

Cofinality of Galois cohomology within purely quadratic graded algebras

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Abstract. Let p be a prime number. For a field F containing a root of unity of order p , let $H^\bullet(F) = H^\bullet(F, \mathbb{F}_p)$ be the mod- p Galois cohomology graded \mathbb{F}_p -algebra of F . By the Norm Residue Theorem, $H^\bullet(F)$ is a purely quadratic graded-commutative algebra, and is therefore determined by the cup product $\cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$. We prove that the class of all Galois cohomology algebras $H^\bullet(F)$ is cofinal in the class of all purely quadratic graded-commutative \mathbb{F}_p -algebras A_\bullet , in the following sense: For every A_\bullet there exists F such that the bilinear map $A_1 \times A_1 \rightarrow A_2$, which determines A_\bullet , embeds in the cup product bilinear map $\cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$.

We further provide examples of \mathbb{F}_p -bilinear maps which are not realizable by fields F in this way. These are related to recent results by Snopce–Zaleskii and Blumer–Quadrelli–Weigel on the Galois theory of pro- p right-angled Artin groups, as well as to a conjecture by Marshall on the possible axiomatization of quadratic form theory of fields.

1. Introduction

We fix a prime number p . Let F be a field containing a root of unity ζ of order p and let $G_F = \text{Gal}(F^{\text{sep}}/F)$ be its absolute Galois group. We write $H^i(F)$ for the i th profinite cohomology group $H^i(G_F, \mathbb{F}_p)$, where G_F acts trivially on \mathbb{F}_p , and consider the cohomology \mathbb{F}_p -algebra $H^\bullet(F) = \bigoplus_{i \geq 0} H^i(F)$ with the cup product. A fundamental property of this graded-commutative algebra is given by the celebrated *Norm Residue Theorem* (formerly the Bloch–Kato conjecture; See Section 4), proved by Voevodsky and Rost with a contribution by Weibel: $H^\bullet(F)$ is a *quadratic* graded algebra, which loosely speaking means that it is generated by its degree 1 elements, with relations originating from the degree 2 component. Moreover, it is *purely quadratic*, meaning that the defining relations can be taken to be pure tensors in $H^1(F)^{\otimes 2}$ (see Section 3 for the precise definition). A major problem in Galois cohomology is to characterize the graded \mathbb{F}_p -algebras that arise as Galois cohomology algebras $H^\bullet(F)$ among all purely-quadratic graded-commutative \mathbb{F}_p -algebras.

The graded algebra $H^\bullet(F)$ has the additional property that there exists a distinguished element $\varepsilon \in H^1(F)$ such that $2\varepsilon = 0$ and $\chi \cup (\varepsilon + \chi) = 0$ for every $\chi \in H^1(F)$. Specifically, $\varepsilon = (-1)_F$ is the Kummer element of -1 , and this identity is the cohomological

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interpretation of the splitting of symbol F -algebras $(a, -a)_{F, \xi}$ (see Section 4). Therefore, we restrict ourselves to purely quadratic graded-commutative \mathbb{F}_p -algebras $A_\bullet = \bigoplus_{i \geq 0} A_i$ with a distinguished element $\varepsilon \in A_1$ satisfying these two conditions. Thus (A_\bullet, ε) is a κ -algebra in the sense of Bass–Tate [4]. Note that in the case where p is odd we have $\varepsilon = 0$, so the structures A_\bullet are just the purely quadratic graded-commutative \mathbb{F}_p -algebras.

Now a purely quadratic κ -algebra (A_\bullet, ε) is determined by the product map $A_1 \times A_1 \rightarrow A_2$ and ε (see Section 3). Therefore, we consider the category of *augmented \mathbb{F}_p -bilinear maps* $(b: V \times V \rightarrow W, \varepsilon)$, where V and W are \mathbb{F}_p -linear spaces, and $\varepsilon \in V$ is a distinguished element with the above properties (see Section 2). Thus our question reduces to identifying within this category the augmented bilinear maps of *field type*, namely, those realizable as augmented bilinear maps $(\cup: H^1(F) \times H^1(F) \rightarrow H^2(F), (-1)_F)$ for some field F containing a root of unity of order p .

Main Theorem. *The augmented \mathbb{F}_p -bilinear maps of field type are cofinal in the category of all augmented \mathbb{F}_p -bilinear maps. That is, for every augmented \mathbb{F}_p -bilinear map $(b: V \times V \rightarrow W, \varepsilon)$ there exists a field F containing a root of unity of order p and a monomorphism (in the natural sense)*

$$(b: V \times V \rightarrow W, \varepsilon) \rightarrow (\cup: H^1(F) \times H^1(F) \rightarrow H^2(F), (-1)_F),$$

where moreover, W is mapped isomorphically onto $H^2(F)$.

See Theorem 8.1 for a precise statement and proof.

In Corollary 8.2 we prove an analogous result in the non-augmented context: The cup product maps $\cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$, for fields F as above, are cofinal in the category of all skew-symmetric \mathbb{F}_p -bilinear maps $b: V \times V \rightarrow W$.

By the Norm Residue Theorem in degree 2 (which is due to Merkurjev and Suslin), augmented bilinear maps of field type are *surjective*, in the sense that the image of b spans $W = H^2(F)$. A natural question is whether in the Main Theorem one can take the monomorphism to be an isomorphism, when the given augmented bilinear map is assumed to be surjective. In Section 10 we answer this question negatively, as follows: For $p = 2$, augmented \mathbb{F}_2 -bilinear maps $(H^1(F), H^2(F), \cup, (-1)_F)$ of field type have an additional property – the ‘*common slot property*’ of the quaternionic pairing. We provide examples of surjective augmented \mathbb{F}_2 -bilinear maps which do not satisfy this property, and therefore are not of field type. In fact, it is a long-standing open problem in the algebraic theory of quadratic forms whether the above properties of ε together with the common slot property already completely characterize the \mathbb{F}_2 -bilinear maps which are realizable as cup product maps over fields of characteristic $\neq 2$ (see Remark 10.7 as well as [18, Open Problem 1 and Thm. 2.2], [16, Ch. XII, §8]).

Our examples are constructed using *pro- p right-angled Artin groups* G_Γ associated with finite simplicial (unoriented) graphs Γ . This means G_Γ is the pro- p group with the vertices of Γ as generators and with commutators of vertices connected by edges as defining relations. The Galois-theoretic properties of the groups G_Γ have been the object of extensive study in recent years. In particular, Snopce and Zalesskii [27], building on earlier

results by Quadrelli [25] and Cassella–Quadrelli [6], proved the following characterization theorem: G_Γ is realizable as the maximal pro- p Galois group $G_F(p)$ of a field F containing a root of unity of order p if and only if Γ does not contain a 4-vertex line L_3 or a 4-vertex circle C_4 as induced subgraphs. See [5] for a generalization to oriented graphs.

Now as we show in Theorem 10.5, when $p = 2$, the equivalent conditions in the Snopce–Zalesskii theorem are also equivalent to the realization of the augmented \mathbb{F}_p -bilinear map $(\cup: H^1(G_\Gamma) \times H^1(G_\Gamma) \rightarrow H^2(G_\Gamma), 0)$ by a field, as well as to the common slot property for this map. Our counterexamples are just the augmented \mathbb{F}_2 -bilinear maps associated in this way with graphs Γ which do not contain L_3 or C_4 as induced graphs.

In [25,27], the groups G_{L_3}, G_{C_4} are ruled out from being maximal pro- p Galois groups by showing that they are not *Bloch–Kato pro- p groups*, that is, they have closed subgroups for which the mod- p graded cohomology ring is not quadratic. Our approach using the common slot property (for $p = 2$) may therefore suggest a possible connection between the common slot property and the Bloch–Kato property.

We remark that there are additional *conjectural* properties of the graded algebras $H^\bullet(F)$ for arbitrary p , notably they are conjectured to be Koszul. See [19,20,24].

Our proof of the Main Theorem is based on Amitsur’s construction of generic splitting fields of central simple algebras, or alternatively, elements of the second cohomology group $H^2(F)$ (see Section 5). This construction is applied iteratively to produce fields whose cohomology groups realize the desired bilinear maps. Transcendental constructions in this spirit were used by Mináč and Ware [21,22], as well as by Nikolov and the first-named author [3], in their works on the realization of pro- p Demuškin groups of infinite rank as maximal pro- p Galois groups of fields.

2. Augmented bilinear maps

Let R be a commutative ring. An *augmented R -bilinear map* (V, W, b, ε) consists of R -modules V and W , an R -bilinear map

$$b: V \times V \rightarrow W,$$

and a distinguished element $\varepsilon \in V$, such that:

- (i) $b(v, \varepsilon + v) = 0$ for every $v \in V$;
- (ii) $2\varepsilon = 0$.

We say that (V, W, b, ε) is *surjective* if the homomorphism $V \otimes_R V \rightarrow W$ induced by b is surjective.

A *morphism* $f = (f_1, f_2): (V, W, b, \varepsilon) \rightarrow (V', W', b', \varepsilon')$ of augmented R -bilinear maps consists of R -module homomorphisms $f_1: V \rightarrow V'$ and $f_2: W \rightarrow W'$ such that $f_1(\varepsilon) = \varepsilon'$, and the following diagram commutes:

$$\begin{array}{ccccc} V & \times & V & \xrightarrow{b} & W \\ f_1 \downarrow & & \downarrow f_1 & & \downarrow f_2 \\ V' & \times & V' & \xrightarrow{b'} & W'. \end{array}$$

We call f a *monomorphism* if both f_1 and f_2 are injective. The augmented R -bilinear maps thus form a category, denoted **Aug- R -Bilin**.

Lemma 2.1. *Let (V, W, b, ε) be an augmented R -bilinear map.*

- (a) *For every $v \in V$ one has $b(v, \varepsilon) = b(v, v) = b(\varepsilon, v)$.*
- (b) *The map b is skew-symmetric, that is, $b(v, v') = -b(v', v)$ for every $v, v' \in V$.*
- (c) *If 2 is invertible in R , then $\varepsilon = 0$.*

Proof. The first equality in (a) follows from (i), (ii) and the bilinearity of b . For the second equality, apply (i) with v replaced by $\varepsilon + v$.

For every $v, v' \in V$ we obtain from (i) and the bilinearity that

$$\begin{aligned} 0 &= b(v + v', \varepsilon + v + v') = b(v, \varepsilon + v) + b(v, v') + b(v', v) + b(v', \varepsilon + v') \\ &= b(v, v') + b(v', v). \end{aligned}$$

This proves (b).

Assertion (c) follows immediately from (ii). ■

Subsequently, we will focus on the case where $R = \mathbb{F}_p$ for a prime p . In this case, we have the following partial converse of Lemma 2.1 (b).

Lemma 2.2. *Let p be a prime number and let $b: V \times V \rightarrow W$ be a skew-symmetric \mathbb{F}_p -bilinear map. Then there exists an augmented \mathbb{F}_p -bilinear map $(\hat{V}, W, \hat{b}, \varepsilon)$ such that V is an \mathbb{F}_p -linear subspace of \hat{V} of codimension at most 1 and \hat{b} extends b .*

Proof. If $p \neq 2$, then b is alternate, i.e., $b(v, v) = 0$ for every $v \in V$. We may therefore take $\hat{V} = V$, $\hat{b} = b$, and $\varepsilon = 0$.

Suppose that $p = 2$. By the skew-symmetry of b , the map $V \rightarrow W$, $v \mapsto b(v, v)$, is \mathbb{F}_2 -linear. Let $\hat{V} = V \oplus \text{Span}(\varepsilon)$ for a new vector ε . We define a map $\hat{b}: \hat{V} \times \hat{V} \rightarrow W$ by

$$\hat{b}(v + a\varepsilon, v' + a'\varepsilon) = b(v, v') + a'b(v, v) + ab(v', v')$$

for $v, v' \in V$ and $a, a' \in \mathbb{F}_2$. A direct computation shows that it is \mathbb{F}_2 -bilinear. We have

$$\hat{b}(v + a\varepsilon, v + a\varepsilon) = b(v, v) = \hat{b}(v + a\varepsilon, \varepsilon).$$

Therefore $(\hat{V}, W, \hat{b}, \varepsilon)$ is an augmented \mathbb{F}_2 -bilinear map. ■

3. κ -Algebras

Following Bass and Tate [4], we define a κ -algebra $(A_\bullet, \varepsilon_A)$ over the ring R to be a graded R -algebra $A_\bullet = \bigoplus_{i \geq 0} A_i$ with a distinguished central element $\varepsilon_A \in A_1$ such that

- (i) $a \cdot (\varepsilon_A + a) = 0$ for every $a \in A_1$;
- (ii) $2\varepsilon_A = 0$;
- (iii) A_\bullet is generated as an R -algebra by A_1 .

A morphism $f: (A_\bullet, \varepsilon_A) \rightarrow (B_\bullet, \varepsilon_B)$ of κ -algebras over R is a graded R -algebra morphism $f: A_\bullet \rightarrow B_\bullet$ such that $f(\varepsilon_A) = \varepsilon_B$. The κ -algebras over R thus form a category, denoted $\kappa\text{-Alg}_R$.

Given an R -module V , let $T_\bullet(V)$ denote the tensor graded R -algebra over V . For a κ -algebra $(A_\bullet, \varepsilon_A)$, there is a unique morphism $f: (T_\bullet(A_1), \varepsilon_A) \rightarrow (A_\bullet, \varepsilon_A)$ of κ -algebras which is the identity on A_1 . One says that A_\bullet is *quadratic* (resp., *purely quadratic*) if f is surjective and $\text{Ker}(f)$ is generated as a two-sided ideal of $T_\bullet(A_1)$ by elements of $A_1 \otimes_R A_1$ (resp., pure tensors $a \otimes a'$ in $A_1 \otimes_R A_1$).

We define a functor

$$\mathbf{F}: \kappa\text{-Alg}_R \rightarrow \text{Aug-}R\text{-Bilin}$$

by mapping a κ -algebra $(A_\bullet = \bigoplus_{i \geq 0} A_i, \varepsilon_A)$ over R to the augmented R -bilinear map $(A_1, A_2, \mu, \varepsilon_A)$, where $\mu: A_1 \times A_1 \rightarrow A_2$ is the multiplication map in A_\bullet .

Remark 3.1. By Lemma 2.1 (b), μ is skew-symmetric. In view of (iii), this implies that A_\bullet is graded-commutative (see [4, Prop. 2.2 (a)]).

Conversely, we define a functor

$$\mathbf{G}: \text{Aug-}R\text{-Bilin} \rightarrow \kappa\text{-Alg}_R$$

by mapping an augmented R -bilinear map (V, W, b, ε) to the purely quadratic κ -structure $(T_\bullet(V)/I, \varepsilon)$, where I is the two-sided (graded) ideal in $T_\bullet(V)$ generated by all pure tensors $v \otimes v'$, with $v, v' \in V$ satisfying $b(v, v') = 0$.

The composition $\mathbf{G} \circ \mathbf{F}$ is the *purely quadratic hull* functor; see [7, §3].

4. Galois cohomology

We now fix a prime number p and take the ring $R = \mathbb{F}_p$. Given a profinite group G , we write $H^i(G) = H^i(G, \mathbb{F}_p)$ for the i th profinite cohomology group of G with respect to its trivial action on \mathbb{F}_p . Then the cup product

$$\cup: H^i(G) \times H^j(G) \rightarrow H^{i+j}(G)$$

is a skew-symmetric \mathbb{F}_p -bilinear map [23, Ch. I, Prop. 1.4.4].

As in the Introduction, we consider the cup product in a Galois-theoretic context as follows: Let F be a field and let $G_F = \text{Gal}(F^{\text{sep}}/F)$ be its absolute Galois group, where F^{sep} denotes the separable closure of F . We will further assume that F contains a root of unity of order p , so in particular, $\text{Char } F \neq p$. We fix an isomorphism between the group μ_p of all roots of unity of order p in F^{sep} and \mathbb{Z}/p . Let $\zeta \in \mu_p$ correspond to $\bar{1} \in \mathbb{Z}/p$. For simplicity we abbreviate $H^i(F) = H^i(G_F)$. We write again $H^\bullet(F) = \bigoplus_{i \geq 0} H^i(F)$ for the cohomology graded \mathbb{F}_p -algebra with the cup product.

Given a field extension E/F , we write $\text{Res}_{E/F} = \text{Res}_{E/F}^i: H^i(F) \rightarrow H^i(E)$ for the functorial (restriction) map. Note that here we do not assume that E/F is algebraic.

One has the Kummer isomorphism

$$F^\times / (F^\times)^p \cong H^1(G_F, \mu_p) = H^1(F).$$

To every $a \in F^\times$ we associate its Kummer element $(a)_F \in H^1(F)$.

Further, the identification $\mathbb{Z}/p = \mu_p$ induces a canonical isomorphism between $H^2(F)$ and the p -torsion part ${}_p\text{Br}(F)$ of the Brauer group of F . Under this isomorphism, a cup product $(a_1)_F \cup (a_2)_F$, where $a_1, a_2 \in F^\times$, corresponds to the similarity class of the *symbol algebra* $(a_1, a_2)_{F, \zeta}$ over F , generated by elements t_1, t_2 subject to the relations $t_1^p = a_1, t_2^p = a_2$, and $t_1 t_2 = \zeta t_2 t_1$.

Recall that the *Milnor K-algebra* of the field F is the graded \mathbb{Z} -algebra

$$\mathbf{K}_\bullet^M(F) = \mathbf{T}_\bullet(F^\times) / J,$$

where $\mathbf{T}_\bullet(F^\times)$ is the tensor \mathbb{Z} -algebra over the multiplicative group F^\times of F , and J is its two-sided (graded) ideal generated by all pure tensors $a \otimes (1 - a)$, with $0, 1 \neq a \in F$. It is a purely quadratic κ -algebra over \mathbb{Z} with the distinguished element $\varepsilon_K = -1$ [4, Ex. 2.1]. Therefore $\mathbf{K}_\bullet^M(F) \otimes (\mathbb{Z}/p)$ is a purely quadratic κ -algebra over \mathbb{F}_p .

By the Norm Residue Theorem [14, 28, 29], the Kummer map $F^\times \rightarrow H^1(F)$, $a \mapsto (a)_F$, induces a graded algebra isomorphism

$$\mathbf{K}_\bullet^M(F) \otimes (\mathbb{Z}/p) \xrightarrow{\sim} H^\bullet(F). \quad (4.1)$$

It follows that $(H^\bullet(F), (-1)_F)$ is a κ -algebra over \mathbb{F}_p .

By applying the functor \mathbf{F} , we deduce that the cup product map

$$\cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$$

gives rise to a surjective augmented \mathbb{F}_p -bilinear map

$$(H^1(F), H^2(F), \cup, (-1)_F). \quad (4.2)$$

Given a field extension E/F , we have a morphism of κ -structures

$$\text{Res} = \bigoplus_{i \geq 0} \text{Res}_{E/F}^i: H^\bullet(F) \rightarrow H^\bullet(E),$$

as well as a morphism of augmented \mathbb{F}_p -bilinear maps

$$\text{Res}_{E/F} = (\text{Res}_{E/F}^1, \text{Res}_{E/F}^2): (H^1(F), H^2(F), \cup, (-1)_F) \rightarrow (H^1(E), H^2(E), \cup, (-1)_E).$$

We define κ -**Alg-Fields** to be the subcategory of κ -**Alg** $_{\mathbb{F}_p}$ consisting of all κ -algebras over \mathbb{F}_p of the form $(H^\bullet(F), (-1)_F)$, and with restriction maps as morphisms. We call its objects κ -*algebras over \mathbb{F}_p of field type*.

Likewise we define **Aug- \mathbb{F}_p -Bilin** to be the subcategory of **Aug- \mathbb{F}_p -Bilin** consisting of all (surjective) augmented \mathbb{F}_p -bilinear maps of the form (4.2) for some F , with restriction maps as morphisms. We call its objects *augmented \mathbb{F}_p -bilinear maps of field type*.

Theorem 4.1. *The restricted functors*

$$\kappa\text{-Alg-Fields} \begin{array}{c} \xrightarrow{\mathbf{F}} \\ \xleftarrow{\mathbf{G}} \end{array} \text{Aug-Bilin-Fields}$$

form an equivalence of categories.

Proof. It is straightforward to verify that \mathbf{F} and \mathbf{G} map restriction morphisms to restriction morphisms.

Next take a field F containing a root of unity of order p . We denote the image under $\mathbf{G} \circ \mathbf{F}$ of the κ -algebra $(H^\bullet(F), (-1)_F)$ by

$$(\hat{H}^\bullet(F) = \bigoplus_{i \geq 0} \hat{H}^i(F), (-1)_F).$$

Thus $\hat{H}^\bullet(F) = \mathbf{T}_\bullet(H^1(F))/I$, where I is the two-sided ideal generated by all pure tensors $(a)_F \otimes (b)_F$ with $(a)_F \cup (b)_F = 0$ in $H^2(F)$. Since $(a)_F \cup (1-a)_F = 0$ for every $0, 1 \neq a \in F$, the isomorphism (4.1) factors as

$$\mathbf{K}_\bullet^M(F) \otimes (\mathbb{Z}/p) \rightarrow \hat{H}^\bullet(F) \rightarrow H^\bullet(F),$$

In view of the Kummer isomorphism, the left map is surjective, and we deduce that both maps are isomorphisms. This implies that $\mathbf{G} \circ \mathbf{F} = \mathbf{Id}$ on $\kappa\text{-Alg-Fields}$.

For the second composition, we observe that

$$(\mathbf{F} \circ \mathbf{G})(H^1(F), H^2(F), \cup, (-1)_F) = (\hat{H}^1(F), \hat{H}^2(F), b, (-1)_F), \quad (4.3)$$

where $b: \hat{H}^1(F) \times \hat{H}^1(F) \rightarrow \hat{H}^2(F)$ is induced by the tensor product. Now $\hat{H}^1(F) = H^1(F)$. Further, for the degree 2 component I_2 of I , we obtain from the previous paragraph that the cup product induces an isomorphism $\hat{H}^2(F) = H^1(F)^{\otimes 2}/I_2 \rightarrow H^2(F)$. Hence the right hand-side of (4.3) is isomorphic to $(H^1(F), H^2(F), \cup, (-1)_F)$. Therefore, $\mathbf{F} \circ \mathbf{G}$ is naturally isomorphic to the identity functor \mathbf{Id} on Aug-Bilin-Fields . ■

Remark 4.2. In the above discussion, we may replace the absolute Galois group G_F by its maximal pro- p quotient $G_F(p)$. Indeed, it is a consequence of the Norm Residue Theorem that the inflation map $\text{inf}: H^i(G_F(p)) \rightarrow H^i(G_F)$ is an isomorphism for all i [7, Rem. 8.2].

5. Generic splitting fields

The following proposition restates in cohomological terms some main facts about Amitsur's construction of generic splitting fields of central simple algebras [1, 2, 26].

Proposition 5.1. *Given a field K and a cohomology element $\omega \in H^2(K)$, there exists a regular extension L of K with finite relative transcendence degree such that the following sequence is exact:*

$$0 \rightarrow \langle \omega \rangle \rightarrow H^2(K) \xrightarrow{\text{Res}_{L/K}} H^2(L).$$

By a slight abuse of notation, we call L a *generic splitting field* of ω over K .

We fix an algebraically closed field Ω of sufficiently large transcendence degree over the ground field. We may assume that all generic splitting fields in the following constructions will be taken inside Ω .

Proposition 5.2. *Let K be a field containing a root of unity of order p and let A be a subgroup of $H^2(K)$. Then there is a field extension L of K such that*

- (i) $\text{Res}_{L/K}: H^1(K) \rightarrow H^1(L)$ is injective; and
- (ii) $\text{Res}_{L/K}: A \rightarrow H^2(L)$ is an isomorphism.

Proof. We consider the set \mathcal{S} of all intermediate fields $K \subseteq L \subseteq \Omega$ such that (i) holds as well as:

- (ii') $\text{Res}_{L/K}: H^2(K) \rightarrow H^2(L)$ is injective on A .

It clearly contains K .

We partially order \mathcal{S} by setting $L \preceq L'$ if and only if $L \subseteq L'$, the map

$$\text{Res}_{L'/L}: H^1(L) \rightarrow H^1(L')$$

is injective, and $\text{Res}_{L'/L}: H^2(L) \rightarrow H^2(L')$ is injective on $\text{Res}_{L/K}(A)$.

For a nonempty totally ordered subset \mathcal{C} of \mathcal{S} , the union $L_{\mathcal{C}}$ of the fields in \mathcal{C} satisfies

$$H^i(L_{\mathcal{C}}) = \lim_{\substack{\longrightarrow \\ L \in \mathcal{C}}} H^i(L) \quad \text{for every } i.$$

Hence $L_{\mathcal{C}} \in \mathcal{S}$, and $L \preceq L_{\mathcal{C}}$ for every $L \in \mathcal{C}$. Zorn's lemma therefore yields a maximal field L in (\mathcal{S}, \preceq) .

It remains to prove that $\text{Res}_{L/K}(A) = H^2(L)$. To this end suppose that $\omega \in H^2(L)$ is not in $\text{Res}_{L/K}(A)$. Let L' be a generic splitting field of ω over L . It is a regular extension of L , so $\text{Res}_{L'/L}: H^1(L) \rightarrow H^1(L')$ is injective. The kernel $\langle \omega \rangle$ of $\text{Res}_{L'/L}: H^2(L) \rightarrow H^2(L')$ has trivial intersection with $\text{Res}_{L/K}(A)$, implying that $\text{Res}_{L'/L}: H^2(L) \rightarrow H^2(L')$ is injective on $\text{Res}_{L/K}(A)$. It follows using (ii') that $\text{Res}_{L'/K}: H^2(K) \rightarrow H^2(L')$ is injective on A . We conclude that $L' \in \mathcal{S}$. The maximality of L in \mathcal{S} implies that $L' = L$, contrary to $\omega \neq 0$. ■

6. Power series fields

Let $\Gamma = (\Gamma, \leq)$ be an ordered abelian group. Let K be a field and let $E = K((\Gamma))$ be the field of all formal power series $z = \sum_{\gamma \in \Gamma} a_{\gamma} t^{\gamma}$ with coefficients a_{γ} in K , exponents in Γ , and well ordered support $\text{Supp}(z) = \{\gamma \in \Gamma \mid a_{\gamma} \neq 0\}$ [8, §2.8]. It is Henselian with respect to the valuation $v: E^{\times} \rightarrow \Gamma$ defined by $v(z) = \min(\text{Supp}(z))$ [8, Cor. 18.4]. We identify the elements of K as constant power series in E , thus making K a subfield of E . For instance, $K((\mathbb{Z}))$, with the canonical order of \mathbb{Z} , is the usual formal power series field $K((t))$ with its canonical valuation.

The cohomology groups of K and E are related as follows [30, §3]: We choose a totally ordered set (Λ, \leq) and elements $\gamma_l, l \in \Lambda$, of Γ whose cosets form an \mathbb{F}_p -linear basis of $\Gamma/p\Gamma$. Then for every $i \geq 0$,

$$H^i(E) = \bigoplus_{j=0}^i \bigoplus_{\substack{l_1, \dots, l_j \in \Lambda \\ l_1 < \dots < l_j}} \text{Res}_{E/K} (H^{i-j}(K)) \cup (t^{l_1})_E \cup \dots \cup (t^{l_j})_E.$$

Moreover, for every $0 \leq j \leq i$ and $l_1 < \dots < l_j$ in Λ , the map

$$H^{i-j}(K) \rightarrow H^i(E), \quad \omega \mapsto \text{Res}_{E/K}(\omega) \cup (t^{l_1})_E \cup \dots \cup (t^{l_j})_E,$$

is injective. In particular,

$$\begin{aligned} H^1(E) &\cong H^1(K) \oplus (\Gamma/p\Gamma), \\ H^2(E) &\cong H^2(K) \oplus \left(H^1(K) \otimes \bigoplus_{l \in \Lambda} (\mathbb{Z}/p) \right) \oplus \bigoplus_{l_1 < l_2} (\mathbb{Z}/p). \end{aligned} \quad (6.1)$$

We immediately obtain the following.

Lemma 6.1. *For $E = K((\Gamma))$, the restriction morphism*

$$\text{Res}_{E/K}: (H^1(K), H^2(K), \cup, (-1)_K) \rightarrow (H^1(E), H^2(E), \cup, (-1)_E)$$

is a monomorphism of augmented \mathbb{F}_p -bilinear maps.

Lemma 6.2. *Denoting $m = \dim_{\mathbb{F}_p}(\Gamma/p\Gamma)$, one has $\dim_{\mathbb{F}_p} H^2(E) \geq m$ unless $H^1(K) = 0$ and $m \leq 2$.*

Proof. This follows from (6.1). ■

We now prove a weak form of the Main Theorem in the special case where $V = \text{Span}(\varepsilon)$.

Proposition 6.3. *Let (V, W, b, ε) be an augmented \mathbb{F}_p -bilinear map with $V = \text{Span}(\varepsilon)$. Then there exists a field E containing a root of unity of order p and a monomorphism*

$$(f_1, f_2): (V, W, b, \varepsilon) \rightarrow (H^1(E), H^2(E), \cup, (-1)_E).$$

Proof. First we take a field K as follows:

- If $\varepsilon = 0$, let $K = \mathbb{C}$. Here $(-1)_K = 0$.
- If $\varepsilon \neq 0$ (so $p = 2$) and $b(\varepsilon, \varepsilon) = 0$, let K be a field such that -1 is a sum of two squares but is not a square (for instance, an algebraic extension of \mathbb{Q} with $G_K \cong \mathbb{Z}_2$ and $\sqrt{-1} \notin K$, which exists e.g., by [13, Thm. 21.5.6]). Here $(-1)_K \neq 0$, but $(-1)_K \cup (-1)_K = 0$.
- If $b(\varepsilon, \varepsilon) \neq 0$ (so $\varepsilon \neq 0$ and $p = 2$), let $K = \mathbb{Q}_2$. Here -1 is not a sum of two squares, so $(-1)_K \cup (-1)_K \neq 0$ (see [16, Ch. VI, Cor. 2.24(4)]).

Note that in all cases, K contains a root of unity of order p .

Next let $m = \max\{\dim_{\mathbb{F}_p}(W), 3\}$. Let $\Gamma = \mathbb{Z}^m$, considered as an ordered abelian group with respect to the lexicographic order induced by the natural order on \mathbb{Z} , and set $E = K((\Gamma))$. Then $(-1)_E = 0$ if $\varepsilon = 0$, and $(-1)_E \cup (-1)_E = 0$ if $b(\varepsilon, \varepsilon) = 0$. By Lemma 6.2, $\dim_{\mathbb{F}_p}(H^2(E)) \geq m$.

We now define an \mathbb{F}_p -linear map $f_1: V = \text{Span}(\varepsilon) \rightarrow H^1(E)$ by $f_1(\varepsilon) = (-1)_E$. We further choose an \mathbb{F}_p -linear monomorphism $f_2: W \rightarrow H^2(E)$ satisfying

$$f_2(b(\varepsilon, \varepsilon)) = (-1)_E \cup (-1)_E.$$

Then (f_1, f_2) is a monomorphism of augmented \mathbb{F}_p -bilinear maps, as desired. \blacksquare

7. A transfinite construction

The following proposition will be the key step in the proof of the Main Theorem.

Proposition 7.1. *Let (U, W, b, ε) be an augmented \mathbb{F}_p -bilinear map. Suppose that \hat{U} is an \mathbb{F}_p -linear space containing U as a subspace of codimension 1, and let $(\hat{U}, W, \hat{b}, \varepsilon)$ be an augmented \mathbb{F}_p -bilinear map with $\hat{b}|_{U \times U} = b$ (but with the same space W). Let K be a field containing a root of unity of order p , and let (f_1, f_2) be a monomorphism as in the lower row of the diagram below.*

Then there exists a field extension \hat{K}/K , with $\text{Res}_{\hat{K}/K}: H^1(K) \rightarrow H^1(\hat{K})$ injective, and a monomorphism (\hat{f}_1, \hat{f}_2) of augmented \mathbb{F}_p -bilinear maps, making the following diagram commutative:

$$\begin{array}{ccc} (\hat{U}, W, \hat{b}, \varepsilon) & \xrightarrow{(\hat{f}_1, \hat{f}_2)} & (H^1(\hat{K}), H^2(\hat{K}), \cup, (-1)_{\hat{K}}) \\ \uparrow & & \uparrow \text{Res}_{\hat{K}/K} \\ (U, W, b, \varepsilon) & \xrightarrow{(f_1, f_2)} & (H^1(K), H^2(K), \cup, (-1)_K). \end{array}$$

Proof. By (6.1), one has

$$H^1(K((t))) = \text{Res}_{K((t))/K}(H^1(K)) \oplus \langle \delta \rangle,$$

where $\delta = (t)_{K((t))}$. We may therefore replace K by $K((t))$ to assume, using Lemma 6.1, that there exists $0 \neq \delta \in H^1(K)$ such that

$$f_1(U) \cap \langle \delta \rangle = \{0\}. \quad (7.1)$$

Next let $v \in \hat{U} \setminus U$, so $\hat{U} = U \oplus \text{Span}(v)$. We further choose a well ordered basis $\{u_\alpha\}_{1 \leq \alpha < \mu}$ for U , where μ is an ordinal. We write

$$V_\alpha = \text{Span}\{u_\beta \mid \beta \leq \alpha\}.$$

We construct by transfinite induction a chain of fields L_α , $\alpha < \mu$, with the following properties:

- (i) For all $\beta \leq \alpha$, $L_\beta \subseteq L_\alpha$.

- (ii) For all $\beta \leq \alpha$, the map $\text{Res}_{L_\alpha/L_\beta}: H^1(L_\beta) \rightarrow H^1(L_\alpha)$ is injective.
 (iii) For all $1 \leq \alpha < \mu$ and $u \in V_\alpha$,

$$\text{Res}_{L_\alpha/K} (f_1(u) \cup \delta) = \text{Res}_{L_\alpha/K} (f_2(\hat{b}(u, v))).$$

- (iv) For all $1 \leq \alpha < \mu$ and $\chi \in H^1(K)$, if

$$\text{Res}_{L_\alpha/K}(\chi \cup \delta) \in \text{Res}_{L_\alpha/K} (\text{Im}(f_2)),$$

then $\chi \in f_1(V_\alpha)$.

- (v) For all $1 \leq \alpha < \mu$, the restriction of $\text{Res}_{L_\alpha/K}$ to $\text{Im}(f_2)$ is injective.

Case I: $\alpha = 0$. We take $L_0 = K$, and there is nothing to prove.

Case II: $\alpha = \gamma + 1$ for some γ . Let L_α be a generic splitting field of

$$\text{Res}_{L_\gamma/K} (f_1(u_\alpha) \cup \delta - f_2(\hat{b}(u_\alpha, v))) \in H^2(L_\gamma).$$

Property (i) is immediate.

For Property (ii) note that since a generic splitting field is a regular extension, the map $\text{Res}_{L_\alpha/L_\gamma}: H^1(L_\gamma) \rightarrow H^1(L_\alpha)$ is injective. Further, by induction, $\text{Res}_{L_\gamma/L_\beta}: H^1(L_\beta) \rightarrow H^1(L_\gamma)$ is injective for all $\beta \leq \gamma$.

Property (iii) holds for $u = u_\alpha$ by the choice of L_α . Since, by induction, it is valid for every $u \in V_\gamma$, we deduce that it holds for all $u \in V_\alpha = V_\gamma \oplus \text{Span}(u_\alpha)$.

To show Property (iv), we take $\chi \in H^1(K)$ and $\omega \in \text{Im}(f_2)$ such that $\text{Res}_{L_\alpha/K}(\chi \cup \delta) = \text{Res}_{L_\alpha/K}(\omega)$. By the properties of a generic splitting field (Proposition 5.1), there exists an integer m such that

$$\text{Res}_{L_\gamma/K}(\chi \cup \delta - \omega) = m \text{Res}_{L_\gamma/K} (f_1(u_\alpha) \cup \delta - f_2(\hat{b}(u_\alpha, v))).$$

It follows that

$$\text{Res}_{L_\gamma/K} ((\chi - m f_1(u_\alpha)) \cup \delta) \in \text{Res}_{L_\gamma/K} (\text{Im}(f_2)).$$

By Property (iv) for L_γ , this implies that $\chi - m f_1(u_\alpha) \in f_1(V_\gamma)$, and therefore $\chi \in f_1(V_\alpha)$, as desired.

To prove Property (v), suppose that $\text{Res}_{L_\alpha/K}(\omega) = 0$ for some $0 \neq \omega \in \text{Im}(f_2)$. Thus $\text{Res}_{L_\gamma/K}(\omega) \in \text{Ker}(\text{Res}_{L_\alpha/L_\gamma})$. The exact sequence in Proposition 5.1 yields $0 \leq m < p$ such that

$$\text{Res}_{L_\gamma/K}(\omega) = m \text{Res}_{L_\gamma/K} (f_1(u_\alpha) \cup \delta - f_2(\hat{b}(u_\alpha, v))).$$

By induction, $\text{Res}_{L_\gamma/K}(\omega) \neq 0$, so necessarily $m \neq 0$. It follows that

$$\text{Res}_{L_\gamma/K} (f_1(u_\alpha) \cup \delta) \in \text{Res}_{L_\gamma/K} (\text{Im}(f_2)).$$

By Property (iv) for L_γ , this implies that $f_1(u_\alpha) \in f_1(V_\gamma)$. Since f_1 is injective, $u_\alpha \in V_\gamma$, contrary to the linear independence of u_β , $\beta < \mu$.

Case III: α is a limit ordinal. Let $L_\alpha = \bigcup_{\gamma < \alpha} L_\gamma$. Then

$$G_{L_\alpha} = \varprojlim_{\gamma < \alpha} G_{L_\gamma},$$

so

$$H^i(L_\alpha) = \varinjlim_{\gamma < \alpha} H^i(L_\gamma), \quad i = 1, 2.$$

Properties (i)–(v) now follow by a standard limit argument.

To complete the proof, we take $\hat{K} = \bigcup_{\alpha < \mu} L_\alpha$. It follows from Property (ii) that

$$\text{Res}_{\hat{K}/K}: H^1(K) \rightarrow H^1(\hat{K}) = \varinjlim_{\alpha < \mu} H^1(L_\alpha)$$

is injective.

Take α such that $\varepsilon \in V_\alpha$. Property (iii) for L_α implies that

$$\text{Res}_{\hat{K}/K} (f_1(\varepsilon) \cup \delta) = \text{Res}_{\hat{K}/K} (f_2(\hat{b}(\varepsilon, v))).$$

Since $(-1)_K = f_1(\varepsilon)$, we obtain using Lemma 2.1(b) that

$$\begin{aligned} \text{Res}_{\hat{K}/K} (\delta \cup \delta) &= \text{Res}_{\hat{K}/K} ((-1)_K \cup \delta) = \text{Res}_{\hat{K}/K} (f_1(\varepsilon) \cup \delta) \\ &= \text{Res}_{\hat{K}/K} (f_2(\hat{b}(\varepsilon, v))) = \text{Res}_{\hat{K}/K} (f_2(\hat{b}(v, v))). \end{aligned} \quad (7.2)$$

We now define \mathbb{F}_p -linear maps

$$\hat{f}_1: \hat{U} = U \oplus \text{Span}(v) \rightarrow H^1(\hat{K}), \quad \hat{f}_2: W \rightarrow H^2(\hat{K})$$

by

$$\hat{f}_1|_U = \text{Res}_{\hat{K}/K} \circ f_1, \quad \hat{f}_1(v) = \text{Res}_{\hat{K}/K}(\delta), \quad \hat{f}_2 = \text{Res}_{\hat{K}/K} \circ f_2.$$

We show that (\hat{f}_1, \hat{f}_2) is a morphism of augmented \mathbb{F}_p -bilinear maps. Indeed,

$$\hat{f}_1(\varepsilon) = \text{Res}_{\hat{K}/K} (f_1(\varepsilon)) = \text{Res}_{\hat{K}/K} ((-1)_K) = (-1)_{\hat{K}}.$$

Furthermore, take $\hat{u}, \hat{u}' \in \hat{U}$, and write $\hat{u} = u + mv$, $\hat{u}' = u' + m'v$ with $u, u' \in U$ and $m, m' \in \mathbb{Z}$. Using Property (iii) and equation (7.2), we compute:

$$\begin{aligned} \hat{f}_1(\hat{u}) \cup \hat{f}_1(\hat{u}') &= \text{Res}_{\hat{K}/K} ((f_1(u) + m\delta) \cup (f_1(u') + m'\delta)) \\ &= \text{Res}_{\hat{K}/K} (f_1(u) \cup f_1(u') + m' f_1(u) \cup \delta - m f_1(u') \cup \delta + m m' \delta \cup \delta) \\ &= \text{Res}_{\hat{K}/K} (f_2(\hat{b}(u, u')) + m' f_2(\hat{b}(u, v)) - m f_2(\hat{b}(u', v)) + m m' f_2(\hat{b}(v, v))) \\ &= \text{Res}_{\hat{K}/K} (f_2(\hat{b}(u, u') + m' \hat{b}(u, v) - m \hat{b}(u', v) + m m' \hat{b}(v, v))) \\ &= \text{Res}_{\hat{K}/K} (f_2(\hat{b}(\hat{u}, \hat{u}'))) \\ &= \hat{f}_2(\hat{b}(\hat{u}, \hat{u}')), \end{aligned}$$

as desired.

The commutativity of the diagram in the assertion of the proposition follows immediately from the definition of \hat{f}_1 and \hat{f}_2 .

Finally, by Property (ii), $\text{Res}_{\hat{K}/K}$ is injective on $H^1(K)$. Therefore the injectivity of \hat{f}_1 follows from (7.1). Furthermore, by Property (v), $\text{Res}_{\hat{K}/K}$ is injective on $\text{Im}(f_2)$, implying the injectivity of \hat{f}_2 . ■

8. The main result

Given augmented \mathbb{F}_p -bilinear pairs (V, W, b, ε) and $(V', W', b', \varepsilon')$, we set $(V, W, b, \varepsilon) \leq (V', W', b', \varepsilon')$ if and only if there exists a monomorphism

$$(f_1, f_2): (V, W, b, \varepsilon) \rightarrow (V', W', b', \varepsilon').$$

Thus our Main Theorem asserts that the subcategory **Aug-Bilin-Fields** is cofinal in **Aug- \mathbb{F}_p -Bilin** with respect to \leq in the following (slightly stronger) sense.

Theorem 8.1. *Let (V, W, b, ε) be an augmented \mathbb{F}_p -bilinear map. Then there exists a field F containing a root of unity of order p , and a monomorphism*

$$(f_1, f_2): (V, W, b, \varepsilon) \rightarrow (H^1(F), H^2(F), \cup, (-1)_F)$$

of augmented \mathbb{F}_p -bilinear maps such that f_2 is an isomorphism.

Proof. Let (V, W, b, ε) be an augmented \mathbb{F}_p -bilinear map. Let \mathcal{S} be the set of all triples (U, E, g) consisting of:

- (1) A subspace U of V containing ε ;
- (2) A field E containing a root of unity of order p ;
- (3) A monomorphism of augmented \mathbb{F}_p -bilinear maps

$$g = (g_1, g_2): (U, W, b|_{U \times U}, \varepsilon) \rightarrow (H^1(E), H^2(E), \cup, (-1)_E).$$

Proposition 6.3 shows that \mathcal{S} is nonempty, since $U = \text{Span}\{\varepsilon\}$ yields a triple in \mathcal{S} .

We partially order \mathcal{S} by setting $(U, E, g) \leq (U', E', g')$ if and only if $U \leq U'$, E is a subfield of E' such that $\text{Res}_{E'/E}: H^1(E) \rightarrow H^1(E')$ is injective, and there is a commutative diagram of morphisms of augmented \mathbb{F}_p -bilinear maps:

$$\begin{array}{ccc} (U', W, b, \varepsilon) & \xrightarrow{g'=(g'_1, g'_2)} & (H^1(E'), H^2(E'), \cup, (-1)_{E'}) \\ \uparrow & & \uparrow \text{Res}_{E'/E} \\ (U, W, b, \varepsilon) & \xrightarrow{g=(g_1, g_2)} & (H^1(E), H^2(E), \cup, (-1)_E). \end{array}$$

Every chain $\mathcal{C} = \{(U_i, E_i, g_i = (g_{i,1}, g_{i,2})) \mid i \in I\}$ in (\mathcal{S}, \leq) admits an upper bound (U, E, g) in \mathcal{S} , where

$$U = \bigcup_{i \in I} U_i, \quad E = \bigcup_{i \in I} E_i, \quad g = \left(\varinjlim_{i \in I} g_{i,1}, \varinjlim_{i \in I} g_{i,2} \right).$$

Zorn's lemma therefore yields a maximal element $(U, K, f = (f_1, f_2))$ in (\mathcal{S}, \leq) .

If $U \neq V$, then we choose an intermediate subspace $U \subset \hat{U} \subseteq V$, where U has codimension 1 in \hat{U} . Proposition 7.1 yields a field extension \hat{K}/K such that $\text{Res}_{\hat{K}/K}: H^1(K) \rightarrow H^1(\hat{K})$ is injective, and a monomorphism $\hat{f} = (\hat{f}_1, \hat{f}_2)$ of augmented \mathbb{F}_p -bilinear maps with a commutative diagram:

$$\begin{array}{ccc} (\hat{U}, W, b|_{\hat{U} \times \hat{U}}, \varepsilon) & \xrightarrow{(\hat{f}_1, \hat{f}_2)} & (H^1(\hat{K}), H^2(\hat{K}), \cup, (-1)_{\hat{K}}) \\ \uparrow & & \uparrow \text{Res}_{\hat{K}/K} \\ (U, W, b|_{U \times U}, \varepsilon) & \xrightarrow{(f_1, f_2)} & (H^1(K), H^2(K), \cup, (-1)_K). \end{array}$$

This contradicts the maximality of (U, K, f) . Consequently, $U = V$.

Finally, Proposition 5.2 yields a field extension F of K such that $\text{Res}_{F/K}: H^1(K) \rightarrow H^1(F)$ is injective and $\text{Res}_{F/K}: f_2(W) \rightarrow H^2(F)$ is an isomorphism. Then the composition

$$(\hat{f}_1, \hat{f}_2) = (\text{Res}_{F/K} \circ f_1, \text{Res}_{F/K} \circ f_2): (V, W, b, \varepsilon) \rightarrow (H^1(F), H^2(F), \cup, (-1)_F)$$

is a monomorphism of \mathbb{F}_p -bilinear maps, and moreover, \hat{f}_2 is an isomorphism. ■

We deduce an analogous result for non-augmented \mathbb{F}_p -bilinear maps, provided that they are skew-symmetric.

Corollary 8.2. *Let V, W be \mathbb{F}_p -linear spaces and let $b: V \times V \rightarrow W$ be a skew-symmetric \mathbb{F}_p -bilinear map. Then there exists a field F containing a root of unity of order p , an \mathbb{F}_p -linear monomorphism $f_1: V \rightarrow H^1(F)$ and an \mathbb{F}_p -linear isomorphism $f_2: W \rightarrow H^2(F)$, such that the following diagram commutes*

$$\begin{array}{ccccc} H^1(F) & \times & H^1(F) & \xrightarrow{\cup} & H^2(F) \\ \uparrow f_1 & & \uparrow f_1 & & \uparrow f_2 \\ V & \times & V & \xrightarrow{b} & W. \end{array}$$

Proof. Lemma 2.2 yields an augmented \mathbb{F}_p -bilinear map $(\hat{V}, W, \hat{b}, \varepsilon)$ such that V is an \mathbb{F}_p -subspace of \hat{V} and \hat{b} extends b . Now apply Theorem 8.1 for this augmented bilinear map to obtain a field F and \mathbb{F}_p -linear maps f_1, f_2 as required. ■

9. Graphs

Our next aim is to show that, in general, the monomorphism in Theorem 8.1 cannot be required to be an isomorphism even for surjective augmented bilinear maps. Examples showing this will be given in the next section. They will be based on a general construction of augmented bilinear maps from graphs, which we now describe.

For the proof we first recall from [15, §7.8] and [23, Ch. III, §3] the duality between the cup product and the relation structure of a general finitely generated pro- p group.

Let S be free pro- p group on finitely many generators x_1, \dots, x_n . We write $S^{(t)}$, $t = 1, 2, \dots$, for the lower p -central filtration of S , defined inductively (in the pro- p sense) by

$$S^{(1)} = S, \quad S^{(t+1)} = (S^{(t)})^p[S, S^{(t)}].$$

We identify $H^1(S)$ with the group of all profinite group homomorphisms $S \rightarrow \mathbb{F}_p$. The substitution map gives a bilinear map $S/S^{(2)} \times H^1(S) \rightarrow \mathbb{F}_p$, which is *perfect*, in the sense that both its left and right kernels are trivial. Let χ_1, \dots, χ_n be an \mathbb{F}_p -linear basis of $H^1(S)$ which is dual to the cosets of x_1, \dots, x_n in $S/S^{(2)}$.

Let R be a closed normal subgroup of S which is contained in the pro- p Frattini subgroup $S^{(2)} = S^p[S, S]$, and let $G = S/R$. We thus have a minimal pro- p presentation

$$1 \rightarrow R \rightarrow S \rightarrow G \rightarrow 1$$

of G . The inflation map $\text{Inf}: H^1(G) \rightarrow H^1(S)$ is an isomorphism, and as S is free pro- p , $H^2(S) = 0$ [23, Prop. 3.5.17]. It therefore follows from the five term exact sequence in profinite cohomology [23, Prop. 1.6.7] that the transgression map $\text{trg}: H^1(R)^G \rightarrow H^2(G)$ is an isomorphism.

There is a well-defined perfect \mathbb{F}_p -bilinear map [11, Cor. 2.2]

$$R/R^p[S, R] \times H^1(R)^G \rightarrow \mathbb{F}_p, \quad (\bar{r}, \psi) \mapsto \psi(r).$$

It gives rise to a perfect \mathbb{F}_p -bilinear map

$$\langle \cdot, \cdot \rangle: R/R^p[S, R] \times H^2(G) \rightarrow \mathbb{F}_p, \quad (\bar{r}, \omega) \mapsto -(\text{trg}^{-1}(\omega))(r).$$

We call it the *transgression pairing*.

Let r_1, \dots, r_m be a minimal system of generators of R as a closed normal subgroup of S . Equivalently, the cosets of r_1, \dots, r_m form an \mathbb{F}_p -linear basis of $R/R^p[S, R]$ [23, Cor. 3.9.3]. By duality, there is an isomorphism

$$H^2(G) \xrightarrow{\sim} \mathbb{F}_p^m, \quad \omega \mapsto ((\bar{r}_i, \omega))_{1 \leq i \leq m}.$$

Every element r of R has a unique presentation modulo $S^{(3)}$ as

$$r \equiv \prod_{i=1}^n x_i^{pa_{ri}} \prod_{1 \leq i < j \leq n} [x_i, x_j]^{a_{rij}} \pmod{S^{(3)}},$$

with $a_{ri}, a_{rij} \in \{0, 1, \dots, p-1\}$. Viewing χ_1, \dots, χ_n also as elements of $H^1(G)$, one has for $r \in R$ and $i < j$ that

$$a_{rij} = \langle \bar{r}, \chi_i \cup \chi_j \rangle, \tag{9.1}$$

by [23, Prop. 3.9.13].

Remark 9.6. In a similar manner, the exponents a_{ri} are related to the images of the χ_i under the Bockstein homomorphism $H^1(G) \rightarrow H^2(G)$ [23, Prop. 3.9.14]. These connections can be extended to higher terms $S^{(l)}$ in the lower p -central filtration using tools from the combinatorics of words [9]. We refer to [10] for the analogous theory in the case of the p -Zassenhaus filtration.

Proof of Proposition 9.5. We denote the vertices of $\Gamma = (V(\Gamma), E(\Gamma))$ by v_1, \dots, v_n . Let again S be the free pro- p group on the basis x_1, \dots, x_n , take $G = G_\Gamma$, and let R be the kernel of the epimorphism $S \rightarrow G_\Gamma$, $x_i \mapsto v_i$. Since R is generated as a closed normal subgroup of S by commutators, it is contained in $S^{(2)}$, and the previous discussion applies.

The commutators $r_e = [x_k, x_l]$ associated with the edges $e = (v_k, v_l) \in E(\Gamma)$, $k < l$, form a minimal system of generators of R as a closed normal subgroup of S . We may therefore identify $H^2(G_\Gamma)$ with $\mathbb{F}_p^{E(\Gamma)}$ as above, with $\omega_e \in \mathbb{F}_p$ denoting the e -th coordinate of $\omega \in H^2(G_\Gamma)$.

For $e = (v_k, v_l) \in E(\Gamma)$, $k < l$, and for $1 \leq i < j \leq n$ we have $a_{reij} = 1$, if $k = i$ and $l = j$, and $a_{reij} = 0$ in all other cases. In view of (9.1), the dual basis χ_1, \dots, χ_n of $H^1(G)$ satisfies

$$(\chi_i \cup \chi_j)_e = \begin{cases} 1, & \text{if } (v_i, v_j) = e, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the bilinear maps $\cup: H^1(G_\Gamma) \times H^1(G_\Gamma) \rightarrow H^2(G_\Gamma)$ and $b_\Gamma: V_\Gamma \times V_\Gamma \rightarrow W_\Gamma$ coincide on basis elements, giving the required isomorphism. ■

10. The common slot property

In [25, Thm. 5.6] Quadrelli proves that the pro- p right-angled Artin group G_{C_4} is not realizable as the maximal pro- p Galois group $G_F(p)$ of any field F containing a root of unity of order p . This result was extended by Snopce and Zalesskii, who used profinite Bass–Serre theory to prove the following remarkable fact.

Theorem 10.1 (Snopce–Zalesskii [27]). *For every prime p and for a finite simplicial graph Γ , the pro- p group G_Γ is realizable as the maximal pro- p Galois group $G_F(p)$ of a field F containing a root of unity of order p if and only if Γ does not contain either of the graphs L_3 or C_4 as an induced subgraph.*

Several other equivalent Galois-theoretic conditions are given in [27], as well as in the earlier work [6] by Cassella and Quadrelli. For instance, this is also equivalent to the Bloch–Kato property for G_Γ , as well as to the universal Koszulity of $H^\bullet(G_\Gamma)$ (see [6] for definitions). Another equivalent condition is that G_Γ is of *elementary type* in the following strong sense: It can be constructed in finitely many steps from the group \mathbb{Z}_p using free pro- p products and direct products with \mathbb{Z}_p .

See [5] for a generalization of Theorem 10.1 to oriented graphs.

Remark 10.2. As pointed out in [5], when $p = 2$ the field F in Theorem 10.1 can be taken to contain $\sqrt{-1}$. In fact, using the strong elementary type structure of G_Γ one can

even show that, for every p , there exists such a field F containing all roots of unity of p -power order. However this fact will not be used in the following discussion.

We now relate Theorem 10.1 to the augmented bilinear map $(V_\Gamma, W_\Gamma, b_\Gamma, 0)$ associated with the graph Γ . Namely, we show that for $p = 2$, the equivalent conditions of this theorem are also equivalent to $(V_\Gamma, W_\Gamma, b_\Gamma, 0)$ being isomorphic to an augmented \mathbb{F}_2 -bilinear map of field type. This gives plenty of examples of augmented bilinear maps which are not of field type (up to isomorphism), as desired – see Example 10.6 below.

Our proof is based on the following notion.

Definition 10.3. Let V and W be sets (at this point not assumed to be linear spaces) and consider a map $b: V \times V \rightarrow W$. We say that b has the *common slot property* if for every $v, u, v', u' \in V$ with $b(v, u) = b(v', u')$ there exists $u'' \in V$ such that $b(v, u) = b(v, u'')$ and $b(v', u') = b(v', u'')$.

Example 10.4. Let $p = 2$ and let F be a field of characteristic $\neq 2$. Then the cup product map

$$\cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$$

has the common slot property. See [16, Ch. III, Thm. 4.13] or [12, Lem. 98.15], and recall the identification of the cup product $(a_1)_F \cup (a_2)_F$ in $H^2(F)$ with the quaternion algebra $(a_1, a_2/F)$ in the 2-torsion subgroup ${}_2\text{Br}(F)$ of the Brauer group.

Theorem 10.5. Let $p = 2$ and let Γ be a finite simplicial graph. The following conditions are equivalent:

(a) *There exists a field F such that*

$$(V_\Gamma, W_\Gamma, b_\Gamma, 0) \cong (H^1(F), H^2(F), \cup, (-1)_F)$$

as augmented \mathbb{F}_2 -bilinear pairs;

(b) *The bilinear map $b_\Gamma: V_\Gamma \times V_\Gamma \rightarrow W_\Gamma$ has the common slot property;*

(c) *There exists a field F such that $G_\Gamma \cong G_F(2)$ and $-1 \in (F^\times)^2$;*

(d) *There exists a field F such that $(H^\bullet(G_\Gamma), 0) \cong (H^\bullet(F), (-1)_F)$ as κ -algebras;*

(e) *Γ does not contain either of the graphs L_3 or C_4 as an induced subgraph.*

Proof. For a field F of characteristic 2, the group $G_F(2)$ is free pro-2 and $H^i(F) = 0$ for every $i \geq 2$ [23, Thm. 6.1.4 and Cor. 3.9.5]. Further, one has $H^i(G_\Gamma) = 0$ for every $i \geq 2$ if and only if G_Γ is a free pro- p group. Therefore, when $\text{Char } F = 2$, the isomorphisms in (a), (c) and (d) mean that Γ has no edges. But then conditions (b) and (e) hold trivially. Consequently, we may restrict ourselves in conditions (a), (c) and (d) to fields F of characteristic $\neq 2$. Thus -1 is a root of unity of order 2 in F .

(a) \Rightarrow (b). This is a consequence of Example 10.4.

(c) \Rightarrow (d). Immediate.

(d) \Rightarrow (a). We apply the functor \mathbf{F} of Section 3 and use Theorem 9.5.

(c) \Leftrightarrow (e). This follows from Theorem 10.1 for $p = 2$ and Remark 10.2.

(b) \Rightarrow (e). Arguing by contradiction, we assume that the subgraph of Γ induced by the vertices v_1, v_2, v_3, v_4 is either L_3 or C_4 . Thus Γ contains the edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$, and no other edges among v_1, v_2, v_3, v_4 , except for $e = (v_1, v_4)$ when the induced subgraph is C_4 . We may assume that e_1, e_2, e_3 correspond to the first three coordinates of W_Γ .

We claim that the common slot property fails for

$$v = v_2 + v_3, \quad u = v_1 + v_2, \quad v' = v_1 + v_2 + v_3 + v_4, \quad u' = v_2.$$

This will yield the desired contradiction. Indeed, one has

$$b_\Gamma(v, u) = (1, 1, 0, 0, \dots, 0) = b_\Gamma(v', u').$$

Now let $a_1, \dots, a_n \in \mathbb{F}_2$. The coordinates at e_1, e_2, e_3 of the vectors

$$b_\Gamma\left(v, \sum_{i=1}^n a_i v_i\right), \quad \text{resp.,} \quad b_\Gamma\left(v', \sum_{i=1}^n a_i v_i\right)$$

are

$$a_1, a_2 + a_3, a_4, \quad \text{resp.,} \quad a_1 + a_2, a_2 + a_3, a_3 + a_4.$$

However, the system of equations

$$\begin{cases} a_1 = a_1 + a_2 = 1, \\ a_2 + a_3 = 1, \\ a_4 = a_3 + a_4 = 0 \end{cases}$$

obviously has no solutions. Therefore the above two vectors cannot be simultaneously equal to $(1, 1, 0, 0, \dots, 0)$, proving our claim. \blacksquare

Theorem 10.5 provides many examples showing that the restriction to monomorphisms in the Main Theorem is necessary, even for surjective augmented bilinear maps, as these cannot be replaced by isomorphisms.

Example 10.6. Let Γ be either L_{n-1} or C_n with $n \geq 4$. Then the augmented \mathbb{F}_2 -bilinear map $(V_\Gamma, W_\Gamma, b_\Gamma, 0)$ is not isomorphic to an augmented \mathbb{F}_2 -bilinear map of field type.

In view of these counterexamples, it would make sense to add the common slot property as an additional requirement to the definition of surjective augmented bilinear maps, in the case where the ground ring is \mathbb{F}_2 . This is closely related to the following notion due to Marshall ([17], [18, §2]).

Definition 10.7. A *quaternionic map* is a surjective map $q: G \times G \rightarrow Q$, where G is a group of exponent 2 with a distinguished element -1 , and Q is a set with a distinguished element 1, satisfying the following conditions for all $a, b, c \in G$:

- (1) $q(a, -a) = 1$;

- (2) $q(a, b) = q(a, c)$ implies that $q(a, bc) = 1$;
- (3) $q(a, b) = q(b, a)$;
- (4) q satisfies the common slot property.

To a field F of characteristic $\neq 2$ we associate a quaternionic map by taking $G = H^1(F)$, $-1 = (-1)_F$, and by taking Q to be the image of the cup product map $q = \cup: H^1(F) \times H^1(F) \rightarrow H^2(F)$, with $1 \in Q$ being the zero element of $H^2(F)$. It is a long-standing open question in quadratic form theory whether every quaternionic map can be realized as the quaternionic map of a field of characteristic $\neq 2$ (see [18, Open Problem 1 and Thm. 2.2], [16, Ch. XII, §8]). Note however that this possible axiomatization is restricted to the case $p = 2$.

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