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Calculus of Variations. – *Homogenization of non-local integral functionals via two-scale Young measures*, by Giacomo Bertazzoni, Andrea Torricelli and Elvira Zappale, accepted on 9 March 2025.

Abstract. – We prove a homogenization result in terms of two-scale Young measures for non-local integral functionals. The result is obtained by means of a characterization of two-scale Young measures.

Keywords. – homogenization, non-local functionals, relaxation, two-scale Young measures, Γ-convergence.

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1. Introduction and main results

Homogenization has attracted much attention in the last five decades, due to the many applications that mixtures and finely heterogeneous media have in applied science and technology; see for instance [17, 30] among a wider pioneering literature. In fact, both classical and more recent theories of ferro-magnetics and mechanics provide an accurate macroscopic description of phenomena occurring in materials which incorporate fine structures at the microscopic level. We refer to [2, 5, 7, 10, 19, 20, 23–25] for applications to the theory of structured deformations, thin structures, non-simple materials, evolution problems with nonstandard growth, elastomers, electromagnetic phenomena and constrained models. Here we want to adopt the homogenization techniques to formalize how the effects at micro-level emerge at the macro-level in the study of more general phenomena, such as peridynamics, where classical gradients are replaced by non-classical ones and, consequently, non-local integral functions take over the role of standard integrals.

Indeed, adopting a different view point than [13], the aim of this paper consists of studying the asymptotic behavior as $\varepsilon \searrow 0^+$ of $I_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to \mathbb{R}^+$ defined as

$$I_\varepsilon(u) := \iint_{\Omega\times\Omega} W\left(x, x', \frac{x}{\varepsilon}, \frac{x'}{\varepsilon}, u(x), u(x')\right) dx\, dx', \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open subset, $u : \Omega \to \mathbb{R}^d$ is in some Lebesgue space L^p, s$1 < p < \infty$ and the integrand $W : \Omega \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ has some measurability and continuity properties to be specified later and $W(x, x', \cdot, \cdot, \xi, \eta)$ is periodic; i.e., we will equivalently denote I_ε by

$$I_\varepsilon(u) = \iint_{\Omega\times\Omega} W\left(x, x,' \left\langle \frac{x}{\varepsilon}\right\rangle, \left\langle \frac{x'}{\varepsilon}\right\rangle, u(x), u(x')\right) dx\, dx',$$

where $\langle t\rangle$ denotes the vector in $Q := (0,1)^N$ whose components are the fractional parts of $t \in \mathbb{R}^N$.

This kind of non-local functional appears in the mathematical modeling of peridynamics [40], ferro-magnetics [39], etc. For instance, in the first setting, i.e., peridynamics, Ω represents the reference configuration, u is the deformation and I_ε is the associated energy.

In general, I_ε is not lower semi-continuous; hence, the existence of minimizers is not guaranteed, and one should look for its relaxed version, for which we refer to [11], for instance, in order to detect the microstructure of the material in the context of nonlinear elasticity. Here we follow an analogous approach in terms of suitable Young measures, extending first L^p to the space of Young measures equipped with the narrow topology.

Thus, we can provide a characterization of the Γ-limit as the minimum of an appropriate functional over a class of two-scale Young measure with controlled mean-value. In fact, the Young measures are so far a crucial tool for obtaining a limiting formula. We refer to [4,22,38], etc., for background on Young measures and to [6,8,9,34] for the two-scale Young measures that are a key tool in this homogenization setting.

As already emphasized in [11, 29, 36, 37], the relaxation and homogenization of (1.1) in L^p is a very delicate issue since it is not clear if the limiting energy (as $\varepsilon \to 0$) has still the same form, i.e., the double integral form.

In our subsequent analysis, we will make use of the two-scale Young measures introduced by [21] and later developed by [43], defined in Definition 2.3, to compute, via Γ-convergence with respect to a suitable topology, the limit energy of (1.1), under the assumption on the energy density W of being an admissible integrand (as introduced in [12]) in the sense of Definition 2.11. We also stress that there is no loss of generality in assuming that W is *symmetric*, as observed in [11, 35], i.e.,

$$W(x, x_0, y, y_0, \xi, \xi_0) = W(x_0, x, y_0, y, \xi_0, \xi) \tag{1.2}$$

for a.e. $x, x_0 \in \Omega$, and all $y, y_0 \in Q$, $\xi, \xi_0 \in \mathbb{R}^d$.

THEOREM 1.1. *Let $p > 1$ and assume $W : \Omega \times \Omega \times Q \times Q \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is a symmetric admissible integrand and there exists $a, \alpha \in L^1(\Omega\times\Omega)$ and $c > 0$ such that*

$$\alpha(x, x') + \frac{1}{c}|\xi|^p \le W(x, x', y, y', \xi, \xi') \le a(x, x') + c\big(|\xi|^p + |\xi'|^p\big) \tag{1.3}$$

for a.e. $x, x' \in \Omega$, *all* $y, y' \in \mathbb{R}^d$ *and* $\xi, \xi' \in \mathbb{R}^d$. *Let* $\varepsilon > 0$ *and let* $\{I_\varepsilon\}_\varepsilon$ *be the family of functionals in* (1.1). *Then,* $\{I_\varepsilon\}_\varepsilon$ Γ*-converges, with respect to the weak* $L^p(\Omega;\mathbb{R}^d)$ *convergence, to* $I_{\hom} : L^p(\Omega, \mathbb{R}^d) \to [0, +\infty)$, *where* $I_{\hom}$ *is defined as*

$$I_{\hom}(u) := \min_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') d\nu_{(x,y)} \times (\xi) d\nu_{(x',y')}(\xi') dy\, dy'\, dx\, dx',$$

where

$$\mathcal{M}_u := \Big\{\nu \in L^\infty_w\big(\Omega \times Q, \mathcal{M}(\mathbb{R}^d)\big) : \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \text{ is a two-scale Young measure such that } \int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi)dy = u(x)\Big\}. \tag{1.4}$$

REMARK 1.2. We would like to underline that in the previous theorem we chose the image set $[0, +\infty]$ for simplicity's sake. Indeed, the result still holds if we assume W bounded from below and having values in $\mathbb{R} \cup \{\infty\}$.

In order to prove Theorem 1.1, we characterize two-scale Young measures by proving the following result. We will use the strategy introduced in [27, 28] and later adopted in [6] (and in [26] for the Sobolev–Orlicz setting). See also [9, 38] where some parts of our results have been proven.

THEOREM 1.3. *Let* $Q := (0, 1)^N$, Ω *be an open and bounded subset of* $\mathbb{R}^N$ *with Lipschitz boundary, and let* $\nu \in L^\infty_w(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ *be such that* $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ *for a.e.* $(x, y) \in \Omega \times Q$. *Then, the family* $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ *is a two-scale Young measure if and only if the following conditions hold:*

(i) *here exists* $1 < p < +\infty$ *such that*

$$(x, y) \mapsto \int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q);$$

(ii) *there exists* $u_1 \in L^p(\Omega, L^p_{\text{per}}(Q))$ *such that*

$$\int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) = u_1(x, y),$$

for a.e. $(x, y) \in \Omega \times Q$;

(iii) *for every* $f \in \mathcal{E}_p$

$$\int_Q \int_{\mathbb{R}^d} f(y, \xi) d\nu_{(x,y)}(\xi) dy \geq f_{\hom}\big(u(x)\big) \quad \text{for a.e. } x \in \Omega,$$

where

$$\mathcal{E}_p := \Big\{ f : \bar{Q} \times \mathbb{R}^d \to \mathbb{R} \text{ continuous and such that exists} \tag{1.5}$$

$$\lim_{|\xi| \to +\infty} \frac{f(y,\xi)}{1+|\xi|^p} \text{ uniformly with respect to } y \in \bar{Q} \Big\}, \tag{1.6}$$

$$u(x) := \int_Q u_1(x,y)dy,$$

and for every $\xi \in \mathbb{R}^d$,

$$f_{\hom}(\xi) := \lim_{T \to +\infty} \frac{1}{T^N} \inf \Big\{ \int_{(0,T)^N} f\big(\langle x \rangle, \xi + v(x)\big)dx : \\ v \in L^p\big((0,T)^N, \mathbb{R}^d\big), \int_{(0,T)^N} v(x)dx = 0 \Big\}. \tag{1.7}$$

For a generalization of the above characterization when $p = 1$, in the constrained case (i.e., u satisfying a suitable PDE, not necessarily curl), we refer to [3], leaving to future investigation the study of non-local models as in (1.1) when W satisfies a linear growth condition (i.e., (1.3) for $p = 1$). To conclude this section, we point out that Theorem 1.1 holds true in the multi-scale case (see [9, 34] for a discussion on the topic). Indeed, the techniques used in the proof (under appropriate symmetry conditions) can be similarly replicated in the case of multiple scales.

The paper is organized as follows. Section 2 is devoted to notation and preliminaries. In Section 3, we prove some technical results and, in particular, Theorem 1.3. Finally, in Section 4, we prove the homogenization result, i.e., Theorem 1.1, and we compare our limiting formula with the relaxed energy obtained in [11] in the homogeneous isotropic setting.

2. Preliminaries

We start by fixing the notation that we will use in the paper.

2.1. *Notation*

Throughout this work, $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, $\mathcal{A}(\Omega)$ denotes the family of the open subsets of Ω, and for any $m \in \mathbb{N}$, $\mathcal{L}^m$ is the Lebsegue measure in $\mathbb{R}^m$. $Q := (0,1)^N$ is the unit cube in $\mathbb{R}^N$. The symbols $\langle \cdot \rangle$ and $[\cdot]$ stand, respectively, for the fractional and integer part of a number, or a vector component-wise. The Dirac mass at a point $a \in \mathbb{R}^m$ is denoted by δ_a. For every Lebesgue measurable set $A \subset \mathbb{R}^m$, the symbol $⨍_A$ stands for the average $(\mathcal{L}^m(A))^{-1} \int_A$.

Let U be an open subset of $\mathbb{R}$. Then, the following notation is adopted in the sequel.

- For any linear space X, by X' we denote its dual space.
- $\mathcal{C}_c(U)$ is the space of the continuous functions $f : U \to \mathbb{R}$ with compact support.
- $\mathcal{C}^\infty_{\text{per}}(Q)$ is the space of Q-periodic functions on $\mathbb{R}^N$.
- $\mathcal{C}^\infty_c(U)$ is the space of the smooth functions $f : U \to \mathbb{R}$ with compact support.
- $\mathcal{C}_0(U)$ is the closure of $\mathcal{C}_c(U)$ with respect to the uniform convergence; it coincides with the space of the continuous functions $f : U \to \mathbb{R}$ such that, for every $\eta > 0$, there exists a compact set $K_\eta \subset U$ with $|f| < \eta$ on $U \setminus K_\eta$.
- $\mathcal{C}_b(U; \mathbb{R}^m)$ is the space of continuous and bounded functions from U to $\mathbb{R}^m$.
- $\mathcal{M}(U)$ is the space of real-valued Radon measures with finite total variation. We recall that, by the Riesz Representation Theorem, $\mathcal{M}(U)$ can be identified with the dual space of $\mathcal{C}_0(U)$ through the duality
$$\langle \mu, \phi \rangle = \int_U \phi \, d\mu, \quad \mu \in \mathcal{M}(U),\ \phi \in \mathcal{C}_0(U).$$
- $\mathcal{P}(U)$ denotes the space of probability measures on U, i.e., the space of all $\mu \in \mathcal{M}(U)$ such that $\mu \geq 0$ and $\mu(U) = 1$.
- $L^1(\Omega, \mathcal{C}_0(U))$ is the space of maps $\phi : \Omega \to \mathcal{C}_0(U)$ such that
  - (i) ϕ is strongly measurable; i.e., there exists a sequence of simple functions $s_n : \Omega \to \mathcal{C}_0(U)$ such that $\|s_n(x) - \phi(x)\|_{\mathcal{C}_0(U)} \to 0$ for a.e. $x \in \Omega$;
  - (ii) $x \mapsto \|\phi(x)\|_{\mathcal{C}_0(U)} \in L^1(\Omega)$.
- $L^\infty_w(\Omega, \mathcal{M}(U))$ is the space of maps $\nu : E \to \mathcal{M}(U)$ such that
  - (i) ν is weak* measurable; i.e., $x \mapsto \langle \nu_x, \phi \rangle$ is measurable for every $\phi \in \mathcal{C}_0(U)$;
  - (ii) $x \mapsto \|\nu_x\|_{\mathcal{M}(U)} \in L^\infty(\Omega)$. The space $L^\infty_w(\Omega, \mathcal{M}(U))$ can be identified with the dual space of $L^1(\Omega, \mathcal{C}_0(U))$ via the duality
  $$\langle \mu, \phi \rangle = \int_\Omega \int_U \phi(x, \xi) d\mu_x(\xi) dx, \quad \mu \in L^\infty_w\big(\Omega, \mathcal{M}(U)\big),\ \phi \in L^1\big(\Omega, \mathcal{C}_0(U)\big),$$
  where $\phi(x, \xi) := \phi(x)(\xi)$ for all $(x, \xi) \in \Omega \times U$.
- The space $L^p_{\text{per}}(Q)$ stands for the L^p-closure of all functions $f \in \mathcal{C}(\mathbb{R}^N)$ which are Q−periodic.

2.2. *Young measures*

Young measures and two-scale Young measures (introduced in [21] and later developed by Pedregal in [32] and Valadier in [43]) are a key tool in our subsequent analysis.

Here we recall the definitions and main properties, starting from the theory of classical Young measures; see [22, 38, 41, 42].

DEFINITION 2.1 (Young measures). Let $\nu \in L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^m))$, and for every $n \in \mathbb{N}$ let $z_n : \Omega \to \mathbb{R}^m$ be a measurable function. The family of measures $\{\nu_x\}_{x\in\Omega}$ is said to be the Young measure generated by the sequence $\{z_n\}$ provided $\nu_x \in \mathcal{P}(\mathbb{R}^m)$ for a.e. $x \in \Omega$ and

$$\delta_{z_n} \overset{*}{\rightharpoonup} \nu \quad \text{in } L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^m)),$$

i.e., for all $\psi \in L^1(\Omega, \mathcal{C}_0(\mathbb{R}^m))$

$$\lim_{n\to+\infty} \int_\Omega \psi(x, z_n(x))dx = \int_\Omega \int_{\mathbb{R}^m} \psi(x,\xi)d\nu_x(\xi)dx.$$

The family $\{\nu_x\}_{x\in\Omega}$ is said to be a homogeneous Young measure if the map $x \mapsto \nu_x$ is independent of x. In this case, the family $\{\nu_x\}_{x\in\Omega}$ is identified with a single element ν of $\mathcal{M}(\mathbb{R}^m)$.

A proof of this result can be found in [22].

THEOREM 2.2 (Fundamental Theorem on Young Measures). *Let $\{z_n\}$ be a sequence of measurable functions $z_n : \Omega \to \mathbb{R}^m$. Then, there exists a subsequence $\{z_{n_k}\}$ and $\nu \in L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^m))$ with $\nu_x \geq 0$ for a.e. $x \in \Omega$, such that $\delta_{z_{n_k}} \overset{*}{\rightharpoonup} \nu$ in $L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^m))$ and the following properties hold:*

(i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \nu_x(\mathbb{R}^m) \leq 1$ *for a.e.* $x \in \Omega$;

(ii) *if* $\operatorname{dist}(z_{n_k}, K) \to 0$ *in measure for some closed set* $K \subset \mathbb{R}^m$, *then* $\operatorname{Supp}(\nu_x) \subset K$ *for a.e.* $x \in \Omega$;

(iii) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ *if and only if there exists a Borel function* $g : \mathbb{R}^m \to [0, +\infty]$ *such that*

$$\lim_{|\xi|\to+\infty} g(\xi) = +\infty \quad \textit{and} \quad \sup_{k\in\mathbb{N}} \int_\Omega g(z_{n_k}(x))dx < +\infty;$$

(iv) *if* $f : \Omega \times \mathbb{R}^m \to [0, +\infty]$ *is a normal integrand, then*

$$\liminf_{k\to+\infty} \int_\Omega f(x, z_{n_k}(x))dx \geq \int_\Omega \int_{\mathbb{R}^m} f(x,\xi)d\nu_x(\xi)dx;$$

(v) *if* (iii) *holds and if* $f : \Omega \times \mathbb{R}^m \to [0, +\infty]$ *is a Carathéodory integrand such that the sequence* $\{f(\cdot, z_{n_k})\}$ *is equi-integrable, then*

$$\lim_{k\to\infty} \int_\Omega f(x, z_{n_k}(x))dx = \int_\Omega \int_{\mathbb{R}^m} f(x,\xi)d\nu_x(\xi)dx.$$

We recall the definition of two-scale Young measure, as introduced by [34] and later developed by [6, 8, 9] and recently extended to the case $p = 1$ (i.e., generalized two-scale Young measures) by [3].

DEFINITION 2.3 (Two-scale Young measures). Let $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^d))$. The family $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is said to be a two-scale Young measure if $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x, y) \in \Omega \times Q$ and if for every sequence $\{\varepsilon_n\} \to 0$ there exists a bounded sequence $\{u_n\}$ in $L^p(\Omega, \mathbb{R}^d)$ such that for every $z \in L^1(\Omega)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times d})$,

$$\lim_{n\to\infty} \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n(x)\right)dx = \int_\Omega \int_Q \int_{\mathbb{R}^d} z(x)\varphi(y,\xi)d\nu_{(x,y)}(\xi)dy\,dx.$$

Namely, $\{(\langle\cdot/\varepsilon_n\rangle, u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x\in\Omega}$ (see Definition 2.1). Finally, we may call $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ the two-scale Young measure associated with the sequence $\{u_n\}$.

We refer to [9, Theorem 3.5] and [6, Section 2.3] to compare the two-scale Young measure associated with $\{u_n\}$ (according to the above definition), with the Young measure associated with $\{(\langle \frac{x}{\varepsilon_n}\rangle, u_n(x))\}$, which, in principle, at a naive look, could seem a different element in $L^\infty_w(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$, where $\mathbb{T}^N$ is the torus in $\mathbb{R}^N$. In fact, for a better understanding of the above notion, it is important to recall the notion of two-scale convergence introduced by [31] and later developed by [1] and exploited by many authors.

DEFINITION 2.4 (Two-scale convergence). Let $\{u_h\}$ be a sequence in $L^1(\Omega, \mathbb{R}^d)$. The sequence $\{u_h\}$ is said to be two-scale convergent to a function $u = u(x, y) \in L^1(\Omega \times Q, \mathbb{R}^d)$ if, for every component u^i_h of u_h and u^i of u, $i = 1, \dots, d$,

$$\lim_{h\to\infty} \int_\Omega \varphi(x)\phi\left(\frac{x}{\varepsilon_h}\right)u^i_h(x)dx = \iint_{\Omega\times Q} \varphi(x)\phi(y)u^i(x, y)dx\,dy,$$

for any $\varphi \in \mathcal{C}^\infty_c(\Omega)$ and any $\phi \in \mathcal{C}^\infty_{\text{per}}(Q)$. We simply write $u_h \rightsquigarrow u$.

We recall [34, Proposition 2.3].

PROPOSITION 2.5. *Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ as in Definition 2.3, then the mapping $u_1 : \Omega \times Q \to \mathbb{R}^d$ defined by*

$$u_1(x, y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda)$$

is the two-scale limit of the sequence $\{u_n\}$ generating $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

Now, we also recall the definition for the underlying deformation and we show that it does not depend on the generating sequence but only on the two-scale Young measures.

REMARK 2.6. Consider $\{\varepsilon_n\}$, $\{u_n\}$ and $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ as in Definition 2.3. Since $\{u_n\}$ is bounded in L^p, then there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightsquigarrow u_1$ (see for instance [1, Theorem 0.1]). By [9, Proposition 3.4], $u_{n_k} \rightharpoonup u$ in L^p, with

$$u(x) = \int_Q u_1(x,y)dy.$$

It is important to notice that u is uniquely defined. Indeed, by Proposition 2.5, it holds that

$$u_1(x,y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda).$$

So, if we consider another subsequence $\{u_{n_j}\}$ two-scale converging to some function v_1, then since ν is the same for both $\{\varepsilon_{n_k}\}$ and $\{\varepsilon_{n_j}\}$, it holds that

$$u_1(x,y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda) = v_1(x,y).$$

In particular, it follows that

$$u(x) = \int_Q \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda)dy,$$

and that u does not depend on the chosen subsequence $\{\varepsilon_{n_k}\}$. The function u is called the "underlying deformation" of $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

The proof of the following result is omitted, being easier than the one of [6, Lemma 3.4], since the generating sequences are only in L^p and not in $W^{1,p}$.

LEMMA 2.7. *Let D be a Lebesgue measurable subset of Ω, and consider μ and ν in $L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ such that $\{\mu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ and $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ are two-scale Young measures with the same underlying deformation $u \in L^p(\Omega, \mathbb{R}^d)$. Let*

$$\sigma_{(x,y)} := \begin{cases} \mu_{(x,y)} & \text{if } (x,y)\in D\times Q, \\ \nu_{(x,y)} & \text{if } (x,y)\in (\Omega\setminus D)\times Q. \end{cases}$$

Then $\sigma \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ and $\{\sigma_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure with underlying deformation u.

Next we consider the homogeneous case.

DEFINITION 2.8 (Homogeneous two-scale Young measure). Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$, we say that it is homogeneous if the map $(x,y)\mapsto \nu_{(x,y)}$ is independent of x. In this case, it can be identified with an element of $L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$ and we may write $\{\nu_y\}_{y\in Q} \equiv \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

This yields the definition of average for a two-scale Young measure and we can also prove that this is a homogeneous two-scale Young measure.

DEFINITION 2.9 (Average). Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$, its average is the family $\{\overline{\nu}_y\}_{y\in Q}$ defined by

$$\langle \overline{\nu}_y, \phi\rangle := \fint_\Omega \int_{\mathbb{R}^d} \phi(\xi) d\nu_{(x,y)}(\xi) dx, \quad \phi \in C_0(\mathbb{R}^d).$$

As observed in [6], if $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure, then it can be seen that $\overline{\mu} := \overline{\nu}_y \otimes dy$ is the average of $\{\mu_x\}_{x\in\Omega}$ with $\mu_x := \nu_{(x,y)} \otimes dy$ and $\mu \in L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^d))$. Thus, $\overline{\mu}$ is a homogeneous Young measure by [33, Theorem 7.1, p. 117]. Following the same strategy as [6, Lemma 2.9], we prove that $\{\overline{\nu}_y\}_{y\in Q}$ is actually a homogeneous two-scale Young measure. We consider the case of constant underlying deformation $F \in \mathbb{R}^d$.

LEMMA 2.10. *Let $\nu \in L^\infty_w(Q\times Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\{\nu_{(x,y)}\}_{(x,y)\in Q\times Q}$ is a two-scale Young measure with underlying deformation F, with $F \in \mathbb{R}^d$. Then $\{\overline{\nu}_y\}_{y\in Q}$ is a homogeneous two-scale Young measure with the same underlying deformation.*

PROOF. First we note that $\overline{\nu} \equiv \{\overline{\nu}_y\}_{y\in Q} \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$; indeed by Fubini's theorem and by Definition 2.9 we have that $y \mapsto \overline{\nu}_y$ is weakly* measurable. Our aim is to show that for every sequence $\{\varepsilon_n\}$ there exists $\{v_n\} \in L^p(Q, \mathbb{R}^d)$ such that the sequence $\{(\langle \frac{\cdot}{\varepsilon_n}\rangle, v_n)\}$ generates the measure $\overline{\mu} = \overline{\nu}_y \otimes dy$ and $v_n \rightharpoonup F$ in L^p. Let $\{u_n\} \subset L^p(Q, \mathbb{R}^d)$ be the sequence generating the two-scale Young measure ν and assume that $u_n \rightharpoonup F$ in L^p. Let $\varepsilon_n \to 0$ and define $\rho_n := \varepsilon_n[\frac{1}{\sqrt{\varepsilon_n}}]$. By construction there exists $m_n \in \mathbb{N}$, $a_i^n \in \rho_n\mathbb{Z} \cap Q$, and a measurable set $E_n \subset Q$ such that $\mathcal{L}^N(E_n) \to 0$ and

$$Q = \bigcup_{i=1}^{m_n} (a_i^n + \rho_n Q) \cup E_n.$$

Let

$$v_n(x) := \begin{cases} u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right), & \text{if } x \in a_i^n + \rho_n Q \text{ and } i \in \{1,\dots,m_n\} \\ F & \text{otherwise.} \end{cases}$$

By construction, $v \in L^p(Q, \mathbb{R}^d)$ and $v_n \rightharpoonup F$. Let $z \in \mathcal{C}_c(Q)$ and $\phi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d)$; then

$$\begin{aligned} &\int_Q z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right) dx \\ &\quad = \sum_{i=1}^{m_n} \int_{a_i^n+\rho_n Q} z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right) dx \\ &\qquad + \int_{E_n} z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, F\right) \end{aligned}$$

$$= \sum_{i=1}^{m_n} z(a_i^n) \int_{a_i^n+\rho_n Q} \phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right)dx$$
$$+ \sum_{i=1}^{m_n} \int_{a_i^n+\rho_n Q} (z(x)-z(a_i^n))\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right)dx$$
$$+ \int_{E_n} z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, F\right).$$

By the uniform continuity of z and the fact that $\mathcal{L}^N(E_n) \to 0$, it follows that

$$\int_Q z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx$$
$$= \sum_{i=1}^{m_n} \rho_n^N z(a_i^n) \int_Q \phi\left(\left\langle \frac{\rho_n x + a_i^n}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}(x)\right)dx + o(1),$$

i.e., that

$$\int_Q z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx = \sum_{i=1}^{m_n} \rho_n^N z(a_i^n) \int_Q \phi\left(\left\langle \frac{x}{\varepsilon_n/\rho_n}\right\rangle, u_{\rho_n/\varepsilon_n}(x)\right)dx + o(1),$$

and so passing to the limit for $n \to \infty$ and by Definition 2.9 we get

$$\lim_{n\to\infty} \int_Q z(x)\phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx = \int_Q z(x)dx \int_Q \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\nu_{(x,y)}(\xi)dy\,dx$$
$$= \int_Q z(x)dx \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\overline{\nu}_y(\xi)dy$$
$$= \langle \overline{\nu}, \phi\rangle \int_Q z(x)dx.$$

By a density argument it is easy to see that the previous identity holds for every $z \in L^1(Q)$ and so that $\{\overline{\nu}_y\}_{y\in Q}$ is the homogeneous two-scale Young measure generated by $\{v_n\}$. ■

Now we introduce our class of functions, according to the notion introduced in [12], and later adopted by [6, 9, 43].

Definition 2.11 (Admissible integrand). A function $f : \Omega \times \Omega \times Q \times Q \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$ is said to be an admissible integrand if, for any $\eta > 0$, there exist compact sets $X^\eta \subset \Omega \times \Omega$, $Y^\eta \subset Q \times Q$, with

$$\mathcal{L}^{2N}\big((\Omega\times\Omega)\setminus X^\eta\big) < \eta, \quad \mathcal{L}^{2N}\big((Q\times Q)\setminus Y^\eta\big) < \eta$$

and such that $f|_{X^\eta\times Y^\eta\times\mathbb{R}^d\times\mathbb{R}^d}$ is continuous.

REMARK 2.12. (i) We observe that by [9, Lemma 4.11], if f is an admissible integrand, then for any fixed $\varepsilon > 0$ the functional

$$(x, x', \xi, \xi') \mapsto f(x, x', x/\varepsilon, x'/\varepsilon, \xi, \xi')$$

is $\mathcal{L}^{2N}(\Omega \times \Omega) \otimes \mathcal{B}(\mathbb{R}^{2d})$-measurable, where $\mathcal{B}(\mathbb{R}^{2d})$ is the σ-algebra of Borel subsets of $\mathbb{R}^{2d}$. In particular, the functional I_ε (defined in (1.1)) is well defined in $L^p(\Omega, \mathbb{R}^d)$.

(ii) Moreover, we recall that by the Scorza–Dragoni theorem it is possible to prove that a Carathéodory function is an admissible integrand.

The next result, stated for generic spaces and dimensions, is a two-scale analog of the [Fundamental Theorem on Young Measures (iv), (v)], and it was proved in [8, Theorem 2.8]. We recall it here since it is a crucial tool to prove Theorem 1.1.

THEOREM 2.13. *Let $k, m \in \mathbb{N}$, $R \subset \mathbb{R}^k$ be open and bounded, and let S be the unit cube in $\mathbb{R}^k$. Let $\{\varepsilon_n\}$ be a vanishing positive sequence. Let $\{u_n\} \subset L^1(R, \mathbb{R}^m)$ be a bounded sequence. Then, there exist non-relabeled subsequences and a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in R\times S}$ such that it holds that*

(i) *if $\mathcal{W} : R \times S \times \mathbb{R}^m \to [0, +\infty]$ is an admissible integrand, then*

$$\liminf_{n\to+\infty} \int_R \mathcal{W}\Big(x, \Big\langle \frac{x}{\varepsilon_n} \Big\rangle, u_n(x)\Big) dx \geq \iint_{R\times S} \overline{\mathcal{W}}(x, y) dx\, dy, \tag{2.1}$$

where

$$\overline{\mathcal{W}}(x, y) := \int_{\mathbb{R}^M} \mathcal{W}(x, y, \lambda) d\nu_{(x,y)}(\lambda);$$

(ii) *if $\mathcal{W} : R \times S \times \mathbb{R}^m \to \mathbb{R}$ is an admissible integrand and $\{\mathcal{W}(\cdot, \langle \frac{\cdot}{\varepsilon_n} \rangle, u_n(\cdot))\}$ is equi-integrable for any $x \in R$, then $\mathcal{W}(x, y, \cdot)$ is $\nu_{(x,y)}$-integrable for a.e. $(x, y) \in R \times S$, the function $\overline{\mathcal{W}}$ is in $L^1(R \times S)$ and*

$$\lim_{n\to+\infty} \int_R \mathcal{W}\Big(x, \Big\langle \frac{x}{\varepsilon_n} \Big\rangle, u_n(x)\Big) dx = \iint_{R\times S} \overline{\mathcal{W}}(x, y) dx\, dy. \tag{2.2}$$

REMARK 2.14. We would like to point out that the statement of Theorem 2.13 is slightly different compared to [8, Theorem 2.8]. Indeed, in [8, Theorem 2.8 (i)] the integrand function $\mathcal{W}$ is assumed to be positive and finite. As far as we understand the proof still works for $\mathcal{W}$ having values in $[0, +\infty]$ as long as the set where $\mathcal{W} = \infty$ has null measure.

REMARK 2.15. (i) We remark that it is possible to look at Young measures from a different perspective than Definition 2.1. In fact, if $\Omega \subset \mathbb{R}^N$ is an open bounded subset and $p > 1$, we can say that a Young measure in $\Omega \times \mathbb{R}^m$ is a positive measure μ in $\Omega \times \mathbb{R}^m$, such that its push-forward measure $\pi_\Omega \# \mu$, obtained through the projection π_Ω, is the $\mathcal{L}^N$ measure on Ω; i.e., for all Borel subsets B of Ω,

$$\pi_\Omega \# \mu(B) := \mu(B \times \mathbb{R}^m) = \mathcal{L}^N(B).$$

This set of measures is denoted by $\mathcal{Y}(\Omega;\mathbb{R}^m)$. By [4, Theorem 4.2.4], $\mathcal{Y}(\Omega;\mathbb{R}^m) \subset L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$, with the identification $\mu = \{\mu_x\}_{x\in\Omega} \otimes \mathcal{L}^N$.

(ii) On the other hand, [4, Theorem 4.3.1] ensures that in $\mathcal{Y}(\Omega;\mathbb{R}^m)$, the weak* convergence in $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ of $\{\mu_n\} \subset \mathcal{Y}(\Omega;\mathbb{R}^m)$ towards $\mu \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ is equivalent to the narrow convergence, where the latter is defined as follows:

$$\mu_n \overset{\text{nar}}{\rightharpoonup} \mu \iff \lim_n \iint_{\Omega\times\mathbb{R}^m} \psi(x,\lambda) d\mu_n(x,\lambda) = \iint_{\Omega\times\mathbb{R}^m} \psi(x,\lambda)\mu(x,\lambda),$$

for every $\psi \in C_b(\Omega;\mathbb{R}^m)$. For more details on the narrow topology we refer to [4, Section 4.3].

Equivalently, by the identification $\mu_n = \{(\mu_n)_x\}_{x\in\Omega}$ and $\mu = \{\mu_x\}_{x\in\Omega}$, we say that $\{\mu_n\}$ narrowly converges to μ if and only if for every $g \in L^1(\Omega)$ and $h \in \mathcal{C}_0(\mathbb{R}^m)$ it holds that

$$\lim_{n\to\infty} \int_\Omega g(x) \int_{\mathbb{R}^m} h(y) d(\mu_n)_x(y) dx = \int_\Omega g(x) \int_{\mathbb{R}^m} h(y) d\mu_x(y) dx.$$

(iii) Given $p \geq 1$, $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ is the subset of $\mathcal{Y}(\Omega;\mathbb{R}^m)$ such that

$$\iint_{\Omega\times\mathbb{R}^m} |y|^p d\mu(x,y) < +\infty. \tag{2.3}$$

As a consequence of Hölder's inequality, $\mathcal{Y}^p(\Omega;\mathbb{R}^m) \subset \mathcal{Y}^q(\Omega;\mathbb{R}^m)$ if $1 \leq q \leq p$. We recall also that $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ is not closed in $\mathcal{Y}(\Omega;\mathbb{R}^m)$ under the narrow topology. Nevertheless, given a bounded sequence $\{v_n\} \subset L^p(\Omega;\mathbb{R}^m)$ there exists $\mu \in \mathcal{Y}(\Omega,\mathbb{R}^m)$ such that $\{v_n\}$ (up to a subsequence) generates μ; and $\mu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$. Conversely, for any $\mu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, there exists a bounded sequence $\{v_n\} \subset L^p(\Omega;\mathbb{R}^m)$, generating μ and such that $\{|v_n|^p\}$ is equi-integrable. We refer to [4, 11] for details.

Moreover, considering a bounded sequence $\{v_n\} \subset L^p(\Omega,\mathbb{R}^m)$ and $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^m)$, up to identifying each v_n with the Dirac mass $\delta_{v_n} \in \mathcal{Y}^p(\Omega,\mathbb{R}^m)$, by [4, Theorem 4.3.1] we have that $\{v_n\}$ generates ν (in the sense of Definition 2.1) if and only if δ_{v_n} narrowly converges to ν.

(iv) Following [9, 43], we can extend the definition of Young measure of finite p moment to parametrized measures defined in Ω with values in S, with $S = \mathbb{T}^N \times \mathbb{R}^d$, equipped with the Borel σ-algebra, i.e., $\mathcal{Y}^p(\Omega; S)$. In this setting the test functions for the narrow convergence can be given by $C_c(\Omega; \mathbb{R}^d) \otimes C_c(S; \mathbb{R}^d)$; see [43, p. 149].

These latter observations motivate the comparison of the homogenized functional I_{hom} of Theorem (1.1) with the functional $\overline{I}_{\text{hom}}$ in Corollary 4.3 computed in terms of suitable Young measures in $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$.

In fact, we also recall that the relaxation of non-local functionals as in (1.1), but not dependent on ε, has been obtained in [11] in terms of narrow convergence in $\mathcal{Y}^p(\Omega; \mathbb{R}^d)$. The analogous stand point for the homogenization will be given in Corollary 4.3, with the use of the above-mentioned subset of $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$.

2.3. Γ-*convergence*

We recall the definition and main properties of Γ-convergence. For a deeper overview of this topic, we refer to [18]. The following is an equivalent definition in the metric setting, which is sufficient for our purposes.

Definition 2.16 ([18, Proposition 8.1]). Let (X, d) be a metric space and $F_k : X \to \mathbb{R} \cup \{+\infty\}$, $\forall k \in \mathbb{N}$, a sequence of functionals. Then $\{F_k\}$ Γ-converges to $F : X \to \mathbb{R} \cup \{+\infty\}$ if and only if

(i) for every $x \in X$ and every sequence $\{x_k\}$ such that $d(x_k, x) \to 0$ as $k \to +\infty$, it is
$$F(x) \le \liminf_{k\to\infty} F_k(x_k);$$

(ii) for every $x \in X$, there exists a sequence $\{x_k\}$ such that $d(x_k, x) \to 0$ as $k \to +\infty$, such that
$$F(x) = \lim_{k\to\infty} F_k(x_k).$$

In fact, we write
$$F(x) : \inf \Big\{ \liminf F_k(x_k) : d(x_k, x) \to 0, \text{ as } k \to +\infty \Big\}.$$

A fundamental property we want to underline is that the Γ-limit is lower semi-continuous with respect to the convergence induced by d; see [18, Proposition 6.8].

We also provide the definition for Γ-convergence for a family of functionals.

Definition 2.17. Let (X, d) be a metric space. For a positive parameter ε, we say that a family $\{F_\varepsilon\}_\varepsilon$ of functionals, with $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$, Γ-converges to $F : X \to \mathbb{R} \cup \{+\infty\}$, with respect to the metric d as $\varepsilon \to 0^+$, if for all vanishing sequences $\{\varepsilon_k\}$, $\{F_{\varepsilon_k}\}$ Γ-converges to F, when $k \to \infty$.

We will write, as for the case of sequences,

$$F(x) : \inf\left\{\liminf F_\varepsilon(x_\varepsilon) : d(x_\varepsilon, x) \to 0, \text{ as } \varepsilon \to 0\right\}.$$

Finally, we state a theorem, whose proof is omitted being very similar to the one of [16, Theorem 1.1] (for related results see also [15]).

THEOREM 2.18. *Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory integrand satisfying the following:*

(H1) *$f(\cdot, b)$ is Q-periodic, for all $b \in \mathbb{R}^d$;*

(H2) *there exist $p > 1$ and a positive constant C such that*

$$\frac{1}{C}|b|^p - C \leq f(x, b) \leq C\left(1 + |b|^p\right),$$

for a.e. $x \in \Omega$ and for every $b \in \mathbb{R}^d$.

For every $\varepsilon > 0$, consider the family of functionals $F_\varepsilon : L^p(\mathbb{R}^d) \to (-\infty, +\infty]$, defined as

$$F_\varepsilon(u) := \int_\Omega f\left(\frac{x}{\varepsilon}, u(x)\right) dx.$$

Let $\mathcal{F}$ be the Γ-limit of $\{F_\varepsilon\}_\varepsilon$ with respect to the weak convergence in $L^p(\Omega; \mathbb{R}^d)$, i.e., $\mathcal{F}(u) := \inf\{\liminf F_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)\}$. Then

$$\mathcal{F}(u) = \int_\Omega f_{\text{hom}}\big(u(x)\big) dx,$$

where f_{hom} is the function in (1.7).

3. CHARACTERIZATION

This section is mainly devoted to the proof of Theorem 1.3 and to the attainment of other related results which will be used in the sequel.

We start by recalling the space $\mathcal{E}_p$ defined by (1.5) and (1.6). Its properties have been presented in [6, Section 3]. We briefly recall the main ones. It is a Banach space under the norm

$$\|f\|_{\mathcal{E}_p} := \sup_{\substack{y \in \bar{Q}, \\ \xi \in \mathbb{R}^d}} \frac{\left|f(y, \xi)\right|}{1 + |\xi|^p}.$$

Moreover, $\mathcal{E}^p$ is isomorphic to the space $\mathcal{C}(\bar{Q} \times (\mathbb{R}^d \cup \{\infty\}))$.

Before proving Theorem 1.3 we start commenting on the formulas (1.7).

Under the assumptions of Theorem 1.3, if f belongs to the class (1.5), in view of standard relaxation results (see [18, Proposition 6.11], [22, Theorem 6.68 and Remark 6.69]),

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega (\mathrm{co}\, f)\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\}, \tag{3.1}$$

where co f stands for the convex envelope of f with respect to the second variable.

Theorem 2.18 (see the proof of [16, Theorem 1.1.] within a slightly more general context) guarantees that

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \int_\Omega f_{\mathrm{hom}}\big(u(x)\big)dx,$$

where f_{hom} is as in (1.7). On the other hand, (3.1) gives

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \int_\Omega (\mathrm{co}\, f)_{\mathrm{hom}}\big(u(x)\big)dx,$$

where, for every $\xi \in \mathbb{R}^d$,

$$(\mathrm{co}\, f)_{\mathrm{hom}}(\xi) = \lim_{T\to+\infty}\frac{1}{T^N}\inf\left\{\int_{(0,T)^N}(\mathrm{co}\, f)\big(\langle x\rangle, \xi + v(x)\big)dx : v \in L^p\big((0,T)^N, \mathbb{R}^d\big),\ \int_{(0,T)^N} v(x)dx = 0\right\}. \tag{3.2}$$

Finally, arguing as in [14, Theorem 14.7], it results that

$$(\mathrm{co}\, f)_{\mathrm{hom}}(\xi) = \inf\left\{\int_{(0,1)^N}(\mathrm{co}\, f)\big(\langle x\rangle, \xi + v(x)\big)dx : v \in L^p\big((0,1)^N, \mathbb{R}^d\big),\ \int_{(0,1)^N} v(x)dx = 0\right\}.$$

Hence, by easier arguments than those in [22, Proposition 6.24], we can conclude that $f_{\mathrm{hom}} = (\mathrm{co}\, f)_{\mathrm{hom}}$.

The next result, whose proof is omitted, as in [6, Corollary 3.8], follows from the fact that conditions (i), (ii) and (iii) in Theorem 1.3 do not depend on the sequence $\{\varepsilon_n\}$.

COROLLARY 3.1. *Let $\{u_n\}$ be a bounded sequence in $L^p(\Omega, \mathbb{R}^d)$ and assume that there exists a vanishing sequence $\{\varepsilon_n\}$ such that $\{(\langle\frac{\cdot}{\varepsilon_n}\rangle, u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x\in\Omega}$. Then $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure.*

We start by addressing the proof of the necessity of Theorem 1.3.

THEOREM 3.2. *Let Ω be a bounded and open subset of $\mathbb{R}^N$. Let*

$$\nu \equiv \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w\big(\Omega\times Q, \mathcal{M}(\mathbb{R}^d)\big)$$

be a two-scale Young measure. Then there exist $u_1 \in L^p(\Omega; L^p_{\rm per}(Q,\mathbb{R}^d))$ and $u \in L^p(\Omega,\mathbb{R}^d)$ such that

$$u_1(x,y) = \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi), \tag{3.3}$$

$$u(x) = \int_Q u_1(x,y)dy \quad \text{for a.e. } x\in\Omega, \tag{3.4}$$

$$f_{\rm hom}\big(u(x)\big) \le \int_Q\int_{\mathbb{R}^d} f(y,\xi)d\nu_{(x,y)}(\xi)dy, \quad \text{for a.e. } x\in\Omega, \text{ for any } f\in\mathcal{E}_p, \tag{3.5}$$

where $f_{\rm hom}(F)$ is defined as in (1.7) and

$$\int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega\times Q). \tag{3.6}$$

PROOF. By definition of two-scale Young measure there exists a sequence $\{u_n\}\subset L^p(\Omega,\mathbb{R}^d)$ which generates $\{\nu_{x,y}\}_{(x,y)\in\Omega\times Q}$, whose two-scale limit is a function u_1 as in the statement, which, in particular, as observed in Proposition 2.5, satisfies (3.3). Also the weak limit of $\{u_n\}$ is a function $u\in L^p(\Omega;\mathbb{R}^d)$ such that $\int_Q u_1(x,y)dy = u(x)$.

Property (3.6) follows from [9, Theorem 3.6 (iv)] applied to the function $|\cdot|^p$, once we replace the original generating sequence $\{u_n\}$ by a p-equi-integrable one, which is always possible, in view of the Decomposition Lemma [22, Lemma 8.13].

The proof of (3.5) is based on the same argument as the one for proving [6, Lemma 3.1 (ii)], with the difference that we use an argument entirely similar to [16, Theorem 1.1] instead of [14, Theorem 14.5]. We write the proof for the readers' convenience.

Since $f\in\mathcal{E}_p$, then, as observed in [6, Section 3], there exists a constant c such that $|f(x,\xi)|\le c(1+|\xi|^p)$ for every $(x,\xi)\in Q\times\mathbb{R}^d$. In order to apply Theorem 2.18, the function $f(x,\cdot)$ must also be p-coercive. Since this is not the case, we introduce an auxiliary function. Fix $M>0$ and consider $f_M(x,\xi) := \max\{-M, f(x,\xi)\}$, with $(x,\xi)\in Q\times\mathbb{R}^d$. Now we fix $\alpha>0$ and we define $f_{M,\alpha}(x,\xi) := f_M(x,\xi)+\alpha|\xi|^p$ for every $(x,\xi)\in Q\times\mathbb{R}^d$. By construction,

$$\alpha|\xi|^p - M \le \big|f_{M,\alpha}(x,\xi)\big| \le (c+\alpha)\big(1+|\xi|^p\big), \quad (x,\xi)\in\bar{Q}\times\mathbb{R}^d.$$

By Theorem 2.18, it follows that for every $A\in\mathcal{A}(\Omega)$,

$$\liminf_{n\to\infty}\int_A f_{M,\alpha}\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n\right)dx \ge \int_A (f_{M,\alpha})_{\rm hom}(u)dx \ge \int_A f_{\rm hom}(u)dx, \tag{3.7}$$

with $f_{\hom}$ defined as in (1.7), since $f_{M,\alpha} \geq f$. By construction we have that

$$\liminf_{n\to\infty} \int_A f_{M,\alpha}\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx \leq \liminf_{n\to\infty} \int_A f_M\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx + \alpha \sup_{n\in\mathbb{N}} \int_A |u_n|^p dx. \tag{3.8}$$

Combining (3.7) and (3.8) and passing to the limit for $\alpha \to 0$ we get

$$\liminf_{n\to\infty} \int_A f_M\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx \geq \int_A f_{\hom}(u) dx. \tag{3.9}$$

Now we consider the set

$$A_n^M := \left\{ x \in A : f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) \leq -M \right\}$$

and we observe that by Chebyshev's inequality it holds that

$$\mathcal{L}^N(A_n^M) \leq \frac{c}{M},$$

for some constant c independent from n and M. By definition of f_M we have that

$$\int_A f_M\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx = -M\mathcal{L}^N(A_n^M) + \int_{A\setminus A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx \leq \int_{A\setminus A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx. \tag{3.10}$$

Since $\{u_n\}$ converges weakly in L^p, then, in view of the Decomposition Lemma [22, Lemma 8.13], up to a subsequence, we can replace $\{u_n\}$ by a p-equi-integrable one, still denoted by $\{u_n\}$, and so, together with the p-growth condition of $f(x,\cdot)$, $\{f(\cdot, u_n)\}$ is also equi-integrable and so

$$\int_{A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx \to 0 \quad \text{as } M \to +\infty \tag{3.11}$$

uniformly with respect to n. Combining (3.9), (3.10) and (3.11), we get

$$\liminf_{n\to+\infty} \int_A f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx \geq \int_A f_{\hom}(u) dx.$$

Since $\{f(\cdot, u_n)\}$ is equi-integrable, by Theorem 2.2 (v), we get

$$\lim_{n\to\infty} \int_A f\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n\right) dx = \int_A \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(x,y)}(\xi) dx,$$

and the proof follows by combining the last two inequalities and using a localization argument. ∎

We start addressing the proof of the sufficiency of Theorem 1.3, by introducing some preliminary notions and by proving some intermediate steps.

For every $F \in \mathbb{R}^d$ let

$$M_F := \Big\{\nu \in L^\infty_w\big(Q, \mathcal{M}(\mathbb{R}^d)\big) : \{\nu_y\}_{y\in Q} \text{ is a homogeneous two-scale Young measure and } \int_Q \int_{\mathbb{R}^d} \xi d\nu_y(\xi)dy = F\Big\}. \tag{3.12}$$

Remark 3.3. Exactly as observed in [6, Remark 3.3], the set M_F is independent of Ω; i.e., if $\nu \in M_F$ and $\Omega' \subset \mathbb{R}^N$ is another domain, then for all vanishing sequences $\{\varepsilon_n\}$ there exists a sequence $\{v_n\} \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle\cdot/\varepsilon_n\rangle, v_n)\}$ generates $\nu_y \otimes dy$. Indeed, let $r > 0$ such that $\Omega' \subset r\Omega$. In fact, given an arbitrary vanishing sequence $\{\varepsilon_n\}$, define $\delta_n := \varepsilon_n/r$. Then there exists a sequence $\{u_n\} \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle\cdot/\delta_n\rangle, u_n)\}$ generates the homogeneous Young measure $\nu_y \otimes dy$. Define now $v_n(x) := u_n(x/r)$ so that $\{v_n\} \subset L^p(r\Omega, \mathbb{R}^d)$ and thus $\subset L^p(\Omega', \mathbb{R}^d)$. By changing variables it follows that the sequence $\{(\langle\cdot/\varepsilon_n\rangle, v_n)\}$ generates the homogeneous Young measure $\nu_y \otimes dy$ as well.

Lemma 3.4. *Let $F \in \mathbb{R}^d$, M_F as in (3.12) and $\mathcal{E}_p$ as in (1.5); then M_F is a convex and weak* closed subset of $(\mathcal{E}_p)'$.*

Proof. First we show that M_F is a subset of $(\mathcal{E}_p)'$. Fix $\nu \in \mathcal{M}_F$ and identify it with the homogeneous Young measure $\nu_y \otimes dy$. Let $\{v_n\} \subset L^p(\Omega, \mathbb{R}^d)$ be the sequence such that $\{(\langle\frac{\cdot}{\varepsilon_n}\rangle, v_n)\}$ generates ν.

Up to a subsequence, one can assume that $\{v_n\}$ is p-equi-integrable; see [22, Lemma 8.12]. Consequently, by [9, Theorem 3.6 (iv)] it holds that

$$K := \iint_{Q\times\mathbb{R}^d} |\xi|^p d\nu_y(\xi)dy < +\infty.$$

By assumption, for a.e. $y \in Q$, the measure ν_y is a probability measure on Ω, so for every $f \in \mathcal{E}_p$ we have

$$\int_Q \int_{\mathbb{R}^d} f(y,\xi)d\nu_y(\xi)dy \le \|f\|_{\mathcal{E}_p} \int_Q \int_{\mathbb{R}^d} \big(1 + |\xi|^p\big)d\nu_y(\xi)dy = (1+K)\|f\|_{\mathcal{E}_p};$$

hence, $M_F \subset (\mathcal{E}_p)'$. Since $M_F \subset \overline{M_F}$, then in order to show that M_F is weakly* closed, it is enough to show that its weak* closure $\overline{M_F} \subset M_F$. Since $\mathcal{E}_p$ is separable, then $(\mathcal{E}_p)'$ is metrizable; hence, for every $\nu \in \overline{M_F}$ there exists a sequence $\{\nu^n\} \subset M_F$ such that $\nu^n \overset{*}{\rightharpoonup} \nu$ in $(\mathcal{E}_p)'$. For every $n \in \mathbb{N}$, let $\{v^n_k\} \subset L^p(\Omega, \mathbb{R}^d)$ be the sequence generating ν^n. Since the map $(x,\xi) \mapsto \xi$ is an element of $\mathcal{E}_p$, then by the definition of weak* convergence it follows that

$$\lim_{n\to+\infty}\int_Q\int_\Omega \xi\, d\nu_y^n(\xi)dy = \int_Q\int_\Omega \xi\, d\nu_y(\xi)dy = F,$$

and thus ν has F as underlying deformation. We have now to show that ν is a homogeneous two-scale Young measure. For every $z \in L^1(Q)$ and $\phi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d)$, we have

$$\begin{aligned}\lim_{k\to+\infty}\lim_{n\to+\infty}&\int_Q z(x)\int_\Omega \phi\Big(\Big\langle\frac{x}{\varepsilon_k}\Big\rangle, v_k^n(x)\Big)dx\\ &= \lim_{n\to+\infty}\int_Q\int_Q\int_{\mathbb{R}^d} z(x)\phi(y,\xi)d\nu_y^n(\xi)dy\,dx\\ &= \int_Q z(x)dx\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\nu_y(\xi)dy,\end{aligned}$$

where in the second equality we used the fact that $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d) \subset \mathcal{E}_p$. By a diagonalization argument we can find a sequence $\{k_n\}$ such that once defined $u_n := v_{k_n}^n$, then it holds that

$$\lim_{n\to+\infty}\int_Q z(x)\int_\Omega \phi\Big(\Big\langle\frac{x}{\varepsilon_n}\Big\rangle, u_n(x)\Big)dx = \int_Q z(x)dx\int_Q\int_\Omega \phi(y,\xi)d\nu_y(\xi)dy.$$

It follows that ν is a homogeneous two-scale Young measure, which implies that M_F is weakly* closed. Now we have to prove that M_F is a convex set. Fix $\nu, \mu \in M_F$ and $t \in (0,1)$. Consider $D := (0,t)\times(0,1)^{N-1}$ and define

$$\sigma_{(x,y)} := \begin{cases}\nu_y & \text{if } (x,y)\in D\times Q,\\ \mu_y & \text{if } (x,y)\in (Q\setminus D)\times Q.\end{cases}$$

By Lemma 2.7, $\{\sigma_{(x,y)}\}_{(x,y)\in Q\times Q}$ is a two-scale Young measure, while by Lemma 2.10 we have that $\{\overline{\sigma}_y\}_{y\in Q}$ is a homogeneous two-scale Young measure, hence an element of M_F.

Finally, observe that for every $\phi \in L^1(Q, \mathcal{C}_0(\mathbb{R}^d))$ it holds that

$$\begin{aligned}\int_Q\int_{\mathbb{R}^d}&\phi(y,\xi)d\overline{\sigma}_y(\xi)dy\\ &= \int_Q\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\sigma_{(x,y)}(\xi)dy\,dx\\ &= \int_D\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\nu_y(\xi)dy\,dx + \int_{Q\setminus D}\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\mu_y(\xi)dy\,dx\\ &= t\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\nu_y(\xi)dy + (1-t)\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\mu_y(\xi)dy,\end{aligned} \tag{3.13}$$

which means that $\overline{\sigma} = t\nu + (1-t)\mu$. In order to conclude that $\overline{\sigma} \in M_F$, we have to show that $\int_Q \int_{\mathbb{R}^d} \xi d\overline{\sigma}_y(\xi)dy = F$. In fact, this is the case: it suffices to apply (3.13) to $\phi(y,\xi) = \xi$, taking into account that

$$\int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\mu_y(\xi)dy = \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\nu_y(\xi)dy = F. \qquad \blacksquare$$

Now we show that conditions (i), (ii) and (iii) of Theorem 1.3 are sufficient to characterize two-scale Young measures. As in [6] we start from the homogeneous case. The non-homogeneous one will be obtained through a suitable approximation of two-scale Young measures by piecewise constant ones.

LEMMA 3.5. *Let $F \in \mathbb{R}$ and $\nu \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $y \in Q$. Assume that*

$$F = \int_Q \int_{\mathbb{R}^d} \xi d\nu_y(\xi)dy, \tag{3.14}$$

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_y(\xi)dy < +\infty, \tag{3.15}$$

and that

$$f_{\text{hom}}(F) \le \int_Q \int_{\mathbb{R}^d} f(y,\xi)d\nu_y(\xi)dy, \tag{3.16}$$

for every $f \in \mathcal{E}_p$ where $f_{\text{hom}}(F)$ is defined in (1.7) and $\mathcal{E}_p$ is the space introduced in (1.5).

Then $\{\nu_y\}_{y\in Q}$ is a homogeneous two-scale Young measure.

PROOF. Let $F \in \mathbb{R}^d$ and $\nu \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $y \in Q$, and that (3.14), (3.15) and (3.16) hold. We will proceed by contradiction using the Hahn–Banach Separation Theorem. Assume that ν is not an element of M_F. By Lemma 3.4, M_F is a convex and weak* closed subset of $(\mathcal{E}_p)'$. Moreover, by (3.15) and the fact that $\{\nu_y\}_{y\in Q}$ is a family of probability measures, we get that $\nu \in (\mathcal{E}_p)'$ as well. Since $\nu \notin M_F$, according to Hahn–Banach Separation Theorem, we can separate ν from M_F; i.e., there exists a linear weak* continuous map $L : (\mathcal{E}_p)' \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\langle L, \nu\rangle_{(\mathcal{E}_p)'',(\mathcal{E}_p)'} < \alpha$ and $\langle L, \mu\rangle_{(\mathcal{E}_p)'',(\mathcal{E}_p)'} \ge \alpha$ for all $\mu \in M_F$. Let $f \in \mathcal{E}_p$ be such that

$$\alpha \le \langle L, \mu\rangle_{(\mathcal{E})''_p,(\mathcal{E}_p)'} = \langle \mu, f\rangle_{(\mathcal{E}_p)',\mathcal{E}_p} = \int_Q \int_{\mathbb{R}^d} f(y,\xi)d\mu_y(\xi)dy \tag{3.17}$$

for all $\mu \in M_F$ and

$$\alpha > \langle L, \nu\rangle_{(\mathcal{E})''_p,(\mathcal{E}_p)'} = \langle \nu, f\rangle_{(\mathcal{E})'_p,\mathcal{E}_p} = \int_Q \int_{\mathbb{R}^d} f(y,\xi)d\nu_y(\xi)dy \ge f_{\text{hom}}(F). \tag{3.18}$$

Let f_H be defined as

$$f_H(F) := \inf_{\mu \in M_F} \int_Q \int_{\mathbb{R}^d} f(y,\xi)d\mu_y(\xi)dy, \quad F \in \mathbb{R}^d.$$

Then by (3.17), we have that $\alpha \leq f_H(F)$. We are going to show that

$$f_H(F) \leq f_{\text{hom}}(F), \tag{3.19}$$

which is in contradiction with (3.18) and proves the lemma.

To prove (3.19), let $T \in \mathbb{N}$ and $\phi \in L^p((0,T)^N;\mathbb{R}^d)$ such that $\int_{(0,T)^N} \phi(x)dx = 0$. Extend ϕ to $\mathbb{R}^N$ by $(0,T)^N$-periodicity and consider the sequence

$$\phi_n(x) := F + \phi\left(\frac{x}{\varepsilon_n}\right),$$

where $\{\varepsilon_n\}$ is an arbitrary vanishing sequence. Let $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d)$ and $z \in L^1(Q)$. Then, since $T \in \mathbb{N}$, the function $F \mapsto \varphi(\langle y\rangle, F + \phi(y))$ is $(0,T)^N$-periodic and according to the Riemann–Lebesgue lemma, we get that

(3.20)
$$\begin{aligned}\lim_{n\to+\infty} \int_Q z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, \phi_n(x)\right)dx &= \lim_{n\to+\infty} \int_Q z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, F + \phi\left(\frac{x}{\varepsilon_n}\right)\right)dx \\ &= \int_Q z(x)dx \fint_{(0,T)^N} \varphi\big(\langle y\rangle, F + \phi(y)\big)dy.\end{aligned}$$

Observe that

$$\begin{aligned}&\fint_{(0,T)^N} \varphi\big(\langle y\rangle, F + \phi(y)\big)dy \\ &\quad = \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0,T)^N} \int_{a_i+Q} \varphi\big(\langle y\rangle, F + \phi(y)\big)dy \\ &\quad = \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0,T)^N} \int_Q \varphi\big(\langle a_i + y\rangle, F + \phi(a_i + y)\big)dy \\ &\quad = \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0,T)^N} \int_Q \varphi\big(y, F + \phi(a_i + y)\big)dy.\end{aligned} \tag{3.21}$$

Thus, from (3.20) and (3.21), the pair $\{(\langle\cdot/\varepsilon_n\rangle;\phi_n)\}$ generates the homogeneous Young measure

$$\mu := \sum_{a_i \in \mathbb{Z}^N \cap [0,T)^N} \frac{1}{T^N}\delta_{F+\phi(a_i+y)} \otimes dy.$$

Then,

$$\int_Q \int_{\mathbb{R}^d} \xi d\mu_y(\xi) dy = F,$$

which implies that $\mu \in M_F$. In addition,

$$\fint_{(0,T)^N} f\big(\langle y\rangle, F + \phi(y)\big) dy = \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\mu_y(\xi) dy,$$

and then

$$\fint_{(0,T)^N} f\big(\langle y\rangle, F + \phi(y)\big) dy \geq f_H(F).$$

As a consequence, taking the infimum over all $\phi \in L^p((0,T)^N, \mathbb{R}^d)$ such that

$$\int_{(0,T)^N} \phi(x) dx = 0$$

and the limit as $T \to +\infty$, we get that $f_H(F) \leq f_{\text{hom}}(F)$ which implies (3.19). ■

The following result is an unconstrained version of [6, Proposition 3.6].

LEMMA 3.6. *Let* $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^d))$ *be such that the family* $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ *is a two-scale Young measure. Then for a.e.* $a \in \Omega$, $\{\nu_{(a,y)}\}_{y\in Q}$ *is a homogeneous two-scale Young measure.*

PROOF. Since $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure, it satisfies (3.3), (3.5) and (3.6). Since $u_1(x,\cdot)$ is Q-periodic for a.e. $x \in \Omega$, integrating (3.3) we get (3.4). Furthermore, (3.6) implies that

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)} dy < +\infty \quad \text{for a.e. } x \in \Omega. \tag{3.22}$$

Let $E \subset \Omega$ be a set of Lebesgue measure zero such that (3.4), (3.5) and (3.22) do not hold. Then for every $a \in \Omega \setminus E$,

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(a,y)}(\xi) dy = u(a),$$

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(a,y)}(\xi) dy < +\infty,$$

and

$$\int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(a,y)}(\xi) dy \geq f_{\text{hom}}\big(u(a)\big)$$

for every $f \in \mathcal{E}_p$. As a consequence of Lemma 3.5, for every $a \in \Omega \setminus E$, the family $\{\nu_{(a,y)}\}_{y\in Q}$ is a homogeneous two-scale Young measure. ■

THEOREM 3.7. *Let Ω be a bounded and open subset of $\mathbb{R}^d$. Let $\nu \in L_w^\infty(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x, y) \in \Omega \times Q$. Assume that* (i)–(iii) *of Theorem* 1.3 *hold; i.e.,*

$$\int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q), \tag{3.23}$$

there exist $u_1 \in L^p(\Omega; (L^p_{\text{per}}(Q; \mathbb{R}^d)))$ and $u \in L^p(\Omega; \mathbb{R}^d)$ such that

$$u_1(x, y) = \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi), \tag{3.24}$$

with $\int_Q u_1(x, y)dy = u(x)$. Assume also that

$$f_{\text{hom}}\big(u(x)\big) \leq \int_Q \int_{\mathbb{R}^d} f(y, \xi) d\nu_{(x,y)}(\xi)dy, \quad \textit{for a.e. } x \in \Omega, \textit{ for any } f \in \mathcal{E}_p, \tag{3.25}$$

where f_{hom} is the function in (1.7). *Then $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure with underlying deformation u.*

PROOF. Let $u = 0$ a.e. on Ω and let (φ, z) be in a countable dense subset of $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d) \times L^1(\Omega)$. Define

$$\overline{\varphi}(x) := \int_Q \int_{\mathbb{R}^d} \varphi(y, \xi) d\nu_{(x,y)}(\xi)dy.$$

Since the average with respect to y of $u_1(x, y)$ is $u(x)$ for a.e. $x \in \Omega$, integrating (3.24) with respect to $y \in Q$, it follows that

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi)dy = 0, \quad \text{for a.e. } x \in \Omega. \tag{3.26}$$

Let $E \subset \Omega$ be the null Lebesgue measure set such that (3.26), (3.25) and (3.23) do not hold. Then for every $a \in \Omega \setminus E$

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(a,y)}(\xi)dy = 0,$$

$$f_{\text{hom}}(0) \leq \int_Q \int_{\mathbb{R}^d} f(y, \xi) d\nu_{(a,y)}(\xi)dy$$

for every $f \in \mathcal{E}_p$, and

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(a,y)}(\xi)dy < +\infty.$$

Now, let $k \in \mathbb{N}$. According to [33, Lemma 7.9, p. 129], there exist points $a_i^k \in \Omega \setminus E$ and positive numbers $\rho_i^k \leq 1/k$ such that $a_k^i + \rho_i^k \Omega$ are pairwise disjoint for each k,

$$\bar{\Omega} = \bigcup_{i \geq 1} (a_i^k + \rho_i^k \bar{\Omega}) \cup E_k, \quad \text{with } \mathcal{L}(E_k) = 0,$$

and

$$\int_\Omega z(x)\bar{\varphi}(x)dx = \lim_{k \to +\infty} \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x)dx. \tag{3.27}$$

For each $k \in \mathbb{N}$, let $m_k \in \mathbb{N}$ be large enough so that

$$\left| \sum_{i=1}^{m_k} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x)dx - \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x)dx \right| < \frac{1}{k}.$$

For fixed i and k, by choice of a_i^k and by Lemma 3.6, the family $\{\nu_{(a_i^k, y)}\}_{y \in Q}$ is a homogeneous two-scale Young measure. Hence, by Remark 3.3, for every vanishing sequence $\{\varepsilon_n\}$, there exist sequences $\{u_n^{i,k}\} \subset L^p(a_i^k + \rho_i^k \Omega, \mathbb{R}^d)$ such that

$$\lim_{n \to +\infty} \int_{a_i^k + \rho_i^k \Omega} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^{i,k}(x)\right)dx = \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x)dx.$$

Summing up,

$$\lim_{n \to +\infty} \sum_{i=1}^{m_k} \int_{a_i^k + \rho_i^k \Omega} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^{i,k}\right)dx = \sum_{i=1}^{m_k} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x)dx.$$

Let us define

$$u_n^k(x) := \begin{cases} u_n^{i,k}(x) & \text{if } x \in a_i^k + \rho_i^k \Omega, \\ 0 & \text{otherwise} \end{cases}$$

and remark that $u_n^k \in L^p(\Omega, \mathbb{R}^d)$. Since the sets $a_i^k + \rho_i^k \Omega$ are pairwise disjoint for each k, we have that

$$\begin{aligned} &\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^k(x)\right)dx \\ &\quad = \sum_{i \geq 1} \int_{a_i^k + \rho_i^k \Omega} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^{i,k}(x)\right)dx \\ &\quad = \sum_{i=1}^{m_k} \int_{a_i^k + \rho_i^k \Omega} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^{i,k}(x)\right)dx \\ &\qquad + \int_{\Omega \cap \bigcup_{i > m_k} (a_i^k + \rho_i^k \Omega)} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}, u_n^k(x) \right\rangle\right)dx. \end{aligned}$$

But as $z \in L^1(\Omega)$ and $\mathcal{L}^N(\Omega \cap \bigcup_{i>m_k}(a_i^k + \rho_i^k \Omega)) \to 0$, as $k \to +\infty$, it follows that

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\left|\int_{\Omega\cap\bigcup_{i>m_k}(a_i^k+\rho_i^k\Omega)} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx\right| = 0. \tag{3.28}$$

Then, gathering (3.27)–(3.28) we obtain that

$$\begin{aligned}
&\lim_{k\to+\infty}\lim_{n\to+\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx\\
&\quad= \lim_{k\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{m_k}\int_{a_i^k+\rho_i^k\Omega} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx\\
&\quad= \lim_{k\to+\infty}\sum_{i=1}^{m_k}\overline{\varphi}(a_i^k)\int_{a_i^k+\rho_i^k\Omega} z(x)dx\\
&\quad= \lim_{k\to+\infty}\sum_{i\geq 1}\overline{\varphi}(a_i^k)\int_{a_i^k+\rho_i^k\Omega} z(x)dx\\
&\quad= \int_\Omega z(x)\overline{\varphi}(x)dx.
\end{aligned}$$

A diagonalization argument implies the existence of a diverging sequence $\{k_n\}$, as $n \to \infty$, such that upon setting $u_n := u_n^{k_n}$, then

$$\lim_{n\to+\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n(x)\right)dx = \int_\Omega z(x)\overline{\varphi}(x)dx$$

and $u_n \rightharpoonup 0$ in $L^p(\Omega, \mathbb{R}^d)$, which completes the proof whenever $u = 0$.

Consider now a general $u \in L^p(\Omega, \mathbb{R}^d)$ and ν satisfying properties (3.23)–(3.25). We define $\widetilde{\nu} \in L^\infty_w(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ by

$$\langle\widetilde{\nu}, \psi\rangle := \int_\Omega\int_Q\int_{\mathbb{R}^d}\psi\big(x, y, \xi - u(x)\big)d\nu_{(x,y)}(\xi)dy\,dx, \tag{3.29}$$

for every $\psi \in L^1(\Omega \times Q, \mathcal{C}_0(\mathbb{R}^d))$. We can easily check that $\widetilde{\nu}$ satisfies the analog properties of (3.23)–(3.25) with $u = 0$. Hence, applying the first step of the proof, for vanishing every sequence $\{\varepsilon_n\}$ we may find a sequence $\{\tilde{u}_n\} \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle\cdot/\varepsilon_n\rangle, \tilde{u}_n)\}$ generates the Young measure $\{\widetilde{\nu}_{(x,y)} \otimes dy\}_{x\in\Omega}$. Define $u_n := \tilde{u}_n + u$. It is easily seen that $\{(\langle\cdot/\varepsilon_n\rangle, u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x\in\Omega}$. Indeed, let $\psi \in L^1(\Omega, \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d))$ and define

$$\widetilde{\psi}(x, y, \xi) := \psi\big(x, y, \xi + u(x)\big)$$

where $\widetilde{\psi} \in L^1(\Omega, \mathscr{C}_0(\mathbb{R}^N \times \mathbb{R}^d))$ as well. Then, by (3.29),

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega \psi\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x)\right)dx &= \lim_{n\to+\infty} \int_\Omega \widetilde{\psi}\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \tilde{u}_n(x)\right)dx \\ &= \int_\Omega \int_Q \int_{\mathbb{R}^d} \widetilde{\psi}(x, y, \xi) d\widetilde{\nu}_{(x,y)}(\xi) dy\, dx \\ &= \int_\Omega \int_Q \int_{\mathbb{R}^d} \psi(x, y, \xi) d\nu_{(x,y)}(\xi) dy\, dx\end{aligned}$$

which completes the proof. ∎

4. Homogenization

This section is devoted to the proof of Theorem 1.1. To this end we start by showing that two-scale Young measures behave as measure products when generated by couples of sequences. This fact is a fundamental tool for achieving our result, and it represents an extension of the pioneering [32, Proposition 2.3].

Proposition 4.1. *Let*

$$\Lambda = \{\Lambda_{(x,x,y,y')}\}_{(x,x',y,y')\in\Omega^2\times Q^2} \subset L^\infty_w\left(\Omega^2 \times Q^2; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)\right)$$

be a family of probability measures supported on $\mathbb{R}^d \times \mathbb{R}^d$. Λ is the two-scale Young measure associated with a sequence $\{(u_n(x), u_n(x'))\}$, with $\{u_n\} \subset L^p(\Omega, \mathbb{R}^d)$ if and only if $\Lambda_{(x,x',y,y')} = \nu_{(x,y)} \otimes \nu_{(x',y')}$, where

$$\nu := \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \subset L^\infty_w\left(\Omega \times Q; \mathcal{M}(\mathbb{R}^d)\right)$$

is the two-scale Young measure associated (in the sense of Definition 2.3) with the sequence $\{u_n\}$.

Proof. Let $\{\varepsilon_n\}$ be any vanishing sequence and let Λ be the two-scale Young measure associated with the sequence $\{(u_n(x), u_n(x'))\}$ for some bounded sequence $\{u_n\} \subset L^p(\Omega, \mathbb{R}^d)$. We test Λ against functions $\theta(x, x')\varphi(y, \xi, y', \xi')$ with $\theta \in L^1(\Omega \times \Omega)$ and $\varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d)$ (in view of the density of linear combinations of such functions in $L^1(\Omega \times \Omega; C_0(\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d))$). In particular, we choose φ such that $\varphi(y, \xi, y', \xi') = \varphi_1(y, \xi)\varphi_2(y', \xi')$ and $\theta(x, x') = \theta_1(x)\theta_2(x')$. Consider

$$\begin{aligned}&\iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x')\bigg(\iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(y', \xi') \\ &\qquad\qquad \times \varphi_2(y', \xi') d\Lambda_{(x,x',y,y')}(\xi, \xi') dy\, dy'\bigg) dx\, dx' \\ &= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} \theta(x, x')\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \left\langle \frac{x'}{\varepsilon_n} \right\rangle, u_n(x), u_n(x')\right) dx\, dx'\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x')\varphi_1\Big(\Big\langle\frac{x}{\varepsilon_n}\Big\rangle, u_n(x)\Big)\varphi_2\Big(\Big\langle\frac{x'}{\varepsilon_n}\Big\rangle, u_n(x')\Big)dx\,dx' \\
&= \lim_{n\to+\infty} \Big(\int_\Omega \theta_1(x)\varphi_1\Big(\Big\langle\frac{x}{\varepsilon_n}\Big\rangle, u_n(x)\Big)dx\Big)\Big(\int_\Omega \theta_2(x')\varphi_2\Big(\Big\langle\frac{x'}{\varepsilon_n}\Big\rangle, u_n(x')\Big)dx'\Big) \\
&= \int_\Omega \theta_1(x)\Big(\int_Q\int_{\mathbb{R}^d} \varphi(y,\xi)d\nu_{(x,y)}(\xi)dy\Big)dx \\
&\quad \times \int_\Omega \theta_2(x')\Big(\int_Q\int_{\mathbb{R}^d} \varphi(y',\xi')d\nu_{(x',y')}(\xi')dy'\Big)dx' \\
&= \iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x')\Big(\iint_{Q\times Q}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(y,\xi) \\
&\qquad\qquad \times \varphi_2(y',\xi')d(\nu_{(x,y)}\otimes\nu_{(x',y')})(\xi,\xi')dy\,dy'\Big)dx\,dx'.
\end{aligned}
$$

The fourth equality is due to the boundedness of the sequence $\{u_n\}$, and the fact that $\{(\langle\frac{x}{\varepsilon_n}\rangle, u_n(x))\}$ generates the two-scale Young measure ν. The thesis follows from the chain of identities. The reverse implication can be proved in a similar way. ■

Now, we prove Theorem 1.1. This result is a generalization of [6, Theorem 1.2]. We observe also that we will refer to the Γ-convergence as it has been introduced in Definition 2.17, since by the growth assumption (1.3), we can work in bounded subsets of $L^p(\Omega;\mathbb{R}^m)$, which are metrizable.

PROOF OF THEOREM 1.1. We recall the family of functionals $I_\varepsilon : L^p(\Omega,\mathbb{R}^d)\to[0,+\infty)$ defined in (1.1) as

$$
I_\varepsilon(u) := \int_\Omega\int_\Omega W\Big(x,x',\Big\langle\frac{x}{\varepsilon}\Big\rangle,\Big\langle\frac{x'}{\varepsilon}\Big\rangle,u(x),u(x')\Big)dx\,dx',
$$

where $\Omega\subset\mathbb{R}^N$, $u\in L^p(\Omega;\mathbb{R}^d)$ and $W:\Omega\times\Omega\times Q\times Q\times\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ is a symmetric admissible integrand, satisfying (1.3).

Let $u\in L^p(\Omega,\mathbb{R}^d)$ and let $\{\varepsilon_n\}$ be such that $\varepsilon_n\to 0$. We start by showing that

$$
\begin{aligned}
(4.1)\qquad &\Gamma-\limsup_{n\to+\infty} I_{\varepsilon_n}(u) \\
&\leq \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega}\iint_{Q\times Q}\iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x,x',y,y',\xi,\xi')d\nu_{(x,y)} \\
&\qquad\qquad \times(\xi)d\nu_{(x',y')}(\xi')dy\,dy'\,dx\,dx',
\end{aligned}
$$

where $\mathcal{M}_u$ is defined in (1.4). Let $\nu\in\mathcal{M}_u$; by Remark 2.6, Theorem 1.3 and Proposition 4.1, there exists a sequence $\{u_n\}\subset L^p(\Omega,\mathbb{R}^d)$ such that $u_n\rightharpoonup u$ in $L^p(\Omega,\mathbb{R}^d)$ generates the two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, and clearly $\{(u_n(x),u_n(x'))\}_n$ is the sequence associated with the two-scale Young measure $\{\nu_{(x,y)}\otimes\nu_{(x',y')}\}_{(x,x',y,y')\in\Omega^2\times Q^2}$. Extract a subsequence $\{\varepsilon_{n_k}\}\subset\{\varepsilon_n\}$ such that

$$\limsup_{n\to\infty} I_{\varepsilon_n}(u_n)=\lim_{k\to\infty} I_{\varepsilon_{n_k}}(u_{n_k})$$

and that $\{|u_{n_k}|^p\}$ is equi-integrable, which is always possible by the Decomposition Lemma [22, Lemma 8.13]. In particular, due to the p-growth condition (1.3), it follows that the sequence $\{W(\cdot,\cdot',\langle\cdot/\varepsilon_{n_k}\rangle,\langle\cdot'/\varepsilon_{n_k}\rangle,u_{n_k}(\cdot),u_{n_k}(\cdot'))\}$ is equi-integrable in $\Omega\times\Omega$ as well; hence, applying Theorem 2.13 (ii) we get that

$$\begin{aligned}&\Gamma-\limsup_{n\to+\infty} I_{\varepsilon_n}(u)\\&\le \lim_{k\to+\infty}\iint_{\Omega\times\Omega} W\left(x,x',\left\langle\frac{x}{\varepsilon_{n_k}}\right\rangle,\left\langle\frac{x'}{\varepsilon_{n_k}}\right\rangle,u_{n_k}(x),u_{n_k}(x')\right)dx\,dx'\\&=\iint_{\Omega\times\Omega}\iint_{Q\times Q}\iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x,x',y,y',\xi,\xi')d\nu_{(x,y)}(\xi)d\nu_{(x',y')}(\xi')dy\,dy'dx\,dx'.\end{aligned}\tag{4.2}$$

Taking the infimum over all $\nu\in\mathcal{M}_u$ in the right-hand side of (4.2) yields (4.1).

Let us prove now that

$$\begin{aligned}\Gamma-\liminf_{n\to+\infty} I_{\varepsilon_n}(u)&\\ \ge \inf_{\nu\in\mathcal{M}_u}\iint_{\Omega\times\Omega}\iint_{Q\times Q}\iint_{\mathbb{R}^d\times\mathbb{R}^d}& W(x,x',y,y',\xi,\xi')d\nu_{(x,y)}\\&\times(\xi)d\nu_{(x',y')}(\xi')dy\,dy'\,dx\,dx'.\end{aligned}\tag{4.3}$$

Let $\eta>0$ and $\{u_n\}\subset L^p(\Omega,\mathbb{R}^d)$ such that $u_n\rightharpoonup u$ in $L^p(\Omega,\mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} I_{\varepsilon_n}(u)\le\Gamma-\liminf_{n\to+\infty} I_{\varepsilon_n}(u)+\eta.\tag{4.4}$$

For a subsequence $\{n_k\}$, still thanks to Proposition 4.1, we can assume that there exists $\nu\in L^\infty_w(\Omega\times Q,\mathcal{M}(\mathbb{R}^d))$ such that $\{(u_{n_k}(x),u_{n_k}(x'))\}$ is the sequence associated with the two-scale Young measure $\{\nu_{(x,y)}\otimes\nu_{(x',y')}\}_{(x,y,x',y')\in(\Omega\times Q)^2}$ and

$$\lim_{k\to+\infty} I_{\varepsilon_{n_k}}(u_{n_k})=\liminf_{n\to+\infty} I_{\varepsilon_n}(u_n).\tag{4.5}$$

We remark that $\{u_{n_k}\}$ is equi-integrable since it is bounded in $L^p(\Omega,\mathbb{R}^d)$ and $p>1$. Thus, by the Fundamental Theorem on Young Measures (Theorem 2.2) we get that for every $A\in\mathcal{A}(\Omega)$,

$$\int_A u(x)dx=\lim_{k\to+\infty}\int_A u_{n_k}(x)dx=\int_A\int_Q\int_{\mathbb{R}^d}\xi d\nu_{(x,y)}(\xi)dy\,dx.$$

By the arbitrariness of the set A, it follows that

$$u(x) = \int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) dy \quad \text{a.e. in } \Omega. \tag{4.6}$$

As a consequence of Corollary 3.1, $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure and, by (4.6), we also have that $\nu \in \mathcal{M}_u$. Applying now Theorem 2.13 (i) we get that

$$\begin{aligned}
&\lim_{k\to+\infty} \iint_{\Omega\times\Omega} W\left(x, x', \left\langle \frac{x}{\varepsilon_{n_k}} \right\rangle, \left\langle \frac{x'}{\varepsilon_{n_k}} \right\rangle, u_{n_k}(x), u_{n_k}(x')\right) dx\, dx' \\
&\geq \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy\, dy' dx\, dx' \\
&\geq \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'.
\end{aligned}$$

Hence, by (4.4), (4.5) and the arbitrariness of η we get the desired result. Gathering (4.1) and (4.3), we obtain that

$$\begin{aligned}
\Gamma - \lim_{n\to+\infty} I_{\varepsilon_n}(u) = \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} & W(x, x', y, y', \xi, \xi') d\nu_{(x,y)} \\
& \times (\xi) d\nu_{(x',y')}(\xi') dy\, dy'\, dx\, dx'.
\end{aligned}$$

To conclude the proof of the upper bound, we have to prove that the minimum is attained. Consider a recovering sequence $\{\bar{u}_n\} \subset L^p(\Omega, \mathbb{R}^d)$ for $\Gamma - \lim_n I_{\varepsilon_n}(u)$. Arguing exactly as before we can assume that (a subsequence of) $\{\bar{u}_n\}$ generates a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, that $\nu \in \mathcal{M}_u$ and the couple $\{(\bar{u}_n(x), \bar{u}_n(x'))\}_n$ generates $\{\nu_{(x,y)} \otimes \nu_{(x',y')}\}_{(x,y,x',y')\in(\Omega\times Q)^2}$, and $\{W(\cdot, \cdot', \langle \cdot/\varepsilon_{n_k}\rangle, \langle \cdot'/\varepsilon_{n_k}\rangle, u_{n_k}(\cdot), u_{n_k}(\cdot'))\}$ is equi-integrable. According to Theorem 2.13 (ii), we get

$$\begin{aligned}
&\Gamma - \lim_{n\to+\infty} I_{\varepsilon_n}(u) \\
&= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} W\left(x, x', \left\langle \frac{x}{\varepsilon_n} \right\rangle, \left\langle \frac{x'}{\varepsilon_n} \right\rangle, \bar{u}_n(x), \bar{u}_n(x')\right) dx dx' \\
&= \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy\, dy'\, dx\, dx',
\end{aligned}$$

which completes the proof. ∎

REMARK 4.2. (i) We observe that if in the above theorem W does not depend on $y, y' \in Q$, the above Γ-convergence result reduces to a relaxation result with respect to the L^p-weak convergence, with I_{hom} therein replaced by

$$\begin{aligned}
I_{\text{hom}}(u) := \min_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} & W(x, x', \xi, \xi') d\nu_{(x,y)} \\
& \times (\xi) d\nu_{(x',y')}(\xi') dy\, dy'\, dx\, dx',
\end{aligned}$$

with $\mathcal{M}_u$ in (1.4). $I_{\rm hom}$, in turn, becomes

$$I_{\rm hom}(u)=\min_{\mu\in\mathcal{M}'_u}\iint_{\Omega\times\Omega}\iint_{\mathbb{R}^d\times\mathbb{R}^d}W(x,x',\xi,\xi')d\mu_x(\xi)d\mu_{x'}(\xi')dx\,dx', \tag{4.7}$$

where

$$\mathcal{M}'_u:=\Big\{\mu\in L^\infty_w\big(\Omega,\mathcal{M}(\mathbb{R}^d)\big):\\ \{\mu_x\}_{x\in\Omega}\text{ is a Young measure such that }\int_{\mathbb{R}^d}\xi d\nu_x(\xi)=u(x)\Big\}.$$

The above equality is easily obtained in view of Definition 2.3, which ensures that any measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}\in\mathcal{M}_u$ is such that $\nu_{(x,y)}\otimes dy=\mu_x\in\mathcal{M}'_u$.

(ii) We observe that the same strategy adopted in the proof of Theorem 1.1 would lead directly to (4.7), without using two-scale Young measures but adopting the classical Young measures generated by sequence in $L^p(\Omega;\mathbb{R}^d)$. It suffices to replace Theorem 2.13 by Theorem 2.2 (iv)–(v) and the characterization of Young measures proved in [33, Theorem 7.7].

(iii) Observe that equation 4.7 provides a statement analogous of [11, Theorem 6.1] replacing the narrow convergence by the L^p-weak convergence, under a slightly more general set of assumptions on W, cf. Remarks 1.2, 2.12 and 2.14.

We point out that analogous arguments as those in the proof of Theorem 1.1 allow us to obtain a Γ-convergence result in terms of a suitable narrow convergence (see Remark 2.15). In order to do so, a preliminary step is needed, i.e., a suitable extension of the functionals I_ε in (1.1) to $\mathcal{Y}^p(\Omega;\mathbb{T}^N\times\mathbb{R}^d)$, where we recall that $\mathbb{T}^N$ represents the N-dimensional torus in $\mathbb{R}^N$.

Corollary 4.3. *Let p, Ω, W and $\{I_\varepsilon\}_\varepsilon$ be defined as in Theorem 1.1. Let $\mathcal{Y}^p(\Omega;\mathbb{T}^N\times\mathbb{R}^d)$ be the space introduced in Remark 2.15 and let $\overline{I}_\varepsilon:\mathcal{Y}^p(\Omega,\mathbb{T}^N\times\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}$,*

$$\overline{I}_\varepsilon(\mu):=\begin{cases}I_\varepsilon(u) & \text{if } \mu=\{\delta_{(\langle\frac{x}{\varepsilon}\rangle,u(x))}\}_{x\in\Omega},\\ +\infty & \text{otherwise.}\end{cases}\tag{4.8}$$

Then, $\{\overline{I}_\varepsilon\}_\varepsilon$ Γ-converges, with respect to the narrow convergence, i.e., testing with functions in $L^1(\Omega;C_0(\mathbb{T}^N\times\mathbb{R}^d))$, to $\overline{I}_{\rm hom}:\mathcal{Y}^p(\Omega;\mathbb{T}^N\times\mathbb{R}^d)\to[0,+\infty]$, where

$$\overline{I}_{\rm hom}(\mu)=\begin{cases}\iint_{\Omega\times\Omega}\iint_{(Q\times\mathbb{R}^d)^2}W\big(x,x',(y,\xi),(y',\xi')\big)d\mu_x(y,\xi)d\mu_{x'}(y',\xi')dx\,dx',\\ \qquad\text{if } \{\mu_x\}_{x\in\Omega}=\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}\otimes dy,\\ +\infty\quad\text{otherwise,}\end{cases}$$

with ν a two-scale Young measure.

Before proving the result we observe that, when $\overline{I}_{\text{hom}}$ is finite, the right-hand side coincides with

$$\iint_{\Omega\times\Omega}\iint_{Q\times Q}\iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x,x',y,y',\xi,\xi')d\nu_{(x,y)}(\xi)d\nu_{(x',y')}(\xi')dy\,dy'\,dx\,dx'.$$

Proof. The proof follows the same argument as [11, Theorem 6.1]. Let $\mu\in\mathcal{Y}^p(\Omega;\mathbb{T}^N\times\mathbb{R}^d)$ and $\{\mu_n\}\subset\mathcal{Y}^p(\Omega;\mathbb{T}^N\times\mathbb{R}^d)$ be a sequence which converges narrowly to μ. With no loss of generality we can assume that

$$\liminf_{n\to\infty}\overline{I}_{\varepsilon_n}(\mu_{\varepsilon_n})<+\infty.$$

Then, up to passing to a suitable subsequence (not relabeled) for which the liminf is a limit, $\mu_n=\delta_{(\langle\frac{x}{\varepsilon_n}\rangle,u_n(x))}$ for a suitable sequence $\{u_n\}\subset L^p(\Omega;\mathbb{R}^d)$.

By (1.3), $\{u_n\}$ is bounded in $L^p(\Omega;\mathbb{R}^d)$, hence (up to a subsequence) weakly convergent; then as observed in [9] $\{(\langle\frac{x}{\varepsilon_n}\rangle,u_n)\}$ is tight and in view of Theorem 1.3 the narrow limit μ is of the type $\nu_{(x,y)}\otimes dy$ for a suitable two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$. Hence, the lower semicontinuity of

$$\iint_{\Omega\times\Omega}\iint_{(Q\times\mathbb{R}^d)^2} W(x,x',(y,\xi),(y',\xi'))d\mu_x(y,\xi)d\mu_{x'}(y',\xi')dx\,dx'$$

with respect to the narrow convergence, observed in [11, Proposition 3.7], proves the lower bound.

For what concerns the upper bound, if μ is of the type $\nu_{(x,y)}\otimes dy$ for a suitable two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, then Theorem 1.3 ensures the existence of a sequence $\{u_n\}\subset L^p(\Omega;\mathbb{R}^d)$, weakly convergent in $L^p(\Omega)$ and such that $\{(\langle\frac{x}{\varepsilon_n}\rangle,u_n)\}$ generates the two-scale Young measure $\nu_{(x,y)}$. The Decomposition Lemma ensures that the $\{u_n\}$ can be chosen p-equi-integrable. Clearly, by [9], the sequence is also narrowly convergent to $\mu_x=\nu_{(x,y)}\otimes dy$. Then (1.3) ensures that the integrand of $I_{\varepsilon_n}(u_n)$ is equi-integrable and, exactly as in the proof of Theorem 1.1, we can invoke Theorem 2.13 (ii) which, in turn, guarantees the convergence towards

$$\iint_{\Omega\times\Omega}\iint_{(Q\times\mathbb{R}^d)^2} W\big(x,x',(y,\xi),(y',\xi')\big)d\mu_x(y,\xi)d\mu_{x'}(y',\xi')dx\,dx',$$

which concludes the proof in this case.

Finally, if μ is not of the type $\nu_{(x,y)}\otimes dy$ for a suitable two-scale Young measure ν, then Theorem 1.3 says that $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ cannot be generated by any $\{\delta_{(\langle\frac{x}{\varepsilon_n}\rangle,u_n(x))}\}$; hence, it cannot be the narrow limit of such a sequence. Consequently, the $\{\mu_n\}$ narrowly converging to μ is such that the energy $\overline{I}_{\varepsilon_n}(\mu_n)=+\infty$ for any $n\in\mathbb{N}$. This concludes the proof. ∎

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