Are there non-trivial theorems about all finitely generated groups?

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We discuss several interpretations and potential counterexamples to the meta-mathematical question in the title of the paper, focussing on geometric group theory and probability on groups.

> *I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description, and perhaps I could never succeed in intelligibly doing so. But I know it when I see it, and the motion picture involved in this case is not that.*
>
> Jacobellis v. Ohio, 378 U.S., p. 197 (Stewart, J., concurring)

1 Thesis

The goal of this article is to give a light-hearted and, hopefully, thought-provoking critique of the following famous aphorism:

"Theorem". *If a property is satisfied by all finitely generated groups, then it must be satisfied for trivial reasons.*

This "theorem" is often attributed[1] to Gromov, the father of the field known as *geometric group theory*, who may have intended it as a joke. In geometric group theory, one seeks to understand groups as *geometric objects*, starting with the observation that if G is a group with finite generating set S (meaning that G does not have any strict subgroups containing S), then G acts by isometries on its *Cayley graph* $\mathrm{Cay}(G,S)$, the graph with vertex set G and where two vertices x and y are connected by an edge if $x = ys$ or $x = ys^{-1}$ for some $s \in S$. (Technically this defines the *right* Cayley graph, on which G acts by *left* multiplication.) The resulting graph metric on G is called a *word metric*. Although different Cayley graphs of G need not be isomorphic, they are always *bi-Lipschitz equivalent* in the sense that for any two such metrics there exist positive constants c and C such that $c d_2(x,y) \leq d_1(x,y) \leq C d_2(x,y)$. (This is analogous to how all norms on $\mathbb{R}^d$ are equivalent.) As such, one can always consider a finitely generated group as a "geometric" object up to bi-Lipschitz equivalence. In fact, one typically works with a larger family of equivalences known as *quasi-isometries* (in which one also allows distances to be distorted by *additive* constants), which has the advantage that finitely generated groups are quasi-isometric to their finite-index subgroups and $\mathbb{Z}^d$ is quasi-isometric to $\mathbb{R}^d$. Flipping this perspective on its head (and slightly extending the scope of geometric group theory from finitely generated groups to compactly generated, locally compact groups), one can alternatively think of geometric group theory as a *language* for the study of arbitrary "homogeneous" geometric spaces, a subject that is of obvious interest to geometers and which one need not be an algebraist to appreciate or contribute to. (It is obligatory here to mention the bridge between these two perspectives provided by the Švarc–Milnor lemma, a.k.a. the fundamental observation of geometric group theory, which states that if a group acts properly discontinuously and cocompactly on a proper length space (X,d), then G is finitely generated and its Cayley graphs are quasi-isometric to X.)

Is the "theorem" not obviously false?

Leaving aside the question of how to define "trivial," the first reaction of a certain kind of mathematician upon reading the above "theorem" will be to point out various obvious "counterexamples," perhaps starting with a "property of a group" defined through some irrelevant but true and difficult-to-prove assertion; at least one of "The Riemann hypothesis is true" or "The Riemann hypothesis is false" will do. A more charitably minded mathematician might contend that one is obviously not supposed to consider "properties" of a group defined via such irrelevancies (that in this example do not even reference the group), but instead point out "counterexamples" that involve either conditional statements ("if P then Q")

[1] Ghys and de la Harpe [22] write *"On attribue à M. Gromov l'affirmation suivant laquelle un théorème valable pour tous les groupes ne peut etre que trivial ou sans importance,"* which is of course a slightly different statement. Interpreters of this quote should bear in mind that geometric group theorists will often leave the words "finitely generated" implicit and talk simply of "groups," especially in oral communication with colleagues.

or dichotomies ("P or Q"), an important example being Gromov's theorem:

If G has polynomial volume growth, then it is virtually nilpotent.

which is certainly a non-trivial statement concerning all finitely generated groups. (Here a group is said to have polynomial volume growth if the radius-r balls of one – and hence all – of its Cayley graphs have cardinality at most $(1+r)^C$ for some constant C, and is said to be virtually nilpotent if it has a nilpotent subgroup of finite index.) This is of course just one particularly prominent example among countless others of a theorem stating that one property of a finitely generated group implies another, and since many of these theorems are clearly non-trivial (and *pas sans importance*), we should be careful to exclude such conditional statements from the "theorem." There are also important, non-trivial theorems showing that some complicated data associated to a group is equivalent to some other complicated data,[2] or similarly that one quantity associated to a group is always bounded by another quantity,[3] which do not always "feel like" "counterexamples" to the "theorem" even though they do arguably pertain to a "property" of all finitely generated groups. (What is a property?) Similarly, there are e.g., several important theorems in percolation theory that hold on general transitive graphs (e.g., the sharpness of the phase transition and the mean-field lower bound on the critical volume tail [1]), but which "feel like" properties of percolation rather than properties of the group and which I do not personally find compelling as "counterexamples" to the "theorem." (There are other theorems from percolation that do "feel like counterexamples" to me, which I discuss below.)

It is not the intention of this article to develop a logical framework specifying precise constraints to be placed on what "properties" the "theorem" applies to (if such a thing is even possible[4]). Instead, I want to ask:

- What interesting examples are there, if any, of "counterexamples" to the "theorem," that is, non-trivial properties that hold for all finitely generated groups?
- What is the value of the "theorem" as *advice* – that is, as a rule of thumb cautioning mathematicians not to waste their time working at an overly great level of generality in a way that is unlikely to lead to interesting results?

For both questions, it will be interesting to examine not just theorems that apply to *all* finitely generated groups, but also those applying to very large classes of such groups, such as all *infinite* finitely generated groups. For the first question, I hope to convince the reader that there are unambiguous counterexamples to the theorem that are of genuine mathematical interest and do not involve any "cheating" via irrelevant or conditional statements as above. For the second, the answer is more complicated: while the implicit advice of the "theorem" does usefully guard against certain naïve mathematical endeavours, it may also lead to an excessively pessimistic outlook that I hope to offer some respite from.

Dreams or nightmares?

I was motivated to write this article partly by an interesting cultural difference[5] I have observed between probabilists and group theorists regarding problems at the intersection of their interests: Many of the problems probabilists find most interesting about "probability on groups" concern very large classes of (finitely generated) groups, such as all infinite groups, all non-amenable groups, all groups of at least quadratic growth, etc. Two prominent open problems of this flavour are the Benjamini–Schramm conjecture [4], which asserts that percolation on any non-amenable Cayley graph has a non-trivial phase in which there are infinitely many infinite clusters, and the Liouville stability problem, which asks whether the Liouville property (all bounded harmonic functions are constant) is independent of the choice of generating set for all finitely generated groups; experts strongly believe the Benjamini–Schramm conjecture to be true (see [28, 29, 53] for partial results), while there is less consensus on the Liouville stability problem.

Two further important open problems coming internally from within group theory that involve putative properties of all finitely generated groups include the question whether all finitely generated groups are *sofic* [33] and Gaboriau's problems of fixed price and cost vs. Betti number [21]. For the first of these problems, it is widely believed that non-sofic groups exist, and preprints have recently appeared arguing that the closely related Connes embedding conjecture and Aldous–Lyons conjecture are false [6, 7, 32]. (Interestingly, the methods of these papers are non-constructive and do not yield explicit counterexamples.) There is less consensus regarding Gaboriau's problems.

[2] A favourite example of mine from GGT being Olshanskii's characterisation of metrics arising on G from word metrics on larger groups H via embeddings $G \hookrightarrow H$ [49, 50], up to bi-Lipschitz equivalence, as exactly the left-invariant metrics of at most exponential volume growth.

[3] This includes the various inequalities relating the volume growth, isoperimetry, heat-kernel decay, entropy, and rate of escape of the random walk as surveyed in [63]. Many of these relationships hold for arbitrary bounded-degree graphs and are not really specific to groups.

[4] Philosophers have attempted to solve similar problems using so-called *relevance logics* (a.k.a. *relevant logics*) [45].

[5] This difference appears to be orthogonal to other well-noted cultural differences between "analysts" and "algebraists" pertaining to, say, the two cultures of mathematics [23] and the manner in which they eat corn on the cob [60].

Regarding problems like this, I have often found probabilists more likely to believe in positive solutions[6] while among group theorists there is a strong tendency, informed in part by the above "theorem," to expect counterexamples to exist to any sufficiently general conjecture that has resisted serious attempts at proof for any length of time. This intuition is hard-won, and stems in part from important negative results obtained via the construction of pathological examples including Grigorchuk's groups of intermediate volume growth [24] and Gromov–Osajda monsters [52]; see also, e.g., [9, 19, 50, 51] for further examples. On the other hand, while there is undoubtedly a degree of naïvety behind the probabilist's attitude to these questions, it has also led to a number of significant results over the years including, e.g., the solutions to the Choquet–Deny problem [20], the $p_c < 1$ problem [12, 17], and the many important results proven about percolation on non-amenable groups by subsets of Benjamini, Lyons, Peres, and Schramm in the late 1990s [3, 40, 44]. My opinion is that optimism and pessimism are both valuable and that mathematicians should endeavour not to neglect either point of view.

2 Antithesis

We now discuss various possible "counterexamples" to the "theorem," i.e., interesting, non-conditional statements applying to all finitely generated groups whose only known proofs are non-trivial. The examples we give are highly biased towards my own research interests at the intersection of geometric group theory and probability.

Cayley graphs with few automorphisms

Let us begin with the following theorem of Leemann and de la Salle [38, 39], which has the advantage that its statement is very clean and easy to appreciate. Every finitely generated group acts by automorphisms on each of its Cayley graphs, but these Cayley graphs may have many further automorphisms. For example, the four-regular tree (which is a Cayley graph of the free group on two generators) has uncountably many automorphisms.

Theorem 2.1. *Every finitely generated group has a Cayley graph with countable automorphism group.*

The proof of this theorem proceeds by case analysis according to whether or not the group is virtually Abelian (i.e., has an Abelian subgroup of finite index): The theorem was proven for virtually Abelian groups (or, more generally, all groups with an element of infinite order) via an elementary argument in the earlier work [56], while for groups that are *not* virtually Abelian the authors use a probabilistic construction relying on a theorem of Tointon characterising virtually Abelian groups in terms of commuting probabilities of random walks [61]. In the latter case the authors show the stronger result that the group has a Cayley graph with no automorphisms other than the group itself; this is not true for infinite[7] Abelian groups, which always have an additional $x \mapsto -x$ symmetry.

Diffusive lower bounds on the random walk

Our next proposed "counterexample" has the caveat that it applies only to *infinite* finitely generated groups. Recall that the simple random walk on a Cayley graph (or any other graph) is a random process which, at each time step, jumps to a uniform random neighbour of its current position independently of everything it has done previously. One of the most basic questions one can ask about random walk is its *rate of escape*, i.e., the large-n asymptotics of the typical distance travelled by the random walk in n steps. This typical distance is of order $\sqrt{n}$ on infinite Abelian groups as a consequence of the central limit theorem, and is of order n on the k-regular tree for $k \geq 3$. (It is a major open problem whether the rate of escape can depend on the choice of generating set for general finitely generated groups.)

It is now known that the rate of escape for the random walk on an infinite, finitely generated group can be essentially any function between $\sqrt{n}$ and n [9]. On the other hand, it is *not* possible for the rate of escape to be slower than $\sqrt{n}$, in contrast to random walk on *fractals* (more accurately "pre-fractals," like the graphical Sierpiński gasket) that often have rate-of-escape of order n^β for $\beta < 1/2$ [36]. The following theorem appeared in the work of Lee and Peres [37] (which also establishes stronger results of the same form), but existed previously as folklore and is sometimes attributed to Virág. Here we write $(X_n)_{n\geq 0}$ for the random walk started at the identity and write $|X_n|$ for the distance between X_n and the identity.

Theorem 2.2. *Let G be a finitely generated group with symmetric generating set S, and consider the simple random walk on the Cayley graph* $\mathrm{Cay}(G,S)$*. If G is infinite, then there exists a constant c such that*

$$\mathbb{E}|X_n|^2 \geq cn$$

for every $n \geq 0$.

[6] For example, the probabilists David Aldous and Russ Lyons are some of the only mathematicians to have publicly supported the conjecture that all groups are sofic, motivated by their more general conjecture about unimodular random graphs [2].

[7] This is needed to rule out groups of the form $(\mathbb{Z}/2\mathbb{Z})^n$, where the map $x \mapsto -x$ is just the identity.

Geometric purists can expunge all reference to probability from this theorem by reformulating it in terms of solutions to the heat equation. See also [30,63] for further results and conjectures related to this theorem. The proof of this theorem relies on a case analysis according to whether or not the group is amenable. When the group is non-amenable, the stronger statement $\mathbb{E}|X_n|^2 \geq cn^2$ holds as a trivial consequence of Kesten's theorem (which states that a group is non-amenable if and only if the random walk return probabilities decay exponentially). On the other hand, it is a (very) special case of a theorem of Mok, Korevaar, and Schoen [35, 46] that if G is amenable (or, more generally, does not have Kazhdan's property (T)), then it admits a non-trivial *equivariant harmonic embedding into Hilbert space* $\Psi\colon G \to \mathbb{H}$. In this case the image $\Psi(X_n)$ is a (possibly infinite-dimensional) martingale and one has

$$\mathbb{E}|X_n|^2 \geq c_1 \mathbb{E}\|\Psi(X_n)\|^2 = c_1 \mathbb{E} \sum_{i=1}^{n} \|\Psi(X_n) - \Psi(X_{n-1})\|^2 = c_2 n$$

by equivariance of the embedding and orthogonality of martingale increments.[8] This theorem is related to the fact that every infinite Cayley graph admits a harmonic function of at most linear growth, another "counterexample" to the "theorem" that plays an important role in Kleiner's proof of Gromov's theorem [34] and can be proven via a similar case analysis.

The Burnside problem

I now describe a purely algebraic counterexample to the "theorem" pointed out to me on MathOverflow by Moishe Kohan. The *Burnside problem* asks for each integer n, must every finitely generated group G such that $g^n = \mathrm{id}$ for every $g \in G$ be finite? An infinite finitely generated group with this property is called a Burnside group of exponent n. The problem was solved for large, odd n in a breakthrough 1968 work of Novikov and Adian [48], who proved that there exist Burnside groups of exponent n for each odd $n > 4381$. While this is not a counterexample to the "theorem" since it concerns specific groups, it is also known that Burnside groups of exponent 1, 2, 3, 4, and 6 do not exist, with the cases $n = 4, 6$ being non-trivial theorems of Sanov and Hall [26, 57]. This yields purely algebraic properties (e.g., "there exists $g \in G$ such that $g^6 \neq \mathrm{id}$") that are true for all *infinite* finitely generated groups for non-trivial reasons. The cases $n = 5, 7$ of the Burnside problem remain open (among other small values).

[8] Russ Lyons has pointed out to us that an alternative simpler proof of this theorem is presented in [42, Section 13.9] that may make the theorem too easy to count as a "counterexample" to the "theorem," depending on what one considers "trivial." This proof still relies on a case analysis according to whether or not the group is amenable.

Coarse regular variation of the inverse growth

I next give a "counterexample" that, while more technical than my other examples, has the strongest credentials of non-triviality in that its proof relies on some of the deepest work in geometric group theory: Gromov's theorem [25], Pansu's theorem [55], and Hrushovski's theorem [27]. It is also arguably the more "purely geometric" of my proposed counterexamples. Recall that a function $f\colon (0,\infty) \to \mathbb{R}$ is said to be *regularly varying* if $f(x) > 0$ for all sufficiently large x and the limit

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)}$$

is well defined for each $\lambda > 0$. The first basic theorem about regularly varying functions is that (subject to very mild additional regularity assumptions such as measurability or Baire measurability that certainly hold for the piecewise-continuous functions we consider), this convergence must hold uniformly for compact sets of λ in $(0,\infty)$ and that there must exist a real number α, known as the *index of regular variation*, such that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha$$

for every $\lambda > 0$. Note that regular variation is a first-order asymptotic property, in the sense that if $f(x) \sim g(x)$ and f is regularly varying, then g is regularly varying of the same index as f. Regularly varying functions of index 0 are known as *slowly varying functions* and include functions such as $\log(1+x)$, $\log(1+\log(1+x))$, $\exp[\sqrt{\log x}\,]$, and so on. Regularly varying functions were introduced by Karamata in the 1930s with the goal of producing a unified framework for Abelian and Tauberian theorems (i.e., conditions under which the first-order asymptotics of a function and its Laplace transform or a sequence and its generating function determine one another), and are now a standard part of the language in many fields of pure mathematics with an emphasis on asymptotic analysis, including, e.g., probability theory and analytic number theory; see, e.g., [5] for an introduction.

Regular variation is one of several natural regularity properties for functions of (sub)polynomial growth; for increasing functions, a much weaker but still useful property is the *doubling property*

$$\limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} < \infty \qquad \text{for each } \lambda > 1.$$

In geometric group theory, regularity properties like doubling or regular variation often arise as hypotheses in theorems relating the asymptotics of one quantity to another, such as in the relationship between random-walk return probabilities and the spectral profile [10], or the relation between the isoperimetric profiles of two groups and their wreath product [18]. Moreover, it is common to consider a rather loose notion of asymptotic equivalence between monotone functions, where if f and g are increasing we say that $f \simeq g$ if there exist positive constants c and C such that

$cg(cx) \leq f(x) \leq Cg(Cx)$ for all sufficiently large x. This is natural, since many functions associated to a finitely generated group, such as the volume growth and the isoperimetric profile (defined below) are well defined (i.e., independently of the choice of finite generating set) modulo this notion of asymptotic equivalence. However, when working at a high level of generality one often comes to realise that this notion of equivalence is not very useful for functions not known to satisfy at least some weak form of regularity such as doubling. These issues came to a head for me in my paper [30], where I was able to prove a conjecture of Lyons, Peres, Sun, and Zheng [43] only for groups whose *isoperimetric profile*

$$\Psi(n) := \min\Big\{\frac{|\partial W|}{|W|} : W \subseteq G,\ |W| \leq n\Big\}$$

is *coarsely regularly varying*, that is, equivalent to a regularly varying function under the equivalence relation $\simeq$. (Here ∂W denotes the set of edges of the Cayley graph with one endpoint in W and the other in $G \setminus W$.) I conjecture, but have so far been unable to prove, that this is *always* the case, for every infinite finitely generated group.

I am now ready to state my next proposed "counterexample." In an underhanded effort to make the theorem I am about to state apply to all finitely generated groups rather than merely all *infinite* finitely generated groups, I will depart from the standard conventions by saying that a function $f\colon (0,\infty) \to [0,\infty]$ is regularly varying if it either takes values in $(0,\infty)$ and is regularly varying in the usual sense or satisfies $f(x) \equiv 0$ or $f(x) \equiv \infty$ for all sufficiently large x, in which case we say f is regularly varying of index $-\infty$ or $+\infty$ as appropriate. We write $\mathrm{Gr}(r)$ for the number of group elements of word length at most r.

Theorem 2.3. *Let G be a group with finite generating set S. Then the inverse growth function* $\mathrm{Gr}^{-1}(x) := \inf\{r \geq 0 : \mathrm{Gr}(r) \geq x\}$ *is coarsely regularly varying.*

Many groups have isoperimetric profile of the same order as the reciprocal of the inverse growth, with a one-sided bound always holding by a theorem of Coulhon and Saloff-Coste [11], so that this theorem supports my conjecture that the isoperimetric profile is always coarsely regularly varying.

Proof of Theorem 2.3. If G is finite, then $\mathrm{Gr}^{-1}(x) = \infty$ for all $x > |G|$, so that $\mathrm{Gr}^{-1}(x)$ is regularly varying of infinite index. If G has polynomial volume growth, it follows from Gromov's theorem [25] that G is virtually nilpotent and hence, by a theorem of Pansu [55], that there exist a constant $A > 0$ and an integer $d \geq 1$ such that

$$\mathrm{Gr}(r) \sim Ar^d \qquad \text{as } r \to \infty$$

and hence that

$$\mathrm{Gr}^{-1}(x) \sim (x/A)^{1/d} \qquad \text{as } x \to \infty,$$

which is more than sufficient for regular variation (with index $1/d$). What if G has superpolynomial growth? It is a consequence of a deep theorem of Hrushovski [27] that if G has superpolynomial volume growth then

$$\frac{\mathrm{Gr}(2r)}{\mathrm{Gr}(r)} \to \infty$$

as $r \to \infty$ (the theorem of [27] states more generally that $\lim_{n\to\infty} |A_n^2|/|A_n| = \infty$ for any exhaustion of G by finite sets $(A_n)_{n\geq 1}$). This is easily seen to imply that the inverse function $\mathrm{Gr}^{-1}(x)$ is coarsely slowly varying (that is, $\simeq$-equivalent to a slowly varying function). ■

Theorems for "most" infinite finitely generated groups

There are several important theorems concerning not quite all infinite finitely generated groups, but instead all infinite finitely generated groups that are not virtually $\mathbb{Z}$. While such theorems are arguably not true "counterexamples" to the "theorem," they are similar in spirit in that they establish non-trivial theorems concerning very large classes of groups. One example is the "$p_c < 1$" problem of Benjamini and Schramm [4], which asked whether an infinite, finitely generated group has a percolation phase transition if and only if it is not virtually $\mathbb{Z}$. This problem was solved by Duminil-Copin, Goswami, Raoufi, Severo, and Yadin [13], who proved that $p_c < 1$ for every bounded-degree graph satisfying a $(4+\varepsilon)$-dimensional isoperimetric inequality; the remaining cases have polynomial growth so that they are virtually nilpotent via Gromov's theorem and can be treated via classical methods. (E.g., using the fact they contain subgraphs quasi-isometric to $\mathbb{Z}^2$.) The proof of the high-dimensional case of the theorem relied on a highly non-trivial comparison theorem relating percolation with another more complicated model defined in terms of the Gaussian free field. A new and much simpler proof of the theorem, still relying on a case analysis between low- and high-dimensional examples, was recently given by Easo, Severo, and Tassion [17]; this new proof also establishes the purely geometric fact that an infinite finitely generated group has the *exponential cut sets property* if and only if it is not virtually $\mathbb{Z}$. Another important example in the same vein is Varopoulos's theorem [62], which states that the random walk on an infinite, finitely generated group is recurrent (returns to the identity infinitely often) if and only if the group is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, which again can be deduced as a consequence of Gromov's theorem.

Theorems requiring uniform *control of all groups*

Let us close this section by mentioning that there has recently been significant progress made on theorems concerning *uniform* properties of all finitely generated groups (or all infinite finitely generated groups, or all infinite finitely generated groups that are not virtually $\mathbb{Z}$, etc.), with constants that are either universal or depend only on, say, the size of the generating set. The problem of proving uniform theorems like this is often closely related to the problem of proving theorems about *finite* groups, where one is typically forced to work with *families* of finite groups in order to make meaningful statements about asymptotic properties. Significant results in this direction include finitary versions include $p_c < 1$ theorems for finite transitive graphs [31], "gap at 1" theorems for the critical probability on infinite vertex transitive graphs [31, 41, 54], and finite-graph versions of Varopoulos's theorem [58]. Many of these results rely on a *structure vs. expansion dichotomy* using the *finitary structure theorem* of Breuillard, Green and Tao [8] (a deep and powerful strengthening of the theorems of Gromov [25] and Hrushovski [27] stating roughly that Hrushovski's theorem holds *uniformly* over all finitely generated groups) and its extension to transitive graphs due to Tessera and Tointon [59]. Using this finitary structure theory, one can attempt to treat "high-dimensional" and "low-dimensional" cases separately and "patch the analysis together" on scales in which the dimension switches from high to low (if such scales exist). Our recent proof of Schramm's locality conjecture with Easo [15] and Easo's extension of this result to finite graphs [14] also rely heavily on these ideas, including in particular a finitary version of the fact that groups of polynomial growth are finitely presented [16].

3 Synthesis

A unifying feature of many of the examples considered in the previous section is that the proofs involve some kind of case analysis. As such, one might propose the following modification to our "theorem":

"Theorem*". *If a property is satisfied by all finitely generated groups* for the same reason*, then this reason must be trivial.*

Again I do not really claim this "theorem" is "true" in any precise mathematical sense,[9] merely that it is *more true* than the original "theorem."

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[9] Wes Wrigley has pointed out to me that intuitionistic logic [47] does not allow for proofs by case analysis, so that one could perhaps interpret "Theorem*" in intuitionistic terms. I am too ignorant of intuitionism to make an informed comment on this, but would prefer to keep the "theorem" both vague and within the framework of standard mathematics (at the cost of making it likely to be false!).

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