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Counting mobiles by integrable systems

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Abstract. Mobiles are a particular class of decorated plane trees which serve as codings for planar maps. Here, we address the question of enumerating mobiles in their most general flavor, in correspondence with planar Eulerian (i.e., bicolored) maps. We show that the generating functions for such mobiles satisfy a number of recursive equations which lie in the field of integrable systems, leading us to explicit expressions for these generating functions as ratios of particular determinants. In particular, we recover known results for mobiles associated with uncolored maps and prove some conjectured formulas for the generating functions of mobiles associated with p-constellations.

Contents

1. Introduction

Mobiles denote a particular class of decorated plane trees carrying integer labels and subject to a number of local rules – to be detailed below – around their vertices.

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Mobiles were introduced in [5], where it was shown that they provide a *bijective coding* of general classes of *planar maps*.

Recall that a planar map is a connected graph drawn on the two-dimensional sphere (or equivalently, on the plane) without edge crossings, and it is considered up to continuous deformations (see [17] for a comprehensive introduction to maps). The problem of enumerating maps has attracted a lot of attention since the seminal work of Tutte in the early 60's [18], and many enumeration techniques for maps with various topologies were developed over the years, ranging from matrix integral techniques [7] to the so-called topological recursion approach [9, 11].

In this quest for enumerative formulas, the discovery of the coding by mobiles was an important step as it allowed one to get a recursive construction of planar maps, transcribed into a set of recursive equations for their *generating functions*. An interesting feature of the coding is that the decorations carried by the mobiles allow one to keep track of some of the geodesic distances between marked "points" (i.e., vertices, edges or faces) on the associated map. This property led to a number of results on the *statistics of distances* between points within random maps [4].

Of particular interest is the *generating function* R_i for mobiles rooted at a vertex labeled by some positive integer i which, via the bijection with maps, enumerates planar maps with two marked points at a prescribed distance constrained to remain less than i – see the next section for a precise definition. Indeed, it was soon realized [4] that, for many map families of interest, the recursive equations obeyed by the R_i's for $i \geq 1$ turn out to be *integrable* [8, 12]. In particular, fully explicit expressions for R_i were obtained for a number of families of planar maps: these include *quadrangulations* – in bijection with so-called well-labeled trees [16] which are a particularly simple example of mobiles –, *triangulations*, and more generally *maps with prescribed face degrees*. In this latter case, an interesting link with continued fractions was established in [6], which allows one to understand the precise form of the R_i's as ratios of Hankel determinants.

The most general family of mobiles that were introduced in [5] are those mobiles which code for the so-called *planar Eulerian maps*. Recall that a planar Eulerian map is a planar map whose all vertices have an even degree, which, if the map is rooted, is equivalent to the condition that the map be canonically face-bicolored, i.e., with faces colored in black and white with no two adjacent faces of the same color. Such maps are also called *hypermaps* in the literature to emphasize the fact that they encompass the case of usual uncolored general maps[1]. A natural question is therefore that of finding

[1]Indeed, starting from a general uncolored map, we get a bicolored one by first coloring all faces in white, and by then inflating all the edges into black bivalent faces separating the original faces.

an explicit expression for the R_i's in this extended family of mobiles. It is precisely the purpose of the present paper to obtain such a formula thanks to the machinery developed for solving integrable systems.

Before we proceed, let us recall that expressions for R_i in the context of genuine Eulerian maps were conjectured in the restricted case of p-constellations [4], which are Eulerian maps with only p-valent black faces and white faces whose degrees are multiples of p. So far, these conjectured formulas were proved only for the case of Eulerian triangulations, in connection with the theory of multicontinued fractions [1]. The approach developed here allows us to: (i) formulate the mobile enumeration problem in the framework of integrable systems (ii) recover known results for general uncolored maps, (iii) prove the conjectured formulas for p-constellations, and (iv) extend these formulas to the general family of mobiles coding for planar Eulerian maps with bounded face degrees.

The paper is organized as follows.

Section 2 is devoted to the definition of mobiles. We first discuss in Section 2.1 the local rules on labels satisfied by mobiles in their most general flavor as obtained from their correspondence with Eulerian maps. We then define in Section 2.2 a number of generating functions for mobiles and half-mobiles and derive the set of recursive equations which determine them. These generating functions can be expressed as the entries of two semi-infinite matrices P and Q which are shown to "quasi-commute" in Section 2.3. We discuss in Section 2.4 the case of mobiles with bounded vertex degrees in correspondence with maps with bounded face degrees and show that the R_i's, as well as the other (half-mobile) generating functions, admit large i limits, which are formal power series in the weights attached to the mobile vertices. We end this section by discussing in Section 2.5 mobiles with a weight g per labeled vertex, corresponding to a weight g per vertex of the associated map: we show, in particular, that our recursive system of equations admits a unique *combinatorial* solution for which R_i and the other generating functions of interest are formal power series in $\sqrt{g}$.

The goal of Section 3 is to place our equations within the framework of integrable systems and to use the associated machinery to obtain a number of properties that must be satisfied by the operators P and Q. We first show that, thanks to their quasi-commutation property, we can build out of the operators P and Q an *isospectral system*: more precisely, we build in Section 3.1 a *discrete Lax equation*. We define and compute in Section 3.2 the associated *spectral curve* $\mathcal{E}(x, y)$ as the characteristic polynomial of a suitable endomorphism expressing the action of P in the eigenvector space of Q associated with an arbitrary eigenvalue x. In Section 3.3, we introduce the notion of branch points and double points of the spectral curve and prove some crucial identity involving these double points (Lemma 3.3). We then construct in Section 3.4 the matrix wave function associated to the discrete Lax equation whose entries are given by common right (resp., left) eigenvectors ψ (resp., ϕ) of P and Q at the branch points of generic x. In Section 3.5, we prove that, as limit of a well-defined matrix determinant, ψ and ϕ take the form of Baker–Akhiezer functions via the so-called reconstruction formula.

Section 4 presents the proof of our main theorem, namely, Theorem 4.3, giving, in particular, an explicit expression for R_i. Note that, apart from Lemma 3.3, this section does not make use of the results of Section 3 and is thus self-contained. Starting from the spectral curve and its double points, we define in Section 4.1 a set of specific Baker–Akhiezer functions ψ and ϕ in the form of appropriate determinants. Thanks to a suitable scalar product introduced in Section 4.2, we reconstruct explicitly in Section 4.3 the entries of the operators P and Q as scalar products of these Baker–Akhiezer functions. These entries are shown to satisfy half of the recursion relations wanted for the mobile generating functions. In order to show that the second half of the wanted equations is also satisfied, we construct in the same way in Section 4.4 the entries of Q and P transposed. We conclude our proof in Section 4.5, in particular, we obtain an explicit expression for the generating functions R_i of mobiles as bi-ratios of determinants.

In Section 5, we present a number of applications of our main result: we first recover in Section 5.1 the expression for R_i found in [6] for the case of mobiles associated to general (uncolored) planar maps with bounded face degrees. We then prove in Section 5.2 the formula for R_i conjectured in [4,12] in the case of p-constellations.

We gather in Section 6 a number of concluding remarks. Some technical points are discussed in various appendices.

2. Mobiles

2.1. From Eulerian maps to mobiles

The rules which define mobiles are directly inherited from their correspondence with maps. Several classes of mobiles may thus be defined, in connection with several classes of maps. In this paper, we discuss mobiles in their most general flavor, as obtained from their bijection with *Eulerian planar maps*. Most of the results of this section and Section 2.2 may already be found in [5] except for the relation (2.5) which is new. Sections 2.3 to 2.5 are also new in their formulation.

Recall that a planar map is a cellular embedding of a graph in the sphere, considered up to continuous deformation. It is therefore made of vertices, edges, and faces, the degree of a vertex (resp., a face) being the number of its incident half-edges (resp., edge sides). A planar map is said Eulerian if *its vertices all have an even degree*. Equivalently, this is a planar map which may be *face-bicolored*, i.e., whose faces may

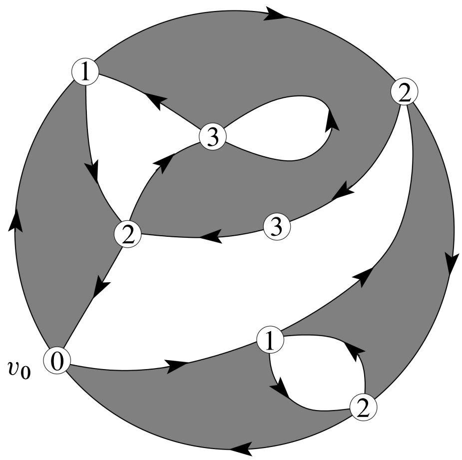

Figure 1. A pointed Eulerian (i.e., face bicolored) planar map with root vertex v_0. Edges are oriented clockwise around black faces and counterclockwise around white faces. Each vertex is labeled by its oriented geodesic distance from the root vertex.

be colored, say in black and white so that no two adjacent faces are of the same color. Note that, for a given Eulerian map, there are exactly two possible such colourings (related to each other by exchanging the colors). In the following, we will always assume that our Eulerian maps are endowed with one of their two bi-colourings, i.e., an Eulerian map will in practice refer to a face-bicolored map. Let us now see how decorated trees emerge as a coding for Eulerian maps.

Since each edge in an Eulerian map separates a black and a white face, we may orient canonically all the edges of the map by demanding that their incident black face lies on their right when following the orientation. In other words, edges are oriented clockwise around the black faces and counterclockwise around the white faces (see Figure 1). If we now consider a *pointed* Eulerian map, i.e., with a distinguished *root vertex* v_0, we may label each vertex v of the map by its *oriented geodesic distance* from the root vertex, defined as the length (= number of edges) of any shortest path from v_0 to v following edges of the map and respecting their orientation. This label is a non-negative integer $i(v)$ satisfying $i(v_0) = 0$ and

$$\text{for each edge } e \text{ oriented from } v \text{ to } v'\text{:} \quad i(v') \leq i(v) + 1. \tag{2.1}$$

A mobile is then constructed from the pointed Eulerian map as follows: we first add a black (resp., white) vertex at the center of each black (resp., white) face. For each edge e oriented from v to v', we do the following construction: if $i(v') = i(v) + 1$, we draw a new edge from v' to the white vertex at the center of the white face incident to e and attach a *bud* (pointing towards e) to the black vertex at the center

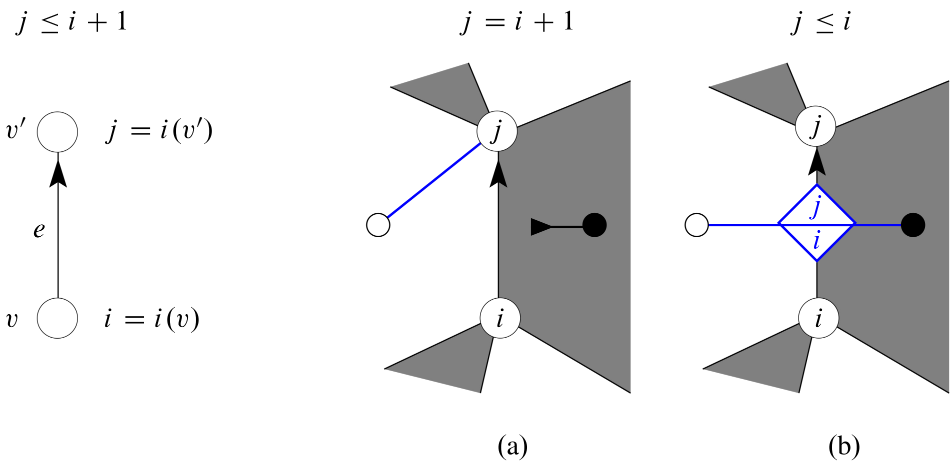

Figure 2. Local rule for the construction of a mobile, applied to each oriented edge e of an Eulerian map, linking a vertex v to a vertex v' with respective labels i and j. Either $j = i + 1$ and we do the construction (a) resulting in a labeled vertex with label j connected by a regular edge to a white vertex and a bud (►—) connected to a black vertex, or $j \le i$ and we do the construction (b) resulting in a flagged edge with labels i, j connecting a white to a black vertex.

of the black face incident to e (see Figure 2 (a)). If $i(v') \le i(v)$, we draw instead a new edge between the black and white vertices at the center of the black and white faces incident to e and put labeled flags on both sides of this edge: a flag with label $i(v)$ on the side of v and a flag with label $i(v')$ on the side of v' (see Figure 2 (b)). We finally remove the original edges of the Eulerian map. It was shown in [5] that the graph formed by the newly drawn edges and the buds form a *connected plane tree* which spans all the original vertices of the map but the root vertex v_0 (see Figure 3 for an example). This tree, endowed with all its decorations (buds, flags) and labels forms the desired mobile.

We arrive at the following definition of mobiles, entirely dictated by their construction from the associated Eulerian maps.

Definition 2.1. A mobile is a plane tree formed of the following:

- three types of vertices: black vertices, white vertices and labeled vertices which carry *positive* integer labels;
- two types of edges: regular edges and flagged edges which carry pairs of *non-negative* integer labels (one on each side of the edge). Regular edges necessarily connect a white vertex to a labeled one. Flagged edges necessarily connect a black vertex to a white one.

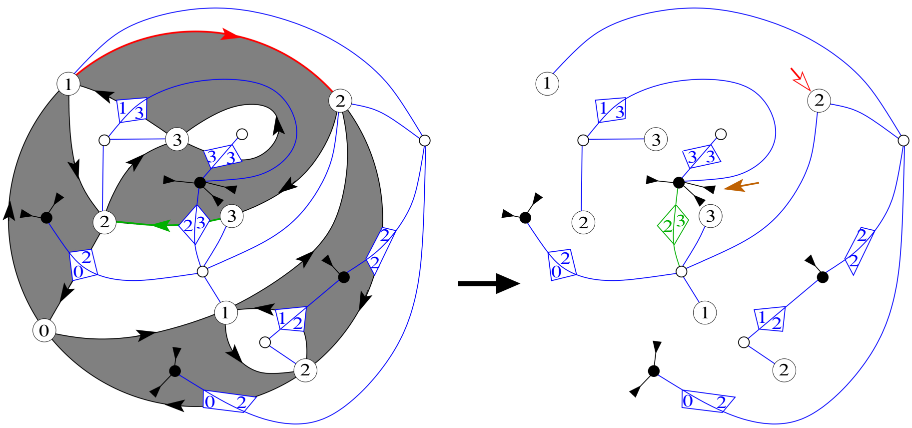

Figure 3. Applying the rules of Figure 2 to all the edges of a pointed Eulerian map results into a mobile (right). In red: an edge of type $i \to i+1$ is in correspondence with a corner at a labeled vertex with label $i+1$. In green: an edge of type $i \to j$ with $j \le i$ is in correspondence with a flagged edge with labels i, j. In brown: the bud associated to the marked red corner (see text). Note that there are exactly $j-i$ buds between any two clockwise consecutive flagged edges with consecutive labels i and j.

The labels obey the following local rules around black and white vertices.

- Given a black vertex, the sequence of labels on its incident flagged edges, as read *clockwise* around this vertex, is non-increasing at the crossing of each flagged edge and non-decreasing between two consecutive flagged edges.
- Given a white vertex, the sequence of labels on its incident flagged edges and adjacent labeled vertices, as read *clockwise* around this vertex, is non-decreasing at the crossing of each flagged edge, constant between a flag and the next flag or labeled vertex, and decreasing by 1 between a labeled vertex and the next flag or labeled vertex.
- Black vertices are decorated by buds by putting exactly $j-i$ buds between two consecutive flagged edges with consecutive labels i and j (recall that $j \ge i$ by definition – see Figure 3 – right for an illustration).

The vertex maps reduced to a single (black, white or labeled) vertex are not considered as mobiles.

The label rules around black and white vertices directly follow from the mobile edge construction rules, as described above, applied to the sequence of edges around a given black or white face in an Eulerian map. Note that a mobile necessarily contains a flagged edge (since the rule around a white vertex cannot be fulfilled if all its incident edges are regular), hence, it contains a black and a white vertex (since a flagged edge connects a black to a white vertex). Note also that, from the label rules, the minimum label $i_{\min}$ on a mobile is necessarily carried by a flag and satisfies $i_{\min} \geq 0$. Note finally that buds clearly constitute a redundant information, but we introduce them to guarantee that the total degree of a black vertex (counting buds as incident half-edges) is identical to that of the associated black face. We have the following bijection [5].

Theorem 2.1 (Bouttier–Di Francesco–Guitter (BDG) bijection). *The above mapping is a bijection between pointed Eulerian maps and mobiles with minimum label* $i_{\min} = 0$ *(or equivalently mobiles having at least a flag labeled* 0*).*

In particular, pointed Eulerian maps with a *marked edge e* oriented from v to v' at respective oriented distance $i(v)$ and $i(v')$ from the root vertex are in bijection with mobiles having at least a flag labeled 0 with a *marked flagged edge* with flag labels $i(v)$ and $i(v')$ if $i(v') \leq i(v)$, or with a *marked corner* at a labeled vertex with label $i(v')$ if $i(v') = i(v) + 1$ (see Figure 3).

Note that the constraint that the mobile has minimum label $i_{\min} = 0$ follows from the fact that any edge oriented towards the root vertex will create a flagged edge with one of its labels equal to 0. For convenience, this constraint is *not* included in the above definition of mobiles, *whose minimal label is only required to be non-negative*. We will see later how to simply restore the constraint when necessary. Clearly, the rules around black and white vertices are invariant by a global shift of the labels. We have thus the following corollary.

Corollary 2.1. *Given some fixed integer* $i_{\min} \geq 0$*, mobiles whose minimum label is* $i_{\min}$ *are in bijection with pointed Eulerian maps.*

The correspondence between these mobiles and pointed Eulerian maps is moreover obtained by exactly the same construction as above provided we now endow the Eulerian maps with a new labeling of vertices equal to their oriented geodesic distance from the root vertex *plus* $i_{\min}$.

To conclude this section, let us mention that the labels around black or white vertices in a mobile are simply *two different but equivalent codings* for sequences of labels satisfying (2.1) along the oriented edges around a black or a white face in a pointed Eulerian map. At a white vertex, the sequence is trivially recovered by going counterclockwise around a vertex and ignoring the first encountered label of each flagged edge (which is identical to the preceding label counterclockwise). In particular, the degree of the associated white face is also, as it should, the degree of the white vertex on the mobile (see Figure 4 for an example). Around a black vertex, this sequence is recovered instead by considering the clockwise cyclic sequence of labels $i_1, i'_1, i_2, i'_2, \ldots, i_p, i'_p$, where $i_k \to i'_k$ denotes the passage from a flagged edge to the next (hence, $i'_k \geq i_k$) and $i'_k \to i_{k+1}$ denotes the crossing of a given flagged edge (hence, $i_{k+1} \leq i'_k$ with the convention $i_{p+1} = i_1$), and replacing each step $i_k \to i'_k$

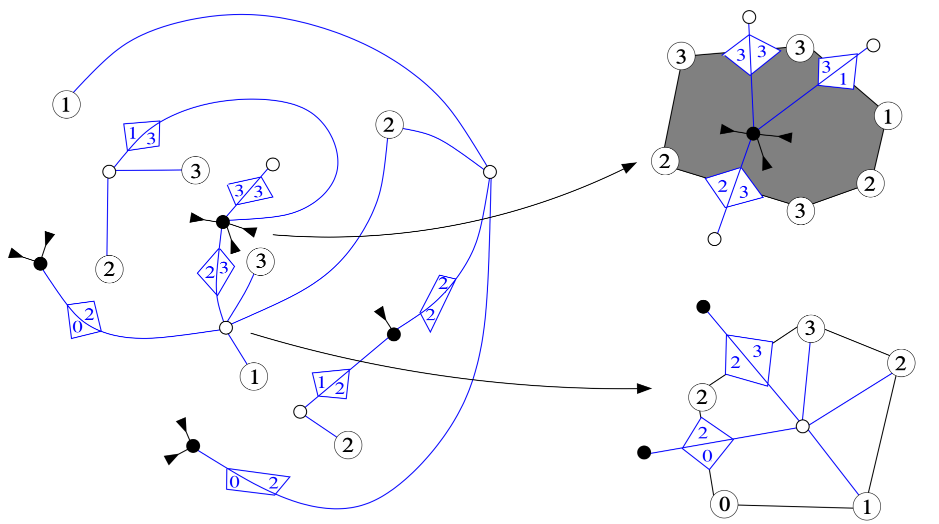

Figure 4. Top right: reconstruction of the vertex labels around a black face from the environment of the associated black vertex in the mobile. Bottom right: reconstruction of the vertex labels around a white face from the environment of the associated white vertex in the mobile.

by a sequence of $i'_k - i_k$ steps, each with increment $+1$ (so as to obtain the sequence $i_1, i_1 + 1, i_1 + 2, \ldots, i'_1, i_2, i_2 + 1, i_1 + 2, \ldots, i'_2, \ldots, i_p, i_p + 1, i'_1 + 2, \ldots, i'_p$). In particular, the degree of the associated black face in the Eulerian map is given by $\sum_{k=1}^{p}(i'_k - i_k + 1)$, which is precisely, as it should, the degree of the black vertex (counting the buds) (see Figure 4 for an example).

2.2. Enumeration of mobiles

We now come to the enumeration of mobiles. More precisely, we will consider generating functions for mobiles with a weight g_k per white vertex of degree k and a weight $\tilde{g}_k$ per black vertex of degree k. From the BDG bijection, this amounts to weigh the Eulerian maps with a weight g_k (resp., $\tilde{g}_k$) per white (resp., black) face of degree k.

As was done in [5], we will consider three families of generating functions:

- the generating functions R_i ($i \geq 1$) for mobiles with a marked corner at a labeled vertex with label i. For convenience, we incorporate also in R_i a conventional additional term 1, which we interpret as enumerating the vertex map made of a single isolated vertex with label i (recall that this vertex map is not considered as a mobile – the generating function for genuine mobiles is thus $R_i - 1$),
- the generating functions $W_{i,j}$ ($0 \leq j \leq i$) for *white half-mobiles* as obtained by cutting a mobile at a flagged edge with labels i and j and keeping only the part of

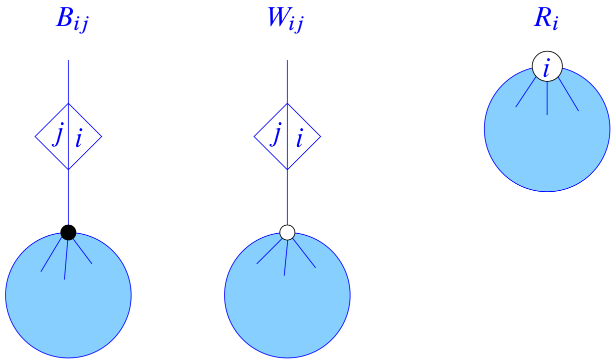

Figure 5. Schematic picture of a black half-mobile enumerated by B_{ij}, a white half-mobile enumerated by W_{ij} and a mobile with a marked corner (or a single labeled vertex), as enumerated by R_i.

the mobile lying on the side of its white incident vertex. Note that we keep also in the half-mobile the flagged edge itself with its labels i and j so that i denotes the flag label on the left when going *towards* the white vertex (see Figure 5),

- the generating functions $B_{i,j}$ $(0 \le i \le j)$ for *black half-mobiles* as obtained by cutting a mobile at a flagged edge with labels i and j and keeping only the part of the mobile lying on the side of its black incident vertex. Again, we keep in the half-mobile the flagged edge itself with its labels i and j so that i denotes the flag label on the left when going *towards* the black vertex (see Figure 5).

The generating functions R_i, $B_{i,j}$, and $W_{i,j}$ satisfy a closed set of equations that determine them completely as power series in the formal weight variables g_k and $\tilde{g}_k$. These equations are best expressed upon introducing the semi-infinite matrices P and Q with entries

$$P_{i,j} = \begin{cases} B_{i,j} & \text{if } j \ge i, \\ R_i & \text{if } j = i-1, \\ 0 & \text{if } j < i-1, \end{cases} \quad Q_{i,j} = \begin{cases} W_{i,j} & \text{if } j \le i, \\ 1 & \text{if } j = i+1, \\ 0 & \text{if } j > i+1 \end{cases} \tag{2.2}$$

for $i, j \ge 0$. We may then write the following two equations[2]:

$$\begin{aligned} B_{i,j} &= \sum_{k\ge1} \tilde{g}_k (Q^{k-1})_{i,j} \quad \text{for } i \le j, \\ W_{i,j} &= \sum_{k\ge1} g_k (P^{k-1})_{i,j} \quad \text{for } i \ge j. \end{aligned} \tag{2.3}$$

[2]These equations are nothing but [5, (3.4) and (3.5)] with the correspondence $(Q, \widetilde{Q}) \to (P, Q)$.

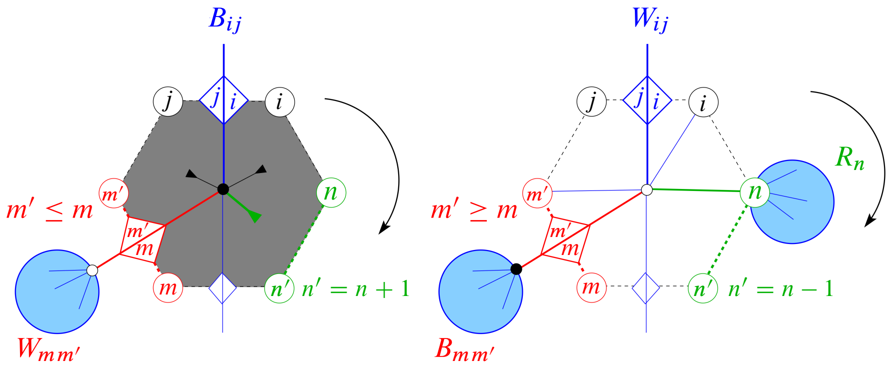

Figure 6. A schematic picture of equations (2.3) giving the expressions of B_{ij} and W_{ij}, respectively.

The first equation relies simply on a census of all possible environments of the first encountered black vertex in a black half-mobile enumerated by $B_{i,j}$ (see Figure 6): if this vertex has degree k in the decorated mobile (hence, receives the weight $\tilde{g}_k$), this environment is coded by a path of length $k-1$ from "height" i to height j (the height being the label around the associated black face). The height at each elementary step either increases by 1, say from n to $n' = n+1$, giving rise to a bud with no weight ($=$ multiplicative factor 1) or decreases weakly, say from m to $m' \leq m$, giving rise to a white half mobile enumerated by $W_{m,m'}$. The sum of factors over all path configurations is clearly counted by $(Q^{k-1})_{i,j}$, hence, the formula. As for the environments of the first encountered white vertex in a white half mobile enumerated by $W_{i,j}$: if this vertex has degree k (hence, receives the weight g_k), it is again coded by path of length $k-1$ from height i to height j, whose height at each elementary step may now either decrease by 1, say from n to $n' = n-1$, giving rise to a mobile with a distinguished corner at a vertex labeled n or to an isolated vertex labeled n (the two possibilities being enumerated by R_n) or it may increase weakly, say from m to $m' \geq m$, giving rise to a black half mobile enumerated by $B_{m,m'}$ (see Figure 6). The sum of these factors over all path configurations is now counted by $(P^{k-1})_{i,j}$.

To close the system, we must also give the equation for R_i. It is obtained by noting that the mobiles counted by R_i form a sequence of an arbitrary number of mobiles planted at a *univalent* labeled vertex with label i (see Figure 7-left). Denoting L_i the generating function for these latter mobiles, we may, therefore, write $R_i = \sum_{k\geq 0}(L_i)^k = 1/(1-L_i)$ (this expression includes a first term 1 accounting for the vertex map with label i, as desired). As for L_i, it is determined along the same line as before upon inspecting the possible environments of the white vertex incident to the univalent vertex i: if this vertex has degree k (hence, receives the weight g_k), the

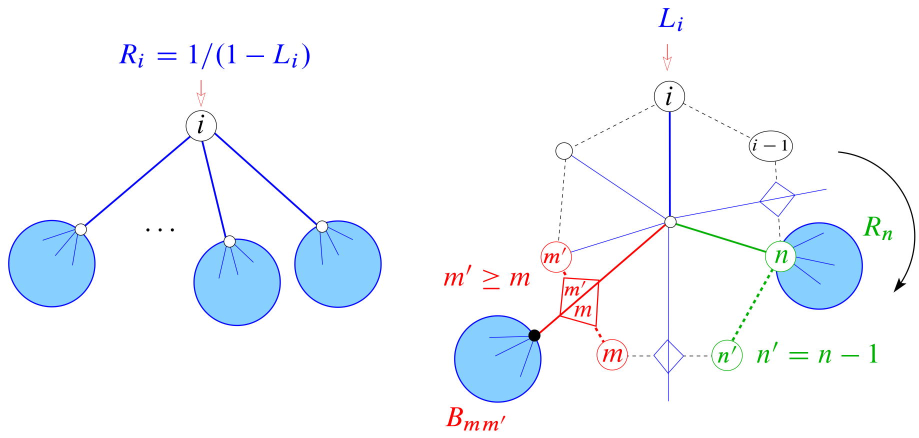

Figure 7. Left: the mobiles enumerated by R_i form a sequence of mobiles planted at a univalent labeled vertex with label i, counted by L_i. This implies the relation $R_i = 1/(1 - L_i)$. Right: a schematic picture of the relation $L_i = \sum_{k \geq 1} g_k (P^{k-1})_{i-1,i}$.

environment is coded by path of length $k - 1$ from height $i - 1$ to height i and the sum over all path configurations is therefore now counted by $(P^{k-1})_{i-1,i}$. We deduce[3]

$$R_i = 1 \Big/ \Big(1 - \sum_{k \geq 1} g_k (P^{k-1})_{i-1,i}\Big) \tag{2.4}$$

for $i \geq 1$. It is interesting to note that R_i may be alternatively obtained by another equation, now depending explicitly on the $\tilde{g}_k$'s instead, namely,

$$R_i = 1 + \sum_{k \geq 1} \tilde{g}_k (Q^{k-1})_{i,i-1} \tag{2.5}$$

for $i \geq 1$. This equation may be understood combinatorially as follows: $R_i - 1$ enumerates mobiles with a marked corner at a vertex with label i. If we call i_{min} the minimum label of any such mobile, this mobile is associated with a pointed Eulerian map having a marked oriented edge e pointing from a vertex v with label $i - 1$ to a vertex v' with label i under the labeling by oriented distances from the root vertex plus i_{min}. This oriented edge selects in turn a bud in the mobile, namely, the bud incident to the black vertex at the center of the black face incident to e and pointing towards e. We may, therefore, *re-root* the mobile at this bud (see Figure 3 for an example of such re-rooting) and recover $R_i - 1$ by enumerating these bud-rooted mobiles. The

[3]This equation is nothing but [5, (3.1) and (3.2)] with the correspondence $(Q, \widetilde{Q}) \to (P, Q)$.

environment of the black vertex incident to the root bud, of arbitrary degree k (hence, weighted by $\tilde{g}_k$) is coded by a path of length $k-1$ from height i to height $i-1$ (as obtained by reading the surrounding labels clockwise around the associated black face in the map, starting from the distinguished bud). Such environments are enumerated by $(Q^{k-1})_{i,i-1}$, leading directly to the above expression for R_i.

2.3. Computation of $[P, Q]$

Equations (2.2)–(2.3) may be rephrased as follows:

$$\Big(P - \sum_{k\geq 1} \tilde{g}_k Q^{k-1}\Big)_+ = 0, \qquad \Big(Q - \sum_{k\geq 1} g_k P^{k-1}\Big)_- = 0,$$

where $(\cdot)_+$ (resp., $(\cdot)_-$) denotes the upper (resp., lower) part of the matrix at hand, namely, the upper (resp., lower triangular) matrix obtained by keeping only those elements $(\cdot)_{i,j}$ with $j \geq i$ (resp., $i \geq j$). This implies that $(P - \sum_k \tilde{g}_k Q^{k-1})$ is a *strictly* lower triangular matrix, whose commutator with Q is therefore a lower triangular matrix. We deduce that $[P, Q]$ is a lower triangular matrix. Similarly, $(Q - \sum_k g_k P^{k-1})$ is a *strictly* upper triangular matrix, whose commutator with P is thus an upper triangular matrix. We deduce that $[P, Q]$ is also an upper triangular matrix, hence, it is diagonal. We arrive at the following property.

Proposition 2.1. *$[P, Q]$ is a diagonal matrix with diagonal elements*

$$[P, Q]_{i,i} = -\delta_{i,0} = -(e_0 e_0^t)_{i,i}, \tag{2.6}$$

where e_k is the vector whose coordinates are $(e_k)_j = \delta_{k,j}$.

Proof. The diagonal elements may be computed via

$$\begin{aligned}
&[P, Q]_{i,i} \\
&= \Big[P - \sum_{k\geq 1} \tilde{g}_k Q^{k-1}, Q\Big]_{i,i} \\
&= \begin{cases} \Big(P - \sum\limits_{k\geq 1} \tilde{g}_k Q^{k-1}\Big)_{i,i-1} Q_{i-1,i} - Q_{i,i+1}\Big(P - \sum\limits_{k\geq 1} \tilde{g}_k Q^{k-1}\Big)_{i+1,i} & \text{for } i \geq 1 \\ -Q_{0,1}\Big(P - \sum\limits_{k\geq 1} \tilde{g}_k Q^{k-1}\Big)_{1,0} & \text{for } i = 0 \end{cases} \\
&= \begin{cases} \Big(R_i - \sum\limits_{k\geq 1} \tilde{g}_k (Q^{k-1})_{i,i-1}\Big) - \Big(R_{i+1} - \sum\limits_{k\geq 1} \tilde{g}_k (Q^{k-1})_{i+1,i}\Big) & \text{for } i \geq 1 \\ -\Big(R_1 - \sum\limits_{k\geq 1} \tilde{g}_k (Q^{k-1})_{1,0}\Big) & \text{for } i = 0 \end{cases} \\
&= -\delta_{i,0},
\end{aligned}$$

where we used (2.5). ∎

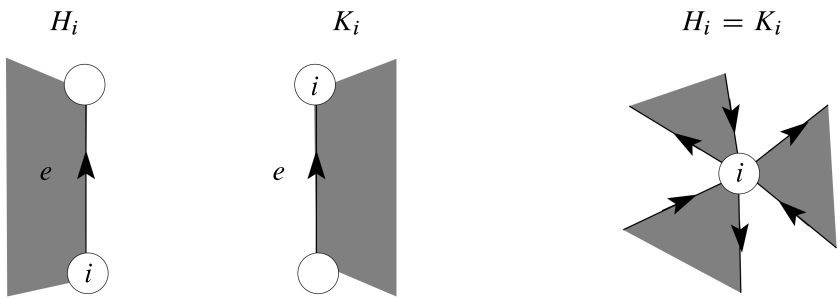

Figure 8. Schematic picture of H_i and K_i (left) and of the relation $H_i = K_i$ (right).

We could have obtained the same result via

$$
\begin{aligned}
&[P,Q]_{i,i}\\
&= \Big[P, Q - \sum_{k\geq 1} g_k P^{k-1}\Big]_{i,i}\\
&= \begin{cases} P_{i,i-1}\Big(Q - \sum\limits_{k\geq 1} g_k P^{k-1}\Big)_{i-1,i} - \Big(Q - \sum\limits_{k\geq 1} g_k P^{k-1}\Big)_{i,i+1} P_{i+1,i} & \text{for } i \geq 1\\ -P_{0,1}\Big(Q - \sum\limits_{k\geq 1} k g_k P^{k-1}\Big)_{1,0} & \text{for } i = 0 \end{cases}\\
&= \begin{cases} R_i\Big(1 - \sum\limits_{k\geq 1} g_k (P^{k-1})_{i-1,i}\Big) - R_{i+1}\Big(1 - \sum\limits_{k\geq 1} g_k (P^{k-1})_{i,i+1}\Big) & \text{for } i \geq 1\\ -R_1\Big(1 - \sum\limits_{k\geq 1} g_k (P^{k-1})_{0,1}\Big) & \text{for } i = 0 \end{cases}\\
&= -\delta_{i,0}
\end{aligned}
$$

upon using now (2.4).

Combinatorial interpretation

Interestingly enough, equation (2.6) has a nice combinatorial explanation as follows: consider the generating function H_i of pointed Eulerian maps (with weights g_k – resp., $\tilde{g}_k$ – per white face – resp., black face of degree k) with a *marked oriented edge e pointing away from a vertex at distance i from the root vertex* (see Figure 8). The marking of this edge is equivalent, in the associated mobile, to either the marking of a flagged edge with labels i and $j \leq i$ if the extremity of e is at distance $j \leq i$ from the root vertex, or to the marking of a corner at a labeled vertex with label $i + 1$ if the extremity of e is at distance $i + 1$ from the root vertex. The first case corresponds to mobiles enumerated by $\sum_{j\leq i} W_{i,j} B_{j,i}$ if we ignore the constraint that the mobiles in bijection with maps must have their minimum label equal to 0, while the second case corresponds to mobiles enumerated by $R_{i+1} - 1$ (recall that R_{i+1} incorporates a conventional term 1 which does not account for any acceptable mobile). The constraint of having minimum label 0 is automatic if $i = 0$ but, if $i \geq 1$, we must eliminate those mobiles having a minimum label larger than 1. Starting from these mobiles, we still get an acceptable mobile by shifting all the labels by -1. In other words, the mobiles to be subtracted are enumerated by $\sum_{j \leq i-1} W_{i-1,j} B_{j,i-1} + R_i - 1$. We end up with

$$H_i = \begin{cases} \Big(R_{i+1} - 1 + \sum\limits_{j \leq i} W_{i,j} B_{j,i}\Big) - \Big(R_i - 1 + \sum\limits_{j \leq i-1} W_{i-1,j} B_{j,i-1}\Big) & \text{for } i \geq 1 \\ R_1 - 1 + \sum\limits_{j \leq 0} W_{0,j} B_{j,0} & \text{for } i = 0 \end{cases}$$
$$= \begin{cases} (QP)_{i,i} - (QP)_{i-1,i-1} & \text{for } i \geq 1 \\ (QP)_{0,0} - 1 & \text{for } i = 0. \end{cases}$$

Consider now the generating function K_i of pointed Eulerian maps with a *marked oriented edge e pointing towards a vertex at distance i from the root vertex* (see Figure 8). The marking of this edge is now equivalent, in the associated mobile, to either the marking of a flagged edge with labels i and $j \geq i$ if the origin of e is at distance $j \geq i$ from the root vertex, or to the marking of a corner at a labeled vertex with label i if the origin of e is at distance $i - 1$ from the root vertex. The first case corresponds to mobiles enumerated by $\sum_{j \geq i} B_{i,j} W_{j,i}$ if we ignore the constraint that the desired mobiles must have their minimum label equal to 0, while the second case corresponds to mobiles enumerated by $R_i - 1$. Note that this second case is possible only if $i \geq 1$. The constraint of having minimum label 0 is taken into account as above and we end up with

$$K_i = \begin{cases} \Big(R_i - 1 + \sum\limits_{j \geq i} B_{i,j} W_{j,i}\Big) - \Big(R_{i-1} - 1 + \sum\limits_{j \geq i-1} B_{i-1,j} W_{j,i-1}\Big) & \text{for } i \geq 2 \\ \Big(R_1 - 1 + \sum\limits_{j \geq 1} B_{1,j} W_{j,1}\Big) - \Big(\sum\limits_{j \geq 0} B_{0,j} W_{j,0}\Big) & \text{for } i = 1 \\ \sum\limits_{j \geq 0} B_{0,j} W_{j,0} & \text{for } i = 0 \end{cases}$$
$$= \begin{cases} (PQ)_{i,i} - (PQ)_{i-1,i-1} & \text{for } i \geq 2 \\ (PQ)_{1,1} - (PQ)_{0,0} - 1 & \text{for } i = 1 \\ (PQ)_{0,0} & \text{for } i = 0. \end{cases}$$

Now, on an Eulerian map with its canonical edge orientation, and for each vertex on the map, there is an equal number of edges oriented away from this vertex and towards this vertex. Indeed, the orientations alternate when turning around the vertex (whose degree is even by definition – see Figure 8). We deduce, in particular, that

$$H_i = K_i.$$

Equating the above expressions for H_i and K_i, we obtain

$$\begin{aligned}[P,Q]_{i,i} - [P,Q]_{i-1,i-1} &= 0 \quad \text{for } i \geq 2,\\ [P,Q]_{1,1} - [P,Q]_{0,0} &= 1,\\ [P,Q]_{0,0} &= -1,\end{aligned}$$

from which (2.6) follows immediately.

2.4. Mobiles with bounded degrees

From now on, we will assume that our mobiles have *black and white vertices with bounded degrees*. More precisely, we impose that the degree of a black vertex is bounded by p and that of a white vertex is bounded by q, where p and q are fixed integers such that $(p-1)(q-1) > 1$. Clearly, those mobiles code for Eulerian maps with bounded *face degrees*, with black faces of degree at most p, and white faces of degree at most q. Note that the degrees of the labeled vertices in the mobiles remain unbounded.

In practice, the degree restriction is achieved in the generating functions R_i, B_{ij}, and W_{ij} by keeping non-vanishing g_k's only for $1 \leq k \leq q$ and non-vanishing $\tilde{g}_k$'s only for $1 \leq k \leq p$. From now on, it is thus implicitly assumed that

$$R_i := R_i(g_1, \ldots, g_q, \tilde{g}_1, \ldots, \tilde{g}_p) \quad \text{for } i \geq 1,$$

and similarly, for B_{ij} $(j \geq i \geq 0)$ and W_{ij} $(i \geq j \geq 0)$.

From (2.2)–(2.3), we immediately see that $B_{i,j} = 0$ for $j > i + p - 1$, while $W_{ij} = 0$ for $j < i - q + 1$: the matrix P, therefore, has one band below the diagonal (with entries R_i) and p bands on or above the diagonal (with entries $B_{i,i+k}$, $0 \leq k \leq p-1$), and the matrix Q has one band above the diagonal (with entries 1) and q bands on or below the diagonal (with entries $W_{i,i-k}$, $0 \leq k \leq q-1$). From their interpretation as mobile generating functions, all these non-vanishing entries are formal power series of the coupling constants g_k and $\tilde{g}_k$ with non-negative integer coefficients.

Recall that, by definition, the minimum label in a mobile or a half-mobile is non-negative: let us denote by $R_i^{(0)}$ (respectively, $W_{ij}^{(0)}$ and $B_{ij}^{(0)}$) the generating function of those mobiles (respectively, white and black half-mobiles) in the set enumerated by R_i (respectively, by W_{ij} and by B_{ij}) and *whose minimum label is* 0. We note that, in a mobile in the set enumerated by R_i $(i \geq 2)$ and whose minimum label is *positive*, we may shift all labels by -1 and get a mobile in the set enumerated by R_{i-1}: we deduce the relations

$$R_1 = R_1^{(0)} + 1,$$
$$R_i = R_i^{(0)} + R_{i-1} \quad \text{for } i \geq 2.$$

Similarly, we have

$$W_{i,i-k} = W_{i,i-k}^{(0)} + W_{i-1,i-1-k} \quad \text{for } i \geq k+1 \text{ and } 0 \leq k \leq q-1,$$
$$B_{i,i+k} = B_{i,i+k}^{(0)} + B_{i-1,i-1+k} \quad \text{for } i \geq 1 \text{ and } 0 \leq k \leq p-1.$$

Now, it is easily seen that if we fix the numbers m_ℓ $(1 \leq \ell \leq q)$ of white vertices of degree ℓ of and the numbers n_ℓ $(1 \leq \ell \leq p)$ of black vertices of degree ℓ in a mobile, the maximal difference of labels in this mobile is *bounded*. This implies that, *for i large enough*,

$$\left[g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}\right] R_i^{(0)} = 0,$$
$$\text{i.e., } \left[g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}\right] R_i = \left[g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}\right] R_{i-1}.$$

More precisely, we have the following proposition.

Proposition 2.2. *For fixed non-negative integers m_ℓ $(1 \leq \ell \leq q)$ and n_ℓ $(1 \leq \ell \leq p)$, there exist a positive integer $i_0 := i_0(m_1, \ldots, m_q, n_1, \ldots, n_p)$ and a non-negative integer $r(m_1, \ldots, m_q, n_1, \ldots, n_p)$ such that*

$$\forall i \geq i_0, \quad \left[g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}\right] R_i = r(m_1, \ldots, m_q, n_1, \ldots, n_p).$$

Introducing the formal power series

$$\begin{aligned} R &:= R(g_1, \ldots, g_q, \tilde{g}_1, \ldots, \tilde{g}_p) \\ &= \sum_{\substack{m_1, \ldots, m_q \geq 0 \\ n_1, \ldots, n_p \geq 0}} r(m_1, \ldots, m_q, n_1, \ldots, n_p) g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}, \end{aligned}$$

we thus have, as formal power series,

$$\lim_{i \to \infty} R_i = R. \tag{2.7}$$

The same argument holds for black or white half-mobiles.

Proposition 2.3. *For fixed k, $0 \leq k \leq q-1$, and for fixed non-negative integers m_ℓ $(1 \leq \ell \leq q)$ and n_ℓ $(1 \leq \ell \leq p)$, there exist a positive integer $j_k := j_k(m_1, \ldots, m_q, n_1, \ldots, n_p)$ and a non-negative integer $a_k(m_1, \ldots, m_q, n_1, \ldots, n_p)$ such that*

$$\forall i \geq j_k, \quad \left[g_1^{m_1} \cdots g_q^{m_q} \tilde{g}_1^{n_1} \cdots \tilde{g}_p^{n_p}\right] W_{i,i-k} = a_k(m_1, \ldots, m_q, n_1, \ldots, n_p).$$

Introducing the formal power series

$$\alpha_k := \sum_{\substack{m_1,\ldots,m_q\geq 0\\ n_1,\ldots,n_p\geq 0}} a_k(m_1,\ldots,m_q,n_1,\ldots,n_p) g_1^{m_1}\cdots g_q^{m_q}\tilde{g}_1^{n_1}\cdots\tilde{g}_p^{n_p},$$

we thus have, as formal power series,

$$\lim_{i\to\infty} W_{i,i-k} = \alpha_k. \tag{2.8}$$

Proposition 2.4. *For fixed k,* $0 \leq k \leq p-1$, *and for fixed non-negative integers* m_ℓ $(1 \leq \ell \leq q)$ *and* n_ℓ $(1 \leq \ell \leq p)$, *there exist a positive integer* $\ell_k := \ell_k(m_1,\ldots,m_q, n_1,\ldots,n_p)$ *and a non-negative integer* $b_k(m_1,\ldots,m_q,n_1,\ldots,n_p)$ *such that*

$$\forall i \geq \ell_k, \quad \left[g_1^{m_1}\cdots g_q^{m_q}\tilde{g}_1^{n_1}\cdots\tilde{g}_p^{n_p}\right]B_{i+k,i} = b_k(m_1,\ldots,m_q,n_1,\ldots,n_p).$$

Introducing the formal power series

$$\beta_k := \sum_{\substack{m_1,\ldots,m_q\geq 0\\ n_1,\ldots,n_p\geq 0}} b_k(m_1,\ldots,m_q,n_1,\ldots,n_p) g_1^{m_1}\cdots g_q^{m_q}\tilde{g}_1^{n_1}\cdots\tilde{g}_p^{n_p},$$

we have, as formal power series,

$$\lim_{i\to\infty} B_{i,i+k} = \beta_k. \tag{2.9}$$

The formal power series R, α_k, and β_k are linked via the following identity.

Proposition 2.5. *We have*

$$R - 1 = \sum_{k=1}^{\min(q-1,p-1)} k\alpha_k\beta_k. \tag{2.10}$$

Proof. From the relation $R_i = R_i^{(0)} + R_{i-1}$, we deduce that, for $i \geq 0$, $R_{i+1} - 1 = \sum_{j=0}^{i} R_{j+1}^{(0)}$, and therefore, from the BDG bijection, $R_{i+1} - 1$ is the generating function for pointed Eulerian planar maps with a marked oriented edge of type $j \to j+1$ (after labeling vertices by their oriented distance from the root vertex) for some j satisfying $0 \leq j \leq i$. Taking the $i \to \infty$ limit, we deduce that $R - 1$ is the generating function for pointed Eulerian planar maps with a marked oriented edge of type $j \to j+1$ for arbitrary $j \geq 0$. By a similar argument, $W_{i,i-k}B_{i-k,i}$ enumerates pointed Eulerian planar maps with a marked oriented edge of type $j \to j-k$ for some j satisfying $k \leq j \leq i$. Taking the $i \to \infty$ limit, we deduce that $\alpha_k\beta_k$ is the generating function for pointed Eulerian planar maps with a marked oriented edge of type $j \to j-k$ for arbitrary $j \geq k$. Consider now a white face f in a pointed Eulerian map and denote by $t(f)$ the number of oriented edges incident to f that are of type $j \to j+1$ for some arbitrary unfixed j (i.e., those oriented edges along which the label increases by 1). Similarly, denote by $s_k(f)$ the number of oriented edges incident to f that are of type $j \to j-k$ for some arbitrary unfixed j (i.e., those oriented edges along which the label decreases by k). In order to recover the same label after one turn around f, we have the consistency relation $t(f) = \sum_{k=1}^{\min(p-1,q-1)} k\, s_k(f)$. Summing over all white faces f and all pointed maps, the left-hand side adds up to $R-1$ (enumerating pointed maps with a marked edge along which the label increases) while the right-hand side adds up to $\sum_{k=1}^{\min(q-1,p-1)} k\alpha_k\beta_k$ (pointed maps with a marked edge along which the label decreases by k, weighted by k and summed over k). The desired identity (2.10) follows immediately. ■

2.5. Mobiles with a weight g per labeled vertex

We will specialize to the case where $g_1 = \tilde{g}_1 = 0$ and

$$\begin{aligned} g_k &= g^{\frac{k-2}{2}}\lambda_k, \quad k = 2,\ldots,q, \\ \tilde{g}_k &= g^{\frac{k-2}{2}}\tilde{\lambda}_k, \quad k = 2,\ldots,p. \end{aligned} \tag{2.11}$$

In the Eulerian map language, this corresponds to forbidding faces of degree 1 and giving a weight λ_k (resp., $\tilde{\lambda}_k$) per white (resp., black) face of degree k. Assuming that there are n_k (resp., m_k) such faces, we have an overall factor $g^{\sum_{k\ge 2}(\frac{k-2}{2})(n_k+m_k)} = g^{E-F} = g^{V-2}$ if E, F, V are the total numbers of edges, faces, and vertices. Here, we used

$$\sum_{k\ge 2}\frac{k-2}{2}(n_k+m_k) = \frac{1}{2}\sum_{k\ge 2}(k\,n_k + k\,m_k) - \sum_{k\ge 2}(n_k+m_k) = \frac{1}{2}(2E) - F,$$

and the Euler relation $V - E + F = 2$ for planar maps. In the mobile language this correspond to assigning the weight λ_k (resp., $\tilde{\lambda}_k$) per white (resp., black) vertex of degree k (where the degree incorporates the number of buds for black vertices), together with an overall factor $g^{\mathcal{V}-1}$, where $\mathcal{V}$ is total number of labeled vertices in the mobile. Recall that we have indeed $\mathcal{V} = V-1$ since the root vertex of the (pointed) map is absent from the mobile.

Let us now show the following proposition.

Proposition 2.6. *Using the scaling* (2.11)*, equations* (2.3) *and* (2.5) *have a unique solution for which* R_i $(i \ge 1)$*,* $W_{i,j}$ $(i \ge j \ge 0)$*, and* $B_{i,j}$ $(j \ge i \ge 0)$ *are formal power series in* $\sqrt{g}$*.*

This solution is precisely the one that we are looking for when we are interested in the enumeration of mobiles.

Proof. The proof is given in Appendix A. ∎

From its definition as a mobile generating function, we can furthermore infer that R_i has an expansion in integer powers of g whose general term is polynomial in $\lambda_2,\dots,\lambda_q,\tilde{\lambda}_2,\dots,\tilde{\lambda}_q$ and $\frac{1}{1-\tilde{\lambda}_2\lambda_2}$ with non-negative integer coefficients. Here, the combination $\frac{1}{1-\tilde{\lambda}_2\lambda_2}$ corresponds to sequences of arbitrary length made of alternating bivalent black and white vertices, connected by flagged edges having all the same labels. Indeed, the weight of such sequences is independent of g since bivalent vertices have weight $g_2=\lambda_2$ or $\tilde{g}_2=\tilde{\lambda}_2$. In the case of half-mobile generating functions $B_{i,j}$ or $W_{i,j}$, it is a simple exercise to show that their overall power in $\sqrt{g}$ is

$$\sqrt{g}^{\,j-i-1+2\mathcal{V}},$$

where $\mathcal{V}$ is now the total number of labeled vertices in the half-mobile. In particular, if $j-i$ is even then the expansion of $B_{i,j}$ and $W_{i,j}$ has only half-integer powers of g while if $j-i$ is odd it has only integer powers. Again, the general terms in these expansions are polynomials in $\lambda_2,\dots,\lambda_q,\tilde{\lambda}_2,\dots,\tilde{\lambda}_q$ and $\frac{1}{1-\tilde{\lambda}_2\lambda_2}$ with non-negative integer coefficients.

Remark 2.1. To summarize, for $g_1=\tilde{g}_1=0$, it will be enough to show that the expressions that we obtain for the generating functions R_i, $B_{i,j}$, and $W_{i,j}$ have power series expansions in $\sqrt{g}$ for the scaling (2.11) to guarantee that they also have power series expansions in all the g_k's and $\tilde{g}_k$'s for $k\geq 2$ with positive integer coefficients.

3. Integrable systems

From the preceding section, the enumeration of mobiles in the restricted context of bounded degrees boils down to finding the solution of the following problem dictated by equations (2.2) to (2.5): find two semi-infinite band matrices, $(Q_{n,m})_{n,m\geq 0}$ having one band[4] above and $q-1$ bands below diagonal, and $(P_{n,m})_{n,m\geq 0}$ having one band below and $p-1$ bands above diagonal such that

$$\begin{aligned} &P_{n,n-1}=R_n, && Q_{n,n+1}=1,\\ &\Big(P-\sum_{k=0}^{p-1}\tilde{g}_{k+1}Q^k\Big)_+=0, && \Big(Q-\sum_{k=0}^{q-1}g_{k+1}P^k\Big)_-=0,\\ &\Big(P-\sum_{k=0}^{p-1}\tilde{g}_{k+1}Q^k\Big)_{n,n-1}=1, && \Big(Q-\sum_{k=0}^{q-1}g_{k+1}P^k\Big)_{n,n+1}=\frac{1}{R_{n+1}}. \end{aligned} \tag{3.1}$$

[4]Recall that the kth band of a matrix M is the set of elements $M_{i,i+k}$ for some fixed integer k. A semi-infinite band matrix is a matrix with non-zero elements only in the kth bands for some bounded set of integers k.

Note that this algebraic problem could possibly have many solutions, but combinatorics of mobiles guarantees that there is a unique one such that $P_{n,m}$ and $Q_{n,m}$ are formal power series of the coupling constants g_k and $\tilde{g}_k$. Moreover, this solution is such that R_n has a limit as $n \to \infty$.

As we have seen in Proposition 2.1, these relations imply that Q and P quasi-commute, namely,

$$[P, Q] = -e_0 e_0^t,$$

which is a Lax equation, much studied in the literature of integrable systems, and whose general solution [2, 13–15] is expressed through algebraic geometry. However, we cannot directly use the general solution here because the mobiles correspond to highly degenerate initial conditions, and although the method is very similar to the general solution of [13–15], a new proof is required here in the context of mobiles.

Our approach uses the general framework of integrable systems: in the present section, we will explain in a sketchy way how necessary conditions lead us to some isospectral integrable system, in Lax form, and Baker–Akhiezer functions. Then, we will proceed backwards in Section 4 and show that specific Baker–Akhiezer functions indeed produce a solution to our combinatorial problem of counting mobiles.

The reader who is not interested in knowing how the solution emerges from the general framework of integrable systems may skip this section and go directly to Section 4.

3.1. Isospectral system

We will define the notion of semi-infinite vector $\vec{\psi}$, which is an eigenvector for the semi-infinite band matrices. We will introduce the nth window $\mathcal{W}_n$ in order to compute the dimension of the eigenvector space and finally prove that the spectrum of the Lax matrix is independent of n, which corresponds to the *isospectral property*.

Let us describe a number of general facts concerning band matrices.

Definition 3.1. Let $A = (A_{i,j})_{i,j \geq 0}$ be a semi-infinite band matrix with a_+ lines above diagonal and a_- lines below. A semi-infinite vector $\vec{\psi}$ is called a right (resp., left) eigenvector of A with eigenvalue x iff

$$A\vec{\psi} = x\vec{\psi} \quad (\text{resp., } \vec{\psi}^t A = x\vec{\psi}^t).$$

We will denote by $\mathcal{V}_x(A)$ (resp., $\tilde{\mathcal{V}}_x(A)$) the vector space of right (resp., left) eigenvectors of A for the eigenvalue x.

We have the following theorem.

Theorem 3.1. *If the upper-most and lower-most diagonals of A have non-vanishing entries, then for any $x \in \mathbb{C}$, the space $\mathcal{V}_x(A)$ (resp., $\tilde{\mathcal{V}}_x(A)$) of right (resp., left)* *eigenvectors of A for the eigenvalue x is a vector space of dimension $a_+ + a_-$, namely,*

$$\dim \mathcal{V}_x(A) = \dim \widetilde{\mathcal{V}}_x(A) = a_+ + a_-.$$

More precisely, let us call the set of $a_+ + a_-$ consecutive integers $\mathcal{W}_n = \{n - a_-, \dots, n + a_+ - 1\}$ the nth "window". Take $n \geq a_-$. Let $\vec{\psi}$ be a right eigenvector; then, the map

$$\mathcal{L}_{\mathcal{W}_n}(x) : \mathcal{V}_x(A) \to \mathbb{C}^{\mathcal{W}_n}, \qquad \vec{\psi} \mapsto \{\psi_i\}_{i \in \mathcal{W}_n}$$

is an isomorphism.

The map $\Lambda_n(x) := \mathcal{L}_{\mathcal{W}_{n+1}}(x)\mathcal{L}^{-1}_{\mathcal{W}_n}(x)$ from $\mathbb{C}^{\mathcal{W}_n}$ to $\mathbb{C}^{\mathcal{W}_{n+1}}$ is a companion matrix[5] *of size $a_+ + a_-$, linear in x. More precisely, we have for $n \geq a_-$:*

$$\Lambda_n(x) = \mathcal{L}_{\mathcal{W}_{n+1}}(x)\mathcal{L}^{-1}_{\mathcal{W}_n}(x) = \begin{pmatrix} 0 & 1 & 0\cdots & & \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & 0 & & 1 \\ \frac{-A_{n,n-a_-}}{A_{n,n+a_+}} & \cdots & \frac{x - A_{n,n}}{A_{n,n+a_+}} & \cdots & \frac{-A_{n,n+a_+-1}}{A_{n,n+a_+}} \end{pmatrix}. \tag{3.2}$$

Proof. If $\vec{\psi}$ is a right eigenvector of A for the eigenvalue x, i.e., $A\vec{\psi} = x\vec{\psi}$, then for every $m > 0$, we have

$$\psi_{m-1} = \frac{1}{A_{m-1+a_-,m-1}}\left(x\psi_{m-1+a_-} - \sum_{j=0}^{a_++a_--1} A_{m-1+a_-,m+j}\,\psi_{m+j}\right), \tag{3.3}$$

and, for every $m \geq a_+ + a_- - 1$, we have

$$\psi_{m+1} = \frac{1}{A_{m+1-a_+,m+1}}\left(x\psi_{m+1-a_+} - \sum_{j=0}^{a_++a_--1} A_{m+1-a_+,m-j}\,\psi_{m-j}\right). \tag{3.4}$$

Using these equations, we see that every ψ_m can be expressed as a linear combination of ψ_i with i restricted to the nth window $\mathcal{W}_n$. Indeed, using equation (3.3) for $m = n - a_-$ shows that ψ_{n-a_--1} is a linear combination of ψ_i with $i \in \mathcal{W}_n$, and by recursion, all ψ_m for $m < n - a_-$ are also linear combinations of the nth window elements. Similarly, taking $m = n + a_+ - 1$ in equation (3.4) shows that ψ_{n+a_+} is a linear combination of elements in the nth window and by recursion the same property holds for all $m > n + a_+ - 1$. This implies that the vector space $\mathcal{V}_x(A)$ of eigenvectors of

[5] We call a companion matrix a matrix whose elements are equal to 1 in the superdiagonal are arbitrary in the last row and with all their other entries equal to zero.

A for the eigenvalue x is isomorphic to $\mathbb{C}^{\mathcal{W}_n}$, hence, it has dimension $a_+ + a_-$. The same is true for left eigenspaces.

The fact that $\Lambda_n(x)$ is a companion matrix of the form (3.2) is obtained by a simple computation. ∎

As an application, since Q is a band matrix with 1-band above and $q-1$ bands below diagonal, we have

$$\dim \mathcal{V}_x(Q) = \dim \widetilde{\mathcal{V}}_x(Q) = q.$$

Similarly, P is a band matrix with $p-1$ bands above and 1 band below diagonal, hence, we have

$$\dim \mathcal{V}_y(P) = \dim \widetilde{\mathcal{V}}_y(P) = p.$$

Let us now discuss the action of P on $\mathcal{V}_x(Q)$. We have the following theorem.

Theorem 3.2. *If $\vec{\psi} \in \mathcal{V}_x(Q)$, then $\vec{\phi} = P\vec{\psi}$ satisfies*

$$Q\vec{\phi} = x\vec{\phi} + \psi_0 e_0,$$

i.e.,

$$\forall n \geq 0, \quad \sum_{m\geq 0} Q_{n,m}\phi_m = x\phi_n + \psi_0\delta_{n,0}.$$

The map

$$\pi : P\mathcal{V}_x(Q) \to \mathcal{V}_x(Q),$$
$$\vec{\phi} \mapsto \mathcal{L}_{\mathcal{W}_n}(x)^{-1}(\phi_{n+1-q}, \dots, \phi_{n-1}, \phi_n)$$

is well defined, and independent of n for $n \geq q$. The map $\pi \circ P$ is an endomorphism of $\mathcal{V}_x(Q)$.

Proof. Let $\vec{\psi} \in \mathcal{V}_x(Q)$; then, we have

$$QP\vec{\psi} = PQ\vec{\psi} + [Q,P]\vec{\psi} = P(x\vec{\psi}) + \psi_0 e_0 = xP\vec{\psi} + \psi_0 e_0.$$

If $\vec{\phi} = P\vec{\psi}$, we thus have

$$\forall n \geq 0, \quad \sum_{m\geq 0} Q_{n,m}\phi_m = x\phi_n + \psi_0\delta_{n,0}.$$

Let $n \geq q$, and let $\vec{\hat{\phi}} = \mathcal{L}_{\mathcal{W}_n}(x)^{-1}(\phi_{n+1-q}, \dots, \phi_{n-1}, \phi_n)$. By definition of $\mathcal{L}_{\mathcal{W}_n}(x)$, we have $\vec{\hat{\phi}} \in \mathcal{V}_x(Q)$. Since $n \geq q$ then $0 \notin \mathcal{W}_n$, and thus, $\hat{\phi}_m = \phi_m$ for all $m \in \mathcal{W}_n$. Moreover, $\hat{\phi}_m$ and ϕ_m satisfy the same recursion relation of order q for all $m > 0$. In particular, they coincide in any other window that does not contain 0. This shows that π is independent of n for all $n \geq q$. ∎

Definition 3.2. For $n \geq q$, we may write $\pi \circ P$ in the canonical basis of $\mathbb{C}^{\mathcal{W}_n}$ as a $q \times q$ matrix, polynomial in x:

$$\mathcal{D}_n(x) := \mathcal{L}_{\mathcal{W}_n}(x)(\pi \circ P)\mathcal{L}_{\mathcal{W}_n}(x)^{-1}.$$

More explicitly, we have

$$\begin{aligned}\mathcal{D}_n(x) &\\ &= \mathrm{diag}(R_{n-q+1},\dots,R_n)\Lambda_{n-1}(x)^{-1} + \mathrm{diag}(P_{n-q+1,n-q+1},\dots,P_{n,n}) \\ &\quad + \sum_{j=1}^{p-1} \mathrm{diag}(P_{n-q+1,n-q+1+j},\dots,P_{n,n+j})\big(\Lambda_{n+j-1}(x)\cdots\Lambda_{n+1}(x)\Lambda_n(x)\big),\end{aligned} \tag{3.5}$$

with $\Lambda_n(x)$ as in equation (3.2) for $A = Q$, $a_+ = 1$ and $a_- = q-1$, namely,

$$\Lambda_n(x) = \begin{pmatrix} 0 & 1 & 0\cdots & \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -Q_{n,n-q+1} & -Q_{n,n-q+2} & \cdots & x - Q_{n,n} \end{pmatrix}. \tag{3.6}$$

Example 3.1 (Quadrangulations ($q = 2$, $p = 4$, and $g_1 = \tilde{g}_1 = \tilde{g}_3 = 0$)). We have

$$\mathcal{D}_n(x) = \begin{pmatrix} x\left(\frac{R_{n-1}}{Q_{n-1,n-2}} - Q_{n,n-1}P_{n-1,n+2}\right) & P_{n-1,n+2}(x^2 - Q_{n+1,n}) + P_{n-1,n} - \frac{R_{n-1}}{Q_{n-1,n-2}} \\ -P_{n,n+3}Q_{n,n-1}(x^2 - Q_{n+2,n+1}) - P_{n,n+1}Q_{n,n-1} + R_n & xP_{n,n+3}(x^2 - Q_{n+2,n+1} - Q_{n+1,n}) + P_{n,n+1}x \end{pmatrix}. \tag{3.7}$$

Lemma 3.1. *Changing the window $n \to n+1$ amounts to performing the conjugation*

$$\mathcal{D}_{n+1}(x) = \Lambda_n(x)\mathcal{D}_n(x)\Lambda_n(x)^{-1}. \tag{3.8}$$

Theorem 3.3 (Isospectral system). *The eigenvalues of $\mathcal{D}_n(x)$ are independent of n for $n \geq q$.*

Proof. This is an immediate consequence of the previous lemma. ■

Equation (3.8) is a discrete *Lax equation*[6], and the matrices $(\mathcal{D}_n(x), \Lambda_n(x))$ form a *Lax pair* with the discrete time n and spectral parameter x. $\mathcal{D}_n(x)$ is an *isospectral* Lax matrix (meaning that its spectrum is independent of n). We start entering the realm of integrable systems.

[6] In a continuous time t, the Lax equation would be $\partial_t \mathcal{D} = [\mathcal{L}', \mathcal{D}]$, which implies that the eigenvalues are conserved, namely, $\partial_t \det(y - \mathcal{D}(x,t)) = 0$. Here, we have instead its discrete time analog.

3.2. Spectral curve

In this subsection, we will study the fundamental aspects of integrable systems. Starting with the spectral curve of our system, we introduce the notion of Newton polytope for a more geometric view of the spectral curve. We then extract the conserved quantities as elements of the Newton polytope that lie inside the convex hull.

Definition 3.3 (Spectral curve). Let $n \geq q$. We define the spectral curve as the characteristic polynomial of the endomorphism $\pi \circ P$, namely,

$$\mathcal{E}(x, y) := \det\big(y \,\mathrm{Id}_{\mathcal{V}_x(Q)} - \pi \circ P_{\mathcal{V}_x(Q)}\big) = \det(y \,\mathrm{Id}_{q\times q} - \mathcal{D}_n(x)). \tag{3.9}$$

The spectral curve is the locus of the eigenvalues of Q and $\pi \circ P$ for their common eigenvectors

$$\mathcal{V}_x(Q) \cap \mathcal{V}_y(\pi \circ P) \neq \{0\} \Longleftrightarrow \mathcal{E}(x, y) = 0.$$

Notice that $\mathcal{E}(x, y)$ is a polynomial of x and y, and it is independent of n since $n \geq q$.

Definition 3.4. From equations (2.3) and (2.7) to (2.9), the following limits exist for the combinatorial solutions that we are looking for:

$$\alpha_k := \lim_{n\to\infty} Q_{n,n-k}$$

and

$$\beta_k := \lim_{n\to\infty} P_{n,n+k}, \quad R := \lim_{n\to\infty} R_n.$$

We define the following Laurent polynomials $\in \mathbb{C}[z, 1/z]$:

$$X(z) := z + \sum_{i=0}^{q-1} \alpha_i z^{-i}, \quad Y(z) := \frac{R}{z} + \sum_{i=0}^{p-1} \beta_i z^i. \tag{3.10}$$

Definition 3.5 (Potentials). We also define the following polynomials, called the "potentials":

$$\tilde{V}(x) := \sum_{k=1}^{p} \frac{\tilde{g}_k}{k} x^k, \quad V(y) := \sum_{k=1}^{q} \frac{g_k}{k} y^k. \tag{3.11}$$

Theorem 3.4. *The α_i, β_i can be found as follows: equations (3.1) imply that the generating functions $X(z)$ and $Y(z)$ satisfy the system of algebraic equations*

$$\begin{gathered}\tilde{V}'(X(z))_+ = Y(z)_+, \quad V'(Y(z))_- = X(z)_-,\\ Y(z) - \tilde{V}'(X(z)) \underset{z\to\infty}{\sim} \frac{1}{z} + O(1/z^2),\\ X(z) - V'(Y(z)) \underset{z\to 0}{\sim} \frac{z}{R} + O(z^2).\end{gathered} \tag{3.12}$$

Here, the notations $(\cdot)_+$ and $(\cdot)_-$ refer to the non-negative and non-positive parts in the expansion in the variable z. The last two conditions are not independent, they can be obtained by computing

$$-\mathrm{Res}_{z\to\infty} Y dX = \mathrm{Res}_{z\to 0} Y dX = 1,$$

and they can be reformulated as follows:

$$\sum_{k=1}^{\min(q-1,p-1)} k\alpha_k\beta_k = R-1. \tag{3.13}$$

This identity is nothing but equation (2.10), whose combinatorial interpretation was given in Section 2.4.

Example 3.2 (Quadrangulations ($q=2$, $p=4$, and $g_1=\tilde{g}_1=\tilde{g}_3=0$)). Take $\widetilde{V}(x)=\frac{\tilde{g}_4}{4}x^4+\frac{\tilde{g}_2}{2}x^2$ and $V(y)=\frac{g_2}{2}y^2$. We have

$$X(z)=z+\alpha_1 z^{-1}, \quad Y(z)=\frac{R}{z}+\beta_1 z+\beta_3 z^3.$$

They have to satisfy

$$\beta_3=\tilde{g}_4, \quad \beta_1=3\tilde{g}_4\alpha_1+\tilde{g}_2, \quad \alpha_1=Rg_2, \quad \alpha_1\beta_1=R-1.$$

This gives

$$Rg_2(\tilde{g}_2+3\tilde{g}_4Rg_2)=R-1,$$

and thus,

$$R=\frac{1}{6\tilde{g}_4g_2^2}\left(1-\tilde{g}_2g_2-\sqrt{(1-\tilde{g}_2g_2)^2-12g_2^2\tilde{g}_4}\right).$$

Here, we choose the unique branch which is a power series of the coupling constants (i.e., the minus sign in front of the square-root), namely,

$$R=1+\tilde{g}_2g_2+\tilde{g}_2^2g_2^2+3\tilde{g}_4g_2^2+\cdots.$$

Substituting $Q_{n,m}\to\alpha_{n-m}$ and $P_{n,m}\to\beta_{m-n}$, in (3.5) and (3.6), we see that $\mathcal{D}_n(x)$ and $\Lambda_n(x)$ have large n limits:

$$\Lambda_\infty(x)=\begin{pmatrix} 0 & 1 & 0\cdots & \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -\alpha_{q-1} & -\alpha_{q-2} & \cdots & x-\alpha_0 \end{pmatrix}, \tag{3.14}$$

$$\mathcal{D}_\infty(x)=R\,\Lambda_\infty(x)^{-1}+\sum_{k=0}^{p-1}\beta_k\Lambda_\infty(x)^k. \tag{3.15}$$

Theorem 3.5 (Spectral curve as a resultant). $(x, y) \in \mathbb{C} \times \mathbb{C}$ *belongs to the spectral curve iff there exists* $z \in \mathbb{C}^*$ *such that* $x = X(z)$ *and* $y = Y(z)$*:*

$$\{(x, y) \mid \mathcal{E}(x, y) = 0\} = \{(X(z), Y(z)) \mid z \in \mathbb{C}^*\}.$$

Since it can be parametrized by a complex variable, this implies that the spectral curve is a genus zero Riemann surface. As a consequence, we have

$$\mathcal{E}(x, y) = \text{Resultant}_z(z^{q-1}(X(z) - x), z(Y(z) - y)),$$

namely,

$$\mathcal{E}(x, y) = \frac{1}{\alpha_{q-1}} \det \begin{pmatrix} 1 & \alpha_0 - x & \alpha_1 & \cdots & \alpha_{q-1} & & \\ & \ddots & \ddots & & & \ddots & \\ & & 1 & \alpha_0 - x & \alpha_1 & \cdots & \alpha_{q-1} \\ \beta_{p-1} & \cdots & \beta_0 - y & R & & & \\ & \ddots & & \ddots & \ddots & & \\ & & \beta_{p-1} & \cdots & \beta_0 - y & R & \\ & & & \beta_{p-1} & \cdots & \beta_0 - y & R \end{pmatrix} \tag{3.16}$$

with p lines in the upper part and q lines in the lower part.

Proof. Since the spectral curve $\mathcal{E}(x, y) = \det(y \,\mathrm{Id}_{q\times q} - \mathcal{D}_n(x))$ is independent of n, we can take $n \to \infty$, and $\mathcal{E}(x, y) = \det(y \,\mathrm{Id}_{q\times q} - \mathcal{D}_\infty(x))$. Because of the form of (3.15), it is easy to see that $\mathcal{E}(x, y) = 0$ iff there exists z such that $x = X(z)$ and $y = Y(z)$. In other words, (x, y) belongs to the spectral curve iff the two polynomials of z: $z^{q-1}(X(z) - x)$ and $z(Y(z) - y)$ admit a common root. By definition, the resultant of two polynomials vanishes if and only if they have a common zero. The determinant form (3.16) is the standard formula for the resultant apart from the factor $\frac{1}{\alpha_{q-1}}$ which we added to ensure that the leading coefficient in y is 1. (We briefly recall why (3.16) is the resultant: notice that if z is a common zero of $X(z) - x$ and $Y(z) - y$, then the vector $(z^{p+q-1}, \ldots, z, 1)$ is in the kernel of the resultant matrix, and vice versa, any vector in the kernel must be of that form. Furthermore, the resultant is a polynomial of x and y of the correct degree, which vanishes exactly when $\mathcal{E}(x, y)$ vanishes, it is thus proportional to $\mathcal{E}(x, y)$). The prefactor is determined by matching the large y behavior of $\mathcal{E}(x, y)$. ∎

Definition 3.6 (Newton polytope). Let us write

$$\mathcal{E}(x, y) = \sum_{(i,j)\in\mathcal{N}} \mathcal{E}_{i,j} x^i y^j,$$

where $\mathcal{N} \subset \{0,\ldots,p\} \times \{0,\ldots,q\}$ is a finite set of integer points in $\mathbb{Z}^2$, called the Newton polytope of $\mathcal{E}$.

We remark that

$$\mathcal{E}_{0,q} = 1, \quad \mathcal{E}_{p,0} = \frac{\tilde{g}_p}{g_q}, \quad \mathcal{E}_{i,q-1} = \frac{g_{q-1}\delta_{i,0} - \tilde{g}_{i+1}}{g_q},$$
$$\mathcal{E}_{p-1,j} = \frac{\tilde{g}_{p-1}\delta_{j,0} - g_{j+1}}{g_q}, \quad \mathcal{E}_{p-2,q-2} = \tilde{g}_p$$

are trivial functions of the coupling constants g_k and $\tilde{g}_k$.

The remaining $N = (p-1)(q-1) - 1$ coefficients $\mathcal{E}_{i,j}$ with $i \le p-2$ and $j \le q-2$ and $(i,j) \ne (p-2,q-2)$, are non-trivial combinations.

In general, the points $(i,j) \in \mathcal{N}$ are called as follows:

- if $(i+1, j+1)$ is on the boundary or outside of the convex envelope of $\mathcal{N}$, then $\mathcal{E}_{i,j}$ is called a Casimir. It is a rational fraction of the coefficients g_i, $\tilde{g}_i$,
- if $(i+1, j+1)$ is strictly inside the convex envelope of $\mathcal{N}$, then $\mathcal{E}_{i,j}$ is called a conserved Hamiltonian.

The general theory of Newton's polygons says that the slopes of sides of the convex envelope are related to poles of $X(z)$ and $Y(z)$. The two poles at $z = \infty$ and $z = 0$ with respective asymptotics $Y \sim \tilde{V}'(X)$ and $X \sim V'(Y)$ imply that the Newton's polygon has two sides with slopes $(1, 1-q)$ and $(p-1, -1)$.

All this implies that the spectral curve can be written in the form

$$\mathcal{E}(x,y) = \frac{-1}{g_q}\left((\tilde{V}'(x) - y)(V'(y) - x) - \sum_{i=0}^{p-2}\sum_{j=0}^{q-2} C_{i,j} x^i y^j \right).$$

Moreover, the condition (3.13) implies that $C_{p-2,q-2} = g_q \tilde{g}_p$.

Proposition 3.1 (Conserved quantities). *The coefficients of $\mathcal{E}_{i,j}$ with $(i+1, j+1)$ strictly inside the convex envelope of $\mathcal{N}$, written as polynomials of the $Q_{i,j}$'s and $P_{i,j}$'s using equation (3.9), i.e., – after elimination – as polynomials of the R_i's, are thus conserved quantities, i.e., although their expression depends explicitly on n via equation (3.9) their value is in fact independent of n.*

In fact, these generate *all the conserved quantities* [2, Chapter 5].

Proof. Since $\mathcal{E}(x,y)$ as given by equation (3.9) is actually independent of n for $n \ge q$, it is clear that all $\mathcal{E}_{i,j}$ are conserved quantities. The restriction of being inside the convex envelope is just because those that are outside (the so-called Casimirs) are independent of the $Q_{i,j}$ or $P_{i,j}$: they are obvious constants. ∎

The fact that these generate all conserved quantities is a classical result in integrable systems and outside the scope of this article, we will take it as given.

Example 3.3 (Quadrangulations ($q = 2$, $p = 4$, and $g_1 = \tilde{g}_1 = \tilde{g}_3 = 0$)). From equation (3.7), we have

$$\begin{aligned}
\operatorname{Tr} \mathcal{D}_n(x) = x\bigg(&\frac{R_{n-1}}{Q_{n-1,n-2}} - Q_{n,n-1}P_{n-1,n+2} \\
&- P_{n,n+3}(Q_{n+2,n+1} + Q_{n+1,n}) + P_{n,n+1}\bigg) + x^3 P_{n,n+3},\\
\det \mathcal{D}_n(x) = \bigg(&R_n P_{n-1,n+2} Q_{n+1,n} - \frac{R_{n-1} P_{n,n+1} Q_{n,n-1}}{Q_{n-1,n-2}} \\
&+ \frac{R_{n-1} P_{n,n+3} Q_{n,n-1} Q_{n+2,n+1}}{Q_{n-1,n-2}} \\
&+ P_{n-1,n} P_{n,n+1} Q_{n,n-1} - P_{n-1,n+2} P_{n,n+1} Q_{n,n-1} Q_{n+1,n} \\
&- P_{n-1,n} P_{n,n+3} Q_{n,n-1} Q_{n+2,n+1} \\
&+ P_{n-1,n+2} P_{n,n+3} Q_{n,n-1} Q_{n+1,n} Q_{n+2,n+1} \\
&- R_n P_{n-1,n} + \frac{R_{n-1} R_n}{Q_{n-1,n-2}}\bigg) \\
&+ \frac{x^2}{Q_{n-1,n-2}}\big(-R_n P_{n-1,n+2} Q_{n-1,n-2} - R_{n-1} P_{n,n+3} Q_{n,n-1} \\
&- R_{n-1} P_{n,n+3} Q_{n+1,n} - R_{n-1} P_{n,n+3} Q_{n+2,n+1} \\
&+ P_{n-1,n} P_{n,n+3} Q_{n-1,n-2} Q_{n,n-1} + R_{n-1} P_{n,n+1}\big) \\
&+ x^4 \frac{R_{n-1} P_{n,n+3}}{Q_{n-1,n-2}}.
\end{aligned}$$

There are thus five invariants (the coefficients of x and x^3 in $\operatorname{Tr} \mathcal{D}_n(x)$ and the coefficients of x^0, x^2, x^4 in $\det \mathcal{D}_n(x)$), but some of them are trivially independent of n, for instance, $P_{n,n+3} = \tilde{g}_4$ and $Q_{n,n-1}/R_n = g_2$.

Proposition 3.2 (Left eigenvectors). *We could have proceeded in the same way with left eigenvectors in $\widetilde{\mathcal{V}}_x(Q)$, and in the windows $\widetilde{\mathcal{W}}_n = \{n-1, n, n+1, \dots, n+q-2\} = \mathcal{W}_{n+q-2}$ for $n \geq 1$, and define*

$$\widetilde{\mathcal{D}}_n(x) = \mathcal{L}_{\widetilde{\mathcal{W}}_n}(x)(\tilde{\pi} \circ P^t)\mathcal{L}_{\widetilde{\mathcal{W}}_n}(x)^{-1},$$

where $\tilde{\pi}$ is defined as follows:

$$\begin{aligned}
\tilde{\pi} : P^t \widetilde{\mathcal{V}}_x(Q) &\to \widetilde{\mathcal{V}}_x(Q),\\
\vec{\phi} &\mapsto \mathcal{L}_{\widetilde{\mathcal{W}}_n(x)^{-1}}(\phi_{n-1}, \dots, \phi_{n+q-2})
\end{aligned}$$

such that the map $\tilde{\pi} \circ P^t$ *is an endomorphism of* $\tilde{\mathcal{V}}_x(Q)$*. This yields the same spectral curve, namely,*

$$\det(y\,\mathrm{Id}_{q\times q} - \tilde{\mathcal{D}}_n(x)) = \det(y\,\mathrm{Id}_{q\times q} - \mathcal{D}_{n+q-2}(x)) = \mathcal{E}(x,y) \tag{3.17}$$

for n *large enough.*

Proof. We only have to prove the first equality in (3.17). The latter is obtained immediately by transposition. ∎

3.3. Branch points and double points

This subsection is dedicated to prove a fundamental lemma (Lemma 3.3), which is a relation satisfied by the double points. For this purpose, we first start by a definition of branch points and double points.

For generic (x, y) on the spectral curve, we have that

$$\dim \mathcal{V}_x(Q) \cap \mathcal{V}_y(\pi \circ P) = 1.$$

Given x, there are generically q distinct values of z, denoted by $z^i(x), i = 0, 1, \ldots, q-1$ (the ordering does not matter, an ordering can be defined locally in each simply connected open domain of the spectral curve not containing singularities of X or X^{-1}) such that

$$X(z^i(x)) = x, \quad \{z^0(x), z^1(x), \ldots, z^{q-1}(x)\} = X^{-1}(x).$$

Branch points. There exist non-generic points at which the $z^i(x)$ are not distinct: these are called *branch points*. There are q branch points, denoted by $a_0, \ldots, a_{q-1}$ on the spectral curve, which are the q solutions of $X'(a_i) = 0$. Generically, branch points are simple, i.e., $X'(z)$ has a simple zero at $z = a_i$. From now on, we assume that the coupling constants g_k and $\tilde{g}_k$ are generic and all branch points are simple. The case of non-simple branch points is similar with more technical details, and anyway, it can be obtained by analytic continuation.

Simple branch points are also solutions of

$$\mathcal{E}_y(X(a_i), Y(a_i)) := \partial_y \mathcal{E}(X(a_i), Y(a_i)) = 0.$$

Double points. There also exist non-generic points at which the $z^i(x)$ are distinct, but the $Y(z^i(x))$ are not distinct, these are called double points. Double points are pairs $(w_a, \bar{w}_a)$ such that

$$X(w_a) = X(\bar{w}_a), \quad Y(w_a) = Y(\bar{w}_a), \quad w_a \neq \bar{w}_a.$$

By convention, we will call w_a the one with the lowest modulus

$$|w_a| \leq |\bar{w}_a|,$$

and for generic g_k's and $\tilde{g}_k$'s, we have $|w_a| < |\bar{w}_a|$, which we will assume in the following.

Double points are also solutions of

$$\mathcal{E}_y(X(w_a), Y(w_a)) = 0 = \mathcal{E}_y(X(\bar{w}_a), Y(\bar{w}_a)),$$

as well as

$$\mathcal{E}_x(X(w_a), Y(w_a)) = 0 = \mathcal{E}_x(X(\bar{w}_a), Y(\bar{w}_a)).$$

Let N be the number of double points and define

$$\Delta(z) := \prod_{a=1}^{N} (z - w_a), \quad \overline{\Delta}(z) := \prod_{a=1}^{N} (z - \bar{w}_a).$$

They satisfy the following lemma.

Lemma 3.2. *The number of double points is*

$$N = (p-1)(q-1) - 1$$

and

$$z^{N-1} \mathcal{E}_y(X(z), Y(z)) = (\tilde{g}_p)^{q-1} \Delta(z) \overline{\Delta}(z) X'(z). \tag{3.18}$$

Proof. The left-hand side of equation (3.18) is a Laurent polynomial of z, it could have negative powers of z. As $z \to 0$, we have $X(z) \sim \alpha_{q-1} z^{1-q}$ and $Y(z) \sim R/z$, and $\mathcal{E}_y(x, y) = (q-1)\mathcal{E}_{p-1,q-1} x^{p-1} y^{q-2}(1 + o(1))$ which behaves as $\mathcal{E}_y(X(z), Y(z)) \sim (q-1)\mathcal{E}_{p-1,q-1} \alpha_{q-1}^{p-1} R^{q-2} z^{-(p-1)(q-1)} z^{-(q-2)}$. If we define $N = (p-1)(q-1) - 1$, we see that

$$z^{N+q-1} \mathcal{E}_y(X(z), Y(z))$$

is a polynomial of z, it has no negative powers and is not vanishing at $z = 0$. Moreover, at $z \to \infty$, we have $X(z) = O(z^1)$ and $Y(z) = O(z^{p-1})$, so it behaves as follows:

$$O(z^{N+q-1} X(z)^{p-1} Y(z)^{q-2}) = O(z^{N+q-1} z^{p-1} z^{(p-1)(q-2)}) = O(z^{2N+q}).$$

In other words, $z^{N+q-1} \mathcal{E}_y(X(z), Y(z))$ is a polynomial of z of degree $2N + q$.

Its zeros must be either the branch points, i.e., the q zeros of the polynomial $z^q X'(z)$, or the double points. This implies that there are N pairs of double points. We have that

$$z^{N+q-1} \mathcal{E}_y(X(z), Y(z)) \propto z^q X'(z) \Delta(z) \overline{\Delta}(z)$$

and

$$\deg \Delta = \deg \overline{\Delta} = N.$$

■

Remark. The number N of double points is the number of interior points in the Newton polytope of $\mathcal{E}$. In fact, this is a general result in algebraic geometry: when the curve has genus zero (it has a rational parametrization), the number of double points is always equal to the number of interior points of the Newton polytope. We refer to [10, Chapter 2, page 42] for more discussions.

Lemma 3.3. *For $a = 1, \dots, N$, we have*

$$\frac{\bar{w}_a^{N-1}}{\Delta(\bar{w}_a)\bar{\Delta}'(\bar{w}_a)} = -\frac{w_a^{N-1}}{\Delta'(w_a)\bar{\Delta}(w_a)}. \tag{3.19}$$

Proof. Let (x, y) be a point on the spectral curve. For a small deviation around this point along the spectral curve, we have

$$\begin{aligned} 0 = \mathcal{E}(x+\delta x, y+\delta y) &= \delta x \mathcal{E}_x(x,y) + \delta y \mathcal{E}_y(x,y) \\ &+ \frac{1}{2}\big((\delta y)^2 \mathcal{E}_{yy}(x,y) + 2\delta y \delta x \mathcal{E}_{xy}(x,y) + (\delta x)^2 \mathcal{E}_{xx}(x,y)\big) + \cdots. \end{aligned}$$

For double points, we have

$$\mathcal{E}_x = \mathcal{E}_y = 0,$$

and the second order expansion vanishes, namely,

$$\begin{aligned} &(\delta y)^2 \mathcal{E}_{yy}(X(w_a), Y(w_a)) + 2\delta y \delta x \mathcal{E}_{xy}(X(w_a), Y(w_a)) \\ &\quad + (\delta x)^2 \mathcal{E}_{xx}(X(w_a), Y(w_a)) = 0. \end{aligned}$$

This is a second order equation for the variable $\frac{\delta y}{\delta x}$, therefore, its two solutions satisfy

$$\left(\frac{\delta y}{\delta x}\right)_1 + \left(\frac{\delta y}{\delta x}\right)_2 = -2\frac{\mathcal{E}_{xy}}{\mathcal{E}_{yy}} \tag{3.20}$$

with $\mathcal{E}_{xy}$ and $\mathcal{E}_{yy}$ evaluated at $(X(w_a), Y(w_a)) = (X(\bar{w}_a), Y(\bar{w}_a))$.

At the double points $(w_a, \bar{w}_a)$, equation (3.20) becomes

$$\frac{Y'(w_a)}{X'(w_a)} + \frac{Y'(\bar{w}_a)}{X'(\bar{w}_a)} = -2\frac{\mathcal{E}_{xy}}{\mathcal{E}_{yy}}. \tag{3.21}$$

Let us calculate the residue of the one form $\frac{dx}{\mathcal{E}_y}$:

$$\operatorname*{Res}_{z\to w_a} \frac{dx}{\mathcal{E}_y} = \frac{X'(w_a)}{X'(w_a)\mathcal{E}_{xy} + Y'(w_a)\mathcal{E}_{yy}}.$$

Using equation (3.21), we obtain the identity

$$\frac{X'(w_a)}{X'(w_a)\mathcal{E}_{xy} + Y'(w_a)\mathcal{E}_{yy}} = -\frac{X'(\bar{w}_a)}{X'(\bar{w}_a)\mathcal{E}_{xy} + Y'(\bar{w}_a)\mathcal{E}_{yy}},$$

and therefore,

$$\operatorname*{Res}_{z\to w_a} \frac{dx}{\mathcal{E}_y} = - \operatorname*{Res}_{z\to \bar{w}_a} \frac{dx}{\mathcal{E}_y}.$$

Using

$$\operatorname*{Res}_{z\to w_a} \frac{dx}{\mathcal{E}_y} = \operatorname*{Res}_{z\to w_a} \frac{X'(z)dz}{\mathcal{E}_y(X(z),Y(z))} = \operatorname*{Res}_{z\to w_a} \frac{z^{N-1}dz}{\Delta(z)\bar{\Delta}(z)} = \frac{w_a^{N-1}}{\Delta'(w_a)\bar{\Delta}(w_a)},$$

and a similar equation for the residue at $\bar{w}_a$, we obtain the desired equation (3.19). ■

3.4. Asymptotic behavior of eigenvectors

At this stage, we know the spectral curve, i.e., the eigenvalues of Q and $\pi \circ P$.

The so-called "reconstruction method" in integrable systems consists in recovering the eigenvectors from the spectral curve. Knowing both the eigenvalues and the eigenvectors, we can recover the full operators Q and $\pi \circ P$ (or $\mathcal{D}_n(x)$).

The reconstruction method relies on the fact that eigenvectors must be (by Cramer's formula), rational functions of z and, if we know their poles and zeros, we can find them explicitly.

Reconstruction by necessary conditions. Let y be an eigenvalue of $\mathcal{D}_n(x)$, i.e., (x, y) is on the spectral curve $\mathcal{E}(x, y) = 0$, and therefore, there exists z such that $x = X(z)$ and $y = Y(z)$. Let $\boldsymbol{\psi}_n(z)$ be the corresponding eigenvector of $\mathcal{D}_n(x)$, i.e.,

$$\mathcal{D}_n(x)\boldsymbol{\psi}_n(z) = y\, \boldsymbol{\psi}_n(z)$$

with

$$\boldsymbol{\psi}_n = \begin{pmatrix} \psi_{n-q+1}(z) \\ \vdots \\ \psi_n(z) \end{pmatrix}.$$

By definition, the $q \times q$ matrix $y \operatorname{Id}_{q\times q} - \mathcal{D}_n(x)$ is not invertible, but, for generic z, one can invert its $q - 1 \times q - 1$ minors, and the formula of eigenvectors is a Cramer-like formula, a ratio of minors.

Proposition 3.3 (Cramer's formula for eigenvectors). *We have*

$$\forall m \in \mathcal{W}_n, \quad \psi_m(z) = (-1)^{n-m} \psi_n(z) \frac{\det_{i\neq n; j\neq m}(y \operatorname{Id}_{q\times q} - \mathcal{D}_n(x))}{\det_{i\neq n; j\neq n}(y \operatorname{Id}_{q\times q} - \mathcal{D}_n(x))}. \tag{3.22}$$

It is possible to normalize the eigenvectors so that for every n, $\psi_n(z)$ is a Laurent polynomial of z.

Proof. Formula (3.22) just comes from solving $(y\,\mathrm{Id}_{q\times q} - \mathcal{D}_n(x))\boldsymbol{\psi}_n = 0$. In the window $\mathcal{W}_n$, one can choose $\psi_n(z) = \det_{i\neq n; j\neq n}(y\,\mathrm{Id}_{q\times q} - \mathcal{D}_n(x))\, f_n(z)$, where $f_n(z)$ is some Laurent polynomial of z. This gives

$$\forall m \in \mathcal{W}_n, \quad \psi_m(z) = (-1)^{n-m} f_n(z) \det_{i\neq n; j\neq m}(y\,\mathrm{Id}_{q\times q} - \mathcal{D}_n(x)),$$

which is a Laurent polynomial of z for all $m \in \mathcal{W}_n$. Then, by the recursion $\psi_{m+1}(z) = X(z)\psi_m(z) - \sum_{k=m-q+1}^{m} Q_{m,k}\psi_k(z)$, we see that ψ_m will remain a Laurent polynomial for all $m \geq n$. ∎

Behavior at double points. If some $z^k(x)$ approaches w_a, there exists another branch $z^{\bar{k}}(x)$ approaching $\bar{w}_a$:

$$z^k(x) \to w_a \quad \Rightarrow \exists \bar{k} \neq k, \quad z^{\bar{k}}(x) \to \bar{w}_a.$$

Notice from formula (3.22) that, up to a multiplicative constant, the eigenvectors are rational functions of x and y, and therefore, at the double points, the two eigenvectors $\vec{\psi}(z^k(x))$ and $\vec{\psi}(z^{\bar{k}}(x))$ tend to be proportional. This implies that when $(x, y) = (X(w_a), Y(w_a)) = (X(\bar{w}_a), Y(\bar{w}_a))$ is a double point, we have

$$\dim \mathcal{V}_x(Q) \cap \mathcal{V}_y(\pi \circ P) = 1.$$

This implies the following lemma.

Lemma 3.4. *We have*

$$\psi_n(\bar{w}_a)/\psi_n(w_a) = \lambda_a$$

is independent of n, for n large enough.

Similarly, for left eigenvectors,

$$\phi_n(\bar{w}_a)/\phi_n(w_a) = \mu_a$$

is independent of n, for n large enough.

Proof. Using equation (3.22) in the window $\mathcal{W}_n$, we see that for all $m \in \mathcal{W}_n$, we have

$$\frac{\psi_m(\bar{w}_a)}{\psi_n(\bar{w}_a)} = \frac{\psi_m(w_a)}{\psi_n(w_a)},$$

and thus, with choosing $m = n-1$, we get

$$\frac{\psi_m(\bar{w}_a)}{\psi_m(w_a)} = \frac{\psi_n(\bar{w}_a)}{\psi_n(w_a)} = \frac{\psi_{n-1}(\bar{w}_a)}{\psi_{n-1}(w_a)}.$$

It then holds in all windows larger than n, and therefore, by recursion on n, it implies that λ_a is independent of n after a certain rank.

The proof for the left eigenvectors ϕ_n is similar, using the matrix $\widetilde{\mathcal{D}}_n(x)$. ∎

Complete basis of eigenvectors. As we already saw, given x, there are generically q distinct values of z, denoted by $z^i(x)$, $i = 0, 1, \ldots, q-1$ such that $X(z^i(x)) = x$. Then, for all i, $Y(z^i(x))$ is an eigenvalue of $\mathcal{D}_n(x)$, which implies that $\psi(z^i(x)) \in \mathcal{V}_x(Q)$. In particular, the $q \times q$ matrix

$$\Psi_n(x) := \begin{pmatrix} \psi_{n-q+1}(z^0(x)) & \psi_{n-q+1}(z^1(x)) & \cdots & \psi_{n-q+1}(z^{q-1}(x)) \\ \vdots & & & \vdots \\ \psi_{n-1}(z^0(x)) & \psi_{n-1}(z^1(x)) & \cdots & \psi_{n-1}(z^{q-1}(x)) \\ \psi_n(z^0(x)) & \psi_n(z^1(x)) & \cdots & \psi_n(z^{q-1}(x)) \end{pmatrix}$$

is a matrix whose columns are eigenvectors of $\mathcal{D}_n(x)$, and thus,

$$\mathcal{D}_n(x)\Psi_n(x) = \Psi_n(x) \cdot \mathrm{diag}(Y(z^0(x)), \ldots, Y(z^{q-1}(x))).$$

For generic x, the $z^i(x)$ are all distinct, the $Y(z^i(x))$ are all distinct, and all those vectors are linearly independent. We thus have $\det \Psi_n(x) \neq 0$, and

$$\mathcal{D}_n(x) = \Psi_n(x) \cdot \mathrm{diag}(Y(z^0(x)), \ldots, Y(z^{q-1}(x)))\Psi_n(x)^{-1}. \tag{3.23}$$

By construction, we have

$$\Psi_{n+1}(x) = \Lambda_n(x)\Psi_n(x), \tag{3.24}$$

and thus, using (3.6) and (2.3), we have

$$\begin{aligned} \frac{\det \Psi_{n+1}(x)}{\det \Psi_n(x)} &= \det \Lambda_n(x) \\ &= (-1)^q Q_{n,n-q+1} \\ &= (-1)^q g_q R_n R_{n-1} \cdots R_{n-q+2}. \end{aligned}$$

Proposition 3.4 (Large n). *From (3.14), we see that Λ_∞ is the companion matrix of the Laurent polynomial $X(z)$, i.e., its eigenvalues are $z^i(x)$ (for generic x) for $i = 0, \ldots, q-1$, and its eigenvectors form the Vandermonde matrix of the $1/z^i(x)$, namely,*

$$\Lambda_\infty(x) = \mathcal{V}(x)Z(x)\mathcal{V}(x)^{-1},$$

where

$$Z(x) = \mathrm{diag}(z^0(x), \ldots, z^{q-1}(x))$$

and

$$\mathcal{V}(x) = \begin{pmatrix} z^0(x)^{1-q} & z^1(x)^{1-q} & & z^{q-1}(x)^{1-q} \\ z^0(x)^{2-q} & z^1(x)^{2-q} & & z^{q-1}(x)^{2-q} \\ \vdots & & & \vdots \\ z^0(x)^{-1} & z^1(x)^{-1} & & z^{q-1}(x)^{-1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This implies that at large n, we have

$$\mathcal{D}_\infty(x) = \mathcal{V}(x)\left(RZ(x)^{-1} + \sum_{k=0}^{q-1}\beta_k Z(x)^k\right)\mathcal{V}(x)^{-1}$$

and

$$\Psi_n(z) \underset{n\to\infty}{=} \mathcal{V}(x)Z(x)^n C(x)(1+o(1)), \tag{3.25}$$

where $C(x) = \mathrm{diag}(C(z^0(x)),\ldots,C(z^{q-1}(x)))$ *is a diagonal matrix, independent of* n*, i.e.,* $C(z)$ *is a function of* z*, independent of* n*. This is equivalent to*

$$\psi_n(z) \underset{n\to\infty}{=} C(z)z^n(1+o(1)).$$

Similarly, we define

$$\tilde{\mathcal{V}}(x) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z^0(x)^{-1} & z^1(x)^{-1} & & z^{q-1}(x)^{-1} \\ \vdots & & & \vdots \\ z^0(x)^{2-q} & z^1(x)^{2-q} & & z^{q-1}(x)^{2-q} \\ z^0(x)^{1-q} & z^1(x)^{1-q} & & z^{q-1}(x)^{1-q} \end{pmatrix},$$

and we get

$$\Phi_n(z) \underset{n\to\infty}{=} \tilde{\mathcal{V}}(x)Z(x)^{-n-N+1}\tilde{C}(x)(1+o(1)),$$

where $\tilde{C}(x) = \mathrm{diag}(\tilde{C}(z^0(x)),\ldots,\tilde{C}(z^{q-1}(x)))$ *is a diagonal matrix, independent of* n*, i.e.,* $\tilde{C}(z)$ *is a function of* z*, independent of* n*. This is equivalent to*

$$\phi_n(z) \underset{n\to\infty}{=} \tilde{C}(z)z^{-n-N+1}(1+o(1)).$$

Proof. Let us prove first equation (3.25). We define

$$\tilde{\Psi}_n(x) = Z(x)^{-n}\mathcal{V}(x)^{-1}\Psi_n(z).$$

From equation (3.23), we get

$$\tilde{\Psi}_n(x)\,\mathrm{diag}(Y(z^0(x)),\ldots,Y(z^{q-1}(x)))\tilde{\Psi}_n(x)^{-1} = Z^{-n}(x)\mathcal{V}^{-1}(x)\mathcal{D}_n(x)\mathcal{V}(x)Z^n. \tag{3.26}$$

Therefore, for large n, we have

$$\lim_{n\to\infty}\tilde{\Psi}_n(x)\,\mathrm{diag}(Y(z^0(x)),\ldots,Y(z^{q-1}(x)))\tilde{\Psi}_n(x)^{-1} = \mathrm{diag}(Y(z^0(x)),\ldots,Y(z^{q-1}(x))). \tag{3.27}$$

In addition, from equation (3.24), we get

$$\tilde{\Psi}_{n+1}(x)\tilde{\Psi}_n(x)^{-1} = Z^{-n-1}(x)\mathcal{V}^{-1}(x)\Lambda_n(x)\mathcal{V}(x)Z^n(x). \tag{3.28}$$

Taking the limit, equation (3.28) implies that

$$\lim_{n\to\infty} \widetilde{\Psi}_{n+1}(x)\widetilde{\Psi}_n(x)^{-1} = \mathrm{Id}_{q\times q}\,. \tag{3.29}$$

Equation (3.27) implies that, for generic x, at large n $\widetilde{\Psi}_n(x)$ is equivalent to a diagonal matrix. In addition, equation (3.29) implies that for large n, $\widetilde{\Psi}_n(x)$ is independent of n. Therefore, $\widetilde{\Psi}_n(x)$ is equivalent to a diagonal matrix independent of n for each generic x. Hence, we can write

$$\widetilde{\Psi}_n = C(x)(1+o(1)),$$

where $C(x) = \mathrm{diag}(C(z^0(x)),\ldots,C(z^{q-1}(x)))$ is a diagonal matrix, independent of n. Equivalently,

$$\psi_m(z^j(x)) = (z^j(x))^m C(z^j(x))(1+o(1))$$

for all $m=n-q+1,\ldots,n$ and $j=0,\ldots,q-1$, recall that the points $(z^j(x))_{0\le j\le q-1}$ are such that $X(z^j(x)) = x$. Since this is true for every generic point x, we deduce that

$$\psi_m(z) = z^m C(z)(1+o(1))$$

for all $z\in\mathbb{C}^*$.

The same works for matrix $\Phi_n(x)$. ■

Remark 3.1. We must have

$$C(\bar{w}_a) = \lim_{n\to\infty} \lambda_a C(w_a) X_a^n = 0,$$

therefore, $C(\bar{w}_a) = 0$, and thus, $C(z)$ must be proportional to $\bar{\Delta}(z)$.

3.5. Reconstruction formula and Baker–Akhiezer functions

Lemma 3.5. *We have*

$$\frac{\psi_n(z)}{\bar{\Delta}(z)} = z^n\left(1+\sum_{a=1}^{N}\frac{H_{n,a}}{z-\bar{w}_a} + S_n(1/z)\right), \tag{3.30}$$

where $S_n(1/z)\in\mathbb{C}[1/z]$. Moreover,

$$1+S_n(1/z) = \sum_{k=0}^{k_n} S_{n,n-k} z^{-k}, \quad S_{n,n}=1,$$

with k_n some fixed integer less than or equal to N.

Similarly, we have

$$\frac{\phi_n(z)}{\Delta(z)} = z^{-n}\left(1 + \sum_{a=1}^{N} \frac{\widetilde{H}_{n,a}}{z - w_a} + \widetilde{S}_n(z)\right), \tag{3.31}$$

where $\widetilde{S}_n(z) \in \mathbb{C}[z]$. *Moreover,*

$$1 + \widetilde{S}_n(z) = \sum_{k=0}^{\tilde{k}_n} \widetilde{S}_{n,n+k} z^k, \quad \widetilde{S}_{n,n} = 1,$$

with $\tilde{k}_n$ *some fixed integer less than or equal to* N.

Proof. We know that $\psi_n(z)$'s highest degree is z^{n+c}, where c is independent of n. Up to multiplying by a power of z, we may assume that the highest degree is z^{n+N}, and thus, $z^{-n}\psi_n(z)/\overline{\Delta}(z) = 1 + O(1/z)$. It follows that $z^{-n}\psi_n(z)/\overline{\Delta}(z)$ is a rational function of z that can have simple poles at $z = \bar{w}_1, \dots, \bar{w}_N$, and possibly a pole at $z = 0$. ∎

Definition 3.7. Let us set $w_0 = z$ and $\bar{w}_0 = z'$, $\lambda_0 = \mu_0 = 1$ and let the $(N+1) \times (N+1)$ matrices $K_n(z', z)$ and $\widetilde{K}_n(z', z)$:

$$K_n(z, z')_{a,b} := \delta_{a,b} - \delta_{a,0}\delta_{b,0} - \lambda_a \frac{w_a^n \bar{w}_b^{-n}}{w_a - \bar{w}_b},$$

$$\widetilde{K}_n(z, z')_{a,b} := \delta_{a,b} - \delta_{a,0}\delta_{b,0} - \mu_a \frac{w_a^n \bar{w}_b^{-n}}{w_a - \bar{w}_b}$$

for $a, b = 0, \dots, N$.

Let us also define

$$h_n := \det\left(\delta_{a,b} - \lambda_a \frac{w_a^N \bar{w}_b^{-n}}{w_a - \bar{w}_b}\right)_{a,b=1,\dots,N}, \tag{3.32}$$

$$\tilde{h}_n := \det\left(\delta_{a,b} - \mu_a \frac{w_a^N \bar{w}_b^{-n}}{w_a - \bar{w}_b}\right)_{a,b=1,\dots,N}. \tag{3.33}$$

Lemma 3.6. *Let us define the so-called Baker–Akhiezer functions:*

$$\begin{aligned}\hat{\psi}_n(z) &:= \lim_{z' \to \infty} \frac{\overline{\Delta}(z)}{h_n} z'^{n+1} \det K_n(z', z) \\ &= \frac{z^n \overline{\Delta}(z)}{h_n} \det \begin{pmatrix} 1 & \frac{\bar{w}_b^{-n}}{z - \bar{w}_b} \\ \lambda_a w_a^n & \delta_{a,b} - \lambda_a \frac{w_a^n \bar{w}_b^{-n}}{w_a - \bar{w}_b} \end{pmatrix},\end{aligned} \tag{3.34}$$

$$\begin{aligned}\hat{\phi}_n(z') &:= \lim_{z \to \infty} \frac{\Delta(z)}{\tilde{h}_n} z^{-n+1} \det \widetilde{K}_n(z', z) \\ &= \frac{z'^{-n} \Delta(z')}{\tilde{h}_n} \det \begin{pmatrix} -1 & \bar{w}_b^{-n} \\ \mu_a \frac{w_a^n}{z' - w_a} & \delta_{a,b} - \mu_a \frac{w_a^n \bar{w}_b^{-n}}{w_a - \bar{w}_b} \end{pmatrix}.\end{aligned} \tag{3.35}$$

They satisfy for all $a = 1, \dots, N$:

$$\widehat{\psi}_n(\bar{w}_a) = \lambda_a \widehat{\psi}_n(w_a), \quad \widehat{\phi}_n(\bar{w}_a) = \mu_a \widehat{\phi}_n(w_a),$$

we have

$$\begin{aligned} \psi_n(z) &= \frac{\sum_{k=0}^{k_n} S_{n,n-k} h_{n-k} \widehat{\psi}_{n-k}(z)}{\sum_{k=0}^{k_n} S_{n,n-k} h_{n-k}}, \\ \phi_n(z) &= \frac{\sum_{k=0}^{\tilde{k}_n} \widetilde{S}_{n,n+k} \tilde{h}_{n+k} \widehat{\phi}_{n+k}(z)}{\sum_{k=0}^{\tilde{k}_n} \widetilde{S}_{n,n+k} \tilde{h}_{n+k}}. \end{aligned} \tag{3.36}$$

Proof. Equation (3.36) is just the solution of the linear equations $\psi_n(\bar{w}_a)=\lambda_a \psi_n(w_a)$ for the residues $H_{n,a}$ in terms of the $S_{n,k}$ of (3.30), written as a Cramer's determinant. The Cramer determinant coefficient of $S_{n,k}$ is $\widehat{\psi}_{n-k}$. Same thing for ϕ_n. ■

Theorem 3.6 (Reconstruction formula). *We have $S_{n,n-k} = \delta_{k,0}$ and $\widetilde{S}_{n,n+k} = \delta_{k,0}$. This implies that*

$$\psi_n(z) = \widehat{\psi}_n(z), \quad \phi_n(z) = \widehat{\phi}_n(z). \tag{3.37}$$

This theorem is rather important in the theory of integrable systems. Here, we will admit the result (in fact, all the steps of the proof will appear in Section 4 below). Let us just give a sketch of the proof.

(1) We will find a scalar product for which $\langle \widehat{\phi}_m, \widehat{\psi}_n \rangle = \delta_{n,m}$. This will be used to show that $X(z)\widehat{\psi}_n(z) = \sum_m \widehat{Q}_{n,m} \widehat{\psi}_m(z)$ and $\widehat{Q}_{n,m} = \langle \widehat{\phi}_m(z), X(z)\widehat{\psi}_n(z) \rangle$ is a band matrix of the size that we want for Q. Similarly, $Y(z)\widehat{\psi}_n(z) = \sum_m \widehat{P}_{n,m} \widehat{\psi}_m(z)$ and $\widehat{P}_{n,m} = \langle \widehat{\phi}_m(z), Y(z)\widehat{\psi}_n(z) \rangle$ is a band matrix of the size that we want for P.

(2) The matrices $S_{n,m}$ (lower triangular) and $\widetilde{S}_{n,m}$ (upper triangular) are a change of basis from $\widehat{\psi}_n$ to ψ_n and $\widehat{\phi}_n$ to ϕ_n. This means that the band matrices Q and P will be equal to conjugations of the band matrices $\widehat{Q}$, $\widehat{P}$ by these triangular matrices, and this will change their band widths unless the triangular matrices are in fact diagonal. This implies S and $\widetilde{S}$ must be diagonal, and thus, identity.

All this scalar product technique is described below in Section 4.

Conclusion of this section. By necessary conditions, we have found that the functions ψ_n and ϕ_n must take the forms of determinants (3.34) and (3.35). This allows us to recover Q and P as we will see below.

However, let us remark that these determinantal formulas depend on $2N$ parameters λ_a and μ_a, which we still need to find. Keeping λ_a and μ_a arbitrary gives the solution to the *doubly infinite* problem with $Q_{n,m}$ and $P_{n,m}$ doubly infinite band matrices, with indices n and m ranging from $-\infty$ to $+\infty$. Let us call it the "general solution" of the eigenvector problem.

In our case, we need a "special solution" that satisfies the following constraint: we must impose that $Q_{n,m}$ and $P_{n,m}$ have entries only for $n \geq 0$ and $m \geq 0$. This is a very strong constraint that fixes the coefficients λ_a and μ_a, as we will see in the next section.

4. Proof of the main theorem

So far, we have seen that the combinatorics of mobiles falls into the framework of integrable systems, and thus, a necessary condition for finding a solution to our combinatorics problem is to find a solution of the integrable system (3.1) with appropriate boundary conditions at $n = 0$. The purpose of this section is to exhibit explicitly this "combinatorial" solution.

Even though we will use some of the concepts introduced in the previous section, we will redefine them when required so that the present section is in large part independent of Section 3, even though we will use the fact that the spectral curve has $N = (p-1)(q-1) - 1$ double points which satisfy Lemma 3.3.

4.1. Spectral curve and Baker–Akhiezer functions

Our first ingredient is given by the spectral curve and its double points. We recall that the spectral curve, and more precisely the functions $X(z)$ and $Y(z)$ can be found from the potentials $\tilde{V}(x)$ and $V(y)$ of (3.11) via equations (3.10), (3.12), and (3.13). In view of these equations, we may alternatively *define from the start* $X(z)$ and $Y(z)$ in self-contained way via the following proposition.

Proposition 4.1. *There exist unique coefficients α_j, β_j, and R, formal series of the g_k's and the $\tilde{g}_k$'s such that the Laurent polynomials*

$$X(z) := z + \sum_{j=0}^{q-1} \alpha_j z^{-j}, \quad Y(z) := \frac{R}{z} + \sum_{j=0}^{p-1} \beta_j z^j$$

satisfy Theorem 3.4, i.e., obey the equations

$$\begin{aligned} \alpha_j &:= \sum_{k=1+j}^{q} g_k \times [z^{-j}] Y(z)^{k-1}, \quad j = 0, \ldots, q-1, \\ \beta_j &:= \sum_{k=1+j}^{p} \tilde{g}_k \times [z^j] X(z)^{k-1}, \quad j = 0, \ldots, p-1, \\ \sum_{j=1}^{\min(p,q)-1} & j \alpha_j \beta_j = R - 1. \end{aligned} \tag{4.1}$$

Proof. We note that there exist in general several solutions to these equations, but exactly one of them is such that R, the α_j's and the β_j's are formal power series of the g_k's and the $\tilde{g}_k$'s, with R having its constant term equal to 1. This is easily seen by setting $g_k = g\lambda_k$, $\tilde{g}_k = g\widetilde{\lambda}_k$ and developing the equations recursively in powers of g, starting with

$$R = 1 + O(g), \quad \alpha_j = g\lambda_{j+1} + O(g^2), \quad \beta_j = g\widetilde{\lambda}_{j+1} + O(g^2). \qquad \blacksquare$$

From now on, it will always be implicitly assumed when referring to the spectral curve that we take for R and for the α_j's and the β_j's this latter "combinatorial" solution.

We recall that the spectral curve has $N = (p-1)(q-1) - 1 > 0$ double points, i.e., pairs $(w_a, \bar{w}_a)$ such that

$$X(\bar{w}_a) = X(w_a) \quad \text{and} \quad Y(\bar{w}_a) = Y(w_a),$$

and we choose for w_a the one with the smallest modulus $|w_a| < |\bar{w}_a|$ (the moduli are generically not equal). We define $X_a = w_a/\bar{w}_a$, and thus, $|X_a| < 1$. We also define

$$\Delta(z) := \prod_{a=1}^{N}(z - w_a), \quad \overline{\Delta}(z) := \prod_{a=1}^{N}(z - \bar{w}_a).$$

We recall that the double points satisfy Lemma 3.3:

$$\frac{w_a^{N-1}}{\Delta'(w_a)\overline{\Delta}(w_a)} = -\frac{\bar{w}_a^{N-1}}{\Delta(\bar{w}_a)\overline{\Delta}'(\bar{w}_a)}. \tag{4.2}$$

The Baker–Akhiezer functions.

Definition 4.1. We define the N-dimensional vectors:

$$\xi_n := (\bar{w}_1^n - w_1^n, \bar{w}_2^n - w_2^n, \ldots, \bar{w}_N^n - w_N^n).$$

(We remark that $\xi_0 = \vec{0}$.) We then define

$$h_n := \det(\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+N}), \tag{4.3}$$

and, for $n \geq 0$, we define the Baker–Akhiezer functions:

$$\begin{aligned} \psi_n(z) &:= \frac{z^{n+1}}{h_n}\det(z\xi_{n+1} - \xi_{n+2}, z\xi_{n+2} - \xi_{n+3}, \ldots, z\xi_{n+N} - \xi_{n+N+1}), \\ \phi_n(z) &:= \psi_{-2-n-N}(z) = \frac{z^{-n-N-1}}{h_{-n-N-2}}\det(z\xi_{-n-N-1} - \xi_{-n-N}, \ldots, z\xi_{-n-2} - \xi_{-n-1}). \end{aligned} \tag{4.4}$$

As we will see later (equation (4.8)) the above definitions of h_n, $\psi_n(z)$, $\phi_n(z)$ match (up to unimportant global factors) the definitions given in equations (3.32) to (3.35) for some appropriate choice of λ_a and μ_a.

We observe that $h_{-1} = \cdots = h_{-N} = 0$ and that ψ_n is ill-defined for $n = -1, \ldots, -N$ while ϕ_n is ill-defined for $n = -2, \ldots, -(N+1)$.

Remark 4.1 (Generically defined). We remark that w_a and $\bar{w}_a$ are algebraic functions of the g_k's and $\tilde{g}_k$'s, therefore, h_0 is an algebraic function of the g_k's and $\tilde{g}_k$'s. It is not identically vanishing, this implies that for generic g_k's and $\tilde{g}_k$'s, we can assume that $h_0 \neq 0$. Otherwise stated, the set of g_k's and $\tilde{g}_k$'s such that $h_0 = 0$ is at least of codimension 1. This is a consequence of Proposition 4.8 below in the particular case $n = 0$.

The leading behaviors of ψ_n and ϕ_n can be read directly from their expression equation (4.4) and the definition (4.3) of h_n.

Proposition 4.2 (Asymptotic behaviors). *We have*

$$\psi_n(z) \underset{z\to\infty}{\sim} z^{n+N+1}, \quad \phi_n(z) \underset{z\to\infty}{\sim} z^{-n-1},$$
$$\psi_n(z) \underset{z\to 0}{\sim} \frac{(-1)^N h_{n+1}}{h_n} z^{n+1}, \quad \phi_n(z) \underset{z\to 0}{\sim} \frac{(-1)^N h_{-n-N-1}}{h_{-n-N-2}} z^{-n-1-N}.$$

Lemma 4.1. *We have $\psi_n(\bar{w}_a) = \psi_n(w_a)$, $\phi_n(\bar{w}_a) = \phi_n(w_a)$.*

Proof. We will prove it for case $a = 1$. The other cases are clearly similar. We have

$$\psi_n(w_1) = \frac{w_1^{n+1}}{h_n} \det(w_1\xi_{n+1} - \xi_{n+2}, \ldots, w_1\xi_{n+N} - \xi_{n+N+1}).$$

Let us denote by D the determinant $\det(w_1\xi_{n+1} - \xi_{n+2}, \ldots, w_1\xi_{n+N} - \xi_{n+N+1})$. We denote also $\xi_{n,a} = (\bar{w}_a^n - w_a^n)$. We have

$$\begin{aligned} D &= \bar{w}_1^{n+1}(w_1 - \bar{w}_1) \\ &\quad \times \begin{vmatrix} 1 & \bar{w}_1 & \ldots & \bar{w}_1^{N-1} \\ w_1\xi_{n+1,2} - \xi_{n+2,2} & w_1\xi_{n+2,2} - \xi_{n+3,2} & \ldots & w_1\xi_{n+N,2} - \xi_{n+N+1,2} \\ \vdots & & & \vdots \\ w_1\xi_{n+1,N} - \xi_{n+2,N} & w_1\xi_{n+2,N} - \xi_{n+3,N} & \ldots & w_1\xi_{n+N,N} - \xi_{n+N+1,N} \end{vmatrix} \\ &= \bar{w}_1^{n+1}(w_1 - \bar{w}_1) \\ &\quad \times \begin{vmatrix} 1 & \bar{w}_1 + w_1 & \ldots & \sum_{i=1}^N w_1^{N-i}\bar{w}_1^{i-1} \\ w_1\xi_{n+1,2} - \xi_{n+2,2} & w_1^2\xi_{n+1,2} - \xi_{n+3,2} & \ldots & w_1^N\xi_{n+1,2} - \xi_{n+N+1,2} \\ \vdots & & & \vdots \\ w_1\xi_{n+1,N} - \xi_{n+2,N} & w_1^2\xi_{n+1,N} - \xi_{n+3,N} & \ldots & w_1^N\xi_{n+1,N} - \xi_{n+N+1,N} \end{vmatrix}, \end{aligned}$$

where, to go from the first to the second line, we performed recursively the operation $C_i \to C_i + w_1 C_{i-1}$ for each column C_i. We then have

$$D = -\bar{w}_1^{n+1} \times \begin{vmatrix} \bar{w}_1 - w_1 & \bar{w}_1^2 - w_1^2 & \dots & \bar{w}_1^N - w_1^N \\ w_1 \xi_{n+1,2} - \xi_{n+2,2} & w_1^2 \xi_{n+1,2} - \xi_{n+3,2} & \dots & w_1^N \xi_{n+1,2} - \xi_{n+N+1,2} \\ \vdots & & & \vdots \\ w_1 \xi_{n+1,N} - \xi_{n+2,N} & w_1^2 \xi_{n+1,N} - \xi_{n+3,N} & \dots & w_1^N \xi_{n+1,N} - \xi_{n+N+1,N} \end{vmatrix}$$

$$= -\bar{w}_1^{n+1} \times \begin{vmatrix} \bar{w}_1 - w_1 & \bar{w}_1^2 - w_1^2 & \dots & \bar{w}_1^N - w_1^N \\ \bar{w}_1 \xi_{n+1,2} - \xi_{n+2,2} & \bar{w}_1^2 \xi_{n+1,2} - \xi_{n+3,2} & \dots & \bar{w}_1^N \xi_{n+1,2} - \xi_{n+N+1,2} \\ \vdots & & & \vdots \\ \bar{w}_1 \xi_{n+1,N} - \xi_{n+2,N} & \bar{w}_1^2 \xi_{n+1,N} - \xi_{n+3,N} & \dots & \bar{w}_1^N \xi_{n+1,N} - \xi_{n+N+1,N} \end{vmatrix}.$$

If we denote by $\widetilde{D}$ the determinant $\det(\bar{w}_1 \xi_{n+1} - \xi_{n+2}, \dots, \bar{w}_1 \xi_{n+N} - \xi_{n+N+1})$, then we have

$$\psi_n(\bar{w}_1) = \frac{\bar{w}_1^{n+1}}{h_n} \widetilde{D},$$

with

$$\widetilde{D} = -w_1^{n+1} \times \begin{vmatrix} \bar{w}_1 - w_1 & \bar{w}_1^2 - w_1^2 & \dots & \bar{w}_1^N - w_1^N \\ \bar{w}_1 \xi_{n+1,2} - \xi_{n+2,2} & \bar{w}_1^2 \xi_{n+1,2} - \xi_{n+3,2} & \dots & \bar{w}_1^N \xi_{n+1,2} - \xi_{n+N+1,2} \\ \vdots & & & \vdots \\ \bar{w}_1 \xi_{n+1,N} - \xi_{n+2,N} & \bar{w}_1^2 \xi_{n+1,N} - \xi_{n+3,N} & \dots & \bar{w}_1^N \xi_{n+1,N} - \xi_{n+N+1,N} \end{vmatrix}.$$

Therefore, we obtain $\psi_n(w_1) = \psi_n(\bar{w}_1)$, and similarly, $\psi_n(w_a) = \psi_n(\bar{w}_a)$ for any $a = 2, \dots, N$. Since $\phi_n = \psi_{-2-n-N}$, we deduce that $\phi_n(\bar{w}_a) = \phi_n(w_a)$ for $a = 1, \dots, N$. ∎

4.2. Scalar product

In classical integrable systems, the existence of a Hirota equation, i.e., a bilinear relation is equivalent to an orthonormal condition satisfied by the Baker–Akhiezer functions. This leads us to define the following.

Definition 4.2. We define the scalar product as follows:

$$\langle f(z), g(z) \rangle := -\operatorname*{Res}_{z \to \infty} \frac{f(z) g(z) z^{N-1} dz}{\Delta(z) \overline{\Delta}(z)}.$$

Lemma 4.2. *For any bivariate polynomial $\mathcal{P}(x,y) \in \mathbb{C}[x,y]$, and for any $n, m \geq 0$, we have*

$$\begin{aligned}\langle \phi_m(z), \mathcal{P}(X(z), Y(z))\psi_n(z)\rangle \\ := -\operatorname*{Res}_{z\to\infty} \frac{\phi_m(z)\mathcal{P}(X(z),Y(z))\psi_n(z)z^{N-1}dz}{\Delta(z)\overline{\Delta}(z)} \\ = \operatorname*{Res}_{z\to 0} \frac{\phi_m(z)\mathcal{P}(X(z),Y(z))\psi_n(z)z^{N-1}dz}{\Delta(z)\overline{\Delta}(z)};\end{aligned}$$

i.e., the scalar product can be calculated either from the residue at ∞ or by that at 0.

Proof. We observe that $\phi_m(z)\mathcal{P}(X(z),Y(z))\psi_n(z)$ is a Laurent polynomial of z, it has poles only at $z=0$ and at $z=\infty$. Since the sum of residues at all poles must vanish, we only need to prove that the residues at all the poles w_a and $\bar{w}_a$ cancel. In other words, it is enough to prove that

$$\begin{aligned}0 = \operatorname*{Res}_{z\to \bar{w}_a} \frac{\phi_m(z)\mathcal{P}(X(z),Y(z))\psi_n(z)z^{N-1}dz}{\Delta(z)\overline{\Delta}(z)} \\ + \operatorname*{Res}_{z\to w_a} \frac{\phi_m(z)\mathcal{P}(X(z),Y(z))\psi_n(z)z^{N-1}dz}{\Delta(z)\overline{\Delta}(z)}.\end{aligned}$$

This is true due to equation (4.2) (Lemma 3.3) and Lemma 4.1. ∎

Proposition 4.3. *We have*

$$\langle \phi_m(z), \psi_n(z)\rangle = \delta_{n,m}.$$

Also, we have

$$\Delta(0)\overline{\Delta}(0) = \frac{h_{n+1}h_{-n-N-1}}{h_n h_{-n-N-2}}. \tag{4.5}$$

Proof. If $n < m$, from the asymptotic behaviors in Proposition 4.2, we see that at large z

$$\frac{\phi_m(z)\psi_n(z)z^{N-1}}{\Delta(z)\overline{\Delta}(z)} \sim O(z^{N-1+n+N+1-m-1-2N}) = O(z^{n-m-1}) = O(z^{-2}),$$

and thus, the residue at ∞ vanishes. On the contrary, if $n > m$, we have at $z \to 0$

$$\frac{\phi_m(z)\psi_n(z)z^{N-1}}{\Delta(z)\overline{\Delta}(z)} \sim O(z^{N-1+n+1-m-1-N}) = O(z^{n-m-1}) = O(z^0),$$

and thus, the residue at 0 vanishes. This implies that $\langle \phi_m(z), \psi_n(z)\rangle = 0$ if $n \neq m$.

If $n = m$, the residue at ∞ gives

$$\langle \phi_n(z), \psi_n(z)\rangle = 1.$$

We can also compute the same residue at $z \to 0$: this gives equation (4.5). ∎

Definition 4.3. Let

$$\mathfrak{W} = \{\pi \in \mathbb{C}[z] \mid \forall a = 1, \ldots, N,\ \pi(\bar{w}_a) = \pi(w_a)\},$$
$$\widetilde{\mathfrak{W}} = \{\pi \in \mathbb{C}[1/z] \mid \forall a = 1, \ldots, N,\ \pi(\bar{w}_a) = \pi(w_a)\}.$$

Lemma 4.3. *We have the following properties:*

$$\forall n \geq 0, \quad \psi_n \in \mathfrak{W} \quad \textit{and} \quad \phi_n \in \widetilde{\mathfrak{W}}.$$

If $\pi \in \mathbb{C}[z]$ is a polynomial of degree $\leq N$ which is vanishing at $z = 0$ and $\pi \in \mathfrak{W}$, then $\pi = 0$, namely,

$$\pi \in \mathfrak{W}, \quad \pi(0) = 0, \quad \deg \pi \leq N \quad \Rightarrow \pi = 0.$$

If $\pi \in \mathbb{C}[1/z]$ is a polynomial of degree $\leq N$ which is vanishing at $1/z = 0$ and $\pi \in \widetilde{\mathfrak{W}}$, then $\pi = 0$, namely,

$$\pi \in \widetilde{\mathfrak{W}}, \quad \pi(\infty) = 0, \quad \deg \pi \leq N \quad \Rightarrow \pi = 0.$$

Finally, we have

$$\mathfrak{W} = \mathbb{C} \oplus \operatorname{span}(\{\psi_n\}_{n\geq 0}), \quad \widetilde{\mathfrak{W}} = \mathbb{C} \oplus \operatorname{span}(\{\phi_n\}_{n\geq 0})$$
$$\dim(\mathbb{C}[z]/\mathfrak{W}) = N, \quad \dim(\mathbb{C}[1/z]/\widetilde{\mathfrak{W}}) = N.$$

Proof. It is obvious from Lemma 4.1 that for $n \geq 0$, $\psi_n \in \mathfrak{W}$. It is obvious also that a polynomial of degree 0, i.e., a constant is in $\mathfrak{W}$. Let now $\pi \in \mathfrak{W}$ such that $\pi(0) = 0$ and $\deg \pi \leq N$. Let us write

$$\pi(z) = \sum_{k=1}^{N} p_k z^k.$$

Since $\pi \in \mathfrak{W}$, we must have

$$\Xi \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = 0, \quad \text{where } \Xi = (\xi_1, \xi_2, \ldots, \xi_N).$$

We have $\det \Xi = h_0 \neq 0$, therefore, Ξ is invertible, and thus, $\pi = 0$.

Then, if $\pi \in \mathfrak{W}$, this implies that $\pi - \pi(0)$ is a polynomial that vanishes at 0. If $\deg \pi > N$, we can recursively subtract from it a linear combination of $\psi_n (n \geq 0)$ that kills recursively the highest-degree terms, and since $\deg \psi_n = N + n + 1$ and $\psi_n(0) = 0$, we end up with a polynomial of degree $\leq N$ and no constant term. By the result above, it is vanishing. This proves that every $\pi \in \mathfrak{W}$ can be uniquely written as a linear combination of ψ_n's and a constant.

The same works for $\widetilde{\mathfrak{W}}$. ∎

4.3. Construction of the operators Q and P

We remark that $X(z)\psi_n(z)$ is a Laurent polynomial of z. If $n \geq q-1$, then $X(z)\psi_n(z)$ has only positive powers of z, hence, it is a polynomial of z, since $X(w_a)\psi_n(w_a) = X(\bar{w}_a)\psi_n(\bar{w}_a)$, it belongs to $\mathfrak{W}$.

Proposition 4.4 (Operator Q). *There exist some coefficients $Q_{n,m}$ and some Laurent polynomial $U_n(z) \in \mathbb{C}[z, 1/z]$ of degree $\leq N$ in z such that*

$$X(z)\psi_n(z) = \sum_{m\geq 0} Q_{n,m}\psi_m(z) + U_n(z),$$

with $U_n(z) = 0$. If $n \geq q-1$, then we have, more precisely,

$$Q_{n,m} = \langle \phi_m(z), X(z)\psi_n(z)\rangle$$

and, in particular,

$$Q_{n,n+1} = 1,$$
$$Q_{n,m} = 0 \quad \text{if } m < n-q+1 \text{ or } m > n+1.$$

In other words,

$$X(z)\psi_n(z) = \sum_{m=\max(0,n-q+1)}^{n+1} Q_{n,m}\psi_m(z) + U_n(z).$$

Proof. If $n \geq q-1$, we see that $X(z)\psi_n(z) = O(z^{n+2-q})$ is a polynomial in $\mathfrak{W}$ that vanishes at $z = 0$, therefore, it is a unique linear combination of ψ_m's, it can be written as $\sum_m Q_{n,m}\psi_m$. For $0 \leq n < q-1$, we subtract from $X(z)\psi_n(z)$ the linear combination of $Q_{n,m}\psi_m$ that makes it of degree $\leq N$. The remainder $U_n(z)$ is then a Laurent polynomial of degree $\leq N$ in z.

In all cases, let us compute the scalar product as the residue at ∞:

$$\begin{aligned}\langle \phi_m(z), X(z)\psi_n(z)\rangle &= \sum_k Q_{n,k}\langle \phi_m(z), \psi_k(z)\rangle - \operatorname*{Res}_{z\to\infty} \frac{z^{N-1}U_n(z)\phi_m(z)}{\Delta(z)\bar{\Delta}(z)}dz \\ &= \sum_k Q_{n,k}\delta_{k,m} - \operatorname*{Res}_{z\to\infty} O(z^{N-1+N-m-1-2N})dz \\ &= Q_{n,m} - \operatorname*{Res}_{z\to\infty} O(z^{-m-2})dz \\ &= Q_{n,m}.\end{aligned}$$

Since $X(z) \sim z$ at large z, we have $X(z)\psi_n(z) \sim \psi_{n+1}(z)$, and thus, $Q_{n,n+1} = 1$ and $Q_{n,m} = 0$ if $m > n+1$. If $n \geq q-1$, we have at $z \to 0$ $X(z)\psi_n(z) \sim \alpha_{q-1} O(z^{1-q+n+1})$, i.e., it can be a linear combination of ψ_m only with $m \geq n-q+1$, therefore, $Q_{n,m} = 0$ if $m < n-q+1$. ∎

Proposition 4.5 (Operator P). *There exist some coefficients $P_{n,m}$ such that*

$$Y(z)\psi_n(z) = \sum_{m\geq 0} P_{n,m}\psi_m(z) + R\frac{(-1)^N h_1}{h_0}\delta_{n,0}.$$

We have, more precisely,

$$P_{n,m} = \langle \phi_m(z), Y(z)\psi_n(z)\rangle$$

and

$$P_{n,m} = 0 \quad \text{if } m < n-1 \text{ or } m > n+p-1;$$

i.e., we may write

$$Y(z)\psi_n(z) = \sum_{m=\max(0,n-1)}^{n+p-1} P_{n,m}\psi_m(z) + R\frac{(-1)^N h_1}{h_0}\delta_{n,0}.$$

Also, we have

$$P_{n,n-1} = R_n = R\frac{h_{n-1}h_{n+1}}{h_n^2}.$$

Proof. We observe that $Y(z)\psi_n(z) \in \mathfrak{W}$. It can be uniquely decomposed as a linear combination of a constant and the ψ_m's. The constant can occur only if $n = 0$, and it is then equal to $R\frac{(-1)^N h_{n+1}}{h_n}\delta_{n,0}$.

We remark that $\langle \phi_m(z), 1\rangle = 0$, so that

$$P_{n,m} = \langle \phi_m(z), Y(z)\psi_n(z)\rangle.$$

Moreover, $Y(z)$ can at most raise the degree by $p-1$ and lower it by 1, which implies that $P_{n,m} = 0$ if $m > n+p-1$ or $m < n-1$.

By computing the residue at $z \to 0$, we get

$$\begin{aligned} R_n = P_{n,n-1} &= \operatorname*{Res}_{z\to 0} \frac{h_{-n-N}h_{n+1}}{h_{-n-N-1}h_n}\frac{z^{N-1}z^{-N-n}z^{n+1}Rz^{-1}(1+O(z))}{\Delta(z)\overline{\Delta}(z)}dz \\ &= R\frac{h_{-n-N}h_{n+1}}{h_{-n-N-1}h_n}\frac{1}{\Delta(0)\overline{\Delta}(0)} \\ &= R\frac{h_{n-1}h_{n+1}}{h_n^2}. \end{aligned}$$

∎

Lemma 4.4. *For $l \geq 1$, we have*

$$Y(z)^l\psi_n(z) = \sum_m (P^l)_{n,m}\psi_m(z) + R\frac{(-1)^N h_1}{h_0}\left(\frac{Y(z)^l - P^l}{Y(z)-P}\right)_{n,0}.$$

Proof. We remark that $\frac{Y(z)^l-P^l}{Y(z)-P}$ is a polynomial of $Y(z)$ and, thus, a Laurent polynomial of z. We prove the lemma by recursion on l. For $l=1$, we have $(\frac{Y(z)^l-P^l}{Y(z)-P})_{n,0} = (\mathrm{Id})_{n,0} = \delta_{n,0}$, so the statement is true for $l=1$. Let us write

$$c = R\frac{(-1)^N h_1}{h_0}.$$

Assume that the statement holds for l. Then, compute

$$\begin{aligned}
Y(z)^{l+1}\psi_n(z) &= Y(z)\left(\sum_k (P^l)_{n,k}\psi_k(z) + c\left(\frac{Y(z)^l-P^l}{Y(z)-P}\right)_{n,0}\right)\\
&= \sum_k (P^l)_{n,k}Y(z)\psi_k(z) + c\left(Y(z)\frac{Y(z)^l-P^l}{Y(z)-P}\right)_{n,0}\\
&= \sum_k (P^l)_{n,k}\left(\sum_m P_{k,m}\psi_m + c\delta_{k,0}\right)\\
&\quad + c\left(\frac{Y(z)^{l+1}-P^{l+1}}{Y(z)-P} - P^l\right)_{n,0}\\
&= \sum_m (P^{l+1})_{n,m}\psi_m + cP^l_{n,0} + c\left(\frac{Y(z)^{l+1}-P^{l+1}}{Y(z)-P} - P^l\right)_{n,0}\\
&= \sum_m (P^{l+1})_{n,m}\psi_m + c\left(\frac{Y(z)^{l+1}-P^{l+1}}{Y(z)-P}\right)_{n,0},
\end{aligned}$$

which is the statement at rank $l+1$. ∎

Theorem 4.1. *Let $T = Q - V'(P)$, we have*

$$(X(z) - V'(Y(z)))\psi_n(z) = \sum_{m=n+1}^{n+N} T_{n,m}\psi_m(z),$$

where

$$T_{n,m} = \langle \phi_m(z), (X(z) - \tilde{V}'(Y(z)))\psi_n(z)\rangle.$$

In particular, the matrix T is a strictly upper triangular matrix, and we have

$$T_{n,n+1} = \frac{1}{R_{n+1}}.$$

Otherwise stated, we have

$$(Q - V'(P))_- = 0 \quad \textit{and} \quad (Q - V'(P))_{n,n+1} = \frac{1}{R_{n+1}},$$

as wanted.

Proof. We observe that $(X(z) - V'(Y(z))) = \frac{z}{R} + O(z^2)$ is a polynomial of z, that vanishes at $z = 0$. Therefore, $(X(z) - V'(Y(z)))\psi_n(z) \in \mathfrak{W}$ and vanishes at $z = 0$, this implies that it must be a linear combination of ψ_m's. We have

$$(X(z) - V'(Y(z)))\psi_n(z) = \sum_m Q_{n,m}\psi_m(z) - \sum_m (V'(P))_{n,m}\psi_m(z) + U_n(z) - c\left(\frac{V'(Y(z)) - V'(P)}{Y(z) - P}\right)_{n,0}.$$

Notice that $\frac{V'(Y(z))-V'(P)}{Y(z)-P}$ is a polynomial of $Y(z)$ of degree at most $q-2$ and, thus, a Laurent polynomial of z of degree at most $(q-2)(p-1) = N+2-p$. If we assume $p \geq 2$, this implies that it can have degree at most N. We have

$$(X(z) - V'(Y(z)))\psi_n(z) - \sum_m T_{n,m}\psi_m(z) = U_n(z) - c\left(\frac{V'(Y(z)) - V'(P)}{Y(z) - P}\right)_{n,0}.$$

Both terms on the right-hand side are Laurent polynomials of z, with possibly negative powers of z, however, since the left-hand side has no negative powers, all the negative powers must cancel. Therefore, the right-hand side is a polynomial of z, of degree $\leq N$. The left-hand side vanishes at $z = 0$ and is in $\mathfrak{W}$, therefore, the right-hand side must be a polynomial in $\mathfrak{W}$, of degree $\leq N$ and vanishing at 0, therefore, it must vanish. This implies

$$(X(z) - V'(Y(z)))\psi_n(z) = \sum_m T_{n,m}\psi_m(z).$$

By taking the scalar product, we have

$$T_{n,m} = \langle \phi_m(z), (X(z) - V'(Y(z)))\psi_n(z)\rangle.$$

It then follows that $T_{n,m}$ is strictly upper triangular.

Finally, we have

$$\begin{aligned}
T_{n,n+1} &= \langle \phi_{n+1}(z), (X(z) - V'(Y(z)))\psi_n(z)\rangle \\
&= \operatorname*{Res}_0 \frac{h_{-n-N-2}h_{n+1}}{h_{-n-N-3}h_n} \frac{z^{N-1}z^{-N-n-2}z^{n+1}R^{-1}z^1(1+O(z))}{\Delta(z)\bar{\Delta}(z)}dz \\
&= \frac{1}{R}\frac{h_{-n-N-2}h_{n+1}}{h_{-n-N-3}h_n}\frac{1}{\Delta(0)\bar{\Delta}(0)} \\
&= \frac{1}{R}\frac{h_{n+1}^2}{h_{n+2}h_n} \\
&= \frac{1}{R_{n+1}}.
\end{aligned}$$

■

4.4. The operators Q and P transposed

We redo the same with the left eigenvectors ϕ_n's.

Proposition 4.6 (Operator P). *There exist some coefficients $\widetilde{P}_{n,m}$ and some Laurent polynomial $\widetilde{U}_n(z) \in \mathbb{C}[z, 1/z]$ of degree $\geq -N$ such that*

$$Y(z)\phi_n(z) = \sum_{m=\max(0,n-p+1)}^{n+1} \widetilde{P}_{m,n}\phi_m(z) + \widetilde{U}_n(z)$$

with $\widetilde{U}_n(z) = 0$ if $n \geq p-1$. We have

$$\widetilde{P}_{n,m} = \langle \phi_m(z)Y(z), \psi_n(z)\rangle = \langle \phi_m(z), Y(z)\psi_n(z)\rangle = P_{n,m},$$

hence, $\widetilde{P}$ is the same matrix as that, P, computed in the previous section.

Proof. If $n \geq p-1$, we see that $Y(z)\phi_n(z) = O(z^{-n+p-2})$ is a polynomial of $1/z$ in $\widetilde{\mathfrak{W}}$ that vanishes at $z = \infty$, therefore, it is a unique linear combination of ϕ_m's, it can be written as $\sum_m \widetilde{P}_{m,n}\phi_m$. For $0 \leq n < p-1$, we subtract from $Y(z)\phi_n(z)$ the unique linear combination $\widetilde{P}_{m,n}\phi_m$ that makes it of degree $\leq N$ in $1/z$. The remainder $\widetilde{U}_n(z)$ is then a Laurent polynomial of degree $\geq -N$.

In all cases, let us compute the scalar product as the residue at 0. The remainder does not contribute, and we get

$$\langle \phi_m(z)Y(z), \psi_n(z)\rangle = \widetilde{P}_{n,m} = \langle \phi_m(z), Y(z)\psi_n(z)\rangle = P_{n,m},$$

therefore, $\widetilde{P}_{n,m} = P_{n,m}$. ■

Proposition 4.7 (Operator Q). *There exist some coefficients $\widetilde{Q}_{n,m}$ such that*

$$X(z)\phi_n(z) = \sum_{m=\max(0,n-q+1)}^{n+1} \widetilde{Q}_{m,n}\phi_m(z) + \delta_{n,0}.$$

We have

$$\widetilde{Q}_{n,m} = \langle \phi_m(z)X(z), \psi_n(z)\rangle = \langle \phi_m(z), X(z)\psi_n(z)\rangle = Q_{n,m},$$

hence, $\widetilde{Q}$ coincides with the matrix Q computed before.

Proof. We remark that $X(z)\phi_n(z) \in \widetilde{\mathfrak{W}}$. It can be uniquely decomposed as a linear combination of a constant and the ϕ_m's. The constant can occur only if $n = 0$, and is then worth $\delta_{n,0}$.

We observe that $\langle 1, \psi_m(z)\rangle = 0$, so that

$$Q_{n,m} = \langle \phi_m(z)X(z), \psi_n(z)\rangle = \langle \phi_m(z), X(z)\psi_n(z)\rangle,$$

which then coincides with the matrix found before. ■

Lemma 4.5. *For $l \geq 1$, we have*

$$X(z)^l \phi_n(z) = \sum_m (Q^l)_{m,n} \phi_m(z) + \left(\frac{X(z)^l - Q^l}{X(z) - Q} \right)_{0,n}.$$

Proof. we remark that $\frac{X(z)^l - Q^l}{X(z)-Q}$ is a polynomial of $X(z)$ and, thus, a Laurent polynomial of z. We prove the lemma by recursion on l. For $l = 1$, we have $(\frac{X(z)^l - Q^l}{X(z)-Q})_{0,n} = (\mathrm{Id})_{0,n} = \delta_{n,0}$, so the statement is true for $l = 1$.

Assume that the statement holds for l. Then, compute

$$\begin{aligned}
X(z)^{l+1}\phi_n(z) &= X(z)\left(\sum_k (Q^l)_{k,n}\phi_k(z) + \left(\frac{X(z)^l - Q^l}{X(z) - Q} \right)_{0,n} \right) \\
&= \sum_k (Q^l)_{k,n} X(z)\phi_k(z) + \left(X(z) \frac{X(z)^l - Q^l}{X(z) - Q} \right)_{0,n} \\
&= \sum_k (Q^l)_{k,n} \left(\sum_m Q_{m,k}\phi_m + \delta_{k,0} \right) \\
&\quad + \left(\frac{X(z)^{l+1} - Q^{l+1}}{X(z) - Q} - Q^l \right)_{0,n} \\
&= \sum_m (Q^{l+1})_{m,n}\phi_m + Q^l_{0,n} + \left(\frac{X(z)^{l+1} - Q^{l+1}}{X(z) - Q} - Q^l \right)_{0,n} \\
&= \sum_m (Q^{l+1})_{m,n}\phi_m + \left(\frac{X(z)^{l+1} - Q^{l+1}}{X(z) - Q} \right)_{0,n},
\end{aligned}$$

which is the statement at rank $l + 1$. ∎

Theorem 4.2. *Let $\widetilde{T} = P - \widetilde{V}'(Q)$. We have*

$$(Y(z) - \widetilde{V}'(X(z)))\phi_n(z) = \sum_{m=n+1}^{n+N} \widetilde{T}_{m,n}\phi_m(z),$$
$$\widetilde{T}_{n,m} = \langle \phi_m(z), (Y(z) - \widetilde{V}'(X(z)))\psi_n(z) \rangle.$$

In particular, $\widetilde{T}$ is a strictly lower triangular matrix, and we have

$$\widetilde{T}_{n,n-1} = 1.$$

Otherwise stated, we have

$$(P - \widetilde{V}'(Q))_+ = 0 \quad \textit{and} \quad (P - \widetilde{V}'(Q))_{n,n-1} = 1,$$

as wanted.

Proof. We remark that $(Y(z) - \tilde{V}'(X(z))) = \frac{1}{z} + O(z^{-2})$ is a polynomial of $1/z$, that vanishes at $z = \infty$. Therefore, $(Y(z) - \tilde{V}'(X(z)))\phi_n(z) \in \tilde{\mathfrak{W}}$ and vanishes at $z = \infty$, this implies that it must be a linear combination of ϕ_m's. We have

$$(Y(z) - \tilde{V}'(X(z)))\phi_n(z) = \sum_m P_{m,n}\phi_m(z) - \sum_m (\tilde{V}'(Q))_{m,n}\phi_m(z) + \tilde{U}_n(z) - \left(\frac{\tilde{V}'(X(z)) - \tilde{V}'(Q)}{X(z) - Q}\right)_{0,n}.$$

Notice that $\frac{\tilde{V}'(X(z))-\tilde{V}'(Q)}{X(z)-Q}$ is a polynomial of $X(z)$ of degree at most $p-2$ and, thus, a Laurent polynomial of z of degree $\geq -(p-2)(q-1) = -(N+2-q)$. If we assume $q \geq 2$, this implies that it can have degree $\geq -N$. We have

$$(Y(z) - \tilde{V}'(X(z)))\phi_n(z) - \sum_m \tilde{T}_{m,n}\phi_m(z) = \tilde{U}_n(z) - \left(\frac{\tilde{V}'(X(z)) - \tilde{V}'(Q)}{X(z) - Q}\right)_{0,n}.$$

Both terms on the right-hand side are Laurent polynomials of z, with possibly positive powers of z, however, since the left-hand side has no positive powers, all the positive powers must cancel. Therefore, the right-hand side is a polynomial of $1/z$, of degree $\geq -N$. The left-hand side vanishes at $z = \infty$ and is in $\tilde{\mathfrak{W}}$, therefore, the right-hand side must be a polynomial of $1/z$ in $\tilde{\mathfrak{W}}$, of degree $\geq -N$ and vanishing at ∞, therefore, it must vanish. This implies

$$(Y(z) - \tilde{V}'(X(z)))\phi_n(z) = \sum_m \tilde{T}_{m,n}\phi_m(z).$$

By taking the scalar product, we have

$$\tilde{T}_{n,m} = \langle \phi_m(z), (Y(z) - \tilde{V}'(X(z)))\psi_n(z)\rangle.$$

It then follows that $\tilde{T}_{n,m}$ is strictly lower triangular.

Finally, we have

$$\begin{aligned}
\tilde{T}_{n,n-1} &= \langle \phi_{n-1}(z), (Y(z) - \tilde{V}'(X(z)))\psi_n(z)\rangle \\
&= -\operatorname*{Res}_{z\to\infty} \frac{z^{N-1}z^{-n}z^{n+1+N}z^{-1}(1+O(1/z))}{\Delta(z)\bar{\Delta}(z)}dz \\
&= -\operatorname*{Res}_{z\to\infty} z^{-1}(1+O(1/z))dz \\
&= 1.
\end{aligned}$$

■

4.5. Conclusion of the proof

The combinatorial mobile enumeration problem has a unique solution for which the mobile or half-mobile generating functions are power series of the g_k's and $\tilde{g}_k$'s, with a well-defined limit when $n \to \infty$.

As in Section 2.5, we consider mobiles weighted by a weight g per labeled vertex by setting $g_1 = \tilde{g}_1 = 0$ and

$$\begin{aligned} g_k &= g^{\frac{k-2}{2}} \lambda_k, \quad k = 2, \dots, q, \\ \tilde{g}_k &= g^{\frac{k-2}{2}} \tilde{\lambda}_k, \quad k = 2, \dots, p. \end{aligned} \tag{4.6}$$

From what precedes, we may now state the following theorem, which is our main result.

Theorem 4.3 (Main theorem). *Assume $g_1 = \tilde{g}_1 = 0$ and g_k, $\tilde{g}_k$ as in equation (4.6). Then, the semi-infinite matrices Q and P whose elements are given by the scalar products*

$$Q_{n,m} = \langle \phi_m(z), X(z)\psi_n(z) \rangle, \quad P_{n,m} = \langle \phi_m(z), Y(z)\psi_n(z) \rangle, \quad n, m \geq 0,$$

are the solution to the combinatorial mobile problem. In particular, we have the expression

$$R_n = R \frac{h_{n-1} h_{n+1}}{h_n^2}, \quad h_n = \det_{1 \leq a,b \leq N} \left(\bar{w}_a^{n+b} - w_a^{n+b} \right). \tag{4.7}$$

Proof. Note that the above scalar products indeed satisfy equations (3.1), which is a direct consequence of Theorems 4.1 and 4.2. In order to prove that they correspond to the wanted combinatorial solution, we only have to verify that they define formal power series in $\sqrt{g}$. In practice, this boils down to prove that the R_n's are indeed power series in g.

Let us first discuss a number of alternative expressions for R_n. We first note that, as proved in Appendix B,

$$h_n = \prod_{a<b} (\bar{w}_a - \bar{w}_b) \prod_{a=1}^{N} \bar{w}_a^{n+1} \det_{1 \leq a,b \leq N} \left(\delta_{a,b} - \frac{\prod_{c \neq b} (w_a - \bar{w}_c)}{\prod_{c \neq a} (\bar{w}_a - \bar{w}_c)} \left(\frac{w_a}{\bar{w}_a} \right)^{n+1} \right), \tag{4.8}$$

and we may, therefore, write

$$R_n = R \frac{\bar{h}_{n-1} \bar{h}_{n+1}}{\bar{h}_n^2}, \quad \bar{h}_n = \det_{1 \leq a,b \leq N} \left(\delta_{a,b} - \frac{\rho_a}{w_a - \bar{w}_b} X_a^{n+1} \right), \tag{4.9}$$

with

$$\rho_a = \frac{\prod_{c \in \{1,\dots,N\}} (w_a - \bar{w}_c)}{\prod_{\substack{c \in \{1,\dots,N\} \\ c \neq a}} (\bar{w}_a - \bar{w}_c)} = \frac{\bar{\Delta}(w_a)}{\bar{\Delta}'(\bar{w}_a)}.$$

From (4.2), we may also write

$$\rho_a = -X_a^{N-1} \frac{\prod_{c \in \{1,\dots,N\}} (\bar{w}_a - w_c)}{\prod_{\substack{c \in \{1,\dots,N\} \\ c \neq a}} (w_a - w_c)} = -X_a^{N-1} \frac{\Delta(\bar{w}_a)}{\Delta'(w_a)}.$$

so that, using the identity

$$\det_{1\le a,b\le N}\left(\delta_{a,b}-\frac{\rho_a}{w_a-\bar{w}_b}X_a^{n+1}\right)=\det_{1\le a,b\le N}\left(\delta_{a,b}-\frac{\rho_a}{w_b-\bar{w}_a}X_a^{n+1}\right), \quad (4.10)$$

we get the alternative expression

$$\bar{h}_n=\det_{1\le a,b\le N}\left(\delta_{a,b}-\frac{\prod_{c\neq b}(\bar{w}_a-w_c)}{\prod_{c\neq a}(w_a-w_c)}\left(\frac{w_a}{\bar{w}_a}\right)^{n+N}\right). \quad (4.11)$$

In order to prove that R_n's are power series in g, it is enough to prove the following proposition.

Proposition 4.8. *For $n\ge 0$, the $\bar{h}_n$'s given by (4.9), or equivalently by (4.11), are formal power series in g.*

This is done in Appendix C. ■

5. Applications

5.1. The case of general planar map

The case of *general*, i.e., non-necessarily face bicolorable maps, is obtained by specializing the black face weights of Eulerian maps to $\widetilde{g_k}=\delta_{k,2}$ so that only black faces of degree 2 are allowed and receive weight 1. These Eulerian maps are clearly in bijection with general maps upon squeezing the bivalent black faces into single edges, keeping only as faces the original white faces. The faces of the general maps are weighted by g_k according to their degree k. Note that the canonical orientation on the Eulerian map is such that, after squeezing, each edge of the general map is oriented *both ways*. In particular, the oriented geodesic distance on the (supposedly pointed) Eulerian map is the true geodesic distance from the root vertex on the associated pointed general map (i.e., the graph distance using paths on un-oriented edges). Let us now discuss the enumeration of the mobiles corresponding to this specialization, with a special emphasis on the generating function R_i. Note that, in the map language, we may now interpret $R_i^{(0)}=R_i-R_{i-1}$ for $i\ge 2$ (or simply, $R_1^{(0)}=R_1$ if $i=1$) as the generating function for pointed general planar maps with a marked edge e connecting a vertex v at distance $i-1$ from the root vertex to a vertex v' at distance i.

5.1.1. Characteristic equation. From (2.3), we deduce that the only non-vanishing elements of P are

$$P_{i,i-1}=R_i,\quad P_{i,i}=B_{i,i}=W_{i,i}=:S_i,\quad P_{i,i+1}=B_{i,i+1}=W_{i,i+1}=1,$$

with R_i and S_i given in terms of P only via (2.4) and

$$S_i = \sum_{k\geq 1} g_k (P^{k-1})_{i,i}. \tag{5.1}$$

This in turn implies that

$$\begin{aligned} Y(z) &= \frac{R}{z} + S + z, \\ X(z) &= z + \sum_{j\geq 0} z^{-j} \sum_{k\geq 1+j} g_k \pi_{-j}(k-1; R, S), \end{aligned}$$

where

$$\pi_m(n; R, S) := [z^m] Y(z)^n$$

denotes the generating function for three-step paths, i.e., lattice paths in the discrete Cartesian plane $\mathbb{Z}\times\mathbb{Z}$, starting at $(0,0)$ and ending at (n,m), and made of elementary up-steps $(1,1)$, level-steps $(1,0)$ and down-steps $(1,-1)$ with a weight R attached to each down-step and a weight S attached to each level-step. Here, R and S are obtained through

$$R = 1\Bigg/\Bigg(1 - \sum_{k\geq 1} g_k \pi_1(k-1; R, S)\Bigg), \quad S = \sum_{k\geq 1} g_k \pi_0(k-1; R, S), \tag{5.2}$$

which are the large i limit counterparts of equations (2.4) and (5.1).

Writing $Y(w) = Y(\bar{w})$ with $w \neq \bar{w}$, we deduce that

$$0 = \frac{Y(w) - Y(\bar{w})}{w - \bar{w}} = -\frac{R}{w\bar{w}} + 1,$$

hence,

$$w\,\bar{w} = R,$$

while writing $X(w) = X(\bar{w})$, we deduce that

$$\begin{aligned} 0 = \frac{X(w) - X(\bar{w})}{w - \bar{w}} &= 1 + \sum_{j\geq 1} \frac{w^{-j} - \bar{w}^{-j}}{w - \bar{w}} \sum_{k\geq 1+j} g_k \pi_{-j}(k-1; R, S) \\ &= 1 - \sum_{j\geq 1} R^{-\frac{j+1}{2}} \Bigg(\sum_{\substack{n=-j+1 \\ n=-j+1[2]}}^{j-1} x^n \Bigg) \sum_{k\geq 1+j} g_k \pi_{-j}(k-1; R, S), \end{aligned}$$

where we have set

$$x = \sqrt{X} = \sqrt{\frac{w}{\bar{w}}}.$$

Multiplying the above equation by R and writing $j = |n| + 1 + 2m$ (with $m \geq 0$), we deduce the characteristic equation

$$0 = \sum_{n \in \mathbb{Z}} B_n x^n$$

$$\text{with } B_n = R\,\delta_{n,0} - \sum_{k \geq 2+|n|} g_k \sum_{m=0}^{\lfloor \frac{k-2-|n|}{2} \rfloor} \pi_{-2m-|n|-1}(k-1; R, S) R^{-m-\frac{|n|}{2}}. \quad (5.3)$$

We may compare this equation with the characteristic obtained in [6] by a totally different approach using continued fractions. Therein, the characteristic equation has exactly the same form as above, but with another expression for B_n, namely,

$$B_n = \sum_{s \geq |n|} A_s \pi_{|n|}(s; R, S) R^{\frac{|n|}{2}}, \quad A_s = R\Big(\delta_{s,0} - \sum_{k \geq s+2} g_k \pi_0(k-s-2; R, S)\Big),$$

or equivalently,

$$B_n = R\,\delta_{n,0} - \sum_{k \geq 2+|n|} g_k \sum_{s=|n|}^{k-2} \pi_0(k-s-2; R, S) \pi_{|n|}(s; R, S) R^{1+\frac{|n|}{2}}.$$

This apparently different expression for B_n turns out to be fully equivalent to our expression, as a consequence of the identity

$$\sum_{s=n}^{k-2} \pi_0(k-s-2; R, S) \pi_{|n|}(s; R, S) = \sum_{m=0}^{\lfloor \frac{k-2-|n|}{2} \rfloor} \pi_{-2m-|n|-1}(k-1; R, S) R^{-m-|n|-1}, \quad (5.4)$$

proved in Appendix D.

5.1.2. Expression for R_i. We will consider here general maps whose face degrees are bounded by a fixed integer, say q, larger that or equal to 3. In other words, we set $g_k = 0$ for $k > q$. Then, $X(z)$ has terms ranging from z to z^{-q+1} and the spectral curve has generically $N = q - 2$ pairs of double-points $(w_a, \bar{w}_a)$, $a = 1, \ldots, q-2$ (with $|w_a/\bar{w}_a| \leq 1$ by convention). These points satisfy $w_a \bar{w}_a = R$ and we may thus write

$$w_a = \sqrt{R}\, x_a, \quad \bar{w}_a = \frac{\sqrt{R}}{x_a}, \quad x_a = \sqrt{\frac{w_a}{\bar{w}_a}},$$

where the x_a, $a = 1, \ldots, q-2$ are the solutions with modulus less than 1 of the characteristic equation (5.3). Note that $B_n = 0$ for $|n| > q - 2$, so the characteristic equation has $2(q-2)$ solutions (the x_a and their inverses $1/x_a$), as expected. Our general expression (4.7) for R_i reads

$$R_i = R \frac{h_{i-1} h_{i+1}}{h_i^2},$$

with R given by (5.2) and with

$$h_i = \det_{1\le a,b\le q-2}\left(\bar{w}_a^{i+b} - w_a^{i+b}\right) = \sqrt{R}^{q-2} \det_{1\le a,b\le q-2}\left(\frac{1}{x_a^{i+b}} - x_a^{i+b}\right).$$

This allows us to write eventually

$$R_i = R\frac{\tilde{h}_{i-1}\tilde{h}_{i+1}}{\tilde{h}_i^2}, \quad \tilde{h}_i = \det_{1\le a,b\le q-2}\left(\frac{1}{x_a^{i+b}} - x_a^{i+b}\right).$$

This form is precisely the general expression obtained in [6] in the framework of continued fractions.

5.2. The case of p-constellations

A p-constellation is an Eulerian map whose black faces are all of degree p and whose white faces have a degree multiple of p. The generating functions for the corresponding mobiles are obtained by setting $\tilde{g}_k = \delta_{k,p}$ (we do not give a non-trivial weight to black faces as their number can be obtained from the numbers of white faces of all allowed degrees) and $g_k = \hat{g}_m$ if $k = pm$ for some m and $g_k = 0$ otherwise.

5.2.1. Characteristic equation. From (2.3), we now deduce the only non-vanishing elements of P are

$$P_{i,i-1} = R_i, \quad P_{i,i+p-1} = B_{i,i+p-1} = (Q^{p-1})_{i,i+p-1} = 1,$$

with R_i in terms of P only via

$$R_i = 1\Big/\Big(1 - \sum_{m\ge 1} \hat{g}_m (P^{pm-1})_{i-1,i}\Big). \tag{5.5}$$

As for Q, its only non-vanishing elements are $Q_{i,i+1} = 1$, $Q_{i,i-pm+1} = W_{i,i-pm+1}$ for $m \ge 1$. This leads to

$$Y(z) = \frac{R}{z} + z^{p-1}, \quad X(z) = z + \sum_{j\ge 1} z^{-pj+1} \sum_{m\ge j} \hat{g}_m \pi^{(p)}_{-pj+1}(pm-1; R),$$

where

$$\pi_m^{(p)}(n; R, S) \equiv [z^m]Y(z)^n$$

denotes the generating function for p-paths, i.e., lattice paths in the discrete Cartesian plane $\mathbb{Z} \times \mathbb{Z}$, starting at $(0,0)$ and ending at (n,m) and made of elementary up-steps $(1, p-1)$ and elementary down-steps $(1,-1)$, with a weight R attached to each down-step. Here, R itself is obtained through

$$R = 1\Big/\Big(1 - \sum_{m\ge 1} \hat{g}_m \pi_1^{(p)}(pm-1; R)\Big), \tag{5.6}$$

which is the large i counterpart of equation (5.5).

If we now consider p-constellations whose white face degrees are bounded, say by $p\,\ell$ (i.e., $\hat{g}_m = 0$ for $m > \ell$, and therefore, $q = p\,\ell$ with the notations of Section 2.4), then $X(z)$ has terms ranging from z to $z^{-p\,\ell+1}$ and the spectral curve has generically $(p-1)(p\,\ell-1)-1 = p(p\ell-\ell-1)$ pairs of double-points. Now, writing $Y(w) = Y(\bar{w})$ and $X(w) = X(\bar{w})$ displays a clear p-fold symmetry, so the double-points may be classified into p-tuples of pairs, namely,

$$(w_a\Omega^s, \bar{w}_a\Omega^s), \qquad \begin{cases} a = 1,\dots,p\ell-\ell-1, \\ s = 0,\dots,p-1, \end{cases} \qquad \Omega = e^{2i\pi/p}$$

with again $|w_a/\bar{w}_a| \le 1$ by convention. Note that we have made implicitly a choice of pair in each p-tuple (the pair corresponding to $s=0$) but this choice is irrelevant in what follows as all quantities involved below are invariant under the p-fold symmetry. For instance, we define

$$X_a = \frac{w_a}{\bar{w}_a},$$

which is also $w_a\Omega^s/\bar{w}_a\Omega^s$ for any s. Writing $Y(w_a) = Y(\bar{w}_a)$ with $w_a \ne \bar{w}_a$, we get

$$R = \bar{w}_a^p(X_a + X_a^2 + \cdots + X_a^{p-1}) = w_a^p(X_a^{-1} + X_a^{-2} + \cdots + X_a^{-p+1}),$$

and we may take the choice

$$w_a = \left(\frac{R}{X_a^{-1} + X_a^{-2} + \cdots + X_a^{-p+1}}\right)^{1/p}, \quad \bar{w}_a = \left(\frac{R}{X_a + X_a^2 + \cdots + X_a^{p-1}}\right)^{1/p}. \tag{5.7}$$

Writing $X(w_a) = X(\bar{w}_a)$, we deduce that

$$\begin{aligned} 0 &= \frac{X(w_a) - X(\bar{w}_a)}{w_a - \bar{w}_a} \\ &= 1 + \sum_{j=1}^{\ell} \frac{w_a^{-p\,j+1} - \bar{w}_a^{-p\,j+1}}{w_a - \bar{w}_a} \sum_{m=j}^{\ell} \hat{g}_m\, \pi^{(p)}_{-p\,j+1}(p\,m-1;R) \\ &= 1 - \sum_{j=1}^{\ell} R^{-j}(X_a + \cdots + X_a^{p-1})^{\frac{j}{2}}(X_a^{-1} + \cdots + X_a^{-p+1})^{\frac{j}{2}} \\ &\quad \times \left(\sum_{\substack{n=-p\,j+2 \\ n=-p\,j+2\,[2]}}^{p\,j-2} X_a^{\frac{n}{2}}\right) \sum_{m=j}^{\ell} \hat{g}_m \pi^{(p)}_{-p\,j+1}(p\,m-1;R) \\ &= 1 - \sum_{j=1}^{\ell} R^{-j}(X_a + \cdots + X_a^{p-1})^{j} \\ &\quad \times \left(\sum_{n=1}^{p\,j-1} X_a^{-n}\right) \sum_{m=j}^{\ell} \hat{g}_m \pi^{(p)}_{-p\,j+1}(p\,m-1;R), \end{aligned}$$

hence, the characteristic equation

$$X_a^{p\ell-\ell-1} = \sum_{j=1}^{\ell} R^{-j}(1+\cdots+X_a^{p-2})^j \times \left(\sum_{n=1}^{pj-1} X_a^{p\ell-\ell-1+j-n}\right) \sum_{m=j}^{\ell} \hat{g}_m \pi_{-pj+1}^{(p)}(pm-1;R),$$

which is a polynomial equation of degree $2(p\ell-\ell-1)$ in X_a (the coefficients in the sum over n are non-negative), hence, gives $2(p\ell-\ell-1)$ solutions, as wanted (the X_a and $1/X_a$ for $a=1,\ldots,p\ell-\ell-1$).

As a simple example, consider Eulerian p-angulations, made of black and white faces of degree p only, with a weight g per white face (the number of black and white faces are necessarily the same). The corresponding mobiles are obtained by setting $\hat{g}_m = g\delta_{m,1}$ (so that $\ell=1$) and the characteristic equation reads

$$X_a^{p-2} = R^{-1}(1+\cdots+X_a^{p-2})\left(\sum_{n=1}^{p-1} X_a^{p-1-n}\right) g\, R^{p-1},$$

or equivalently,

$$(1+X_a+\cdots+X_a^{p-2})(1+X_a^{-1}+\cdots+X_a^{-p+2}) = \frac{1}{g\, R^{p-2}},$$

with $R = 1/(1-g(p-1)R^{p-2})$ from (5.6). Using this last equation, we may write the following alternative form for the characteristic equation:

$$\sum_{n=1}^{p-2}(p-1-n)\big(X_a^n + X_a^{-n}\big) = \frac{1}{g\, R^{p-1}}.$$

5.2.2. Expression for R_i. Let us denote by $N_0 = p\ell-\ell-1$ the number of p-tuples of pairs of double-points of the spectral curve. From the general expressions (4.11) and (4.9), respectively, we have $R_i = R\bar{h}_{i-1}\bar{h}_{i+1}/(\bar{h}_i^2)$ with the following two equivalent expressions for $\bar{h}_i$:

$$\bar{h}_i = \det_{\substack{1\le a,b\le N_0\\ 0\le s,t\le p-1}} \left(\delta_{a,b}\delta_{s,t} - \frac{\prod_{(c,u)\neq(b,t)}(\bar{w}_a\Omega^s - w_c\Omega^u)}{\prod_{(c,u)\neq(a,s)}(w_a\Omega^s - w_c\Omega^u)} X_a^{i+N_0 p}\right) \tag{5.8}$$

and

$$\bar{h}_i = \det_{\substack{1\le a,b\le N_0\\ 0\le s,t\le p-1}} \left(\delta_{a,b}\delta_{s,t} - \frac{\prod_{(c,u)\neq(b,t)}(w_a\Omega^s - \bar{w}_c\Omega^u)}{\prod_{(c,u)\neq(a,s)}(\bar{w}_a\Omega^s - \bar{w}_c\Omega^u)} X_a^{i+1}\right). \tag{5.9}$$

Using the first expression (5.8) and performing the products, we deduce that

$$\bar{h}_i = \det_{\substack{1\le a,b\le N_0 \\ 0\le s,t\le p-1}} \left(\delta_{a,b}\delta_{s,t} - U_{a,b}^{(i)} \times \frac{1}{p} \sum_{r=0}^{p-1} (\Omega^{t-s} V_{a,b})^r \right), \tag{5.10}$$

where

$$U_{a,b}^{(i)} = \frac{\prod_{c\neq b}(\bar{w}_a^p - w_c^p)}{\prod_{c\neq a}(w_a^p - w_c^p)} X_a^{i+1+p(N_0-1)}, \quad V_{a,b} = \frac{w_b}{\bar{w}_a}.$$

Here, we used

$$\begin{aligned}
\frac{\prod_{u\neq t}(\bar{w}_a\Omega^s - w_b\Omega^u)}{\prod_{u\neq s}(w_a\Omega^s - w_a\Omega^u)} &= X_a^{1-p} \frac{\prod_{u\neq t}(\Omega^s - \frac{w_b}{\bar{w}_a}\Omega^u)}{\prod_{u\neq s}(\Omega^s - \Omega^u)} \\
&= \frac{X_a^{1-p}}{p} \prod_{u\neq t} (1 - \frac{w_b}{\bar{w}_a}\Omega^{u-s}) \\
&= \frac{X_a^{1-p}}{p} \sum_{r=0}^{p-1} \left(\Omega^{t-s} \frac{w_b}{\bar{w}_a} \right)^r .
\end{aligned}$$

Now, the matrix in equation (5.10) is a block-circulating matrix, so we may write its determinant as a product of determinants (see Appendix E for details):

$$\begin{aligned}
\bar{h}_i &= \prod_{s=0}^{p-1} \det_{1\le a,b\le N_0} (\delta_{a,b} - U_{a,b}^{(i)} V_{a,b}^s) \\
&= \prod_{s=0}^{p-1} \det_{1\le a,b\le N_0} \left(\delta_{a,b} - U_{a,b}^{(i)} \left(\frac{w_b}{\bar{w}_a} \right)^s \right) \\
&= \prod_{s=0}^{p-1} \frac{\prod_{1\le b\le N_0} w_b^s}{\prod_{1\le a\le N_0} \bar{w}_a^s} \det_{1\le a,b\le N_0} \left(\delta_{a,b} \left(\frac{w_a}{\bar{w}_a} \right)^{-s} - U_{a,b}^{(i)} \right) \\
&= \prod_{s=0}^{p-1} \det_{1\le a,b\le N_0} (\delta_{a,b} - U_{a,b}^{(i)} X_a^s) = \prod_{s=1}^{p} u_{i+s},
\end{aligned} \tag{5.11}$$

with

$$u_i = \det_{1\le a,b\le N_0} \left(\delta_{a,b} - \frac{\prod_{c\neq b}(\bar{w}_a^p - w_c^p)}{\prod_{c\neq a}(w_a^p - w_c^p)} X_a^{i+p(N_0-1)} \right).$$

Using this factorization of $\bar{h}_i$, we deduce that

$$R_i = R \frac{u_i u_{i+p+1}}{u_{i+1} u_{i+p}},$$

with u_i as above.

Using now the second expression (5.9) for $\bar{h}_i$, we get instead the alternative expression

$$\bar{h}_i = \prod_{s=1}^{p} v_{i+s}, \quad R_i = R\frac{v_i v_{i+p+1}}{v_{i+1}v_{i+p}},$$
$$\text{with } v_i = \det_{1\le a,b\le N_0}\left(\delta_{a,b} - \frac{\prod_{c\neq b}(w_a^p - \bar{w}_c^p)}{\prod_{c\neq a}(\bar{w}_a^p - \bar{w}_c^p)} X_a^i\right).$$

Using the explicit form (5.7) of w_a and $\bar{w}_a$ in terms of X_a, we may express u_i and v_i in terms of X_a only, namely,

$$u_i = \det_{1\le a,b\le N_0}\left(\delta_{a,b} - \frac{\prod_{c\neq b}(\xi_a - \chi_c)}{\prod_{c\neq a}(\chi_a - \chi_c)} X_a^i\right),$$
$$v_i = \det_{1\le a,b\le N_0}\left(\delta_{a,b} - \frac{\prod_{c\neq b}(\chi_a - \xi_c)}{\prod_{c\neq a}(\xi_a - \xi_c)} X_a^{i+p(N_0-1)}\right),$$
$$\xi_a \equiv X_a + \cdots + X_a^{p-1}, \quad \chi_a \equiv X_a^{-1} + \cdots + X_a^{-p+1}.$$

In particular, u_i reads

$$\begin{aligned} u_i &= \det_{1\le a,b\le N_0}\left(\delta_{a,b} - \frac{\sigma_a}{(\xi_a - \chi_b)} X_a^i\right) \\ &= \sum_{K\subset\{1,\dots,N_0\}} \prod_{a\in K}(-\sigma_a X_a^i) \times \det_{a,b\in K}\left(\frac{1}{\xi_a - \chi_b}\right) \\ &= \sum_{K\subset\{1,\dots,N_0\}} \prod_{a\in K}(-\tau_a X_a^i) \prod_{\substack{a,b\in K \\ a<b}} \frac{(\xi_a - \xi_b)(\chi_a - \chi_b)}{(\xi_a - \chi_b)(\chi_a - \xi_b)}, \end{aligned}$$

with

$$\sigma_a \equiv \frac{\prod_c(\xi_a - \chi_c)}{\prod_{c\neq a}(\chi_a - \chi_c)}, \quad \tau_a \equiv \frac{\prod_{c\neq a}(\xi_a - \chi_c)}{\prod_{c\neq a}(\chi_a - \chi_c)},$$

and similarly,

$$\begin{aligned} v_i &= \det_{1\le a,b\le N_0}\left(\delta_{a,b} - \frac{\tilde{\sigma}_a}{(\chi_a - \xi_b)} X_a^i\right) \\ &= \sum_{K\subset\{1,\dots,N_0\}} \prod_{a\in K}(-\tilde{\sigma}_a X_a^i) \times \det_{a,b\in K}\left(\frac{1}{\chi_a - \xi_b}\right) \\ &= \sum_{K\subset\{1,\dots,N_0\}} \prod_{a\in K}(-\tilde{\tau}_a X_a^i) \prod_{\substack{a,b\in K \\ a<b}} \frac{(\xi_a - \xi_b)(\chi_a - \chi_b)}{(\xi_a - \chi_b)(\chi_a - \xi_b)}, \end{aligned}$$

with

$$\widetilde{\sigma}_a \equiv X_a^{p(N_0-1)} \frac{\prod_c (\chi_a - \xi_c)}{\prod_{c \neq a} (\xi_a - \xi_c)}, \quad \widetilde{\tau}_a \equiv X_a^{p(N_0-1)} \frac{\prod_{c \neq a} (\chi_a - \xi_c)}{\prod_{c \neq a} (\xi_a - \xi_c)},$$

which is precisely the form that was conjectured in [4, 12].

6. Conclusion

In this paper, we showed that the system of equations satisfied by the mobile generating functions falls into the category of integrable systems. In particular, we exhibited an explicit formula for the generating function R_i of mobiles rooted at a labeled vertex i as the ratio of appropriate $(N \times N)$ determinants involving the double points of the spectral curve. Note that, in our derivation, the fact that the operators P and Q take the form of band matrices was crucial in our analysis. In our case, this property was guaranteed by the fact that our mobiles have unlabeled vertices of bounded degrees. We know, however, from the continued fraction analysis of [6] that for $p = 2$ this result can be extended to white unlabeled vertices of unbounded degree: in that case, the expression for R_i involves Hankel determinants which are typically of size $i \times i$, hence, growing with i. It is therefore likely that our results may be extended to the case of mobiles with unbounded black and white vertex degrees, but the explicit expression for R_i is yet to be determined. A first step in this direction would be to study mobile enumeration problems for which the derivatives of the potentials ($\widetilde{V}(x)$ and $V(y)$) are rational fractions of x and y. Indeed, we expect in this case that the P and Q operators still remain band matrices, allowing us to extend our construction in a quite straightforward way.

Another direction of study would be to consider the case of (face-bicolored) maps drawn on higher genus and/or non-orientable surfaces, in correspondence with unicellular mobiles [3].

A. Proof of Proposition 2.6

Let us rewrite equations (2.3) and (2.5) in a way which shows that R_i, $W_{i,j}$, and $B_{i,j}$ have an expansion as power series in $\sqrt{g}$.

We first use (2.5) to write

$$R_i = 1 + \widetilde{\lambda}_2 W_{i,i-1} + \sum_{k \geq 3} \widetilde{\lambda}_k \sqrt{g}^{k-2} (Q^{k-1})_{i,i-1},$$

while from (2.3), we get

$$W_{i,i-1} = \lambda_2 R_i + \sum_{k\geq 3} \lambda_k \sqrt{g}^{k-2} \left(P^{k-1}\right)_{i,i-1}.$$

From these two equations, we deduce that

$$\begin{aligned} R_i &= \frac{1}{1-\tilde{\lambda}_2\lambda_2} + \frac{1}{1-\tilde{\lambda}_2\lambda_2} \\ &\quad \times \sum_{k\geq 3} \sqrt{g}^{k-2} \{\tilde{\lambda}_k \left(Q^{k-1}\right)_{i,i-1} + \tilde{\lambda}_2\lambda_k \left(P^{k-1}\right)_{i,i-1}\}, \\ W_{i,i-1} &= \frac{\lambda_2}{1-\tilde{\lambda}_2\lambda_2} + \frac{1}{1-\tilde{\lambda}_2\lambda_2} \\ &\quad \times \sum_{k\geq 3} \sqrt{g}^{k-2} \{\tilde{\lambda}_k \lambda_2 \left(Q^{k-1}\right)_{i,i-1} + \lambda_k \left(P^{k-1}\right)_{i,i-1}\}. \end{aligned} \tag{A.1}$$

As for $B_{i,i+1}$, we immediately obtain from (2.3) that

$$B_{i,i+1} = \tilde{\lambda}_2 + \sum_{k\geq 3} \tilde{\lambda}_k \sqrt{g}^{k-2} \left(Q^{k-1}\right)_{i,i+1}. \tag{A.2}$$

We now use

$$\begin{aligned} W_{i,i} &= \lambda_2 B_{i,i} + \sum_{k\geq 3} \lambda_k \sqrt{g}^{k-2} \left(P^{k-1}\right)_{i,i}, \\ B_{i,i} &= \tilde{\lambda}_2 W_{i,i} + \sum_{k\geq 3} \tilde{\lambda}_k \sqrt{g}^{k-2} \left(Q^{k-1}\right)_{i,i} \end{aligned}$$

to deduce that

$$\begin{aligned} W_{i,i} &= \frac{1}{1-\tilde{\lambda}_2\lambda_2} \sum_{k\geq 3} \sqrt{g}^{k-2} \{\tilde{\lambda}_k \lambda_2 \left(Q^{k-1}\right)_{i,i} + \lambda_k \left(P^{k-1}\right)_{i,i}\}, \\ B_{i,i} &= \frac{1}{1-\tilde{\lambda}_2\lambda_2} \sum_{k\geq 3} \sqrt{g}^{k-2} \{\tilde{\lambda}_k \left(Q^{k-1}\right)_{i,i} + \tilde{\lambda}_2 \lambda_k \left(P^{k-1}\right)_{i,i}\}. \end{aligned} \tag{A.3}$$

Finally, for $j < i-1$, we may use the property that $(P^{k-1})_{i,j} = 0$ for $k-1 < i-j$ to write

$$W_{i,j} = \sum_{k\geq i-j+1} \sqrt{g}^{k-2} \lambda_k (P^{k-1})_{i,j}, \tag{A.4}$$

which involves only strictly positive power of $\sqrt{g}$ since $i-j+1 > 2$.

Similarly, for $j > i+1$, we may use the property that $(Q^{k-1})_{i,j} = 0$ for $k-1 < j-i$ to write

$$B_{i,j} = \sum_{k\geq j-i+1} \sqrt{g}^{k-2} \tilde{\lambda}_k (Q^{k-1})_{i,j}, \tag{A.5}$$

which involves only strictly positive power of $\sqrt{g}$ since $j-i+1 > 2$.

From (A.1), (A.2), (A.3), (A.4), and (A.5) it is clear that the generating functions R_i, $B_{i,j}$ and $W_{i,j}$ have formal series expansions in powers of $\sqrt{g}$ and their coefficients $\sqrt{g}^n$ are uniquely determined recursively from the knowledge of the coefficients $\sqrt{g}^m$ of these generating functions for all $m < n$. In other words these equations determine uniquely the expansions of $B_{i,j}$, $W_{i,j}$, and R_i to all orders in $\sqrt{g}$ from the initial values given by

$$\begin{aligned}
R_i &= \frac{1}{1-\tilde{\lambda}_2\lambda_2} + O(\sqrt{g}),\\
W_{i,i-1} &= \frac{\lambda_2}{1-\tilde{\lambda}_2\lambda_2} + O(\sqrt{g}),\\
B_{i,i+1} &= \tilde{\lambda}_2 + O(\sqrt{g}),\\
B_{i,j} &= O(\sqrt{g}) \quad \text{for } j = i \text{ or } j > i+1,\\
W_{i,j} &= O(\sqrt{g}) \quad \text{for } j = i \text{ or } j < i-1.
\end{aligned}$$

It is clear from the form of the recursive equations (A.1), (A.2), (A.3), (A.4), and (A.5) that the coefficients in these expansions are polynomial in $\lambda_2,\dots,\lambda_q,\tilde{\lambda}_2,\dots,\tilde{\lambda}_q$ and $\frac{1}{1-\tilde{\lambda}_2\lambda_2}$ with non-negative integer coefficients.

B. Proof of the equality (4.8)

We adapt and extend here a similar proof given in [4, Appendix B]. Consider the $N \times N$ matrices $M^{(n)}$ and H with entries

$$\begin{aligned}
M^{(n)}_{a,b} &= \delta_{a,b} - \frac{\prod_{c\neq b}(w_a - \bar{w}_c)}{\prod_{c\neq a}(\bar{w}_a - \bar{w}_c)}\left(\frac{w_a}{\bar{w}_a}\right)^{n+1},\\
H_{a,b} &= \frac{\bar{w}_a^{b-1}}{\prod_{c\neq a}(\bar{w}_a - \bar{w}_c)},
\end{aligned}$$

where $a, b = 1,\dots,N$. We have

$$(M^{(n)}H)_{a,b} = \frac{\bar{w}_a^{b-1}}{\prod_{c\neq a}(\bar{w}_a - \bar{w}_c)}\left(1 - \left(\frac{w_a}{\bar{w}_a}\right)^{n+1}\frac{1}{\bar{w}_a^{b-1}}\sum_{e=1}^{N}\bar{w}_e^{b-1}\prod_{c\neq e}\frac{(w_a-\bar{w}_c)}{(\bar{w}_e-\bar{w}_c)}\right).$$

Now, the function

$$f_b(x) = \sum_{e=1}^{N}\bar{w}_e^{b-1}\prod_{c\neq e}\frac{(w-\bar{w}_c)}{(\bar{w}_e-\bar{w}_c)}$$

is a polynomial in w of degree at most $N-1$ which satisfies $f_b(\bar{w}_a) = \bar{w}_a^{b-1}$, so that the polynomial $f_b(w) - w^{b-1}$ (of degree at most $N-1$ since $1 \le b \le N$) vanishes at the N distinct points $\bar{w}_a$, $a = 1, \dots, N$. We deduce that $f_b(w) = w^{b-1}$ for $1 \le b \le N$ and

$$(M^{(n)} H)_{a,b} = \frac{\bar{w}_a^{-n-1}}{\prod_{c \ne a} (\bar{w}_a - \bar{w}_c)} (\bar{w}_a^{n+b} - w_a^{n+b}).$$

The identity (4.8) follows immediately since

$$\prod_a \prod_{c \ne a} (\bar{w}_a - \bar{w}_c) \times \det_{1 \le a,b \le N} H_{a,b} = \prod_{1 \le a < b \le N} (\bar{w}_a - \bar{w}_b).$$

C. Proof of Proposition 4.8

Let us set $g_1 = \tilde{g}_1 = 0$ and

$$\begin{aligned} g_k &= g^{\frac{k-2}{2}} \lambda_k, \quad k = 2, \dots, q, \\ \tilde{g}_k &= g^{\frac{k-2}{2}} \tilde{\lambda}_k, \quad k = 2, \dots, p, \end{aligned} \tag{C.1}$$

where it is implicitly assumed that $\lambda_q \ne 0$ and $\tilde{\lambda}_p \ne 0$ and $|\lambda_2 \tilde{\lambda}_2| < 1$.

From equation (4.1), we first deduce that

$$\begin{aligned} \alpha_0 &= \lambda_2 \beta_0 + O(\sqrt{g}), \\ \beta_0 &= \tilde{\lambda}_2 \alpha_0 + O(\sqrt{g}), \end{aligned} \tag{C.2}$$

hence, $\alpha_0 = O(\sqrt{g})$ and $\beta_0 = O(\sqrt{g})$. We also deduce that, for $j > 0$,

$$\begin{aligned} \alpha_j &= \lambda_{j+1} \sqrt{g}^{\,j-1} R^j + O(\sqrt{g}^{\,j}), \\ \beta_j &= \tilde{\lambda}_{j+1} \sqrt{g}^{\,j-1} + O(\sqrt{g}^{\,j}). \end{aligned} \tag{C.3}$$

From the third equation in (4.1), we deduce that

$$R - 1 = \lambda_2 \tilde{\lambda}_2 R + O(\sqrt{g})$$

and

$$R = \frac{1}{1 - \lambda_2 \tilde{\lambda}_2} + O(\sqrt{g}). \tag{C.4}$$

In practice, (C.1), (C.2), (C.3), and (C.4) are the first terms of a systematic expansion of α_j, β_j, and R in power of $\sqrt{g}$. We now look for double points w_a, $\bar{w}_a$ solutions of $X(w_a) = X(\bar{w}_a)$ and $Y(w_a) = Y(\bar{w}_a)$, i.e.,

$$\begin{aligned} w_a + \sum_{j=0}^{q-1} \alpha_j w_a^{-j} &= \bar{w}_a + \sum_{j=0}^{q-1} \alpha_j \bar{w}_a^{-j}, \\ \frac{R}{w_a} + \sum_{j=0}^{p-1} \beta_j w_a^j &= \frac{R}{\bar{w}_a} + \sum_{j=0}^{p-1} \beta_j \bar{w}_a^j, \end{aligned} \tag{C.5}$$

where we impose by convention that $|w_a| < |\bar{w}_a|$. From these equations, we deduce that

$$w_a \sim \frac{\sqrt{g}}{\eta_a} R, \quad \bar{w}_a \sim \frac{\xi_a}{\sqrt{g}},$$

where η_a and ξ_a are solutions of the system of equations

$$\sum_{j=1}^{q-1} \lambda_{j+1} \eta_a^j = \xi_a, \quad \sum_{j=1}^{p-1} \tilde{\lambda}_{j+1} \xi_a^j = \eta_a$$

as obtained by equating the leading order of the left-hand side and right-hand side of equations in (C.5) (all of order $\frac{1}{\sqrt{g}}$). Finding the solutions of the system is equivalent to finding the roots of polynomial of degree $(p-1)(q-1) = N+1$. Removing the unwanted trivial solution $(\xi_a, \eta_a) = (0,0)$, we are left with exactly N solutions (generically) which yield the desired double points $w_a, \bar{w}_a$. We have, in particular,

$$X_a = \frac{w_a}{\bar{w}_a} = \frac{gR}{\eta_a \xi_a}(1 + O(\sqrt{g})).$$

We finally deduce that

$$\begin{aligned} \bar{h}_n &= \det_{1 \le a,b \le N} \left(\delta_{a,b} - \frac{\prod_{c \ne b}(w_a - \bar{w}_c)}{\prod_{c \ne a}(\bar{w}_a - \bar{w}_c)} X_a^{n+1} \right) \\ &= 1 - \left(\frac{g}{1 - \lambda_2 \tilde{\lambda}_2} \right)^{n+1} \sum_{a=1}^{N} \frac{1}{(\xi_a \eta_a)^{n+1}} \prod_{c \ne a} \frac{1}{1 - \frac{\xi_a}{\xi_c}} (1 + O(\sqrt{g})), \end{aligned}$$

which is a first term of a systematic expansion in powers of $\sqrt{g}$. Let us now see in detail how this expansion works.

From equation (4.1), we have

$$\begin{aligned} \alpha_j &= \lambda_{j+1} g^{\frac{j-1}{2}} R^j + \lambda_{j+2} g^{\frac{j}{2}} (j+1) \beta_0 R^j + o\big(g^{\frac{j+1}{2}}\big), \\ \beta_j &= \tilde{\lambda}_{j+1} g^{\frac{j-1}{2}} + \tilde{\lambda}_{j+2} g^{\frac{j}{2}} (j+1) \alpha_0 + o\big(g^{\frac{j+1}{2}}\big), \end{aligned}$$

with

$$\begin{aligned} \alpha_0 &= \sqrt{g} \frac{2\lambda_2 \lambda_3}{1 - \lambda_2 \tilde{\lambda}_2} + o(\sqrt{g}), \\ \beta_0 &= \sqrt{g} \frac{2\tilde{\lambda}_2 \tilde{\lambda}_3}{1 - \lambda_2 \tilde{\lambda}_2} + o(\sqrt{g}); \end{aligned}$$

then, we may thus write

$$\begin{aligned} \alpha_j &= \lambda_{j+1} g^{\frac{j-1}{2}} R^j + \gamma_j g^{\frac{j+1}{2}} R^j + o\big(g^{\frac{j+1}{2}}\big), \\ \beta_j &= \tilde{\lambda}_{j+1} g^{\frac{j-1}{2}} + \tilde{\gamma}_j g^{\frac{j+1}{2}} + o\big(g^{\frac{j+1}{2}}\big). \end{aligned}$$

Let us set $w_a = g^{\frac{1}{2}} z_a$ and $\bar{w}_a = g^{\frac{-1}{2}} \bar{z}_a$: the system of equations (C.5) becomes

$$
\begin{aligned}
& g z_a + g \sum_{j=1}^{q-1} \gamma_j R^j z_a^{-j} + \sum_{j=1}^{q-1} \lambda_{j+1} R^j z_a^{-j} \\
& \quad = \bar{z}_a + \sum_{j=1}^{q-1} g^j \lambda_{j+1} R^j \bar{z}_a^{-j} + \sum_{j=1}^{q-1} \gamma_j g^{j+1} R^j \bar{z}_a^{-j} + O(g^2), \\
& \frac{R}{z_a} + \sum_{j=1}^{p-1} g^j \widetilde{\lambda}_{j+1} z_a^j + \sum_{j=1}^{p-1} g^{j+1} \widetilde{\gamma}_j z_a^j \\
& \quad = g \frac{R}{\bar{z}_a} + \sum_{j=1}^{p-1} \widetilde{\lambda}_{j+1} \bar{z}_a^j + g \sum_{j=1}^{p-1} \widetilde{\gamma}_j \bar{z}_a^j + O(g^2).
\end{aligned}
$$

This can be written as

$$
\begin{aligned}
\sum_{j=1}^{q-1} \lambda_{j+1} R^j z_a^{-j} - \bar{z}_a &= -g z_a - g \sum_{j=1}^{q-1} \gamma_j R^j z_a^{-j} + g \lambda_2 R \bar{z}_a^{-1} + O(g^2), \\
\sum_{j=1}^{p-1} \widetilde{\lambda}_{j+1} \bar{z}_a^j - \frac{R}{z_a} &= g \widetilde{\lambda}_2 z_a - g \frac{R}{\bar{z}_a} - g \sum_{j=1}^{p-1} \widetilde{\gamma}_j \bar{z}_a^j + O(g^2).
\end{aligned}
\tag{C.6}
$$

Define the functions f and h as

$$
\begin{aligned}
f(z, \bar{z}) &= \sum_{j=1}^{q-1} \lambda_{j+1} R^j z^{-j} - \bar{z}, \\
h(z, \bar{z}) &= \sum_{j=1}^{p-1} \widetilde{\lambda}_{j+1} \bar{z}^j - \frac{R}{z}.
\end{aligned}
$$

As we have seen above, $f(z, \bar{z}) = h(z, \bar{z}) = 0$ fixes the leading order to $(z_a^*, \bar{z}_a^*) = (\frac{R}{\eta_a}, \xi_a)$. Writing $z = z_a^* + \delta z_a$, $\bar{z}_a = \bar{z}_a^* + \delta \bar{z}_a$ and linearizing (C.6) to first order in δz_a and $\delta \bar{z}_a$, we get a linear system involving the matrix

$$
D = \begin{pmatrix} \frac{\partial f}{\partial z}\Big|_{\substack{z = z_a^* \\ \bar{z} = \bar{z}_a^*}} & \frac{\partial f}{\partial \bar{z}}\Big|_{\substack{z = z_a^* \\ \bar{z} = \bar{z}_a^*}} \\ \frac{\partial h}{\partial z}\Big|_{\substack{z = z_a^* \\ \bar{z} = \bar{z}_a^*}} & \frac{\partial h}{\partial \bar{z}}\Big|_{\substack{z = z_a^* \\ \bar{z} = \bar{z}_a^*}} \end{pmatrix}.
$$

Its determinant is

$$
\det(D) = -\frac{1}{R} \sum_j j \lambda_{j+1} \eta_a^{j+1} \sum_j j \widetilde{\lambda}_{j+1} \xi_a^{j-1} + \frac{\eta_a^2}{R}.
$$

For $\lambda_a = \lambda_q \delta_{q,a}$ and $\widetilde{\lambda}_a = \widetilde{\lambda}_p \delta_{p,a}$, we get

$$\det(D) = -\frac{N}{R}\eta_a^2 \neq 0.$$

We may deduce from this special case that the determinant does not vanish generically. Therefore, the matrix D is invertible, which means that there is a second order expansion of w_a and $\bar{w}_a$ and recursively there is an expansion to all orders. Furthermore, since the right-hand side of equation (C.6) is an integer power of g, the expansion of z_a and $\bar{z}_a$ is also an integer power of g.

Let us calculate the second order expansion. Defining f_1 and h_1 as

$$f_1(z,\bar{z}) = -z - \sum_{j=1}^{q-1} \gamma_j R^j z^{-j} + \lambda_2 R \bar{z}^{-1},$$

$$h_1(z,\bar{z}) = \widetilde{\lambda}_2 z - \frac{R}{\bar{z}} - \sum_{j=1}^{p-1} \widetilde{\gamma}_j \bar{z}^j,$$

we have

$$\begin{pmatrix} \delta z_a \\ \delta \bar{z}_a \end{pmatrix} = g D^{-1} \begin{pmatrix} f_1(z_a^*, \bar{z}_a^*) \\ h_1(z_a^*, \bar{z}_a^*) \end{pmatrix}.$$

We can see now that the expansion of $\bar{h}_n$ is given by integer powers of g.

D. Proof of the identity (5.4)

To prove (5.4), we first use $\pi_m(n; R, S) = R^{-m}\pi_{-m}(n; R, S)$ (as obtained by reversing the height of the enumerated paths) to write the r.h.s. of this equation as

$$\begin{aligned}
&\sum_{m=0}^{\lfloor \frac{k-2-|n|}{2} \rfloor} \pi_{-2m-|n|-1}(k-1; R, S) R^{-m-|n|-1} \\
&= \sum_{m=0}^{\lfloor \frac{k-2-|n|}{2} \rfloor} \pi_{2m+|n|+1}(k-1; R, S) R^m \\
&= \sum_{s=n}^{k-2} \pi_{|n|}(s; R, S) \sum_{m=0}^{\lfloor \frac{k-s-2}{2} \rfloor} \pi_{2m}^{+}(k-s-2; R, S) R^m,
\end{aligned}$$

where $\pi_m^+(n; R, S)$ denote tree-step paths from $(0,0)$ to (n,m) *which remain above height* 0. Indeed, the r.h.s. in the first line above enumerates three-step paths from $(0,0)$ to $(k-1, 2m+|n|+1)$ for all possible positive m, with an extra weight R^m.

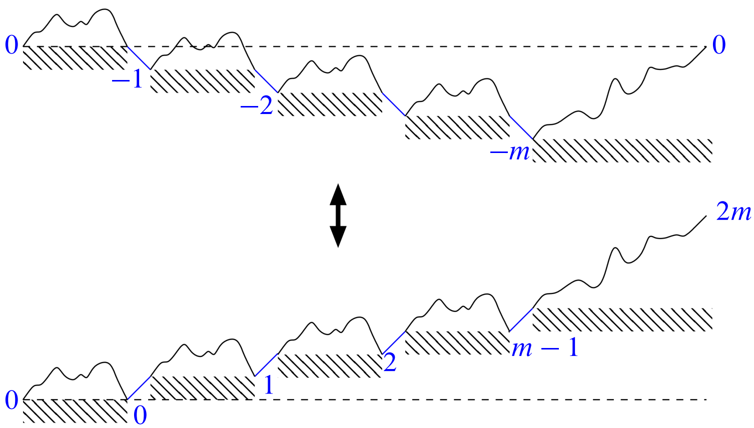

Figure 9. Sketch of the proof of equation (D.1) for a given fixed r: a path enumerated by $\pi_0(r; R, S)$ and minimum height $-m$ (top) is bijectively mapped onto a path enumerated by $\pi^+_{2m}(r; R, S)$ (bottom). The weight of the former path is R^m times that of the latter.

By marking the *last passage* at height $|n|$ (which is a step $(s, |n|) \to (s + 1, |n|)$ for some s between 0 and $k - 2$), the path is decomposed into a first part enumerated by $\pi_{|n|}(s; R, S)$ and a last part enumerated by $\pi^+_{2m}(k - s - 2; R, S)$ (paths of length $k - s - 2$ from height $|n| + 1$ to height $|n| + 1 + 2m$ which remain above height $|n| + 1$). Proving equation (5.4), therefore, reduces to proving

$$\pi_0(r; R, S) = \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \pi^+_{2m}(r; R, S) R^m \tag{D.1}$$

for some arbitrary r, which is done as follows (see Figure 9 for an illustration): consider a path enumerated by $\pi_0(r; R, S)$ and mark the step before the *first* passage at height -1 (which is a step $(s_1, 0) \to (s_1 + 1, -1)$ for some $s_1 \geq 0$), then the step before the first passage at height -2 (which is a step $(s_1 + 1 + s_2, -1) \to (s_1 + 1 + s_2 + 1, -2)$ for some $s_2 \geq 0$), and so on up to the step before the first passage at height $-m$, where $-m$ is the minimum height reached by the path (this last marked step is of type $(s_1 + 1 + s_2 + 1 + \cdots + s_m, -m + 1) \to (s_1 + 1 + s_2 + 1 + \cdots + s_m + 1, -m)$ for some $s_m \geq 0$). The last part of the path is a path from height $-m$ to height 0 which remains above height $-m$. The m marked steps are down steps, hence, contribute R^m to $\pi_0(r; R, S)$. Replacing these steps by up steps creates paths of length r from height 0 to height $2m$ which remain above height 0, as enumerated by $\pi^+_{2m}(r; R, S)$. This mapping is bijective since the steps which have been reversed are easily identified as the steps just after the *last* passage at the heights $0, 1, \ldots, m - 1$ in the image path. Equation (D.1) follows immediately.

E. Proof of the factorization (5.11)

Consider the matrix

$$M := \left(\delta_{a,b}\delta_{s,t} - U^{(i)}_{a,b} \times \frac{1}{p}\sum_{r=0}^{p-1}(\Omega^{t-s}V_{a,b})^r\right)_{\substack{1\le a,b\le N_0\\ 0\le s,t\le p-1}}$$

whose determinant gives the desired $\bar{h}_i$. This matrix is block-circulating (recall that $\Omega^p = 1$), i.e., may be written as

$$M = \begin{pmatrix} (m_{a,b}(0))_{1\le a,b\le N_0} & (m_{a,b}(1))_{1\le a,b\le N_0} & (m_{a,b}(2))_{1\le a,b\le N_0} & \cdots & (m_{a,b}(p-1))_{1\le a,b\le N_0} \\ (m_{a,b}(p-1))_{1\le a,b\le N_0} & (m_{a,b}(0))_{1\le a,b\le N_0} & (m_{a,b}(1))_{1\le a,b\le N_0} & \cdots & (m_{a,b}(p-2))_{1\le a,b\le N_0} \\ (m_{a,b}(p-2))_{1\le a,b\le N_0} & (m_{a,b}(p-1))_{1\le a,b\le N_0} & (m_{a,b}(0))_{1\le a,b\le N_0} & \cdots & (m_{a,b}(p-3))_{1\le a,b\le N_0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m_{a,b}(1))_{1\le a,b\le N_0} & (m_{a,b}(2))_{1\le a,b\le N_0} & (m_{a,b}(3))_{1\le a,b\le N_0} & \cdots & (m_{a,b}(0))_{1\le a,b\le N_0} \end{pmatrix}$$

with blocks formed of $N_0 \times N_0$ matrices $m(t)$ with elements $(m(t))_{a,b} := m_{a,b}(t)$ given by

$$m_{a,b}(t) = \delta_{a,b}\delta_{t,0} - U^{(i)}_{a,b} \times \frac{1}{p}\sum_{r=0}^{p-1}(\Omega^t V_{a,b})^r.$$

Introducing the $p \times p$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

we may write

$$M = \sum_{t=0}^{p-1} m(t) \otimes A^t.$$

Now, A is easily diagonalized into $A = P \cdot D \cdot P^{-1}$ with D the $p \times p$ diagonal matrix with diagonal elements Ω^s for $s = 0, \ldots, p-1$, and some unimportant invertible matrix P. We deduce $M = (1 \otimes P) \cdot (\sum_{t=0}^{p-1} m(t) \otimes D^t) \cdot (1 \otimes P)^{-1}$ and $\det(M) = \det(\sum_{t=0}^{p-1} m(t) \otimes D^t)$, which is the determinant of a block-diagonal matrix, hence,

$$\det(M) = \prod_{s=0}^{p-1} \det\left(\sum_{t=0}^{p-1} m(t)(\Omega^s)^t\right).$$

Now, we have

$$\sum_{t=0}^{p-1} m_{a,b}(t)(\Omega^s)^t = \delta_{a,b} - \sum_{t=0}^{p-1} U_{a,b}^{(i)} \times \frac{1}{p}\sum_{r=0}^{p-1}\left(\Omega^t V_{a,b}\right)^r (\Omega^s)^t$$
$$= \delta_{a,b} - U_{a,b}^{(i)} \sum_{r=0}^{p-1} V_{a,b}^r \times \frac{1}{p}\sum_{t=0}^{p-1} \Omega^{(r+s)t}$$
$$= \begin{cases} \delta_{a,b} - U_{a,b}^{(i)} & \text{if } s = 0 \\ \delta_{a,b} - U_{a,b}^{(i)} V_{a,b}^{p-s} & \text{if } s = 1, \dots, p-1 \end{cases}$$

and, upon reorganizing the terms, we end up with

$$\det(M) = \prod_{s=0}^{p-1} \det_{1\le a,b\le N_0}\left(\delta_{a,b} - U_{a,b}^{(i)} V_{a,b}^s\right)$$

as wanted.

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