigma\Big), \tag{4.8}$$

$$\sigma \sim \sum_{j=0}^{\infty}\Big(\sum_{[\beta]=j}\ \sum_{[\beta']=[\beta]} c^{\mathrm{KN},\tau}_{\beta',\beta}\, \Delta^{\beta'} X^{\beta} \sigma_\tau\Big), \tag{4.9}$$

in the sense that for given $M, N \in \mathbb{N}_0$

$$R^{\tau,\mathrm{KN}}_M := \sigma_\tau - \sum_{[\alpha]\leq M}\ \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \Delta^{\alpha'} X^{\alpha} \sigma \in S^{m-(\rho-\delta)(M+1)}_{\rho,\delta}(G),$$

$$R^{\mathrm{KN},\tau}_N := \sigma - \sum_{[\beta]\leq N}\ \sum_{[\beta']=[\beta]} c^{\mathrm{KN},\tau}_{\beta',\beta}\, \Delta^{\beta'} X^{\beta} \sigma_\tau \in S^{m-(\rho-\delta)(N+1)}_{\rho,\delta}(G).$$

Moreover, the coefficients $c^{\tau,\mathrm{KN}}_{\alpha',\alpha}, c^{\mathrm{KN},\tau}_{\beta',\beta} \in \mathbb{R}$ are uniquely determined by the equations

$$q_\alpha\big(\tau(y)\big) = \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \tilde{q}_{\alpha'}(y), \tag{4.10}$$

$$q_\beta\big(\tau(y)^{-1}\big) = \sum_{[\beta']=[\beta]} c^{\mathrm{KN},\tau}_{\beta',\beta}\, \tilde{q}_{\beta'}(y).$$

Proof. Our proof crucially relies on the relation between the associated kernels. We will see that it suffices to prove our statement for the dense subset of $\sigma \in S^{-\infty}(G)$ with $\kappa_\sigma \in \mathcal{S}(G \times G)$, and conversely for $\sigma_\tau \in S^{-\infty}(G)$ with $\kappa_{\sigma_\tau} \in \mathcal{S}(G \times G)$, and extend the obtained seminorm estimates to $\sigma \in S^m_{\rho,\delta}(G)$, and to $\sigma_\tau \in S^m_{\rho,\delta}(G)$, respectively. Thus, let $m \in \mathbb{R}$, $0 \leq \delta < \min\{\frac{1}{v_n}, \rho\}$ and let $\tau: G \to G$ be a quantizing function which satisfies (HP). To show the necessary condition and the asymptotic expansion (4.8), let us therefore suppose that $\sigma \in S^{-\infty}(G)$ with associated kernel satisfying $\kappa_\sigma \in \mathcal{S}(G \times G)$. Then also $\kappa_{\sigma_\tau} \in \mathcal{S}(G \times G)$ due to (HP) and the relation

$$\kappa_{\sigma_\tau}(x, y) = \kappa_\sigma\big(x\tau(y), y\big), \tag{4.11}$$

which we deduce from the identity

$$\begin{aligned}(Tf)(x) = \big(\mathrm{Op}(\sigma) f\big)(x) &= \iint_{\widehat{G}\times G} \mathrm{Tr}\big(\pi(y^{-1}x)\sigma(x,\pi) f(y)\big)\, dy\, d\mu(\pi) \\ &= \big(\mathrm{Op}^\tau(\sigma_\tau) f\big)(x) = \iint_{\widehat{G}\times G} \mathrm{Tr}\big(\pi(y^{-1}x)\sigma_\tau(x\tau(y^{-1}x)^{-1}, \pi) f(y)\big)\, dy\, d\mu(\pi)\end{aligned}$$

and an application of the pullback $(CV^\tau)^{-1}$ defined by (3.5) to the associated kernels κ_σ and κ_{σ_τ}. Moreover, the corresponding symbol

$$\sigma_\tau(x,\pi) = \mathcal{F}_G\big(\kappa_{\sigma_\tau}(x,\cdot)\big)(\pi) \tag{4.12}$$

is an element of $S^{-\infty}(G)$ by [28, Lem. 5.5.20]. As in the proofs of Theorems 4.14 and 4.19, we employ a Taylor expansion to reveal the asymptotic expansion of σ_τ in terms of σ. To do so, we observe that the condition (HP) implies that for any $\alpha \in \mathbb{N}_0^n$ there exist coefficients $c_{\alpha',\alpha} \in \mathbb{R}$, $\alpha' \in \mathbb{N}_0^n$, $[\alpha'] = [\alpha]$, such that

$$(q_\alpha \circ \tau)(y) = \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \tilde{q}_{\alpha'}(y),$$

for the homogeneous polynomials q_α and $\tilde{q}_{\alpha'} = x \mapsto q_{\alpha'}(x^{-1})$ determined by (2.4) for all $\alpha, \alpha' \in \mathbb{N}_0^n$. So, we may expand

$$\begin{aligned}\kappa_{\sigma_\tau}(x,y) = \kappa_\sigma\big(x\tau(y),y\big) &\underbrace{\sum_{[\alpha]\le M} \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \tilde{q}_{\alpha'}(y) X^\alpha_{x_1=x}\kappa_\sigma(x_1,y)}_{:=\kappa_{T_0}(x,y)} \\ &+ \sum_{M+1\le[\alpha]\le M_0} \sum_{[\alpha']=[\alpha]} c^{\tau,\mathrm{KN}}_{\alpha',\alpha}\, \tilde{q}_{\alpha'}(y) X^\alpha_{x_1=x}\kappa_\sigma(x_1,y) + R^{\kappa_\sigma(\cdot,y)}_{x,M_0}\big(\tau(y)\big)\end{aligned} \tag{4.13}$$

for some $M_0 \ge M+1$, where the Taylor remainder is controlled by (2.5). Since

$$\Delta^{\alpha'} X^\alpha \sigma \in S^{m-(\rho-\delta)[\alpha]}_{\rho,\delta}(G), \tag{4.14}$$

we immediately obtain that $T_0 := \mathcal{F}_G(\kappa_{T_0})$ yields the asymptotic expansion (4.8) provided that $T_1(x,\pi) := \int_G R^{\kappa_\sigma(\cdot,y)}_{x,M_0}(\tau(y))\pi(y)^*\,dy$ belongs to $S^{m-(\rho-\delta)(M+1)}_{\rho,\delta}(G)$. To prove this, we will show that

$$\sup_{(x,\pi)\in G\times\widehat{G}} \Big\| (\Delta^{\alpha_0} X^{\beta_0} T_1)(x,\pi)\big(I+\pi(\mathcal{R})\big)^{-\frac{m-(\rho-\delta)(M+1)-\rho[\alpha_0]+\delta[\beta_0]}{\nu}} \Big\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty \tag{4.15}$$

for arbitrary, but fixed $\alpha_0, \beta_0 \in \mathbb{N}_0^n$. Thus, let $\alpha_0, \beta_0 \in \mathbb{N}_0^n$ and let M_1 be the least non-negative integer such that

$$-m + (\rho-\delta)(M+1) + \rho[\alpha_0] - \delta[\beta_0] - \nu M_1 \le 0.$$

Since

$$\begin{aligned}\sup_{\pi\in\widehat{G}} &\Big\| \big(I+\pi(\mathcal{R})\big)^{-\frac{m-(\rho-\delta)(M+1)-\rho[\alpha_0]+\delta[\beta_0]}{\nu}-M_1} \Big\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ &= \Big\| (I+\mathcal{R})^{-\frac{m-(\rho-\delta)(M+1)-\rho\alpha_0+\delta\beta_0}{\nu}-M_1} \Big\|_{L^2\to L^2} =: C < \infty,\end{aligned}$$

we can now bound (4.15) by estimating

$$\begin{aligned}
&\left\|(\Delta^{\alpha_0}X^{\beta_0}T_1)(x,\pi)\big(I+\pi(\mathcal{R})\big)^{-\frac{m-(\rho-\delta)(M+1)-\rho\alpha_0+\delta\beta_0}{\nu}}\right\|_{\mathcal{L}(\mathcal{H}_\pi)}\\
&\quad\le C\left\|(\Delta^{\alpha_0}X^{\beta_0}T_1)(x,\pi)\big(I+\pi(\mathcal{R})\big)^{M_1}\right\|_{\mathcal{L}(\mathcal{H}_\pi)}\\
&\quad= C\left\|\int_G X^{\beta_0}_{x_1=x}R^{\kappa_\sigma(\cdot,y)}_{x_1,M_0}\big(\tau(y)\big)\tilde{q}_{\alpha_0}(y)\pi(y)^*\big(I+\pi(\mathcal{R})\big)^{M_1}\,dy\right\|_{\mathcal{L}(\mathcal{H}_\pi)}\\
&\quad= C\Bigg\|\overline{\sum_{[\beta_1]+[\beta_2]+[\beta_3]\le\nu M_1}}\int_G \widetilde{X}^{\beta_1}_{y_1=y}\widetilde{X}^{\beta_2}_{y_2=y}X^{\beta_0}_{x_1=x}R^{\kappa_\sigma(\cdot,y_1)}_{x_1,M_0}\\
&\qquad\qquad\qquad\times\big(\tau(y_2)\big)(\widetilde{X}^{\beta_3}\tilde{q}_{\alpha_0})(y)\pi(y)^*dy\Bigg\|_{\mathcal{L}(\mathcal{H}_\pi)}\\
&\quad= C\Bigg\|\overline{\sum_{[\beta_1]+[\beta_2]+[\beta_3]\le\nu M_1}}\int_G \widetilde{X}^{\beta_2}_{y_2=y}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M_0}\\
&\qquad\qquad\qquad\times\big(\tau(y_2)\big)(\widetilde{X}^{\beta_3}\tilde{q}_{\alpha_0})(y)\pi(y)^*\,dy\Bigg\|_{\mathcal{L}(\mathcal{H}_\pi)}.
\end{aligned}$$

Because of $\|\pi(y)^*\|_{\mathcal{L}(\mathcal{H}_\pi)}=1$, so we can rewrite

$$\begin{aligned}
&\widetilde{X}^{\beta_2}_{y_2=y}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M}\big(\tau(y_2)\big)\\
&\quad=\sum_{\substack{|\beta_2'|\le|\beta_2|\\ [\beta_2']\ge[\beta_2]}} Q_{\beta_2',\beta_2}(y)X^{\beta_2'}_{y_2=y}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M_0}\big(\tau(y_2)\big)\\
&\quad=\sum_{\substack{|\beta_2'|\le|\beta_2|\\ [\beta_2']\ge[\beta_2]}} Q_{\beta_2',\beta_2}(y)\overline{\sum_{\substack{[\beta_2']\le[\beta_2''']\le\upsilon_n|\beta_2'|\\ [\beta_2''']-[\beta_2'']=[\beta_2']}}} q_{\beta_2''}(y)X^{\beta_2'''}_{y_2=\tau(y)}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M_0}(y_2)\\
&\quad=\overline{\sum_{[\beta_2''']\le\upsilon_n[\beta_2]}} q_{\beta_2'''-\beta_2}(y)X^{\beta_2'''}_{y_2=\tau(y)}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M_0}(y_2),
\end{aligned}$$

where in the second step we have used Lemma 4.8. We further estimate

$$\begin{aligned}
&\left\|(\Delta^{\alpha_0}X^{\beta_0}T_1)(x,\pi)\big(I+\pi(\mathcal{R})\big)^{-\frac{m-(\rho-\delta)(M+1)-\rho\alpha_0+\delta\beta_0}{\nu}}\right\|_{\mathcal{L}(\mathcal{H}_\pi)}\\
&\le C\overline{\sum_{\substack{[\beta_1]+[\beta_2]+[\beta_3]\le\nu M_1\\ [\beta_2''']\le\upsilon_n[\beta_2]}}}\int_G\big|(\widetilde{X}^{\beta_3}\tilde{q}_{\alpha_0})(y)\big|\big|q_{\beta_2'''-\beta_2}(y)\big|\Big|X^{\beta_2'''}_{y_2=\tau(y)}R^{X^{\beta_0}_{x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_1,y_1)}_{0,M_0}(y_2)\Big|dy\\
&\le C\overline{\sum_{\substack{[\beta_1]+[\beta_2]+[\beta_3]\le\nu M_1\\ [\beta_2''']\le\upsilon_n[\beta_2]}}}\int_G\big|q_{\beta_2'''-\beta_2}(y)\big|\sum_{\substack{|\gamma|\le\lceil(M_0-[\beta_2'''])\rfloor_++1\\ [\gamma]\ge\lceil(M_0-[\beta_2'''])\rfloor_+}}\big|\tau(y)\big|^{[\gamma]}\\
&\qquad\qquad\qquad\times\sup_{x_1\in G}\Big|X^\gamma_{x_1}X^{\beta_2'''}_{x_2=x_1}\widetilde{X}^{\beta_1}_{y_1=y}\kappa_\sigma(x_2,y_1)\Big|dy.
\end{aligned}$$

Away from $y = 0$, the integral above converges absolutely since $X_{x_1}^{\gamma} X_{x_2=x_1}^{\beta_2'''} \widetilde{X}_{y_1=y}^{\beta_1} \kappa_\sigma$ has Schwartz decay and $|\tau(y)|$ is of polynomial growth due to (HP). To see that it also converges in a neighborhood of the origin, we notice that there exist $a, b \in \mathbb{N}_0$ and a constant $C_0 > 0$ such that

$$\sup_{x_1 \in G} \left| X_{x_1}^{\gamma} X_{x_2=x_1}^{\beta_2'''} \widetilde{X}_{y_1=y}^{\beta_1} \kappa_\sigma(x_2, y_1) \right| \le C_0 \|\sigma\|_{S^m_{\rho,\delta}, a, b} |y|^{-\frac{m+[\beta_1]+\delta([\gamma]+[\beta_2'''])}{\rho}}.$$

Hence, the integral converges in this neighborhood, since we can bound $|\tau(y)| \lesssim |y|$ and pick $M_0 \in \mathbb{N}$, $M_0 > \nu M_1 \ge [\beta_2''']$, large enough so that

$$\begin{aligned} \rho\big(Q + [\gamma]\big) \ge \rho(Q + M_0) &> Q + m + \nu M_1(1 + v_n) + \delta M_0 \\ &> Q + m + \nu M_1(1 + v_n) + \delta[\gamma], \end{aligned}$$

because the latter is greater or equal $Q + m + [\beta_1] + \delta([\gamma] + [\beta_2'''])$. Note that such a choice of M_0 is possible due to $\frac{\delta}{\rho} < 1$. So, we obtain

$$\sup_{(x,\pi) \in G \times \widehat{G}} \left\| (\Delta^{\alpha_0} X^{\beta_0} T_1)(x, \pi) \big(I + \pi(\mathcal{R})\big)^{-\frac{m-(\rho-\delta)(M+1)-\rho\alpha_0+\delta\beta_0}{\nu}} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} \lesssim \|\sigma\|_{S^m_{\rho,\delta}, a, b},$$

and, in view of (4.14), also

$$\|\sigma_\tau\|_{S^m_{\rho,\delta}, [\alpha_0], [\beta_0]} \lesssim \|\sigma\|_{S^m_{\rho,\delta}, a, b}. \tag{4.16}$$

This completes the proof of (4.8).

To prove the sufficient condition and the asymptotic expansion (4.9), our line of arguments applies essentially verbatim. From the identity (4.11) we immediately obtain the converse identity

$$\kappa_\sigma(x, y) = \kappa_{\sigma_\tau}\big(x\tau(y)^{-1}, y\big). \tag{4.17}$$

Since the condition (HP) implies the condition (CV), it follows that the kernel $\kappa_\sigma \in \mathcal{S}(G \times G)$ whenever $\kappa_{\sigma_\tau} \in \mathcal{S}(G \times G)$, and we may follow precisely the same steps as above to show (4.9) and

$$\|\sigma\|_{S^m_{\rho,\delta}, [\alpha_0], [\beta_0]} \lesssim \|\sigma_\tau\|_{S^m_{\rho,\delta}, a, b}. \tag{4.18}$$

This proves the converse direction.

To pass to general symbols $\sigma \in S^m_{\rho,\delta}(G)$, we employ an argument based on the approximation of σ by a net of smoothing symbols. By [28, Lem. 5.4.11] and the equivalence of seminorms [28, Thm. 5.5.20], there exists a net $\{\sigma_\varepsilon\}_{\varepsilon \in (0,1)} \subseteq S^{-\infty}(G)$, with $\kappa_{\sigma_\varepsilon} \in \mathcal{S}(G \times G)$, such that for every $\theta > 0$ and every seminorm $\|\cdot\|_{S^m_{\rho,\delta}, a, b}$ there exist a constant $C = C(a, b, c, m, \theta, \rho, \delta) > 0$ and $a', b' \in \mathbb{N}_0$ such that

$$\begin{aligned} \|\sigma - \sigma_\varepsilon\|_{S^{m+\theta}_{\rho,\delta}, a, b} &\le C \|\sigma\|_{S^m_{\rho,\delta}, a', b'} \, \varepsilon^{\frac{\theta}{\nu}}, \\ \|\sigma_\varepsilon\|_{S^m_{\rho,\delta}, a, b} &\le C \|\sigma\|_{S^m_{\rho,\delta}, a', b'} \, \varepsilon^{\frac{\theta}{\nu}}. \end{aligned} \tag{4.19}$$

Now, let $a, b \in \mathbb{N}_0$ and $\eta \geq 0$. Since $\sigma \mapsto \sigma_\tau$ is a linear map from $S^{-\infty}(G)$ into itself with the property that $\kappa_{\sigma_\tau} \in \mathcal{S}(G \times G)$ precisely when $\kappa_\sigma \in \mathcal{S}(G \times G)$, the estimates (4.16) and (4.19) yield the existence of integers $a', b', a'', b'' \in \mathbb{N}$ and a constant $C > 0$ such that

$$\left\|(\sigma_\varepsilon)_\tau - (\sigma_{\varepsilon'})_\tau\right\|_{S^{m+\theta}_{\rho,\delta},a,b} \leq \|\sigma_\varepsilon - \sigma_{\varepsilon'}\|_{S^{m+\theta}_{\rho,\delta},a',b'} \leq C\|\sigma\|_{S^m_{\rho,\delta},a'',b''}\left(\varepsilon^{\frac{\theta}{\nu}} + \varepsilon'^{\frac{\theta}{\nu}}\right) \leq \eta,$$

for all $\varepsilon, \varepsilon' \in (0, \varepsilon_0)$ and a sufficiently small $\varepsilon_0 \in (0,1)$. Let us denote the by $(\sigma_\tau)_\varepsilon$ the sequence $(\sigma_\varepsilon)_\tau$. Since $a, b \in \mathbb{N}_0$ and $\eta > 0$ were arbitrary, the net $\{(\sigma_\tau)_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq S^m_{\rho,\delta}(G)$ is Cauchy in the coarser topology of $S^{m+\theta}_{\rho,\delta}(G) \supseteq S^{m_1+m_2}_{\rho,\delta}(G)$, but its limit $\sigma'_\tau := \lim_\varepsilon(\sigma_\tau)_\varepsilon$ nevertheless lies in a closed bounded subset of $S^m_{\rho,\delta}$, due to (4.16). On the one hand, by [28, Lem. 5.4.11 (3)] and Proposition 4.9, this implies that $\mathrm{Op}^\tau((\sigma_\tau)_\varepsilon)$ converges strongly on $\mathcal{S}(G)$ to $\mathrm{Op}^\tau(\sigma'_\tau)$ with $\sigma'_\tau \in S^m_{\rho,\delta}(G)$.[21] On the other hand, since $\mathrm{Op}^\tau((\sigma_\tau)_\varepsilon) = \mathrm{Op}^\tau((\sigma_\varepsilon)_\tau) = \mathrm{Op}(\sigma_\varepsilon)$ strongly converges on $\mathcal{S}(G)$ to $\mathrm{Op}(\sigma)$ and $\sigma \in S^m_{\rho,\delta}(G)$, this gives

$$\mathrm{Op}^\tau(\sigma'_\tau)f = \lim_\varepsilon \mathrm{Op}^\tau\big((\sigma_\tau)_\varepsilon\big)f = \lim_\varepsilon \mathrm{Op}(\sigma_\varepsilon)f = \mathrm{Op}(\sigma)f$$

for all $f \in \mathcal{S}(G)$. By the Schwartz kernel theorem, the kernels, and hence the associated kernels, in $\mathcal{S}'(G \times G)$ of these two continuous operators coincide. The extension of (4.11) to $\mathcal{S}'(G \times G)$ by duality thus yields $\sigma'_\tau = \sigma_\tau \in S^m_{\rho,\delta}(G)$. This proves the continuity of $\sigma \mapsto \sigma_\tau\colon S^m_{\rho,\delta}(G) \to S^m_{\rho,\delta}(G)$.

Due to (4.18), we can apply these arguments verbatim to show that also

$$\sigma_\tau \mapsto \sigma\colon S^m_{\rho,\delta}(G) \to S^m_{\rho,\delta}(G)$$

is continuous.

Clearly, the identities (4.17) and (4.11) establish a one-to-one correspondence between the associated kernels and by applying the Fourier transform in y also between the symbols (cf. (3.4) and (4.12)). Together with the continuity of the map, this yields a vector space isomorphism $\sigma \mapsto \sigma_\tau$ from $S^{m-(\rho-\delta)(M+1)}_{\rho,\delta}(G)$ onto itself, which completes the proof. ■

Example 4.12. For the Weyl quantization on $G = \mathbb{R}^n$, the asymptotic expansion (4.8) recovers the well-known relation

$$\sigma_{\mathrm{W}} \sim \sum_{|\alpha|\leq M} \frac{2^{|\alpha|}}{\alpha!}(i\partial_\xi)^\alpha \partial_x^\alpha \sigma_{\mathrm{KN}},$$

since the coefficients of both series are immediately seen to coincide due to the Abelian structure of G and (4.10). (Cf., e.g., [31, Thm. 2.41] or [40, Lem. 4.1.5].)

In addition to the continuity of τ-quantized operators on $\mathcal{S}(G)$, and consequently on $\mathcal{S}'(G)$, due to Proposition 4.9, we can immediately extend several crucial continuity results

[21] To employ Proposition 4.9, we require $\delta < \frac{1}{v_n}$, unlike in the preceding part of the proof.

of the Kohn–Nirenberg calculus. For the definitions and properties of the Hardy space $H^1(G)$ and its dual space $\mathrm{BMO}(G)$ we refer to [33]; for the definitions and embedding properties of the inhomogeneous Besov spaces $B^s_{p,q}(G)$, $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$ we refer to [9].

Theorem 4.13. *Let $m \in \mathbb{R}$, $0 \le \delta < \min\{\rho, \frac{1}{v_n}\} \le 1$,*[22] *and let τ satisfy (HP). Then for all $s \in \mathbb{R}$, $p \in (1,\infty)$, $q \in (0,\infty]$, and $m_p := Q(1-\rho)|\frac{1}{2} - \frac{1}{p}|$ the operator $T = \mathrm{Op}^\tau(\sigma)$ lies in*

(i) $\mathcal{L}(L^2_s(G), L^2_{s-m}(G))$ *for each* $\sigma \in S^m_{\rho,\delta}(G)$;

(ii) $\mathcal{L}(L^p_s(G), L^p_{s-m}(G))$ *for each* $\sigma \in S^m_{1,0}(G)$;

(iii) $\mathcal{L}(H^1(G), L^1(G))$ *for each* $\sigma \in S^{-m}_{\rho,\delta}(G)$ *with* $m = \frac{Q(1-\rho)}{2}$;

(iv) $\mathcal{L}(\mathrm{BMO}(G), L^\infty(G))$ *for each* $\sigma \in S^{-m}_{\rho,\delta}(G)$ *with* $m = \frac{Q(1-\rho)}{2}$;

(v) $\mathcal{L}(L^p(G))$ *for each* $\sigma \in S^{-m}_{\rho,\delta}(G)$ *with* $m \ge m_p$;

(vi) $\mathcal{L}(B^s_{p,q}(G))$ *for each* $\sigma \in S^{-m}_{\rho,\delta}(G)$ *with* $m \ge m_p$.
In particular, this holds for $L^p_s(G) = B^s_{p,2}(G)$.

Proof. Since by Theorem 4.11, every T in (i) and (ii) can be uniquely written as an operator $T = \mathrm{Op}(\sigma')$ with σ' in $S^m_{\rho,\delta}(G)$ and $S^m_{1,0}(G)$, respectively, the statements (i)–(vi) follow directly from [28, Cor. 5.7.2], [28, Cor. 5.7.4], [7, Thm. 1.2], and [7, Thm. 4.18], respectively. ■

In a similar fashion, one can extend the Gårding inequality for elliptic Kohn–Nirenberg quantized operators, due to [8, Thm. 4.5], to general τ-quantizations. We refer the interested reader to [21, Thm. 6.3].

4.3. The symbol of the adjoint

In this subsection, we show that the symbol of the formal adjoint of an operator $\mathrm{Op}^\tau(\sigma)$ with $\sigma \in S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, $0 \le \delta < \min\{\rho, \frac{1}{v_n}\}$, is in the same symbol class, and we provide an explicit asymptotic expansion for it. Below we shall denote by $\sigma^{(*)}_\tau$ the symbol of the adjoint of $\mathrm{Op}^\tau(\sigma)$, and by σ^* the pointwise adjoint of the symbol σ. When $\tau = e_G$, we shall simply write $\sigma^{(*)}$ for $\sigma^{(*)}_{\tau=e_G}$.

Theorem 4.14. *Let $m \in \mathbb{R}$, $0 \le \delta < \min\{\rho, \frac{1}{v_n}\} \le 1$, and let $\tau: G \to G$ be a quantizing function which satisfies (HP). Let T be a continuous linear operator from $\mathcal{S}(G)$ to $\mathcal{S}'(G)$ given by $T = \mathrm{Op}^\tau(\sigma)$ for some $\sigma \in S^m_{\rho,\delta}(G)$. Then its formal adjoint T^*, defined with respect to the $\mathcal{S}(G)$-$\mathcal{S}'(G)$-duality, is given by $T^* = \mathrm{Op}^\tau(\sigma^{(*)}_\tau)$ for a uniquely determined symbol $\sigma^{(*)}_\tau \in S^m_{\rho,\delta}(G)$.*

[22] The assumption $\delta < \rho$ can be weakened to $\delta \le \rho$, $\delta \ne 1$, for the Kohn–Nirenberg calculus in several cases.

The map $\sigma \mapsto \sigma^{(*)}_\tau$ is a Fréchet space isomorphism from $S^m_{\rho,\delta}(G)$ onto itself, and the symbols are related to one another by the asymptotic expansion

$$\sigma^{(*)}_\tau \sim \sum_{j=0}^{\infty}\Big(\sum_{[\alpha]=j} \sum_{[\alpha']=[\alpha]} c^{(*),\tau}_{\alpha',\alpha}\, \Delta^{\alpha'} X^\alpha \sigma^* \Big), \tag{4.20}$$

in the sense that for given $M \in \mathbb{N}_0$

$$R^{\sigma^{(*)}_\tau}_M := \sigma^{(*)}_\tau - \sum_{[\alpha]\leq M} \sum_{[\alpha']=[\alpha]} c^{(*),\tau}_{\alpha',\alpha}\, \Delta^{\alpha'} X^\alpha \sigma^* \in S^{m-(\rho-\delta)(M+1)}_{\rho,\delta}(G).$$

The coefficients $c^{(*),\tau}_{\alpha',\alpha} \in \mathbb{R}$ are uniquely determined by the equations

$$q_\alpha\big(\tau(y)y^{-1}\tau(y^{-1})^{-1}\big) = \sum_{[\alpha']=[\alpha]} c^{(*),\tau}_{\alpha',\alpha}\, \tilde{q}_{\alpha'}(y).$$

For the quantizing function $\tau = e_G$ the expansion (4.20) recovers the asymptotic expansion

$$\sigma^{(*)} \sim \sum_{j=0}^{\infty}\Big(\sum_{[\alpha]=j} \Delta^\alpha X^\alpha \sigma^* \Big) \tag{4.21}$$

in the Kohn–Nirenberg calculus, which was established in [28, Cor. 5.5.17].

Moreover, the quantizing function τ is symmetric, i.e., it satisfies

$$T^* = \mathrm{Op}^\tau(\sigma)^* = \mathrm{Op}^\tau(\sigma^*) \tag{4.22}$$

and (4.20) collapses to $\sigma^{(*)}_\tau = \sigma^*$, if and only if τ satisfies

$$\tau(x) = \tau(x^{-1})x \tag{4.23}$$

for all $x \in G$.

In particular, if $T = \mathrm{Op}^\tau(\sigma)$ for some $\sigma \in S^0_{\rho,\delta}(G)$ and a symmetry function τ, then $T = T^* \in \mathcal{L}(L^2(G))$ if and only if $\sigma^{(*)}_\tau = \sigma^* = \sigma$.

Proof. As in the proof of Theorem 4.11 our proof crucially relies on the relation between the associated kernels. Note that it suffices to prove our statement for the dense subset of symbols $S^{-\infty}(G)$ whose associated kernels lie in $\mathcal{S}(G \times G)$ since the extension to symbols in $S^m_{\rho,\delta}(G)$ is identical to the one in the proof of Theorem 4.11.

Thus, let $m \in \mathbb{R}$, $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\}$, and let $\tau\colon G \to G$ be a quantizing function which satisfies (HP). If $\sigma \in S^{-\infty}(G)$ with $\kappa_\sigma \in \mathcal{S}(G \times G)$, then also the integral kernel Ker^τ_σ of $T = \mathrm{Op}^\tau(\sigma)$ is a member of $\mathcal{S}(G \times G)$. The integral kernel of its formal adjoint T^*, which we denote by $\mathsf{Ker}^\tau_{\sigma^{(*)}} \in \mathcal{S}'(G \times G)$, is explicitly related to Ker^τ_σ by $\mathsf{Ker}^\tau_{\sigma^{(*)}}(x, y) = \overline{\mathsf{Ker}^\tau_\sigma}(y, x)$, essentially due to the Schwartz kernel theorem. It immediately follows that

$\mathrm{Ker}^{\tau}_{\sigma^{(*)}} \in \mathcal{S}(G\times G)$ and that we can use $\mathrm{Ker}^{\tau}_{\sigma} = \kappa_\sigma \circ \mathrm{cv}^\tau$ to express the associated kernel by

$$\begin{aligned}\kappa_{\sigma_\tau^{(*)}}(x,y) &= \mathrm{Ker}^{\tau}_{\sigma_\tau^{(*)}}\big(x\tau(y), x\tau(y)y^{-1}\big) = \overline{\mathrm{Ker}^{\tau}_{\sigma}}\big(x\tau(y)y^{-1}, x\tau(y)\big)\\ &= \overline{\kappa_\sigma}\big(x\tau(y)y^{-1}\tau\big(\tau(y)^{-1}x^{-1}x\tau(y)y^{-1}\big)^{-1}, \tau(y)^{-1}x^{-1}x\tau(y)y^{-1}\big)\\ &= \overline{\kappa_\sigma}\big(x\tau(y)y^{-1}\tau(y^{-1})^{-1}, y^{-1}\big) = \kappa_{\sigma^*}\big(x\tau(y)y^{-1}\tau(y^{-1})^{-1}, y\big), \end{aligned} \tag{4.24}$$

where we recall from [28, Thm. 5.2.22] that the kernel associated to σ^* is given $\kappa_{\sigma^*}(x,y) = \overline{\kappa_\sigma}(x,y^{-1})$. We immediately notice that if τ satisfies (4.23), then

$$\tau(y)y^{-1}\tau(y^{-1})^{-1} = e_G \tag{4.25}$$

and (4.24) yields

$$\kappa_{\sigma_\tau^{(*)}}(x,y) = \kappa_{\sigma^*}(x,y) \iff \sigma_\tau^{(*)} = \sigma^*, \tag{4.26}$$

hence τ is symmetric. Conversely, if τ is symmetric, that is, if the condition (4.26) holds, then by the above computation the Schwartz functions $\mathrm{Ker}^{\tau}_{\sigma^{(*)}}$ and $\mathrm{Ker}^{\tau}_{\sigma^*}$ coincide only if (4.23) holds true. By duality, this argument extends to distributions in $\mathcal{S}'(G\times G)$. Now, since any operator $T = \mathrm{Op}^\tau(\sigma)$ with $\sigma \in S^m_{\rho,\delta}(G)$ is continuous from $\mathcal{S}(G)$ into $\mathcal{S}(G)$ by Proposition 4.9, the associated kernel lies in $\mathcal{S}'(G\times G)$ and the isomorphism property follows essentially verbatim as the last part of the proof of Theorem 4.11. To finish the part of the proof concerning symmetric τ, note that $T = \mathrm{Op}^\tau(\sigma)$ with $\sigma \in S^0_{\rho,\delta}(G)$ is continuous from $L^2(G)$ into itself (cf. Theorem 4.13 (i) below). Hence, if T is self-adjoint, then the kernels, and hence the associated kernels and also the symbols, of T and T^* coincide.

Let us continue the proof for a τ which is not necessarily symmetric. In order to derive an asymptotic expansion of $\sigma_\tau^{(*)}$ in terms of σ^*, we will employ a Taylor expansion of its associated kernel based on (4.24). Since we need not consider the trivial case (4.25), we may exclude symmetry functions. For any non-symmetric quantizing function τ, the condition (HP) and the gradation of $\mathfrak{g}$ ensure that for at least one $j = 1,\dots,n$, the j-th coordinate of $\tau(y)y^{-1}\tau(y^{-1})^{-1}$ is given by

$$(2C_j^\tau - 1)y_j + p_j(y_1,\dots,y_{j-1}) \tag{4.27}$$

for some $C_j^\tau \neq \frac{1}{2}$ and some homogeneous polynomial p_j of degree v_j, which only depends on the variables $y_1,\dots,y_{j-1}$, while the other coordinates may vanish. This yields a non-trivial Taylor expansion

$$\begin{aligned}\kappa_{\sigma_\tau^{(*)}}(x,y) = &\sum_{[\alpha]\le M}\sum_{[\alpha']=[\alpha]} c^{(*),\tau}_{\alpha',\alpha}\,\tilde{q}_{\alpha'}(y) X^\alpha_{x_1=x}\kappa_{\sigma^*}(x_1,y)\\ &+ \sum_{M+1\le[\alpha]\le M_0}\sum_{[\alpha']=[\alpha]} c^{(*),\tau}_{\alpha',\alpha}\,\tilde{q}_{\alpha'}(y) X^\alpha_{x_1=x}\kappa_{\sigma^*}(x_1,y)\\ &+ R^{\kappa_{\sigma^*}(\cdot,y)}_{x,M_0}\big(\tau(y)y^{-1}\tau(y^{-1})^{-1}\big)\end{aligned}$$

for some $M_0 \leq M + 1$. Since $|\tau(y)y^{-1}\tau(y^{-1})^{-1}| \lesssim |y|$ due to (4.27), the rest of the proof now follows verbatim the last part of the proof of Theorem 4.11 from (4.13) on.

Finally, note that for $\tau = e_G$ the identity (4.24) reduces to $\kappa_{\sigma_\tau^{(*)}}(x, y) = \kappa_{\sigma^*}(xy, y)$. Consequently, the Taylor coefficients satisfy

$$c_{\alpha',\alpha}^{(*),\tau} = \delta_{\alpha',\alpha} = \begin{cases} 1 & \text{if} \alpha' = \alpha, \\ 0 & \text{otherwise}, \end{cases}$$

which yields the asymptotic expansion (4.21) as expected. This completes the proof. ∎

Remark 4.15. We wish to remark that the result of Theorem 4.14 can also be proved by using an alternative strategy. For instance, one could use the change of quantization from Theorem 4.11 in combination with the asymptotic formula for the symbol of the adjoint operator in the Kohn–Nirenberg quantization in [28], very similarly to how this is done in [31, Thm. 2.52] for the Euclidean Weyl and Kohn–Nirenberg quantizations. This indirect proof would shorten the current work substantially. For this reason, to avoid lengthy calculations, we present the two strategies to get the asymptotic formula for the composite symbol in the equivalent results in Theorems 4.18 and 4.19, sketching the long one based on the use of associated kernels. Details can be found in the self-contained version of this paper in [21].

Example 4.16. Since for $G = \mathbb{R}^n$ the quantizing function $\tau(x) = \frac{x}{2}$ clearly satisfies (4.23), the identity (4.22) simply recovers the property (1.2) of the Weyl quantization. Note that Theorem 4.14 also recovers the classical fact that among the τ-quantizations (1.4), defined by the quantizing functions $\tau(x) = \tau x$, $\tau \in [0, 1]$, the Weyl quantization $\mathrm{Op}^{\mathrm{w}} = \mathrm{Op}^{\frac{1}{2}}$ is the only one that satisfies (4.22).

Example 4.17. On $G = \mathbb{H}_n$, Theorem 4.14 holds true for all members of the family of symmetry functions (1.16), which by Example 4.5 is precisely the set of all symmetry functions $\tau: \mathbb{H}_n \to \mathbb{H}_n$ that satisfy (HP) with respect to the canonical homogeneous structure.

For the sake of completeness, let us recall the inner product formula for τ-quantized Hilbert–Schmidt operators on $L^2(G)$ established in [41, §3.2] in the context of Theorem 4.14. If σ_1, σ_2 are symbols on $G \times \widehat{G}$ given by measurable fields of operators whose associated kernels $\kappa_{\sigma_1}, \kappa_{\sigma_2}$ lie in $L^2(G \times G)$, then the operators $\mathrm{Op}^\tau(\sigma_1), \mathrm{Op}^\tau(\sigma_2)$ are Hilbert–Schmidt operators with $\sigma_j(x, \pi) \in (\mathtt{HS})(\mathcal{H}_\pi)$ for $j = 1, 2$ and a.e. $x \in G, \pi \in \widehat{G}$. The Hilbert–Schmidt inner product of such operators can be expressed as

$$\big\langle \mathrm{Op}^\tau(\sigma_1), \mathrm{Op}^\tau(\sigma_2) \big\rangle_{\mathtt{HS}} = \iint\limits_{\widehat{G} \times G} \mathrm{Tr}\big(\sigma_1(x,\pi)\sigma_2^*(x,\pi)\big)\, dx\, d\mu(\pi) = \langle \sigma_1, \sigma_2 \rangle_{L^2(G \times \widehat{G})} \quad (4.28)$$

for any measurable $\tau: G \to G$. Hence, this formula holds in particular for any τ admissible in sense of Section 4.1 and any two Hilbert–Schmidt operators $\mathrm{Op}^\tau(\sigma_1), \mathrm{Op}^\tau(\sigma_2)$ with symbols $\sigma_1, \sigma_2 \in S^\infty(G)$.

4.4. The composition of symbols

In this subsection, we prove the main result of the paper. It states that if the quantizing function τ satisfies the condition (HP), then the associated composition of symbols $\circ_\tau$ is continuous on the Hörmander symbol classes $S^m_{\rho,\delta}(G)$, $m \in \mathbb{R}$, $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq 1$, and the composite symbol is approximately given by an asymptotic expansion. We stress once again that the fundamental case $(\rho, \delta) = (1, 0)$ is covered by our result.

We will present two versions of the asymptotic formula for the composite symbol, whose main difference lies in their respective derivations. Given a sufficiently long asymptotic expansion in each case, their respective coefficients coincide for all multi-indices of length less or equal any arbitrary but fixed natural number, but the remainder terms in question may well differ. A short proof that combines the Kohn–Nirenberg composite formula, established in [28], and the continuous change of quantization, given by Theorem 4.11, suffices to prove the first version of the composite formula. The second version, which does not rely on previously established results for the Kohn–Nirenberg quantization, is based on tedious calculations and kernel estimates. Due to the considerable length of the proof, we shall merely explain the key proof steps. For a complete proof, we refer the interested reader to the unabridged preprint version of this paper [21]. Since these two approaches rely on very different proof strategies, the corresponding asymptotic formulas hold for somewhat different ranges of ρ and δ. To be precise, the admissible range is larger in the first case because a more relaxed restriction of $0 \leq \delta < \rho \leq 1$ is inherited from the proof of Theorem 4.11. This advantage of the first approach is, however, counterbalanced by the fact that the second approach allows for a straight-forward computation of the asymptotic coefficients of the composite symbol, which is clearly of great importance for all applications of the calculus. Let us therefore emphasize that whenever we employ the second version of theorem, we shall restrict the range accordingly so that we can use the much more explicit version of the asymptotic coefficients.

Theorem 4.18. *Let $m_1, m_2 \in \mathbb{R}$, $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq 1$, and let the quantizing function τ satisfy* (HP). *Let T_1 and T_2 be continuous linear operators from $\mathcal{S}(G)$ to $\mathcal{S}(G)$ given by $T_1 = \mathrm{Op}^\tau(\sigma_1)$ and $T_2 = \mathrm{Op}^\tau(\sigma_2)$ for some $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$, respectively. Then there exists a uniquely determined symbol $\sigma \in S^{m_1+m_2}_{\rho,\delta}(G)$ such that $\mathrm{Op}^\tau(\sigma) = T_1 T_2$. If $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$, then the symbol σ is given by the absolutely convergent integral* (4.34).

The τ-composition of symbols

$$\circ_\tau \colon S^{m_1}_{\rho,\delta}(G) \times S^{m_2}_{\rho,\delta}(G) \to S^{m_1+m_2}_{\rho,\delta}(G),$$
$$(\sigma_1, \sigma_2) \mapsto \sigma =: \sigma_1 \circ_\tau \sigma_2$$

is a bilinear and continuous map, and the symbol σ is asymptotically given by

$$\sigma \sim \sum_{i,j,k=1}^{\infty} \Big(\sum_{[\alpha]=k} \sum_{\substack{[\beta_1]=i,\\ [\beta_2]=j}} \sum_{\substack{[\beta_1']=[\beta_1],\\ [\beta_2']=[\beta_2]}} c^{\mathrm{KN},\tau}_{\beta_1',\beta_1} c^{\mathrm{KN},\tau}_{\beta_2',\beta_2} \big(\Delta^\alpha \Delta^{\beta_1'} X^{\beta} \sigma_1\big) \big(\Delta^{\beta_2'} X^\alpha X^{\beta_2} \sigma_2\big)\Big),$$

in the sense that for given $M, N \in \mathbb{N}_0$ and $L := \min\{M, N\}$

$$R_L^\sigma := \sigma - \sum_{\substack{[\alpha]\le L,\\ [\beta_1]\le N\\ [\beta_2]\le M}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c^{\mathrm{KN},\tau}_{\beta_1',\beta_1} c^{\mathrm{KN},\tau}_{\beta_2',\beta_2} \big(\Delta^\alpha \Delta^{\beta_1'} X^\beta \sigma_1\big)\big(\Delta^{\beta_2'} X^\alpha X^{\beta_2}\sigma_2\big) \in S^{m_1+m_2-(\rho-\delta)(L+1)}_{\rho,\delta}(G).$$

Moreover, the coefficients $c^{\mathrm{KN},\tau}_{\beta_1',\beta_1} c^{\mathrm{KN},\tau}_{\beta_2',\beta_2} \in \mathbb{R}$ are uniquely determined by the equation in (4.10).

Proof. The proof is a direct consequence of the change of quantization formula in Theorem 4.11 and the asymptotic expansion for Kohn–Nirenberg composite symbols in [28, Cor. 5.5.8]. In particular, we can pass from the τ-symbols σ_1, σ_2 to the associated Kohn–Nirenberg symbols η_1, η_2 by Theorem 4.11. In other words, by (4.9), for $M, N \in \mathbb{N}$, we can find two symbols $\eta_1 \in S^{m_1}_{\rho,\delta}$, $\eta_2 \in S^{m_2}_{\rho,\delta}$ such that

$$\mathrm{Op}^\tau(\sigma_1) = \mathrm{Op}(\eta_1), \quad \mathrm{Op}^\tau(\sigma_2) = \mathrm{Op}(\eta_2),$$

where

$$\eta_1 = \sum_{j_1=0}^{N} \sum_{\substack{[\beta_1]=j_1,\\ [\beta_1']=[\beta_1]}} c^{\mathrm{KN},\tau}_{\beta_1',\beta_1} \Delta^{\beta_1'} X^{\beta_1}\sigma_1(x,\pi) + r_{m_1-(N+1)} := \eta_1' + r_{m_1-(N+1)} \in S^{m_1}_{\rho,\delta}(G),$$

$$\eta_2 = \sum_{j_2=0}^{M} \sum_{\substack{[\beta_2]=j_2,\\ [\beta_2']=[\beta_2]}} c^{\mathrm{KN},\tau}_{\beta_2',\beta_2} \Delta^{\beta_2'} X^{\beta_2}\sigma_2(x,\pi) + r_{m_2-(M+1)} := \eta_2' + r_{m_2-(M+1)} \in S^{m_1}_{\rho,\delta}(G),$$

and where the remainder terms r_{m_1-N-1}, r_{m_2-M-1} are symbols of order $m_1 - N, m_2 - N$, respectively. If we now employ the asymptotic formula in [28, Cor. 5.5.8], we get, for $L \in \mathbb{N}$, $L = \min\{N, M\}$,

$$\begin{aligned}
\sigma =: \sigma_1 \circ_\tau \sigma_2 &= (\eta_1) \circ_{\mathrm{KN}} (\eta_2) \\
&= \sum_{[\alpha]=0}^{L} \big(\Delta^\alpha(\eta_1' + r_{m_1-N})\big)\big(X^\alpha(\eta_2' + r_{m_2-M})\big) + r_{m_1+m_2-(L+1)} \\
&= \sum_{[\alpha]=0}^{L} (\Delta^\alpha \eta_1')(X^\alpha \eta_2') + r_{m_1+m_2-(\min\{N,M\}+1)} + r_{m_1+m_2-(L+1)},
\end{aligned}$$

and, by inserting the above identities for η_1 and η_2, the desired result. ■

The second, yet equivalent, version of our τ-asymptotic composition formula is given in Theorem 4.19 below. Apparently, the coefficients of the two formulas, the one above and the one below, look different. However, it is not hard to check, at least when some explicit calculations can be performed, for instance, when the group is stratified (so that one can exploit the knowledge of the group law and derive the homogeneous polynomials q_α), that the two results return the same expansion.

Theorem 4.19. *Let $m_1, m_2 \in \mathbb{R}$, $0 \le \delta < \frac{\rho}{v_n} \le 1$, and let the quantizing function τ satisfy* (HP). *Let T_1 and T_2 be continuous linear operators from $\mathcal{S}(G)$ to $\mathcal{S}(G)$ given by $T_1 = \mathrm{Op}^\tau(\sigma_1)$ and $T_2 = \mathrm{Op}^\tau(\sigma_2)$ for some $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$, respectively. Then there exists a uniquely determined symbol $\sigma \in S^{m_1+m_2}_{\rho,\delta}(G)$ such that $\mathrm{Op}^\tau(\sigma) = T_1T_2$. If $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G\times G)$, then the symbol σ is given by the absolutely convergent integral* (4.34).

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$$\begin{aligned}\circ_\tau\colon S^{m_1}_{\rho,\delta}(G) \times S^{m_2}_{\rho,\delta}(G) &\to S^{m_1+m_2}_{\rho,\delta}(G),\\ (\sigma_1,\sigma_2) &\mapsto \sigma =: \sigma_1 \circ_\tau \sigma_2\end{aligned}$$

is a bilinear and continuous map, and the symbol σ is asymptotically given by

$$\sigma \sim \sum_{i,j=1}^{\infty} \Big(\sum_{\substack{[\alpha]=i,\\ [\beta]=j}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \big(\Delta^{\alpha_2}\Delta^{\beta_1} X^\alpha \sigma_1\big)\big(\Delta^{\beta_2}\Delta^{\alpha_1} X^\beta \sigma_2\big)\Big), \tag{4.29}$$

in the sense that for given $M, N \in \mathbb{N}_0$ and $L := \min\{M,N\}$

$$\begin{aligned} R^\sigma_L := \sigma - \sum_{\substack{[\alpha]\le M,\\ [\beta]\le N}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} & c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \big(\Delta^{\alpha_2}\Delta^{\beta_1} X^\alpha \sigma_1\big)\big(\Delta^{\beta_2}\Delta^{\alpha_1} X^\beta \sigma_2\big)\\ &\in S^{m_1+m_2-(\rho-\delta)(L+1)}_{\rho,\delta}(G). \end{aligned} \tag{4.30}$$

Moreover, the coefficients $c_{\alpha_1,\alpha_2}, c_{\beta_1,\beta_2} \in \mathbb{R}$ are uniquely determined by the equations

$$q_\alpha\big(p_1(y,z)\big) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2} \tilde{q}_{\alpha_1}(z)\tilde{q}_{\alpha_2}(z^{-1}y), \tag{4.31}$$

$$q_\beta\big(p_2(y,z)\big) = \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2} \tilde{q}_{\beta_1}(z^{-1}y)\tilde{q}_{\beta_2}(z). \tag{4.32}$$

The proof of Theorem 4.14 requires some auxiliary lemmas.[23]

Lemma 4.20. *Let G be a graded group and let σ_1, σ_2 be smooth symbols whose associated kernels $\kappa_{\sigma_1}, \kappa_{\sigma_2}$ are Schwartz functions. If the quantizing function τ satisfies* (CV), *then*

$$\mathrm{Op}^\tau(\sigma_1)\mathrm{Op}^\tau(\sigma_2) = \mathrm{Op}^\tau(\sigma)$$

for a smooth symbol σ. The integral kernel and the associated kernel of $\mathrm{Op}^\tau(\sigma)$ are Schwartz functions on $G \times G$, given by the absolutely convergent integrals

$$\begin{aligned}\mathsf{Ker}^\tau_\sigma(x,y) &= \kappa_\sigma\big(x\tau(y^{-1}x)^{-1}, y^{-1}x\big)\\ &= \int_G \kappa_{\sigma_1}\big(x\tau(z^{-1}x)^{-1}, z^{-1}x\big)\kappa_{\sigma_2}\big(z\tau(y^{-1}z)^{-1}, y^{-1}z\big)\, dz,\end{aligned}$$

[23] Their proofs can be found in [21].

and

$$\kappa_\sigma(x,y) = \int_G \kappa_{\sigma_1}\big(x\tau(y)\tau(z^{-1}y)^{-1}, z^{-1}y\big)\kappa_{\sigma_2}\big(x\tau(y)y^{-1}z\tau(z)^{-1}, z\big)\,dz, \tag{4.33}$$

respectively. Explicitly, we have

$$\sigma(x,\pi) = \big(\mathcal{F}\big(\kappa_\sigma(x,\cdot)\big)\big)(\pi) \tag{4.34}$$

for all $x \in G$ and almost all $\pi \in \widehat{G}$.

In particular, for the symmetry function $\tau(x) = \exp(\frac{1}{2}\log(x))$ the identity (4.33) simplifies to

$$\kappa_\sigma(x,y) = \int_G \kappa_{\sigma_1}\big(x\tau(y)\tau(y^{-1}z), z^{-1}y\big)\kappa_{\sigma_2}\big(x\tau(y)^{-1}\tau(z), z\big)\,dz,$$

while for the constant quantizing function $\tau = e_G$ it recovers the composite kernel in the Kohn–Nirenberg quantization:

$$\kappa_\sigma(x,y) = \int_G \kappa_{\sigma_1}(x, z^{-1}y)\kappa_{\sigma_2}(xy^{-1}z, z)\,dz = \kappa_{\sigma_1\circ_{\mathrm{KN}}\sigma_2}(x,y). \tag{4.35}$$

We now define $p_1, p_2 \colon G \times G \to G$ as

$$\begin{aligned} p_1(y,z) &:= \tau(y)\tau(z^{-1}y)^{-1}, \\ p_2(y,z) &:= \tau(y)y^{-1}z\tau(z)^{-1}. \end{aligned} \tag{4.36}$$

These functions, that appear in the following crucial lemmas, will play a prominent role in the proof of Theorem 4.19.

Lemma 4.21. *Let σ_1, σ_2 be smooth symbols with $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G\times G)$, and let $m_1, m_2 \in \mathbb{R}$. For every arbitrary but fixed $M_0 \in \mathbb{N}$, let τ_{M_0} be the symbol defined by*

$$\tau_{M_0}(x,\pi) := \iint_{G\times G} R^{\kappa_{\sigma_1}(\cdot,s)}_{x,M_0}\big(p_1(y, ys^{-1})\big)\kappa_{\sigma_2}(x, ys^{-1})\pi(y)^* dy\, ds.$$

Then for all $\alpha_0, \beta_0 \in \mathbb{N}_0^{2n+1}$ and for all $\rho, \delta \in [0,1]$ such that $0 \le \delta < \min\{\rho, \frac{1}{v_n}\}$ there exist a constant $C > 0$ and two seminorms $\|\cdot\|_{S^{m_1}_{\rho,\delta},a_1,b_1}$, $\|\cdot\|_{S^{m_2}_{\rho,\delta},a_2,b_2}$ such that

$$\sup_{(x,\pi)\in G\times\widehat{G}} \Big\|(X_x^{\beta_0}\Delta^{\alpha_0}\tau_{M_0})(x,\pi)\pi(I+\mathcal{R})^{-\frac{m_1+m_2-(\rho-\delta)(M_0+1)-\rho[\alpha_0]+\delta[\beta_0]}{\nu}}\Big\|_{\mathcal{L}(\mathcal{H}_\pi)} \le C\,\|\sigma_1\|_{S^{m_1}_{\rho,\delta},a_1,b_1}\|\sigma_2\|_{S^{m_2}_{\rho,\delta},a_2,b_2}.$$

Lemma 4.22. *Let σ_1, σ_2 be smooth symbols with $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G\times G)$, and let $m_1, m_2 \in \mathbb{R}$. For every arbitrary but fixed $M_0 \in \mathbb{N}$, let τ_{M_0} be the symbol defined by*

$$\tau_{M_0}(x,\pi) := \iint_{G\times G} R^{\kappa_{\sigma_2}(\cdot,s)}_{x,M_0}\big(p_2(y,s)\big)\kappa_{\sigma_1}(x, s^{-1}y)\pi(y)^* dy\, ds.$$

Then for all $\alpha_0, \beta_0 \in \mathbb{N}_0^n$ *and* $\rho, \delta \in [0,1]$ *such that* $0 \le \delta < \min\{\rho, \frac{1}{v_n}\} \le 1$ *there exist a constant* $C > 0$ *and two seminorms* $\|\cdot\|_{S^{m_1}_{\rho,\delta},a_1,b_1}$, $\|\cdot\|_{S^{m_2}_{\rho,\delta},a_2,b_2}$ *such that*

$$\big\|(X^{\beta_0}\Delta^{\alpha_0}\tau_{M_0})(x,\pi)\pi(I+\mathcal{R})^{-\frac{m_1+m_2-(\rho-\delta)(M_0+1)-\rho[\alpha_0]+\delta[\beta_0]}{\nu}}\big\|_{\mathcal{L}(\mathcal{H}_\pi)} \le C\|\sigma_1\|_{S^{m_1}_{\rho,\delta},a_1,b_1}\|\sigma_2\|_{S^{m_2}_{\rho,\delta},a_2,b_2}.$$

Lemma 4.23. *Let* σ_1, σ_2 *be smooth symbols with* $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$*, and let* $m_1, m_2 \in \mathbb{R}$ *and* $M_0, N_0 \in \mathbb{N}$*. Let also* r_{M_0,N_0} *be the symbol defined by*

$$r_{M_0,N_0}(x,\pi) := \int_{G\times G} R^{\kappa_{\sigma_1}(\cdot,s^{-1}y)}_{x,M_0}\big(p_1(y,s)\big)R^{\kappa_{\sigma_2}(\cdot,s)}_{x,N_0}\big(p_2(y,s)\big)\pi(y)^*\,ds\,dy.$$

Then for all $\alpha_0, \beta_0 \in \mathbb{N}_0^n$ *and for all* $\rho, \delta \in [0,1]$ *such that* $0 \le \delta < \frac{\rho}{v_n} \le 1$*, there exist a constant* $C > 0$ *and two seminorms* $\|\cdot\|_{S^{m_1}_{\rho,\delta},a_1,b_1}$, $\|\cdot\|_{S^{m_2}_{\rho,\delta},a_2,b_2}$ *such that*

$$\big\|(X^{\beta_0}\Delta^{\alpha_0}r_{M_0,N_0})(x,\pi)\pi(I+\mathcal{R})^{-\frac{m_1+m_2-(\rho-\delta)(\min\{M_0,N_0\}+1)-\rho[\alpha_0]+\delta[\beta_0]}{\nu}}\big\|_{\mathcal{L}(\mathcal{H}_\pi)} \le C\|\sigma_1\|_{S^{m_1}_{\rho,\delta},a_1,b_1}\|\sigma_2\|_{S^{m_2}_{\rho,\delta},a_2,b_2}.$$

Remark 4.24. We would like to highlight the curious fact that our asymptotic formula depends on two delimiting parameters, M and N, in contrast to the asymptotic expansions for the Kohn–Nirenberg quantization in [28] and the Weyl quantization on $\mathbb{R}^n$. The absence of a second parameter in the case of the Kohn–Nirenberg quantization is a direct consequence of (4.35). In the case of the Euclidean Weyl quantization, the asymptotic series of the composite symbol can be written as a power series of Poisson brackets, due to the symplectic nature of phase space, which allows the two parameters M and N to be merged into one.

This difference motivates the arguments in the proof of Part II of Theorem 4.19 and in the proof of Step 2 of Lemma 4.21, Lemma 4.22 and Lemma 4.23, where we prove that the remainder terms belong to symbol classes whose order depends on both M and N, since $L = L(M,N) := \min\{M,N\}$, and not just on one parameter L independent of M and N. Hence, to make the proof work one has to carefully take into account both the parameters.

Proof of Theorem 4.19*, Part I.* To prove the theorem, we will make crucial use of the relation (3.8) between the symbol σ of a given τ-quantized operator $\mathrm{Op}^\tau(\sigma)$ and its associated kernel κ_σ in combination with the identity (4.33) for the composite associated kernel. Since for $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$ the integrals in (3.8) and (4.33) converge absolutely and we may differentiate under the integral sign, we will first prove the asymptotic expansion and accompanied estimates for the dense subset of symbols $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ with associated kernels $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$.[24] In particular, we will show that for such σ_1, σ_2,

[24]This set is dense in each $S^m_{\rho,\delta}(G)$ with $m \in \mathbb{R}$, $0 \le \delta < \rho \le 1$, due to [28, Lem. 5.4.11].

the $S^{m_1+m_2-(\rho-\delta)(\min\{M,N\}+1)}_{\rho,\delta}$ seminorms of the remainder term R^σ_L are bounded by products of suitable $S^{m_1}_{\rho,\delta}$ and $S^{m_2}_{\rho,\delta}$ seminorms of σ_1 and σ_2, respectively, thus showing that $R^\sigma_L \in S^{m_1+m_2-(\rho-\delta)(\min\{M,N\}+1)}_{\rho,\delta}$. This will turn out to be crucial for the final part of the proof, an approximation argument that ensures that the continuity of the composition $\circ_\tau$ and the asymptotic expansion (4.29) hold on the whole space $S^{m_1}_{\rho,\delta}(G) \times S^{m_2}_{\rho,\delta}(G)$.

Thus, let $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ with associated kernels $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$. In order to employ the homogeneous Taylor expansions of κ_{σ_1} and κ_{σ_2} to derive (4.29), we observe that by (HP) the functions $p_1, p_2 \colon G \times G \to G$ defined by (4.36) are smooth and such that, for every $j = 1, \ldots, n$, their j-th coordinate is a homogeneous polynomial of degree v_j in two variables. So, for arbitrary but fixed orders $M, N \in \mathbb{N}$, we can employ (2.3) to obtain the homogeneous Taylor expansions

$$\kappa_{\sigma_1}\big(xp_1(y,z), z^{-1}y\big) = \sum_{[\alpha]\le M} q_\alpha\big(p_1(y,z)\big) X^\alpha_{x_1=x}\kappa_{\sigma_1}(x_1, z^{-1}y) + R^{\kappa_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}\big(p_1(y,z)\big),$$

$$\kappa_{\sigma_2}\big(xp_2(y,z), z\big) = \sum_{[\beta]\le N} q_\beta\big(p_2(y,z)\big) X^\beta_{x_2=x}\kappa_{\sigma_2}(x_2, z) + R^{\kappa_{\sigma_2}(\cdot,z)}_{x,N}\big(p_2(y,z)\big),$$

where $R^{\kappa_{\sigma_1}}_{x,M}$ and $R^{\kappa_{\sigma_2}}_{x,N}$ satisfy (2.5). Since we know that $q_\alpha \circ p_1, q_\beta \circ p_2 \colon G \times G \to \mathbb{R}$ are homogeneous polynomials of degree $[\alpha]$ and $[\beta]$, respectively, we may rewrite[25] $q_\alpha \circ p_1$ as in (4.31) and $q_\beta \circ p_2$ as in (4.32). With this at hand, we can now expand the composite kernel (4.33) as

$$\begin{aligned}\kappa_\sigma(x,y) = &\sum_{\substack{[\alpha]\le M,\\ [\beta]\le N}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2} c_{\beta_1,\beta_2} \int_G \tilde{q}_{\alpha_2}(z^{-1}y)\tilde{q}_{\beta_1}(z^{-1}y) X^\alpha_{x_1=x}\kappa_{\sigma_1}(x_1, z^{-1}y) \\ &\qquad\qquad\qquad\qquad \times \tilde{q}_{\beta_2}(z)\tilde{q}_{\alpha_1}(z) X^\beta_{x_2=x}\kappa_{\sigma_2}(x_2,z)\, dz \\ &+ \sum_{[\beta]\le N} \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2} \int_G \tilde{q}_{\beta_1}(z^{-1}y)\tilde{q}_{\beta_2}(z) X^\beta_{x_2=x}\kappa_{\sigma_2}(x_2,z) R^{\kappa_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}\big(p_1(y,z)\big)\, dz \\ &+ \sum_{[\alpha]\le N} \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2} \int_G \tilde{q}_{\alpha_1}(z)\tilde{q}_{\alpha_2}(z^{-1}y) X^\alpha_{x_1=x}\kappa_{\sigma_1}(x_1, z^{-1}y) R^{\kappa_{\sigma_2}(\cdot,z)}_{x,N}\big(p_2(y,z)\big)\, dz \\ &+ \int_G R^{\kappa_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}\big(p_1(y,z)\big) R^{\kappa_{\sigma_2}(\cdot,z)}_{x,N}\big(p_2(y,z)\big)\, dz.\end{aligned}$$

These very distinct types of singular integrals suggest to split the associated kernel into four terms and to treat each of them as an associated kernel of a symbol of specific type. To see that the three latter contributions together form the remainder $R^\sigma_L(x,\pi)$ defined by (4.30), we observe that

$$\int_G \tilde{q}_{\alpha_2}(z^{-1}y)\tilde{q}_{\beta_1}(z^{-1}y) X^\alpha_x \kappa_{\sigma_1}(x, z^{-1}y)\pi(z^{-1}y)^*\, dy = \big(\Delta^{\alpha_1}\Delta^{\beta_1} X^\alpha \sigma_1\big)(x,\pi),$$

[25] Note that we choose the homogeneous polynomials $\tilde{q}_\gamma(z) = q_\gamma(z^{-1}) = (-1)^{[\gamma]} q_\gamma(z)$, $\gamma = \alpha_1, \alpha_2, \beta_1, \beta_2$, over the polynomials q_γ in the expansions because they "quantize" difference operators. Alternatively, one can work with the q_γ, replace them afterwards by the $\tilde{q}_\gamma$ and absorb the additional factors $(-1)^{[\gamma]}$, a posteriori, via the constants $c_{\alpha_1,\alpha_2}, c_{\beta_1,\beta_2}$.

and that

$$\int_G \tilde{q}_{\beta_2}(z)\tilde{q}_{\alpha_1}(z)\tilde{q}_{\beta_2}(z)X_x^\beta \kappa_{\sigma_2}(x,z)\pi(z)^*\,dz = \big(\Delta^{\alpha_2}\Delta^{\beta_2}X^\beta\sigma_2\big)(x,\pi),$$

since this implies that the Fourier transform of the associated kernel

$$\kappa_{T_0}(x,y) := \sum_{\substack{[\alpha]\le M,\\ [\beta]\le N}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}c_{\beta_1,\beta_2}\int_G \tilde{q}_{\alpha_2}(z^{-1}y)\tilde{q}_{\beta_1}(z^{-1}y)X^\alpha_{x_1=x} \times \kappa_{\sigma_1}(x_1,z^{-1}y)\tilde{q}_{\beta_2}(z)\tilde{q}_{\alpha_1}(z)\tilde{q}_{\beta_2}(z)X^\beta_{x_2=x}\kappa_{\sigma_2}(x_2,z)\,dz$$

equals

$$\begin{aligned} T_0(x,\pi) &= \int_G \kappa_{T_0}(x,y)\pi(y)^*\,dy = \iint\limits_{G\times G} \kappa_{T_0}(x,y)\pi(z^{-1}y)^*\,dy\,\pi(z)^*\,dz \\ &= (\sigma - R_L^\sigma)(x,\pi), \end{aligned}$$

as desired. Since this is a finite sum of (products of) symbols of decreasing order (cf. Remark 3.2 for the product of symbols), nothing more has to be said about the symbol T_0 until the final part of the proof, where we justify the extension from $\sigma_1,\sigma_2 \in S^{-\infty}(G)$ with $\kappa_{\sigma_1},\kappa_{\sigma_2} \in \mathcal{S}(G\times G)$ to $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$, $\sigma_2 \in S^{m_1}_{\rho,\delta}(G)$. The seminorm estimates for the remaining symbols

$$T_1(x,\pi) := \sum_{\substack{[\beta]\le N,\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\beta_1,\beta_2} \iint\limits_{G\times G} \tilde{q}_{\beta_1}(z^{-1}y)\tilde{q}_{\beta_2}(z)X^\beta_{x_2=x}\kappa_{\sigma_2}(x_2,z) \times R^{\kappa_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}\big(p_1(y,z)\big)\,dz\,\pi(y)^*dy, \quad (4.37)$$

$$T_2(x,\pi) := \sum_{\substack{[\alpha]\le M,\\ [\alpha_1]+[\alpha_2]=[\alpha]}} c_{\alpha_1,\alpha_2} \iint\limits_{G\times G} \tilde{q}_{\alpha_1}(z)\tilde{q}_{\alpha_2}(z^{-1}y)X^\alpha_{x_1=x}\kappa_{\sigma_1}(x_1,z^{-1}y) \times R^{\kappa_{\sigma_2}(\cdot,z)}_{x,N}\big(p_2(y,z)\big)\,dz\,\pi(y)^*dy, \quad (4.38)$$

$$T_3(x,\pi) := \iint\limits_{G\times G} R^{\kappa_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}\big(p_1(y,z)\big)R^{\kappa_{\sigma_2}(\cdot,z)}_{x,N}\big(p_2(y,z)\big)\,dz\pi(y)^*\,dy, \quad (4.39)$$

which will justify the validity of (4.29), will constitute Part II of the proof. ■

Proof of Theorem 4.19*, Part II.* In this part of the proof we will show that each of the $S^{m_1+m_2-(\rho-\delta)(L+1)}(G)$-seminorms of the symbols T_j, $j=1,2,3$, defined by (4.37)–(4.39), are finite and bounded by the product $\|\sigma_1\|_{S^{m_1}_{\rho,\delta,a_1,b_1}}\|\sigma_2\|_{S^{m_2}_{\rho,\delta,a_2,b_2}}$ for suitable a_1,a_2, $b_1,b_2 \in \mathbb{R}$. As a byproduct, this will give $R_L^\sigma \in S^{m_1+m_2-(\rho-\delta)(L+1)}(G)$ when σ_1,σ_2 are smooth symbols with Schwartz kernels. Moreover, the estimates we obtain here will be fundamental in the final part of the proof, where we deal with symbols $\sigma_j \in S^{m_j}_{\rho,\delta}(G)$, $j=1,2$.

After an auxiliary result, the necessary estimates of the remainder terms T_j, $j=1,2,3$, in the asymptotic expansion (4.29) are provided by Lemmas 4.21, 4.22 and 4.23, which are deeply based on kernel estimates.

Let us begin with the auxiliary lemma.

Lemma 4.25. *For any quantizing function $\tau\colon G \to G$ that satisfies (HP), the polynomial functions $p_1(y,z) = \tau(y)\tau(z^{-1}y)^{-1}$ and $p_2(y,z) = \tau(y)y^{-1}z\tau(z)^{-1}$ satisfy the following pointwise estimates for all $y, z \in G$:*

$$\big|p_1(y,z)\big| \lesssim |z| + \sum_{j=1}^{n} \sum_{\substack{[\alpha]+[\beta]=v_j\geq 2\\ \alpha,\beta\in\mathbb{N}^n}} |z|^{\frac{[\alpha]}{v_j}} |z^{-1}y|^{\frac{[\beta]}{v_j}}, \tag{4.40}$$

$$\big|p_2(y,z)\big| \lesssim |z^{-1}y| + \sum_{j=1}^{n} \sum_{\substack{[\alpha]+[\beta]=v_j\geq 2\\ \alpha,\beta\in\mathbb{N}^n}} |z|^{\frac{[\alpha]}{v_j}} |z^{-1}y|^{\frac{[\beta]}{v_j}}. \tag{4.41}$$

Remark 4.26. Note that (4.40) implies the useful estimate

$$\big|p_1(y,z)\big| \lesssim \big(|z| + |z|^{\frac{1}{v_n}}\big)\big(1 + |y^{-1}z| + |y^{-1}z|^{\frac{1}{v_n}}\big), \tag{4.42}$$

which we will use in the proof of Lemma 4.21. To see this, it suffices to observe that

$$|z| \lesssim \big(|z| + |z|^{\frac{1}{v_j}}\big) \quad \forall z \in G,$$

and that, for $\alpha, \beta \neq 0$ satisfying $[\alpha] + [\beta] = v_j \geq 2$, we have

$$|z|^{\frac{[\alpha]}{v_j}} |y|^{\frac{[\beta]}{v_j}} \lesssim \big(|z| + |z|^{\frac{1}{v_j}}\big)\big(|y^{-1}z| + |y^{-1}z|^{\frac{1}{v_j}}\big) \quad \forall y, z \in G.$$

Hence, using the latter inequalities in (4.40), we get (4.42).

An analogous estimate, based on (4.41), holds for p_2 with the roles of z and $y^{-1}z$ exchanged in (4.42).

Remark 4.27. Note that when $G = \mathbb{R}^n$ and $v_n = 1$, we have $v_1 = \cdots = v_n = 1$. So, the condition (HP) implies that τ is linear, hence $|p_1(y,z)| \lesssim |z|$, $|p_2(y,z)| \lesssim |y - z|$.

We can now start proving the estimates for the remainder terms T_1, T_2, T_3 defined by (4.37), (4.38) and (4.39), respectively.

The symbol T_1. We begin with the symbol T_1 defined by (4.37). For technical reasons that will become obvious soon, we employ the measure-preserving change of variables $(y,z) =: (y, ys^{-1})$ to rewrite $T_1(x,\pi)$ as

$$T_1(x,\pi) = \sum_{\substack{[\beta]\leq N,\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\beta_1,\beta_2} \iint_{G\times G} \tilde{q}_{\beta_1}(s)\tilde{q}_{\beta_2}(ys^{-1}) X^{\beta}_{x_2=x}\kappa_{\sigma_2}(x_2, ys^{-1}) \times R^{\kappa_{\sigma_1}(\cdot,s)}_{x,M}\big(p_1(y, ys^{-1})\big)\, ds\, \pi(y)^*\, dy.$$

For T_1 written in this form, the desired estimate is now provided by Lemma 4.21 above. Notice that the absence of the integrands $\tilde{q}_{\beta_2}$ is justified by the fact that while homogeneous polynomials in the variables of integration may improve the oscillatory behavior close to the origin, they never worsen it. To apply the same argument to the integrands $\tilde{q}_{\beta_1}$, we simply rewrite $\tilde{q}_{\beta_1}(s) = \overline{\sum}_{[\beta_1']+[\beta_1'']=[\beta_1']} \tilde{q}_{\beta_1'}(y)\tilde{q}_{\beta_1''}(y^{-1}s)$, due to (2.6).

The symbol T_2. The estimate of the symbol T_2 defined by (4.38) is completely analogous to the case of T_1 and based on Lemma 4.22 above.

The symbol T_3. To estimate the seminorms of the remaining symbol T_3, defined by (4.39), we need to employ a very different strategy, that is Lemma 4.23.

To summarize, we have proved the following so far: if $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ and such that $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$, then for every fixed $M, N \in \mathbb{N}$, $L := \min\{M, N\}$, and for all $a, b \in \mathbb{N}_0$, there exist $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$ such that

$$\|R_L^\sigma\|_{S^{m_1+m_2-(\rho-\delta)(L+1)}_{\rho,\delta,a,b}} = \Big\| \sum_{j=1}^{3} T_j \Big\|_{S^{m_1+m_2-(\rho-\delta)(L+1)}_{\rho,\delta,a,b}} \lesssim \|\sigma_1\|_{S^{m_1}_{\rho,\delta,a_1,b_1}} \|\sigma_2\|_{S^{m_2}_{\rho,\delta,a_2,b_2}},$$

where

$$R_L^\sigma := \sigma - \sum_{\substack{[\alpha]\le M,\\ [\beta]\le N}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \big(\Delta^{\alpha_2}\Delta^{\beta_1} X^\alpha \sigma_1\big)\big(\Delta^{\beta_2}\Delta^{\alpha_1} X^\beta \sigma_2\big).$$

This, in particular, implies that $R_L^\sigma \in S^{m_1+m_2-(\rho-\delta)(L+1)}_{\rho,\delta}(G)$, which is the desired conclusion of Part II.

Our final goal now is the extension of this result to the general case when $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$, $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$. This will be done in the third and final part of the proof. ∎

Proof of Theorem 4.19, Part III. To pass from $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ with $\kappa_{\sigma_1}, \kappa_{\sigma_2} \in \mathcal{S}(G \times G)$ to general $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$, $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$, we employ an argument based on the approximation of σ_1 and σ_2 by nets of smoothing symbols (see [28, Lem. 5.4.11]) as in the proof of Theorem 4.11. We refer to [21] for a complete proof in full detail. ∎

In order to recover in an easy way the well-known classical asymptotic expansions for the composite symbol in, e.g., the Weyl calculus on $\mathbb{R}^n$ or the Kohn–Nirenberg calculus on $\mathbb{H}_n$, it is convenient to rewrite (4.29) as

$$\sigma \sim \sum_{j=0}^{\infty} \Big(\sum_{[\alpha]+[\beta]=j} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \big(\Delta^{\alpha_2}\Delta^{\beta_1} X^\alpha \sigma_1\big)\big(\Delta^{\beta_2}\Delta^{\alpha_1} X^\beta \sigma_2\big)\Big). \tag{4.43}$$

Introducing the notations $\omega_j := \omega_j(\sigma_1, \sigma_2)$ and

$$\omega_j(\sigma_1, \sigma_2) := \sum_{[\alpha]+[\beta]=j} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha],\\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \big(\Delta^{\alpha_2}\Delta^{\beta_1} X^\alpha \sigma_1\big)\big(\Delta^{\beta_2}\Delta^{\alpha_1} X^\beta \sigma_2\big) \tag{4.44}$$

and

$$R_M = R_M(\sigma_1, \sigma_2) := \sigma - \sum_{j=0}^{M} \omega_j(\sigma_1, \sigma_2)$$

for $\omega_j(\sigma_1,\sigma_2) \in S^{m_1+m_2-(\rho-\delta)j}_{\rho,\delta}(G)$ and $R_M(\sigma_1,\sigma_2) \in S^{m_1+m_2-(\rho-\delta)(M+1)}_{\rho,\delta}(G)$ by Theorem 4.19, we may abbreviate the asymptotic expansion as

$$\sigma \sim \sum_{j=0}^{\infty} \omega_j .$$

Note that each symbol appearing in this expression belongs to $S^{m_1+m_2-(\rho-\delta)j}_{\rho,\delta}(G)$.

In what follows we shall use the symbolic asymptotic composition formula as written in (4.43) to recover some well-known classical results.

Example 4.28. Let G be an arbitrary graded group. Choosing the constant quantizing function $\tau = e_G$, we get $p_1(y,z) = e_G$ and $p_2(y,z) = y^{-1}z$. It follows that $c_{\alpha_1,\alpha_2} = \delta_{\alpha_1,0}\,\delta_{\alpha_2,0}$ for all $\alpha \in \mathbb{N}_0$, and that $c_{\beta_1,\beta_2} = \delta_{\beta_2,0}$ for all $\beta \in \mathbb{N}_0$, thus (4.43) recovers the asymptotic expansion

$$\sigma_1 \circ_{\mathrm{KN}} \sigma_2 \sim \sum_{j=0}^{\infty} \sum_{[\beta]=j} (\Delta^\beta \sigma_1)(X^\beta \sigma_2)$$

in the *Kohn–Nirenberg quantization* due to [28, Cor. 5.58]. For the special case $G = \mathbb{R}^n$, this gives the well-known expansion

$$\sigma_1 \circ_{\mathrm{KN}} \sigma_2 \sim \sum_{j=0}^{\infty} \sum_{|\beta|=j}^{\infty} \big((i^{-1}\partial_\xi)^\beta \sigma_1\big)(\partial_x^\beta \sigma_2)$$

for all symbols $\sigma_1 \in S^{m_1}_{\rho,\delta}(\mathbb{R}^n)$, $\sigma_2 \in S^{m_1}_{\rho,\delta}(\mathbb{R}^n)$ with $m_1, m_2 \in \mathbb{R}$, $0 \le \delta < \rho \le 1$.

Example 4.29. When $G = \mathbb{R}^n$ and the symmetry function $\tau(x) = \frac{x}{2}$, i.e., the group is Abelian and τ symmetric and linear, we have

$$\begin{aligned} p_1(y,z) &= \tau(y)\tau(z^{-1}y)^{-1} = \frac{y}{2} - \frac{y-z}{2} = \frac{z}{2} = \tau(z), \\ p_2(y,z) &= \tau(y)y^{-1}z\tau(z)^{-1} = \frac{y}{2} - y + z - \frac{z}{2} = \frac{z-y}{2} = \tau(z^{-1}y)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} q_\alpha\big(p_1(y,z)\big) &= q_\alpha\left(\frac{z}{2}\right) = \frac{z^\alpha}{2^{|\alpha|}\alpha!}, \\ q_\beta\big(p_2(y,z)\big) &= q_\beta\left(\frac{z-y}{2}\right) = (-1)^{|\beta|}\frac{(y-z)^\beta}{2^{|\beta|}\beta!}, \end{aligned}$$

so the expansion (4.43) recovers the classical asymptotic expansion

$$\sigma_1 \circ_{\mathrm{w}} \sigma_2 \sim \sum_{j=0}^{\infty} \sum_{|\alpha|+|\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!2^{|\alpha|+|\beta|}} \big((i^{-1}\partial_\xi)^\beta \partial_x^\alpha \sigma_1\big)\big((i^{-1}\partial_\xi)^\alpha \partial_x^\beta \sigma_2\big)$$

in the *Weyl quantization* (1.1) for all symbols $\sigma_1 \in S^{m_1}_{\rho,\delta}(\mathbb{R}^n)$, $\sigma_2 \in S^{m_1}_{\rho,\delta}(\mathbb{R}^n)$ with $m_1, m_2 \in \mathbb{R}$, $0 \le \delta < \rho \le 1$.

Example 4.30. A curious example on $G = \mathbb{R}^2$ arises from $\tau(x_1, x_2) = (0, \frac{x_2}{2})$, which defines a sort of *hybrid Kohn–Nirenberg-Weyl quantization*, which is not included in the classical τ-quantizations (1.4). By the observations made in Examples 4.28 and 4.29, the corresponding asymptotic expansion (4.29) is given by

$$\sigma_1 \circ_\tau \sigma_2 \sim \sum_{j=0}^{\infty} \sum_{l_1+l_2+k_2=j} \frac{(-1)^{l_2}}{k_2! l_2! 2^{k_2+l_2}} \big((i^{-1}\partial_{\xi_1})^{l_1}(i^{-1}\partial_{\xi_2})^{l_2}\partial_{x_2}^{k_2}\sigma_1\big)\big((i^{-1}\partial_{\xi_2})^{k_2}\partial_{x_1}^{l_1}\partial_{x_2}^{l_2}\sigma_2\big).$$

Example 4.31. Let G be the Heisenberg group $\mathbb{H}_n$, equipped with the canonical homogeneous dilations, and let τ be the symmetry function (1.15), which in exponential coordinates is given by $\tau(x) = \big(\frac{x_1}{2}, \dots, \frac{x_{2n+1}}{2}\big)$. In the following, we will write out the asymptotic expansion for the homogeneous orders greater or equal $m_1 + m_2 - 2(\rho - \delta)$ and compare the resulting version of (4.29) with its ((2n + 1)-dimensional) Euclidean counterpart from Example 4.29. To determine the desired coefficients, we have to solve

$$q_\alpha\big(p_1(y,z)\big) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\tilde{q}_{\alpha_1}(z)\tilde{q}_{\alpha_2}(z^{-1}y),$$
$$q_\beta\big(p_2(y,z)\big) = \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2}\tilde{q}_{\beta_1}(z^{-1}y)\tilde{q}_{\beta_2}(z)$$

for $[\alpha], [\beta] = 1$. To do so, we observe that

$$\begin{aligned} p_1(y,z) &= \tau(y)\tau(y^{-1}z) \\ &= \Bigg(\frac{z_1}{2}, \dots, \frac{z_{2n}}{2}, \frac{z_{2n+1}}{2} - \frac{1}{8}\sum_{j=1}^{n}\big(z_j(y_{n+j} - z_{n+j}) - z_{n+j}(y_j - z_j)\big)\Bigg), \\ p_2(y,z) &= \tau(y^{-1})\tau(z) \\ &= \Bigg(\frac{z_1 - y_1}{2}, \dots, \frac{z_{2n} - y_{2n}}{2}, \frac{z_{2n+1} - y_{2n+1}}{2} \\ &\qquad + \frac{1}{8}\sum_{j=1}^{n}\big((y_{n+j} - z_{n+j})z_j - (y_j - z_j)z_{n+j}\big)\Bigg), \end{aligned}$$

and we recall from [28, Ex. 5.2.4] that the homogeneous polynomials q_γ of degree $[\gamma] = 1$ are precisely the monomials $q_{e_j}(x) = x_j$, $j = 1, \dots, 2n$. It follows that

$$\begin{aligned} q_{e_j}\big(p_1(y,z)\big) &= \frac{z_j}{2} = -\frac{1}{2}\tilde{q}_{e_j}(z), \\ q_{e_k}\big(p_2(y,z)\big) &= \frac{y_k - z_k}{2} = +\frac{1}{2}\tilde{q}_{e_k}(z^{-1}y), \end{aligned} \qquad j,k = 1, \dots, 2n.$$

We can now approximate (4.29) by

$$\sigma \sim \sigma_1\sigma_2 + \omega_1 \quad \mathrm{mod}\ S^{m_1+m_2-2(\rho-\delta)}(\mathbb{H}_n),$$

with[26]

$$\omega_1 = \frac{1}{2}\sum_{j=1}^{2n}\big((X^{e_j}\sigma_1)(\Delta^{e_j}\sigma_2) - (\Delta^{e_j}\sigma_1)(X^{e_j}\sigma_2)\big) \in S^{m_1+m_2-(\rho-\delta)}(\mathbb{H}_n).$$

4.5. The G-Poisson bracket

Despite the striking differences between the asymptotic expansions in Examples 4.29 and 4.31, we observe that the term ω_1 can in both cases be interpreted as a kind of Poisson bracket (up to a multiplicative constant) defined by the first strata of the respective groups. As we will see in the following, this property is actually shared by all the symmetric quantizations on stratified groups.

Definition 4.32. Let G be a stratified group and let it be equipped with the canonical dilations, i.e., $v_j = j$ for all $j = 1, \dots, n$. Let $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta \le \rho \le 1$. Then for any two symbols $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$ we define the *G-Poisson bracket* of σ_1 and σ_2 to be the symbol

$$\{\sigma_1, \sigma_2\}_G := \sum_{[\alpha]=1} \big((X^\alpha\sigma_1)(\Delta^\alpha\sigma_2) - (\Delta^\alpha\sigma_1)(X^\alpha\sigma_2)\big) \in S^{m_1+m_2+(\rho-\delta)}_{\rho,\delta}(G).$$

Remark 4.33. The fact that pointwise products of symbols belong to the expected symbol classes (even for $\delta = \rho$) was proved in [28, Thm. 5.2.22 (ii)] (see also Remark 3.2).

Remark 4.34. Due to the general noncommutative nature of our setting, the G-Poisson bracket does not satisfy the anticommutativity, the Leibniz rule and the Jacobi identity which characterize Poisson brackets. However, when G is commutative, hence $G = \mathbb{R}^n$, our definition agrees with the standard one and we have a Poisson bracket in the usual sense (see Remark 4.35). The compatibility with the Euclidean case and the possible applications in Hamiltonian mechanics in the group setting lead us to call this object the G-Poisson bracket.

Remark 4.35. On $G = \mathbb{R}^n$, where the classical Poisson bracket is defined by

$$\{\sigma_1, \sigma_2\} := \sum_{j=1}^{n}\big((\partial_{x_j}\sigma_1)(-i\,\partial_{\xi_j}\sigma_2) - (-i\,\partial_{\xi_j}\sigma_1)(\partial_{x_j}\sigma_2)\big),$$

we have

$$\omega_1 = \frac{1}{2}\{\sigma_1, \sigma_2\} = \frac{1}{2}\{\sigma_1, \sigma_2\}_G.$$

Since the first stratum of the Abelian Lie algebra $\mathbb{R}^n$ (equipped with the trivial Lie bracket) coincides with the whole Lie algebra, ω_1 can be fully expressed by the G-Poisson bracket. This is no coincidence as we will shortly show in Proposition 4.36.

[26] The multi-index $\alpha = e_j \in \mathbb{N}_0^n$ denotes the Euclidean coordinate vector whose j-th coordinate equals 1, while all others vanish.

Before we prove the proposition, we briefly make an interesting observation about the symbols ω_j in (4.44), in particular about ω_1, when $\sigma_1 \in S^{m_1}_{1,0}(G)$, $\sigma_2 \in S^{m_2}_{1,0}$ are additionally homogeneous in the sense of [23, Def. 4.1]. We recall that any symbol $\sigma = \{\sigma(x,\pi)\colon \mathcal{H}^\infty_\pi \to \mathcal{H}_\pi \mid x \in G,\ \pi \in \widehat{G}\}$ on a graded group is *homogeneous* of degree $m \in \mathbb{R}$ (or m-homogeneous) if

$$\sigma(x, r\cdot\pi) = r^m \sigma(x,\pi)$$

for all $x \in G$, a.e. $\pi \in \widehat{G}$ and a.e. $r \in \mathbb{R}^+$, where

$$\sigma(x, r\cdot\pi) := r^{-Q}\mathcal{F}_{y\mapsto\pi}\big(\kappa_\sigma\big(x, D_{r^{-1}}(y)\big)\big). \tag{4.45}$$

An m-homogeneous symbol is called regular if it is smooth in the x-variable and satisfies the seminorm estimates (3.2), and the space of all m-homogeneous symbols is denoted by $\dot{S}^m$. Clearly, $\dot{S}^m \subseteq S^m_{1,0}(G)$.

Now observe that, given two symbols $\sigma_1 \in \dot{S}^{m_1}, \sigma_2 \in \dot{S}^{m_2}$, then for all $\alpha, \beta \in \mathbb{N}_0^n$,

$$\Delta^\alpha\sigma_1(x, r\cdot\pi) = r^{m-[\alpha]}(\Delta^\alpha\sigma_1)(x,\pi)$$

and

$$\big(\Delta^\alpha\sigma_1(x,r\cdot\pi)\big)\big(\Delta^\beta\sigma_2(x,r\cdot\pi)\big) = r^{m_1+m_2-[\alpha]-[\beta]}(\Delta^\alpha\sigma_1)(x,\pi)(\Delta^\beta\sigma_2)(x,\pi), \tag{4.46}$$

that is, the two symbols above are homogeneous of degree $m_1 - [\alpha]$ and $m_1 + m_2 - [\alpha] - [\beta]$, respectively. This immediately implies that for any graded group G, any τ satisfying (HP) and any two symbols $\sigma_1 \in \dot{S}^{m_1}$ and $\sigma_2 \in \dot{S}^{m_2}$, the symbol ω_j defined by (4.44) belongs to $\dot{S}^{m_1+m_2-j}$. These arguments can now easily be used to generalize well-known facts about classical pseudo-differential symbols on $\mathbb{R}^n$ to the setting of general graded groups.

Proposition 4.36. *Let G be a stratified group, equipped with the canonical dilations, and let τ be a symmetric quantizing function that satisfies* (HP). *Then, for any $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_2 \in S^{m_1}_{\rho,\delta}(G)$, with $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta < \frac{\rho}{v_n} \le 1$, the uniquely determined summand ω_1 of order $m_1 + m_2 - (\rho - \delta)$ in the asymptotic expansion* (4.43) *of their composite symbol is given by $\frac{1}{2}\{\sigma_1,\sigma_2\}_G$.*

In particular, this holds true for the quantizing functions (1.12) *and* (1.13) *on general stratified groups and the family* (1.16) *on $G = \mathbb{H}_n$.*

In the special case of $\sigma_1 \in \dot{S}^{m_1}$ and $\sigma_2 \in \dot{S}^{m_2}$ we even have $\{\sigma_1,\sigma_2\}_G \in \dot{S}^{m_1+m_2-1}$.

Proof. Denote by $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ the stratification of $\mathfrak{g}$ and by $\{X_{1_i}, \ldots, X_{n_i}\}$ the arbitrary but fixed bases of its strata $\mathfrak{g}_i$, $i = 1, \ldots, k$, chosen to begin with (cf. Section 2.1). Since τ satisfies (HP) with respect to the basis $\{X_{1_1}, \ldots, X_{n_1}\}$ by assumption, we have $c^\tau_{j_1}(x) = C^\tau_{j_1} x_{j_1}$, with $C^\tau_{j_1} \in \mathbb{R}$, for all coordinates $j_1 = 1, \ldots, n_1$ of the first stratum. These are precisely the components c^τ_j of τ which are homogeneous of order $v_1 = 1$. Moreover, by Theorem 4.14, the quantizing function τ is symmetric if and only if $\tau(x) = \tau(x^{-1})x$

for all $x \in G$. In combination with the previous observation, this formula immediately implies that $C^{\tau}_{j_1} = \frac{1}{2}$ for all $j_1 = 1, \ldots, n_1$ (cf. Example 4.5). Now, it is easily seen that because of the stratification of $\mathfrak{g}$, the monomials x_{j_1} are precisely the ones satisfying the defining condition (2.4) for the chosen basis $\{X_{1_1}, \ldots, X_{n_1}\}$ of $\mathfrak{g}_1$. Hence, they are precisely the homogeneous polynomials which determine the difference operators of homogeneous order 1 in the τ-calculus under consideration. A calculation identical to the one in Example 4.31 finally shows that the summand ω_1 of order $m_1 + m_2 - (\rho - \delta)$. The property for homogeneous symbols now follows directly from (4.46). This completes the proof. ∎

For a recent extension result on KN-quantized classical pseudo-differential operators on graded groups based on tangent groupoid techniques, we refer to [20].

5. Invariance properties of symmetric quantizations

In this section we present an appropriately restricted version of the symplectic invariance of the Euclidean Weyl quantization that generalizes to all graded groups and show that

- it is always satisfied by the quantization arising from $\tau(x) = \exp\big(\frac{1}{2}\log(x)\big)$;
- it singles out the latter among all symmetric quantizations on $\mathbb{R}^n$ and $\mathbb{H}_n$.

5.1. The Euclidean case

To start with, we give a quick review of symplectic invariance on $\mathbb{R}^n$, following the presentation in [31, Ch. 4]. We recall that the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is the group of all $2n \times 2n$ real matrices which preserve the standard symplectic form on $\mathbb{R}^{2n}$, given by

$$\omega(v,w) := \sum_{j=1}^{n} (v_j w_{n+j} - v_{n+j} w_j),$$

for $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. The group $\mathrm{Sp}(2n,\mathbb{R})$ is generated by either of the unions of subgroups $\mathrm{D} \cup \mathrm{N} \cup \{J\}$ and $\mathrm{D} \cup \bar{\mathrm{N}} \cup \{J\}$, where

$$\mathrm{N} := \left\{ \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \mid C = C^* \in \mathrm{M}(n \times n, \mathbb{R}) \right\}, \quad \mathrm{D} := \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \mid A \in \mathrm{GL}(n,\mathbb{R}) \right\},$$
$$\bar{\mathrm{N}} := \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \mid C = C^* \in \mathrm{M}(n \times n, \mathbb{R}) \right\}, \quad J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The symplectic invariance (1.3) now arises from the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$, a double cover of $\mathrm{Sp}(2n,\mathbb{R})$ which is characterized by the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R}) \to 0.$$

The metaplectic group can be represented as a group of unitary operators on $L^2(\mathbb{R}^n)$, which are intimately related to the Schrödinger representations $\pi_\lambda \in \widehat{\mathbb{H}}_n$, $\lambda \in \mathbb{R} \setminus \{0\}$, whose definition was recalled in Section 2.4. Namely, for any arbitrary but fixed $\lambda \in \mathbb{R} \setminus \{0\}$ and any $S \in \mathrm{Sp}(2n, \mathbb{R})$ there exists an operator $\eta_\lambda(S) \in \mathcal{U}(L^2(\mathbb{R}^n))$, uniquely determined up to a factor ± 1, such that for the block matrix

$$\widetilde{S} := \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2n+1; \mathbb{R}) \tag{5.1}$$

the identity

$$\pi_\lambda(\widetilde{S}x) = \eta_\lambda(S)\, \pi_\lambda(x)\, \eta_\lambda(S)^{-1} \tag{5.2}$$

holds true for all $x \in \mathbb{H}_n$. Moreover, for any two $S_1, S_2 \in \mathrm{Sp}(2n, \mathbb{R})$, we have $\eta_\lambda(S_1)\, \eta_\lambda(S_2) = \pm \eta_\lambda(S_1 S_2)$.[27] In the special case when $\lambda = 1$, the invariance property (5.2) is easily shown to yield the symplectic invariance of the Weyl quantization (1.3):

$$\mathrm{Op}^{\mathrm{w}}(\sigma \circ S) = U_S^{-1} \mathrm{Op}^{\mathrm{w}}(\sigma) U_S$$

for all $S \in \mathrm{Sp}(2n, \mathbb{R})$ and the unitary operators $U_S = \eta_1(S^*)^{-1}$. This invariance a priori holds for all symbols $\sigma \in \mathcal{S}(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$ but extends to all $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$ since the operators $\eta_\lambda(S)$ preserve $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

5.2. The general graded case

For an arbitrary graded group G we cannot in general expect the existence of continuous maps from $G \times \widehat{G}$ into itself that generalize in a meaningful way the action of the symplectomorphism $S \in \mathrm{N} \cup \{J\}$ since the spaces G and $\widehat{G}$ coincide if and only if $G = \mathbb{R}^n$ for some $n \in \mathbb{N}$. However, a straight-forward generalization is possible for the action of the symplectomorphisms $S \in \mathrm{D}$.

To show this, we recall that $\mathrm{GL}(n, \mathbb{R})$ is precisely the group of automorphisms of the Lie group $(\mathbb{R}^n, +)$. Now, it is easy to check that for any $\mathcal{A} \in \mathrm{Aut}(G)$ and any $\pi \in [\pi] \in \widehat{G}$ of a given graded group G, the composite map $\pi_{\mathcal{A}} := \pi \circ \mathcal{A}$ defines a unitary irreducible representation of G. It follows that $\pi_{\mathcal{A}}$ is an element of some equivalence class $[\pi'] \in \widehat{G}$, which, depending on the type of automorphism $\mathcal{A}$, may or may not coincide with $[\pi]$.

Let us illustrate this dichotomy for two particular types of subgroups of $\mathrm{Aut}(G)$:

(i) the normal subgroup of inner automorphisms

$$\mathrm{conj}_y := x \mapsto yxy^{-1}, \quad y \in G;$$

(ii) any group of homogeneous dilations $\{D_r\}_{r>0}$ on G, for which the direct summands of the given gradation $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ form eigenspaces of the matrix $\log(D_1)$, with eigenvalues $\upsilon_1, \ldots, \upsilon_n \in \mathbb{N}$ (cf. Section 2.1).

On the one hand, if $\mathcal{A} = \mathrm{conj}_y$ for some $y \in G$, then $[\pi_{\mathrm{conj}_y}] = [\pi]$ for any $\pi \in \widehat{G}$ since the representations are intertwined by

$$\pi_{\mathrm{conj}_y}(x) = \pi(y)\pi(x)\pi(y)^{-1} \tag{5.3}$$

[27] Explicit formulas for these operators can be found in [31, p. 179]

for all $x \in G$. On the other hand, if $\mathcal{A} = D_r$ for some $r \in \mathbb{R}^+ \setminus \{1\}$, then $[\pi_{\mathcal{A}}] \neq [\pi]$ unless $[\pi] = [1] \in \widehat{G}$. This can be seen by realizing π as an induced representation π_l, which by Kirillov's orbit method is uniquely determined by the (any) representative $l \in \mathfrak{g}^*$ of the corresponding co-adjoint orbit $\mathcal{O}_\pi = \mathrm{Ad}^*(G)l \subseteq \mathfrak{g}^*$: by a routine computation, one can show that $\pi_l \circ D_r$ coincides with the induced representation $\pi_{D_r^*(l)}$, where $D_r^*: \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the adjoint of $D_r: \mathfrak{g} \to \mathfrak{g}$, $r > 0$. Since the representatives l and $D_r^*(l)$ necessarily determine disjoint co-adjoint orbits, it follows that

$$[\pi] = [\pi_l] \neq [\pi_{D_r^*(l)}] = [\pi_{D_r}].$$

(Compare with the closely related (4.45).)

In order to show now that any $\mathcal{A} \in \mathrm{Aut}(G)$ yields an invariance relation of the type (1.3) for the symmetric quantization Op^τ with $\tau(x) = \exp(\frac{1}{2}\log(x))$, let us settle the necessary notation. Thus, let us denote by $\mathcal{A}^*$ the induced dual automorphism $\pi \mapsto \pi_{\mathcal{A}}$ of $\widehat{G}$ and by $\mathcal{S} = \mathcal{S}(\mathcal{A})$ the automorphism

$$\mathcal{S}: G \times \widehat{G} \to G \times \widehat{G}: (x, \pi) \mapsto (\mathcal{A}x, \mathcal{A}^{*-1}\pi). \tag{5.4}$$

With this at hand, we can show the following statement.

Theorem 5.1. *Let G be a graded group and let τ be the symmetry function $\tau(x) = \exp(\frac{1}{2}\log(x))$. Then for any $\mathcal{A} \in \mathrm{Aut}(G)$ and $\mathcal{S}$ as in (5.4) there exists a unitary map $U_{\mathcal{S}}$ such that*

$$\mathrm{Op}^\tau(\sigma \circ \mathcal{S})f = U_{\mathcal{S}}^{-1}\mathrm{Op}^\tau(\sigma)U_{\mathcal{S}}f \tag{5.5}$$

for all $\sigma \in S^0_{\rho,\delta}(G)$, with $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq 1$, and all $f \in L^2(G)$.

Proof. Let $\mathcal{A} \in \mathrm{Aut}(G)$ and choose an arbitrary $\sigma \in S^0_{\rho,\delta}(G)$, with $0 \leq \delta < \min\{\rho, \frac{1}{v_n}\} \leq 1$, and $f \in L^2(G)$. For the sake of explicit computations involving absolutely convergent integrals, we will assume that f lies in the dense subspace $\mathcal{S}(G)$. This is possible without loss of generality since by Corollary 4.13 both $\sigma \circ S$ and σ quantize continuous operators on $L^2(G)$, therefore the computations extend to all of $L^2(G)$.

To begin with, observe that

$$\begin{aligned}\mathrm{Op}^\tau(\sigma \circ S)f(x) &= \iint\limits_{\widehat{G}\times G} \mathrm{Tr}\Big(\pi(y^{-1}x)\sigma\big(\mathcal{A}\big(x\tau(y^{-1}x)^{-1}\big), \pi_{\mathcal{A}^{-1}}\big)f(y)\Big)\,dy\,d\mu(\pi)\\ &= \iint\limits_{\widehat{G}\times G} \mathrm{Tr}\Big(\pi_{\mathcal{A}}(y^{-1}x)\sigma\Big(\big(\mathcal{A}(x)\mathcal{A}\big(\tau(y^{-1}x)\big)^{-1}\Big), \pi\Big)f(y)\Big)\,dy\,d\mu(\pi_{\mathcal{A}})\end{aligned}$$

for all $x \in G$, but in order to continue our computation, we need to show that

$$\mathcal{A}\big(\tau(y^{-1}x)\big) = \tau\big(\mathcal{A}(y)^{-1}\mathcal{A}(x)\big) \tag{5.6}$$

holds for all $x, y \in G$ for our choice of τ. However, since the Lie algebra automorphism $\mathcal{A}' := d\mathcal{A}(e_G)$ not only preserves the Lie bracket on $\mathfrak{g}$ but also the Baker–Campbell–Hausdorff product

$$\begin{aligned} X \star Y &:= \log\bigl(\exp(X)\exp(Y)\bigr) \\ &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\bigl[X,[X,Y]\bigr] - \frac{1}{12}\bigl[Y,[X,Y]\bigr] + \cdots, \end{aligned}$$

the identity (5.6) follows from

$$\begin{aligned} \log\bigl(\mathcal{A}\bigl(\tau(y^{-1}x)\bigr)\bigr) &= \mathcal{A}'\log\bigl(\tau(y^{-1}x)\bigr) = \mathcal{A}'\Bigl(\frac{1}{2}\bigl((-Y)\star X\bigr)\Bigr) \\ &= \frac{1}{2}\bigl((-\mathcal{A}'Y)\star \mathcal{A}'X\bigr) = \log\bigl(\tau\bigl((\mathcal{A}y)^{-1}\mathcal{A}(x)\bigr)\bigr), \end{aligned}$$

where $X := \log(x)$ and $Y := \log(y)$. Using the change of variables $y' := \mathcal{A}(y)$, we get

$$\begin{aligned} &\mathrm{Op}^\tau(\sigma\circ S)f(x) \\ &= \int_{\widehat{G}} \mathrm{Tr}\Bigl(\pi_{\mathcal{A}}(x)\int_G \sigma\bigl(\bigl(\mathcal{A}(x)\tau\bigl(\mathcal{A}(y)^{-1}\mathcal{A}(x)\bigr)^{-1}\bigr),\pi\bigr)\pi_{\mathcal{A}}(y)^* f(y)\,dy\Bigr)d\mu(\pi_{\mathcal{A}}) \\ &= \frac{1}{\bigl|\det(\mathcal{A}')\bigr|}\int_{\widehat{G}} \mathrm{Tr}\Bigl(\pi\bigl(\mathcal{A}(x)\bigr)\int_G \sigma\bigl(\bigl(\mathcal{A}(x)\tau\bigl(y'^{-1}\mathcal{A}(x)\bigr)^{-1}\bigr),\pi\bigr) \\ &\qquad\qquad \times \pi(y')^* f\bigl(\mathcal{A}^{-1}(y')\bigr)\,dy'\Bigr)d\mu(\pi). \end{aligned}$$

Since the identity $d\mu(\pi_{\mathcal{A}}) = |\det(\mathcal{A}')|\,d\mu(\pi)$ is an immediate consequence of formula (2.8), the above identity can be rewritten as

$$\mathrm{Op}^\tau(\sigma\circ S)f = \bigl(\mathrm{Op}^\tau(\sigma)(f\circ\mathcal{A}^{-1})\bigr)\circ\mathcal{A} = U_{\mathcal{S}}^{-1}\mathrm{Op}^\tau(\sigma)U_{\mathcal{S}}f$$

for the unitary operator

$$U_{\mathcal{S}}\colon L^2(G)\to L^2(G)\colon f\mapsto \bigl|\det(\mathcal{A}')\bigr|^{-\frac{1}{2}} f\circ\mathcal{A}^{-1}.$$

Since $\sigma \in S^0_{\rho,\delta}(G)$ and $f \in \mathcal{S}(G)$ were arbitrary, this completes the proof. ■

At this point we ought to make a few relevant observations. The proof of Theorem 5.1 not only works for $\tau(x) = \exp(\frac{1}{2}\log(x))$, but in fact any admissible τ that commutes with the group automorphisms $\mathcal{A} \in \mathrm{Aut}(G)$. On general graded G this is obviously the case for $\tau = e_G$, the quantizing function of the Kohn–Nirenberg quantization, while on $G = \mathbb{R}^n$ all linear quantizing functions $\tau(x) = \tau x$, $\tau \in [0,1]$, commute with all $A \in \mathrm{GL}(n,\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^n)$.

So, unlike the full symplectic invariance (1.3) on $\mathbb{R}^n$, on a generic graded group the invariance (5.5) under group automorphisms does not necessarily single out any specific τ-quantization. In particular, the homogeneous dilations $\{D_r\}_{r>0}$ clearly commute with all τ

that satisfy (HP). However, a combination of (5.5) and the preservation of involution (1.10) suffice already on $\mathbb{R}^n$: even if we admit all symmetry functions $\tau : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy (HP) for any given admissible homogeneous structure with weights $v_1, \dots, v_n \in \mathbb{N}$, only the linear symmetry function $\tau(x) = \frac{1}{2}x$ commutes with all $A \in \mathrm{GL}(n, \mathbb{R})$.

Admittedly, the group of automorphisms can vary wildly among general graded groups, and canonical subgroups like the inner automorphism may not suffice to single out specific symmetry functions, as we will see in the case of $G = \mathbb{H}_n$ in Section 5.3, but a relatively explicit description of $\mathrm{Aut}(G)$ combined with the use of a symmetric function τ may nevertheless do the job, as this is the case on both $\mathbb{R}^n$ and $\mathbb{H}_n$.

Let us also point out that a related, yet different type of invariance [39] has recently been shown in the context of semi-classical calculus on graded groups (cf. [22, 24, 27]).

5.3. The Heisenberg case

In this subsection we illustrate the general arguments of Section 5.2 for the special case of the Heisenberg group $\mathbb{H}_n$ and prove the following sharper version of Theorem 5.1.

Theorem 5.2. *Let $\mathbb{H}_n$ be the Heisenberg group, equipped with the canonical dilations. Then among all quantizing functions $\tau\colon \mathbb{H}_n \to \mathbb{H}_n$ which satisfy (HP) only $\tau(x) = \exp(\frac{1}{2}\log(x))$ satisfies the following two conditions for all $\sigma \in S^0_{\rho,\delta}(G)$, $0 \le \delta < \min\{\rho, \frac{1}{v_{2n+1}}\} \le \frac{1}{2} \le 1$, and all $f \in L^2(\mathbb{H}_n)$:*

- *Preservation of involution:*

$$\mathrm{Op}^\tau(\sigma)^* f = \mathrm{Op}^\tau(\sigma^*) f;$$

- *Automorphic invariance:*
  for each $\mathcal{A} \in \mathrm{Aut}(\mathbb{H}_n)$ and $\mathcal{S}$ as in (5.4) there exists an operator $U_{\mathcal{S}} \in \mathcal{U}(L^2(\mathbb{H}_n))$ such that

$$\mathrm{Op}^\tau(\sigma \circ \mathcal{S}) f = U_{\mathcal{S}}^{-1} \mathrm{Op}^\tau(\sigma) U_{\mathcal{S}} f.$$

That is, among all admissible symmetric quantizations on $\mathbb{H}_n$, only the one defined by $\tau(x) = \exp(\frac{1}{2}\log(x))$ is invariant under all automorphic changes of variables.

To set the stage for the proof of Theorem 5.2, we will recall the classification of $\mathrm{Aut}(\mathbb{H}_n)$, following the exposition in [31, §1.2]. For each type of $\mathcal{A} \in \mathrm{Aut}(\mathbb{H}_n)$ we will additionally provide an explicit description of the map $\pi \mapsto \pi_{\mathcal{A}}$, which we used in Section 5.2.

Thus, let us recall that each $\mathcal{A} \in \mathrm{Aut}(\mathbb{H}_n)$, when viewed as a map on the underlying vector space $\mathbb{R}^{2n+1}$, is a linear isomorphism which can be uniquely written as[28]

$$\mathcal{A} = \widetilde{S} \circ \mathrm{conj}_y \circ D_r \circ \Theta$$

[28] The fourth map in [31, §1.2] is $\iota = \widetilde{J} \circ \Theta$, whereas one may equivalently use Θ, which proved very convenient in [28, Ch. 6].

for some $\widetilde{S}$ as in (5.1), some $y \in \mathbb{H}_n$ and some $r > 0$, where

$$\operatorname{conj}_y(x) = \Big(x_1, \dots, x_{2n}, x_{2n+1} + \sum_{j=1} (y_j x_{n+j} - y_{n+j} x_j)\Big),$$
$$D_r(x) = (r x_1, \dots, r x_{2n}, r^2 x_{2n+1}),$$
$$\Theta(x_1, \dots, x_{2n}, x_{2n+1}) := (x_1, \dots, x_n, -x_{n+1}, \dots, -x_{2n}, -x_{2n+1}).$$

In the computation of the map $\mathcal{A}^* = \pi \mapsto \pi_{\mathcal{A}} = \pi \circ \mathcal{A}$ for each type of automorphism, we restrict ourselves to the set of Schrödinger representations since the remaining one-dimensional representations have Plancherel measure zero.

(i) For any symplectic map $S \in \mathrm{Sp}(2n, \mathbb{R})$ the invariance property (5.2) implies that the dual automorphism $(\widetilde{S})^*$ is given by $\pi_\lambda \mapsto \eta_\lambda(S) \pi_\lambda \eta_\lambda(S^{-1}) \in [\pi_\lambda]$ for all $\pi_\lambda \in \widehat{\mathbb{H}}_n$, and $d\mu(\pi_{\widetilde{S}}) = d\mu(\pi)$ since all symplectic matrices are of determinant 1.

(ii) A general formula for the conjugations is given by (5.3), and by the bi-invariance of the Haar measure on nilpotent Lie groups we have $d\mu(\pi_{\operatorname{conj}_y}) = d\mu(\pi)$.

(iii) For the homogeneous dilations $\{D_r\}_{r>0}$ we compute

$$\begin{aligned}\big(\mathcal{F}(f \circ D_r)\big)(\pi_\lambda) &= \int_{\mathbb{H}_n} f\big(D_r(x)\big)\, \pi_\lambda(x)^*\, dx = r^{-Q} \int_{\mathbb{H}_n} f(x)\, \pi_\lambda\big(\big(D_{r^{-1}}(x)\big)\big)^*\, dx \\ &= r^{-Q} \int_{\mathbb{H}_n} f(x)\, \pi_{r^{-2}\lambda}(x)^*\, dx = r^{-Q}\, \hat{f}(\pi_{r^{-2}\lambda})\end{aligned}$$

for any $f \in \mathcal{S}(\mathbb{H}_n)$ and the homogeneous dimension $Q = 2n + 2$. The Plancherel formula (2.8) then implies that the dual automorphism $(D_r)^*$ is given by $\pi_\lambda \mapsto \pi_{r^2\lambda}$ for all $\pi_\lambda \in \widehat{\mathbb{H}}_n$, and that $d\mu(\pi_{D_r}) = r^Q\, d\mu(\pi)$.

(iv) For the Haar measure-preserving, self-inverse map $\Theta = \Theta^{-1}$ one immediately gets $\pi_\lambda(\Theta x) = \pi_{-\lambda}(x)$ for all $x \in \mathbb{H}_n$. So, the dual automorphism Θ^* is given $\pi_\lambda \mapsto \pi_{-\lambda}$ for all $\pi_\lambda \in \widehat{\mathbb{H}}_n$, and $d\mu(\pi_\Theta) = d\mu(\pi)$.

Proof of Theorem 5.2. Suppose that the quantizing function τ satisfies (HP). Then by Theorem 4.14, the associated τ-quantization preserves the involution if and only if τ is symmetric. So, for the rest of the proof we may limit ourselves to symmetric quantizations. Among the latter, the quantization defined by $\tau(x) = \exp(\frac{1}{2}\log(x))$ is invariant under automorphic changes of variables by Theorem 5.1. To complete the proof, it therefore suffices to show that no other symmetry function τ that satisfies (HP) commutes with all $\mathcal{A} \in \mathrm{Aut}(\mathbb{H}_n)$. By the definition of (HP), all such τ commute with the homogeneous dilations. To check the remaining types of automorphisms, we recall from Example 4.5 that the symmetry functions which satisfy (HP) for the homogeneous dilation structure on $\mathbb{H}_n$ are precisely the functions of the form

$$\tau(x) = \left(\frac{x_1}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\right)$$

for any choice of $c_{j,k} \in \mathbb{R}$, $j, k = 1, \dots, 2n$. One easily checks that all of these functions also commute with the inner automorphisms conj_y, $y \in \mathbb{H}_n$, so we may pass to checking the commutation with the automorphism Θ. Note that due to its form τ commutes with Θ if and only if $\tau(\Theta(x))_{2n+1} = \Theta(\tau(x))_{2n+1}$, i.e., if the matrix $C := \{c_{j,k}\}_{j,k=1}^{2n}$ of the quadratic form $q(x) = \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k$ satisfies certain properties: Since the antisymmetric part of the quadratic form

$$q\colon (x_1, \dots, x_{2n}) \mapsto \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k$$

always vanishes, it is not restrictive to assume for the rest of the proof that C is symmetric. Now it is easy to check that $\tau(\Theta(x))_{2n+1} = \Theta(\tau(x))_{2n+1}$ if and only if C is of the form

$$C := \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix},$$

where C_1, C_2 are two symmetric $n \times n$-matrices. With this information at hand, we can focus on the maps $\widetilde{S}$ with $S \in \mathrm{Sp}(2n, \mathbb{R})$. Note that τ commutes with all $\widetilde{S}$, $S \in \mathrm{Sp}(2n, \mathbb{R})$, precisely when the invariance $S^*CS = C$, where, recall, C is symmetric and has a block anti-diagonal form. If we now pick $S = J$, then we must also have $J^*CJ = -C^*$, since this holds for all $C \in \mathrm{M}_{2n}(\mathbb{R})$.[29] The two conditions above and the symmetry of C then yield

$$C = S^*CS = J^*CJ = -C^* = -C,$$

which implies that C is the null matrix. This leaves only $\tau(x) = \exp(\frac{1}{2}\log(x))$, and the proof is complete. ∎

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[29] See, e.g., [31, Prop. 4.1 (f)]

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