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Schröder trees and antipode formulas: An application to non-commutative probability and Wick polynomials

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Abstract. The double tensor Hopf algebra has been introduced by Ebrahimi-Fard and Patras to provide an algebraic framework for cumulants in non-commutative probability theory. In this paper, we obtain a cancellation-free formula, represented in terms of Schröder trees, for the antipode in the double tensor Hopf algebra. We apply the antipode formula to recover cumulant-moment formulas as well as a new expression for Anshelevich's free Wick polynomials in terms of Schröder trees.

1. Introduction

In *Quantum Field Theory*, Alain Connes and Dirk Kreimer [16,17] presented a method to compute certain divergent integrals associated with Feynman diagrams using the theory of *Hopf algebras* [11, Chap. 3]. In their approach, Connes and Kreimer introduced a Hopf algebra of rooted trees such that each Feynman diagram has an associated rooted tree that encodes the divergence of the corresponding integral. Through a process called *renormalisation*, it is possible to remove the divergences of the integrals such that the remaining finite quantity is still meaningful. A central part of their approach is to apply the *antipode* of the Hopf algebra of rooted trees in order to compute the inverse of a character, providing the counterterms needed to cancel the divergences of the integrals.

In addition to classical methods for computing antipodes in general graded connected Hopf algebras, such as the Dyson–Salam formula and the Bogoliubov recursion, there is interest in finding more efficient formulas that minimise cancellations. One such method is the Zimmermann forest formula, which is particularly effective for the Connes–Kreimer Hopf algebra. This formula significantly reduces the number of terms in the antipode expression. In recent work, Menous and Patras [29]

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generalised the forest formula to apply to right-handed polynomial Hopf algebras, which are precisely obtained as the enveloping algebras of pre-Lie algebras.

Hopf algebra theory has recently found applications in the field of *non-commutative probability*. One fundamental example of a non-commutative probability theory is the so-called *free probability theory* [33, 44], which was introduced by Dan Voiculescu in the 1980s, aiming to solve the problem of isomorphism between von Neumann algebras generated by free groups. Since then, free probability has found numerous applications and connections with various branches of mathematics, including random matrix theory, operator algebras, combinatorics, representation theory, and quantum information theory. Subsequently, multiple notions of non-commutative independence have been explored, including *Boolean independence* [40] and *monotone independence* [30], each providing a sufficiently rich framework for defining a corresponding theory of non-commutative probability.

Roughly speaking, the initial idea in non-commutative probability is to look at random variables as abstract elements inside a unital algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ and to replace the usual expectation with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, with $\varphi(1_{\mathcal{A}}) = 1$, called the *non-commutative expectation*. In this abstract setting, we can consider non-commutative analogues of classical notions in probability theory, such as distributions, moments, and independence. In the classical case, the independence of random variables can be regarded as a recipe to compute moments of mixed products of random variables. In other words, if two random variables X and Y are independent, then $\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n)$, for any $m, n \geq 0$. In the non-commutative setting, the independence of non-commutative random variables $a, b \in \mathcal{A}$ can also be defined as a recipe to compute moments of the form $\varphi(a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k})$ in terms of the marginal distributions of a and b, i.e., the sequences $\{\varphi(a^n)\}_{n\geq 0}$ and $\{\varphi(b^n)\}_{n\geq 0}$.

One of the most significant concepts in the combinatorial study of non-commutative probability is the notion of *cumulants*. In the 1990s, Roland Speicher introduced in [39] the notion of *free cumulants* and proved that, from a combinatorial point of view, the transition from classical probability to free probability consists in replacing the lattice of all set partitions with the lattice of *non-crossing partitions*. Moreover, the analogues of *Boolean cumulants* [40] and *monotone cumulants* [24] have been defined, using the lattices of interval and monotone non-crossing partitions, respectively. More recently, the works of [5, 8, 25] have shown connections between the combinatorics of non-commutative probability and *Schröder trees*. In particular, in [5, 25], the authors provide an alternative description of the cumulant-moment formulas by using Schröder trees instead of non-crossing partitions. Furthermore, Schröder trees in free probability have also been shown to be useful in the enumerative study of morphisms in a category of non-crossing partitions (cf. [15]).

Another non-commutative analogue of classical probability notions in free probability is the concept of *free Wick polynomials*, introduced by Anshelevich [3]. In both classical and quantum stochastic analysis, the classical Wick polynomials [6] play a fundamental role, particularly in the study of orthogonal polynomials and operator product renormalisation. In classical probability, they normalise random variable products to create orthogonal systems, acting as multi-dimensional analogues of Hermite polynomials linked to Wiener–Itô integrals. In free probability, free Wick polynomials extend these ideas in the context of free independence. Anshelevich's work connects free Wick products to stochastic calculus, using free Brownian motion and non-crossing partitions for combinatorial interpretations (cf. [2, 4]). These polynomials also relate to operator theory via free Fock spaces, resembling creation and annihilation operators in quantum field theory.

A connection between Hopf algebras and non-commutative probability theory was developed by Ebrahimi-Fard and Patras in a series of papers [19–21]. In this framework, the free, Boolean, and monotone cumulants are interpreted as linear functionals on a certain word Hopf algebra, and the relations between moments and the different types of cumulants can be explained analogously to the link between a Lie algebra and its corresponding Lie group. Ebrahimi-Fard and Patras' point of view has proven to be a powerful framework for understanding cumulants and their combinatorial relations in a unified way, as well as the different notions of additive convolutions originated by each type of independence.

In more precise words, the Hopf algebra considered by Ebrahimi-Fard and Patras in [19] is based on the *double tensor module on a vector space* V, that is, the vector space $\mathsf{T}(\mathsf{T}_+(V))$ defined by

$$\mathsf{T}(\mathsf{T}_+(V)) := \bigoplus_{n\geq 0} \mathsf{T}_+(V)^{\otimes n}, \quad \text{where } \mathsf{T}_+(V) := \bigoplus_{n>0} V^{\otimes n}.$$

The authors of [19] introduced a product and a coproduct which provide $\mathsf{T}(\mathsf{T}_+(V))$ with a structure of connected graded bialgebra. By the general theory of Hopf algebras, an antipode can be recursively defined on each graded component of $\mathsf{T}(\mathsf{T}_+(V))$, providing it automatically with a Hopf algebra structure called the *double tensor Hopf algebra on* V. The link with non-commutative probability arises by identifying cumulants as the solutions of certain linear fixed-point equations in the dual of $\mathsf{T}(\mathsf{T}_+(V))$ (Theorem 4.3).

Motivated by the forest formulas for antipodes in [29] and the fact that the coproduct in the double tensor Hopf algebra can be considered as a decorated and non-commutative version of the Connes–Kreimer coproduct, the main objective of the present manuscript is to address the problem of computing a cancellation-free combinatorial formula for the antipode in $\mathsf{T}(\mathsf{T}_+(V))$, expressed in terms of Schröder trees. We apply this method to first compute the antipode in $\mathsf{S}(\mathsf{T}_+(V))$, the *symmetric algebra on* $\mathsf{T}_+(V)$, that is, the Hopf algebra of symmetric tensors over $\mathsf{T}_+(V)$ which can be thought of as the commutative analogue of $\mathsf{T}(\mathsf{T}_+(V))$. The method in [29] provides a combinatorial formula for the iterations of the reduced coproduct in $\mathsf{S}(\mathsf{T}_+(V))$ based on certain decorated non-planar trees, where the non-planarity of the trees reflects the fact that the symmetric algebra is commutative. Our combinatorial analysis shows that the decorated trees of Menous and Patras can be replaced by Schröder trees whose internal vertices are labelled increasingly, respecting the poset structure. It turns out that Schröder trees effectively describe the actions of the iterated coproduct and the antipode in $\mathsf{S}(\mathsf{T}_+(V))$ (Theorem 5.12).

With the antipode formula for the commutative case in hand, we provide a combinatorial analysis that shows how cancellations are performed in the non-commutative case in order to arrive at the combinatorial formula for the antipode in $\mathsf{T}(\mathsf{T}_+(V))$ expressed in terms of Schröder trees (Theorem 6.10). With the antipode formula, we examine its implications in non-commutative probability through the framework of Ebrahimi-Fard and Patras. In particular, we can recover cumulant-moment formulas in terms of Schröder trees, which have recently appeared in [5, 25], as well as a new combinatorial formula that writes free Wick polynomials in terms of Schröder trees (Theorem 7.8).

It is worth mentioning that a similar description of the antipode formula has been given in [1, 28]. In the former article, the antipode formula is expressed in terms of *tubings* of linear orders in the context of *Hopf monoids of paths*. In the latter article, an antipode formula appears in the context of *natural Hopf algebras over an operad*. In future work, we plan to develop a closer relation between several Hopf monoids and notions of independence in non-commutative probability.

Organisation of the paper

Besides the present section that serves as an introduction, the paper is organised in the following way. In Section 2, we present a description of the main combinatorial objects used in this manuscript: trees, Schröder trees, and non-crossing partitions. We also explain a natural map that associates a canonical non-crossing partition with any Schröder tree. In Section 3, we precisely describe the Hopf algebra structure on the double tensor module $\mathsf{T}(\mathsf{T}_+(V))$ defined by Ebrahimi-Fard and Patras. We also explain how the coproduct in this Hopf algebra can be split, producing two non-coassociative coproducts, which will allow us to define three exponential-type maps on the dual of the double tensor Hopf algebra. Section 4 contains the definition of cumulants in non-commutative probability and describes the link between the double tensor Hopf algebra and non-commutative probability. Section 5 aims to describe the general method in [29] to compute a forest-type formula for the antipode in a particular class of Hopf algebras. We then apply this method to obtain a combinatorial formula for the antipode in $\mathsf{S}(\mathsf{T}_+(V))$ in terms of Schröder trees. In Section 6, we present in Theorem 6.10 the main result of our manuscript: a cancellation-free formula for the antipode in the double tensor algebra in terms of Schröder trees. Finally, in Section 7, we apply Theorem 6.10 in the Ebrahimi-Fard and Patras' framework for non-commutative probability and recover recently known cumulant-moment formulas in terms of Schröder trees, as well as a combinatorial expression for the inverse of a character in terms of non-crossing partitions. We conclude by obtaining a new formula for Anshelevich's free Wick polynomials, also in terms of Schröder trees.

2. Preliminaries on Schröder trees and non-crossing partitions

The aim of this section is to describe the combinatorial objects that are fundamental for the present manuscript: Schröder trees, non-crossing partitions, and their relations. To this end, we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ as the set of non-negative integers. For every $n \in \mathbb{N}$, we denote by $[n]$ the set $[n] := \{1, 2, \ldots, n\}$, with $[0] := \emptyset$. Given a finite set I, its cardinality is denoted by $|I|$. For every integer n, $\mathrm{Abs}(n) \in \mathbb{N}$ denotes its absolute value.

2.1. Trees

A *tree* is a connected graph that has no cycles. A *rooted tree* is a tree with a distinguished vertex called the *root*. All trees in this work are rooted, so we will not distinguish between trees and rooted trees. The set of edges in a rooted tree possesses a natural orientation, following the direction opposite to the root. Given a vertex v of a tree t, the vertex connected to v in the direction to the root is called the *parent* of v, and any vertex connected to v by an edge oriented towards the root is called a *child* of v. We say that a tree t is *planar* if, for every vertex v of t, the set of children of t is endowed with a total order. *Non-planar trees* refer to trees that are not planar.

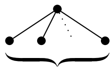
A *leaf* of a tree is a vertex with no children, and an *internal vertex* of a tree is a vertex that is not a leaf. We denote by $\mathrm{Vert}(t)$ and $\mathrm{Int}(t)$ the set of vertices and internal vertices of t, respectively. Also, for $v \in \mathrm{Vert}(t)$, we denote by $\mathrm{succ}(v)$ the set of children of v. A *non-planar forest* is a set of non-planar rooted trees. Similarly, an *ordered planar forest* is an ordered sequence of planar trees.

Any (planar or non-planar) rooted tree t can be regarded as a poset in the following way. The elements of the poset are given by $\mathrm{Vert}(t)$. Moreover, for $v, w \in \mathrm{Vert}(t)$, we define the partial order $v \leq w$ if and only if the unique path from the root of t towards w passes through v. In the case that $v \leq w$, we say that w is a *descendant* of v. Notice that, in this poset structure, the root of t is the unique minimal element, and the leaves of t are the maximal elements.

Given a forest f, the *branching* of f is the new tree $B_+(f)$ obtained by joining all roots of every tree in f to a new vertex r, so the root of $B_+(f)$ is r. Every tree t can be obtained as the branching $B_+(f)$ of a (possibly empty) forest f. For planar trees, we will also denote by $B_+(f)$ the planar tree obtained by grafting from left to right the elements of the sequence $f = (t_1, \dots, t_s)$ to a new common root r.

The *single-vertex tree* $\circ$ is the unique tree with no internal vertex and just one leaf. Given $m \geq 1$, the *m-th corolla* $C_{(m)}$ is the unique tree with one internal vertex and m leaves:

$$C_{(m)} := B_+(\underbrace{\circ, \circ, \dots, \circ}_{m \text{ times}}) =$$



m leaves

For any rooted tree t, the *size* $|t|$ of t is its number of vertices, i.e., $|t| := |\mathrm{Vert}(t)|$. In the same way, we denote by $i(t) := |\mathrm{Int}(t)|$ the number of internal vertices of t. Therefore, $|t| - i(t)$ is the number of leaves of the tree t. If $t = B_+(t_1, t_2, \dots, t_k)$, then

$$|t| = 1 + |t_1| + \cdots + |t_k| \quad \text{and} \quad i(t) = 1 + i(t_1) + \cdots + i(t_k).$$

The *tree factorial* $t!$ is recursively defined by setting

$$t! := \begin{cases} 1 & \text{if } |t| = 1, \\ |t|\, t_1! t_2! \cdots t_k! & \text{if } t = B_+(t_1, t_2, \dots, t_k). \end{cases}$$

Finally, if f is a forest formed by the rooted trees $t_1, \dots, t_m$, we set $f! := t_1! \cdots t_m!$.

2.2. Schröder trees

A *Schröder tree* is a planar rooted tree for which each internal vertex has at least two children. For every $n, k \in \mathbb{N}$ such that $1 \leq k \leq n$, the set of Schröder trees with k internal vertices and $n+1$ leaves is denoted by $\mathrm{Sch}_k(n)$. Also, we denote by $\mathrm{Sch}(n) = \bigcup_{k \geq 1} \mathrm{Sch}_k(n)$ the set of Schröder trees with $n+1$ leaves and $\mathrm{Sch} = \bigcup_{n \geq 0} \mathrm{Sch}(n)$, with Sch(0) to be the set only containing the single-vertex tree $\circ$, whose unique vertex is considered as a leaf. The *degree* of a Schröder tree t is the integer n such that $t \in \mathrm{Sch}(n)$, and it is denoted by $\deg(t)$. In this work, every Schröder tree is represented with the root at the top and the leaves at the bottom. Internal vertices will be depicted as black nodes, while leaves will be depicted as white nodes. The set Sch(3), consisting of nine elements, is depicted in Figure 1.

Analogously, a *Schröder forest* is an ordered sequence of Schröder trees $F = (t_1, \dots, t_m)$. In addition, the *degree of* F is given by the sum of the degrees of its underlying trees, i.e.,

$$\deg(F) = \deg(t_1) + \cdots + \deg(t_m).$$

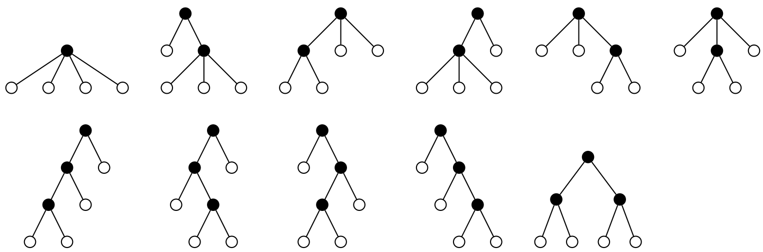


Figure 1. Schröder trees in Sch(3).

The set of Schröder forest of degree n is then denoted by $\mathrm{FSch}(n)$, and the set of Schröder forest of the form $F = (t_1, \ldots, t_m)$, where $\deg(t_i) = n_i$ for $1 \leq i \leq m$, is denoted by $\mathrm{FSch}(n_1, \ldots, n_m)$.

Schröder trees are known in the literature under several names, such as *reduced planar trees* (see [18]). For every $n \geq 0$, the number of elements in $\mathrm{Sch}_k(n)$ is given by

$$|\mathrm{Sch}_k(n)| = \frac{1}{n+k+3}\binom{n+k+3}{k+1}\binom{n}{k}, \quad 0 \leq k \leq n.$$

This number corresponds to [34, A033282]. When $k = n$, a tree in $\mathrm{Sch}_n(n)$ corresponds to a *binary tree*, a planar tree for which every internal vertex has exactly two children. The above formula leads to $|\mathrm{Sch}_n(n)| = \mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$, the *$n$-th Catalan number*. In general, Schröder trees are equinumerous to several objects in many contexts in mathematics. For instance, the set $\mathrm{Sch}_k(n)$ is in bijection with k-dimensional faces of the n-th dimensional associahedron, with k non-crossing diagonals in a convex $(n+2)$-gon, or standard Young tableaux of shape $(k+1, k+1, 1^{n-k-1})$, just to name a few (see [38] for an extensive list).

The *skeleton* of a Schröder tree t is the planar rooted subtree of t generated by the internal vertices of t. We denote the skeleton of t by $\mathrm{sk}(t)$. The skeleton of t can also be regarded as a poset whose elements are given by $\mathrm{Int}(t)$. In the same way, if $F = (t_1, \ldots, t_m)$ is a Schröder forest, we define the *skeleton of F*, denoted by $\mathrm{sk}(F)$, as the ordered forest $(\mathrm{sk}(t_1), \ldots, \mathrm{sk}(t_m))$. Moreover, $\mathrm{sk}(F)$ can also be regarded as the poset formed by the union of posets $\bigcup_{i=1}^{m} \mathrm{sk}(t_i)$.

For our purposes, it is convenient to introduce two special subsets of $\mathrm{Sch}(n)$.

Definition 2.1. Let t be a Schröder tree.

(1) We say that t is a *prime Schröder tree* if its leftmost subtree is a leaf. In other words, if $t = B_+(t_1, \ldots, t_k)$, then $t_1 = \circ$ (a single-vertex tree). We denote by $\mathrm{PSch}(n)$ the set of prime Schröder trees with $n+1$ leaves.

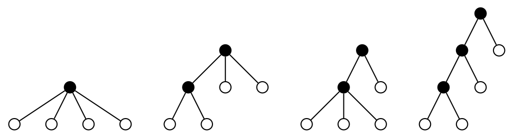


Figure 2. Boolean Schröder trees in BSch(3).

(2) We say that t is a *Boolean Schröder tree* if t is a Schröder tree such that, for every $v \in \mathrm{Int}(t)$, the children of v are all leaves except possibly its leftmost child. For each $n \geq 1$, we denote by $\mathrm{BSch}(n)$ the set of Boolean Schröder trees with $n+1$ leaves. Figure 2 shows the four elements of the set BSch(4).

It is known that prime Schröder trees are counted by the *large Schröder numbers* [34, A006318]. In particular, for any $n \geq 1$, there exists a two-to-one surjective function $P : \mathrm{PSch}(n) \to \mathrm{Sch}(n-1)$ so that $|\mathrm{PSch}(n)| = 2|\mathrm{Sch}(n-1)|$. For the case of Boolean Schröder trees, it can be shown that $|\mathrm{BSch}(n)| = 2^{n-1}$, for any $n \geq 1$.

Let $t \in \mathrm{Sch}(n)$ be a Schröder tree with $n+1$ leaves. Since t is planar, we can naturally label the set of leaves of t with the numbers $1, 2, \ldots, n+1$, reading the leaves from left to right. Hence, two leaves of t are *consecutive* if they are labelled with i and $i+1$, for some $1 \leq i \leq n$. A *sector*[1] of t is a minimal subtree of t containing two consecutive leaves. Roughly speaking, a sector of t is the area encompassed by two consecutive leaves of t. Also, notice that the root of a sector is the intersection of the unique paths from the consecutive leaves to the root of t. In addition, if v is an internal vertex of t and s is a sector of t whose root is v, we say that s is *adjacent to the vertex* v.

We denote by $\mathrm{Sect}(t)$ the set of all sectors of t. It is clear that $|\mathrm{Sect}(t)| = n$. The set $\mathrm{Sect}(t)$ is naturally endowed with a total order, which consists of enumerating each sector of t from left to right, bottom to top. Formally, we have the following.

- If $t = \circ$, we have an empty set of sectors. If $t = C_{(\ell+1)}$ is a corolla with exactly $\ell + 1$ leaves, for some $\ell \geq 1$, then t possesses ℓ sectors. We denote by $s_1, s_2, \ldots, s_m$ the sectors of t, by order of appearance from left to right. Then, we define the following total order between the sectors:

$$s_1 < s_2 < \cdots < s_m.$$

- More generally, assume that

$$t = B_+(t_0, t_1, \ldots, t_m),$$

[1] Our definition comes from [35], where sectors are called "branchings".

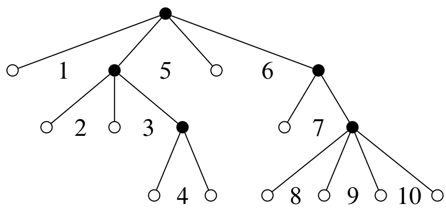


Figure 3. A Schröder tree $t \in \mathrm{Sch}(11)$ and its natural labelling.

where $t_0, t_1, \dots, t_m$ are Schröder trees. If s is a sector of t adjacent to the root of t and located between the trees t_i and t_{i+1}, we extend inductively the orders of $\mathrm{Sect}(t_0), \mathrm{Sect}(t_1), \dots, \mathrm{Sect}(t_m)$ to an order on $\mathrm{Sect}(t)$, via

$$s' < s < s'' \quad \text{for all } s' \in \mathrm{Sect}(t_i),\ s'' \in \mathrm{Sect}(t_{i+1}).$$

Let $t \in \mathrm{Sch}(n)$, with $n \geq 1$. There exists a unique increasing bijection

$$\iota_t : ([n], \leq_{\mathbb{N}}) \to (\mathrm{Sect}(t), <),$$

where $\leq_{\mathbb{N}}$ is the usual order on $\mathbb{N}$. We call ι_t the *natural labelling* of t. Figure 3 depicts an example of a Schröder tree together with its natural labelling.

2.3. Set partitions

Let I be a finite set. A *partition* of I (or *set partition*) is a finite set formed by non-empty sets $\pi = \{B_1, B_2, \dots, B_k\}$ such that the union of all elements in π is I, and every pair of distinct elements in π are disjoint. Every element $B \in \pi$ is called a *block* of π. We denote by $\Pi(I)$ the set of all partitions of I. If $\pi \in \Pi(I)$, we write $\pi \vdash I$. When $I = [n]$, we simply denote by $\Pi(n)$ the set of non-crossing partitions of $[n]$. There is a unique partition of $\Pi(\emptyset)$, with no blocks.

Let $\pi = \{B_1, B_2, \dots, B_k\}$ be a partition of I. Given a subset S of I, the *restriction* of π to S is the partition given by $\pi_S := \{B_1 \cap S, B_2 \cap S, \dots, B_k \cap S\}$, where we omit empty intersections. The *length* of π, $|\pi| := k$, is defined as its number of blocks. The *factorial* of the partition π is given by

$$\pi! := \prod_{B \in \pi} |B|!.$$

For every finite set I, the set $\Pi(I)$ is partially ordered under *reverse refinement*: we set $\pi \leq \sigma$ if every block of π is contained in a block of σ. In other words, $\pi \leq \sigma$ if every block of σ can be expressed as a union of blocks of π. The partition 0_I of I into singletons is the unique minimum, and $1_I := \{I\}$ is the unique maximum under the refinement order. Therefore, $(\Pi_I, \leq)$ is a lattice.

Let $\pi, \sigma \vdash I$ such that $\pi \leq \sigma$. If $B \in \sigma$, let $n_B := |\pi_B|$. That is, n_B is the number of blocks of π that refine the block B of σ. The Möbius function of $\Pi(I)$ is then given by [41, Chap. 3]

$$\mathrm{Moeb}(\pi, \sigma) = (-1)^{|\pi|-|\sigma|} \prod_{B \in \sigma} (n_B - 1)!$$

for every $\pi, \sigma \vdash I$ such that $\pi \leq \sigma$.

2.4. Non-crossing partitions

Let I be a finite totally ordered set. A *non-crossing partition* of I is a partition $\pi \in \Pi(I)$ for which whenever four elements $a < b < c < d$ in I are such that if a, c are in the same block of π and b, d are in another block of π, then both blocks coincide. The set of non-crossing partitions of I is denoted by $\mathrm{NC}(I)$. When $I = [n]$, we simply denote by $\mathrm{NC}(n)$ the set of non-crossing partitions of $[n]$.

The reverse refinement order $\leq$ on $\Pi(n)$ induces a poset structure on $\mathrm{NC}(n)$. Since the finest partition 0_n and the coarsest partition 1_n in $\Pi(n)$ are non-crossing partitions, then $(\mathrm{NC}(n), \leq)$ is a lattice. This poset is closed under meet operation, but not under join. Hence, it is not a sublattice of $(\Pi(n), \leq)$, but rather a meet-sublattice.

Non-crossing partitions belong to a vast family of combinatorial objects, namely, the *Catalan family*. The number of non-crossing partitions in $\mathrm{NC}(n)$ is the n-th Catalan number Cat_n. Several objects are related to non-crossing partitions in many contexts (see [38] for a compendium of some related objects and [42] for a more extensive list).

One of the first studies on non-crossing partitions dates back to 1952, in the work of Becker (see [7]), where non-crossing partitions are referred to as "*planar rhyme schemes*." In 1972, Kreweras [26] and then Poupard [36] continued with a combinatorial analysis of these objects. For instance, Kreweras obtained a formula for the Möbius function of $\mathrm{NC}(n)$ between its minimal and maximal elements:

$$\mathrm{Moeb}_{\mathrm{NC}(n)}(0_n, 1_n) = (-1)^{n-1}\, \mathrm{Cat}_{n-1}\,.$$

Several types of non-crossing partitions are considered in this work, arising from non-commutative probability theory. An *interval partition of* $[n]$ is a non-crossing partition $\pi \in \mathrm{NC}(n)$ such that all its blocks are of the form $\{k, k+1, \ldots, k+l\}$ for some $1 \leq k \leq n$ and $0 \leq l \leq n-k$. We denote by $\mathrm{NCInt}(n)$ the set of interval partitions of $[n]$. It is straightforward to verify that the reverse refinement order also provides $\mathrm{NCInt}(n)$ with a poset structure, which is isomorphic to the poset of subsets of a set of cardinality $n-1$. Figure 4 provides a graphical representation of a non-crossing partition and an interval partition.

(a) Non-crossing partition (b) Interval partition

Figure 4. Different types of set partitions of the set $[6] = \{1, \ldots, 6\}$. The blocks of the partitions are represented by arcs. The non-crossing condition of the blocks means that there are no intersections of the arcs.

2.5. Nesting forests from non-crossing partitions

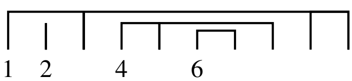
To every non-crossing partition $\pi \in \mathrm{NC}(n)$ with k blocks we associate to it a (non-planar) forest f_π with k vertices in total, constructed from a hierarchy law of the blocks of π. More precisely, given a non-crossing partition π and two blocks $B, B' \in \pi$, we say that B is *nested* on B' if $\min B' \leq x \leq \max B'$, for every $x \in B$. This relation gives a poset structure on the set of blocks of π. The minimal elements are called *outer blocks* of π. By construction, the Hasse diagram of the poset has no cycles and may have several connected components. Hence, the nesting poset associated with π is a (non-planar) forest, denoted by f_π, and called the *forest of nestings* of π. In particular, f_π is a single tree if and only if π is an *irreducible non-crossing partition*; i.e., 1 and n belong to the same block in π.

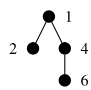
If $\{B_1, \ldots, B_s\}$ is the set of outer blocks of a non-crossing partition π, we can construct $\pi_1, \ldots, \pi_s$ irreducible non-crossing partitions as

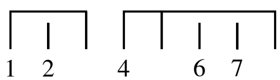
$$\pi_i := \{B \in \pi : B \text{ is nested in } B_i\},$$

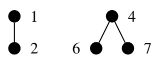
for each $1 \leq i \leq s$. The non-crossing partitions $\pi_1, \ldots, \pi_s$ are called the *irreducible components of* π. Hence, the nesting forest associated with π is given by $f_\pi = \{f_{\pi_1}, \ldots, f_{\pi_s}\}$.

In the following picture, we draw two non-crossing partitions π and σ together with their respective forest of nesting f_π and f_σ. For clarity, each vertex in the forest is decorated with the minimal element of its associated block in the partition.

$\pi =$ 

, $f_\pi =$ 

$\sigma =$ 

, $f_\sigma =$ 

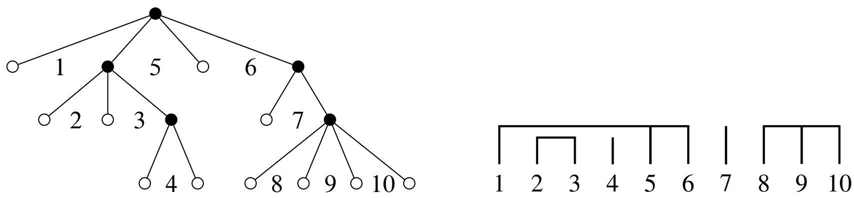


Figure 5. Schröder tree t with 10 sectors and its associated non-crossing partition $\pi(t) = \{\{1,5,6\},\{2,3\},\{4\},\{7\},\{8,9,10\}\} \in \mathrm{NC}(10)$.

It is convenient to consider the case in which the partial order described by the forest of nestings of a non-crossing partition can be extended to a total order. More precisely, we say that (π,λ) is a *monotone partition on* $[n]$ if $\pi \in \mathrm{NC}(n)$ and $\lambda : \pi \to [|\pi|]$ is an order-preserving bijection; i.e., if B and B' are distinct blocks of π such that B' is nested in B, then $\lambda(B) < \lambda(B')$. The set of monotone partitions on $[n]$ is denoted by $\mathcal{M}(n)$.

2.6. From Schröder trees to non-crossing partitions

Let $t \in \mathrm{Sch}(n)$. Recall that the set $\mathrm{Sect}(t)$ of all sectors of t is naturally endowed with a total order. Moreover, the Schröder tree t has a natural labelling $\iota_t : [n] \to \mathrm{Sect}(t)$, which is a bijection between totally ordered sets. The next definition presents a natural way to associate a non-crossing partition to any Schröder tree.

Definition 2.2. Let $t \in \mathrm{Sch}(n)$ with a set of internal vertices $\mathrm{Int}(t) = \{v_1, v_2, \dots, v_k\}$. The *non-crossing partition associated with t* is the set partition

$$\pi(t) := \{B_1, \dots, B_k\} \in \mathrm{NC}(n)$$

formed by k blocks, given by

$$B_i := \{\iota_t^{-1}(s) : s \text{ is a sector adjacent to } v_i\}, \tag{2.1}$$

for each $1 \le i \le k$. In the context of (2.1), we also say that B_i is *associated with* v_i.

Figure 5 illustrates a Schröder tree along with its associated non-crossing partition. It is straightforward to verify that the map $\mathrm{Sch}(n) \ni t \mapsto \pi(t) \in \mathrm{NC}(n)$ is a surjection. In fact, if we restrict this map to the subset of prime Schröder trees $\mathrm{PSch}(n)$, we still get a surjection that is related to the Möbius function on $\mathrm{NC}(n)$.

Proposition 2.3 ([25, (111)]). *Let* $n \ge 1$. *For any* $\pi \in \mathrm{NC}(n)$, *we have*

$$|\{t \in \mathrm{PSch}(n) : \pi(t) = \pi\}| = \mathrm{Abs}(\mathrm{Moeb}_{\mathrm{NC}(n)}(\pi, 1_n)).$$

To finish this section, we remark that the map $t \mapsto \pi(t)$ actually provides a bijection between BSch(n) and NCInt(n), for any $n \geq 1$. As a consequence, we can conclude that

$$|\mathrm{BSch}(n)| = |\mathrm{NCInt}(n)| = 2^{n-1}.$$

3. Hopf algebra structure on the double tensor module

In this section, we present the definition of the double tensor Hopf algebra introduced by Ebrahimi-Fard and Patras in [19]. We also describe how the coproduct can be split into two parts, generating a shuffle (or dendriform) structure on the dual of the double tensor Hopf algebra. Finally, we explain how the shuffle structure allows us to construct three different bijections that resemble the relationship between a Lie group and its Lie algebra.

We denote by $\mathbb{K}$ a base field of characteristic 0 and by $\mathbb{C}$ the field of complex numbers. The vector spaces considered in this work are $\mathbb{C}$-vector spaces. We mention that the constructions presented in this section can also be done, considering a base field $\mathbb{K}$.

3.1. Definition of the double tensor Hopf algebra

Let V be a vector space. If $k \geq 0$, we write elementary tensors from $V^{\otimes k}$ as words, $w = u_1 u_2 \cdots u_k$, with $u_i \in V$ for $1 \leq i \leq k$. In this case, we say that $|w| := k$ is the *length* of w. Also, we identify $V^{\otimes 0}$ with $\mathbb{C}$. Then, we call the vector spaces of linear combinations of words and non-empty words

$$\mathsf{T}(V) := \bigoplus_{k \geq 0} V^{\otimes k} \quad \text{and} \quad \mathsf{T}_+(V) := \bigoplus_{k \geq 1} V^{\otimes k}$$

as the *tensor module* and the *augmented tensor module* generated by V, respectively.

It also makes sense to consider the tensor module of $\mathsf{T}_+(V)$, denoted by $\mathsf{T}(\mathsf{T}_+(V))$. To avoid confusion, the second tensor product appearing in the construction of the tensor module of $\mathsf{T}_+(V)$ is denoted by a bar; i.e., if $w_1, w_2, \ldots, w_n \in \mathsf{T}_+(V)$, then we write

$$w_1 | w_2 | \cdots | w_n \in \mathsf{T}_+(V)^{\otimes n}.$$

As a vector space, $\mathsf{T}(\mathsf{T}_+(V))$ is graded, where for any $n \geq 0$, the n-th graded component is given by

$$\mathsf{T}(\mathsf{T}_+(V))_n := \bigoplus_{\substack{n_1 + \cdots + n_k = n \\ n_i \geq 0}} V^{\otimes n_1} \oplus \cdots \oplus V^{\otimes n_k}. \tag{3.1}$$

It turns out that the tensor module $\mathsf{T}(\mathsf{T}_+(V))$ can be endowed with a product and a coproduct respecting the grading (3.1) as follows:

- *Product rule*: if $v = v_1|\cdots|v_r \in \mathsf{T}_+(V)^{\otimes r}$ and $w = w_1|\cdots|w_s \in \mathsf{T}_+(V)^{\otimes s}$, then the product of v and w is the element
$$v|w := v_1|\cdots|v_r|w_1|\cdots|w_s \in \mathsf{T}_+(V)^{\otimes(r+s)}.$$
In addition, if $\mathbb{1}$ stands for the empty word, then we identify $w|\mathbb{1} = w = \mathbb{1}|w$ for any $w \in \mathsf{T}_+(V)$ so that $\mathbb{1}$ works as a unit.
- *Coproduct rule*: the definition of the coproduct involves extra technical notions. Given a word $w = u_1\cdots u_n \in V^{\otimes n}$ and $I = \{i_1,\ldots,i_k\} \subset \mathbb{N}$, where $i_1 < \cdots < i_k$, we write the *restriction of w to I* as
$$w_I := u_{i_1}\cdots u_{i_k}.$$
If $I \cap [n] = \emptyset$, we set $w_I := \mathbb{1}$.

Recall that a set $I \subseteq \mathbb{N}$ is an *interval* if I is a set of consecutive positive integers. For a pair of nested intervals $I \subseteq J \subseteq [n]$, the set $J \setminus I$ can be decomposed (uniquely) into the disjoint union
$$J \setminus I = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r,$$
where $K_1,\ldots,K_r$ are intervals of $[n]$ such that no union of two such blocks is an interval. Additionally, the above list is ordered in increasing order according to the minimal elements. Therefore, $\min K_1 < \cdots < \min K_r$. We call the sequence $K(I,J) := \{K_1,\ldots,K_r\}$ the *decomposition of $J \setminus I$ into its connected components*. With this, we set
$$w^{(I,J)} := w_{K_1}|\cdots|w_{K_r} \in \mathsf{T}(\mathsf{T}_+(V)).$$
If $J = [n]$, we set $K(I) := K(I,[n])$ and $w^{(I)} := w^{(I,J)}$.

Example 3.1. Let V be a vector space. For $n = 9$ and $w = u_1\cdots u_9 \in V^{\otimes 9}$, let $J = \{2,3,4,6,7,8,9\}$ and $I = \{4,7,9\}$ so that $J \setminus I = \{2,3,6,8\}$. Then, the elements $w^{(I,J)}$, $w^{(I)}$ and $w^{(J)}$ are, respectively,
$$w^{(I,J)} = u_2u_3|u_6|u_8, \quad w^{(I)} = u_1u_2u_3|u_5u_6|u_8, \quad \text{and} \quad w^{(J)} = u_1|u_5.$$

With this notation, define now the map $\Delta : \mathsf{T}_+(V) \to \mathsf{T}(V) \otimes \mathsf{T}(\mathsf{T}_+(V))$ by
$$\begin{aligned}\Delta(w) :&= \sum_{I\subseteq[n]} w_I \otimes w^{(I)}\\ &= \sum_{I\subseteq[n]} w_I \otimes w_{K_1}|\cdots|w_{K_r},\end{aligned} \tag{3.2}$$
where $K(I) = \{K_1,\ldots,K_r\}$.

To complete the definition of the coproduct on the double tensor algebra, we extend the map Δ multiplicatively to all of $\mathsf{T}(\mathsf{T}_+(V))$ by setting $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ and

$$\Delta(w_1|\cdots|w_k) := \Delta(w_1)|\cdots|\Delta(w_k),$$

for any $w_1, \ldots, w_k \in \mathsf{T}_+(V)$. Finally, we define the counit map $\varepsilon : \mathsf{T}(\mathsf{T}_+(V)) \to \mathbb{C}$ as the algebra morphism given by $\varepsilon(\mathbb{1}) = 1$ and $\varepsilon(w) = 0$ for any $w \in \mathsf{T}_+(\mathsf{T}_+(V))$.

Example 3.2. Let V be a vector space and $a_1, a_2, a_3 \in V$. Then, we have the following computations for the coproduct in (3.2):

$$\begin{aligned}
\Delta(a_1) &= a_1 \otimes \mathbb{1} + \mathbb{1} \otimes a_1,\\
\Delta(a_1a_2) &= a_1a_2 \otimes \mathbb{1} + a_1 \otimes a_2 + a_2 \otimes a_1 + \mathbb{1} \otimes a_1a_2,\\
\Delta(a_1a_2a_3) &= a_1a_2a_3 \otimes \mathbb{1} + a_1a_2 \otimes a_3 + a_1a_3 \otimes a_2 + a_2a_3 \otimes a_1\\
&\qquad + a_1 \otimes a_2a_3 + a_2 \otimes a_1 \mid a_3 + a_3 \otimes a_1a_2 + \mathbb{1} \otimes a_1a_2a_3.
\end{aligned}$$

The iterated coproduct is recursively defined by $\Delta^{[m+1]} := (\mathrm{id}^{\otimes(m-1)} \otimes \Delta) \circ \Delta^{[m]}$ for $m \geq 1$, where id stands for the identity map on $\mathsf{T}(\mathsf{T}_+(V))$, $\Delta^{[1]} = \mathrm{id}$, and $\Delta^{[2]} = \Delta$. Then, for $m \geq 2$ and a non-empty word $w = u_1 \cdots u_n \in \mathsf{T}_+(V)$, we have

$$\Delta^{[m]}(w) = \sum_{I=I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{m-1} \subseteq [n]} w_I \otimes w^{(I_1,I_2)} \otimes w^{(I_2,I_3)} \otimes \cdots \otimes w^{(I_{m-1})}.$$

It turns out that the tensor module $\mathsf{T}(\mathsf{T}_+(V))$ endowed with the bar (concatenation) product and the coproduct described in (3.2) is a graded bialgebra which in addition is *connected*, i.e., $\mathsf{T}(\mathsf{T}_+(V))_0 \cong \mathbb{K}$. Since any connected bialgebra is a Hopf algebra (see, for instance, [37, Lem. 7.6.2]), we have the following result.

Theorem 3.3 ([19]). *The tensor module $\mathsf{T}(\mathsf{T}_+(V))$ endowed with the bar product and the coproduct in (3.2) is a graded connected non-commutative and non-cocommutative Hopf algebra.*

Remark 3.4. The Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$ in the previous theorem is called the *double tensor Hopf algebra on V*. The grading and the connectedness of the Hopf algebra allow us to prove the existence of its antipode S from the so-called *Bogoliubov formula*[2]

$$0 = S(w) + w + m_{\mathsf{T}(\mathsf{T}_+(V))} \circ (S \otimes \mathrm{id}) \circ \overline{\Delta}(w),$$

for any $w \in \mathsf{T}_+(V)$, where $m_{\mathsf{T}(\mathsf{T}_+(V))}$ stands for the multiplication on $\mathsf{T}(\mathsf{T}_+(V))$ and $\overline{\Delta}$ stands for the *reduced coproduct*, i.e., $\overline{\Delta}(w) := \Delta(w) - \mathbb{1} \otimes w - w \otimes \mathbb{1}$, for any $w \in \mathsf{T}_+(\mathsf{T}_+(V))$.

[2]This terminology is in correspondence with Bogoliubov–Shirkov's recursive formula for renormalisation, as in [9].

3.2. Shuffle algebra structure on $\mathrm{Lin}(\mathsf{T}(\mathsf{T}_+(V)), \mathbb{C})$

A fundamental algebraic property of the coproduct on the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$ is that it can be split into two half-coproducts. More precisely, given a non-empty word $w = a_1 \cdots a_n \in V^{\otimes n}$, we consider the maps

$$\Delta_{\prec}(w) = \sum_{\substack{A \subseteq [n] \\ 1 \in A}} w_A \otimes w^{(A)},$$
$$\Delta_{\succ}(w) = \sum_{\substack{A \subseteq [n] \\ 1 \notin A}} w_A \otimes w^{(A)}.$$

Clearly, we have that $\Delta(w) = \Delta_{\prec} + \Delta_{\succ}$. The previous two maps are extended to $\mathsf{T}_+(\mathsf{T}_+(V))$ by declaring

$$\Delta_{\prec}(w_1|w_2|\cdots|w_n) = \Delta_{\prec}(w_1)\Delta(w_2)\cdots\Delta(w_n),$$
$$\Delta_{\succ}(w_1|w_2|\cdots|w_n) = \Delta_{\succ}(w_1)\Delta(w_2)\cdots\Delta(w_n),$$

for any non-empty words $w_1, w_2 \ldots, w_n \in \mathsf{T}_+(V)$.

It is not difficult to see that neither map is coassociative. Instead, the authors of [19] proved that $\mathsf{T}(\mathsf{T}_+(V))$ together with $\Delta_{\prec}$ and $\Delta_{\succ}$ satisfy the definition of a *counital unshuffle bialgebra* (also known as *codendriform coalgebra*, see [19, Thm. 5] and reference [23] for a precise definition). This property translates to the dual space $\mathrm{Lin}(\mathsf{T}(\mathsf{T}_+(V)), \mathbb{C})$ as follows: for the coassociative coproduct Δ as well as $\Delta_{\prec}$ and $\Delta_{\succ}$, which from now on we will call *left and right half-unshuffle coproducts*, respectively, we consider the dual maps

$$f * g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta,$$
$$f \prec g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta_{\prec},$$
$$f \succ g = m_{\mathbb{C}} \circ (f \otimes g) \circ \Delta_{\succ},$$

for any $f, g \in \mathrm{Lin}(\mathsf{T}_+(\mathsf{T}_+(V)), \mathbb{C})$, where $m_{\mathbb{C}}$ stands for the associative product on $\mathbb{C}$. We call $\prec$ and $\succ$ the *left and right half-shuffle product*, respectively. Neither of them is associative; nevertheless, they satisfy the *shuffle identities*:

$$\begin{aligned}(f \prec g) \prec h &= f \prec (g * h),\\ (f \succ g) \prec h &= f \succ (g \prec h),\\ (f * g) \succ h &= f \succ (g \succ h),\end{aligned} \tag{3.3}$$

for any $f, g, h \in \mathrm{Lin}(\mathsf{T}_+(\mathsf{T}_+(V)), \mathbb{C})$. In general, any vector space D together with two non-associative products $\prec$ and $\succ$ satisfying the shuffle identities with $* = \prec + \succ$ is called a *shuffle algebra* (also known as *dendriform algebra*, see [27]). By setting

$f \prec \varepsilon = f = \varepsilon \succ f$ and $\varepsilon \prec f = 0 = f \succ f$, where ε is the counit on $\mathsf{T}(\mathsf{T}_+(V))$, we have the following theorem.

Theorem 3.5 ([19]). $(\mathrm{Lin}(\mathsf{T}(\mathsf{T}_+(V)), \mathbb{C}), \prec, \succ)$ *is a unital shuffle algebra.*

From the general theory of Hopf algebras, the exponential convolution defines a bijection between two special subsets in $\mathrm{Lin}(\mathsf{T}(\mathsf{T}_+(V)), \mathbb{C})$. More precisely, recall that a linear functional Φ on $\mathsf{T}(\mathsf{T}_+(V))$ is a *character* if it is a unital algebra morphism. On the other hand, a linear functional α on $\mathsf{T}(\mathsf{T}_+(V))$ is an *infinitesimal character* if $\alpha(\mathbb{1}) = 0$ and $\alpha(w_1|w_2) = 0$ for any $w_1, w_2 \in \mathsf{T}_+(V)$. We denote by G and $\mathfrak{g}$ the sets of characters and infinitesimal characters, respectively. The coassociativity of Δ implies that G is a group with respect to the convolution product $*$. Besides, one can also show that $\mathfrak{g}$ is a Lie algebra with respect to the Lie bracket $[\alpha, \gamma] := \alpha * \gamma - \gamma * \alpha$. With these notions, one has that the exponential convolution

$$\mathfrak{g} \ni \alpha \mapsto \exp^*(\alpha) = \sum_{n \geq 0} \frac{\alpha^{*n}}{n!} \in G$$

defines a bijection between $\mathfrak{g}$ and G. Furthermore, the half-shuffle products motivate the exponential-type maps

$$\mathcal{E}_\prec(\alpha) = \sum_{n \geq 0} \alpha^{\prec n} \quad \text{and} \quad \mathcal{E}_\succ(\alpha) = \sum_{n \geq 0} \alpha^{\succ n},$$

for any $\alpha \in \mathfrak{g}$, where $\alpha^{\prec 0} = \alpha^{\succ 0} = \varepsilon$, $\alpha^{\prec (n+1)} = \alpha \prec \alpha^{\prec n}$ for any $n \geq 0$, and analogously, $\alpha^{\succ (n+1)} = \alpha^{\succ n} \succ \alpha$. Following the nomenclature, we call $\mathcal{E}_\prec(\alpha)$ and $\mathcal{E}_\succ(\alpha)$ the *left and right half-shuffle exponential maps of* α, respectively. It turns out that the half-shuffle exponential maps also provide bijections between $\mathfrak{g}$ and G.

Theorem 3.6 ([19, 20]). *For a character $\Phi \in G$, there exists a unique triple of infinitesimal characters $(\kappa, \beta, \rho) \in \mathfrak{g}^3$ such that*

$$\Phi = \mathcal{E}_\prec(\kappa) = \mathcal{E}_\succ(\beta) = \exp^*(\rho). \tag{3.4}$$

The infinitesimal characters κ and β are the unique solutions of the fixed point equations

$$\Phi = \varepsilon + \kappa \prec \Phi \quad \textit{and} \quad \Phi = \varepsilon + \Phi \succ \beta, \tag{3.5}$$

respectively. Conversely, $\mathcal{E}_\prec(\alpha)$, $\mathcal{E}_\succ(\beta)$, and $\exp^*(\alpha)$ are characters, for any $\alpha \in \mathfrak{g}$.

The previous theorem says, in particular, that $\mathcal{E}_\prec(\alpha)$ is invertible with respect to the convolution product $*$, for any $\alpha \in \mathfrak{g}$. Even more, the following lemma exhibits a relation between both half-shuffle exponentials through the inverse with respect to $*$.

Lemma 3.7 ([19, Lem. 2]). *For any $\alpha \in \mathfrak{g}$, we have that $\mathcal{E}_\prec(\alpha)^{*-1} = \mathcal{E}_\succ(-\alpha)$.*

4. Algebraic approach to non-commutative probability

The objective of this section is to explain how the double tensor Hopf algebra construction provides a useful framework for non-commutative probability, in particular for non-commutative cumulants. We begin by providing the combinatorial definition of cumulants for the free, Boolean, and monotone cases. Then, we exhibit how cumulants can be considered as linear functionals on a double tensor Hopf algebra and indicate how shuffle-algebraic relations effectively describe relations between moments and cumulants.

4.1. Cumulants in non-commutative probability

Recall that a pair $(\mathcal{A}, \varphi)$ is a *non-commutative probability space* if $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$, where $1_{\mathcal{A}}$ is the unit of $\mathcal{A}$. Elements $a \in \mathcal{A}$ are called random variables, and $\varphi(a)$ is called the moment of a with respect to φ.

A non-commutative analogue of independence may be considered algebraically as a rule to compute the joint distribution of independent random variables from their marginal distributions. Muraki in [31] proved that there are only five such notions of independence that satisfy certain natural conditions. For each notion of independence, a non-commutative probability theory can be constructed – one can define analogous notions of convolutions, central limit theorems, and cumulants.

The notions of cumulants for *free, Boolean*, and *monotone independence* can be introduced by the so-called moment-cumulant formulas. Before stating the next definition, we fix the following notation: given a family $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ of multilinear functionals, elements $a_1, \dots, a_n \in \mathcal{A}$ and $\pi \in \mathrm{NC}(n)$, we write

$$f_\pi(a_1, \dots, a_n) := \prod_{B\in\pi} f_{|B|}(a_1, \dots, a_n|B),$$

where $f_{|B|}(a_1,\dots,a_n|B) := f_{|B|}(a_{i_1},\dots,a_{i_\ell})$ if $B = \{i_1 < \cdots < i_\ell\}$. The linear functional φ provides an example of a family of multilinear functionals $\{\varphi_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ by writing $\varphi_n(a_1, \dots, a_n) := \varphi(a_1 \cdot_{\mathcal{A}} \dots \cdot_{\mathcal{A}} a_n)$, where $\cdot_{\mathcal{A}}$ stands for the associative product of the algebra $\mathcal{A}$.

Definition 4.1. The *free* [39], *Boolean* [40], *and monotone* [24] *functionals cumulants* form, respectively, the families of multilinear functionals $\{k_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, and $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, implicitly defined by the equations

$$\varphi_n(a_1, \dots, a_n) = \sum_{\pi\in\mathrm{NC}(n)} k_\pi(a_1, \dots, a_n), \tag{4.1}$$

$$\varphi_n(a_1,\ldots,a_n) = \sum_{\pi\in\mathrm{NCInt}(n)} b_\pi(a_1,\ldots,a_n), \tag{4.2}$$

$$\varphi_n(a_1,\ldots,a_n) = \sum_{\pi\in\mathrm{NC}(n)} \frac{1}{f_\pi!} h_\pi(a_1,\ldots,a_n), \tag{4.3}$$

for any $n \geq 1$ and $a_1,\ldots,a_n \in \mathcal{A}$, where f_π stands for the forest of nestings of π.

Remark 4.2. It is not difficult to see that the previous combinatorial formulas indeed recursively define the so-called free, Boolean, and monotone cumulants. Actually, for the free and Boolean cases, we can take advantage of the Möbius inversion formulas in the posets $\mathrm{NC}(n)$ and $\mathrm{NCInt}(n)$, respectively, in order to obtain the corresponding cumulant-moment relations. However, this approach cannot be directly followed in the monotone case since the coefficients $\frac{1}{f_\pi!}$ in (4.3) do not satisfy the multiplicativity condition to apply Möbius inversion.

4.2. Shuffle-algebraic framework for non-commutative cumulants

In [19, 20], the authors established a group-theoretical framework for the moment-cumulants relations based on a double tensor Hopf algebra described in Section 3. More precisely, given a non-commutative probability space $(\mathcal{A},\varphi)$, consider $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ the double tensor Hopf algebra on $\mathcal{A}$. Then, we extend the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ to a character $\Phi : \mathsf{T}(\mathsf{T}_+(\mathcal{A})) \to \mathbb{C}$ by setting

$$\Phi(w) = \varphi_n(a_1,\ldots,a_n), \tag{4.4}$$

for any word $w = a_1\cdots a_n \in \mathcal{A}^{\otimes n}$, and extending multiplicatively and linearly for all the elements of $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$. With these notions, the connection with non-commutative probability is stated in the following theorem.

Theorem 4.3 ([19, 20]). *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and let $\Phi : \mathsf{T}(\mathsf{T}_+(\mathcal{A})) \to \mathbb{C}$ be the extension of φ to a character as described above. Let (κ,β,ρ) be the unique triple of infinitesimal characters given in Theorem 3.6. Then, κ, β, and ρ correspond with the free, Boolean, and monotone cumulants, respectively,*

$$\kappa(w) = k_n(a_1,\ldots,a_n), \quad \beta(w) = b_n(a_1,\ldots,a_n), \quad \rho(w) = h_n(a_1,\ldots,a_n),$$

for any $w = a_1\cdots a_n \in \mathcal{A}^{\otimes n}$.

The strategy to prove the previous theorem is to observe that the three equations in (3.4) recover the moment-cumulant relations (4.1), (4.2), and (4.3) when evaluating on a word $w \in \mathsf{T}_+(\mathcal{A})$.

More precisely, we have for any $w = a_1 \cdots a_n \in \mathsf{T}_+(\mathcal{A})$:

$$\mathcal{E}_{\prec}(\kappa)(w) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{B \in \pi} \kappa(w_B),$$

$$\mathcal{E}_{\succ}(\beta)(w) = \sum_{\pi \in \mathrm{NCInt}(n)} \prod_{B \in \pi} \beta(w_B),$$

$$\exp^*(\rho)(w) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{f_\pi !} \prod_{B \in \pi} \rho(w_B).$$

Theorem 4.3 provides a shuffle-algebraic point of view for moments and non-commutative cumulants, which have been recently studied to understand in a unified way several notions in non-commutative probability theory, such as additive convolutions [20], conditional free cumulants [21], and recently, solving the open problems of finding combinatorial relations to write monotone cumulants in terms of moments [5], and free and Boolean cumulants [12].

Let us recall one context where the shuffle-algebraic framework for non-commutative probability is applied. Recall that, in the classical case, if X is a classical random variable with finite moments of all orders, there exists a particular family $\{W_n(x)\}_{n\geq 0}$ of polynomials associated with X closely related with the classical cumulants of X, called *Wick polynomials*, which are determined by the properties $W_0(x) = 1$ and

$$\mathbb{E}(W_n(X)) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x} W_n(x) = n W_{n-1}(x), \quad \text{for any } n > 0.$$

In [3], the author introduced the free counterpart of the Wick polynomials, which satisfy properties analogous to those of classical Wick polynomials. Roughly speaking, the free Wick polynomials appear by considering the family of free cumulants of a random variable instead of its classical cumulants. This situation is described in Hopf-algebraic terms as follows. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and consider the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ as well as the character Φ on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ extending φ. Since Φ is a character, then Φ^{*-1} exists and we can define the *free Wick map* $W : \mathsf{T}(\mathsf{T}_+(\mathcal{A})) \to \mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ by the recipe

$$W := (\mathrm{id} \otimes \Phi^{*-1}) \circ \Delta. \tag{4.5}$$

The map W was introduced in [22], where the authors proved that the evaluation $W(a_1 \cdots a_n)$ coincides with the free Wick polynomials introduced in [3]. This can be proved by using the Hopf-algebraic framework in the following way by taking advantage of algebraic relations and obtaining a combinatorial formula by evaluating on a word $w = a_1 \cdots a_n$.

Proposition 4.4 ([3, Prop. 3.12]). *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space with a family of free cumulants $\{k_n\}_{n\geq 1}$. Consider the double tensor Hopf algebra*

$\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ *and the character* Φ *on* $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ *extending* φ. *Also, consider the free Wick map* W *defined in* (4.5). *Then, for a word* $w = a_1 \cdots a_n \in \mathsf{T}_+(\mathcal{A})$, *we have that*

$$W(a_1 \cdots a_n) = \sum_{\substack{B \subseteq [n] \\ B=\{i_1<\cdots<i_s\}}} a_{i_1} \cdots a_{i_s} \sum_{\substack{\pi \in \mathrm{NCInt}([n]\setminus B) \\ \{B\}\cup\pi \in \mathrm{NC}(n)}} (-1)^{|\pi|} k_\pi(a_1, \ldots, a_n \mid [n]\setminus B), \tag{4.6}$$

for any $n \geq 1$ *and* $a_1, \ldots, a_n \in \mathcal{A}$.

Proof. Let κ be the infinitesimal character such that $\Phi = \mathcal{E}_\prec(\kappa)$. By Theorem 4.3, κ identifies with the free cumulants when evaluating on a word $w \in \mathsf{T}_+(\mathcal{A})$. Also, Lemma 3.7 implies that $\Phi^{*-1} = \mathcal{E}_\prec(\kappa)^{*-1} = \mathcal{E}_\succ(-\kappa)$. In the language of non-commutative probability, this means that $-\kappa$ is the infinitesimal character extending the Boolean cumulant functionals of Φ^{*-1}. Then, we have

$$\Phi^{*-1}(w) = \sum_{\pi \in \mathrm{NCInt}(n)} \prod_{B \in \pi} (-\kappa(w_B)) = \sum_{\pi \in \mathrm{NCInt}(n)} (-1)^{|\pi|} k_\pi(a_1, \ldots, a_n), \tag{4.7}$$

for $n \geq 1$ and any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. On the other hand, by the definition of W and the fact that Φ^{*-1} is multiplicative, we have for a word $w = a_1 \cdots a_n$:

$$\begin{aligned} W(w) &= (\mathrm{id} \otimes \Phi^{*-1}) \circ \Delta(w) \\ &= \sum_{B \subseteq [n]} w_B \prod_{i=1}^{s} \Phi^{*-1}(w_{K_i}), \end{aligned}$$

where $K(B) = \{K_1, \ldots, K_s\}$ is the decomposition of $[n] \setminus B$ into its connected components. Notice that we can use (4.7) on each $\Phi^{*-1}(w_{K_i})$ in order to write each factor in terms of free cumulants. Furthermore, given $B \subseteq [n]$, observe that the fact that K_i and K_{i+1} are not consecutive intervals, for every $1 \leq i < s$, implies that the conditions $\pi \in \mathrm{NCInt}([n]\setminus B)$ and $\{B\} \cup \pi \in \mathrm{NC}(n)$ mean that $\pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_s$, where $\pi_i \in \mathrm{NCInt}(K_i)$ for each $1 \leq i \leq s$. Therefore, we get

$$\begin{aligned} W(w) &= \sum_{B \subseteq [n]} w_B \prod_{i=1}^{s} \Phi^{*-1}(w_{K_i}) \\ &= \sum_{B \subseteq [n]} w_B \prod_{i=1}^{s} \sum_{\pi_i \in \mathrm{NCInt}(K_i)} (-1)^{|\pi_i|} k_{\pi_i}(a_1, \ldots, a_n | K_i) \\ &= \sum_{B \subseteq [n]} w_B \sum_{\substack{\pi \in \mathrm{NCInt}([n]\setminus B) \\ \{B\}\cup\pi \in \mathrm{NC}(n)}} (-1)^{|\pi|} k_\pi(a_1, \ldots, a_n \mid [n]\setminus V), \end{aligned}$$

as we wanted to show. ■

5. The antipode of $\mathsf{S}(\mathsf{T}_+(V))$ and forest formulas

As a preliminary step towards finding the antipode in a double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$, we begin by addressing the same issue in its commutative counterpart, the symmetric algebra of the augmented tensor algebra, $\mathsf{S}(\mathsf{T}_+(V))$. The goal of this section is to understand the antipode formula using a general method that was previously developed in [29]. This method involves calculating a formula for the iterated coproduct using a specific class of decorated non-planar rooted trees, which naturally incorporates Schröder trees into the antipode formula.

5.1. Right-handed polynomial Hopf algebras and forest formulas

In [29], the authors introduced a method to produce combinatorial tree-indexed formulas for the iterated coproduct and the antipode for a particular class of Hopf algebras that are enveloping algebras of pre-Lie algebras (see [11, Chap. 6]). The precise definition of this class of Hopf algebras can be stated as follows.

Definition 5.1. Let V be a vector space. A *right-handed polynomial Hopf algebra* is a polynomial algebra $\mathsf{S}(V)$ together with a coproduct δ that makes $\mathsf{S}(V)$ a Hopf algebra and such that δ satisfies

$$\overline{\delta}(V) \subset V \otimes \mathsf{S}(V),$$

where $\overline{\delta}$ is the reduced coproduct $\overline{\delta}(x) = \delta(x) - x \otimes 1_{\mathsf{S}(V)} - 1_{\mathsf{S}(V)} \otimes x$.

Example 5.2. Let V be a finite-dimensional vector space, and consider $\mathsf{S}(\mathsf{T}_+(V))$ the *symmetric algebra* on the tensor algebra on V, i.e., $\mathsf{S}(\mathsf{T}_+(V))$ is defined to be the quotient of $\mathsf{T}(\mathsf{T}_+(V))$ by the Hopf ideal generated by elements of the form $w|w' - w'|w$, for any $w, w' \in \mathsf{T}_+(V)$. Notice that $\mathsf{T}_+(V)$ has a countable basis given by the set $\mathcal{W}$ of non-empty words with letters in V. By symmetrizing the expression for the coproduct in $\mathsf{T}(\mathsf{T}_+(V))$, the reduced coproduct in $\mathsf{S}(\mathsf{T}_+(V))$ on a word $w \in \mathcal{W}$ of length $|w| = n$, denoted by $\overline{\Delta}_{\mathsf{S}}$, is given by

$$\overline{\Delta}_{\mathsf{S}}(w) = \sum_{\emptyset \neq A \subsetneq [n]} w_A \otimes \tilde{w}^{(A)}, \tag{5.1}$$

where if $K(A) = \{K_1, \ldots, K_s\}$, then $\tilde{w}^{(A)}$ denotes the commutative monomial in $\mathsf{S}(\mathsf{T}_+(V))$ given by the polynomial product $w_{K_1} \cdot \ldots \cdot w_{K_s}$. In particular, we have that $\mathsf{S}(\mathsf{T}_+(V))$ is a right-handed polynomial Hopf algebra.

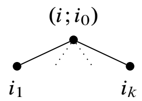
We now describe the method developed in [29] to find an antipode formula for right-handed polynomial Hopf algebras. First, assume that $\mathsf{S}(V)$ is a conilpotent right-handed polynomial Hopf algebra and that V has a countable basis $\mathcal{B} = \{b_i\}_{i\geq 1}$. Then, the reduced coproduct of an element $b_i \in \mathcal{B}$ can be expanded as

$$\overline{\delta}(b_i) = \sum_{i_0, I \neq \emptyset} \lambda_I^{i;i_0} b_{i_0} \otimes b_I, \tag{5.2}$$

for coefficients $\lambda_I^{i;i_0} \in \mathbb{K}$, where the above sum runs over the integers $i_0 \geq 1$ and non-empty multisets $I \subseteq \mathbb{N}$, where $b_I := b_{i_1} \cdot b_{i_2} \cdot \ldots \cdot b_{i_s}$ is the monomial given by the (commutative) polynomial product of the basis elements indexed by $I = \{i_1, i_2, \ldots, i_s\}$. Notice that the $\lambda_I^{i;i_0}$-coefficients completely determine the coproduct, its action on products of elements of $\mathcal{B}$, as well as the action of the iterated reduced coproducts $\overline{\delta}^{[k]}$.

A key observation in [29] is that the summation in (5.2) can be indexed by non-planar decorated corollas in the following way:

$$\overline{\delta}(b_i) = \sum \lambda\Big(\qquad \Big) b_{i_0} \otimes b_{i_1} \cdots b_{i_k}. \tag{5.3}$$



The decoration of the corollas is given by a pair of positive integers $(i;i_0)$ associated with the root, and leaves are decorated by positive integers $i_1, \ldots, i_k$, with $k \geq 1$. The commutativity of the elements in $\mathsf{S}(V)$ is reflected by the fact that we are considering non-planar trees. Furthermore, expression (5.3) suggests that iterations of the reduced coproduct can be encoded in a family of more general non-planar decorated trees. To this end, we describe the following notions associated with decorated trees.

Definition 5.3 ([29]). Let T be a non-planar finite rooted tree whose internal vertices are decorated by pairs $(p_1; p_2)$ of positive integers, and whose leaves are decorated by positive integers. In the case of T being a single-vertex tree, the vertex is considered a leaf.

(i) For any $x \in \mathrm{Int}(T)$ internal vertex of T, we denote by $d(x) = (d_1(x); d_2(x))$ its decoration. If x is a leaf of T, we denote its decoration by $d(x) = d_1(x) = d_2(x)$.

(ii) If the root of T is decorated by i or $(i;i_0)$, we say that T is *associated with* $b_i \in \mathcal{B}$. The set of decorated trees associated with b_i is denoted by $\mathcal{T}_i$.

(iii) For a given pair of positive integers $(i;i_0)$, we denote by $B_+^{(i;i_0)}(T_1, \ldots, T_s)$ the decorated tree obtained by adding a common root decorated by $(i;i_0)$ to the decorated trees $T_1, \ldots, T_s$. If $T = B_+^{(i;i_0)}(T_1, \ldots, T_s)$, we denote by $B_-(T)$ the multiset of trees $\{T_1, \ldots, T_s\}$.

(iv) Let F be a multiset of decorated trees given by

$$F = \{T_{1,i_1}^{k_{1,1}}, \ldots, T_{s_1,i_1}^{k_{s_1,1}}\} \cup \cdots \cup \{T_{1,i_p}^{k_{1,p}}, \ldots, T_{s_p,i_p}^{k_{s_p,p}}\},$$

where

- the tree T_{j,i_q} is associated with i_q, for every $1 \leq q \leq p$;
- the trees T_{j,i_q} are all distinct, for every $1 \leq j \leq s_q$;
- the notation $T_{j,i_q}^{k_{j,q}}$ means that the tree T_{j,i_q} appears with multiplicity $k_{j,q}$ in the multiset F, for every $1 \leq j \leq s_q$ and $1 \leq q \leq p$.

The *symmetry coefficient* of F, denoted by $\mathrm{sym}(F)$, is then given by

$$\mathrm{sym}(F) := \prod_{j=1}^{p} \binom{k_{1,j} + \cdots + k_{s_j,j}}{k_{1,j}, \ldots, k_{s_j,j}}.$$

(v) We define the coefficient $\lambda(T)$ as follows: if $\bullet_i$ stands for the single-vertex tree with decoration i, then $\lambda(\bullet_i) := 1$. In general, if $T = B_+^{(i;i_0)}(T_1, \ldots, T_s)$, then

$$\lambda(T) := \lambda_{i_1,\ldots,i_s}^{i;i_0} \,\mathrm{sym}(F)\lambda(T_1) \cdots \lambda(T_s),$$

when $T_1, \ldots, T_s$ are trees, respectively, associated with $b_{i_1}, \ldots, b_{i_s}$, and F is the multiset $\{T_1, \ldots, T_s\}$.

It was observed in [29] that the tensors of length k arising from the k-fold iterated reduced coproduct can be described using the notion of k-linearisation of a poset.

Definition 5.4. Let P be a finite poset. We say that $f : P \to [k]$ is a *k-linearisation of P* if f is a surjective, strictly order-preserving map $f : P \to [k]$. We denote by $k\text{-lin}(P)$ the set of k-linearisations of P.

Recall that any tree can be regarded as a poset where the root is a minimal element. In particular, if T is a decorated tree with a given decoration $d = (d_1; d_2)$ and f is a k-linearisation of T, we can write

$$C(f) := \Big(\prod_{x_1 \in f^{-1}(1)} b_{d_2(x_1)} \Big) \otimes \cdots \otimes \Big(\prod_{x_k \in f^{-1}(k)} b_{d_2(x_k)} \Big),$$

where we recall that the product in the above expression is the commutative polynomial product.

Theorem 5.5 ([29, Lem. 12], [13]). *Let $\mathsf{S}(V)$ be a right-handed polynomial Hopf algebra with coproduct δ such that V has a countable basis $\mathcal{B} = \{b_i\}_{i \geq 1}$. Then, for any $b_i \in \mathcal{B}$, we have for the action of the k-fold iterated reduced coproduct:*

$$\overline{\delta}^{[k]}(b_i) = \sum_{T \in \mathcal{T}_i} \sum_{f \in k\text{-lin}(T)} \lambda(T)C(f). \tag{5.4}$$

The formula in (5.4) is called the *forest formula for iterated coproducts*. Recalling *Takeuchi's formula* [43] for the antipode S_H in a Hopf algebra H with product m_H and reduced coproduct $\overline{\Delta}_H$, we get

$$S_H(b) = \sum_{k\geq 1}(-1)^k m_H^{[k]} \circ \overline{\Delta}_H^{[k]}(b) \quad \text{for all } b \in H. \tag{5.5}$$

Applying Takeuchi's formula, we obtain the following cancellation-free forest formula for the antipode of $\mathsf{S}(V)$.

Theorem 5.6 ([29, Thm. 8]). *With the notation of Theorem 5.5, the evaluation of the antipode S_V of the right-handed polynomial Hopf algebra $\mathsf{S}(V)$ on an element $b_i \in \mathcal{B}$ is given by the cancellation-free formula*

$$S_V(b_i) = \sum_{T\in\mathcal{T}_i} (-1)^{|T|}\lambda(T)\tau(T), \tag{5.6}$$

where $\tau(T) = \prod_{x\in\mathrm{Vert}(T)} b_{d_2(x)}$, for any $T \in \mathcal{T}_i$.

5.2. Application to the case of $\mathsf{S}(\mathsf{T}_+(V))$

We are now interested in applying the above machinery for the case of the symmetric algebra of the augmented tensor algebra of a finite-dimensional vector space. From Example 5.2, we have that $\mathsf{S}(\mathsf{T}_+(V))$ is a right-handed polynomial Hopf algebra such that the set of non-empty words $\mathcal{W}$ on letters in V is a countable basis of $\mathsf{T}_+(V)$. For notational convenience, we give an enumeration $\mathcal{W} = \{w_i\}_{i\geq 1}$.

Remark 5.7. The formula for the antipode in $\mathsf{S}(\mathsf{T}_+(V))$, in the general case in which V is not finite dimensional, will also follow from the finite-dimensional case. This can be noticed by the fact that, for a word $w = a_1 \cdots a_n \in V^{\otimes n}$, the coproduct $\Delta(w)$ is a linear combination of elements of the form $w' \otimes \mathfrak{w}$, where w' is a subword of w and $\mathfrak{w}$ is a product of subwords of w. Therefore, we can obtain $S(w)$ by computing the formula for the antipode in $\mathsf{S}(\mathsf{T}_+(V_w))$, where V_w is the finite-dimensional vector space generated by $\{a_1, \dots, a_n\}$.

We begin by writing the expression for $\overline{\Delta}_{\mathsf{S}}$ in (5.1) as a sum indexed by decorated corollas as in (5.3). To this end, take $A \subsetneq [n]$ a non-empty subset; i.e., $K(A) = \{K_1, \dots, K_s\}$ is non-empty and does not contain $[n]$. Now, consider a corolla C_0 with $|A| + 1$ leaves, and for any $K_i \in K(A)$, consider a corolla C_i with $|K_i| + 1$ leaves. Then, we label the sectors of C_0 with the elements of A increasingly from left to right, i.e., by composing the natural labelling of the sectors of C_0 with the unique increasing bijection from A to $|A|$. Moreover, we decorate C_i with the elements of K_i in the same way. Notice that the definition of $K(A)$ implies that its elements are non-consecutive

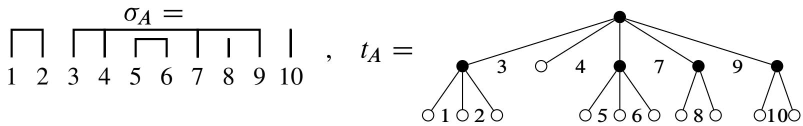


Figure 6. For $A = \{3, 4, 7, 9\} \subsetneq [10]$ and the non-crossing partition σ_A, there is a unique Schröder tree t_A with its root-sectors labelled by A such that $\pi(t_A) = \sigma_A$.

intervals. This implies that we can graft $C_1, \ldots, C_s$ to different leaves of C_0 in order to obtain a Schröder tree $t_A \in \mathrm{Sch}(n)$, in such a way that the labellings of the sectors of the corollas coincide with the natural labelling of t_A.

Alternatively, given $\emptyset \neq A \subsetneq [n]$, we can describe $t_A \in \mathrm{Sch}(n)$ as follows: consider the non-crossing partition $\sigma_A = \{A\} \cup K(A) \in \mathrm{NC}(n)$. Then, t_A is the unique Schröder tree with $n + 1$ leaves such that $\pi(t_A) = \sigma_A$ and the block $A \in \sigma_A$ is determined by the labels of the sectors adjacent to the root of t_A, given by the natural labelling of t_A. For instance, for $n = 10$ and $A = \{3, 4, 7, 9\}$, it is clear that $\pi(t_A) = \sigma_A$ (see Figure 6). On the other hand, notice that the planarity of Schröder trees describes the order between the elements of $K(A)$. In other words, given a Schröder tree t of *height* 2, i.e., t is such that the largest path from the root to any leaf contains exactly two internal vertices, we can recover a non-empty subset $A \subsetneq [n]$ indexing (5.1). Hence, we can state the following lemma.

Lemma 5.8. *Let $w \in \mathcal{W} \subset \mathsf{T}_+(V)$ be a word of length $|w| = n$. Then, the reduced coproduct on $\mathsf{S}(\mathsf{T}(V))$ can be written as*

$$\overline{\Delta}_{\mathsf{S}}(w) = \sum_{\substack{t \in \mathrm{Sch}(n) \\ t \text{ has height } 2}} w_A \otimes \tilde{w}^{(A)}, \tag{5.7}$$

where A is the block of $\pi(t)$ associated with the root of t, and $w^{(A)} = w_{K_1} \cdot \ldots \cdot w_{K_s}$ and $K(A) = \{K_1, \ldots, K_s\}$ is the list of blocks associated with the internal vertices different from the root.

Even more, the terms in the above sum can be equivalently indexed by a decorated version of the *non-planar skeleton of* t, denoted by $\widetilde{\mathrm{sk}}(t)$, which is defined to be the non-planar rooted tree generated by the $\mathrm{Int}(t)$. Notice that, for the case of Schröder trees of height 2, its skeleton is precisely a corolla with exactly $i(t) - 1$ leaves. In addition, the decoration of $\widetilde{\mathrm{sk}}(t)$ is given by labelling the root with $(i; i_0)$, where, i and i_0 are the indexes in the list $\mathcal{W} = \{w_j\}_{j \geq 1}$ such that $w = w_i$ and $w_{i_0} = w_A$, respectively, where A is the block of $\pi(t)$ associated with the root of t, and the leaves are decorated with $i_1, \ldots, i_s$, where for each $1 \leq j \leq s$, i_j is the index such that $w_{i_j} = w_{K_j}$, with $K(A) = \{K_1, \ldots, K_s\}$. Thus, we have shown that (5.7) can be written in the form (5.3), where the respective $\lambda^{i;i_0}_{i_1,\dots,i_s}$-coefficients are precisely described below.

Lemma 5.9. *Let $\mathcal{W} = \{w_j\}_{j\geq 1}$ be the set of non-empty words on a finite-dimensional vector space V. For any indexes $i, i_0, i_1, \dots, i_s \geq 1$, the coefficient $\lambda^{i;i_0}_{i_1,\dots,i_s}$ is the number of ways in which it is possible to select subwords $v_1, v_2, \dots, v_{s+1}$ of w_i such that the following hold*

- *v_j is an* interval subword, *for any $1 \leq j \leq s+1$, i.e., $v_j = w_{J_j}$ for some interval $J_j \subseteq [|w_i|]$;*
- *v_j is a non-empty word, for $2 \leq j \leq s$;*
- *$w_{i_0} = v_1 v_2 \cdots v_{s+1}$; i.e., the concatenation of $v_1, \dots, v_{s+1}$ is w_{i_0};*
- *there exists a permutation σ of the set $[s]$ such that*

$$w_i = v_1 w_{i_{\sigma(1)}} v_2 w_{i_{\sigma(2)}} \cdots v_s w_{i_{\sigma(s)}} v_{s+1}.$$

Proof. Assume that there are $s+1$ interval subwords $v_1, \dots, v_{s+1}$ of $w_i = a_1 \cdots a_n \in V^{\otimes n}$ such that $v_2, \dots, v_s$ are non-empty, $w_{i_0} = v_1 \cdots v_{s+1}$, and

$$w_i = v_1 w_{i_{\sigma(1)}} v_2 w_{i_{\sigma(2)}} \cdots v_s w_{i_{\sigma(s)}} v_{s+1}$$

for a permutation σ of $[s]$. Now, for each $1 \leq j \leq s+1$, define $B_j \subseteq [n]$ such that $v_j = w_{B_j}$. This produces a list of pairwise disjoint subsets $B_1, \dots, B_{s+1} \subseteq [n]$ such that $B_2, \dots, B_s \neq \emptyset$. We now set $A = B_1 \cup \cdots \cup B_{s+1}$. The remaining indexes $[n] \backslash A$ can be grouped into s non-empty, pairwise disjoint, and non-consecutive intervals $K_1, \dots, K_s$. By construction, it is clear that $K(A) = \{K_1, \dots, K_s\}$, $w_{i_0} = w_A$, and $\tilde{w}^{(A)} = w_{i_1} \cdot \dots \cdot w_{i_s}$ so that every selection of $s+1$ internal subwords $v_1, \dots, v_{s+1}$ as described at the beginning will produce term in the sum (5.1). Since different selection of the internal subwords will produce different subsets A that give the same term $w_{i_0} \otimes w_{i_1} \cdot \dots \cdot w_{i_s}$, we have that $\lambda^{i;i_0}_{i_1,\dots,i_s}$ counts the number of the desired decompositions of w_i, as we wanted. ■

Now, we are ready to apply the above machinery to compute the antipode of $\mathsf{S}(\mathsf{T}_+(V))$ via the forest formula in Theorem 5.6. In order to write the forest formulas for a fixed $w_i \in \mathcal{W}$, we need to describe $\mathcal{T}_i$, the set of decorated trees associated with w_i.

Proposition 5.10. *Let $w_i \in \mathcal{W} \subset \mathsf{T}_+(V)$ be a non-empty word such that $|w_i| = n$ and consider $\mathcal{T}_i'$ the subset of $T \in \mathcal{T}_i$ such that $\lambda(T) \neq 0$. Then, there exists a surjective map $\Lambda : \mathrm{Sch}(n) \to \mathcal{T}_i'$ such that $|\Lambda^{-1}(T)| = \lambda(T)$, for any $T \in \mathcal{T}_i'$.*

Proof. First, we will give the description of the map Λ, which is given by a decorated version of the non-planar skeleton of t, $\widetilde{\mathrm{sk}}(t)$. Assume that $w := w_i = a_1 \cdots a_n$ with

$a_1, \ldots, a_n \in V$. We generalise the decoration of corollas to more general trees as follows. Take a vertex v of $\widetilde{\mathrm{sk}}(t)$ and consider it as an internal vertex of t. Then, let B_v be the block of $\pi(t)$ associated with v. Then, we set $d_2(v)$ to be the index such that $w_{d_2(v)} = w_{B_v}$. For $d_1(v)$, note that v is the root of a sub-Schröder tree t' of t consisting of v and all its descendants. By setting $C_v = \bigcup_{B \in \pi(t')} B$, we set $d_1(v)$ to be the index such that $w_{d_1(v)} = w_{C_v}$. With this construction, define $\Lambda(t)$ as the decorated tree T given by the non-planar skeleton of t together with the previous decoration of the vertices of $\widetilde{\mathrm{sk}}(t)$. It is clear that if r is the root of T, then $w_{d_1(r)} = w_i$. Furthermore, the previous construction implies that, for any $v \in \mathrm{Vert}(\widetilde{\mathrm{sk}}(t))$, $w_{d_2(v)}$ is a subword of $w_{d_1(v)}$. Also, the set of indices $d_1(\mathrm{succ}(v)) := \{d_1(x) : x \in \mathrm{succ}(v)\}$ is such that

$$\{w_{d_1(x)} : x \in \mathrm{succ}(v)\} = \{w_K : K \in K(B_v, C_v)\}.$$

Hence, $\lambda^{d_1(v);d_2(v)}_{d_1(\mathrm{succ}(v))} \neq 0$ for any $v \in \mathrm{Vert}(\widetilde{\mathrm{sk}}(t))$, which implies that $\lambda(T) \neq 0$ so that $\Lambda(T) \in \mathcal{T}_i'$ and $\Lambda : \mathrm{Sch}(n) \to \mathcal{T}_i'$ is well defined.

Now, we prove that the map Λ is surjective. Take $T \in \mathcal{T}_i'$. The proof will be done by induction on $k = |T|$, the number of vertices of T. If T is a single-vertex tree, i.e., $k = 1$, we take $t \in \mathrm{Sch}(n)$ to be the corolla with $n + 1$ leaves, which is the unique Schröder tree such that $\Lambda(t) = T$. Otherwise, assume that $T = B_+^{(i;i_0)}(T_1, \ldots, T_s)$, where T_j is associated with w_{i_j} for any $1 \leq j \leq s$. Since $\lambda(T) \neq 0$, then we have that $m = \lambda^{i;i_0}_{i_1,\ldots,i_s} > 0$. By Lemma 5.9, we can find m non-empty subsets $A \subsetneq [n]$ such that $w_{i_0} = w_A$ and $\tilde{w}^{(A)} = w_{i_1} \cdot \ldots \cdot w_{i_s}$. Due to the possible repetitions in the subwords w_{i_j}, there are $\mathrm{sym}(B_-(T))$ ways to allocate the decorated trees $T_1, \ldots, T_s$ to the subwords $w_{K_1}, \ldots, w_{K_s}$, where $K(A) = \{K_1, \ldots, K_s\}$. Moreover, since $|T_j| < k$, we can find $\lambda(T_j)$ different Schröder trees $t_j \in \mathrm{Sch}(|K_j|)$ such that their corresponding decorated skeletons are given by the same T_j, for each $1 \leq j \leq s$. Hence, by grafting properly the Schröder trees $t_1, \ldots, t_s$ to a corolla with $|A| + 1$ leaves, we can construct

$$m\, \mathrm{sym}(B_-(T))\lambda(T_1) \cdots \lambda(T_s) = \lambda(T)$$

Schröder trees $t \in \mathrm{Sch}(n)$ with k internal vertices such that $\Lambda(t) = T$, which completes the proof. ∎

The main application of the previous proposition is that it allows us to rewrite the forest formulas (5.4) and (5.6) in terms of Schröder trees. In particular, the second sum in (5.4) can be indexed by k-linearisations of the skeleton of Schröder trees. For the purpose of writing the formulas, we introduce the following notation.

Notation 5.11. Let $w \in \mathsf{T}_+(V)$ be a non-empty word on V such that $|w| = n$. For $t \in \mathrm{Sch}(n)$ such that $\pi(t) = \{B_1, \ldots, B_r\}$ and $f \in k\text{-lin}(\widetilde{\mathrm{sk}}(t))$, we define

$$\tilde{w}_t := \prod_{B \in \pi(t)} w_B = w_{B_1} \cdot \ldots \cdot w_{B_r} \in \mathsf{S}(\mathsf{T}_+(V))$$

and

$$\tilde{c}(w,t,f) := \Big(\prod_{B\in\pi_1} w_B\Big) \otimes \cdots \otimes \Big(\prod_{B\in\pi_k} w_B\Big) \in \mathsf{S}(\mathsf{T}_+(V))^{\otimes k},$$

where π_j is the collection of blocks of $\pi(t)$ associated with the internal vertices v of t such that, regarded as elements of the poset $\widetilde{\mathrm{sk}}(t)$, $v \in f^{-1}(j)$, for $1 \le j \le k$.

Finally, we combine the description of the decorated trees in Proposition 5.10 with the forest formulas in Theorems 5.5 and 5.6 in order to obtain the advertised formulas for the iterated reduced coproduct and the antipode in $\mathsf{S}(\mathsf{T}_+(V))$ in terms of Schröder trees.

Theorem 5.12. *The iterated reduced coproduct* $\overline{\Delta}_{\mathsf{S}}^{[k]}$ *and the antipode* S_{S} *in* $\mathsf{S}(\mathsf{T}_+(V))$ *can be written as*

$$\overline{\Delta}_{\mathsf{S}}^{[k]}(w) = \sum_{t\in \mathrm{Sch}(n)} \sum_{f\in k\text{-lin}(\widetilde{\mathrm{sk}}(t))} \tilde{c}(w,t,f), \tag{5.8}$$

$$S_{\mathsf{S}}(w) = \sum_{t\in \mathrm{Sch}(n)} (-1)^{i(t)} \tilde{w}_t \tag{5.9}$$

for any word $w = a_1\cdots a_n \in \mathsf{T}_+(V)$ *and* $k \ge 1$.

Remark 5.13. The identity (5.9) can also be obtained in the context of algebraic structures on *combinatorial species*. It appears in [1, Prop. 25.2] as the antipode formula for the *commutative Hopf monoid of paths*, and in [28, Thm. 1] as the antipode for the *natural Hopf algebra over an operad*. More precisely, the latter reference discusses the construction of a graded Hopf algebra on the free algebra of *types of combinatorial structures* of a species **p**, where the coproduct is defined directly from a (symmetric) operad on **p**. In particular, when **p** corresponds to the *species of linear orders* and we consider the *associative operad* Ass, the Hopf algebra $\mathsf{S}(\mathsf{T}_+(V))$ is a decorated version (decorated by elements in V) of the resulting natural Hopf algebra associated with Ass, up to twisting the components of the tensors in the coproduct. The advantage of our presented approach is that we are able to provide a formula in terms of Schröder trees, given in (5.8), for the iterated reduced coproduct.

6. The antipode formula of $\mathsf{T}(\mathsf{T}_+(V))$ in terms of Schröder trees

Motivated by the formulas obtained in Theorem 5.12 for the antipode in $\mathsf{S}(\mathsf{T}_+(V))$, it is natural to ask if analogous formulas hold for the case of $\mathsf{T}(\mathsf{T}_+(V))$. In case of a positive answer, one should give a proper order in which words are multiplied due to the non-commutativity of the double tensor Hopf algebra. In this section, we study this issue in detail and also present and prove combinatorial formulas for the iterated reduced coproducts and the antipode S in $\mathsf{T}(\mathsf{T}_+(V))$ in terms of Schröder trees. More precisely, we obtain a Schröder tree-type formula for the action of the reduced coproduct on products of words, which will be used to obtain a combinatorial formula for the iterated reduced coproduct. Finally, by analysing the cancellations in Takeuchi's formula (5.5), we will then arrive at a formula for the antipode in the double tensor Hopf algebra.

Remark 6.1. Let t be a Schröder tree. Recall that the set of internal vertices of t, $\mathrm{Int}(t)$, i.e., the set of vertices of $\mathrm{sk}(t)$, can be regarded as a poset. Furthermore, the planarity of t induces a natural total order on $\mathrm{Int}(t)$. More precisely, if $v, w \in \mathrm{Int}(t)$, we have that $v < w$ if w is a descendant of v, or if v is to the left of w in the planar representation of t. This total order is called the *planar order on* $\mathrm{Int}(t)$.

6.1. Action of the reduced coproduct on products of words

We are interested in computing the antipode by using Takeuchi's formula

$$S(w) = \sum_{k \geq 1} (-1)^k m^{[k]} \circ \overline{\Delta}^{[k]}(w)$$

for $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$. By the definition of the iterated reduced coproduct, it will be convenient to have a combinatorial formula, in terms of Schröder trees, for the action of $\overline{\Delta}$ on products of words. To this end, recall that a Schröder forest is an ordered sequence of Schröder trees $F = (t_1, \ldots, t_m)$. Also, the skeleton of F, $\mathrm{sk}(F)$, can be regarded as a poset. In particular, it makes sense to consider k-linearisations of $\mathrm{sk}(F)$. For the particular case of 2-linearisations, notice that a Schröder forest F that has a 2-linearisation f is either a forest formed by two corollas, or a forest formed by corollas and at least a Schröder tree of height 2. Observe that the elements in $f^{-1}(1)$ form a subset of the roots of the Schröder trees in F.

Now, let $\mathfrak{w} = w_1 | \cdots | w_m \in \mathsf{T}_+(V)^{\otimes m}$ be a product of non-empty words with $|w_i| = n_i$ for any $1 \leq i \leq m$. Also, let $F = (t_1, \ldots, t_m) \in \mathrm{FSch}(n_1, \ldots, n_m)$ be a Schröder forest, and consider a 2-linearisation $f \in 2\text{-lin}(\mathrm{sk}(F))$. Since F is an ordered forest, we can order the elements in $f^{-1}(1) = \{v_1, \ldots, v_l\}$ from left to right. Now, for each $i \in [m]$, we set $B_i \subseteq [n_i]$ as follows: first, label considering the natural labelling of t_i by the elements of $[n_i]$. Then, if the root of t_i is not in $f^{-1}(1)$, we set $B_i := \emptyset$. Otherwise, if $v_{j_i} \in f^{-1}(1)$ is the root of t_i, then B_i is given by the labels of the sectors adjacent to v_{j_i}. With this, we set

$$d_1(\mathfrak{w}, F, f) := (w_1)_{B_1} | \cdots | (w_m)_{B_m}.$$

On the other hand, let $u_{i_1}, \ldots, u_{i_{s_i}}$ be the internal vertices of t_i such that $f(u_{i_j}) = 2$, for $1 \leq i \leq m$. If the root if t_i is in $f^{-1}(2)$, then t_i is necessarily a corolla and $s_i = 1$.

In this case, we set $D_{i_1} := [n_i]$. Otherwise, we can order the vertices $u_{i_1} < \cdots < u_{i_{s_i}}$ according to the planar order of t_i. Next, by labelling the sectors of t_i with its natural labelling, we define D_{i_j} to be the subset of $[n_i]$ given by the labels of the sectors adjacent to u_{i_j}, for $1 \leq j \leq s_i$. Hence, we can define

$$d_2(\mathfrak{w}, F, f) := (w_1)_{D_{1_1}}|\cdots|(w_1)_{D_{1_{s_1}}}|(w_2)_{D_{2_1}}|\cdots|(w_2)_{D_{2_{s_2}}}|\cdots|(w_m)_{D_{m_1}}|\cdots|(w_m)_{D_{m_{s_m}}}.$$

From this notation, we can finally state a Schröder forest-type formula for the reduced coproduct of products of words as follows.

Lemma 6.2. *Let $\mathfrak{w} = w_1|\cdots|w_m \in \mathsf{T}_+(V)^{\otimes m}$ be a product of non-empty words in $\mathsf{T}_+(V)$. If $n_i := |w_i|$ for any $1 \leq i \leq m$, then the action of the reduced coproduct on $\mathfrak{w}$ is given by*

$$\overline{\Delta}(\mathfrak{w}) = \sum_{F \in \mathrm{FSch}(n_1,\ldots,n_m)} \sum_{h \in 2\text{-lin}(\mathrm{sk}(F))} d_1(\mathfrak{w}, F, h) \otimes d_2(\mathfrak{w}, F, h). \tag{6.1}$$

Proof. The proof is an adaptation of the proof of [29, Lem. 11]. For the reader's convenience, we present a complete proof. The proof is done by induction on $m \geq 1$. The base case $m = 1$ follows using the same argument that in Lemma 5.8 and taking into account the order of the product of subwords that appears in the second component of $\overline{\Delta}$. Now, assume that the relation in (6.1) is true for a fixed $m \geq 1$. We will prove that the equation is also valid for $m + 1$.

For the proof, let $w_1, \ldots, w_m, w_{m+1}$ be non-empty words in $T_+(V)$ and consider the product $\mathfrak{w} = \mathfrak{w}'|w_{m+1}$, where $\mathfrak{w}' = w_1|\cdots|w_m$. Since Δ is an algebra morphism, we have

$$\begin{aligned}
\overline{\Delta}(\mathfrak{w}) &= \Delta(\mathfrak{w}'|w_{m+1}) - \mathfrak{w} \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{w} \\
&= (\overline{\Delta}(\mathfrak{w}') + \mathfrak{w}' \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{w}')|(\overline{\Delta}(w_{m+1}) + w_{m+1} \otimes \mathbb{1} + \mathbb{1} \otimes w_{m+1}) \\
&\quad - \mathfrak{w} \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{w} \\
&= \overline{\Delta}(\mathfrak{w}')|\overline{\Delta}(w_{m+1}) + \overline{\Delta}(\mathfrak{w}')|(w_{m+1} \otimes \mathbb{1}) + \overline{\Delta}(\mathfrak{w}')|(\mathbb{1} \otimes w_{m+1}) \\
&\quad + (\mathfrak{w}' \otimes \mathbb{1})|\overline{\Delta}(w_{m+1}) + (\mathbb{1} \otimes \mathfrak{w}')|\overline{\Delta}(w_{m+1}) \\
&\quad + \mathfrak{w}' \otimes w_{m+1} + w_{m+1} \otimes \mathfrak{w}'.
\end{aligned} \tag{6.2}$$

Let us analyse the first of the seven terms on the right-hand side of the above equation. By the induction hypothesis, the term $\overline{\Delta}(\mathfrak{w}')|\overline{\Delta}(w_{m+1})$ is equal to

$$\begin{aligned}
\overline{\Delta}(\mathfrak{w}')|\overline{\Delta}(w_{m+1}) = \Big(&\sum_{F' \in \mathrm{FSch}(n_1,\ldots,n_m)} \sum_{h \in 2\text{-lin}(\mathrm{sk}(F'))} d_1(\mathfrak{w}', F', h) \otimes d_2(\mathfrak{w}', F', h)\Big) \\
&| \Big(\sum_{t \in \mathrm{Sch}(n_{m+1})} \sum_{f \in 2\text{-lin}(\mathrm{sk}(t))} d_1(w_{m+1}, t, f) \otimes d_2(w_{m+1}, t, f)\Big).
\end{aligned}$$

First, observe that, given $F' \in \mathrm{FSch}(n_1, \ldots, n_m)$ and $t \in \mathrm{Sch}(n_{m+1})$, we can construct $F \in \mathrm{FSch}(n_1, \ldots, n_{m+1})$ by adding t at the end of the sequence F. Furthermore, given $h \in$ 2-lin(sk(F')) and $f \in$ 2-lin(sk(t)), we construct $g \in$ 2-lin(sk(F)) by setting $g(v) = h(v)$ if $v \in \mathrm{Vert}(\mathrm{sk}(F'))$ and $g(v) = f(v)$ if $v \in \mathrm{Vert}(\mathrm{sk}(t))$. Notice that, in particular, g is a 2-linearisation such that the sets $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(F'))$ and $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(t))$ are non-empty, for $i = 1, 2$. Hence, the above sum can be indexed by the described pair (F, g), and the corresponding term in the sum is precisely

$$\begin{aligned} &d_1(\mathfrak{w}', F', h)|d_1(w_{m+1}, t, f) \otimes d_2(\mathfrak{w}', F', h)|d_2(w_{m+1}, t, f) \\ &\quad = d_1(\mathfrak{w}, F, g) \otimes d_2(\mathfrak{w}, F, g). \end{aligned} \tag{6.3}$$

Working in the same way, it turns out that each of the remaining six terms on the right-hand side of the last equation in (6.2) can be rewritten as a double sum indexed by Schröder forests

$$F \in \mathrm{FSch}(n_1, \ldots, n_{m_1})$$

given in the same way as in the first case, and a 2-linearisation g of sk(F) given as follows:

- for $\overline{\Delta}(\mathfrak{w}')|(w_{m+1} \otimes \mathbb{1})$, g is such that $g(v) = 1$ for any $v \in \mathrm{Vert}(\mathrm{sk}(t))$, and $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(F')) \neq \emptyset$ for $i = 1, 2$;
- for $\overline{\Delta}(\mathfrak{w}')|(\mathbb{1} \otimes w_{m+1})$, g is such that $g(v) = 2$ for any $v \in \mathrm{Vert}(\mathrm{sk}(t))$, and $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(F')) \neq \emptyset$ for $i = 1, 2$;
- for $(\mathfrak{w}' \otimes \mathbb{1})|\overline{\Delta}(w_{m+1})$, g is such that $g(v) = 1$ for any $v \in \mathrm{Vert}(\mathrm{sk}(F'))$, and $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(t)) \neq \emptyset$ for $i = 1, 2$;
- for $(\mathbb{1} \otimes \mathfrak{w}')|\overline{\Delta}(w_{m+1})$, g is such that $g(v) = 2$ for any $v \in \mathrm{Vert}(\mathrm{sk}(F'))$, and $g^{-1}(i) \cap \mathrm{Vert}(\mathrm{sk}(t)) \neq \emptyset$ for $i = 1, 2$;
- for $\mathfrak{w}' \otimes w_{m+1}$, g is such that $g(v) = 1$ for any $v \in \mathrm{Vert}(\mathrm{sk}(F'))$, and $g(u) = 2$ for any $u \in \mathrm{Vert}(\mathrm{sk}(t))$;
- for $w_{m+1} \otimes \mathfrak{w}'$, g is such that $g(v) = 2$ for any $v \in \mathrm{Vert}(\mathrm{sk}(F'))$, and $g(u) = 1$ for any $u \in \mathrm{Vert}(\mathrm{sk}(t))$.

Conversely, it is clear that

$$\begin{aligned} &\mathrm{FSch}(n_1, \ldots, n_m) \times \mathrm{Sch}(n_{m+1}) \ni ((t_1, \ldots, t_m), t) \\ &\quad \mapsto F = (t_1, \ldots, t_m, t) \in \mathrm{FSch}(n_1, \ldots, n_m, n_{m+1}) \end{aligned}$$

is a bijection, and any $g \in$ 2-lin(sk(F)) can be obtained in a unique way as one of the previous seven cases discussed. Hence, we have that the right-hand side of (6.2) can be re-indexed as a sum of pairs $(F, g) \in \mathrm{FSch}(n_1, \ldots, n_m, n_{m+1}) \times$ 2-lin(sk(F)) and, by the definition of d_i, the corresponding term indexed by (F, g) is as on the right-hand side of (6.3). Therefore, we finally conclude that

$$\overline{\Delta}(\mathfrak{w}) = \sum_{F \in \mathrm{FSch}(n_1,\ldots,n_{m+1})} \sum_{g \in 2\text{-lin}(\mathrm{sk}(F))} d_1(\mathfrak{w}, F, g) \otimes d_2(\mathfrak{w}, F, g)$$

so that the induction and the proof are now complete. ■

6.2. Action of the iterated reduced coproduct

The formula obtained in equation (6.1) is one of the ingredients to obtain a Schröder tree-type formula for the iterated reduced coproduct. In the same way as the commutative case, the formula will be indexed by k-linearisation. However, in order to take into account the non-commutativity in $\mathsf{T}(\mathsf{T}_+(V))$, it will be convenient to introduce the following notation.

Notation 6.3. Let $t \in \mathrm{Sch}(n)$ and $f \in k\text{-lin}(\mathrm{sk}(t))$. Also, recall that $i(t)$ stands for the number of internal vertices of t. Then, we denote by $\bar{f} : \mathrm{sk}(t) \to [i(t)]$ the unique order-preserving bijection given by the following conditions:

(1) if $f(v) < f(w)$, then $\bar{f}(v) < \bar{f}(w)$;

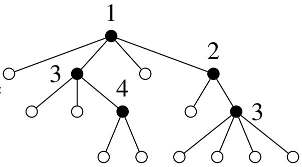
(2) if $f^{-1}(j) = \{v_1, \ldots, v_s\}$, where $v_1 < \cdots < v_s$ according to the planar order, then $\bar{f}(v_i) = \bar{f}(v_1) + i - 1$, for every $i = 1, \ldots, s$.

Let $t \in \mathrm{Sch}(n)$ and consider its associated non-crossing partition $\pi(t) \in \mathrm{NC}(n)$. In addition, if f is a k-linearisation of $\mathrm{sk}(t)$, then the map $\bar{f}$ induces a total order on the blocks of $\pi(t)$ which respects the partial order given by the nestings of the blocks. In other words, the pair $(t, \bar{f})$ can be equivalently regarded as a monotone partition $\pi(t, \bar{f}) = (C_1, \ldots, C_{i(t)})$. By setting $j_1, \ldots, j_{k-1}$ to be the indexes such that $C_{j_{i-1}+1}, C_{j_{i-1}+2}, \ldots, C_{j_i}$ (with $j_0 = 0$ and $j_k = i(t)$) are the blocks whose associated internal vertices satisfy that $f(v) = i$, for $i = 1, \ldots, k$, we can also write $\pi(t, \bar{f}) = (C_{j_0+1}, \ldots, C_{j_k})$. Thus, for a word $w = a_1 \cdots a_n \in V^{\otimes n}$, we define the element $c(w, t, f) \in \mathsf{T}(\mathsf{T}_+(V))^{\otimes k}$ by the recipe

$$c(w, t, f) := w_{C_{j_0+1}} | \cdots | w_{C_{j_1}} \otimes w_{C_{j_1+1}} | \cdots | w_{C_{j_2}} \otimes \cdots \otimes w_{C_{j_{k-1}+1}} | \cdots | w_{C_{j_k}}.$$

For instance, for the following Schröder tree with 11 leaves and with labelled vertices representing a linearisation, we have the construction for a word $w = a_1 \cdots a_{10}$ given in Figure 7.

Observe that, by definition, the element $c(w, t, f)$ is simply a k-order pure tensor obtained from the Schröder tree t, whose components are indexed by the k-linearisation f and, in the case that there are two or more elements with the same index, we multiply them from left to right, following the definition of the coproduct Δ. With this notation, we can now state and prove the following formula for the k-th iteration of the reduced coproduct in terms of Schröder trees and k-linearisations.

$(t, f) =$ 

$\pi(t) = \{\{1,5,6\},\{2,3\},\{4\},\{7\},\{8,9,10\}\}$

$\pi(t,\bar{f})=(\{1,5,6\},\{7\},\{2,3\},\{8,9,10\},\{4\})$ $c(w,t,f)=a_1a_5a_6 \otimes a_7 \otimes a_2a_3|a_8a_9a_{10} \otimes a_4$

Figure 7. Construction of $c(w,t,f)$ given a Schröder tree t with $f \in 4\text{-lin}(\mathrm{sk}(t))$ and a word w.

Theorem 6.4. *For a word $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$ and $k \geq 1$, we have*

$$\overline{\Delta}^{[k]}(w) = \sum_{t \in \mathrm{Sch}(n)} \sum_{f \in k\text{-lin}(\mathrm{sk}(t))} c(w,t,f). \tag{6.4}$$

Proof. Analogously to the case of Lemma 6.2, the present proof is an adaptation of the proof of [29, Lem. 12], and we also provide a complete proof. The proof is done by induction on $k \geq 2$. The base case $k = 2$ follows from Lemma 5.8. Now, assume that the relation in (6.4) is true for a fixed $k \geq 2$. We will prove that the (6.4) is valid for $k + 1$.

Let $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$. By the definition of the iterated reduced coproduct and the induction hypothesis, we have that

$$\begin{aligned}\overline{\Delta}^{[k+1]}(w) &= (\mathrm{id}^{\otimes(k-1)} \otimes \overline{\Delta}) \circ \overline{\Delta}^{[k]}(w) \\ &= \sum_{t \in \mathrm{Sch}(n)} \sum_{f \in k\text{-lin}(\mathrm{sk}(t))} c_1^f \otimes \cdots \otimes c_{k-1}^f \otimes \overline{\Delta}(c_k^f),\end{aligned}$$

where we have written $c(w,t,f) = c_1^f \otimes \cdots \otimes c_k^f$. If $c_k^f = w_1|\cdots|w_m$, where $n_i := |w_i|$ for $1 \leq i \leq m$, we can use Lemma 6.2 in order to write

$$\overline{\Delta}(c_k^f) = \sum_{F \in \mathrm{FSch}(n_1,\ldots,n_m)} \sum_{h \in 2\text{-lin}(\mathrm{sk}(F))} d_1^h \otimes d_2^h$$

so that we have

$$\overline{\Delta}^{[k+1]}(w) = \sum_{\substack{t \in \mathrm{Sch}(n) \\ f \in k\text{-lin}(\mathrm{sk}(t))}} \sum_{\substack{F \in \mathrm{FSch}(n_1,\ldots,n_m) \\ h \in 2\text{-lin}(\mathrm{sk}(F))}} c_1^f \otimes \cdots \otimes c_{k-1}^f \otimes d_1^h \otimes d_2^h.$$

Our objective is to rearrange the above sum as a sum indexed by pairs (t', g), where $t' \in \mathrm{Sch}(n)$ and $g \in (k+1)\text{-lin}(\mathrm{sk}(t'))$. First, consider $t \in \mathrm{Sch}(n)$, $f \in k\text{-lin}(\mathrm{sk}(t))$, $F \in \mathrm{FSch}(n_1,\ldots,n_m)$, and $h \in 2\text{-lin}(\mathrm{sk}(F))$. By definition, a forest F that provides a non-zero contribution in the above sum is either a forest formed by two corollas, or a forest formed by corollas and at least a Schröder tree of height 2. Every Schröder tree in F appears by applying the coproduct on a factor of c_k^f, which is indexed by an internal vertex of t. Each of these internal vertices, together with its adjacent sectors, defines a corolla. Thus, we can replace each of those corollas with the corresponding tree in F, obtaining in this way a Schröder tree t' with $n+1$ leaves. On the other hand, observe that the vertices in the set $f^{-1}(k)$ form the subset of leaves of $\mathrm{sk}(t)$ that coincides with the roots of the trees in F. In particular, $h^{-1}(1) \subseteq f^{-1}(k)$. Thus, we can define $g \in (k+1)\text{-lin}(\mathrm{sk}(t'))$ by setting $g(v) = m$ if $v \in f^{-1}(m)$ and $1 \le m < k$, $g(v) = k$ if $v \in h^{-1}(1)$, and $g(v) = k+1$ if $v \in h^{-1}(2)$.

Conversely, let $t' \in \mathrm{Sch}(n)$ and $g \in (k+1)\text{-lin}(\mathrm{sk}(t'))$. Then, we can consider

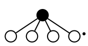
- F to be the forest of Schröder trees given by the subtrees generated by the internal vertices $g^{-1}(k) \cup g^{-1}(k+1)$ as well as their children;
- $h : \mathrm{sk}(F) \to [2]$ to be the 2-linearisation given by the standardisation of the restriction of g to $g^{-1}(k) \cup g^{-1}(k+1)$;
- t to be the Schröder tree with $n+1$ leaves obtained from t' and F as follows: if t_v is a Schröder tree in F with j_v leaves, we replace t_v in t' by a corolla with j_v leaves;
- $f : \mathrm{sk}(t) \to [k]$ to be the k-linearisation given as follows: $f(v) = g(v)$ if $g(v) < k$, and $f(v) = k$ if $g(v) \ge k$.

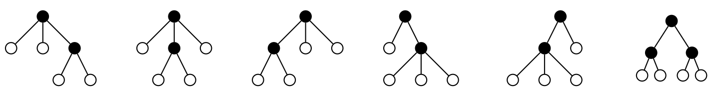
In particular, notice that f is a k-linearisation and $f^{-1}(k)$ are the leaves of $\mathrm{sk}(t)$ that coincide with the roots of the trees in F. Finally, observe that the above construction provides a bijection so that we can write

$$\begin{aligned}\overline{\Delta}^{[k+1]} &= \sum_{t \in \mathrm{Sch}(n)} \sum_{g \in (k+1)\text{-lin}(t)} c_1^g \otimes \cdots c_k^g \otimes c_{k+1}^g \\ &= \sum_{t \in \mathrm{Sch}(n)} \sum_{g \in (k+1)\text{-lin}(t)} c(w,t,g),\end{aligned}$$

as we wanted to prove. ∎

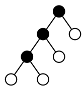
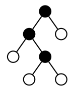
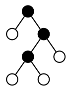
Example 6.5. In the context of the previous theorem, let us compute the k-th iteration of the reduced coproduct on a word $w = a_1a_2a_3$ for $k = 1, 2, 3$.

- $k = 1$. In this case, we have $\overline{\Delta}^{[1]}(a_1a_2a_3) = a_1a_2a_3$, which corresponds to the 1-linearisation of the corolla .
- $k = 2$. In this case, the reduced coproduct is indexed by the following Schröder trees with 4 leaves and 2-linearisations:

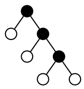
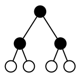
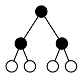


$$\overline{\Delta}^{[2]}(a_1a_2a_3) = a_1a_2 \otimes a_3 + a_1a_3 \otimes a_2 + a_2a_3 \otimes a_1 + a_1 \otimes a_2a_3 + a_3 \otimes a_1a_2 + a_2 \otimes a_1|a_3$$

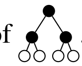
- $k = 3$. Finally, we have that the next iteration of the reduced coproduct is indexed by the following trees with 3-linearisations:

$$\bar{\Delta}^{[3]}(a_1a_2a_3) = a_3 \otimes a_2 \otimes a_1 + a_3 \otimes a_1 \otimes a_2 + a_1 \otimes a_3 \otimes a_2$$

$$+ a_1 \otimes a_2 \otimes a_3 + a_2 \otimes a_1 \otimes a_3 + a_2 \otimes a_3 \otimes a_1$$

Observe that the last two terms correspond to the two 3-linearisations of .

6.3. Ascent-free linearisations

Our final step is to describe which k-linearisations will contribute to Takeuchi's formula for the antipode in $\mathsf{T}(\mathsf{T}_+(V))$.

Definition 6.6. Let $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$ be a word. We define

$$w_{(t,f)} := m^{[k]} \circ c(w, t, f) = w_{C_1}|w_{C_2}| \cdots |w_{C_{i(t)}},$$

for any $t \in \mathrm{Sch}(n)$ and $f \in k\text{-lin}(\mathrm{sk}(t))$. Here, $\pi(t, \bar{f}) = (C_1, \ldots, C_{i(t)})$ is the monotone partition associated with the pair $(t, \bar{f})$ with $\bar{f}$ is as defined in Notation 6.3.

It is readily to see that $w_{(t,f)} = w_{(t,\bar{f})}$; i.e., $w_{(t,f)}$ only depends on f via $\bar{f}$. We can also see that, in Takeuchi's formula, we will have cancellations given by different k-linearisations f that have the same $i(t)$-linearisation $\bar{f}$. More precisely, we have

$$\begin{aligned} S(w) &= \sum_{k \geq 1} (-1)^k \sum_{t \in \mathrm{Sch}(n)} \sum_{f \in k\text{-lin}(\mathrm{sk}(t))} w_{(t,f)} \\ &= \sum_{t \in \mathrm{Sch}(n)} \sum_{g \in i(t)\text{-lin}(\mathrm{sk}(t))} w_{(t,g)} \Big(\sum_{k \geq 1} \sum_{\substack{f \in k\text{-lin}(\mathrm{sk}(t)) \\ \bar{f} = g}} (-1)^k \Big). \end{aligned} \tag{6.5}$$

In order to analyse which $i(t)$-linearisations will have a non-zero contribution to the above sum, we introduce the following notion.

Definition 6.7. Let t be a Schröder tree, and let $(v_1, \ldots, v_{i(t)})$ be the sequence of internal vertices of t ordered increasingly according to the planar order. Also, let $f \in k\text{-lin}(\mathrm{sk}(t))$ as well as its associated $\bar{f} \in i(t)\text{-lin}(\mathrm{sk}(t))$. For $1 \leq j < i(t)$, we say that f has an *ascent on* j if v_j is not the parent of v_{j+1} and $\bar{f}(v_j) < \bar{f}(v_{j+1})$.

The quantity

$$m(t, f) := |\{1 \leq j < i(t) : f \text{ has an ascent on } j\}|$$

is called the *ascent number of* t.

We notice that the definition of the ascent number $m(t, f)$ only depends on f via $\bar{f}$ so that $m(t, f) = m(t, \bar{f})$. In addition, for any Schröder tree t, we can construct an $i(t)$-linearisation f such that $m(t, f) = 0$. Indeed, we can take the labelling induced by a mirrored pre-order traversal on $\mathrm{sk}(t)$: if $\mathrm{sk}(t) = B_+(t_1, \ldots, t_m)$, we visit the root of $\mathrm{sk}(t)$, then orderly visit $t_m, t_{m-1}, \ldots, t_1$ by mirrored pre-order traversal. It is not difficult to show that there is no other f such that $m(t, f) = 0$, which motivates the following definition.

Definition 6.8. Let t be a Schröder tree. We say that $g \in i(t)\text{-lin}(\mathrm{sk}(t))$ is *ascent-free* if $m(t, g) = 0$. The unique $f_t^{\mathrm{op}} \in i(t)\text{-lin}(\mathrm{sk}(t))$ such that $m(t, f_t^{\mathrm{op}}) = 0$ is called the *ascent-free linearisation of* t.

It turns out that the only $g \in i(t)\text{-lin}(\mathrm{sk}(t))$ that have a non-zero contribution in (6.5) are those that are ascent-free. More precisely, we have the following lemma.

Lemma 6.9. *Let t be a Schröder tree and consider $g \in i(t)\text{-lin}(\mathrm{sk}(t))$. Then,*

$$\sum_{k \geq 1} \sum_{\substack{f \in k\text{-lin}(\mathrm{sk}(t)) \\ \bar{f} = g}} (-1)^k = \begin{cases} (-1)^{i(t)} & \text{if } g \text{ is ascent-free}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.6}$$

Proof. We will prove the lemma by induction on $i(t)$, the number of internal vertices of t. The case $i(t) = 1$ is trivial since the only 1-linearisation that exists is ascent-free.

Now, assume that (6.6) is valid for Schröder trees with at most m internal vertices, and take a Schröder tree t with $i(t) = m + 1$. The idea is to describe the set of k-linearisations of $\mathrm{sk}(t)$ in terms of a smaller tree. More precisely, let v_x be the internal vertex of t such that $g(v_x) = i(t)$. Since g is order-preserving, we have that v_x is a leaf of $\mathrm{sk}(t)$ so that if c stands for the corolla determined by v_x and their children in t, we have that $t' = t \backslash c$ is a Schröder tree (replacing the root of c by a leaf) with $i(t') = m$. In addition, g restricted to $\mathrm{Int}(t')$, denoted by g', is also an $i(t')$-linearisation of $\mathrm{sk}(t')$. We then have the following two cases.

(i) *g does not have an ascent on $i(t') = i(t) - 1$.* In this case, for any f k-linearisation such that $\bar{f} = g$, we can equivalently find a $(k-1)$-linearisation such that $\overline{f'} = g'$ by taking the restriction of f to $\mathrm{Int}(t') = \mathrm{Vert}(\mathrm{sk}(t'))$. Conversely, given $f' \in (k-1)\text{-lin}(\mathrm{sk}(t'))$ such that $\overline{f'} = g'$, we can find $f \in k\text{-lin}(\mathrm{sk}(t))$ such that $\bar{f} = g$ by defining $f(v_x) = k$. Observe that we cannot define $f(v_x) = k - 1$. Otherwise, if v_y is the internal vertex of t' such that $g'(v_y) = i(t')$, then $f'(v_y) = k - 1$.

Since f' is strictly order-preserving, we have that v_y is not the parent of v_x. In addition, since g does not have an ascent on $i(t)-1$, then $v_x < v_y$ in the planar order on $\mathrm{Int}(t)$. However, $f(v_x) = f(v_y)$ would imply that $g(v_x) < g(v_y)$, which contradicts the fact that v_x is maximal. Therefore, we have

$$\begin{aligned}\sum_{k\geq 1}\sum_{\substack{f\in k\text{-lin}(\mathrm{sk}(t))\\ \bar{f}=g}}(-1)^k &= -\sum_{k\geq 1}\sum_{\substack{f'\in k\text{-lin}(\mathrm{sk}(t'))\\ \overline{f'}=g'}}(-1)^k\\ &= \begin{cases}-(-1)^{i(t')} & \text{if } g' \text{ is ascent-free,}\\ -0 & \text{otherwise}\end{cases}\\ &= \begin{cases}(-1)^{i(t)} & \text{if } g \text{ is ascent-free,}\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where we used the induction hypothesis in the second equality, and in the last equality, we used the fact that if g does not have an ascent on $i(t)-1$, then g is ascent-free if and only if g' is ascent-free.

(ii) *g has an ascent on $i(t')$*. In this case, we have that $m(t,g) > 0$ so that g is not ascent-free. In a similar way to the previous case, an element $f' \in (k-1)\text{-lin}(\mathrm{sk}(t'))$ produces the following two maps:

(1) $f_1 \in k\text{-lin}(\mathrm{sk}(t))$ by setting $f_1(v_x) = k$ and $f_1(v) = f'(v)$ for any $v \in \mathrm{Vert}(\mathrm{sk}(t'))$;

(2) $f_2 \in (k-1)\text{-lin}(\mathrm{sk}(t))$ by setting $f_2(v_x) = k-1$ and $f_2(v) = f'(v)$ for any $v \in \mathrm{Vert}(\mathrm{sk}(t'))$.

Unlike the previous case, the fact that g has an ascent on $i(t)-1$ implies that f_2 in case (2) above is well defined. Conversely, any $f \in k\text{-lin}(\mathrm{sk}(t))$ can be obtained in a unique way from $f' = f|_{\mathrm{Int}(t')}$ as f_1 or f_2 depending on if $f(v_y) = f(v_x)$ or $f(v_y) < f(v_x)$, respectively, where v_y is the internal vertex of t' such that $g'(v_y) = i(t')$. Thus, we have

$$\begin{aligned}\sum_{k\geq 1}\sum_{\substack{f\in k\text{-lin}(\mathrm{sk}(t))\\ \bar{f}=g}}(-1)^k &= \sum_{k\geq 1}\Bigg(\sum_{\substack{f'\in (k-1)\text{-lin}(\mathrm{sk}(t'))\\ \overline{f'}=g'}}(-1)^k + \sum_{\substack{f'\in k\text{-lin}(\mathrm{sk}(t'))\\ \overline{f'}=g'}}(-1)^k\Bigg)\\ &= \begin{cases}-(-1)^{i(t')} + (-1)^{i(t')} & \text{if } g' \text{ is ascent-free,}\\ -0 & \text{otherwise.}\end{cases}\\ &= 0,\end{aligned}$$

where we used the induction hypothesis in the second equality. This completes the induction, and therefore, the lemma is proved. ■

By combining (6.5) with the previous lemma, we finally arrive at one of the main results of the paper: a formula for the antipode in the double tensor Hopf algebra indexed by Schröder trees.

Theorem 6.10 (Antipode formula in the double tensor Hopf algebra). *The action of antipode S in $\mathsf{T}(\mathsf{T}_+(V))$ can be written as*

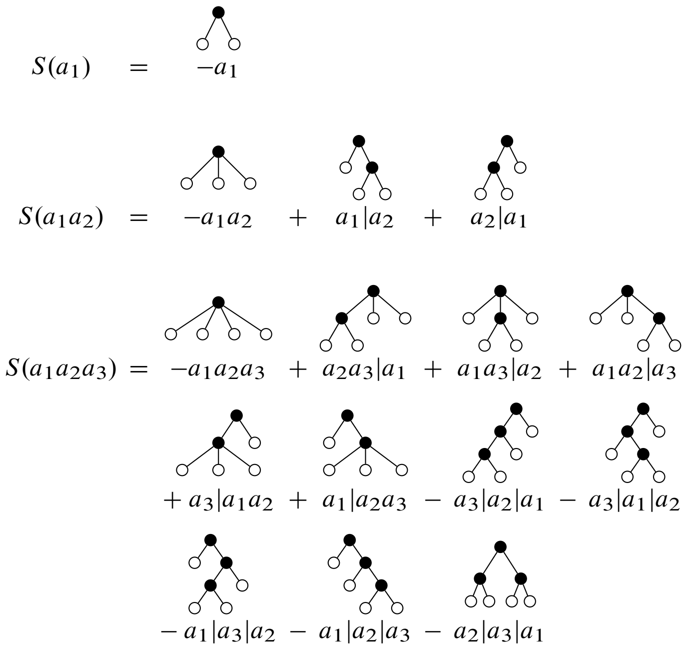
$$S(w) = \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)}\, w_t, \tag{6.7}$$

for any word $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$, where $w_t := w_{(t, f_t^{\mathrm{op}})}$ as in Definition 6.6 and

$$f_t^{\mathrm{op}} : \mathrm{sk}(t) \to [i(t)]$$

is the ascent-free linearisation of t.

Example 6.11. The following computations show how Schröder trees $\mathrm{Sch}(n)$ index the formula for the antipode on words of lengths $n = 1, 2, 3$.



$$S(a_1) = -a_1$$

$$S(a_1a_2) = -a_1a_2 + a_1|a_2 + a_2|a_1$$

$$\begin{aligned} S(a_1a_2a_3) = {} & -a_1a_2a_3 + a_2a_3|a_1 + a_1a_3|a_2 + a_1a_2|a_3 \\ & + a_3|a_1a_2 + a_1|a_2a_3 - a_3|a_2|a_1 - a_3|a_1|a_2 \\ & - a_1|a_3|a_2 - a_1|a_2|a_3 - a_2|a_3|a_1 \end{aligned}$$

As an immediate consequence of Theorem 6.10, we obtain a formula for the inverse of a character on $\mathsf{T}(\mathsf{T}_+(V))$ in terms of Schröder trees.

Corollary 6.12. *For a character Φ on $\mathsf{T}(\mathsf{T}_+(V))$, we have*

$$\Phi^{*-1}(w) = (\Phi \circ S)(w) = \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)}\, \Phi_t(w),$$

for any word $w = a_1 \cdots a_n \in \mathsf{T}_+(V)$, *where*

$$\Phi_t(w) := \Phi(w_t) = \prod_{B \in \pi(t)} \Phi(w_B) = \varphi_{\pi(t)}(a_1, \ldots, a_n).$$

Proof. The result follows immediately from the fact that Φ is multiplicative and (6.7). ■

Remark 6.13. As in the case of $\mathsf{S}(\mathsf{T}_+(V))$, our formula for the antipode of the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$ can be obtained from a particular case of the *non-commutative natural Hopf algebra* on a *non-symmetric operad*. Here, we consider the operad Ass as a non-symmetric operad. In [28, Thm. 4], Liendo and Méndez give a non-commutative analogue of the natural Hopf algebra over an operad, which yields a bialgebra structure on the free non-commutative algebra generated by isomorphism classes of a combinatorial species defined over sets endowed with a linear order (also called *rigid species*). In particular, the Hopf algebra $\mathsf{T}(\mathsf{T}_+(V))$ is a decorated version of the non-commutative natural Hopf algebra generated by Ass up to twisting the components of the tensors in the coproduct. While the proof given in [28] establishes the result by verifying that the proposed formula satisfies a recursion for the antipode, our proof has the additional advantage of providing explicit formulas for the iterated reduced coproducts in terms of Schröder trees.

6.4. A relation with the Schröder operad

The antipode formula obtained in Theorem 6.10 resembles the inverse of the series of corollas in the group associated with the operad of Schröder trees in [25]. More precisely, the *Schröder operad* is given by the collection of vector spaces $\mathcal{S} = \{\mathcal{S}_n\}_{n \geq 1}$, where $\mathcal{S}_n$ is the vector space generated by $\mathrm{Sch}(n-1)$ for any $n \geq 1$, together with a composition map

$$\tilde{\circ} : \mathcal{S}_n \otimes \mathcal{S}_{k_1} \otimes \cdots \otimes \mathcal{S}_{k_n} \to \mathcal{S}_{k_1 + \cdots + k_n}$$

such that $t_0 \otimes t_1 \otimes \cdots \otimes t_n$ is mapped to the tree $t_0 \tilde{\circ} (t_1, \ldots, t_n)$ obtained by replacing the leaves of t_0, from right to left, by the trees $t_1, \ldots, t_n$. The composition map $\tilde{\circ}$ satisfies several properties; for the general definition of an operad, we refer the reader to [11, Appx. B].

Associated with any operad, we can construct a group using a construction due to Chapoton and Livernet [14, Sec. 4]. For the Schröder operad, consider the space

$$G_{\mathcal{S}} = \Big\{ \sum_{t \in \mathrm{Sch}} p_t t : p_t \in \mathbb{K},\ p_\circ = 1 \Big\},$$

where $\circ$ stands for the single-vertex tree, and if

$$p = \sum_{t \in \mathrm{Sch}} p_t t, \quad q = \sum_{t \in \mathrm{Sch}} q_t t$$

are elements in $G_{\mathcal{S}}$, then the product

$$p \odot q := r = \sum_{t \in \mathrm{Sch}} r_t t$$

is given by the coordinates

$$r_t = \sum_{\substack{t_0, t_1, \dots, t_n \in \mathrm{Sch} \\ t = t_0 \tilde{\circ} (t_1, \dots, t_n)}} p_{t_0} q_{t_1} \cdots q_{t_n}.$$

Recalling that $C_{(n+1)}$ stands for the corolla of n leaves, it is possible to show that the series of corollas

$$f_c = \circ + \sum_{n \geq 1} C_{(n+1)} \in G_{\mathcal{S}}$$

has an inverse given by

$$g_c = \sum_{t \in \mathrm{Sch}} (-1)^{i(t)} t \tag{6.8}$$

[25, eq. (65)]. In other words, we have that $f_c \odot g_c = \circ$.

Now, let V be a vector space. For any element $p \in G_{\mathcal{S}}$, we can construct the map $\theta(p) : \mathsf{T}(V) \to \mathsf{T}(\mathsf{T}_+(V))$ by

$$p = \sum_{t \in \mathrm{Sch}} p_t t \mapsto \Big(w \mapsto \theta(p)(w) := \sum_{t \in \mathrm{Sch}(|w|)} p_t w_t, \Big),$$

for any $w \in \mathsf{T}(V)$, where w_t is as in the statement of Theorem 6.10. In the particular case of the series of corollas f_c, we have that $\theta(f_c)$ is the identity map on $\mathsf{T}(V)$. The following result provides an interpretation of the antipode S of $\mathsf{T}(\mathsf{T}_+(V))$ in terms of the Schröder operad via the θ map.

Proposition 6.14. *Let V be a vector space and S the antipode of $\mathsf{T}(\mathsf{T}_+(V))$. Moreover, let g_c be the series given in (6.8) and consider the map $\theta(g_c)$ and extend it to an anti-homomorphism of algebras*

$$\theta(g_c) : \mathsf{T}(\mathsf{T}_+(V)) \to \mathsf{T}(\mathsf{T}_+(V)).$$

Then, $\theta(g_c) = S$.

Proof. In general, let

$$q = \sum_{t \in \mathrm{Sch}} q_t t$$

be an element in $G_{\mathcal{S}}$, where $\mathcal{S}$ is the Schröder operad. Also, we extend $\theta(q)$ to $\mathsf{T}(\mathsf{T}_+(V))$ as an anti-homomorphism of algebras. Then, we have for a word $w \in V^{\otimes n}$ the following convolution product on $\mathrm{Lin}(\mathsf{T}(\mathsf{T}_+(V)), \mathsf{T}(\mathsf{T}_+(V)))$:

$$\begin{aligned}
&\mathrm{id}_{\mathsf{T}(\mathsf{T}_+(V))} * \theta(q)(w) \\
&\quad = \sum_{A \subseteq [n]} w_A \mid \theta(q)(w_{K_1}|w_{K_2}|\cdots|w_{K_s}) \\
&\quad = \sum_{A \subseteq [n]} w_A \mid \theta(q)(w_{K_s}) \mid \cdots \mid \theta(q)(w_{K_2}) \mid \theta(q)(w_{K_1}) \\
&\quad = \sum_{A \subseteq [n]} w_A \mid \Big(\sum_{t_1 \in \mathrm{Sch}(|w_{K_s}|)} q_{t_1} (w_{K_s})_{t_1} \Big) \mid \cdots \mid \Big(\sum_{t_s \in \mathrm{Sch}(|w_{K_1}|)} q_{t_s} (w_{K_1})_{t_s} \Big) \\
&\quad = \sum_{\substack{A \subseteq [n] \\ t_i \in \mathrm{Sch}(|w_{K_{s+1-i}}|) \\ 1 \le i \le s}} q_{t_1} \cdots q_{t_s} w_A \mid (w_{K_s})_{t_1} \mid \cdots \mid (w_{K_1})_{t_s}.
\end{aligned}$$

It is not difficult to see that the previous sum can be indexed by the elements of $\mathrm{Sch}(n)$. Indeed, given $A \subseteq [n]$ and $t_i \in \mathrm{Sch}(|w_{K_{s+1-i}}|)$ for $1 \le i \le s$, we set t_0 to be the corolla with $\ell := |A| + 1$ leaves. Moreover, we augment the list $t_1, \dots, t_s$ to a list $t'_1, \dots, t'_\ell$ by adding single-vertex trees in a such a way that the tree $t := t_0 \tilde{\circ}(t'_1, \dots, t'_\ell) \in \mathrm{Sch}(n)$ satisfies that $w_t = w_A|(w_{K_s})_{t_1}|\cdots|(w_{K_1})_{t_s}$. It is clear that t is uniquely defined so that we can write for any $q \in G_{\mathcal{S}}$:

$$\begin{aligned}
\mathrm{id}_{\mathsf{T}(\mathsf{T}_+(V))} * \theta(q)(w) &= \sum_{t \in \mathrm{Sch}(n)} \Big(\sum_{\substack{t_0, \dots, t_m \in \mathrm{Sch} \\ t_0 \text{ is a corolla} \\ t = t_0 \tilde{\circ}(t_1, \dots, t_m)}} q_{t_1} \cdots q_{t_m} \Big) w_t \\
&= \theta(f_c \odot q)(w).
\end{aligned}$$

In particular, taking $q = g_c$ in the above equation leads to

$$\mathrm{id}_{\mathsf{T}(\mathsf{T}_+(V))} * \theta(g_c)(w) = \theta(f_c \odot g_c) = \theta(\circ)(w) = \varepsilon(w)\mathbb{1} \quad \text{for any } w \in V^{\otimes n}.$$

The previous relation implies that $\theta(g_c)$ is an anti-homomorphism of algebras which coincides with the antipode S of $\mathsf{T}(\mathsf{T}_+(V))$ on $\mathsf{T}(V)$, and hence, $\theta(g_c) = S$. ∎

7. Applications of the antipode formula in moment-cumulant relations

In this final section, we apply the formula for the antipode in the double tensor Hopf algebra provided in Theorem 6.10 in the context of non-commutative probability theory. First, we recover the alternative presentation, given in [5, 25], of the formulas for the free, Boolean, and monotone cumulants-to-moments formula in terms of prime, Boolean, and general Schröder trees, respectively. We also present a new formula, expressing the inverse of the moment map in the double tensor algebra as a signed sum of moments. Finally, we obtain a new formula for the free Wick map in terms of Schröder trees.

7.1. Free cumulants in terms of Schröder trees

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider its free and Boolean cumulants $\{k_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, respectively. By Möbius inversion, the combinatorial moment-cumulant relations (4.1) and (4.2) can be inverted to write cumulants in terms of moments:

$$k_n(a_1, \dots, a_n) = \sum_{\pi \in \mathrm{NC}(n)} \mathrm{Moeb}_{\mathrm{NC}(n)}(\pi, 1_n)\varphi_\pi(a_1, \dots, a_n), \tag{7.1}$$

$$b_n(a_1, \dots, a_n) = \sum_{\pi \in \mathrm{NCInt}(n)} (-1)^{|\pi|-1}\varphi_\pi(a_1, \dots, a_n), \tag{7.2}$$

for any $a_1, \dots, a_n \in \mathcal{A}$. From the shuffle-algebraic point of view, the above formulas should be encompassed when inverting the respective shuffle fixed-point equations in (3.5).

More precisely, consider the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ as well as the character Φ on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ extending φ, as we considered in (4.4). From Theorem 4.3, the infinitesimal character κ given by $\Phi = \varepsilon + \kappa \prec \Phi$ agrees with the free cumulant $k_n(a_1, \dots, a_n)$ when we evaluate it on a word $w = a_1 \cdots a_n \in \mathsf{T}_+(\mathcal{A})$. The previous equation can be written as

$$\Phi - \varepsilon = \kappa \prec \Phi.$$

Using the shuffle identities (3.3), we have

$$\begin{aligned}\kappa &= \kappa \prec \varepsilon \\ &= \kappa \prec (\Phi * \Phi^{*-1}) \\ &= (\kappa \prec \Phi) \prec \Phi^{*-1} \\ &= (\Phi - \varepsilon) \prec \Phi^{*-1}.\end{aligned}$$

If S stands for the antipode in $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$, then Corollary 6.12 suggests the connection between Schröder trees and the cumulant-moments relations. More precisely, we can directly recover [25, Thm. 4.2], which was proved initially employing non-commutative symmetric functions and an unshuffle bialgebra of decorated Schröder trees.

Proposition 7.1 ([25, Thm. 4.2]). *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $\{k_n\}_{n\geq 1}$ its free cumulants. Then, for any $a_1, \ldots, a_n \in \mathcal{A}$, we have*

$$k_n(a_1, \ldots, a_n) = \sum_{t \in \mathrm{PSch}(n)} (-1)^{i(t)-1} \varphi_{\pi(t)}(a_1, \ldots, a_n). \tag{7.3}$$

The proof of the above proposition relies on the following simple yet useful bijection.

Lemma 7.2. *For every $n \geq 0$, the set of prime Schröder trees* $\mathrm{PSch}(n)$ *is in bijection with the set*

$$\mathcal{R} := \{(A, t_1, \ldots, t_{|K(A)|}) : 1 \in A \subseteq [n],\ t_j \in \mathrm{Sch}(|K_j|),\ \textit{for } 1 \leq j \leq |K(A)|\}.$$

Proof. First, we start with a prime Schröder tree $t \in \mathrm{PSch}(n)$ and consider its associated non-crossing partition $\pi(t) \in \mathrm{NC}(n)$. We take A to be the block in $\pi(t)$ associated with the root of t. Since t is prime, we have that $1 \in A$. Moreover, if $K(A) = \{K_1, \ldots, K_s\}$ is the decomposition of $[n] \backslash A$ into its connected components, then for each $1 \leq j \leq s$ we define $t_j \in \mathrm{Sch}(|K_j|)$ to be the Schröder sub-tree of t determined by the internal vertices of t such that the union of their associated blocks in $\pi(t)$ is K_j, as well as all the descendants of such vertices. In this way, it is clear that $(A, t_1, \ldots, t_s) \in \mathcal{R}$. Conversely, take $(A, t_1, \ldots, t_s) \in \mathcal{R}$. For every $1 \leq j \leq s$, we consider $\pi_{K_j}(t_j)$ to be the non-crossing partition in $\mathrm{NC}(K_j)$ obtained by applying the unique increasing bijection from $[|K_j|]$ to K_j to the blocks of $\pi(t_j)$. Then, we construct $t \in \mathrm{PSch}(n)$ by grafting $t_1, \ldots, t_s$ to a corolla with $|A| + 1$ leaves in such a way that

$$\pi(t) = \{A\} \sqcup \pi_{K_1}(t_1) \sqcup \cdots \sqcup \pi_{K_s}(t_s).$$

Since $1 \in A$, we have that t is indeed prime. It is easy to see that both constructions are invertible to each other so that we have a bijection between $\mathrm{PSch}(n)$ and $\mathcal{R}$. ■

Proof of Proposition 7.1. Consider the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ as well as the character Φ on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ extending φ. We also consider κ to be the infinitesimal character such that $\Phi = \mathcal{E}_{\prec}(\kappa)$, or equivalently, $\kappa = (\Phi - \varepsilon) \prec \Phi^{*-1}$. Let $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Using the definition of the left half-shuffle product and Corollary 6.12, we have

$$\begin{aligned}
\kappa(w) &= \sum_{1 \in A \subseteq [n]} \Phi(w_A)(\Phi \circ S)(w_{K_1}|\cdots|w_{K_s}) \\
&= \sum_{1 \in A \subseteq [n]} \Phi(w_A) \prod_{j=1}^{s} \sum_{t_j \in \mathrm{Sch}(|K_j|)} (-1)^{i(t_j)} \Phi_{t_i}(w_{K_i}).
\end{aligned}$$

Finally, by applying Lemma 7.2 to construct $t \in \mathrm{PSch}(n)$ of a given $(A, t_1, \dots, t_s)$ and noticing that $i(t) = 1 + i(t_1) + \cdots + i(t_s)$, we obtain

$$\kappa(w) = \sum_{t \in \mathrm{PSch}(n)} (-1)^{i(t)-1} \Phi_t(w),$$

where we can conclude by recalling that $\Phi_t(w) = \varphi_{\pi(t)}(a_1, \dots, a_n)$. ∎

7.2. Boolean cumulants in terms of Schröder trees

The authors of [25] proved that (7.3) implies the free cumulant-moment formula (7.1) so that the expansion in terms of Schröder trees is finer than the expansion in terms of non-crossing partitions. The corresponding formula for the Boolean counterpart was approached in [5] by similar methods to those in the free case. Unlike the free case, the expansion in terms of Schröder trees for the Boolean cumulant-moment formula is equivalent to the expansion in terms of interval partitions. For the sake of completeness, we present and prove the following result, which exhibits the mentioned relation between Boolean cumulants and Schröder trees, analogously to Proposition 7.1.

Proposition 7.3 ([5, Prop. 6.3]). *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $\{b_n\}_{n \geq 1}$ its Boolean cumulants. Then, for any $a_1, \dots, a_n \in \mathcal{A}$, we have*

$$b_n(a_1, \dots, a_n) = \sum_{t \in \mathrm{BSch}(n)} (-1)^{i(t)-1} \varphi_{\pi(t)}(a_1, \dots, a_n).$$

As in the previous section, we present a bijection that will be useful for proving the above proposition.

Lemma 7.4. *For every $n \geq 0$, there is a bijection between*

$$\mathcal{R}_1 := \{(A, t) : \emptyset \neq A \subseteq \{2, \dots, n\},\ t \in \mathrm{Sch}(|A|)\}$$

and the set $\mathcal{R}_2$ of the pairs (t', d) satisfying the following conditions:

(i) $t' \in \mathrm{Sch}(n) \setminus \mathrm{PSch}(n)$*;*

(ii) *the block $B \in \pi(t')$ such that $1 \in B$ is the block associated with the leftmost leaf of* $\mathrm{sk}(t')$*;*

(iii) $d : \mathrm{Leaf}(\mathrm{sk}(t')) \to \{0, 1\}$ *is a map such that $d(v^{\ell}_{t'}) = 1$, where* $\mathrm{Leaf}(\mathrm{sk}(t'))$ *stands for the set of leaves of* $\mathrm{sk}(t')$*, and $v^{\ell}_{t'}$ stands for the leftmost leaf of* $\mathrm{sk}(t')$.

Proof. Let $A \subseteq [n]$ be a non-empty set such that $1 \notin A$, and $t \in \mathrm{Sch}(|A|)$. We can construct an element $t' \in \mathrm{Sch}(n)$ as follows: let $K(A) = \{K_1, \dots, K_s\}$ be the decomposition of $[n] \backslash A$ into its connected components. Then, for each $1 \leq j \leq s$, let c_j

be a corolla with $|K_j| + 1$ leaves. Also, let $\pi_A(t)$ be the non-crossing partition in $\mathrm{NC}(A)$ obtained by applying the unique increasing bijection from $[|A|]$ to A to the blocks of $\pi(t)$. Next, construct $t' \in \mathrm{Sch}(n)$ by grafting $c_1, \dots, c_s$ to t in such a way that $\pi(t') = \pi_A(t) \sqcup \{K_1, \dots, K_s\}$. The fact that $1 \notin A$ implies that $1 \in K_1$ so that $t' \in \mathrm{Sch}(n) \setminus \mathrm{PSch}(n)$, with K_1 being the block of $\pi(t')$ associated with the leftmost internal vertex of t', denoted by $v_{t'}^{\ell}$. Furthermore, we can encode the corollas that are grafted to t in order to obtain t' by considering a colouring function $d : \mathrm{Leaf}(\mathrm{sk}(t')) \to \{0, 1\}$. By identifying $\mathrm{Leaf}(\mathrm{sk}(t))$ as a subset of $\mathrm{Leaf}(\mathrm{sk}(t'))$, we set $d(v) = 0$ if $v \in \mathrm{Leaf}(\mathrm{sk}(t))$ and $d(v) = 1$ if $v \in \mathrm{Leaf}(\mathrm{sk}(t')) \setminus \mathrm{Leaf}(\mathrm{sk}(t))$. In particular, we have that $d(v_{t'}^{\ell}) = 1$.

The above discussion shows that we have a map from $\mathcal{R}_1$ to $\mathcal{R}_2$. Furthermore, note that the described procedure is reversible. This means, given $t' \in \mathrm{Sch}(n) \setminus \mathrm{PSch}(n)$ and $d : \mathrm{Leaf}(\mathrm{sk}(t')) \to \{0, 1\}$ with $d(v_{t'}^{\ell}) = 1$, and such that the block associated with $v_{t'}^{\ell}$ contains 1, we can produce a subset $1 \notin A \subseteq [n]$ and a Schröder tree $t \in \mathrm{Sch}(|A|)$ by deleting all the corollas given by the leaves v of $\mathrm{sk}(t')$ such that $d(v) = 1$, and setting A to be the union of the blocks of $\pi(t')$ associated with the vertices in $\mathrm{Int}(t) \subset \mathrm{Int}(t')$. The facts that t' is not a prime Schröder tree and $d(v_{t'}^{\ell}) = 1$ imply that the corolla given by $v_{t'}^{\ell}$ will be deleted so that $1 \notin A$. The above procedure establishes a bijection between $\mathcal{R}_1$ and $\mathcal{R}_2$, as desired. ■

Proof of Proposition 7.3. As in the proof of Proposition 7.1, let Φ be the character extending φ on the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$. From Theorem 4.3, we know that Boolean cumulants identify with the infinitesimal character β which satisfies the right fixed-point equation $\Phi = \varepsilon + \Phi \succ \beta$. Equivalently, the shuffle identities (3.3) imply that the previous fixed-point equation is equivalent to

$$\beta = (\Phi \circ S) \succ (\Phi - \varepsilon).$$

Let $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Using the definition of the right half-shuffle product and Corollary 6.12, we have

$$\begin{aligned}
\beta(w) &= \sum_{1 \notin A \subseteq [n]} (\Phi \circ S)(w_A) \Phi(w_{K_1} | \cdots | w_{K_s}) \\
&= \sum_{1 \notin A \subseteq [n]} \Big(\sum_{t \in \mathrm{Sch}(|A|)} (-1)^{i(t)} \Phi_t(w_A) \Big) \prod_{j=1}^{s} \Phi(w_{K_j}) \\
&= \Phi(w) + \sum_{\substack{1 \notin A \subseteq [n] \\ A \neq \emptyset}} \Big(\sum_{t \in \mathrm{Sch}(|A|)} (-1)^{i(t)} \Phi_t(w_A) \Big) \prod_{j=1}^{s} \Phi(w_{K_j}),
\end{aligned}$$

where the first term in the last equality above follows from considering $A = \emptyset$ in the second equality. By applying the bijection provided by Lemma 7.4 to the indexing set of the double sum above, we have

$$\beta(w) = \sum_{\substack{t \in \mathrm{Sch}(n) \setminus \mathrm{PSch}(n) \\ B_{v_t^\ell} \ni 1}} \Phi_t(w) \Big(\sum_{\substack{d : \mathrm{Leaf}(\mathrm{sk}(t)) \to \{0,1\} \\ d(v_t^\ell) = 1}} (-1)^{i(t) - |d^{-1}(1)|} \Big), \tag{7.4}$$

where v_t^ℓ stands for the leftmost vertex of $\mathrm{sk}(t)$ and $B_{v_t^\ell}$ stands for the block in $\pi(t)$ associated with v_t^ℓ. By rearranging according to the index $k = |d^{-1}(1)|$ and counting the number of colourings d that satisfy this condition, we obtain

$$\begin{aligned}
\sum_{\substack{d : \mathrm{Leaf}(\mathrm{sk}(t)) \to \{0,1\} \\ d(v_t^\ell) = 1}} (-1)^{i(t) - |d^{-1}(1)|} &= (-1)^{i(t)-1} \sum_{k=0}^{|\mathrm{Leaf}(\mathrm{sk}(t))|-1} \binom{|\mathrm{Leaf}(\mathrm{sk}(t))| - 1}{k} (-1)^k \\
&= (-1)^{i(t)-1} (1-1)^{|\mathrm{Leaf}(\mathrm{sk}(t))|-1} \\
&= \begin{cases} (-1)^{i(t)-1} & \text{if } |\mathrm{Leaf}(\mathrm{sk}(t))| = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, the only Schröder trees $t \in \mathrm{Sch}(n) \setminus \mathrm{PSch}(n)$ with $B_{v_t^\ell} \ni 1$ that contribute to the sum in (7.4) are those such that

$$|\mathrm{Leaf}(\mathrm{sk}(t))| = 1,$$

and these trees are precisely the elements of $\mathrm{BSch}(n)$. Therefore, by recalling that

$$\Phi_t(w) = \varphi_{\pi(t)}(a_1, \ldots, a_n),$$

we finally obtain

$$\beta(w) = \sum_{t \in \mathrm{BSch}(n)} (-1)^{i(t)-1} \varphi_{\pi(t)}(a_1, \ldots, a_n),$$

which completes the proof. ∎

7.3. Monotone cumulants in terms of Schröder trees

In [5], the authors obtained the monotone cumulant-moment formula via the Hopf algebra of decorated Schröder trees described in [25] and the so-called *Murua coefficients* [32], which are defined by the expression

$$\omega(t) := \sum_{k=1}^{|t|} \frac{(-1)^{k-1}}{k} \omega_k(t),$$

for any rooted tree t, where $\omega_k(t)$ stands for the number of k-linearisations of t, i.e., $\omega_k(t) = |k\text{-lin}(t)|$. Besides, in the work of Murua, these coefficients have also appeared in [32, Rem. 12] in the context of numerical analysis of PDEs, in [10] in the computation of the pre-Lie Magnus expansion in the free pre-Lie algebra on one generator, and recently in [12] in the framework of cumulant-cumulant relations in non-commutative probability.

We now present a proof of the monotone cumulant-moment formula, which is not based on the calculation of the logarithm in the Hopf algebra of Schröder trees but on the Schröder trees-type coproduct formula used to calculate the antipode in $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ obtained in Theorem 6.4.

Theorem 7.5 ([5, Thm. 1.1]). *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $\{h_n\}_{n\geq 1}$ be the monotone cumulants. Then, for any $a_1, \dots, a_n \in \mathcal{A}$, we have*

$$h_n(a_1, \dots, a_n) = \sum_{t \in \mathrm{Sch}(n)} \omega(\mathrm{sk}(t)) \varphi_{\pi(t)}(a_1, \dots, a_n).$$

Proof. By Theorem 3.6, if Φ is the character extending φ on the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$, then the evaluation of $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$ on infinitesimal character

$$\rho = \log^*(\Phi)$$

is the monotone cumulant $h_n(a_1, \dots, a_n)$. On the other hand, we have the expansion

$$\rho = \log^*(\Phi) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (\Phi - \varepsilon)^{*k}.$$

Let $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$. Since $(\Phi - \varepsilon)(\mathbb{1}) = 0$, we can use the iterated reduced coproduct to compute its k-fold convolution power:

$$(\Phi - \varepsilon)^{*k}(w) = m_{\mathbb{C}}^{[k]} \circ \Phi^{\otimes k} \circ \overline{\Delta}^{[k]}(w),$$

where $m_{\mathbb{C}}^{[k]}$ stands for the associative product of k complex numbers. Hence, by Theorem 6.4 and using the fact that Φ is an algebra morphism, we have the following expression in terms of Schröder trees:

$$\begin{aligned}(\Phi - \varepsilon)^{*k}(w) &= \sum_{t \in \mathrm{Sch}(n)} \sum_{f \in k\text{-lin}(\mathrm{sk}(t))} m_{\mathbb{C}}^{[k]} \circ \Phi^{\otimes k}(c(w, t, f)) \\ &= \sum_{t \in \mathrm{Sch}(n)} \sum_{f \in k\text{-lin}(\mathrm{sk}(t))} \Phi_t(w) \\ &= \sum_{t \in \mathrm{Sch}(n)} \omega_k(\mathrm{sk}(t)) \Phi_t(w),\end{aligned}$$

where we write $\Phi_t(w) = \prod_{B\in\pi(t)} \Phi(w_B)$. By replacing the above equation in the expansion for $\log^*(\Phi)$, we obtain

$$\begin{aligned}\rho(w) &= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{t\in \mathrm{Sch}(n)} \omega_k(\mathrm{sk}(t))\Phi_t(w)\\ &= \sum_{t\in \mathrm{Sch}(n)} \left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\omega_k(\mathrm{sk}(t))\right)\varphi_{\pi(t)}(a_1,\dots,a_n)\\ &= \sum_{t\in \mathrm{Sch}(n)} \omega(\mathrm{sk}(t))\varphi_{\pi(t)}(a_1,\dots,a_n),\end{aligned}$$

as we wanted to prove. ■

7.4. A formula for Φ^{*-1} in terms of non-crossing partitions

Following the approach of the combinatorial formulas for non-commutative cumulants, the next result offers a formula for the evaluation of the inverse of a character on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ in terms of non-crossing partitions.

Proposition 7.6. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and consider Φ the character0 extending φ on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$. Then, for a word $w = a_1\cdots a_n \in \mathcal{A}^{\otimes n}$, we have that*

$$\Phi^{*-1}(w) = \sum_{\pi\in \mathrm{NC}(n)} \mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1})\varphi_\pi(a_1,\dots,a_n),$$

where $\hat{\pi}$ is the non-crossing partition in $\mathrm{NC}(\{0,1,\dots,n\}) \cong \mathrm{NC}(n+1)$ *given by*

$$\hat{\pi} := \{\{0\}\} \sqcup \pi.$$

Proof. By Corollary 6.12, we know that

$$\Phi^{*-1}(w) = (\Phi\circ S)(w) = \sum_{t\in \mathrm{Sch}(n)} (-1)^{i(t)}\varphi_{\pi(t)}(a_1,\dots,a_n).$$

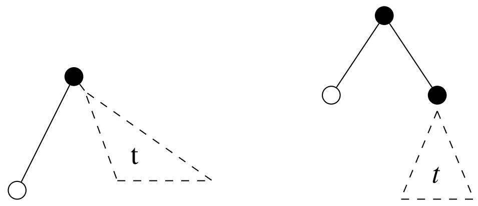
Since $\varphi_{\pi(t)}$ depends only on the non-crossing partition $\pi(t)$ associated with t, we can group the above sum as

$$\begin{aligned}\Phi^{*-1}(w) &= \sum_{\pi\in \mathrm{NC}(n)} \sum_{\substack{t\in \mathrm{Sch}(n)\\ \pi(t)=\pi}} (-1)^{|\pi(t)|}\varphi_{\pi(t)}(a_1,\dots,a_n)\\ &= \sum_{\pi\in \mathrm{NC}(n)} |\{t\in \mathrm{Sch}(n) : \pi(t)=\pi\}|(-1)^{|\pi|}\varphi_\pi(a_1,\dots,a_n).\end{aligned}$$

Hence, given $\pi \in \mathrm{NC}(n)$, we need to count the number of $t \in \mathrm{Sch}(n)$ such that $\pi(t)=\pi$. To this end, we recall that there is a two-to-one surjective map

$$P : \mathrm{PSch}(n+1) \to \mathrm{Sch}(n)$$

such that, for any $t \in \mathrm{Sch}(n)$, $P^{-1}(t)$ contains exactly the following prime Schröder trees:



Type 1 Type 2

Now, consider $\hat{\pi} \in \mathrm{NC}(\{0, 1, \ldots, n\})$ as defined above. Observe that the prime Schröder trees $\hat{t}$ such that $\pi(\hat{t}) = \hat{\pi}$ can be only of Type 1, since otherwise, $\hat{t}$ being Type 2 would imply that 0 belongs to a block of size larger than 1. Then, $\hat{t}$ is the grafting of a single-vertex tree and $t \in \mathrm{Sch}(n)$, and one can see that

$$\pi(t) = \pi.$$

In other words, we have a bijection between

$$\{t \in \mathrm{Sch}(n) : \pi(t) = \pi\} \quad \text{and} \quad \{s \in \mathrm{PSch}(n+1) : \pi(s) = \hat{\pi}\}.$$

By Proposition 2.3, the latter set contains exactly $\mathrm{Abs}(\mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1}))$, and thus,

$$\begin{aligned}\Phi^{*-1}(w) &= \sum_{\pi \in \mathrm{NC}(n)} (-1)^{|\hat{\pi}|-1} \mathrm{Abs}(\mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1}))\varphi_\pi(a_1, \ldots, a_n) \\ &= \sum_{\pi \in \mathrm{NC}(n)} \mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1})\varphi_\pi(a_1, \ldots, a_n),\end{aligned}$$

as we wanted to show. For a proof of the fact that $(-1)^{|\hat{\pi}|-1}$ is the suitable sign in order to obtain $\mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1})$ from its absolute value, the reader can check [25, Sec. 5] for a relation between Schröder trees and the *Kreweras complement* of a non-crossing partition (see [33, Def. 9.21]). ∎

A combinatorial consequence from the proof of the previous proposition is a derivation of the number of Schröder trees associated with the same non-crossing partition.

Proposition 7.7. *Let $n \geq 1$. For any $\pi \in \mathrm{NC}(n)$, we have*

$$|\{t \in \mathrm{Sch}(n) : \pi(t) = \pi\}| = \mathrm{Abs}(\mathrm{Moeb}_{\mathrm{NC}(n+1)}(\hat{\pi}, 1_{n+1})),$$

where $\hat{\pi} = \{\{0\}\} \sqcup \pi \in \mathrm{NC}(\{0, \ldots, n\}) \cong \mathrm{NC}(n+1)$.

7.5. Free Wick polynomials in terms of Schröder trees

As a final application of the antipode formula in $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$, we present a formula that establishes the connection between the free Wick polynomials (4.6) and Schröder trees. To the best of our knowledge, the formula presented in the following theorem represents a new and previously unexplored result.

Theorem 7.8 (Free Wick Polynomials). *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and consider the double tensor Hopf algebra $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$. If Φ is the character on $\mathsf{T}(\mathsf{T}_+(\mathcal{A}))$ extending φ, then the free Wick map W (4.5) of Φ satisfies, for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$,*

$$\begin{aligned} W(w) &= \Phi^{*-1}(w)\mathbb{1} + \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} w_{B_r} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B) \\ &= \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} (w_{B_r} - \Phi(w_{B_r})\mathbb{1}) \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B), \end{aligned}$$

where, for each $t \in \mathrm{Sch}(n)$, B_r stands for the block in $\pi(t)$ associated with the root of t.

Proof. By using the definition of the free Wick map W in equation (4.5) and Corollary 6.12, we have, for any word $w = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$,

$$\begin{aligned} W(w) &= (\mathrm{id} \otimes \Phi^{*-1}) \circ \Delta(w) \\ &= \sum_{A \subseteq [n]} w_A (\Phi \circ S)(w_{K_1} | \cdots | w_{K_s}) \\ &= \Phi^{*-1}(w)\mathbb{1} + \sum_{\emptyset \neq A \subseteq [n]} w_A \prod_{j=1}^{s} \sum_{t_j \in \mathrm{Sch}(|K_j|)} (-1)^{i(t_j)} \Phi_{t_j}(w_{K_j}). \end{aligned}$$

Proceeding similarly as in the proof of Lemma 7.2, we have a bijection between $\mathrm{Sch}(n)$ and the set

$$\{(A, t_1, \ldots, t_{|K(A)|}) : \emptyset \neq A \subseteq [n],\ t_j \in \mathrm{Sch}(|K_j|) \text{ for } 1 \leq j \leq |K(A)|\},$$

where, given $t \in \mathrm{Sch}(n)$, A is the block of $\pi(t)$ associated with the root of t, and for each $1 \leq j \leq s$, $t_j \in \mathrm{Sch}(|K_j|)$ is the Schröder sub-tree of t determined by the internal vertices of t such that the union of their associated blocks in $\pi(t)$ is K_j, as well as all the descendants of such vertices.

Next, given $t \in \mathrm{Sch}(n)$ with corresponding tuple $(B_r, t_1, \dots, t_s)$, we can use the fact that Φ is multiplicative in order to obtain

$$\prod_{j=1}^{s} \Phi_{t_j}(w_{K_j}) = \prod_{j=1}^{s} \prod_{B \in \pi_{K_j}(t_j)} \Phi(w_B) = \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B),$$

where $\pi_{K_j}(t_j)$ stands for the element in $\mathrm{NC}(K_j)$ obtained by applying the unique increasing bijection from $[|K_j|]$ to K_j to every block of $\pi(t_j)$. Hence, we can write

$$W(w) = \Phi^{*-1}(w)\mathbb{1} + \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} w_{B_r} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B),$$

where we have also used the fact that

$$i(t) = 1 + i(t_1) + \cdots + i(t_s).$$

For the second equation, we use Corollary 6.12 to express $\Phi^{*-1}(w)$ as a sum in terms of Schröder trees and obtain

$$\begin{aligned} W(w) &= \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)} \Phi_t(w)\mathbb{1} + \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} w_{B_r} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B) \\ &= \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} \Big(w_{B_r} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B) - \Phi_t(w)\mathbb{1} \Big) \\ &= \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} \Big(w_{B_r} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B) - \Phi(w_{B_r})\mathbb{1} \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B) \Big) \\ &= \sum_{t \in \mathrm{Sch}(n)} (-1)^{i(t)-1} (w_{B_r} - \Phi(w_{B_r})\mathbb{1}) \prod_{\substack{B \in \pi(t) \\ B \neq B_r}} \Phi(w_B), \end{aligned}$$

where we have used the fact that Φ is multiplicative in the third equality. ■

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