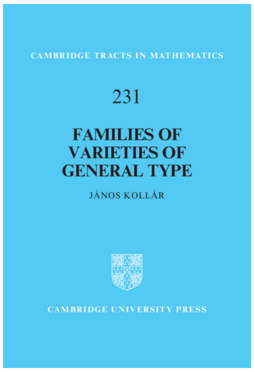
Book reviews

Families of Varieties of General Type by János Kollár

Reviewed by Stefano Filipazzi



János Kollár's *Families of Varieties of General Type*, written with the collaboration of Klaus Altmann and Sándor J. Kovács, is a monumental contribution to modern algebraic geometry. It completes a decades-long endeavor to generalize the moduli theory of algebraic curves to higher-dimensional varieties. The book concludes and brings to fruition a complete moduli theory for varieties of general type, a 30-year work by many mathematicians, where Kollár's vision and contribution stand out. The importance of this monograph has been recognized by the János Bolyai International Mathematics Award, a prestigious prize given every five years by the Hungarian Academy of Sciences.

The work takes the moduli theory of hyperbolic (marked) curves as a starting point and guides the reader through the theory needed to generalize it to their higher-dimensional counterparts, namely, stable varieties and stable pairs. Throughout the text, the main focus is the notion of a family of varieties or pairs, consistently with the modern emphasis on functorial and stack-theoretic approaches to moduli. There are many subtleties in generalizing this notion from curves to higher-dimensional varieties. Step by step, the author addresses the natural difficulties that arise in the process and thus motivates the many new notions and constructions needed to develop the moduli theory for higher-dimensional varieties.

In the case of smooth and projective algebraic curves of genus $g \geq 2$, Deligne and Mumford developed a complete and satisfactory moduli theory: To construct a compact moduli space for curves of a fixed genus $g \geq 2$, it suffices to consider families of nodal curves of (arithmetic) genus g. Furthermore, families of nodal curves over a base scheme S are simple to describe: They are flat morphisms $C \to S$ such that every geometric fiber is a nodal curve. Then, this description can be readily extended to curves of genus g with n marked points.

Already in dimension 2, if we consider smooth projective surfaces with ample canonical line bundle, it is not immediately clear what singular spaces should be allowed to compactify this moduli problem, nor what notion of family shall be used. The first question has been settled for some time and leads to the theory of stable varieties and stable pairs, which is thoroughly discussed in an earlier book by Kollár, *Singularities of the Minimal Model Program*. Yet, an answer to the first question only clarifies *what* geometric objects shall be parametrized, but not *how*. This book provides the answer to this latter question, thus completing the quest for a moduli theory generalizing the one of curves to higher dimensions.

In 1988, Kollár and Shepherd-Barron observed that the notion of family utilized for curves is not well-behaved already in the case of surfaces: relevant discrete invariants are indeed allowed to jump in a flat family. The core of the book (Chapters 2–7) discusses how the notion of flatness should be refined to obtain a well-behaved theory of families of stable varieties or pairs.

The first step is to understand what a family of stable varieties (or pairs) over a smooth curve is. In this case, since the base variety is smooth and 1-dimensional, the correct notion can be fully characterized with the language of pairs. Then, as we generalize this notion to arbitrary bases, more subtleties arise. To define a family of stable varieties $\pi\colon X \to S$ over a reduced scheme S, it suffices to ask that π is flat and the relative canonical divisor $K_{X/S}$ is $\mathbb{Q}$-Cartier. Then, if S is not reduced, it is necessary to analyze all the sheaves $\omega_{X/S}^{[m]}$ and ask that they are flat and with S_2 fibers for every $m \in \mathbb{Z}$. This suffices to define the notion of family of stable varieties; for the case of pairs, the situation is even more intricate. Indeed, if we consider a candidate family of pairs $\pi\colon (X, D) \to S$, explicit examples show that we cannot expect the support of D to be flat over S. Therefore, a refined understanding of the possible alternatives to flatness is needed; this quest then leads Kollár to introduce the notions of C-flatness (inspired by Cayley hypersurfaces), well suited for families over reduced bases, and K-flatness, which generalizes the notion of C-flatness and provides the final notion needed to complete the theory.

Once the notion of family of stable varieties and pairs is fully understood, we are ready to harvest the desired results: The book culminates with the proof that the Kollár–Shepherd-Barron–Alexeev (KSBA) stability condition for pairs leads to a well-behaved moduli theory in characteristic 0, inclusive of a projective coarse moduli space.

The book is structured into eleven chapters, each progressively building toward the main results. A summary of the key sections includes:

- Chapter 1: Provides historical background on moduli theory, tracing developments from Riemann and Cayley to Deligne and Mumford. It outlines the transition from curves to higher-dimensional varieties and highlights the main challenges therein.
- Chapter 2: Develops the theory of 1-parameter families of (locally) stable varieties and pairs, including detailed definitions of local stability and the introduction of slc singularities.
- Chapter 3: Treats the notion of a family of (locally) stable varieties over an arbitrary base. Approaches with the theory of Chow varieties (Cayley–Chow families) and Hilbert schemes (Hilbert–Grothendieck families) are compared.
- Chapter 4: Establishes the core theory of stable pairs over reduced base schemes. The notions of Mumford divisor and C-flatness are introduced here as a crucial geometric condition for managing families of divisors.
- Chapter 5: Provides numerical criteria for flatness and stability, most notably Theorem 5.1, which relates the constancy of the volume of the canonical divisor to the stability of families.
- Chapter 6: Explores moduli problems where the divisorial part is flat and compares several notions of stability, including those due to Viehweg and Alexeev.
- Chapter 7: Introduces K-flatness and its formal properties, which strengthen the framework for working with families of divisors independently of projective embeddings. This chapter provides the final notion of family of (locally) stable pairs over an arbitrary base.
- Chapter 8: Synthesizes previous developments and the work of Chapters 2–7 into a general moduli theory for stable pairs, culminating in Theorem 8.1—the main result establishing that Kollár–Shepherd-Barron–Alexeev stability yields a good moduli theory with coarse projective moduli spaces.
- Chapters 9–11: Contain auxiliary material such as hulls and husks, miscellaneous ancillary results, and a handbook about the Minimal Model Program.

To sum up, *Families of Varieties of General Type* is a landmark work that settles a central question in algebraic geometry. The clear and precise style will certainly make this book the main reference on moduli of algebraic varieties, both for learners and experienced researchers.

The exposition follows Kollár's characteristic and enjoyable writing style: Abstract constructions or definitions go hand in hand with explicit and concrete examples that provide motivation and support for the reader. Another strength of the book is its historical account of moduli theory in algebraic geometry, which allows the reader to appreciate how the theory grew over the past two centuries and provides the more curious with many references to previous works.

This book's contribution to algebraic geometry is both foundational and forward-looking. This is reflected by the multiple uses and audiences the book is suited for. On the one hand, its exhaustive content will make it an indispensable resource for researchers in moduli theory and birational geometry. On the other hand, graduate students and researchers interested in learning more about this field will find a reference rich in examples and motivations that will invite the reader to dive into moduli theory.

János Kollár, *Families of Varieties of General Type.* Cambridge Tracts in Mathematics 231, Cambridge University Press, 2023, xviii+472 pages, Hardback ISBN 978-1-009-34610-8, eBook ISBN 978-1-009-34611-5.

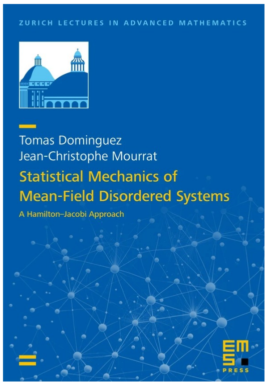
Stefano Filipazzi is an assistant professor at Duke University. He works in algebraic geometry. He obtained his PhD in 2019 at the University of Utah under the supervision of Christopher D. Hacon and spent time at UCLA (2019–2021) and EPFL (2021–2025).

stefano.filipazzi@duke.edu

DOI 10.4171/MAG/279

Statistical mechanics of mean-field disordered systems: A Hamilton–Jacobi approach by Tomas Dominguez and Jean-Christophe Mourrat

Reviewed by Roland Bauerschmidt



The book provides a beautifully written introduction to the topic of mean-field spin glasses from the point of view of infinite-dimensional Hamilton–Jacobi equations – a perspective that has been extensively developed by Mourrat and collaborators in the last years. The prototypical example of a mean-field spin glass is the Sherrington–Kirkpatrick model, whose free energy is defined as the limit

$$f(\beta) = \lim_{N\to\infty} -\frac{1}{N}\mathbb{E}\log \sum_{\sigma\in\{\pm1\}^N} \exp\Big(\frac{\beta}{\sqrt{N}}\sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j\Big),$$

where the g_{ij} are independent standard Gaussian random variables. In physics terminology, the g_{ij} are an example of quenched disorder of a spin system with configurations $\sigma \in \{\pm 1\}^N$ having probability weight proportional to

$$\exp\Big(\frac{\beta}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j\Big).$$

The term mean-field refers to the aspect that all pairs of sites $i, j \in \{1, \dots, N\}$ play an equivalent role (as opposed to a lattice system, for instance, in which the interaction depends on the distance of the sites). As prototypical examples of disordered systems, mean-field spin glasses have become a very active research area within probability theory, with motivation and connections from statistical physics to computer science and statistical inference. Some of this motivation is explained in the book without assuming any prior knowledge.

The non-rigorous but ingenious computation of the free energy by Parisi (using what is known as the "replica method") was a major achievement in theoretical physics that opened the way to the understanding of a broad class of disordered systems. The result is given by a somewhat mysterious variational formula that is now known as the Parisi formula:

$$-f(\beta) - \log 2 = \inf_{\zeta}\Big(\Phi_\zeta(0,0) - \beta^2 \int_0^1 t\zeta(t)\,dt\Big),$$

where the infimum is over probability distribution functions ζ on $[0,1]$ and Φ_ζ is the solution on $[0,1] \times \mathbb{R}$ of a certain PDE, namely

$$-\partial_t \Phi_\zeta(t,x) = \beta^2\big(\partial_x^2 \Phi_\zeta(t,x) + \zeta(t)(\partial_x \Phi_\zeta(t,x))^2\big),$$

with terminal condition $\Phi_\zeta(1,x) = \log\cosh(x)$. The Parisi formula was eventually proved by Guerra and Talagrand, using an approach different from the replica method of Parisi.

The book by Dominguez and Mourrat systematically develops another perspective on the free energy of the Sherrington–Kirkpatrick model (and more general spin glasses), which is that it also turns out to be given in terms of the solution of an infinite-dimensional Hamilton–Jacobi equation. Indeed, up to a trivial constant, the free energy $f(\beta)$ with $\beta = \sqrt{2t}$ turns out to be given by $f(t,0)$ where f solves

$$\partial_t f(t,q) = \int_0^t \partial_q f(t,q,u)^2\,du,$$

and q takes values in the space of square integrable increasing paths from $[0,1)$ to $\mathbb{R}_{\geq 0}$. As a Hamilton–Jacobi equation, the (viscosity) solution of this equation is given by the Hopf–Lax formula:

$$f(t,q) = \sup_{q'}\Big(f(0,q+q') - \frac{1}{4t}\int_0^1 (q'(u))^2\,du\Big).$$

One of the main results presented in the book under review is that this variational formula is in fact equivalent to the Parisi formula. The Hamilton–Jacobi formulation provides a conceptually natural (perhaps less mysterious) perspective on the free energy.

The main motivation for the Hamilton–Jacobi approach to spin glasses is to understand more general models for which the analogue of the Parisi formula is not yet understood. This includes the situation where the quadratic nonlinearity in the Hamilton–Jacobi equation for the Sherrington–Kirkpartrik model is replaced by a nonconvex function. The book concludes with an outlook on this topic of current research by discussing the example of a bipartitive version of the Sherrington–Kirkpatrick model.

The results in the book are presented with a lot of intuition and background along the way. In addition to the motivation of mean-field spin glasses, both from the point of view of statistical physics and from that of statistical inference, the book by Dominguez and Mourrat includes concise, yet essentially self-contained introductions to the necessary mathematical background topics. This includes chapters on convex analysis, the required background on Hamilton–Jacobi equations including the theory of viscosity solutions, and an introduction to Poisson point processes. These concepts are illustrated in examples relevant for the problem of mean-field spin glasses at hand and complemented with a number of exercises.

The book would be an excellent reference for an advanced topics course or a student seminar, by providing an introduction to the active research area of spin glasses in probability as well as introductions to various topics of general mathematical relevance. The book is a real pleasure to read and therefore also an excellent reference for anyone who would like to learn more about this fascinating subject.

Tomas Dominguez and Jean-Christophe Mourrat, *Statistical mechanics of mean-field disordered systems: A Hamilton–Jacobi approach.* Zurich Lectures in Advanced Mathematics, EMS Press, 2024, vi+361 pages, Softcover ISBN 978-3-98547-074-7, eBook ISBN 978-3-98547-574-2.

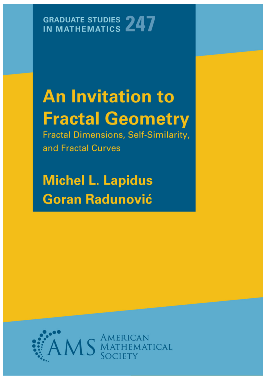
Roland Bauerschmidt is a professor of mathematics at the Courant Institute of Mathematical Sciences. His research interests focus on probabilistic aspects of statistical physics and quantum field theory.

bauerschmidt@cims.nyu.edu

DOI 10.4171/MAG/277

An Invitation to Fractal Geometry: Fractal Dimensions, Self-Similarity, and Fractal Curves by Michel L. Lapidus and Goran Radunović

Reviewed by Lars Olsen



This is a nice and well-written textbook on fractal geometry at the advanced undergraduate level. In order to study fractal geometry, some knowledge of measure theory is inevitably needed, and every author of a textbook on fractal geometry at the undergraduate level must decide how many technical details in measure theory he/she believes are needed and should be included. To answer this question, one is reminded of Tolstoy's short story "How much land does a man need?" in which the protagonist, in the final paragraph, learns that the answer to the eponymous question is: "Enough to get buried in." Similarly, many students have the impression that the answer to the question "How much measure theory do you need to study fractal geometry?" is also: "Enough to get buried in." It is the duty of any author of a textbook on fractal geometry aimed at undergraduate students to carefully make sure that the students are not buried in measure theoretical details before starting the study of fractal geometry. Some authors achieve this by avoiding a technical discussion of measure theory, see, for example [2, 5, 6, 11], whereas others include carefully and appropriately designed chapters discussing the technical foundations of measure theory, see, for example, [1, 3, 10, 12]. The present book belongs to the latter category and includes a thorough discussion of measure theory.

The book consists of four parts.

Part 1 ("Preliminary Material") consists of Chapters 1–3. Chapter 1 contains an interesting introduction to fractal geometry with many examples illustrating a wide variety of fractals. Chapter 2 and Chapter 3 contain technical material covering basic theory of metric space (20 pages) and measure theory including construction of measures using Carathéodory's approach (40 pages), respectively.

Part 2 ("Dimension Theory") is the core and central part of the book and consists of Chapters 4–8. Chapter 4 provides a thorough introduction to Iterated Function Systems, including symbolic dynamics. Chapters 5–7 provide an introduction to Hausdorff measures with respect to arbitrary gauge functions, including a detailed discussion of the δ-approximative Hausdorff measures with respect to different covering systems. Chapter 8 gives a thorough discussion of the Minkowski dimension and the box dimensions in addition to a detailed discussion of the Hausdorff dimension, the packing dimension and the box dimensions of self-similar sets satisfying the open set condition.

Part 3 ("Fractal Curves and Their Complex Dimensions") consists of one long chapter (120 pages), namely, Chapter 9. This chapter provides an introduction to the theory of zeta functions and complex dimensions. While the material covered in Part 2 is standard and can be found in almost all undergraduate textbooks on fractal geometry, the material in Part 3 is less standard and was developed by Lapidus and various collaborators during the past 30 years, cf. [8, 9] and the references therein. This is the first time this material is presented in a textbook for undergraduate students and this makes the present textbook unique amongst other undergraduate textbooks on fractal geometry. Because of this, it seems appropriate to explain the material in Part 3 in slightly more detail. Loosely speaking, the work presented in Part 3 says that the Minkowski dimension of a fractal string can be written as a series involving the poles of the zeta function of the fractal string. More precisely, for a bounded Borel set subset A of $\mathbb{R}^m$, the distance zeta function ζ_A of A is defined by

$$\zeta_A(s) = \int_{A_\delta} \operatorname{dist}(x, A)^{s-m}\, dx$$

for the complex variable $s \in \mathbb{C}$, where $\delta > 0$ and A_δ is the δ-neighbourhood of A (the choice of δ is, in a precise technical sense explained in the book, unimportant), and the complex dimensions of A are by definition the poles of the meromorphic extension of ζ_A. The authors are particularly interested in the following special case. Namely, fix an open bounded subset Ω of the real line $\mathbb{R}$ and let $A = \partial\Omega$ be the boundary of Ω; the set $\partial\Omega$ is called a fractal string. A fractal string is typically a fractal set and fractal geometers are interested in studying the Minkowski dimension and the Minkowski content of $\partial\Omega$. Lapidus' key thesis is that the complex dimensions of $\partial\Omega$ provides an "explicit" formula for those quantities. More precisely, for $\varepsilon > 0$, let $V(\varepsilon) = \operatorname{vol}\{x \in \mathbb{R} \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$ denote the (1-dimensional) volume of the ε-neighbourhood of $\partial\Omega$. The Minkowski dimension, D, of $\partial\Omega$ is defined by $D = 1 - \liminf_{\varepsilon \searrow 0} \frac{\log V(\varepsilon)}{\log \varepsilon}$, and the lower and upper Minkowski contents of $\partial\Omega$ are defined by

$$\mathcal{M}_* = \liminf_{\varepsilon \searrow 0} \varepsilon^{-(1-D)} V(\varepsilon), \quad \mathcal{M}^* = \limsup_{\varepsilon \searrow 0} \varepsilon^{-(1-D)} V(\varepsilon).$$

One of the key results in Part 3 says that (under suitable conditions on the string $\partial\Omega$) we have the following explicit formula for $V(\varepsilon)$:

$$V(\varepsilon) = \sum_\omega c_\omega \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} + R(\varepsilon), \tag{1}$$

where the sum is over all the complex dimensions ω of $\partial\Omega$, the number c_ω is essentially the residue of $\zeta_{\partial\Omega}$ at the pole ω, and $R(\varepsilon)$ is an error term of lower order. It follows from (1) that $\varepsilon^{-(1-D)} V(\varepsilon) = g(\varepsilon) + \varepsilon^{-(1-D)} R(\varepsilon)$, where $g(\varepsilon)$ is a function defined explicitly in terms of the complex dimensions and whose oscillatory behaviour determines the values of the Minkowski contents $\mathcal{M}_*$ and $\mathcal{M}^*$ of $\partial\Omega$.

Part 4 ("Appendices") consists of two short appendices, A and B. Appendix A explain lower and upper limits, and Appendix B provides a more detailed and technical discussion of Carathéodory's extension theorems.

The presentation is clear. Useful motivations and examples are presented before important definitions and details of all proofs are given. A very large number of further interesting historical notes and references are spread out through the text. Each section ends with a fairly large collection of useful exercises. Most of the exercises are not of a computational nature, but require that the reader provides proofs of various mathematical statements. Finally, the book contains a very long (633 entries) and useful list of references including many recent entries (i.e., after 2000).

There are numerous other textbooks in fractal geometry at the undergraduate level, including, [1–7, 10–12]. The book under review is more advanced than Falconer's 1990 texts [5, 6] and the texts by Y. Pesin & V. Climenhaga [11] and Zähle [12] but less ambitious than Mattila's graduate textbook [10]. Whereas Falconer's popular textbooks [5, 6] avoids technical measure theoretical details and presents a large number of examples of fractal sets taken from many parts of mathematics, the book under review provides the reader with the proper measure theoretical foundations for the subject (but at a level that is more accessible to the beginning graduate student than the treatment found in Mattila's text [10]) and concentrates on a more limited number of topics. The book is suited for an advanced undergraduate course in fractal geometry stressing the measure theoretical foundations of the subject. If supplemented with [5], the students will learn the rigorous measure theoretical foundations for the subject and also encounter numerous interesting examples.

Michel L. Lapidus and Goran Radunović, *An Invitation to Fractal Geometry: Fractal Dimensions, Self-Similarity, and Fractal Curves.* Graduate Studies in Mathematics **247**, American Mathematical Society, 2024, xxvii+600 pages, Hardcover ISBN 978-1-4704-7623-6, Softcover ISBN 978-1-4704-7895-7, eBook ISBN 978-1-4704-7896-4.

References

[1] M. Barnsley, *Fractals everywhere*. Academic Press, Boston, MA (1988)

[2] G. A. Edgar, *Measure, topology, and fractal geometry*. Undergrad. Texts Math., Springer, New York (1990)

[3] G. A. Edgar, *Integral, probability, and fractal measures*. Springer, New York (1998)

[4] K. J. Falconer, *The geometry of fractal sets*. Cambridge Tracts in Math. 85, Cambridge University Press, Cambridge (1986)

[5] K. Falconer, *Fractal geometry: Mathematical foundations and applications*. John Wiley & Sons, Chichester (1990)

[6] K. Falconer, *Techniques in fractal geometry*. John Wiley & Sons, Chichester (1997)

[7] J. M. Fraser, *Assouad dimension and fractal geometry*. Cambridge Tracts in Math. 222, Cambridge University Press, Cambridge (2020)

[8] M. L. Lapidus, G. Radunović and D. Žubrinić, *Fractal zeta functions and fractal drums: Higher-dimensional theory of complex dimensions*. Springer Monogr. Math., Springer, Cham (2017)

[9] M. L. Lapidus and M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions: Geometry and spectra of fractal strings*. Springer Monogr. Math., Springer, New York (2006)

[10] P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*. Cambridge Stud. Adv. Math. 44, Cambridge University Press, Cambridge (1995)

[11] Y. Pesin and V. Climenhaga, *Lectures on fractal geometry and dynamical systems*. Stud. Math. Libr. 52, American Mathematical Society, Providence, RI (2009)

[12] M. Zähle, *Lectures on Fractal Geometry.* Fractals Dyn. Math. Sci. Arts Theory Appl. 8, World Scientific, Singapore (2024)

Lars Olsen graduated with a master's degree (cand.scient.) in mathematics and physics from the University of Copenhagen in 1992. He then went to the University of North Texas for his doctoral studies and graduated with his PhD in 1994. Since then, he has been working at the University of St Andrews in the beautiful coastal town of St Andrews in Scotland. Lars works on fractal geometry, and is, in particularly, interested in multifractal analysis and multifractal zeta-functions. When he is not thinking about fractal geometry, he enjoys relaxing and working in the garden together with his wife and their two cats, Ronja and Klaus. In fact, Klaus belongs to Lars and his wife's daughter, but she could not take him with her when she left for university four years ago. This is not really a problem because Klaus loves to live with Lars and his wife, and enjoys catching birds in the garden.

lo@st-and.ac.uk

DOI 10.4171/MAG/278