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Best initial criterion for a degenerate Keller–Segel system with rotational flux terms in even-dimensional spaces

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Abstract. This paper presents a rigorous analysis for a degenerate diffusion Keller–Segel type system featuring rotational flux terms in even-dimensional spaces. The proposed model synthesizes fundamental principles from mathematical biology, fluid dynamics, and electrokinetic phenomena. Through discussion of the diffusion exponent $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$ and rotation angle $\alpha \in (2k\pi-\pi, 2k\pi+\pi]$, $k \in \mathbb{Z}$, we obtain results on the existence and blow-up of solutions. In particular, we derive a sharp initial criterion to distinguish the global existence and the finite-time blow-up of solutions for the case $\alpha \in (2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2})$, $k \in \mathbb{Z}$ and $m \in (\frac{2n}{n+2}, 2-\frac{2}{n})$, and show the existence of solutions under any initial condition for the case $\alpha \in (2k\pi-\pi, 2k\pi-\frac{\pi}{2}] \cup [2k\pi+\frac{\pi}{2}, 2k\pi+\pi]$, $k \in \mathbb{Z}$. This result reveals that repulsive effects caused by rotation effectively prevent solution blow-up.

1. Introduction

In this paper, we study the following degenerate Keller–Segel type system with rotational flux terms and diffusion exponent $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$ in even-dimensional spaces:

$$\rho_t = \Delta\rho^m - \chi \operatorname{div}(\rho A \nabla c), \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.1}$$

$$-\Delta c = \rho, \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.2}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}^n. \tag{1.3}$$

Here, $\rho(x,t)$ and $c(x,t)$ represent the bacterial density and chemical concentration, respectively, with χ being a positive constant and A an $n \times n$ matrix given by

$$A = \begin{pmatrix} R_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & R_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & R_k \end{pmatrix}, \quad R_i = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad i = 1,2,\ldots,k,\ k = \frac{n}{2}, \tag{1.4}$$

where the constant parameter $\alpha \in (-\pi, \pi]$ is called the rotation angle. In fact, due to the periodicity of the cosine function, we can take any angle $\alpha \in (-\pi + 2k\pi, \pi + 2k\pi]$, $k \in \mathbb{Z}$. For convenience, we only consider one period $(-\pi, \pi]$.

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The Keller–Segel system with linear diffusion was first proposed by Patlak [28] and later extended by Keller and Segel [22, 23]. This fundamental model has emerged as the cornerstone in mathematical biology to characterize chemotactic behavior, especially in modeling collective cellular migration and bacterial population patterning.

For the degenerate Keller–Segel system of the porous medium type in $\mathbb{R}^n$ $(n \geq 3)$, substantial studies have conclusively established global existence results. Specifically, most existing studies have focused on analyzing the special case of $A = I$ (corresponding to $\alpha = 0$) in the system (1.1)–(1.3) with arbitrary spatial dimensions as follows.

$$\rho_t = \Delta\rho^m - \chi\,\mathrm{div}(\rho\nabla c), \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.5}$$

$$-\Delta c = \rho, \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.6}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}^n. \tag{1.7}$$

In 2006, Sugiyama [29] conducted a complete classification framework for solution behavior in such a system with the non-linear exponent m serving as the key classifying parameter. The analysis reveals a best critical exponent $m_c = 2 - \frac{2}{n}$ that sharply delineates the solution behavior into two distinct regimes. In the subcritical regime where $m > m_c$, diffusion effects prevail, guaranteeing the existence of global solutions for any initial data. In contrast, when $m < m_c$ (the supercritical case), the aggregation mechanism prevails and leads to finite-time blow-up when the initial data are sufficiently large. Building upon these works, significant research efforts have since been devoted to investigating deeper analytical properties of solutions to the Keller–Segel system in porous media types, including boundedness of the solutions, long-time behavior, and optimal initial criterion. For a detailed discussion, see [1, 3, 8]. Several important studies have since been established in the literature. Notable developments include

- Supercritical case $m < 2 - \frac{2}{n}$. The system (1.5)–(1.6) exhibits both global existence and blow-up phenomena. Chen, Liu and Wang [9] in 2012, Bian and Liu [2] in 2013 established a sharp threshold in the space L^p (where $p = \frac{n(2-m)}{2}$) through application of the Hardy–Littlewood–Sobolev inequality to distinguish between the global existence and finite-time blow-up of weak solutions. But they still did not give an explicit critical value. In 2014, Chen and Wang [10] proved that there exists a best initial criterion to distinguish the global existence and the blow-up of solutions in the parabolic-elliptic case with $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$. The best initial criterion is dominated by the sharp constant of the HLS inequality. For the parabolic-parabolic case, the best initial criterion is dominated by the sharp constant of the Sobolev inequality [31].
- Critical case $m = 2 - \frac{2}{n}$. In 2009, Blanchet, Carrillo and Laurençot [4] proved that the system (1.5)–(1.6) possesses a critical mass M_c. Their work established two fundamental results (i) the global existence of solutions for subcritical masses ($M \leq M_c$), and (ii) finite-time blow-up phenomena for supercritical masses ($M > M_c$). In particular, they discovered significantly more complex finite-time blow-up at $M = M_c$ compared to classical Keller–Segel systems. Moreover, in 2024 the global existence of solutions to the problem for two species was deduced in [7].

- Subcritical case $m > 2 - \frac{2}{n}$. The system admits global-in-time solutions that remain uniformly bounded for arbitrarily large initial data [30]. Furthermore, unstable stationary states emerged for specific values of m, the more detailed analysis can be found in [8, 20].

This paper focuses primarily on the global existence and blow-up of solutions for the model (1.1)–(1.3) with diffusion exponents $\frac{2n}{n+2} \le m < 2 - \frac{2}{n}$ and the rotational flux term. We are interested in the influence of the rotational flux term on the properties of solutions to the degenerate model. The tensor sensitivity coefficient χA emerges as our most interesting parameter in our system, critically determining the characteristics of the chemotactic response. Recent works have extended the Keller–Segel model by making the chemotactic sensitivity a matrix (denoted χA) instead of a single number. This allows cells to respond differently when moving in different directions. This generalization is motivated by experimental evidence of direction-dependent chemotaxis observed in various bacterial species in [16,19]. Experimental observations demonstrate that near surfaces, the combination of bacterial swimming orientation bias and endogenous cellular rotation generates a net torque, leading to distinctive spiral trajectories [34]. As a result, the observed cellular trajectories mainly reflect rotational transport mechanisms rather than classical chemotaxis. These developments have prompted extensive studies of solution behaviors in chemotaxis systems with rotational sensitivity terms in [6,17,27,32,33,35]. These rotational sensitivity terms can depend on spatial and temporal domains, which may exhibit functional dependence on both cell density $\rho(x,t)$ and chemical concentration $c(x,t)$. This paper focuses on constant rotation matrices that are independent of $\rho(x,t)$, $c(x,t)$, x, t in the algebraic sense. Reference [21] indicates that the n-dimension rotation matrix R must satisfy the following properties:

- Orthogonality: $R^T R = RR^T = I_n$ (that is, R is an orthogonal matrix).
- Determinant of 1: $\det(R) = 1$ (ensures a pure rotation without reflection).

The set of all matrices that meet the above conditions is called the Special Orthogonal Group, denoted by

$$SO(n) = \left\{R \in \mathbb{R}^{n\times n} \mid R^T R = I_n,\ \det(R) = 1\right\}.$$

According to the properties of the rotation matrix R, there exists a real orthogonal matrix Q such that

$$Q^T R Q = \begin{pmatrix} R_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & R_r & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & I_{n-2r} \end{pmatrix}_{n\times n}. \tag{1.8}$$

Given that the matrix (1.8) is equivalent to R, we only need to analyze (1.8) when discussing rotation matrices. Here, I_{n-2r} is an identity matrix and each R_i, $i = 1\cdots r$ is a

2-dimension rotation matrix given by

$$R_i = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix} = \cos\alpha_i I - \sin\alpha_i J \quad \text{with } \alpha_i \in (-\pi, \pi], \tag{1.9}$$

where α_i, $i = 1, \ldots, r$ are called rotation angles. I is an identity matrix 2×2 and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a skew-symmetric matrix satisfying $J^T = -J$ and $\det(J) = 1$.

In recent years, numerous results have been established for the Keller–Segel system with rotational sensitivity in (1.9). In 2020, Espejo and Wu in [18] investigated the Cauchy problem of the following generalized Keller–Segel system with rotational sensitivity in $\mathbb{R}^2$:

$$\begin{cases} \rho_t = \Delta\rho - \chi \operatorname{div}(\rho A \nabla c), & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ -\Delta c = \rho, & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \end{cases} \tag{1.10}$$

where χ is a positive constant and A denotes a 2×2 matrix given by

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{with } \alpha \in (-\pi, \pi].$$

For the system (1.10), they established the optimal critical mass $m_c = \frac{8\pi}{\chi\cos\alpha}$ that delineates between global existence and finite-time blow-up of solutions in two dimensions. The global existence of solutions is guaranteed when the initial mass satisfies $m < m_c$ for the rotation angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In contrast, for angle $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$, free-energy solutions globally exist in time. In particular, the rotation angle α in (1.10) serves as a key parameter that bridges several fundamental physical models. When $\alpha = 0$, the system simplifies to the standard Keller–Segel framework to model biological chemotactic movement. The classical drift-diffusion model with Poisson coupling for an electron gas (without confining potential) is recovered in this system when $\alpha = \pi$, as established in [26]. It should be noted that when $\chi = 1$ and $\alpha = -\frac{\pi}{2}$, the system transforms into the vorticity formulation of the incompressible Navier–Stokes equations, as demonstrated in [11]. Thus, the parameter α consequently creates a mathematical bridge between disparate systems spanning biological aggregation, electron transport, and fluid dynamics.

Subsequent investigations have further explored the role of the 2×2 matrix A. A notable contribution was made by Li and Wang (2022) in [24], who established the finite-time blow-up of non-radial solutions while demonstrating the global existence for radial solutions in a two-dimensional space within bounded domains. Furthermore, Cuentas and Espejo (2024) in [13] presented a complete characterization of the solution behavior in a multi-species Keller–Segel model with rotational flux terms, establishing precise conditions for both global regularity and finite-time singularity formation. Extending this work in 2025, they in [15] proved finite-time blow-up results for solutions to a Neumann problem. In parallel development, Zhang, Xu, and Han (2025) in [36] established the existence of classical global bounded solutions in two dimensions.

Naturally, several significant results have also been established in n-dimensional spaces in recent years. Cuentas, Espejo and Suzuki (2024) in [14] conducted a rigorous investigation of Keller–Segel models with a tensorial flux term A, where A denotes a nonsingular matrix $n \times n$ whose components satisfy the positive-definiteness condition

$$x^T \left((AA^T)^{\frac{1}{2}}\right)^{-1} Ax > 0.$$

In particular, the three-dimensional matrix

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

satisfies the condition of the matrix A above. They established the existence and finite-time blow-up of solutions to this system. Chen, Liu and Xiang (2025) in [12] studied the coupled Keller–Segel system with indirect signal production and matrix-valued sensitivity in high dimensions, the authors established a sharp critical mass $m_c = \frac{(8\pi)^2}{\chi\cos\alpha}$ in a four-dimensional space. The 4×4 matrix S was given by

$$S = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{with } \alpha \in (-\pi, \pi].$$

When the rotation angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there exist global solutions provided that the initial mass of the L^1 norm of the initial data is below the initial mass. This result extends to n-dimensional cases where the sensitivity matrix takes the form $S = \cos\theta \cdot I + A$ with $A^T = -A$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Under appropriate smallness conditions on the initial data, the system admits globally bounded solutions that exhibit special exponential convergence properties.

While the n-dimensional matrices in these two references is not rotation matrices themselves, they offer valuable insights for our study of the degenerate Keller–Segel system with a n-dimensional rotation matrix. To preserve the energy structure essential for our analysis, this work discusses only a special case when A is an even-dimensional matrix and any rotation angles α_i are equal to α in (1.4). For odd dimensions, non-rotational components (the identity matrix blocks I_{n-2r} in the matrix structure) introduce significant challenges in constructing the functional of free energy. Consequently, this study does not provide conclusive results for this case.

Our research focuses on the degenerate Keller–Segel model (1.1)–(1.3) with rotation matrices at an angle $\alpha \in (-\pi, \pi]$, specifically considering matrices (1.4). In this model, the chemical concentration $c(x,t)$ is determined by the fundamental solution $\Phi(x)$ of the Laplace equation as follows:

$$c(x,t) = \Phi(x) * \rho(x,t) = \frac{1}{(n-2)\omega(n)} \int_{\mathbb{R}^n} \frac{\rho(y,t)}{|x-y|^{n-2}} dy, \tag{1.11}$$

where $\omega(n) = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the surface area of the n-dimensional unit sphere. The initial data $\rho_0(x)$ satisfies that

$$\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n), \quad \|\rho_0\|_{L^1(\mathbb{R}^n)} =: M_0. \tag{1.12}$$

A fundamental property of this system is mass conservation, which arises directly from the divergence form of the equations,

$$\int_{\mathbb{R}^n} \rho(x,t)dx = \int_{\mathbb{R}^n} \rho(x,0)dx = M_0.$$

The system (1.1) possesses a functional of naturally associated free energy given by

$$\mathcal{F}_\alpha(\rho, c) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m dx - \frac{\chi}{2}\cos\alpha \int_{\mathbb{R}^n} \rho c dx. \tag{1.13}$$

Substituting equation (1.11) into (1.13), the free energy functional can be reformulated as

$$\mathcal{F}_\alpha(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m dx - \frac{\chi\cos\alpha}{2(n-2)\omega(n)}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dx\,dy. \tag{1.14}$$

In our analysis to the free energy functional, the following Hardy–Littlewood–Sobolev (HLS) inequality plays a pivotal role. The complete statement and the derivation of this fundamental inequality are available in [25].

Proposition 1.1 (Hardy–Littlewood–Sobolev (HLS) inequality). *Let $\rho \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dx\,dy \le C(n)\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},$$

where the best constant $C(n)$ is given by

$$C(n) = \pi^{\frac{(n-2)}{2}}\frac{1}{\Gamma(\frac{n}{2}+1)}\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{-\frac{2}{n}}.$$

The key to studying the properties of solutions is to obtain the boundedness of the $L^{\frac{2n}{n+2}}$-norm of ρ (see Lemma 2.3) by using the HLS inequality and decomposing the free energy $\mathcal{F}_\alpha(\rho)$. Here, we define an auxiliary function

$$f(s) := \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}s - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}s^{\frac{n-2}{n(m-1)}}, \tag{1.15}$$

and give that $f(s)$ is a strictly concave function in $s > 0$ for $m < 2 - \frac{2}{n}$ and reaches maximum at

$$s^* = \left(\frac{2(n-2)\omega(n)}{\chi\cos\alpha(m-1)C(n)}M_0^{\frac{2n-m(n+2)}{n-2}}\right)^{\frac{n(m-1)}{2n-2-nm}}. \tag{1.16}$$

Using the properties of the matrix (1.4) and combining techniques to handle the degenerate Keller–Segel equation, we establish complete results regarding the global existence and the finite-time blow-up of solutions for the model (1.1)–(1.3) with the diffusion exponent $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$. Specifically, for angles within $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the drift combines attraction and pure rotation. Therefore, for different initial conditions, both existence and blow-up may occur. Outside this interval, it is repulsive to rotation. Thus, the solutions are global for any initial data. The main results are shown in the table below.

Rotational angle	Diffusion exponent	Initial data	Existence condition	Blow-up condition
$\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$m \in \left(\frac{2n}{n+2}, 2-\frac{2}{n}\right)$	$\rho_0 \in L^1_+ \cap L^m$ $\mathcal{F}_\alpha(\rho_0) < f(s^*)$	$\\|\rho_0\\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} < s^*$	$\\|\rho_0\\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} > s^*$
$\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$m = \frac{2n}{n+2}$	$\rho_0 \in L^1_+ \cap L^m$ $\mathcal{F}_\alpha(\rho_0) < f(s^*)$	$\\|\rho_0\\|_{L^m} < C_s < s^*$	$\\|\rho_0\\|_{L^m}^m > s^*$
$\alpha \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$	$m \in \left[\frac{2n}{n+2}, 2-\frac{2}{n}\right)$	$\rho_0 \in L^1_+ \cap L^m$	any initial data	no exist

Here, $f(s^*)$ and s^* are given by (1.15) and (1.16), respectively. The constant C_s is defined as

$$C_s = \left(\frac{4m^2}{(\chi \cos\alpha)(2m-1)^2 G} \right)^{\frac{1}{2-m}}, \tag{1.17}$$

where G is a constant in the Gagliardo–Nirenberg–Sobolev (GNS) inequality. As shown in the table, there is the notable finding that blow-up is prevented when $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$. When $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there is a best initial criterion depending on the rotation angle. In this paper, we discuss only an even-dimensional matrix A, which is used to maintain the energy structure. The analysis of odd-dimensional rotation matrices presents two principal challenges: (i) difficulties in constructing free energy functionals, and (ii) all current results, see [12], are studied under the assumption of radial symmetry, with no conclusions available for the general case. Consequently, this study does not provide any results for this case.

This paper is arranged as follows. In Section 2, we investigate the second moment estimates for the rotation equation in arbitrary dimensions. We prove the monotonic decay of free energy over time and identify the initial criterion through the decomposition of the free energy and using the HLS inequality. Furthermore, we establish finite-time blow-up behavior under supercritical initial conditions. Section 3 is primarily devoted to studying the global existence of solutions under subcritical initial conditions. In particular, we discover that when the rotation angle $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$, there exist global solutions for arbitrary initial values.

2. Blow-up of solutions

In this section, we consider the finite-time blow-up of solutions to the system (1.1)–(1.3) with rotation angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We begin by establishing generalized second moment estimates for arbitrary dimensions ($n \geq 3$) under the matrix A specified in (1.8).

Lemma 2.1 (Moment estimate). *Let $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$ and $\rho_0|x|^2 \in L^1(\mathbb{R}^n)$. Suppose that the matrix A is defined by (1.8). Then the second moment satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2\rho(x,t)dx \le 2n\|\rho\|^m_{L^m(\mathbb{R}^n)} - \chi(n-2)\min_{i=1,\dots,r}\{\cos\alpha_i, 1\}\int_{\mathbb{R}^n}\rho(x,t)c(x,t)dx,$$

where $m \ge 1$, $r \le n/2$ and $\alpha_i \in (-\pi/2, \pi/2)$.

Proof. Multiplying equation (1.1) by $|x|^2$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}\frac{dm_2(t)}{dt} &= \int_{\mathbb{R}^n}|x|^2\Delta\rho^m dx - \chi\int_{\mathbb{R}^n}|x|^2\operatorname{div}(\rho A\nabla c)dx\\ &= 2n\int_{\mathbb{R}^n}\rho^m dx + 2\chi\int_{\mathbb{R}^n} x\cdot\rho A(\rho*\nabla\Phi)dx\\ &= 2n\int_{\mathbb{R}^n}\rho^m dx - 2\chi\int_{\mathbb{R}^n} x\cdot\left(\rho(x,t)A\frac{1}{\omega(n)}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\rho(y,t)dy\right)dx\\ &= 2n\int_{\mathbb{R}^n}\rho^m dx - \frac{2\chi}{\omega(n)}\int_{\mathbb{R}^n\times\mathbb{R}^n} x\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy. \end{aligned}\tag{2.1}$$

For the second term of the last equality in (2.1), interchanging x and y gives

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} x\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy\\ &\qquad = -\int_{\mathbb{R}^n\times\mathbb{R}^n} y\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} x\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy\\ &\qquad = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}(x-y)\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy.\end{aligned}\tag{2.2}$$

Moreover, since $\alpha_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ implies $\cos\alpha_i > 0$ for all $i = 1,\dots,r$, the matrix A is positive definite, that is, for all non-zero vectors $x \in \mathbb{R}^n$,

$$\begin{aligned}x^TAx &= x^T\left(\frac{1}{2}(A+A^T)\right)x\\ &= x^T\begin{pmatrix}\cos\alpha_1 I_2 & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \cdots & \cos\alpha_r I_2 & \mathbf{0}\\ \mathbf{0} & \cdots & \mathbf{0} & I_{n-2r}\end{pmatrix}x\\ &\ge \min_{i=1,\dots,r}\{\cos\alpha_i, 1\}x^Tx = \min_{i=1,\dots,r}\{\cos\alpha_i, 1\}|x|^2.\end{aligned}$$

Thus, (2.2) can be shown

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} x\cdot A\frac{x-y}{|x-y|^n}\rho(x,t)\rho(y,t)dx\,dy\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{x-y}{|x-y|^{\frac{n}{2}}}\cdot A\frac{x-y}{|x-y|^{\frac{n}{2}}}\rho(x,t)\rho(y,t)dx\,dy\\
&\quad\geq\frac{1}{2}\min_{i=1,\dots,r}\{\cos\alpha_i,1\}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{1}{|x-y|^{n-2}}\rho(x,t)\rho(y,t)dx\,dy\\
&\quad=\frac{(n-2)\omega(n)}{2}\min_{i=1,\dots,r}\{\cos\alpha_i,1\}\int_{\mathbb{R}^n}\rho(x,t)c(x,t)dx.
\end{aligned}\tag{2.3}$$

Substituting (2.3) into (2.1) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}|x|^2\rho(x,t)dx\\
&\quad\leq 2n\|\rho\|^m_{L^m(\mathbb{R}^n)}-\chi(n-2)\min_{i=1,\dots,r}\{\cos\alpha_i,1\}\int_{\mathbb{R}^n}\rho(x,t)c(x,t)dx.
\end{aligned}$$

Specifically, taking the matrix A as (1.4), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}|x|^2\rho(x,t)dx\leq 2n\|\rho\|^m_{L^m(\mathbb{R}^n)}-\chi(n-2)\cos\alpha\int_{\mathbb{R}^n}\rho(x,t)c(x,t)dx.\tag{2.4}$$

■

We then prove that the free energy satisfies the fundamental energy law, guaranteeing its non-increasing monotonic behavior for the rotation angle $\alpha\in(-\pi,\pi]$. Our current analysis focuses on the specific form of A given in (1.4). This is because when A follows the general representations in (1.8) with a different form of α_i, $i=1,2,\dots,r$, the construction of the free energy structure is not easily achieved. Consequently, this problem also becomes our next research subject. Without additional explanation, all subsequent analyses in this work assume that the matrix A adopts the form in (1.4).

Lemma 2.2 (Energy and energy dissipation). *Let $\rho_0\in L^1_+\cap L^m(\mathbb{R}^n)$. Then, for any $m>1$ and rotation angle $\alpha\in(-\pi,\pi]$, the energy functional satisfies*

$$\frac{d}{dt}\mathcal{F}_\alpha(\rho,c)=-\int_{\mathbb{R}^n}\rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1}-\chi\cos\alpha c\right)\right|^2dx\leq 0,\quad\forall t>0.\tag{2.5}$$

Proof. Referring to [18, (2.4)], where the two-dimensional rotation matrix is expressed as a linear combination of the identity matrix and a skew-symmetric matrix, the matrix A can be written

$$A=\cos\alpha\cdot I_{n\times n}-\sin\alpha\cdot\begin{pmatrix}J_1&\cdots&\mathbf{0}\\\vdots&\ddots&\vdots\\\mathbf{0}&\cdots&J_k\end{pmatrix}_{n\times n},\tag{2.6}$$

where $I_{n\times n}$ is the identity matrix and J_i, $i=1,\dots,k$ is the 2×2 skew-symmetric matrix.

In addition, we define the differential operator $\nabla^{\perp}$ as

$$\nabla^{\perp} c = \begin{pmatrix} J_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & J_k \end{pmatrix}_{n\times n} \cdot \nabla c,$$

namely, $\nabla^{\perp} c$ is an orthogonal operator to ∇c satisfying

$$\langle \nabla c, \nabla^{\perp} c\rangle = 0, \quad \operatorname{div}(\nabla^{\perp} c) = 0. \tag{2.7}$$

Thus, we rewrite the equation (1.1) as

$$\begin{aligned}\rho_t &= \Delta \rho^m - \chi \operatorname{div}(\rho A \nabla c) \\ &= \Delta \rho^m - \chi \operatorname{div}(\cos\alpha \rho \nabla c - \sin\alpha \rho \nabla^{\perp} c) \\ &= \operatorname{div}(\nabla \rho^m - \chi \cos\alpha \rho \nabla c) + \chi \sin\alpha \operatorname{div}(\rho \nabla^{\perp} c) \\ &= \operatorname{div}\left(\rho \nabla \left(\frac{m}{m-1}\rho^{m-1} - \chi \cos\alpha c\right)\right) + \chi \sin\alpha \operatorname{div}(\rho \nabla^{\perp} c).\end{aligned} \tag{2.8}$$

Multiplying (2.8) by $\frac{m}{m-1}\rho^{m-1} - \chi\cos\alpha c$ and integrating with $\mathbb{R}^n$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \rho_t \left(\frac{m}{m-1}\rho^{m-1} - \chi \cos\alpha c\right) dx \\ &\qquad = -\int_{\mathbb{R}^n} \rho \left|\nabla \left(\frac{m}{m-1}\rho^{m-1} - \chi\cos\alpha c\right)\right|^2 dx \\ &\qquad\quad - \chi \sin\alpha \int_{\mathbb{R}^n} \rho \nabla^{\perp} c \cdot \nabla \left(\frac{m}{m-1}\rho^{m-1} - \chi \cos\alpha c\right) dx.\end{aligned} \tag{2.9}$$

The second integral on the right-hand side of equality in (2.9) vanishes since

$$\begin{aligned}&-\int_{\mathbb{R}^n} \rho \nabla^{\perp} c \cdot \nabla \left(\frac{m}{m-1}\rho^{m-1} - \chi\cos\alpha c\right) dx \\ &\qquad = -\int_{\mathbb{R}^n} \rho \nabla^{\perp} c \cdot \nabla \left(\frac{m}{m-1}\rho^{m-1}\right) dx + \chi \cos\alpha \int_{\mathbb{R}^n} \rho \nabla^{\perp} c \cdot \nabla c \, dx \\ &\qquad = -\int_{\mathbb{R}^n} \nabla \rho^m \cdot \nabla^{\perp} c \, dx + \chi\cos\alpha \int_{\mathbb{R}^n} \rho \nabla^{\perp} c \cdot \nabla c \, dx \\ &\qquad = \int_{\mathbb{R}^n} \rho^m \operatorname{div}(\nabla^{\perp} c) dx = 0.\end{aligned} \tag{2.10}$$

Moreover, the left-hand side term of equality in (2.9) can be written in

$$\begin{aligned}&\int_{\mathbb{R}^n} \rho_t \left(\frac{m}{m-1}\rho^{m-1} - \chi\cos\alpha c\right) dx \\ &= \int_{\mathbb{R}^n} \left(\frac{m}{m-1}\rho^{m-1}\rho_t\right) dx - \chi \cos\alpha \int_{\mathbb{R}^n} \rho_t(x,t) \int_{\mathbb{R}^n} \rho(y,t)\Phi(x-y) dy\, dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt}\left(\frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m dx - \frac{\chi}{2}\cos\alpha\int_{\mathbb{R}^n\times\mathbb{R}^n}\rho(x,t)\rho(y,t)\Phi(x-y)dx\,dy\right)\\
&= \frac{d}{dt}\mathcal{F}_\alpha(\rho,c). \qquad (2.11)
\end{aligned}
$$

Concerning the second term in the second equality of (2.11), its expression follows from the symmetry property

$$
\int_{\mathbb{R}^n\times\mathbb{R}^n}\rho_t(x,t)\rho(y,t)\Phi(x-y)dx\,dy = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n\times\mathbb{R}^n}\rho(x,t)\rho(y,t)\Phi(x-y)dx\,dy.
$$

Therefore, substituting (2.10) and (2.11) into (2.9), we obtain

$$
\frac{d}{dt}\mathcal{F}_\alpha(\rho,c) = -\int_{\mathbb{R}^n}\rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1} - \chi\cos\alpha c\right)\right|^2 dx \le 0, \quad \forall t>0. \qquad \blacksquare
$$

Next, since free energy decreases monotonically with respect to time, we identify the initial criteria s^* by decomposition of free energy. We then analyze the solution behavior of the system (1.1)–(1.3) in space $L^{\frac{2n}{n+2}}$ under certain initial conditions. The methodology used here is similar to that in [9, 10]. For completeness, we present the detailed procedure in the following.

Lemma 2.3. *Assume the initial free energy $\mathcal{F}_\alpha(\rho_0) < \mathcal{F}_\alpha^* := f(s^*)$, which is defined in* (1.15). *For any $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $m\in[\frac{2n}{n+2}, 2-\frac{2}{n})$,*

(i) *if the initial data $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a constant $0<\mu_1<1$ such that*

$$
\|\rho\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (\mu_1 s^*)^{\frac{n-2}{2n(m-1)}}, \quad \textit{for all } t>0;
$$

(ii) *if the initial data $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a constant $\mu_2>1$ such that*

$$
\|\rho\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (\mu_2 s^*)^{\frac{n-2}{2n(m-1)}}, \quad \textit{for all } t>0,
$$

where s^* is defined in (1.16).

Proof. Firstly, we separate the free energy (1.14) into two parts. For any $t>0$,

$$
\begin{aligned}
\mathcal{F}_\alpha(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m(x,t)dx - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\\
&\quad + \frac{\chi\cos\alpha}{2(n-2)\omega(n)}\left(C(n)\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} - \int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dx\,dy\right)\\
&=: \mathcal{F}_1(\rho) + \mathcal{F}_2(\rho),
\end{aligned}
$$

where $C(n)$ represents the best constant in the HLS inequality. By Proposition 1.1, we know the second part $\mathcal{F}_2(\rho)\ge 0$ for $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$. For $m=\frac{2n}{n+2}$, the functional $\mathcal{F}_1(\rho)$ can be expressed

$$\mathcal{F}_1(\rho) = \frac{1}{m-1}\|\rho\|^{\frac{2n}{n+2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}. \tag{2.12}$$

For $m \in (\frac{2n}{n+2}, 2-\frac{2}{n})$, we give $\mathcal{F}_1(\rho)$ that satisfies

$$\begin{aligned}\mathcal{F}_1(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m(x,t)dx - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\\ &\geq \frac{1}{m-1}\|\rho\|^{\frac{(\theta-1)m}{\theta}}_{L^1(\mathbb{R}^n)}\|\rho\|^{\frac{m}{\theta}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\\ &= \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.\end{aligned} \tag{2.13}$$

Notice that the first inequality is shown by using the interpolation inequality

$$\|\rho\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq \|\rho\|^{1-\theta}_{L^1(\mathbb{R}^n)}\|\rho\|^{\theta}_{L^m(\mathbb{R}^n)}, \tag{2.14}$$

where $\theta = \frac{m(n-2)}{2n(m-1)} \in (0,1)$ due to $m > \frac{2n}{n+2}$. Thus, for $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$, combining (2.12) and (2.13) and depending on $\mathcal{F}_2(\rho) \geq 0$, we find that for any $t > 0$, $\mathcal{F}_\alpha(\rho)$ satisfies

$$\mathcal{F}_\alpha(\rho) \geq \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} - \frac{\chi\cos\alpha C(n)}{2(n-2)\omega(n)}\|\rho\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} = f(s),$$

where $s := \|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, and $f(s)$ is defined by (1.15). Since $f(s)$ is strictly concave in $s > 0$, it reaches its unique maximum at s^* given in (1.16). Through simple calculations, we obtain that $f(s)$ is monotonically increasing for $0 < s < s^*$, while $f(s)$ is monotonically decreasing for $s > s^*$. Based on the assumption, the initial free energy $\mathcal{F}_\alpha(\rho_0)$ satisfies $\mathcal{F}_\alpha(\rho_0) < \mathcal{F}^*_\alpha = f(s^*)$, namely, there exists a $0 < \delta < 1$ such that $\mathcal{F}_\alpha(\rho_0) < \delta f(s^*)$. By the monotonicity of the free energy in (2.5), we conclude for all $t > 0$,

$$f\left(\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\right) \leq \mathcal{F}_\alpha(\rho) \leq \mathcal{F}_\alpha(\rho_0) < \delta f(s^*).$$

Thus, if the initial data satisfies $\|\rho_0\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < s^*$, the increasing behavior of $f(s)$ in the interval $0 < s < s^*$ guaranties the existence of a constant $0 < \mu_1 < 1$ such that

$$\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq \mu_1 s^*.$$

In contrast, if the initial data satisfies $\|\rho_0\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > s^*$, the decrease of $f(s)$ for $s > s^*$ implies the existence of a constant $\mu_2 > 1$ such that

$$\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \geq \mu_2 s^*.$$

Thus, the proof of the lemma is completed. ■

By investigating the free energy, we find that the blow-up of solutions to the problem (1.1)–(1.3) can be suppressed when the rotation angle $\alpha \in (-\pi, -\frac{\pi}{2}] \cap [\frac{\pi}{2}, \pi]$, which is undoubtedly the desired outcome. A brief proof on this is presented in Theorem 3.8. In the following, we mainly prove the finite-time blow-up of solutions when the angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Theorem 2.4. *Assume $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\frac{2n}{n+2} \le m < 2 - \frac{2}{n}$. If the initial data ρ_0 satisfies $m_2(0) = \int_{\mathbb{R}^n} |x|^2 \rho_0(x)dx < \infty$, $\mathcal{F}_\alpha(\rho_0) < \mathcal{F}_\alpha^* = f(s^*)$, and $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a time $T^* > 0$ such that*

$$\lim_{t \to T^*} \|\rho\|_{L^m(\mathbb{R}^n)} = \infty.$$

Here, s^* is the maximum point of $f(s)$.

Proof. First, we focus on the time evolution of the second moment. By substituting the free energy (1.13) into the time derivative of the second moment (2.4), we obtain

$$\begin{aligned} \frac{dm_2(t)}{dt} &\le 2n \int_{\mathbb{R}^n} \rho^m(x,t)dx - \chi(n-2)\cos\alpha \int_{\mathbb{R}^n} \rho(x,t)c(x,t)dx \\ &= \left(2n - \frac{2(n-2)}{m-1}\right) \int_{\mathbb{R}^n} \rho^m(x,t)dx + 2(n-2)\mathcal{F}_\alpha(\rho), \end{aligned}$$

where $2n - \frac{2(n-2)}{m-1} < 0$ due to $m < 2 - \frac{2}{n}$. Using the monotonicity of the free energy in Lemma 2.2 and the interpolation inequality as (2.14), there is $\theta = \frac{m(n-2)}{2n(m-1)} \in (0,1)$ such that

$$\frac{dm_2(t)}{dt} \le \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} \|\rho\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{m}{\theta}} + 2(n-2)\mathcal{F}_\alpha(\rho_0).$$

According to the assumptions given $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{2n(m-1)}{n-2}} > s^*$ and $\mathcal{F}_\alpha(\rho_0) = f(s^*)$, and using Lemma 2.3, we know that there exists a constant $\mu_2 > 1$ such that

$$\begin{aligned} \frac{dm_2(t)}{dt} &\le \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} \|\rho\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{2n(m-1)}{(n-2)}} + 2(n-2)\mathcal{F}_\alpha(\rho_0) \\ &\le \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} \mu_2 s^* + 2(n-2) f(s^*). \end{aligned} \tag{2.15}$$

Plugging (1.15) into (2.15), we have

$$\begin{aligned} \frac{dm_2(t)}{dt} &\le \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* + \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} s^* \\ &\quad + 2(n-2)\left(\frac{1}{m-1} M_0^{\frac{(\theta-1)m}{\theta}} s^* - \frac{\chi \cos\alpha C(n)}{2(n-2)\omega(n)} (s^*)^{\frac{2\theta}{m}}\right) \\ &= \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* + 2n M_0^{\frac{(\theta-1)m}{\theta}} s^* - \frac{\chi \cos\alpha C(n)}{\omega(n)} (s^*)^{\frac{2\theta}{m}} \\ &\le \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1)s^* < 0, \end{aligned}$$

where the second-to-last inequality follows from the definition of s^*. Thus, we know that there exists a finite time T such that

$$\lim_{t\to T} m_2(t) = 0. \tag{2.16}$$

Subsequently, using the property (2.16) that the second moment disappears in finite time, we derive the blow-up of solution ρ in the norm $L^m(\mathbb{R}^n)$. For clarity, we present this derivation as follows. There is a sphere B_R with a center of R, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \rho(x)dx &= \int_{B_R} \rho(x)dx + \int_{B_R^c} \rho(x)dx \\ &\le \left(\int_{B_R} dx\right)^{\frac{m-1}{m}} \|\rho\|_{L^m(B_R)} + \int_{B_R^c} \frac{1}{|x|^2}\cdot |x|^2\rho(x)dx \\ &\le CR^{n(m-1)/m}\|\rho\|_{L^m(\mathbb{R}^n)} + \frac{1}{R^2}m_2(t).\end{aligned}$$

Choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^m(\mathbb{R}^n)}})^{\frac{m}{(m-1)n+2m}}$, there is

$$\int_{\mathbb{R}^n} \rho(x)dx \le C\|\rho\|_{L^m(\mathbb{R}^n)}^{\frac{2m}{(m-1)n+2m}} m_2(t)^{\frac{n(m-1)}{(m-1)n+2m}}.$$

Thus, there exists $T^* \le T$ such that

$$\lim_{t\to T^*} \|\rho\|_{L^m(\mathbb{R}^n)}^{\frac{2m}{(m-1)n+2m}} \ge \frac{\|\rho\|_{L^1(\mathbb{R}^n)}}{Cm_2(t)^{\frac{n(m-1)}{(m-1)n+2m}}} = \infty.$$

Therefore, the proof of the theorem is completed. ■

3. Global existence of weak solutions

In this section, we investigate the existence of global weak solutions to the system (1.1)–(1.3), with separate discussions for the rotation angle α and the degenerate exponent m. We focus on the impact of the rotation angle α on the properties of the solution to the system (1.1)–(1.3). More detailed information is as follows:

- (i) for $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m \in (\frac{2n}{n+2}, 2-\frac{2}{n})$. Assume that the initial data $\rho_0 \in L^1_+ \cap L^p(\mathbb{R}^n)$, $p \ge m$ satisfies $\mathcal{F}_\alpha(\rho_0) < \mathcal{F}_\alpha^*$, if $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$, then there exists a global weak solution.
- (ii) for $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m = \frac{2n}{n+2}$. Assume that the initial data $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$, if $\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s$, then there exists a global weak solution. Here, C_s is a universal constant less than s^*.
- (iii) for $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ and $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$. If the initial data ρ_0 satisfies $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$, then there exists a global weak solution.

In what follows, we provide the principal proof process for the existence of global weak solutions. First, we introduce a regularization of the system (1.1)–(1.3), following the method in [10], which preserves the energy structure of the original system. The detailed form is as follows.

$$\partial_t \rho_\varepsilon = \Delta\big[(\rho_\varepsilon+\varepsilon)^m - \varepsilon^m\big] - \chi \operatorname{div}\big[(\rho_\varepsilon+\varepsilon)A\nabla c_\varepsilon\big], \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{3.1}$$

$$-\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon, \quad x \in \mathbb{R}^n,\ t \geq 0, \tag{3.2}$$

$$\rho_\varepsilon(x,0) = \rho_{0\varepsilon}(x), \quad x \in \mathbb{R}^n, \tag{3.3}$$

where $J(x)$ is a mollifier associated with $J(x) = \frac{n}{\omega(n)}(1+|x|^2)^{-\frac{n+2}{2}}$, $J_\varepsilon(x) = \frac{1}{\varepsilon^n}J(\frac{x}{\varepsilon})$ satisfying $\int_{\mathbb{R}^n} J_\varepsilon dx = 1$ and $\operatorname{spt}(J_\varepsilon) \subset B(0,\varepsilon)$. A direct computation yields

$$c_\varepsilon(x,t) = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{\big(|x-y|^2+\varepsilon^2\big)^{\frac{n-2}{2}}} \rho_\varepsilon(y) dy.$$

The free energy on the regularized solution ρ_ε is

$$\mathcal{F}_\alpha(\rho_\varepsilon, c_\varepsilon) = \frac{1}{m-1}\int_{\mathbb{R}^n} \big((\rho_\varepsilon+\varepsilon)^m - \varepsilon^m\big)dx - \frac{\chi\cos\alpha}{2}\int_{\mathbb{R}^n} \rho_\varepsilon c_\varepsilon dx. \tag{3.4}$$

Equivalently, the free energy can be expressed as

$$\begin{aligned}\mathcal{F}_\alpha(\rho_\varepsilon) = {}& \frac{1}{m-1}\int_{\mathbb{R}^n} \big((\rho_\varepsilon+\varepsilon)^m - \varepsilon^m\big)dx \\ & - \frac{\chi\cos\alpha}{2(n-2)\omega(n)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho_\varepsilon(x,t)\rho_\varepsilon(y,t)}{\big(|x-y|^2+\varepsilon^2\big)^{\frac{n-2}{2}}} dx\, dy.\end{aligned} \tag{3.5}$$

The classical parabolic theory ensures the existence of a global smooth non-negative solution ρ_ε for $t > 0$ to the regularized problem (3.1)–(3.3) when the initial data is non-negative. Notice that this regularized solution preserves the total mass, which is consistent with the system (1.1)–(1.3).

The initial data are smoothed by convolution with J_ε, i.e., $\rho_{0\varepsilon} = \rho_0 * J_\varepsilon$. The assumption in (1.12) guaranties the non-negativity ($\rho_{0\varepsilon} \geq 0$), with $\rho_{0\varepsilon}$ possessing the further properties that

$$\rho_{0\varepsilon} \in L^1_+ \cap L^p(\mathbb{R}^n),\ p \geq m, \quad \|\rho_{0\varepsilon}\|_{L^1(\mathbb{R}^n)} = \|\rho_0\|_{L^1(\mathbb{R}^n)} = M_0.$$

If $\rho_{0\varepsilon} \in L^r(\mathbb{R}^n)$ for some $r > 1$, then $\|\rho_{0\varepsilon} - \rho_0\|_{L^r(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$, especially,

$$\int_{\mathbb{R}^n} |x|^2 \rho_{0\varepsilon} dx \to \int_{\mathbb{R}^n} |x|^2 \rho_0 dx, \quad \mathcal{F}_\varepsilon(\rho_{0\varepsilon}) \to \mathcal{F}(\rho_0) \quad \text{as } \varepsilon \to 0,$$

where $\mathcal{F}_\varepsilon(\rho_{0\varepsilon})$ is the initial regularized entropy in (3.5).

In the following, we discuss the existence of solutions classified by angle α and degenerate exponent m.

3.1. The case of $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$

We now prove the existence of solutions for the model (1.1)–(1.3) with the rotation angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the degenerate exponent $m \in (\frac{2n}{n+2}, 2 - \frac{2}{n})$, establishing the following theorem.

Theorem 3.1. *Let $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and suppose that m satisfies $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$. Assume that the initial density $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$ is such that the free energy $\mathcal{F}_\alpha(\rho_0) < \mathcal{F}^*_\alpha$. If $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$, then for any $T > 0$ and $1 < r, s < 2$, there exists a global weak solution $\rho(x,t)$ to the problem (1.1)–(1.3) such that*

$$\rho \in L^\infty\big(0,T; L^1_+ \cap L^m(\mathbb{R}^n)\big) \cap L^{m+1}\big(0,T; L^{m+1}(\mathbb{R}^n)\big),$$
$$\nabla\rho \in L^2\big(0,T; L^r(\mathbb{R}^n)\big), \quad \partial_t\rho \in L^2\big(0,T; W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n)\big).$$

For this theorem, we mainly use the Lions–Aubin lemma in [5]. First, we give some uniform estimates of the solutions ρ_ε to the regularized problem (3.1)–(3.3).

Lemma 3.2. *Let $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\frac{2n}{n+2} < m < 2 - \frac{2}{n}$. Suppose that the initial density $\rho_0 \in L^1_+ \cap L^p(\mathbb{R}^n)$, $p \geq m$, $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$ and $\mathcal{F}_\alpha(\rho_0) \leq \mathcal{F}^*_\alpha$. Then the solutions of the regularized problem (3.1)–(3.3) satisfy the following uniform estimates*

$$\|\rho_\varepsilon\|_{L^\infty(0,\infty;L^p(\mathbb{R}^n))\cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^n))} \leq C, \quad \left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C,$$
$$\|\nabla c_\varepsilon\|_{L^\infty(0,T;L^s(\mathbb{R}^n))} \leq C, \quad s \in \left(\frac{n}{n-1}, \frac{np}{n-p}\right],$$

where C is a constant independent of ε.

Proof. Taking $p\rho_\varepsilon^{p-1}$ as a test function in (3.1) and using integration by parts and the Hölder inequality, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^p dx &= -p\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^{p-1}\cdot\nabla\big((\rho_\varepsilon+\varepsilon)^m-\varepsilon^m\big)dx \\
&\quad + p\chi\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^{p-1}\cdot\big((\rho_\varepsilon+\varepsilon)A\nabla c_\varepsilon\big)dx \\
&= -pm(p-1)\int_{\mathbb{R}^n}(\rho_\varepsilon+\varepsilon)^{m-1}\rho_\varepsilon^{p-2}|\nabla\rho_\varepsilon|^2dx \\
&\quad + p\chi\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^{p-1}\cdot\big((\rho_\varepsilon+\varepsilon)A\nabla c_\varepsilon\big)dx \\
&\leq -pm(p-1)\int_{\mathbb{R}^n}\rho_\varepsilon^{m+p-3}|\nabla\rho_\varepsilon|^2dx \\
&\quad + (p-1)\chi\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^p\cdot A\nabla c_\varepsilon dx + \varepsilon p\chi\int_{\mathbb{R}^n}\nabla\rho_\varepsilon^{p-1}\cdot A\nabla c_\varepsilon dx \\
&= -\frac{4pm(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^n}\left|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right|^2dx \\
&\quad + (p-1)\chi\cos\alpha\int_{\mathbb{R}^n}\rho_\varepsilon^{p+1}dx + \varepsilon p\chi\cos\alpha\int_{\mathbb{R}^n}\rho_\varepsilon^p dx. \qquad (3.6)
\end{aligned}$$

For the last two terms of the last equation, by utilizing the transformation (2.6) of the matrix A and the properties of $\nabla^{\perp} c_\varepsilon$ in (2.7), we obtain

$$\begin{aligned}\operatorname{div}(A\nabla c_\varepsilon) &= \operatorname{div}(\cos\alpha \nabla c_\varepsilon) - \operatorname{div}\left(\sin\alpha \begin{pmatrix} J_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & J_k \end{pmatrix}_{n\times n} \cdot \nabla c_\varepsilon\right) \\ &= \cos\alpha \Delta c_\varepsilon - \sin\alpha \operatorname{div}(\nabla^{\perp} c_\varepsilon) = \cos\alpha\Delta c_\varepsilon.\end{aligned}$$

Thus, there is

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\rho_\varepsilon^p \cdot A\nabla c_\varepsilon dx &= -\int_{\mathbb{R}^n} \rho_\varepsilon^p \operatorname{div}(A\nabla c_\varepsilon)dx \\ &= -\cos\alpha \int_{\mathbb{R}^n} \rho_\varepsilon^p \Delta c_\varepsilon dx \\ &= \cos\alpha \int_{\mathbb{R}^n} \rho_\varepsilon^{p+1} dx. \qquad (3.7)\end{aligned}$$

Then, by the GNS inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_\varepsilon^{p+1} dx &= \left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(p+1)}{m+p-1}}(\mathbb{R}^n)}^{\frac{2(p+1)}{m+p-1}} \\ &\le G^{\frac{2(p+1)}{m+p-1}} \left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(\mathbb{R}^n)}^{\beta\frac{2(p+1)}{m+p-1}} \left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^r(\mathbb{R}^n)}^{(1-\beta)\frac{2(p+1)}{m+p-1}} \\ &\le G^{\frac{2(p+1)}{m+p-1}} \left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(\mathbb{R}^n)}^{\beta\frac{2(p+1)}{m+p-1}} \|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{(1-\beta)(p+1)} \\ &\le G^{\frac{2(p+1)}{m+p-1}} \left(\varepsilon\left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(\mathbb{R}^n)}^2 + C(\varepsilon)\|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{(1-\beta)(p+1)}{1-\beta\frac{p+1}{m+p-1}}}\right), \qquad (3.8)\end{aligned}$$

where $0 < \beta < 1$ and G is the constant from the GNS inequality,

$$\frac{m+p-1}{2} r = \frac{2n}{n+2}, \qquad \frac{m+p-1}{2(p+1)} = \frac{\beta(n-2)}{2n} + \frac{1-\beta}{r}.$$

Simple computations give $\beta\frac{2(p+1)}{m+p-1} < 2$ due to $m > \frac{2n}{n+2}$, thus demonstrating that the last inequality (3.8) is satisfied by Young's inequality. Bringing (3.8) into (3.6), choosing ε such that

$$\chi(p-1)\cos\alpha G^{\frac{2(p+1)}{m+p-1}}\varepsilon = \frac{pm(p-1)}{(m+p-1)^2},$$

we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^p dx + \frac{2pm(p-1)}{(m+p-1)^2}\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right|^2 dx \le \varepsilon p\chi\cos\alpha \int_{\mathbb{R}^n} \rho_\varepsilon^p dx + C. \qquad (3.9)$$

Moreover, we establish the following L^p-estimate. By the GNS inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_\varepsilon^p dx &= \left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2p}{m+p-1}}(\mathbb{R}^n)}^{\frac{2p}{m+p-1}} \le G^{\frac{2p}{m+p-1}} \left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(\mathbb{R}^n)}^{\frac{2p\gamma}{m+p-1}} \left\|\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^\nu(\mathbb{R}^n)}^{\frac{2p(1-\gamma)}{m+p-1}} \\ &\le G^{\frac{2p}{m+p-1}} \left\|\nabla\rho_\varepsilon^{\frac{m+p-1}{2}}\right\|_{L^2(\mathbb{R}^n)}^{\frac{2p\gamma}{m+p-1}} \|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{(1-\gamma)p},\end{aligned}$$

where $0 < \gamma < 1$ and

$$\frac{m+p-1}{2}\nu = \frac{2n}{n+2}, \quad \frac{m+p-1}{2p} = \frac{\gamma(n-2)}{2n} + \frac{1-\gamma}{\nu}.$$

And since $m < 2 - \frac{2}{n}$,

$$\frac{2p\gamma}{m+p-1} = \frac{4n - 2p(n+2)}{2(n-2) - (n+2)(m+p-1)} < 2.$$

Thus, by using the Young inequality, we obtain

$$\int_{\mathbb{R}^n} \rho_\varepsilon^p dx \le G^{\frac{2p}{m+p-1}} \left(\delta \left\| \nabla \rho_\varepsilon^{\frac{m+p-1}{2}} \right\|_{L^2(\mathbb{R}^n)}^2 + C(\delta) \|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{(1-\gamma)p}{1-\frac{\gamma p}{m+p-1}}} \right). \tag{3.10}$$

Choose δ such that

$$\varepsilon \chi p \cos\alpha G^{\frac{2p}{m+p-1}} \delta = \frac{pm(p-1)}{(m+p-1)^2}.$$

By substituting (3.10) into (3.9), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^p dx + \frac{pm(p-1)}{(m+p-1)^2} \int_{\mathbb{R}^n} \left| \nabla \rho_\varepsilon^{\frac{m+p-1}{2}} \right|^2 dx \le C(p, n, \chi, \alpha). \tag{3.11}$$

Furthermore, (3.10) can be reformulated as

$$\int_{\mathbb{R}^n} \rho_\varepsilon^p dx - C(\delta) G^{\frac{2p}{m+p-1}} \|\rho_\varepsilon\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^{\frac{(1-\gamma)p}{1-\frac{\gamma p}{m+p-1}}} \le \delta G^{\frac{2p}{m+p-1}} \left\| \nabla \rho_\varepsilon^{\frac{m+p-1}{2}} \right\|_{L^2(\mathbb{R}^n)}^2. \tag{3.12}$$

Again choosing the above δ such that

$$\delta G^{\frac{2p}{m+p-1}} = \frac{2pm(p-1)}{(m+p-1)^2}.$$

Substituting (3.12) into (3.11), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^p dx + \int_{\mathbb{R}^n} \rho_\varepsilon^p dx \le C(p, n, \chi, \alpha).$$

By Grönwall's inequality, we obtain

$$\|\rho_\varepsilon\|_{L^\infty(0,\infty;L^p(\mathbb{R}^n))} \le C(p, n, \chi, \alpha).$$

Easily accessible from (3.11) that

$$\left\| \nabla \rho_\varepsilon^{\frac{m+p-1}{2}} \right\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C(T, p, n, \chi, \alpha).$$

Thus, from (3.8), we immediately deduce that

$$\|\rho_\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^n))} \le C(T, p, n, \chi, \alpha).$$

At last, a simple calculation of (1.11) immediately gives

$$\nabla c_\varepsilon = -\frac{1}{\omega(n)} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} J_\varepsilon * \rho_\varepsilon(y) dy.$$

Applying the weak Young inequality, we obtain

$$\|\nabla c_\varepsilon\|_{L^s(\mathbb{R}^n)} \le C \left\| \frac{x-y}{|x-y|^n} \right\|_{L^q_\omega(\mathbb{R}^n)} \|J_\varepsilon * \rho_\varepsilon\|_{L^p(\mathbb{R}^n)} \le \|\rho_\varepsilon\|_{L^p(\mathbb{R}^n)} \le C,$$

where $q = \frac{n}{n-1}$ satisfies $1 + \frac{1}{s} = \frac{1}{q} + \frac{1}{p}$. Hence, the lemma is proved. ∎

Remark 3.3. Assume $p = m$, there is the following result

$$\|\rho_\varepsilon\|_{L^\infty(0,\infty;L^m(\mathbb{R}^n))\cap L^{m+1}(0,T;L^{m+1}(\mathbb{R}^n))} \le C, \tag{3.13}$$

$$\left\|\nabla \rho_\varepsilon^{\frac{2m-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C, \tag{3.14}$$

$$\|\nabla c_\varepsilon\|_{L^\infty(0,T;L^s(\mathbb{R}^n))} \le C, \quad s \in \left(\frac{n}{n-1}, \frac{nm}{n-m}\right]. \tag{3.15}$$

Next, using the skill of (3.7) in Lemma 3.2 and referring to the results of [10, Lemmas 2.3 and 2.4], we derive the following estimates for the time and spatial derivative of ρ_ε.

Lemma 3.4. *Under the assumptions of Lemma 3.2, it follows that*

$$\|\nabla \rho_\varepsilon\|_{L^2(0,T;L^{\frac{2m}{3-m}}(\mathbb{R}^n))} \le C, \quad \text{in the case of } m < \frac{3}{2}, \tag{3.16}$$

$$\|\nabla \rho_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C, \quad \text{in the case of } m \ge \frac{3}{2}. \tag{3.17}$$

Moreover,

$$\|\partial_t \rho_\varepsilon\|_{L^2(0,T;W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n))} \le C, \quad s = \min\left\{\frac{2m}{m+1}, \frac{nm(m+1)}{nm+(n-m)(m+1)}\right\} > 1. \tag{3.18}$$

Furthermore, using formulas (3.13), (3.16)–(3.18), we deduce that $\|\rho_\varepsilon\|_{L^2(0,T;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))}$ and $\|\partial_t \rho_\varepsilon\|_{L^2(0,T;W^{-1,s}_{\mathrm{loc}}(\mathbb{R}^n))}$ are bounded for $p = \min\{\frac{2m}{3-m}, 2\}$. For the ball $\mathbb{B}_k$ of radius R_k satisfying $\mathbb{B}_k \subset \mathbb{R}^n$, since $W^{1,p}(\mathbb{B}_k) \hookrightarrow\hookrightarrow L^q(\mathbb{B}_k)$, $1 < q < \min\{\frac{2nm}{3n-(n+2)m}, \frac{2n}{n-2}\}$, and $L^q(\mathbb{B}_k) \hookrightarrow W^{-1,s}(\mathbb{B}_k)$, the Lions–Aubin lemma implies that there is a subsequence of $\{\rho_\varepsilon\}$ (still denoted by ρ_ε without relabel) such that, as $\varepsilon \to 0$, the following convergence holds

$$\rho_\varepsilon \to \rho \quad \text{in } L^2\big(0,T;L^q(\mathbb{B}_k)\big),$$

where $q \in (1, \min\{\frac{2nm}{3n-(n+2)m}, \frac{2n}{n-2}\})$. Furthermore, Lemma 2.1 provides a second moment estimate that allows us to construct a sequence of balls $\{\mathbb{B}_k\}_{k=1}^\infty \subset \mathbb{R}^n$, centered at the origin with radius $R_k \to \infty$. Then, by a standard diagonal argument, this leads

$$\rho_\varepsilon \to \rho \quad \text{in } L^2\big(0,T;L^q(\mathbb{R}^n)\big).$$

Therefore, Theorem 3.1 is proved.

3.2. The case of $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m = \frac{2n}{n+2}$

In this subsection, our analysis concerns the existence of weak solutions to (1.1)–(1.3) for the rotation angle $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and the exponent $m = \frac{2n}{n+2}$. According to [9], this boundedness of L^m-norm of solutions cannot guarantee the existence of the solutions, thus we only prove the existence for the following small initial data as [9], i.e., the initial data satisfy

$$\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s,$$

where C_s is defined in (1.17).

Theorem 3.5. *When $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m = \frac{2n}{n+2}$. If the initial data $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$ and $\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s$, then there is a global weak solution ρ with*

$$\begin{aligned}
&\rho \in L^\infty\big(0,T;L^1_+ \cap L^m(\mathbb{R}^n)\big) \cap L^{m+1}\big(0,T;L^{m+1}(\mathbb{R}^n)\big),\\
&\nabla\rho \in L^r\big(0,T;L^r(\mathbb{R}^n)\big), \quad r = \min\Big(2, \frac{3n+2}{n+4}\Big),\\
&\partial_t\rho \in L^2\big(0,T;W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^n)\big), \quad p = \frac{2(m+1)}{m+3} > 1.
\end{aligned}$$

The argument parallels the proof of Theorem 3.1, although there are differences in the critical value. After a brief discussion, we establish the following lemmas.

Lemma 3.6. *If the initial data $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$ and $\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s$, ρ_ε is a solution of the regularized problem (3.1)–(3.3), then*

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^m(\mathbb{R}^n))\cap L^{m+1}(0,T;L^{m+1}(\mathbb{R}^n))} \le C, \quad \big\|\nabla\rho_\varepsilon^{\frac{2m-1}{2}}\big\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C, \tag{3.19}$$

$$\|c_\varepsilon\|_{L^\infty(0,T;L^s(\mathbb{R}^n))} + \|\nabla c_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \le C, \quad s \in \Big(\frac{n}{n-2}, \frac{2n}{n-2}\Big]. \tag{3.20}$$

Proof. Taking $m\rho_\varepsilon^{m-1}$ as a test function in (3.1), which corresponds to the case $p = m = \frac{2n}{n+2}$ in (3.6), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^m dx \le &-\frac{4m^2(m-1)}{(2m-1)^2}\int_{\mathbb{R}^n}\big|\nabla\rho_\varepsilon^{\frac{2m-1}{2}}\big|^2 dx\\
&+(m-1)\chi\cos\alpha\int_{\mathbb{R}^n}\rho_\varepsilon^{m+1}dx + \varepsilon m\chi\cos\alpha\int_{\mathbb{R}^n}\rho_\varepsilon^m dx.
\end{aligned}\tag{3.21}$$

Consider the second term to the right of (3.21), it can be estimated by the GNS inequality

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho_\varepsilon^{m+1}dx &= \Big\|\rho_\varepsilon^{m-\frac12}\Big\|^{\frac{m+1}{m-\frac12}}_{L^{\frac{m+1}{m-\frac12}}(\mathbb{R}^n)} \le G\big\|\nabla\rho_\varepsilon^{\frac{2m-1}{2}}\big\|^2_{L^2(\mathbb{R}^n)}\Big\|\rho_\varepsilon^{m-\frac12}\Big\|^{\frac{2(2-m)}{2m-1}}_{L^{\frac{m}{m-\frac12}}(\mathbb{R}^n)}\\
&= G\big\|\nabla\rho_\varepsilon^{\frac{2m-1}{2}}\big\|^2_{L^2(\mathbb{R}^n)}\|\rho_\varepsilon\|^{2-m}_{L^m(\mathbb{R}^n)},
\end{aligned}\tag{3.22}$$

where G is the constant in the GNS inequality. Then we can obtain the estimate by plugging (3.22) into (3.21)

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_\varepsilon^m dx + M\int_{\mathbb{R}^n}\big|\nabla\rho_\varepsilon^{\frac{2m-1}{2}}\big|^2 dx \le \varepsilon m\chi\cos\alpha\int_{\mathbb{R}^n}\rho_\varepsilon^m dx.$$

Under the initial condition $\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s$, we establish

$$M := (m-1)\Big(\frac{4m^2}{(2m-1)^2} - \chi\cos\alpha G\|\rho_0\|^{2-m}_{L^m(\mathbb{R}^n)}\Big) > 0.$$

Building upon this key assumption, we obtain the following estimates

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^m(\mathbb{R}^n))} \le C, \quad \left\|\nabla \rho_\varepsilon^{\frac{2m-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^n))} \le C,$$

combining the inequality (3.22), obtain

$$\|\rho_\varepsilon\|_{L^{m+1}(0,T;L^{m+1}(\mathbb{R}^n))} \le C.$$

For c_ε and ∇c_ε, refer to [9, Theorem 3.1, Step 2], which we do not repeat here. ∎

Furthermore, referring to Steps 3 and 4 of [9, Theorem 3.1], we directly give the following lemma.

Lemma 3.7. *The assumptions of Lemma 3.6 hold, then*

$$\|\partial_t \rho_\varepsilon\|_{L^2(0,T;W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^n))} \le C, \quad p = \frac{2(m+1)}{m+3} > 1, \tag{3.23}$$

$$\|\nabla \rho_\varepsilon\|_{L^r(0,T;L^r(\mathbb{R}^n))} \le C, \quad r = \min\left(2, \frac{3n+2}{n+4}\right). \tag{3.24}$$

Thus, combining (3.19), (3.23), (3.24) and the second moment (2.4), applying the Aubin–Lions lemma, without relabel, we obtain a subsequence of ρ_ε satisfying

$$\rho_\varepsilon \to \rho \quad \text{in } L^r\big(0,T;L^k(\mathbb{R}^n)\big),$$

where $r = \min(2, \frac{3n+2}{n+4})$ and $k = \min\{\frac{(3n+2)n}{n^2+n-2}, \frac{2n}{n-2}\} > 2$.

Hence, the proof of Theorem 3.5 is complete.

3.3. The case of $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ and $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$

In this subsection, we establish the global existence of solutions to system (1.1)–(1.3) for any initial data satisfying $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$, provided that $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ and $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$. This result thereby precludes the occurrence of finite-time blow-up in the considered parameter regime.

Theorem 3.8. *If $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ and $m \in [\frac{2n}{n+2}, 2-\frac{2}{n})$. Assume the initial density $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$. Then there exist global weak solutions for system (1.1)–(1.3).*

Proof. According to Lemma 2.2, it is easy to check that $\mathcal{F}_{\alpha_\varepsilon}(\rho, c)$ is non-increasing:

$$\frac{d}{dt}\mathcal{F}_{\alpha_\varepsilon}(\rho_\varepsilon) \le -\int_{\mathbb{R}^n} (\rho_\varepsilon + \varepsilon)\left|\nabla\left(\frac{m}{m-1}(\rho_\varepsilon+\varepsilon)^{m-1} - \chi\cos\alpha c_\varepsilon\right)\right|^2 dx \le 0, \quad \forall t > 0.$$

For $\alpha \in (-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ (which implies $\cos\alpha \le 0$), we obtain the following inequality for the free energy:

$$\begin{aligned}\mathcal{F}_{\alpha_\varepsilon}(\rho_0) \ge \mathcal{F}_{\alpha_\varepsilon}(\rho, c) &= \frac{1}{m-1}\int_{\mathbb{R}^n} \big((\rho_\varepsilon+\varepsilon)^m - \varepsilon^m\big)dx - \frac{\chi}{2}\cos\alpha \int_{\mathbb{R}^n} \rho_\varepsilon c_\varepsilon dx \\ &\ge 0.\end{aligned} \tag{3.25}$$

This energy estimate ensures that the energy does not diverge. Thus, the energy dissipation and the lower bound of the free energy eliminate the possibility of finite-time blow-up, consequently ensuring the existence of global solutions. The main idea involves establishing time-space estimates for a smooth solution ρ_ε, which are then combined with the Aubin–Lions lemma to derive the global existence of the solution. To avoid duplication, we present here only the final result. For a detailed proof, it is similar to Theorems 3.1 and 3.5. ∎

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