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Statistical Mechanics. – *On the local Maxwellians solving the Boltzmann equation with boundary condition*, by Théophile Dolmaire, accepted on 2 December 2025.

Abstract. – We derive the expressions of the local Maxwellians that solve the Boltzmann equation in the interior of a regular open domain, without assuming the boundedness of the domain. We investigate separately, on the one hand, the case of the bounce-back boundary condition in any dimension, and, on the other hand, the case of the specular reflection boundary condition, in dimension $d = 2$ and $d = 3$. In the case of the bounce-back boundary condition, we prove that the only local Maxwellians solving the Boltzmann equation with boundary condition are the global Maxwellians. In the case of the specular reflection, we provide a complete classification of the domains for which only the global Maxwellians solve the Boltzmann equation with boundary condition, and we describe all the local Maxwellians that solve the equation for the domains presenting symmetries.

Keywords. – Boltzmann equation, local Maxwellian, boundary condition.

Mathematics Subject Classification 2020. – 35Q20 (primary); 82C40, 82D05 (secondary).

1. Introduction

The Boltzmann equation and the H-theorem. In 1872, Ludwig Boltzmann [1] introduced the following equation, in order to describe dilute gases composed with particles that interact with each other via binary collisions:

$$\text{(1.1)}\quad \partial_t f + v\cdot\nabla_x f = \int_{v_*\in\mathbb{R}^d}\int_{\omega\in\mathbb{S}^{d-1}} B\left(|v-v_*|,\left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right|\right)\big[f(v')f(v'_*)-f(v)f(v_*)\big]\,\mathrm{d}\omega\,\mathrm{d}v_*.$$

Here, the unknown $f = f(t,x,v)$ of the Boltzmann equation (1.1) is a probability density for all time t on the space $\Omega\times\mathbb{R}^d \ni (x,v)$ (the particles constituting the gas are assumed to evolve in the domain $\Omega\subset\mathbb{R}^d$, where d is an integer, representing the dimension of the physical space that is considered), and

$$f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$$

represents the number of particles of the gas lying in the infinitesimal volume $x + \mathrm{d}x$ and moving with a velocity in $v + \mathrm{d}v$ at time t. v' and v'_*, the *post-collisional velocities*, are defined as

$$v' = v - (v - v_*) \cdot \omega\omega, \quad v'_* = v_* + (v - v_*) \cdot \omega\omega, \tag{1.2}$$

ω is the *angular parameter*, and $B(|v - v_*|, |\frac{v-v_*}{|v-v_*|} \cdot \omega|)$, the *collision kernel*, describes the rate at which collisions involving two particles with relative velocities $v - v_*$ and colliding with an angular parameter ω take place.

For instance, if we assume that the particles interact as hard spheres, the angular parameter ω represents the direction of the line joining the respective centers of the two colliding particles, and in such a case the collision kernel B is

$$B\left(|v - v_*|, \left|\frac{v - v_*}{|v - v_*|} \cdot \omega\right|\right) = \big|(v - v_*) \cdot \omega\big|.$$

To complete the description of the model, boundary conditions need to be added to (1.1). In the present article, we will assume that the domain Ω is an open set, with a boundary $\partial\Omega$ regular enough such that at any point $x \in \partial\Omega$ we can define an outgoing unitary normal vector $n(x)$ to the boundary. In the present article, we will consider the two following boundary conditions: the *bounce-back boundary condition*, defined as

$$\forall t \in [0, T],\ x \in \partial\Omega,\ v \in \mathbb{R}^d, \quad f(t, x, v) = f(t, x, -v), \tag{1.3}$$

and the *specular reflection boundary condition*, defined as

$$\forall t \in [0, T],\ x \in \partial\Omega,\ v \in \mathbb{R}^d, \quad f(t, x, v) = f(t, x, v'), \tag{1.4}$$

with

$$v' = v - 2\big(v \cdot n(x)\big)n(x). \tag{1.5}$$

These two conditions, written at the mesoscopic level (that is, for the unknown f of the Boltzmann equation (1.1)), reflect the model chosen to describe the dynamics of the particles that compose the gas, when one of these particles collides with the boundary $\partial\Omega$. In the case of the bounce-back boundary condition, the velocity v of a particle about to leave the domain Ω is immediately changed into $-v$, while in the specular reflection case, the reflected velocity v' is defined by (1.5). For other possible boundary conditions and discussions concerning the relevance of the laws (1.3) and (1.4), the reader may refer to [4, 5, 13].

The collision law (1.2) describes elastic collisions since momentum and kinetic energy are conserved when two particles collide. These conserved quantities can be recovered at the level of the Boltzmann equation. Indeed, if f is a solution of (1.1) (at least, in the absence of boundary), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x\int_v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

More surprisingly, if we define the *entropy* H as the functional:

$$H(f)(t) = \int_x\int_v f(t,x,v)\ln f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x,$$

then if f is a solution of the Boltzmann equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f)(t) = -\int_x D(f)(x)\,\mathrm{d}x \le 0,$$

with D the *entropy production* defined as

$$D(f) = \frac{1}{4}\int_v\int_{v_*}\int_\omega B\big[f(v')f(v'_*) - f(v)f(v_*)\big]\ln\frac{f(v')f(v'_*)}{f(v)f(v_*)}\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v.$$

In addition, we have that $\int_x D(f)(x)\,\mathrm{d}x = 0$ if and only if there exist three functions $\rho, a : \Omega \to \mathbb{R}_+$ and $u : \Omega \to \mathbb{R}^d$ such that

$$f(x,v) = \rho(x)\exp\big(-a(x)\big|v-u(x)\big|^2\big).$$

These two statements constitute the celebrated *H-theorem*, discovered by Boltzmann [1]. In addition, let us observe that any function of the form:

$$m(t,x,v) = \rho_0\exp\big(-a_0|v-u_0|^2\big), \tag{1.6}$$

where $\rho_0, a_0 \in \mathbb{R}_+$, $u_0 \in \mathbb{R}^d$ are constant, is a solution to the Boltzmann equation. A function of the form (1.6) is called a *global Maxwellian*, whereas a function of the form:

$$m(t,x,v) = \rho(t,x)\exp\big(-a(t,x)\big|v-u(t,x)\big|^2\big), \tag{1.7}$$

with $\rho, a : I\times\Omega \to \mathbb{R}_+$ and $u : I\times\Omega \to \mathbb{R}^d$ ($I \subset \mathbb{R}$), is called a *local Maxwellian*.

The role of the local Maxwellians. If we can show that the only local Maxwellians solving the Boltzmann equation (1.1) are global Maxwellians, then the H-theorem would indicate the long-time behavior of the solutions of the Boltzmann equation. Indeed, since the entropy is strictly decreasing for solutions that are not local Maxwellians, and since the local Maxwellians are the only critical points of the entropy, in the case when there are no non-trivial local Maxwellians, we would expect that any solution f converges towards the only global Maxwellian that has the same mass, momentum and kinetic energy as f. For this reason, determining the local Maxwellians that solve the Boltzmann equation is of central importance in order to understand the long-time behavior of the solutions of (1.1).

Since a local Maxwellian cancels the collision term of the Boltzmann equation (that is, the right-hand side of (1.1)), a local Maxwellian m solves the Boltzmann equation if and only if m solves the free transport equation: that is,

$$\partial_t m + v \cdot \nabla_x m = 0.$$

It is well known how to determine the local Maxwellians that solve the free transport equation, and such a result was already obtained by Boltzmann himself [2]. The problem is also discussed by Cercignani in [4]. In this last reference, an external field acting on the particles is also considered.

As a matter of fact, in the case of a domain without boundary, many local Maxwellians, which are not global, solve the free transport equation. More generally, the question of the long-time behavior of the solutions of the Boltzmann equation is a very challenging problem that remains an open question nowadays. The difficulties arise from the mixing of the effects of the collision term of the Boltzmann equation on the one hand, and of the transport term $-v \cdot \nabla_x$ on the other hand. Under the action of the collision term, the solutions converge to local Maxwellians (as it can be rigorously proved in the case of the homogeneous Boltzmann equation). However, the collision term is local in x, whereas the transport term $-v \cdot \nabla_x f$ acts on the two variables x and v of the phase space, introducing major complications.

Nevertheless, let us mention that important results were obtained in [8] concerning the long-time behavior of the solutions of the non-homogeneous Boltzmann equation (1.1). For more information on the Boltzmann equation, the reader may refer to the classical references [4, 5, 13].

Back to the question of determining the local Maxwellians solving (1.1), the case of a domain with a boundary reduces dramatically the possibilities. A discussion concerning the geometry of the boundary can already be found in the article [9] of Grad. A classification of the boundaries as well as the complete proof of this classification can be found in [6]. Essentially, the result is that if the boundary of the domain does not present any symmetry, then only the global Maxwellians solve the free transport equation in such a bounded domain. However, the results of [6] are obtained under the assumption that the domain Ω is bounded. Besides, to the best of our knowledge, there exists no analogous result for more general domains. Therefore, and considering the central role of the local Maxwellians in the trend to converge to equilibrium for solutions of the Boltzmann equation, it seemed relevant to provide a discussion similar to the one presented in [6] in the case when the domain Ω is not necessarily bounded.

Main results. In the present article, we determine completely the local Maxwellians that solve the Boltzmann equation in a general domain. Our results recover the classification of the local Maxwellians solving the Boltzmann equation, already known in the case when the domain is bounded [6, 11]. The novelty of our work consists in considering domains that are not necessarily bounded. When the domain presents a boundary, we classify the local Maxwellians in the case of the bounce-back boundary condition, and in the case of the specular reflection boundary condition.

In the case of the bounce-back boundary condition, we prove that the only local Maxwellians solving the Boltzmann equation are necessarily global as soon as the boundary of the domain is non-empty (Theorem 2).

In the case of the specular reflection, we describe all the local Maxwellians solving the Boltzmann equation, in dimension $d = 2$ (Theorem 3) and $d = 3$ (Theorem 4). In particular, we determine local Maxwellians solving the Boltzmann equation in the case when the boundary of the domain presents a helical symmetry. To the best of our knowledge, it is the first description of such solutions. In the case of the half-plane or the half-space, we also determine local Maxwellians solving the Boltzmann equation that have finite mass despite the unboundedness of the domain.

Our approach relies on the following arguments. Starting from the explicit expression of the local Maxwellians solving the Boltzmann equation in the interior of any domain, given in terms of a small number of parameters, we deduce constraints on these parameters coming from the boundary conditions. In the case of the bounce-back boundary condition, we prove in particular that the boundary of the domain is contained in an affine space. The dimension of this affine space can be determined, leading to a contradiction if the local Maxwellian we consider is not global. In the case of the specular reflection, we adapt the strategy of [6], where the constraints on the parameters defining the admissible local Maxwellians are used to construct integral curves that belong to the boundary $\partial\Omega$ of the domain, and that can be explicitly described. Such explicit curves imply that the boundary $\partial\Omega$ exhibits particular symmetries. By carefully considering all the possible symmetries that can coexist, we complete the characterization of the local Maxwellians solving the Boltzmann equation in Ω, with specular reflection on $\partial\Omega$.

The plan of the article is as follows. In the second section, we recall the general expression of the local Maxwellians that solve the free transport equation in the interior of an open domain, and for the sake of completeness, we provide a synthetic proof of the derivation of this expression. This section follows the references [4, 6]. The third and the fourth sections are devoted to the new results. In the third section, we study the local Maxwellians solving (1.1) in a domain with a boundary, considering the bounce-back boundary condition. In particular, we prove that only the global Maxwellians are admissible solutions. Finally, in the fourth section, we turn to the case of the specular reflection boundary condition. In this section, we determine local Maxwellians solving the Boltzmann equation that were not discussed in [6].

Notations. Throughout this article, we will use the following notations.

For $d > 1$ any integer larger than 1, we will denote the $d \times d$ identity matrix by I_d:

$$I_d = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

For $m, n, p \geq 1$ three positive integers, the product of the two matrices $M \in \mathcal{M}_{m \times n}$ and $N \in \mathcal{M}_{n \times p}$ will be denoted by

$$MN.$$

In particular, the product of a $d \times d$ square matrix M with a vector $u \in \mathbb{R}^d$ will be denoted by Mu (the vectors of $\mathbb{R}^d$ are seen as $d \times 1$ matrices).

The transpose of a matrix M will be denoted by ${}^tM$.

The scalar product of two vectors $u, v \in \mathbb{R}^d$ will be denoted by $u \cdot v$.

Finally, in dimension $d = 3$, the cross-product of two vectors $u = (a, b, c) \in \mathbb{R}^3$ and $v = (x, y, z) \in \mathbb{R}^3$ will be denoted by $u \wedge v$: that is,

$$u \wedge v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}.$$

If A is a 3×3 skew-symmetric matrix of the form:

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

we say that A *is represented by the vector* u, or that *the vector* u *represents the matrix* A, with $u = (a, b, c)$. In other words, u is the only vector of $\mathbb{R}^3$ such that

$$Ax = u \wedge x \quad \forall x \in \mathbb{R}^3.$$

Assumptions on the domain. In all the rest of the article, the domain will be denoted by Ω and will be a non-empty, connected open set of $\mathbb{R}^d$. When we will say that Ω *has a boundary*, we will mean that $\partial\Omega$ is non-empty, and $\partial\Omega$ is a $\mathscr{C}^1$ manifold of codimension 1.

2. Obtaining the expressions of the admissible local Maxwellians

In this section, we recall the characterization of the local Maxwellians that solve the Boltzmann equation in dimension $d \geq 2$, on the time interval $[0, T]$, in the interior of a domain $\Omega \subset \mathbb{R}^d$, where Ω is open and regular. This result is classical and might be found under different forms, already in [1], but also in [4, 6, 11]. Since the characterization of the local Maxwellians solving the Boltzmann equation is the starting point of the investigations we conduct in this work, for the sake of completeness we also recall briefly the main steps of the proof of such a characterization. We will follow mainly the approach developed in [6].

We remark that in the present section, Ω might be also the whole Euclidean space $\mathbb{R}^d$. In other words, the boundary of the domain Ω plays no role at this step.

Under its most general form, the expression of a local Maxwellian is written as

$$m(t,x,v) = \rho(t,x)\exp\big(-a(t,x)\big|v-u(t,x)\big|^2\big), \tag{2.1}$$

where we assume that $\rho, a : \mathbb{R}\times\Omega \to \mathbb{R}_+$ are strictly positive almost everywhere, and $u : \mathbb{R}\times\Omega \to \mathbb{R}^d$. We assume also that ρ, a and u are differentiable.

It is well known that $Q(m,m) = 0$ (where $Q(m,m)$ is the quadratic collision operator of the Boltzmann equation; that is, $Q(m,m)$ is the right-hand side of (1.1)) if and only if m is a local Maxwellian (see for instance [1] or [13]), that is, if m is of the form of (2.1). Therefore, a local Maxwellian solves the Boltzmann equation in the domain Ω if and only if

$$\forall\,(t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^d, \quad \partial_t m + v\cdot\nabla_x m = 0; \tag{2.2}$$

that is, the local Maxwellian m solves the free transport equation. It is a classical result to determine completely the local Maxwellians that solve the free transport equation. Theorem 1 below provides the description of the local Maxwellians solving the free transport equation.

Theorem 1 (Local Maxwellians solving the free transport equation). *Let*

$$m : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}_+$$

be a local Maxwellian of the form:

$$m(t,x,v) = \rho(t,x)\exp\big(-a(t,x)\big|v-u(t,x)\big|^2\big),$$

with ρ, a and u differentiable and u twice differentiable in x, with ρ and a positive almost everywhere, which solves on $]0,T[\times\Omega\times\mathbb{R}^d$ the free transport equation:

$$\partial_t m + v\cdot\nabla_x m = 0.$$

Then, there exist four real numbers r_0, α, β and $\gamma \in \mathbb{R}$, two vectors w_1 and $w_2 \in \mathbb{R}^d$ and a skew-symmetric matrix $\Lambda_0 \in \mathcal{M}_d(\mathbb{R})$ such that

$$\begin{aligned}
&\forall\, (t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^d,\\
&m(t,x,v) = r_0 \exp\big(-\alpha|x-tv|^2 + \beta(x-tv)\cdot v - \gamma|v|^2\\
&\qquad\qquad + 2\big(\Lambda_0(x-tv)\big)\cdot v - 2w_1\cdot(x-tv) + 2w_2\cdot v\big).
\end{aligned} \tag{2.3}$$

Remark 1. The expression (2.3) is consistent with the fact that m is a solution of the free transport equation since it depends only on v and $x - tv$.

Let us observe that the expression can be simplified as

$$\begin{aligned}
m(t,x,v) = r_0 \exp\big(&-\alpha|x-tv|^2 + \beta(x-tv)\cdot v - \gamma|v|^2 + 2(\Lambda_0 x)\cdot v\\
&- 2w_1\cdot(x-tv) + 2w_2\cdot v\big),
\end{aligned} \tag{2.4}$$

using that Λ_0 is a skew-symmetric matrix, so that $(\Lambda_0 v)\cdot v = 0$.

Proof of Theorem 1. To prove that a local Maxwellian m of the form (2.1) which solves the free transport equation (2.2) has necessarily the form prescribed by (2.3), we proceed according to the following steps.

First, by substituting the expression (2.1) into the free transport equation (2.2), we obtain a polynomial equation of third degree in $v - u(t,x)$, where the coefficients are expressed in terms of the parameters ρ, a, u, and their respective derivatives. Equating all the coefficients to zero, we obtain the following system:

$$\begin{cases}
\rho(\partial_{x_i} a) = 0 \quad \forall\, 1 \leq i \leq d, & \text{(2.5a)}\\
\rho\big[u\cdot\nabla_x a + \partial_t a - 2a(\partial_{x_i} u_i)\big] = 0 \quad \forall\, 1 \leq i \leq d, & \text{(2.5b)}\\
\rho a\big[\partial_{x_i} u_j + \partial_{x_j} u_i\big] = 0 \quad \forall\, 1 \leq i < j \leq d, & \text{(2.5c)}\\
2\rho a \nabla_x u_i \cdot u + 2\rho a(\partial_t u_i) + \partial_{x_i}\rho = 0 \quad \forall\, 1 \leq i \leq d, & \text{(2.5d)}\\
\partial_t \rho + u\cdot\nabla_x \rho = 0. & \text{(2.5e)}
\end{cases}$$

In particular, we deduce that the coefficient a depends only on the time variable; that is, there exists a function $\sigma_0 : [0,T] \to \mathbb{R}$ which is differentiable, and such that

$$\forall (t,x) \in [0,T]\times\Omega, \quad a(t,x) = \sigma_0(t). \tag{2.6}$$

Second, substituting the new expression (2.6) of a, (2.5a) and (2.5b) provide in particular that the matrix A defined as

$$A = -\frac{\sigma_0'}{2\sigma_0} I_d + \mathrm{Jac}_x(u)$$

is skew symmetric, where I_d denotes the $d \times d$ identity matrix, and $\mathrm{Jac}_x(u)$ is the Jacobian matrix of the velocity field u. Therefore, the Jacobian matrix (in x, at t fixed) $\mathrm{Jac}_x W(t,x) = A(t,x)$ of the vector field

$$(t,x) \mapsto W(t,x) = u(t,x) - \frac{\sigma_0'}{2\sigma_0} x$$

is skew symmetric. So, provided that u is twice differentiable with respect to the position variable x, we deduce that W is an affine mapping in x (see Remark 2 below). In other words, there exist two differentiable functions $t \mapsto \Lambda(t)$ with $\Lambda(t) \in \mathcal{M}_d(\mathbb{R})$ and $t \mapsto C(t)$ with $C(t) \in \mathbb{R}^d$ such that

$$\forall\, (t,x) \in [0,T] \times \Omega, \quad u(t,x) = \Lambda(t)x + \frac{\sigma_0'}{2\sigma_0} x + C(t). \tag{2.7}$$

Third, turning to the coefficient ρ, relying on (2.7) to compute $\nabla_x u$, we can rewrite (2.5d) as

$$\nabla_x(\ln \rho) = Mx - 2\sigma_0 \Lambda C - \sigma_0' C - 2\sigma_0 C' \tag{2.8}$$

with

$$M = -2\big[\sigma_0 \Lambda^2 + \sigma_0' \Lambda + \sigma_0 \Lambda'\big] - 2\sigma_0 \left[\left(\frac{\sigma_0'}{2\sigma_0} \right)^2 + \left(\frac{\sigma_0'}{2\sigma_0} \right)' \right] I_d. \tag{2.9}$$

Since $\nabla_x(\ln \rho)$ is a gradient, the Jacobian matrix with respect to the variable x of this quantity is symmetric. We deduce therefore that the skew-symmetric part of the matrix M has to be zero, that is, $\sigma_0' \Lambda + \sigma_0 \Lambda' = 0$. We find then that $\sigma_0 \Lambda$ is a time-independent matrix. There exist therefore a (constant) skew-symmetric matrix $\Lambda_0 \in \mathcal{M}_d(\mathbb{R})$ and a differentiable function $t \mapsto \rho_0(t)$ such that

$$u(t,x) = \frac{1}{\sigma_0(t)} \Lambda_0 x + \left(\frac{\sigma_0'(t)}{2\sigma_0(t)} \right) x + C(t), \tag{2.10}$$

and

(2.11)

$$\begin{aligned} \rho(t,x) = \rho_0(t) \exp\Bigg((-x) \cdot \Bigg(\Bigg[\frac{1}{\sigma_0(t)} \Lambda_0^2 + \sigma_0(t) \Bigg(\left(\frac{\sigma_0'(t)}{2\sigma_0(t)} \right)^2 + \left(\frac{\sigma_0'(t)}{2\sigma_0(t)} \right)' \Bigg) I_d \Bigg] x \Bigg) \\ - 2\big(\Lambda_0 C(t)\big) \cdot x - \sigma_0'(t)\big(C(t) \cdot x\big) - 2\sigma_0(t)\big(C'(t) \cdot x\big) \Bigg). \end{aligned}$$

Fourth, we can now determine the functions σ_0, C and ρ_0. To do so, we substitute in (2.5e) the expressions (2.10) and (2.11) of u and ρ. We obtain a polynomial equation of second degree in x, with coefficients given in terms of σ_0, C, ρ_0 and their respective derivatives. Equating these coefficients to zero, we find the system:

$$\begin{cases} (\sigma_0\phi)' + \sigma_0'\phi = 2\sigma_0'\phi + \sigma_0\phi' = 0, & (2.12a)\\ 2\sigma_0 C'' + 4\sigma_0' C' + \Big(\sigma_0'' + \dfrac{(\sigma_0')^2}{2\sigma_0} + 2\sigma_0\phi\Big)C = 0, & (2.12b)\\ \rho_0' + \rho_0\big(-\sigma_0'|C|^2 - 2\sigma_0 C'\cdot C\big) = 0, & (2.12c)\end{cases}$$

where ϕ denotes the function: $\phi(t) = \big(\frac{\sigma_0'(t)}{2\sigma_0(t)}\big)^2 + \big(\frac{\sigma_0'(t)}{2\sigma_0(t)}\big)'$.

We start with determining σ_0: we observe that (2.12a) provides $(\sigma_0\phi)' = 0$. Therefore, $2\sigma_0''(t)\sigma_0(t) - (\sigma_0'(t))^2$ is constant, which provides by differentiating that $\sigma_0''' = 0$. So there exist α, β and $\gamma \in \mathbb{R}$ such that

$$\forall\,(t,x) \in [0,T]\times\Omega, \quad a(t,x) = \sigma_0(t) = \alpha t^2 + \beta t + \gamma. \tag{2.13}$$

Turning now to C, substituting (2.13), we can rewrite (2.12b) as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[(\alpha t^2 + \beta t + \gamma)C(t)\big] = 0,$$

and so there exist two vectors $w_1, w_2 \in \mathbb{R}^d$ such that

$$\forall\, t \in [0,T], \quad C(t) = \frac{t w_1 + w_2}{\alpha t^2 + \beta t + \gamma}. \tag{2.14}$$

Finally, we determine ρ_0. To do so, we remark that (2.12c) can be written as

$$\rho_0' + \frac{\mathrm{d}}{\mathrm{d}t}\big(-\sigma_0|C|^2\big)\rho_0 = 0.$$

We deduce then that there exists a positive real number $r_0 \in \mathbb{R}$ such that

$$\forall\, t \in [0,T], \quad \rho_0(t) = r_0 \exp\left(\frac{|t w_1 + w_2|^2}{\alpha t^2 + \beta t + \gamma}\right). \tag{2.15}$$

Substituting (2.15), (2.11), (2.13), (2.10) and (2.14) into the original expression (2.1) of the local Maxwellian m, a direct but tedious computation allows us to conclude the proof of Theorem 1. ∎

REMARK 2. It is a classical and elementary result to establish that if a smooth vector valued function $W : \mathbb{R}^d \to \mathbb{R}^d$ is such that its Jacobian matrix is everywhere skew symmetric, then W is an affine mapping. Such a result means that the movement described by the mapping W corresponds to a rigid motion since W can be seen as an infinitesimal rotation.

Let us observe that such a result can be seen as the "simplest possible" version of a Korn-type inequality:

$$\|\mathrm{Jac}(W)\|_{L^p} \leq C_p \|\mathrm{Jac}^s(W)\|_{L^p},$$

where $\mathrm{Jac}^s(W) = \frac{1}{2}[\mathrm{Jac}(W) + {}^t\mathrm{Jac}(W)]$ is the symmetric part of the Jacobian matrix of W.

One of the first versions of such a result can be found in [10] and was obtained in order to study the deformation of elastic materials. Korn's inequalities are supplemented with appropriate boundary conditions, motivated by physical considerations.

It is worthwhile to mention that versions of Korn's inequality have been established in kinetic theory (such as in [7], or more recently in [3]) and used in a crucial manner to study the long-time behavior of the solutions of the Boltzmann equation [8].

Thanks to Theorem 1, we can now describe completely the local Maxwellians that solve the Boltzmann equation, inside particular domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ (even $\mathbb{R}^d$ in the bounce-back case), depending also on the choice of the boundary condition we prescribe. This is the object of the two following sections.

3. Local Maxwellians and boundary condition I: The bounce-back case

Let us start with the case of the *bounce-back boundary condition*. We consider a gas evolving in a regular open set Ω of $\mathbb{R}^d$, where $d \geq 2$ is an arbitrary integer larger than 2. This boundary condition, which may not look natural at the first glance, is defined as follows.

Definition 1 (Bounce-back boundary condition (BBBC)). Let Ω be a regular, open set in $\mathbb{R}^d$ with $d \geq 2$, and let $f : \overline{\Omega} \to \mathbb{R}$ be any function. f is said to satisfy the *bounce-back boundary condition* (abbreviated as *BBBC*) on the boundary $\partial\Omega$ of Ω if

$$\forall t \in [0, T],\ x \in \partial\Omega,\ v \in \mathbb{R}^d, \quad f(t, x, v) = f(t, x, -v). \tag{3.1}$$

At the level of the particles, this means that the particles are reflected on the boundary of the domain Ω, in such a way that the particles travel back, following backwards exactly the path they took before they collided with the wall.

Let us now characterize the local Maxwellians solving the free transport equation with bounce-back boundary condition.

Theorem 2 (Characterization of the local Maxwellians solving the free transport equation in a domain with BBBC). *Let T be a strictly positive number (possibly $+\infty$),* *let Ω be a regular open set of $\mathbb{R}^d$ with a non-empty boundary with $d \geq 2$ and let us consider a local Maxwellian m, that is, a function of the form (2.3), which solves the transport equation (2.2) on $[0,T] \times \Omega \times \mathbb{R}^d$, with the bounce-back boundary condition, that is, such that*

$$\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^d, \quad m(t,x,v) = m(t,x,-v).$$

Then, there exist two real numbers r_0 and $\gamma \in \mathbb{R}$ such that

$$\forall t \in [0,T],\ x \in \Omega,\ v \in \mathbb{R}^d, \quad m(t,x,v) = r_0 \exp\big(-\gamma|v|^2\big). \tag{3.2}$$

REMARK 3. Equation (3.2) shows in particular that for a mass and a kinetic energy fixed, there exists only one local Maxwellian that solves the free transport equation with BBBC. Such a Maxwellian has zero bulk (the momentum of m is zero), and it is actually global (none of its coefficient depends on the time t nor on the position variable x).

REMARK 4. Provided that the measure of Ω is finite, the Maxwellian given by the expression (3.2) has finite mass for any $\gamma > 0$. In the case when the measure of Ω is not finite, there exists no local Maxwellian solving the Boltzmann equation that has a finite mass.

REMARK 5. In the literature, it is usual to find the assumption that the domain Ω is bounded (see for instance [6, 8] or [11]). This assumption simplifies the proof of Theorem 2. Nevertheless, we claim that the boundedness assumption is not necessary, in the sense that it is possible to obtain the same result, without any assumption concerning the boundedness of the domain.

PROOF OF THEOREM 2. Writing the local Maxwellian m in its general form (2.2), the BBBC (3.1) writes:

$$\rho(t,x)\exp\big(-a(t,x)\big|v-u(t,x)\big|^2\big) = \rho(t,x)\exp\big(-a(t,x)\big|-v-u(t,x)\big|^2\big)$$

for all $t \in [0,T]$, $x \in \partial\Omega$ and $v \in \mathbb{R}^d$. This implies that $u(t,x)$ has to be zero for all time t, and all x that belongs to the boundary $\partial\Omega$ of the domain.

Remembering now that we obtained, along the proof of Theorem 1, that $u(t,x)$ writes:

$$u(t,x) = \frac{1}{\sigma_0(t)}\Lambda_0 x + \frac{\sigma_0'(t)}{2\sigma_0(t)}x + \frac{tw_1 + w_2}{\sigma_0(t)}$$

(see in particular (2.10) and (2.14)), and using the fact that $\sigma_0(t)$ was the polynomial $\alpha t^2 + \beta t + \gamma$ of degree 2 in t (see (2.13)), we obtain

$$\Lambda_0 x + (\alpha t + \beta/2)x + tw_1 + w_2 = 0, \tag{3.3}$$

which implies that the first-order coefficient (in t) in (3.3) has to vanish; that is, we have

$$\alpha x + w_1 = 0 \tag{3.4}$$

for all point $x \in \partial\Omega$. If $\alpha \neq 0$, (3.4) cannot hold because $\partial\Omega$ was assumed to be non-empty, and so it cannot be reduced to a single point by regularity. Therefore, we have $\alpha = 0$.

Using then the expression (2.3) and applying again the boundary condition, we obtain the new equation:

$$4v \cdot (\Lambda_0 x) + 4v \cdot w_2 + 4tv \cdot w_1 + 2\beta v \cdot x = 0,$$

for all $t \in [0, T]$, $x \in \partial\Omega$ and $v \in \mathbb{R}^d$. We deduce immediately the new condition:

$$2\Lambda_0 x + 2w_2 + 2tw_1 + \beta x = 0,$$

for all $t \in [0, T]$, $x \in \partial\Omega$. The first-order term in t provides directly that

$$w_1 = 0, \tag{3.5}$$

which can also be deduced from (3.4) together with the fact that $\alpha = 0$, while the constant term gives

$$2\Lambda_0 x + \beta x + 2w_2 = 0, \tag{3.6}$$

everywhere on the boundary $\partial\Omega$ of the domain. We notice now that (3.6) cannot hold if $\beta \neq 0$: indeed, in that case, we would have that the mapping $\phi : x \mapsto 2\Lambda_0 x + \beta x$ is invertible because for any non-zero vector x, we find $x \cdot \phi(x) = 2x \cdot (\Lambda_0 x) + \beta |x|^2 = \beta |x|^2 \neq 0$. Therefore, we have

$$\beta = 0. \tag{3.7}$$

But then, (3.6) can be rewritten as

$$\Lambda_0 x + w_2 = 0 \tag{3.8}$$

for all $x \in \partial\Omega$. We show now that if $\Lambda_0 \neq 0$, we obtain a contradiction. In that case, the matrix Λ_0 is not trivial, and so its kernel cannot be of dimension d. Let us fix a point $x_0 \in \partial\Omega$. In particular, we have $\Lambda_0 x_0 + w_2 = 0$, and so, for any other point $x \in \partial\Omega$, we have by linearity $\Lambda_0(x - x_0) = 0$; that is, $x - x_0$ is contained in the kernel of Λ_0, or again:

$$\partial\Omega \subset x_0 + \operatorname{Ker} \Lambda_0. \tag{3.9}$$

This last equation shows already that if the boundary $\partial\Omega$ is not an affine hyperplane of $\mathbb{R}^d$, then we cannot have $\Lambda_0 \neq 0$. Let us show that even when the boundary of the domain Ω is flat, we obtain a contradiction anyway, due to the dimension of the kernel of Λ_0. Let us assume that Λ_0 is not the zero matrix. We will prove that the dimension of the kernel of Λ_0 is at most $d-2$.

The assumption that $\Lambda_0 \neq 0$ implies that its spectrum cannot be reduced to zero. Indeed, a skew-symmetric matrix is a particular case of a normal matrix (that is, it commutes with its Hermitian conjugate Λ_0^*), so there exist a unitary matrix U and a diagonal matrix D (both with complex entries) such that

$$\Lambda_0 = UDU^* \quad \text{and} \quad UU^* = I_d.$$

Therefore, there exists a non-zero eigenvector $z \in \mathbb{C}^d$ of Λ_0 associated with a non-zero eigenvalue $\lambda \in \mathbb{C}$. Let us decompose z into its real and imaginary parts:

$$z = u_1 + iu_2 \quad \text{with } u_1, u_2 \in \mathbb{R}^d.$$

Λ_0 being skew-symmetric, we know in addition that λ has to be purely imaginary. Besides, z being non-zero, either its real part u_1 or its imaginary part u_2 is non-zero. Actually both u_1 and u_2 are non-zero. Indeed, if u_2 would be zero, z would be a real-valued vector, which would lead to a contradiction, because Λ_0 is also real valued and $\lambda \in i\mathbb{R}$. In the same way, u_1 cannot be zero.

In addition, the real and imaginary parts u_1 and u_2 of the eigenvector z are linearly independent in $\mathbb{R}^d$. Let us indeed assume that there exists a real number μ (necessarily non-zero) such that $u_2 = \mu u_1$. Using that Λ_0 is a real matrix, we obtain another eigenvector, associated with a different eigenvalue, considering simply the eigenequation:

$$\overline{(\Lambda_0 \cdot z)} = \Lambda_0 \cdot \bar{z} = \overline{(\lambda z)} = \bar{\lambda}\bar{z}.$$

We would have then $z = u_1 + iu_2 = (1+i\mu)u_1$, and $\bar{z} = u_1 - iu_2 = (1-i\mu)u_1$. But this would imply that z and $\bar{z}$ are linearly dependent (in $\mathbb{C}^d$), which cannot be, since they are associated with the respective distinct eigenvalues λ and $\bar{\lambda}$. Therefore, the real and imaginary parts of z are linearly independent. In the end, observing that

$$\Lambda_0 u_1 = -\operatorname{sgn}(\lambda)|\lambda| u_2 \neq 0 \quad \text{and} \quad \Lambda_0 u_2 = \operatorname{sgn}(\lambda)|\lambda| u_1 \neq 0,$$

we can in particular conclude that $\operatorname{Span}(u_1, u_2)$ is of dimension 2, and that this space is in direct sum with $\operatorname{Ker}\Lambda_0$. In other words, the kernel of Λ_0 has a dimension that is at most $d-2$. Therefore, in the case $\Lambda_0 \neq 0$, (3.9) can never hold, regardless of the dimension d, or the geometry of the boundary $\partial\Omega$, because no hypersurface can be contained in a vector space of codimension 2.

This implies, according to (3.8), that $w_2 = 0$, and so Theorem 2 is proved. ∎

REMARK 6. This is equation (3.9) that is used by Desvillettes in [6] to deduce that if Ω is bounded, then we obtain a contradiction if $\Lambda_0 \neq 0$. Our addition here is the discussion concerning the dimension of the kernel of Λ_0, which enables us to obtain the result of Theorem 2, without using any boundedness assumption on Ω.

4. LOCAL MAXWELLIANS AND BOUNDARY CONDITION II: THE SPECULAR REFLECTION CASE

Let us now turn to the case of the specular reflection, which is much more relevant regarding the physical motivation. In this case, the particles are assumed to be reflected against the boundary of the domain, exactly as a billiard ball would do: during the collision, the velocity of the particle is reflected; that is, the velocity is obtained as the orthogonal symmetry, with respect to the tangent hyperplane to the obstacle, at the point of bounce. As for the BBBC, this assumption on the behavior of the particles is translated into an equality on the density function f.

DEFINITION 2 (Specular reflection boundary condition (SRBC)). Let Ω be a regular, open set in $\mathbb{R}^d$, and let $f : \bar{\Omega} \to \mathbb{R}$ be any function. f is said to satisfy the *specular reflection boundary condition* (abbreviated as *SRBC*) on the boundary $\partial\Omega$ of Ω if

$$\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^d, \quad f(t,x,v) = f(t,x,v'), \tag{4.1}$$

where v' is defined as

$$v' = v - 2\big(v \cdot n(x)\big)n(x), \tag{4.2}$$

with $n(x)$ a unitary normal vector to the boundary $\partial\Omega$ of the domain, at $x \in \partial\Omega$.

We will now see that, contrary to the BBBC, there can be much more local Maxwellians solving the free transport equation with these new boundary conditions. More precisely, we will see that, the more the domain is "symmetric", the more we can find local Maxwellians verifying the free transport with SRBC.

We will proceed first when the dimension d is 2, and then when $d = 3$. However, the starting point, namely, exploiting the boundary condition (4.1), is independent from the dimension and provides the equation

$$\big|v - u(t,x)\big|^2 = \big|v' - u(t,x)\big|^2$$

for all $t \in [0,T]$ and $x \in \partial\Omega$ (where v' is defined in (4.2)), as a consequence of the expression (2.1). The last equation can be rewritten as

$$\big(v \cdot n(x)\big)n(x) \cdot u(t,x) = 0,$$

for all $t \in [0, T]$, $x \in \partial\Omega$ and $v \in \mathbb{R}^d$. Since v can be chosen freely in $\mathbb{R}^d$, we have

$$n(x) \cdot u(t, x) = 0 \quad \forall t \in [0, T],\ x \in \partial\Omega,$$

which we rewrite again, taking advantage of the explicit expression of $u(t, x)$ we derived in (2.10), (2.14) and (2.13):

$$n(x) \cdot \big[\Lambda_0 x + (\alpha t + \beta/2)x + t w_1 + w_2\big] = 0 \quad \forall t \in [0, T],\ x \in \partial\Omega. \tag{4.3}$$

Equation (4.3) will then be the central object of our study in the present section. $\alpha, \beta \in \mathbb{R}$, $w_1, w_2 \in \mathbb{R}^d$ and Λ_0 skew symmetric, all have to be determined, depending on the shape of the domain, and the dimension d.

In the case of the specular reflection case, the condition (4.3) we obtained on the boundary $\partial\Omega$ is less easy to study than (3.3) (obtained in the BBBC case). Indeed, this time the information provided by (4.3) is given only along the direction of the normal vector $n(x)$, which depends itself on x. In order to establish the proof of Theorems 3 and 4, we will make use of the following results that describe solutions of some particular linear ODEs in general dimensions.

These results will be applied to describe which type of curves the boundary $\partial\Omega$ of an open set Ω can contain if one knows particular conditions of the form of (4.3) holding on the normal vector $n(x)$ to the boundary $\partial\Omega$. The idea to consider curves drawn on the boundary $\partial\Omega$ can already be found in [6].

LEMMA 1. *Let $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$, and let $w_1 \in \mathbb{R}^d$ (with $d = 2$ or $d = 3$).*

Then the solution $t \mapsto x(t)$ of the following Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(t) = \alpha x(t) + w_1, \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{4.4}$$

is globally well defined and is given by the following expression:

$$x(t) = \frac{1}{\alpha}(\alpha x_0 + w_1)e^{\alpha t} - \frac{w_1}{\alpha} \quad \forall t \in \mathbb{R}.$$

LEMMA 2. *Let $\Lambda_0 \in \mathcal{M}_d(\mathbb{R})$ be a non-zero skew-symmetric matrix, let $\beta \in \mathbb{R}$ be a real number and let $w_2 \in \mathbb{R}^d$ (with $d = 2$ or $d = 3$).*

Then the solution $t \mapsto x(t)$ of the following Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(t) = \Lambda_0 x(t) + \dfrac{\beta}{2} x(t) + w_2, \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{4.5}$$

is globally well defined. In addition, the solution $t \mapsto x(t)$ can be described explicitly as follows.

- *If $\beta \neq 0$ or if the dimension d is equal to 2, the matrix $\Lambda_0 + \frac{\beta}{2} I_d$ is invertible, and the solution $t \mapsto x(t)$ is given by the expression:*

$$x(t) = e^{\frac{\beta}{2}t} e^{t\Lambda_0}\big(x(0) + y\big) - y, \tag{4.6}$$

where y is the preimage of the vector w_2 by the matrix $\Lambda_0 + \frac{\beta}{2} I_d$.
In such a case, the graph of $t \mapsto x(t)$ is a logarithmic spiral inscribed in a plane that contains the point $-y$ if $\beta \neq 0$, and a circle inscribed in a plane if $\beta = 0$.

- *If $d = 3$ and $\beta = 0$, the solution $t \mapsto x(t)$ is given by the expression:*

$$x(t) = e^{t\Lambda_0}\big(x(0) + y\big) - y + \lambda u t, \tag{4.7}$$

where $u = z/|z|$, with z being the unique non-zero vector such that $\Lambda_0 x = z \wedge x$ for all $x \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$, $y \in \mathbb{R}^3$ are such that $w = \Lambda_0 y$, with

$$w_2 = \lambda u + w = \lambda \frac{z}{|z|} + w, \quad \textit{with } w \perp u.$$

In such a case, the graph of $t \mapsto x(t)$ is a helix of axis $\mathrm{Span}(u) = \mathrm{Span}(z/|z|)$ *if $\lambda \neq 0$, and a circle inscribed in a plane if $\lambda = 0$.*

PROOF. In the case when $\beta \neq 0$, the matrix $\Lambda_0 + \frac{\beta}{2} I_d$ is invertible. In the case when $\beta = 0$ and $d = 2$, then this matrix can be written explicitly as $\left(\begin{smallmatrix} 0 & -a \\ a & 0 \end{smallmatrix}\right)$, with $a \neq 0$ (because Λ_0 was assumed to be not zero), which is also invertible. In both cases, we can define the vector y, preimage of w_2 by the matrix $\Lambda_0 + \frac{\beta}{2} I_d$. Therefore, posing

$$\varphi(t) = (x(t) + y)e^{-\frac{\beta}{2}t},$$

we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \Big(\Lambda_0 x(t) + \frac{\beta}{2} x(t) + w_2\Big) e^{-\frac{\beta}{2}t} - \frac{\beta}{2}\big(x(t) + y\big) e^{-\frac{\beta}{2}t} \\ &= \big(\Lambda_0 x(t) + \Lambda_0 y\big) e^{-\frac{\beta}{2}t} = \Lambda_0 \varphi(t); \end{aligned}$$

hence, $\varphi(t) = e^{t\Lambda_0}\varphi(0) = e^{t\Lambda_0}(x(0) + y)$. We deduce then the expression (4.6).

In the second case, since Λ_0 was assumed to be non-zero and the dimension d is assumed to be equal to 3, there exists a non-zero vector z such that the action of Λ_0 can be represented by the cross-product with the vector z. Denoting $z/|z|$ by u, we decompose

$$w_2 = \lambda u + w,$$

for some $\lambda \in \mathbb{R}$ and $w \in \mathbb{R}^3$, such that $w \perp u$. Such a decomposition satisfies that w belongs to the image of Λ_0; let us denote by y one of its preimages (considering for instance $y = ((-z)/|z|^2) \wedge w$, we find $\Lambda_0 y = w$).

This time, we pose

$$\varphi(t) = x(t) + y - \lambda u t,$$

and in this case we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \Lambda_0 x(t) + w_2 - \lambda u = \Lambda_0 x(t) + \Lambda_0 y = \Lambda_0\big(x(t) + y - \lambda u t\big)$$

(because u is colinear to z, so that $\Lambda_0 u = 0$); that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \Lambda_0 \varphi(t).$$

We deduce therefore the expression (4.7) for the solution $x(t)$.

Finally, the geometric description of the graph of the solution $t \mapsto x(t)$ is a consequence of the explicit computation of the exponential of the matrix $t\Lambda_0$. More precisely, in the 2-dimensional case $d = 2$, since we have $\Lambda_0^2 = -a^2 I_2$, we find

$$\Lambda_0^{2k} = (-a^2)^k I_2 \quad \text{and} \quad \Lambda_0^{2k+1} = (-a^2)^k \Lambda_0 \quad \forall k \in \mathbb{N}.$$

Then, we obtain

$$\begin{aligned}
\exp(t\Lambda_0) &= \sum_{k=0}^{+\infty} \frac{t^{2k}}{(2k)!}(-a^2)^k I_2 + \sum_{k=0}^{+\infty} \frac{t^{2k+1}}{(2k+1)!}(-a^2)^k \Lambda_0 \\
&= \begin{pmatrix} \sum_{k=0}^{+\infty}(-1)^k \frac{t^{2k}}{(2k)!} a^{2k} & \sum_{k=0}^{+\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!} a^{2k+1} \\ -\sum_{k=0}^{+\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!} a^{2k+1} & \sum_{k=0}^{+\infty}(-1)^k \frac{t^{2k}}{(2k)!} a^{2k} \end{pmatrix} \\
&= \begin{pmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{pmatrix};
\end{aligned}$$

that is, the matrix $e^{t\Lambda_0}$ is the matrix of a rotation.

The 3-dimensional case $d = 3$ is exactly similar since there is always a choice of coordinate such that z is colinear to the last vector of the canonical basis, providing in such a case that

$$\Lambda_0 = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

leading to the same conclusion. ∎

4.1. *Local Maxwellians and specular reflection, in dimension $d = 2$*

Before stating the main theorem of this section, we introduce the notion of rotational symmetry, which we will use to write the main result of this section.

DEFINITION 3 (Rotational symmetry, planar version). Let $\mathcal{C}$ be a $\mathcal{C}^1$ curve of the Euclidean plane $\mathbb{R}^2$. $\mathcal{C}$ is said to present a *rotational symmetry* if there exists a point $x_0 \in \mathbb{R}^2$ of the plane such that, for any point $x \in \mathcal{C}$ and any affine rotation $\mathcal{R}_{x_0}$ around x, the image $\mathcal{R}_{x_0}(x)$ of x by the rotation $\mathcal{R}_{x_0}$ belongs also to the curve $\mathcal{C}$.

REMARK 7. If the connected curve $\mathcal{C}$ presents a rotational symmetry around a certain point x_0, then $\mathcal{C}$ is a circle of center x_0. Therefore, if Ω is a connected regular open set of the plane, such that its (non-empty) boundary presents a rotational symmetry, then Ω is either a disk, the complement of a disk or an annulus contained between two concentric circles.

We can now turn to the characterization of the local Maxwellians solving the Boltzmann equation in the 2-dimensional case, in the case of the specular reflection boundary condition.

THEOREM 3 (Characterization of the local Maxwellians solving the free transport equation in a domain with SRBC, case $d = 2$). *Let the dimension d be equal to 2. Let T be a strictly positive number (possibly $+\infty$), let Ω be a regular, connected open set of $\mathbb{R}^2$ and let m be a local Maxwellian that is a function of the form (2.3), which solves the transport equation (2.2) on $[0,T] \times \Omega \times \mathbb{R}^2$, with the SRBC, that is, such that*

$$\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \quad m(t,x,v) = m(t,x,v'),$$

where

$$v' = v - 2\big(v \cdot n(x)\big)n(x)$$

and with $n(x)$ a unitary normal vector to $\partial\Omega$ at x.

Then,

- *if Ω is a half-plane, with a boundary of the form $\partial\Omega = \{x_0 n + \lambda\tau / \lambda \in \mathbb{R}\}$, with $x_0 \in \mathbb{R}$, $n, \tau \in \mathbb{S}^1$ such that $n \perp \tau$ (that is, $x_0 n$ is the closest point of boundary $\partial\Omega$ to the origin, and $\partial\Omega$ is orientated by the unit vector τ), there exist six real numbers $r_0 > 0$, $\alpha > 0$, β, $\gamma > 0$, ℓ_1, $\ell_2 \in \mathbb{R}$ such that*

$$\begin{aligned} m(t,x,v) = r_0 \exp\big(&-\alpha(x - tv - 2x_0 n)\cdot(x - tv) + \beta(x - tv - x_0 n)\cdot v \\ &- \gamma|v|^2 - 2\ell_1\tau\cdot(x - tv) + 2\ell_2\tau\cdot v\big) \\ &\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \end{aligned} \tag{4.8}$$

- *if Ω is a slab, with a boundary of the form*

$$\partial\Omega = \{x_1 n + \lambda\tau / \lambda \in \mathbb{R}\} \cup \{x_2 n + \mu\tau / \mu \in \mathbb{R}\},$$

with $x_0 \in \mathbb{R}$*,* $n, \tau \in \mathbb{S}^1$ *such that* $n \perp \tau$*, there exist four real numbers* $r_0 > 0$*,* $\gamma > 0$*,* $\ell_1, \ell_2 \in \mathbb{R}$ *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 - 2\ell_1\tau\cdot(x-tv) + 2\ell_2\tau\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \tag{4.9}$$

- *if* Ω *is a disk centered on* $y \in \mathbb{R}^2$*, or the complement of a disk centered on* $y \in \mathbb{R}^2$*, or an annulus contained between two concentric circles centered on* $y \in \mathbb{R}^2$*, there exist two strictly positive real numbers* $r_0 > 0$ *and* $\gamma > 0$*, and a skew-symmetric matrix* Λ_0 *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2\big[\Lambda_0(x-y)\big]\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \tag{4.10}$$

- *finally, if* Ω *is* not *a half-plane, nor a slab, nor a disk, the complement of a disk, nor an annulus contained between two concentric circles, there exist two strictly positive real numbers* $r_0 > 0$ *and* $\gamma > 0$ *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2. \tag{4.11}$$

REMARK 8. It is interesting to observe that, in the case when the domain Ω is not bounded, we find for the local Maxwellians the expressions (4.8) and (4.9) that are not steady states of the Boltzmann equation.

We also remark that if the measure of Ω is infinite, and if Ω is not a half-plane, then any non-trivial local Maxwellian that solves the Boltzmann equation has an infinite mass. Nevertheless, when Ω is a half-plane, there exist local Maxwellians with finite mass, as it can be seen by choosing $\alpha, \gamma > 0$ and $\beta^2 < 4\alpha\gamma$ in (4.8).

PROOF OF THEOREM 3. By Theorem 1, a local Maxwellian that solves the free transport equation in an open domain is necessarily of the form (2.3). As for the bounce-back case, we start with exploiting what implies the boundary condition on the vector field $u(t,x)$, starting from the expression (4.3).

At x fixed, the left-hand side of (4.3) is a polynomial expression in t, which is zero for the non-trivial interval $[0,T] \ni t$; therefore, all the coefficients of the expression have to be zero, so that we deduce the following system: $\forall x \in \partial\Omega$ we have

$$\begin{cases} [\alpha x + w_1]\cdot n(x) = 0, & \text{(4.12a)} \\ \left[\Lambda_0 \cdot x + \dfrac{\beta}{2}x + w_2\right]\cdot n(x) = 0. & \text{(4.12b)} \end{cases}$$

Therefore, we start with (4.12a), and we consider the Cauchy problem (4.4), such that the initial datum x_0 belongs to the boundary $\partial\Omega$ of the domain Ω. In such a case, the curve $x(t)$ always lies on the boundary $\partial\Omega$ of the domain. Indeed, the following result holds: if a parametric curve $I \subset \mathbb{R} \to \mathbb{R}^d$, $t \mapsto x(t)$ is such that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) \cdot \nabla g(x(t)) = 0$ for a given function $\mathbb{R}^d \to \mathbb{R}$, $x \mapsto g(x)$ and for every $t \in I$, then the curve $x(t)$ lies in the surface S defined implicitly as $S = \{x \in \mathbb{R}^d / g(x) = 0\}$. Observe also that, here, we do not require the boundary $\partial\Omega$ to be of the form:

$$\partial\Omega = \left\{x \in \mathbb{R}^d / g(x) = 0\right\}$$

globally, that is, defined using a single function g. It is enough to proceed locally, which is always possible for a $\mathscr{C}^1$ hypersurface.

Let us assume in addition that $\alpha \neq 0$. In such a case, we can consider the initial datum $x_0 \in \partial\Omega$ such that $x_0 \neq -w_1/\alpha$ (because the boundary $\partial\Omega$ is non-empty and cannot be reduced to a single point). We deduce by the formula (4.4) of Lemma 1 that the boundary $\partial\Omega$ contains the half-line with initial point $-w_1/\alpha$, which is passing through the point x_0. Considering that $\partial\Omega$ is a regular curve (and considering in particular its tangent space at the point $-w_1/\alpha$), we deduce that $\partial\Omega$ has to be reduced to a straight line through $-w_1/\alpha$. Therefore, Ω has to be a half-plane, such that $-w_1/\alpha$ belongs to its boundary.

If now $\alpha = 0$ and $w_1 \neq 0$, (4.12a) implies that $\partial\Omega$ has to be the union of straight lines parallel to w_1. Since in addition Ω was assumed to be connected, $\partial\Omega$ is either a single straight line, or the union of two parallel lines, and so Ω is either a half-plane, or a stripe.

We turn now to the second equation (4.12b). We consider a solution of the Cauchy problem (4.5), with initial datum $x_0 \in \partial\Omega$, so that the whole curve described by the solution belongs to the boundary $\partial\Omega$.

In a first time, let us assume that $\Lambda_0 \neq 0$. According to Lemma 2, the curve of the solution is either a logarithmic spiral, or a circle. In the first case, one would reach a contradiction. Indeed, $\partial\Omega$ being the boundary of the domain Ω, $\partial\Omega$ is in particular closed by assumption, so that the center of the spiral has to belong to $\partial\Omega$. However, at this point the curve does not admit a tangent line, which would violate the assumption that Ω is a regular open set. Therefore, $\partial\Omega$ cannot be a logarithmic spiral, and so if $\Lambda_0 \neq 0$, then $\beta = 0$, and the boundary $\partial\Omega$ is the union of concentric circles, centered on $-y$. Since Ω was assumed to be connected, $\partial\Omega$ is either reduced to a single circle, or to two concentric circles, and Ω is either a disk, or the complement of a disk, or an annulus contained between two concentric circles.

If now $\Lambda_0 = 0$, then the same conclusions hold for the boundary as the conclusions obtained from the first equation (4.12a).

In summary, we obtained the following results.

- if $\alpha \neq 0$, then Ω is a half-plane, with $-w_1/\alpha \in \partial\Omega$,
- if $\alpha = 0$ and $w_1 \neq 0$, then Ω is either a half-plane or a stripe, with a boundary $\partial\Omega$ constituted with one or two straight lines parallel to w_1,
- if $\Lambda_0 \neq 0$, then $\beta = 0$ and $\partial\Omega$ is either a disk, or the complement of a disk, or an annulus contained between two concentric disks, all centered on $-y = (\Lambda_0)^{-1}(w_2)$,
- if $\Lambda_0 = 0$ and $\beta \neq 0$, then Ω is a half-plane, with $-2w_2/\beta \in \partial\Omega$,
- if $\Lambda_0 = 0$, $\beta = 0$ and $w_2 \neq 0$, then Ω is either a half-plane or a stripe, with a boundary $\partial\Omega$ constituted with one or two straight lines parallel to w_2.

The result of Theorem 3 follows. ∎

It is usual in the literature [6, 11] to consider only bounded domains Ω that are in addition simply connected. This particular case forbids to consider domains such as stripes or half-planes, and the case of Ω being an annulus is also excluded. We can then recover the following result, classical in the literature, as a direct consequence of Theorem 3.

COROLLARY 1 (Characterization of the local Maxwellians solving the free transport equation in a bounded domain with SRBC, case $d = 2$). *Let the dimension d be equal to 2. Let T be a strictly positive number (possibly $+\infty$), let Ω be a regular open set of $\mathbb{R}^2$, bounded and simply connected and let m be a local Maxwellian that is a function of the form (2.3), which solves the transport equation (2.2) on $[0,T] \times \Omega \times \mathbb{R}^2$, with the SRBC, that is, such that*

$$\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \quad m(t,x,v) = m(t,x,v'),$$

where

$$v' = v - 2\big(v \cdot n(x)\big)n(x)$$

and with $n(x)$ a unitary normal vector to $\partial\Omega$ at x.

Then,

- *if Ω is* not *a disk, there exist two strictly positive real numbers $r_0 > 0$ and $\gamma > 0$ such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2, \tag{4.13}$$

- *if Ω is a disk centered on $y \in \mathbb{R}^2$, there exist two strictly positive real numbers $r_0 > 0$ and $\gamma > 0$, and a skew-symmetric matrix Λ_0 such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2\big[\Lambda_0(x-y)\big]\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^2. \tag{4.14}$$

REMARK 9. Corollary 1 states that if the boundary $\partial\Omega$ of the domain does not present a rotational symmetry in the sense of Definition 3, then, there exists a single local Maxwellian solving the free transport equation with specular reflection boundary condition (for mass and kinetic energy fixed), and such a Maxwellian is actually global.

On the other hand, if the domain is rotationally symmetric, then there exist other local Maxwellians solving the free transport equation with SRBC. Such Maxwellians do not depend on time, but the coefficients do depend on the position variable x.

Let us also remark that the additional degree of freedom obtained in the symmetric case, that is, the possibility to choose any skew-symmetric matrix Λ_0 to define m as in (4.10), corresponds actually to a single additional dimension of freedom. Indeed, a skew-symmetric matrix in dimension 2 necessarily writes:

$$\Lambda_0 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix},$$

where a is an arbitrary real number.

4.2. *Local Maxwellians and specular reflection, in dimension $d = 3$*

We conclude this study with the second physically relevant case (and maybe the most relevant of the two): the 3-dimensional case.

Before stating the result, we start with introducing definitions concerning particular symmetries that we will use in the following.

DEFINITION 4 (Rotational symmetry, spatial version). Let $\mathcal{S}$ be a $\mathcal{C}^1$ surface of the Euclidean space $\mathbb{R}^3$. $\mathcal{S}$ is said to present a *rotational symmetry* if there exists an affine straight line $\Delta \subset \mathbb{R}^3$ of the space such that, for any point $x \in \mathcal{S}$ and any affine rotation $\mathcal{R}_{x_0}$ around Δ, the image $\mathcal{R}_\Delta(x)$ of x by the rotation $\mathcal{R}_\Delta$ belongs also to the surface $\mathcal{S}$.

REMARK 10. A surface $\mathcal{S}$ that presents a rotational symmetry is the union of circles that are all inscribed in planes orthogonal to the axis of symmetry Δ.

DEFINITION 5 (Helical symmetry). Let $\mathcal{S}$ be a $\mathcal{C}^1$ surface of the Euclidean space $\mathbb{R}^3$. $\mathcal{S}$ is said to present a *helical symmetry* if there exists an affine straight line $\Delta \subset \mathbb{R}^3$ of the space, orientated by a unit vector $v \in \mathbb{S}^2$, and a real number $\lambda \in \mathbb{R}$ such that, for any point $x \in \mathcal{S}$ and any affine rotation $\mathcal{R}_{\Delta,\theta}$ around Δ of angle θ, the image $\mathcal{R}_{\Delta,\theta}(x) + \lambda\theta u$ of x by the screw motion $\mathcal{R}_{\Delta,\theta} + \lambda\theta u$ belongs also to the surface $\mathcal{S}$. The number $2\pi\lambda$ is called the *shift* of the helical symmetry.

REMARK 11. In the case when a surface $\mathcal{S}$ presents a helical symmetry, the surface is the union of helices, that all have the same axis and the same shift (that is, the same distance between two consecutive coils).

In the case when $\lambda = 0$, the notion of helical symmetry coincides with the notion of rotational symmetry.

We introduce a precise nomenclature for remarkable surfaces that will appear in the main result of this section.

Definition 6 (Generalized cylinder). A surface $\mathcal{S}$ of the Euclidean space $\mathbb{R}^3$ is said to be a *generalized cylinder* if there exists a family $(d_s)_{s \in I}$ of straight lines d_s, all parallel, such that $\mathcal{S}$ is the union of all of the lines d_s, for $s \in I$. We call the *direction* of the generalized cylinder $\mathcal{S}$ any unit vector that orientates any of the lines d_s.

Remark 12. A generalized cylinder is in particular a ruled surface.

Definition 7 (Cylinder of revolution). A surface $\mathcal{S}$ of the Euclidean space $\mathbb{R}^3$ is said to be a *cylinder of revolution* if $\mathcal{S}$ is the union of a family of parallel straight lines obtained from all the possible rotations of a given line d around an axis Δ parallel to d.

Remark 13. In particular, a cylinder of revolution is a generalized cylinder such that all the lines d_s intersect a given circle inscribed in a plane orthogonal to the axis Δ.

Finally, we will make use of the two following results concerning surfaces that are generalized cylinders or presenting a helical symmetry.

The first result describes the intersection between the class of the surfaces presenting a helical symmetry, with the class of the generalized cylinders.

Lemma 3. *Let $\mathcal{S}$ be a $\mathcal{C}^1$ connected surface of the Euclidean space $\mathbb{R}^3$, that is, a generalized cylinder. Then, if in addition $\mathcal{S}$ presents a helical symmetry:*

- *if the shift λ of the helical symmetry is zero, $\mathcal{S}$ is either a plane, or a cylinder of revolution,*
- *if the shift λ of the helical symmetry is non-zero, $\mathcal{S}$ is a cylinder of revolution.*

Proof. By assumption, $\mathcal{S}$ presents a helical symmetry (say, of axis Δ) and contains at least one straight line d.

Our definition of helical symmetry covers also the case of a rotational symmetry. Let us start with the first case; that is, we assume in a first time that the shift of the helical symmetry is zero.

Let us assume first that d intersects the axis of symmetry Δ, in which case these two lines are either orthogonal or parallel, or none of these two cases. In the two first cases, $\mathcal{S}$ is a plane, or a cylinder of revolution. In the latter case, that is, when d and Δ intersect with an angle different from $k\pi/2$, $k = 0$ or 1, the rotational motion of

d around Δ generates a cone, which is not a $\mathcal{C}^1$ surface in the neighborhood of the intersection between d and Δ.

If now we assume that d does not intersect the axis Δ, if d and Δ are parallel or orthogonal, the surface generated by the rotational motion of d around the axis Δ is either a cylinder of revolution, or is contained in a plane. If now d and Δ are neither parallel nor orthogonal, the surface generated by the rotational motion of d is a one-sheeted hyperboloid of revolution, which is not a generalized cylinder. This case is therefore excluded.

The case of a helical symmetry with a zero shift is then completely addressed.

Let us now assume that the helical symmetry has a non-zero shift. We will make use of the following result, which the reader may find for instance in [12], to eliminate most of the cases: if the line d is not colinear to Δ, nor orthogonal to Δ, then the helicoid generated by the screw motion of d around the axis Δ presents a self-intersection. Such a surface is called a *helicoid with directrix cone*, or a *helicoid of oblique type*. Therefore, $\mathcal{S}$, which would contain such an oblique helicoid, cannot be a $\mathcal{C}^1$ surface. This case is then excluded.

If d and Δ are parallel, the helical motion of d around Δ generates a cylinder of revolution.

The only remaining case is then when d and Δ are orthogonal. The helical motion of d around Δ generates a *right helicoid* (*closed* or *open*, depending if d intersects Δ or not). Such a helicoid is by definition a ruled surface; however, we will now show that it cannot be a generalized cylinder. If a right helicoid would be a generalized cylinder, its direction could not be parallel to Δ because it would contain a plane such that its normal is orthogonal to Δ, which is not the case. For the same reason, its direction cannot be orthogonal to Δ because the helicoid would contain a plane that is orthogonal to Δ. Finally, if its direction is not parallel nor orthogonal to Δ, then the helicoid would be of oblique type, which is not the case (because a right helicoid is a $\mathcal{C}^1$ surface, but an oblique helicoid is not). Therefore, a right helicoid is never a generalized cylinder, and so the case d and Δ are orthogonal is also excluded.

Since we covered all the possible cases, the proof of Lemma 3 is complete. ■

The second result states that if a surface presents two different helical symmetries, then such a surface has to be either a plane, a cylinder of revolution or a sphere.

LEMMA 4. *Let $\mathcal{S}$ be a $\mathcal{C}^1$ connected surface of the Euclidean space $\mathbb{R}^3$ that is not a plane, and that presents two different helical symmetries (that is, such that either the two axes of the two symmetries are different, or the two shifts of the two symmetries are different, or both the two axes and the two shifts are different).*

Then, $\mathcal{S}$ is either a cylinder of revolution, or a sphere.

Proof. By assumption, $\mathcal{S}$ presents a first helical symmetry. Up to change the coordinates, there exists a real number $p \in \mathbb{R}$ such that $\mathcal{S}$ presents the helical symmetry of axis Δ_1 : Span(e_3) and of shift $2\pi p$, where e_3 is the third vector of the canonical basis.

By assumption, $\mathcal{S}$ presents a second helical symmetry, and so, up to change again the coordinates, there exist three real numbers ρ, θ and λ such that the axis of the second helical symmetry is

$$\Delta_2 : \begin{pmatrix} \rho \\ 0 \\ 0 \end{pmatrix} + \mathrm{Span}\left(\begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}\right)$$

and the shift of this second symmetry is $2\pi\lambda$.

Therefore, for any point of coordinates (x, y, z) belonging to $\mathcal{S}$, the two vector fields:

$$F_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}, \quad F_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} \wedge \begin{pmatrix} x-\rho \\ y \\ z \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix},$$

that is,

$$F_1(x, y, z) = \begin{pmatrix} -y \\ x \\ p \end{pmatrix}, \quad F_2(x, y, z) = \begin{pmatrix} \cos\theta z - \sin\theta y \\ \sin\theta(x - \rho) + \lambda\cos\theta \\ -\cos\theta(x - \rho) + \lambda\sin\theta \end{pmatrix}$$

are tangent to the surface at the point (x, y, z). Therefore, the Lie bracket $[F_1, F_2]$ of F_1 and F_2 is also tangent to the surface at the point (x, y, z). $[F_1, F_2]$ can be written in coordinates as

$$\begin{aligned}
[F_1, F_2](x, y, z) &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta z - \sin\theta y \\ \sin\theta(x - \rho) + \lambda\cos\theta \\ -\cos\theta(x - \rho) + \lambda\sin\theta \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & -\sin\theta & \cos\theta \\ \sin\theta & 0 & 0 \\ -\cos\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} -y \\ x \\ p \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\rho - \cos\theta(\lambda + p) \\ \cos\theta z \\ -\cos\theta y \end{pmatrix}.
\end{aligned}$$

The fact that $[F_1, F_2]$ is tangent to the surface at (x, y, z) can be written as

$$[F_1, F_2](x, y, z) \cdot \big(F_1(x, y, z) \wedge F_2(x, y, z)\big) = 0,$$

or again, in coordinates:

$$\begin{aligned} p\cos^2\theta x^2 + \cos\theta\big(\rho\cos\theta + (\lambda - p)\sin\theta\big)yz + p\cos^2\theta z^2 \\ + \big(\rho\sin\theta - (\lambda + p)\cos\theta\big)\big(\rho\cos\theta + (\lambda - p)\sin\theta\big)x \\ + (-\cos\theta)(\rho\sin\theta - \lambda\cos\theta)(x^2 + y^2) \\ + p\big(\rho\sin\theta - (\lambda + p)\cos\theta\big)(\rho\sin\theta - \lambda\cos\theta) = 0. \end{aligned} \tag{4.15}$$

$\mathcal{S}$ being a $\mathcal{C}^1$ surface, it cannot be entirely contained in the axis Δ_1 of the first helical symmetry. Besides, since by assumption $\mathcal{S}$ was assumed not to be a plane, there exist three real numbers r, t_0 and z_0, with $r \neq 0$ and $z_0 \neq 0$ such that the helix described by the following parametric equation:

$$\big(r\cos(t - t_0), r\sin(t - t_0), z_0 + pt\big) \quad \forall t \in \mathbb{R} \tag{4.16}$$

is contained in the surface $\mathcal{S}$. Let us observe that the plane $z = 0$ corresponds to the plane, orthogonal to the first axis Δ_1, for which the distance between this axis and the other axis Δ_2 is minimal. The information that the surface $\mathcal{S}$ is not entirely contained in this plane will be used in a crucial manner in the following computation.

Replacing in the expression (4.15) of the scalar product $[F_1, F_2] \cdot (F_1 \wedge F_2)$ the variables x, y and z by (4.16), describing a family of points of the surface $\mathcal{S}$, we obtain

$$\begin{aligned} p\cos^2\theta r^2\cos^2(t - t_0) + \cos\theta\big(\rho\cos\theta + (\lambda - p)\sin\theta\big)r\sin(t - t_0)(z_0 + pt) \\ + \big(\rho\sin\theta - (\lambda + p)\cos\theta\big)\big(\rho\cos\theta + (\lambda - p)\sin\theta\big)r\cos(t - t_0) \\ + p\cos^2\theta(z_0 + pt)^2 + (-\cos\theta)(\rho\sin\theta - \lambda\cos\theta)r^2 \\ + p\big(\rho\sin\theta - (\lambda + p)\cos\theta\big)(\rho\sin\theta - \lambda\cos\theta) = 0. \end{aligned} \tag{4.17}$$

Equation (4.17) is of the form:

$$\begin{aligned} c_1 t^2 + c_2 t\sin(t - t_0) + c_3\cos^2(t - t_0) + c_4 t \\ + c_5\cos(t - t_0) + c_6\sin(t - t_0) + c_7, \end{aligned}$$

holding for all $t \in \mathbb{R}$, and where the coefficients c_i, with $1 \leq i \leq 7$, are fixed real numbers. Therefore, all these coefficients have to be zero (this can be seen by choosing infinitely many $t - t_0$ such that $\cos(t - t_0) = \pm 1$, so that $\sin(t - t_0) = 0$, and vice versa, providing four polynomial equations in t, which enables us to conclude). We find therefore the system:

$$
(4.18)\quad \begin{cases} p^3\cos^2\theta = 0, \\ rp\cos\theta\big(\rho\cos\theta + (\lambda - p)\sin\theta\big) = 0, \\ r^2 p\cos^2\theta = 0, \\ 2z_0 p^2\cos^2\theta = 0, \\ r\big(\rho\sin\theta - (\lambda + p)\cos\theta\big)\big(\rho\cos\theta + (\lambda - p)\sin\theta\big) = 0, \\ rz_0\cos\theta\big(\rho\cos\theta + (\lambda - p)\sin\theta\big) = 0, \\ r^2(-\cos\theta)(\rho\sin\theta - \lambda\cos\theta) \\ \quad + p\big(\rho\sin\theta - (\lambda + p)\cos\theta\big)(\rho\sin\theta - \lambda\cos\theta) + z_0 p\cos^2\theta = 0. \end{cases}
$$

We consider now the following cases to study the system (4.18).

First, if $p = 0$ (that is, $\mathcal{S}$ is a surface of revolution around the axis Δ_1), (4.18) becomes

$$
\begin{cases} (\rho\sin\theta - \lambda\cos\theta)(\rho\cos\theta + \lambda\sin\theta) = 0, \\ \cos\theta(\rho\cos\theta + \lambda\sin\theta) = 0, \\ (-\cos\theta)(\rho\sin\theta - \lambda\cos\theta) = 0. \end{cases}
$$

If in addition $\cos\theta = 0$, we find

$$
\rho\lambda\sin^2\theta = \rho\lambda = 0.
$$

If $\rho = 0$, the two axes Δ_1 and Δ_2 are the same, and then $\mathcal{S}$ is a surface of revolution around Δ_1, as well as a surface presenting a helical symmetry around the same axis, with a non-trivial shift (because we assumed that $\mathcal{S}$ presents two distinct helical symmetries). Therefore, $\mathcal{S}$ is a cylinder of revolution.

If on the contrary $\rho \neq 0$, then $\lambda = 0$, but then $\mathcal{S}$ would be a surface of revolution, around two distinct axes Δ_1 and Δ_2 that are parallel. In such a case, $\mathcal{S}$ cannot be a $\mathcal{C}^1$ surface; this case leads to a contradiction.

Finally, if we still assume $p = 0$, but $\cos\theta \neq 0$, (4.18) becomes

$$
\begin{cases} \rho\cos\theta + \lambda\sin\theta = 0, \\ \rho\sin\theta - \lambda\cos\theta = 0. \end{cases}
$$

This last system implies $\rho = \lambda = 0$. In this case, $\mathcal{S}$ is a surface of revolution around two different axes that intersect each other. $\mathcal{S}$ is therefore a sphere.

Second, if $p \neq 0$, the first line of (4.18) implies that $\cos\theta = 0$, and so the system becomes

$$
\begin{cases} \cos\theta = 0, \\ \rho\sin\theta(\lambda - p)\sin\theta = 0, \\ \rho^2\sin^2\theta = 0. \end{cases}
$$

We deduce in particular that $\rho = 0$. In this case, $\mathcal{S}$ is a surface presenting two different helical symmetries, around the same axis. Therefore, the shifts of these two helical symmetries are different, and so $\mathcal{S}$ is a cylinder of revolution.

Since we studied all the possible cases, the proof of Lemma 4 is complete. ■

We are now in position to state the main result of this section.

THEOREM 4 (Characterization of the local Maxwellians solving the free transport equation in a domain with SRBC, case $d = 3$). *Let the dimension d be equal to 3. Let T be a strictly positive number (possibly $+\infty$), let Ω be a regular, connected open set of $\mathbb{R}^3$ and let m be a local Maxwellian that is a function of the form (2.3), which solves the transport equation (2.2) on $[0,T] \times \Omega \times \mathbb{R}^3$, with the SRBC, that is, such that*

$$\forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \quad m(t,x,v) = m(t,x,v'),$$

where

$$v' = v - 2\big(v \cdot n(x)\big)n(x)$$

and with $n(x)$ a unitary normal vector to $\partial\Omega$ at x.

Then,

- *if Ω is a half-space, with a boundary of the form $\partial\Omega = \{x_0 n + \lambda\tau / \lambda \in \mathbb{R},\ \tau \in \mathbb{S}^2,\ \tau \cdot n = 0\}$, with $x_0 \in \mathbb{R}$, $n \in \mathbb{S}^2$ (that is, $x_0 n$ is the closest point of boundary $\partial\Omega$ to the origin, and $\partial\Omega$ is normal to the unit vector n), there exist seven real numbers $r_0 > 0$, $\alpha > 0$, β, $\gamma > 0$, a, ℓ_1, $\ell_2 \in \mathbb{R}$ and two unit vectors $\tau_1, \tau_2 \in \mathbb{S}^2$ orthogonal to n such that*

$$\begin{aligned}(4.19)\quad & m(t,x,v) \\ &= r_0 \exp\big(-\alpha(x-tv-2x_0 n)\cdot(x-tv) + \beta(x-tv-x_0 n)\cdot v \\ &\qquad - \gamma|v|^2 + 2(an \wedge x)\cdot v - 2\ell_1\tau_1\cdot(x-tv) + 2\ell_2\tau_2\cdot v\big) \\ &\qquad\qquad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3,\end{aligned}$$

- *if Ω is the volume contained between two parallel planes, with a boundary of the form $\partial\Omega = \{x_1 n + \lambda\tau_1 / \lambda \in \mathbb{R},\ \tau \in \mathbb{S}^2,\ \tau_1 \cdot n = 0\} \cup \{x_2 n + \mu\tau_2 / \mu \in \mathbb{R},\ \tau_2 \in \mathbb{S}^2,\ \tau_2 \cdot n = 0\}$, with $x_0 \in \mathbb{R}$, $n \in \mathbb{S}^1$, there exist five real numbers $r_0 > 0$, $\gamma > 0$, a, ℓ_1, $\ell_2 \in \mathbb{R}$ and two unit vectors $\tau_1, \tau_2 \in \mathbb{S}^2$ orthogonal to n such that*

$$\begin{aligned}(4.20)\quad & m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2(an\wedge x)\cdot v - 2\ell_1\tau_1\cdot(x-tv) + 2\ell_2\tau_2\cdot v\big) \\ &\qquad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3,\end{aligned}$$

- *if Ω is the volume contained in a cylinder of revolution of affine axis $y + \mathrm{Span}(n)$, or the complement of such a volume, or the volume contained between two cylinders* *of revolution with the same affine axis* $y + \mathrm{Span}(n)$*, with* $y \in \mathbb{R}^3$ *and* $n \in \mathbb{S}^2$*, there exist five real numbers* $r_0 > 0$*,* $\gamma > 0$*,* a*,* ℓ_1 *and* ℓ_2*, and a skew-symmetric matrix* Λ_0 *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2a\big(n \wedge (x-y)\big)\cdot v - 2\ell_1 n\cdot(x-tv) + 2\ell_2 n\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \tag{4.21}$$

- *if* Ω *is a sphere centered on* $y \in \mathbb{R}^3$*, there exist three real numbers* $r_0 > 0$*,* $\gamma > 0$*,* a*, and a unit vector* $n \in \mathbb{S}^2$ *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2a\big(n \wedge (x-y)\big)\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \tag{4.22}$$

- *if* $\partial\Omega$ *presents a helical symmetry of axis* $y + \mathrm{Span}(n)$ *and of shift* $2\pi p$ *with* $y \in \mathbb{R}^3$*,* $n \in \mathbb{S}^2$ *and* $p \in \mathbb{R}$*, but such that* $\partial\Omega$ *is not a sphere nor a generalized cylinder, there exist three real numbers* $r_0 > 0$*,* $\gamma > 0$*,* $a \neq 0$ *and a vector* $w \in \mathbb{R}^3$ *orthogonal to* n *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 + 2a\big(n \wedge (x-y) + pn\big)\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \tag{4.23}$$

- *if* $\partial\Omega$ *is a generalized cylinder of direction* n *with* $n \in \mathbb{S}^2$*, but such that* $\partial\Omega$ *does not present a helical symmetry, there exist four real numbers* $r_0 > 0$*,* $\gamma > 0$*,* ℓ_1*,* ℓ_2 *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2 - 2\ell_1 n\cdot(x-tv) + 2\ell_2 n\cdot v\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3, \tag{4.24}$$

- *if finally* $\partial\Omega$ *is not a generalized cylinder and does not present a helical symmetry, then there exist two strictly positive real numbers* $r_0 > 0$ *and* $\gamma > 0$ *such that*

$$m(t,x,v) = r_0 \exp\big(-\gamma|v|^2\big) \quad \forall t \in [0,T],\ x \in \partial\Omega,\ v \in \mathbb{R}^3. \tag{4.25}$$

REMARK 14. The case when the boundary $\partial\Omega$ of the domain is a plane is the "most symmetric" case. Indeed, the domain is not only invariant by translations parallel to the plane or by rotations around the normal lines to the plane, but it is also invariant by dilatations.

As a consequence, we obtain in this case a large choice of local Maxwellians that solve the Boltzmann equation with the specular reflection boundary condition. Equation (4.19) describes a family of local Maxwellians depending on 9 real parameters.

We observe that, similarly as in the 2-dimensional case, we found in the case of the half-space local Maxwellians that solve the Boltzmann equation with a finite mass even though the measure of the domain Ω is not finite.

REMARK 15. In the literature [6, 11], the result presented in Theorem 4 is restricted to the case when the domain Ω is bounded. Therefore, the three first cases ($\partial\Omega$ being the union of one or two planes, or of one or two coaxial cylinders) and the sixth case ($\partial\Omega$ is a generalized cylinder) are by definition not considered. In addition, among all the possible helical symmetries, only the case of the rotational symmetry can be considered if Ω is bounded. In the case when $\partial\Omega$ is bounded, only three possible cases remain, in agreement with the result presented in [6].

PROOF OF THEOREM 4. As for the 2-dimensional case, we start from the expression (2.3) of a local Maxwellian that solves the free transport equation, and the condition (4.3), which expresses the specular reflection boundary condition. As for the 2-dimensional case, we obtain the same system (4.12) on the coefficients α, w_1, Λ_0, β and w_2 of the local Maxwellian m.

We start with the first equation (4.12a). If $\alpha \neq 0$, by Lemma 1, the boundary $\partial\Omega$ contains all the half lines of starting point $-w_1/\alpha$ and through x_0, for all $x_0 \in \partial\Omega$. Since $\partial\Omega$ is a $\mathscr{C}^1$ surface, $\partial\Omega$ is a plane through the point $-w_1/\alpha$, and therefore Ω is a half-space.

If on the contrary $\alpha = 0$, and if $w_1 \neq 0$, then $\partial\Omega$ is the union of a family of straight lines, all parallel to w_1. In other words, $\partial\Omega$ is a generalized cylinder of direction w_1.

We now turn to the second equation (4.12b). In a first time, we assume that $\Lambda_0 \neq 0$, and $\beta \neq 0$. Then, from Lemma 2, for any $x_0 \in \partial\Omega$, $\partial\Omega$ contains a spiral through x_0 and inscribed in the plane $\mathscr{P} = \mathscr{P}(\Lambda_0, w_2)$, where $\mathscr{P}$ is the plane through $-y$ and orthogonal to z, where y is the preimage of w_2 by $\Lambda_0 + \frac{\beta}{2} I_3$ and $z \in \mathbb{R}^3$ is the unique non-zero vector such that $\Lambda_0 x = z \wedge x$ for all $x \in \mathbb{R}^3$. Therefore, the point x_0 belongs also to the plane $\mathscr{P}$, so that we deduce $\partial\Omega \subset \mathscr{P}$. Since the only $\mathscr{C}^1$ surface contained in a plane is the plane itself, we deduce that $\partial\Omega$ is the plane through $-y$ and orthogonal to z.

If now $\Lambda_0 \neq 0$ and $\beta = 0$, Lemma 2 provides that $\partial\Omega$ is a surface that presents a helical symmetry, of axis orientated by z (where $\Lambda_0 x = z \wedge x \ \forall x \in \mathbb{R}^3$), and of shift determined by w_2.

If $\Lambda_0 = 0$ and $\beta \neq 0$, applying Lemma 1 we deduce that $\partial\Omega$ is a plane through $-2w_2/\beta$, and therefore that Ω is a half-space.

Finally, if $\Lambda_0 = 0$, $\beta = 0$ and $w_2 \neq 0$, $\partial\Omega$ is a generalized cylinder of direction w_2.

Therefore, according to the previous discussion, and the general results of Lemmas 3 and 4, we separate the following cases: either $\partial\Omega$ presents no helical symmetry and is not a generalized cylinder, or it is a generalized cylinder without a helical symmetry, or it presents a helical symmetry without being a generalized cylinder, or finally it is a generalized cylinder presenting a helical symmetry, in which case $\partial\Omega$ is either the union of one or two parallel planes, or the union of one or two coaxial cylinders. In addition, when $\partial\Omega$ presents a helical symmetry without being a general cylinder, we will separate the cases when $\partial\Omega$ presents two different helical symmetries or not. In the end, we consider the following cases that exclude each other and that cover all the possible situations:

- $\partial\Omega$ is a single plane,
- $\partial\Omega$ is the union of two parallel planes,
- $\partial\Omega$ is the union of one or two coaxial cylinders of revolution,
- $\partial\Omega$ is a sphere,
- $\partial\Omega$ is a surface that presents a helical symmetry, but is not a sphere nor a generalized cylinder,
- $\partial\Omega$ is a generalized cylinder which does not present a helical symmetry,
- $\partial\Omega$ does not present a helical symmetry and is not a generalized cylinder neither.

$\partial\Omega$ is a plane. Let us assume first that $\partial\Omega$ is a single plane of the form $\{x_0 n + \lambda\tau / \lambda \in \mathbb{R}, \tau \in \mathbb{S}^2,\ \tau\cdot n = 0\}$, for a certain $x_0 \in \mathbb{R}^3$ that belongs to $\partial\Omega$, and $n \in \mathbb{S}^2$ being normal to $\partial\Omega$.

Then, if $\alpha \neq 0$, $-w_1/\alpha \in \partial\Omega$, so that

$$w_1 = -\alpha x_0 n + \ell_1 \tau_1, \tag{4.26}$$

for some $\ell_1 \in \mathbb{R}$ and $\tau_1 \in \mathbb{S}^2$ with $\tau_1 \cdot n = 0$. If on the contrary $\alpha = 0$ and $w_1 \neq 0$, then w_1 has to be orthogonal to n, hence the general expression (4.26) for w_1.

If $\Lambda_0 \neq 0$ and $\beta \neq 0$, we write $\Lambda_0 x = z \wedge x$ for all $x \in \mathbb{R}^3$ with $z \in \mathbb{R}^3$. According to Lemma 2, $\partial\Omega$ contains a spiral inscribed in a plane $\mathscr{P}$ normal to z, and such that $-y$ belongs to the spiral with $\left[\Lambda_0 + \frac{\beta}{2} I_3\right] y = w_2$. Therefore, we have $z = an$ for some $a \in \mathbb{R}$, $a \neq 0$, and there exist $\lambda \in \mathbb{R}$ and $\tau_y \in \mathbb{S}^2$ with $\tau_y \cdot n = 0$ such that

$$y = -x_0 n + \lambda \tau_y.$$

As a consequence, we have necessarily

$$w_2 = a\lambda n \wedge \tau_y - \frac{\beta}{2} x_0 n + \frac{\beta}{2}\lambda\tau.$$

Since the matrix $\Lambda_0 + \frac{\beta}{2} I_3$ restricted to $n^\perp$ is invertible, w_2 can be written in general as

$$w_2 = -\frac{\beta}{2} x_0 n + \ell_2 \tau_2,$$

for some $\ell_2 \in \mathbb{R}$ and $\tau_2 \in \mathbb{S}^2$ such that $\tau_2 \cdot n = 0$.

If now $\Lambda_0 \neq 0$ (represented as a vector $z \in \mathbb{R}^3$, $z \neq 0$) and $\beta = 0$, then according to Lemma 2, $\partial\Omega$ contains a helix of axis $-y + \mathrm{Span}(z)$, and of shift $2\pi \frac{\lambda}{|z|}$, with

$$w_2 = \lambda \frac{z}{|z|} + \Lambda_0 y.$$

Since $\partial\Omega$ is a plane normal to n, then necessarily $\lambda = 0$, and there exist $a \in \mathbb{R}$, $a \neq 0$, $\ell_2 \in \mathbb{R}$ and $\tau_2 \in \mathbb{S}^2$, $\tau_2 \cdot n = 0$, such that

$$z = an, \quad \text{and} \quad w_2 = -\Lambda_0 y = \ell_2 \tau_2.$$

If $\Lambda_0 = 0$ and $\beta \neq 0$, then, according to Lemma 1, $-2\frac{w_2}{\beta}$ belongs to $\partial\Omega$; that is, there exist $\ell_2 \in \mathbb{R}$ and $\tau_2 \in \mathbb{S}^2$, $\tau_2 \cdot n = 0$, such that

$$w_2 = -\frac{\beta}{2} x_0 n + \ell_2 \tau_2.$$

Finally, if $\Lambda_0 = 0$, $\beta = 0$ and $w_2 \neq 0$, then $\partial\Omega$ is a generalized cylinder of direction w_2, and so w_2 is necessarily orthogonal to n.

We conclude therefore that if $\partial\Omega$ is a single plane, only the local Maxwellians of the form (4.19) solve the Boltzmann equation with specular reflection boundary condition.

$\partial\Omega$ is the union of two parallel planes. In this case, there exist by assumption one unit vector $n \in \mathbb{S}^2$ and two different real numbers $x_1 \neq x_2$ such that

$$\partial\Omega = \{x_1 n + \ell_1 \tau_1 / \ell_1 \in \mathbb{R},\ \tau_1 \in \mathbb{S}^2,\ \tau_1 \cdot n = 0\} \\ \cup \{x_2 n + \ell_2 \tau_2 / \ell_2 \in \mathbb{R},\ \tau_2 \in \mathbb{S}^2,\ \tau_2 \cdot n = 0\}.$$

In this case, according to Lemma 1, $\alpha = 0$ (because $\partial\Omega$ is not composed of a single plane, so $\alpha \neq 0$ leads to a contradiction). If $w_1 \neq 0$, then w_1 is necessarily orthogonal to n.

If $\Lambda_0 \neq 0$ and $\beta \neq 0$, the same conclusion holds as in the case when $\partial\Omega$ is a single plane. In particular, we obtain a contradiction since the component of w_2 along the direction n should be $-\frac{\beta}{2} x_1$ and $-\frac{\beta}{2} x_2$, but $x_1 \neq x_2$.

Similarly, if $\Lambda_0 = 0$ and $\beta \neq 0$, $\partial\Omega$ should be a single plane through $-2\frac{w_2}{\beta}$, which is not the case.

We deduce that $\beta = 0$.

If $\Lambda_0 \neq 0$, then, according to Lemma 2, $\partial\Omega$ presents a helical symmetry of axis orientated by z (where z represents Λ_0). We deduce therefore that $z = an$, for $a \in \mathbb{R}$, $a \neq 0$. In addition, $\partial\Omega$ being the union of two planes, the shift of this helical symmetry has to be zero; therefore, $w_2 = \ell_2 \tau_2$, for some $\ell_2 \in \mathbb{R}$, $\tau_2 \in \mathbb{S}^2$ and $\tau_2 \cdot n = 0$.

If finally $\Lambda_0 = 0$ and $\beta = 0$, $\partial\Omega$ is a generalized cylinder of direction w_2; therefore, w_2 has to be orthogonal to n.

In summary, we deduce that if $\partial\Omega$ is the union of two different, parallel planes, then only the local Maxwellians of the form (4.20) solve the Boltzmann equation with the boundary condition (4.1).

$\partial\Omega$ is the union of one or two coaxial cylinders of revolution. Let us assume that the two cylinders of revolution have the common axis of rotation $y + \mathrm{Span}(n)$, for $y \in \mathbb{R}^3$ and $n \in \mathbb{S}^2$.

The following cases lead to a direct contradiction:

$$\alpha \neq 0, \quad [\Lambda_0 \neq 0 \text{ and } \beta \neq 0], \quad [\Lambda_0 = 0 \text{ and } \beta \neq 0].$$

In particular, we deduce that $\alpha = \beta = 0$.

In the case when $w_1 \neq 0$, we deduce that w_1 has to be colinear to n. If $\Lambda_0 \neq 0$, represented by the vector z, $\partial\Omega$ presents a helical symmetry of axis $-y' + \mathrm{Span}(z)$, where $w_2 = -\Lambda_0 y' + pn$. Therefore, this implies $z = an$ for a certain $a \in \mathbb{R}$, $a \neq 0$, and $\Lambda_0 y' = \Lambda_0 y$, so that

$$w_2 = -\Lambda_0 y + pn.$$

If finally $\Lambda = 0$ and $w_2 \neq 0$, then w_2 is colinear to n. In general, Λ_0 and w_2 have then the forms:

$$\Lambda_0 x = an \wedge x \ \forall x \in \mathbb{R}^3, \quad \text{and} \quad w_2 = -an \wedge y + \ell_2 n,$$

for $a, \ell_2 \in \mathbb{R}$.

Therefore, if $\partial\Omega$ is the union of one or two coaxial cylinders of revolution, any local Maxwellian solving the Boltzmann equation with the boundary condition (4.1) is necessarily of the form (4.21).

$\partial\Omega$ is a sphere. Let us assume that the sphere $\partial\Omega$ is centered on $y \in \mathbb{R}^3$. In this case, the following assumptions lead to a direct contradiction:

$$\alpha \neq 0, \quad [\alpha = 0 \text{ and } w_1 \neq 0], \quad [\Lambda_0 \neq 0 \text{ and } \beta \neq 0],$$
$$[\Lambda_0 = 0 \text{ and } \beta \neq 0], \quad [\Lambda_0 = 0,\ \beta = 0, \text{ and } w_2 \neq 0].$$

We deduce in particular that $\alpha = 0$, $\beta = 0$, $w_1 = 0$, and if $\Lambda_0 = 0$, then $w_2 = 0$.

In addition, if $\Lambda_0 \neq 0$, represented by the vector $z \in \mathbb{R}^3$, according to Lemma 2, $\partial\Omega$ presents a helical symmetry, of axis $y + \mathrm{Span}(z)$, with $w_2 = -\Lambda_0 y + pn$. In this case, since $\partial\Omega$ is a sphere, the shift of the helical symmetry has to be zero, so that

$$w_2 = -\Lambda_0 y.$$

Therefore, if $\partial\Omega$ is a sphere, only the local Maxwellians of the form (4.22) can solve the Boltzmann equation with the boundary condition (4.1).

<u>$\partial\Omega$ presents a helical symmetry, but is not a sphere nor a generalized cylinder</u>. Let us denote by $y + \mathrm{Span}(n)$ the axis of the helical symmetry of $\partial\Omega$, with $y \in \mathbb{R}^3$ and $n \in \mathbb{S}^2$, and let us denote by $2\pi p$ the shift of this helical symmetry, with $p \in \mathbb{R}$. As for the previous case, the following cases lead to a direct contradiction:

$$\alpha \neq 0, \quad [\alpha = 0 \text{ and } w_1 \neq 0], \quad [\Lambda_0 \neq 0 \text{ and } \beta \neq 0],$$
$$[\Lambda_0 = 0 \text{ and } \beta \neq 0], \quad [\Lambda_0 = 0,\ \beta = 0, \text{ and } w_2 \neq 0].$$

We deduce then that $\alpha = 0$, $\beta = 0$, $w_1 = 0$, and if $\Lambda_0 = 0$ and $\beta = 0$, then $w_2 = 0$ as well.

According to Lemma 4, since $\partial\Omega$ is not a plane (which is a particular case of a generalized cylinder), nor a cylinder of revolution (another particular case of a generalized cylinder), nor a sphere, we deduce that $\partial\Omega$ presents a single helical symmetry. Therefore, the direction of z, representing Λ_0 is fixed by n, and the component of w_2 along n is fixed as well by the shift of the helical symmetry. More precisely, if $\Lambda_0 \neq 0$, there exists $a \in \mathbb{R}$, $a \neq 0$, such that

$$\Lambda_0 c = a n \wedge x \quad \forall x \in \mathbb{R}^3,$$

and

$$w_2 = -\Lambda_0 y + apn = a(-n \wedge y + pn).$$

As a consequence, if $\partial\Omega$ presents a helical symmetry, but is not a sphere nor a generalized cylinder, then only the local Maxwellians of the form (4.24) solve the Boltzmann equation with the specular reflection boundary condition.

<u>$\partial\Omega$ is a generalized cylinder which does not present a helical symmetry</u>. Let us assume that the direction of boundary $\partial\Omega$, as a generalized cylinder, is given by a vector $n \in \mathbb{S}^2$. In the present case, the following cases lead to a direct contradiction:

$$\alpha \neq 0, \quad [\Lambda_0 \neq 0 \text{ and } \beta \neq 0], \quad [\Lambda_0 \neq 0 \text{ and } \beta = 0], \quad [\Lambda_0 = 0 \text{ and } \beta \neq 0].$$

In particular, we deduce that $\alpha = 0$, $\beta = 0$ and $\Lambda_0 = 0$. In addition, if w_1 or w_2 are not zero, then they have to be colinear to n. Hence the conclusion: only the local Maxwellians of the form (4.24) can be solutions of the Boltzmann equation with the boundary condition (4.1).

<u>$\partial\Omega$ does not present an helical symmetry, and is not a generalized cylinder</u>. In this final case, if any of the parameters α, β, Λ_0, w_1 or w_2 is not zero, we obtain a contradiction. Therefore, only the parameter γ can be non-zero if a local Maxwellian m solves the Boltzmann equation with the specular reflection boundary condition in such a domain, hence (4.25).

All the cases were investigated, concluding the proof of Theorem 4. ∎

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