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Permutation invariant tensor models and partition algebras

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Abstract. Matrix models with continuous symmetry are powerful tools for studying quantum gravity and holography. Tensor models have also found applications in holographic quantum gravity. Matrix models with discrete permutation symmetry have been shown to satisfy large N factorisation properties relevant to holography, while also having applications to the statistical analysis of ensembles of real-world matrices. Here, we develop 3-index tensor models in dimension D with a discrete symmetry of permutations in the symmetric group S_D. We construct the most general permutation invariant Gaussian tensor model using the representation theory of symmetric groups and associated partition algebras. We define a representation basis for the 3-index tensors, where the two-point function is diagonalised. Inverting the change of basis gives an explicit formula for the two-point function in the tensor basis for general D.

1. Introduction

Since its introduction by Wigner and Dyson [14, 49] random matrix theory, based on matrix models having continuous symmetries, has been fruitfully applied to diverse areas of science, including nuclear physics, chaos, condensed matter physics, biological networks, feature-matrices in bio-statistics, data science, financial correlations, and quantum gravity [9, 15, 20, 22, 33, 37]. Permutation invariant matrix models have been developed and applied to real-world matrix ensembles [30, 42, 43]. Permutation invariant random matrix theory follows a similar approach to traditional random matrix theory but enriches and extends traditional areas of application as the matrix model observables include more general observables than those invariant under continuous symmetries. This has been demonstrated in the case of permutation invariant Gaussian matrix (PIGM) models already in a variety of computational linguistic applications: the Gaussianity analysed in [43] was based on the construction of matrices by linear regression in [30] while [28] extended the analysis of [43] and also analysed the matrices constructed by neural network methods in [50]. It is natural to consider the generalisation of these matrix models to models containing higher-rank objects and tensors.

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In quantum gravity, tensor models with continuous group symmetry generalise the connection between matrix models and random two-dimensional geometries to higher dimensions. The Feynman diagram expansion of tensor models with d indices can be organised in terms of d-dimensional geometries [2, 21, 23, 24, 38, 46]. More recently, quantum mechanical tensor models have been studied as toy models of black hole holography [51]. This connection exploits similarities between tensor models and SYK models, which are believed to be dual to black holes in $1+1$ dimensions [32, 44]. In this paper, we present a framework for studying tensor models with discrete permutation symmetry.

The Linguistic Matrix Theory programme [30,42] proposed random matrix theory with discrete permutation symmetries as a way to model the statistics of ensembles of matrices arising within the context of compositional distributional semantics [8, 11]. Distributional models of meaning in natural language are based on the simple idea that the meaning of a word can be deduced from its co-occurrence with other context words [17, 27]. The use of vectors to represent words is a well-established technique in computational linguistics. Compositional distributional semantics assigns words to vectors, matrices or higher-rank tensors, depending on their grammatical role and the grammatical composition of words corresponds to the contraction of indices.

In [30, 42], a permutation invariant statistical model of $D \times D$ matrices M was defined through a quadratic polynomial function $\mathcal{S}(M_{ij})$ satisfying

$$\mathcal{S}(M_{\sigma(i)\sigma(j)}) = \mathcal{S}(M_{ij}) \quad \forall \sigma \in S_D.$$

In physics parlance, such a polynomial function is called an action. The most general quadratic permutation invariant action has 13 free parameters. It defines a statistical model with partition function

$$Z = \int \mathrm{d}M \exp\{-\mathcal{S}(M)\}.$$

Conventionally, solving a model of this type involves giving an algorithm for computing expectation values

$$\langle f(M_{ij}) \rangle = \frac{1}{Z} \int \mathrm{d}M \exp\{-\mathcal{S}(M)\} f(M)$$

of polynomial functions $f(M_{ij})$. For models defined by a quadratic action, general expectation values can be computed from one-point and two-point functions

$$\langle M_{ij} \rangle, \quad \langle M_{ij} M_{kl} \rangle. \tag{1.1}$$

This uses Wick's theorem, which expresses higher-order expectation values/moments as sums of products of one- and two-point functions. Elegant expressions for (1.1) were given in [42] in terms of S_D representation-theoretic quantities. Observables in this model were defined to be general polynomial functions $f(M)$ satisfying

$$f(M_{\sigma(i)\sigma(j)}) = f(M_{ij}) \quad \forall \sigma \in S_D.$$

Closed-form expressions, as functions of D, for a selection of expectation values of observables were given.

In [5], the theoretical results in [42] were extended to 2-matrix models, and a combinatorial algorithm was developed for computing expectation values of general observables. In [4], permutation invariant matrix observables were described in terms of partition algebras which are actively studied in representation theory (e.g., [25]), and this was exploited to derive a large N factorisation property of expectation values in permutation invariant matrix models analogous to similar large N factorisation for matrix models with continuous symmetry. The factorisation property in the continuous symmetry case has known applications in the AdS/CFT correspondence [1]; see, e.g., [3, 12, 18]. Partition algebras were discovered by Jones [29] and Martin [36] in the context of classical statistical mechanics. Recently, permutation invariant matrix models and partition algebras were connected to quantum many-body physics and holography [6], using mechanisms of hidden symmetry for large N matrix/tensor systems based on Schur–Weyl duality [40,41]. Permutation invariant matrix models were developed for the case of matrices obeying constraints $M^T = M$ and having vanishing (or constant) diagonal entries. This development was motivated by applications to correlation matrices in statistical finance [7].

Building on these developments in matrix systems of general size N with discrete permutation symmetries, we initiate the study of tensor systems for tensors of general size N with discrete permutation symmetry. We will develop a statistical model for a three-index tensor Φ_{ijk} with $i, j, k = 1, \dots, D$. (We are using D instead of N here.) In compositional distributional semantics setting, three-index tensors are needed to encode the grammatical role of transitive verbs, while along the lines of [6] we expect this study will have implications for solvable quantum mechanical tensor systems.

In Section 2, we describe the general structure of the permutation invariant Gaussian tensor model and describe the permutation invariant observables formed from Φ_{ijk}. Representation theory of the symmetric group is used to find expressions for the one-point and two-point functions

$$\langle \Phi_{ijk} \rangle, \quad \langle \Phi_{ijk} \Phi_{pqr} \rangle,$$

in terms of S_D invariant endomorphism tensors which are defined in equation (2.11) in terms of symmetric group Clebsch–Gordan coefficients.

Calculating the invariant endomorphism tensors by explicitly constructing the Clebsch–Gordan coefficients for general D is highly complex. Section 3 is devoted to explicitly constructing the invariant endomorphism tensors without having to calculate Clebsch–Gordan coefficients. These invariant endomorphism tensors are elements of a partition algebra and admit a labelling in terms of graphs with S_D representation theory data. The construction involves diagonalising matrices made out of partition algebra structure constants and the size of the matrices is independent of D. This can be done with the help of open-source software SageMath for partition algebra calculations [48] (see Appendix D) The construction works efficiently for general large D of interest. (There is a minor technical restriction to $D \geq 6$.) For concreteness, we end the section with some explicit examples of invariant endomorphism tensors.

In Section 4, we give a general formula for the counting of permutation invariant tensor observables using characters of permutations in the natural representation V_D. We then show that there is a basis of invariant tensor observables which is in one-to-one correspondence with a family of bipartite 3-coloured graphs. We prove that the counting of these graphs agrees with the representation theory counting. We give a summary in Section 5 along with a discussion of future directions.

2. Permutation invariant Gaussian tensor model

In this section, we will define and construct the permutation invariant Gaussian tensor model. We solve the model using a representation-theoretic change of basis which diagonalises the two point function. We also give formulas for the one-point and two point function in the tensor basis.

The symmetric group S_D can be defined to be the set of bijective maps

$$\sigma : \{1, \ldots, D\} \to \{1, \ldots, D\}$$

with multiplication given by composition of maps. The degrees of freedom in the problem are packaged into a tensor Φ_{ijk} with $i, j, k = 1, \ldots, D$. The symmetric groups acts diagonally on Φ_{ijk}

$$\Phi_{ijk} \to \Phi_{\sigma(i)\sigma(j)\sigma(k)} \quad \forall \sigma \in S_D,\ i, j, k \in \{1, 2, \ldots, D\}. \tag{2.1}$$

A permutation invariant Gaussian tensor model is defined by a Gaussian/quadratic action $\mathcal{S}(\Phi_{ijk})$ satisfying

$$\mathcal{S}(\Phi_{ijk}) = \mathcal{S}(\Phi_{\sigma(i)\sigma(j)\sigma(k)}) \quad \forall \sigma \in S_D. \tag{2.2}$$

The corresponding partition function is

$$Z = \int \mathrm{d}\Phi \exp(-\mathcal{S}(\Phi)).$$

We take observables of the model to be S_D-invariant polynomial functions $f(\Phi_{ijk})$ of general degree:

$$f(\Phi_{ijk}) = f(\Phi_{\sigma(i)\sigma(j)\sigma(k)}) \quad \forall\sigma \in S_D.$$

Expectation values of these observables are then given by

$$\langle f(\Phi)\rangle \equiv \frac{1}{Z}\int d\Phi f(\Phi) e^{-\mathcal{S}(\Phi)}. \tag{2.3}$$

They can be computed using Wick's theorem because the action is Gaussian. To use Wick's theorem, we need expressions for

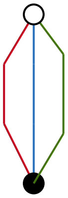
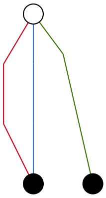
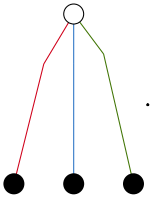
$$\left\langle \Phi_{ijk} \right\rangle, \quad \left\langle \Phi_{ijk}\Phi_{pqr} \right\rangle. \tag{2.4}$$

For a Gaussian action, computing (2.4) requires the inversion of a $D^3 \times D^3$ matrix of quadratic couplings.

Given this definition, the first hurdle in the construction and solution of the permutation invariant Gaussian tensor model is to give a parametrisation of $\mathcal{S}(\Phi_{ijk})$. Note that the condition (2.2) implies that $\mathcal{S}(\Phi_{ijk})$ is a linear combination of invariant polynomials of degree one and two. In the language of [30, 42], there exists a graph basis of invariant polynomials. A basis of invariant polynomials of degree m is labelled by set partitions of $3m$ objects. For example, there are a total of five degree one invariant combinations:

$$\sum_{i=1}^{D}\Phi_{iii}, \quad \sum_{i,j=1}^{D}\Phi_{iij}, \quad \sum_{i,j=1}^{D}\Phi_{iji}, \quad \sum_{i,j=1}^{D}\Phi_{jii}, \quad \sum_{i,j,k=1}^{D}\Phi_{ijk}. \tag{2.5}$$

They correspond to bipartite graphs (that is, graphs with black and white vertices, edges joining black vertices to white), with edges having three colours which we will draw as red, blue, and green. The tensors Φ correspond to white vertices, each index is a coloured edge and black vertices correspond to the identification of indices. This description is used in Section 4 to count observables of general degree, for general D. For example, three of the linear invariants correspond to the following graphs:

$\sum_{i=1}^{D}\Phi_{iii} \Leftrightarrow$  , $\sum_{i,j=1}^{D}\Phi_{iij} \Leftrightarrow$  , $\sum_{i,j,k=1}^{D}\Phi_{ijk} \Leftrightarrow$  .

The invariants in (2.5) also correspond to the set partitions

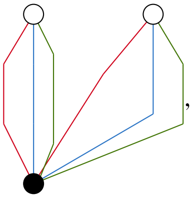
$$\{\{1,2,3\}\}, \quad \{\{1,2\},\{3\}\}, \quad \{\{1,3\},\{2\}\}, \quad \{\{2,3\},\{1\}\}, \quad \{\{1\},\{2\},\{3\}\}.$$

A subset of degree two invariants is

$$\sum_{i=1}^{D} \Phi_{iii}\Phi_{iii}, \quad \sum_{i,j=1}^{D} \Phi_{iii}\Phi_{iij}, \quad \sum_{i,j=1}^{D} \Phi_{iii}\Phi_{jjj},$$
$$\sum_{i,j,k=1}^{D} \Phi_{ijk}\Phi_{jkk}, \quad \sum_{i,j,k,p,q,r=1}^{D} \Phi_{ijk}\Phi_{pqr}. \tag{2.6}$$

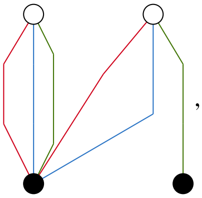
Some examples of the corresponding graphs are

$$\sum_{i=1}^{D} \Phi_{iii}\Phi_{iii} \Leftrightarrow$$



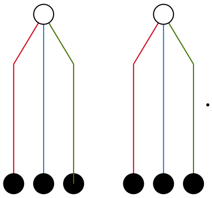
,

$$\sum_{i,j=1}^{D} \Phi_{iii}\Phi_{iij} \Leftrightarrow$$



,

$$\sum_{i,j,k,p,q,r=1}^{D} \Phi_{ijk}\Phi_{pqr} \Leftrightarrow$$



.

Note that bosonic symmetry implies a redundancy in the set partition description corresponding to the permutation symmetry $1 \leftrightarrow 4$, $2 \leftrightarrow 5$, $3 \leftrightarrow 6$. Quadratic invariants also have a corresponding set partition description:

$$\{\{1,2,3,4,5,6\}\}, \quad \{\{1,2,3,4,5\},\{6\}\}, \quad \{\{1,2,3\},\{4,5,6\}\},$$
$$\{\{1\},\{2,4\},\{3,5,6\}\}, \quad \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}.$$

Each of the invariant polynomials in (2.5) and (2.6) will appear in the Gaussian action, along with all other possible degree two invariant combinations, each with an independent coupling parameter. However, (2.4) are difficult to compute, for large D, in the graph basis used in (2.5) and (2.6) because the calculation involves the inversion of a $D^3 \times D^3$ matrix of couplings with no obvious structure that is useful in computing the inverse.

This motivates us to find a better parametrisation of the action, and we will now build towards a representation-theoretic construction of $\mathcal{S}(\Phi_{ijk})$. This will lead to a parametrisation where the quadratic coupling matrix is block diagonal, making the computation of (2.4) much simpler – involving the inverse of seven symmetric matrices of dimensions 5, 10, 6, 6, 1, 2, 1, respectively.

2.1. The natural representation of S_D and its tensor products

To describe the construction, we will need some facts about the representation theory of symmetric groups. The natural representation of S_D is given by a D-dimensional vector space

$$V_D = \mathrm{Span}(e_1, e_2, \ldots, e_D),$$

together with the following map ρ from S_D to the set of linear maps on V_D

$$\rho(\sigma)e_i = e_{\sigma^{-1}(i)}.$$

It can be verified that this is a homomorphism: ρ satisfies

$$\rho(\sigma)\rho(\sigma')e_i = \rho(\sigma\sigma')e_i$$

for all $\sigma, \sigma' \in S_D$.

The natural representation is a reducible representation; it contains two subspaces that are closed under the action of $\rho(\sigma)$:

$$V_D \cong V_{[D]} \oplus V_{[D-1,1]}.$$

We have labelled irreducible representations of S_D by integer partitions of D, as is common in the literature on representation theory of symmetric groups. See [42] for further details about the decomposition of V_D and [26] or [45] for further details on the general representation theory of S_D.

The subspace $V_{[D]}$ is the trivial representation

$$V_{[D]} = \mathrm{Span}(E_0 = e_1 + e_2 + \cdots + e_D).$$

Note that

$$\rho(\sigma)E_0 = E_0 \quad \forall \sigma \in S_D.$$

The subspace $V_{[D-1,1]}$ is a $(D-1)$-dimensional irreducible representation with basis

$$E_a = \sum_{i=1}^{D} C_{a,i} e_i,$$

where

$$C_{a,i} = \frac{1}{\sqrt{a(a+1)}}\left(-a\delta_{i,a+1} + \sum_{j=1}^{a} \delta_{ji}\right).$$

This basis is orthonormal with respect to the inner product

$$(e_i, e_j) = \delta_{ij}.$$

From equation (2.1), we have the isomorphism

$$\mathrm{Span}(\Phi_{ijk}) \cong V_D \otimes V_D \otimes V_D.$$

The representation $V_D \otimes V_D \otimes V_D$ has the following decomposition into irreducible representations:

$$\begin{aligned} &V_D \otimes V_D \otimes V_D \\ &\quad\cong 5V_{[D]} \oplus 10V_{[D-1,1]} \oplus 6V_{[D-2,2]} \oplus 6V_{[D-2,1,1]} \oplus V_{[D-3,3]} \\ &\qquad \oplus 2V_{[D-3,2,1]} \oplus V_{[D-3,1,1,1]}, \end{aligned} \tag{2.7}$$

where we have, again, used partitions of D to label the irreducible representations on the right-hand side. This decomposition is derived in Appendix B and the multiplicities are represented in terms of graphs with edges labelled by irreducible representations. For the familiar reader, they are equivalently thought of as Young diagrams with D boxes. In (2.7), we used the rule [26, Section 7.13] for decomposing tensor products of the form $V_R \otimes V_{[D-1,1]}$ (also see [13]). Equation (2.7) says that there exists a basis for $V_D \otimes V_D \otimes V_D$ spanned by elements

$$\begin{aligned} S_a^{V_\Lambda;\alpha} \quad \text{with } \Lambda \in \{&[D], [D-1,1], [D-2,2], [D-2,1,1], [D-3,3], \\ &[D-3,2,1], [D-3,1,1,1]\}, \end{aligned}$$

with α ranging over the multiplicity of V_Λ, and the a's label an orthonormal basis in each irreducible subspace. For the sake of brevity, we define the following shorthand for each of the irreducible representations appearing on the right-hand side of (2.7):

$$\begin{aligned} V_{[D]} \equiv V_0, \quad V_{[D-1,1]} \equiv V_H, \quad V_{[D-2,2]} \equiv V_2, \quad V_{[D-2,1,1]} \equiv V_3, \\ V_{[D-3,3]} \equiv V_4, \quad V_{[D-3,2,1]} \equiv V_5, \quad V_{[D-3,1,1,1]} \equiv V_6. \end{aligned} \tag{2.8}$$

An explicit change of basis,

$$S_a^{\Lambda,\alpha} = \sum_{i,j,k=1}^{D} C_{a,ijk}^{\Lambda,\alpha} \Phi^{ijk},$$

implementing the isomorphism (2.7), defines a set of coefficients $C_{a,ijk}^{\Lambda,\alpha}$ known as Clebsch–Gordan coefficients. The statement that they implement an isomorphism of representations implies that the coefficients satisfy the equivariance property

$$C_{a,\sigma(i)\sigma(j)\sigma(k)}^{\Lambda,\alpha} = \sum_{b=1}^{\dim V_\Lambda} D_{ab}^{\Lambda}(\sigma) C_{b,ijk}^{\Lambda,\alpha},$$

where $D_{ab}^{\Lambda}(\sigma)$ is an irreducible representation of S_D labelled by Λ.

2.2. Representation basis for linear part of action

Having defined the representation basis, we can now give a good parametrisation of the action, starting with the linear part. According to (2.7), the vectors

$$S^{V_{[D]},1}, \quad S^{V_{[D]},2}, \quad S^{V_{[D]},3}, \quad S^{V_{[D]},4}, \quad S^{V_{[D]},5}$$

form a basis for the 5-dimensional subspace of invariant vectors in $V_D \otimes V_D \otimes V_D$,

$$\rho(\sigma) S^{V_{[D]},\alpha} = S^{V_{[D]},\alpha}. \tag{2.9}$$

The action decomposes into a linear and quadratic part

$$\mathcal{S}(\Phi) = \mathcal{S}_{\text{linear}}(\Phi) + \mathcal{S}_{\text{quadratic}}(\Phi),$$

and from (2.9),

$$\mathcal{S}_{\text{linear}}(\Phi) = \sum_{\alpha=1}^{5} \mu_{[D],\alpha} S^{[D],\alpha} = \sum_{\alpha} \mu_{[D],\alpha} C_{ijk}^{[D],\alpha} \Phi^{ijk},$$

where $\mu_{[D],\alpha}$ are five independent parameters.

2.3. Representation basis for quadratic part of action

The quadratic invariants are in one-to-one correspondence with the trivial representations in the irreducible decomposition of

$$\mathrm{Sym}^2(V_D^{\otimes 3}) \tag{2.10}$$

which is the symmetric part of $V_D^{\otimes 3} \otimes V_D^{\otimes 3}$. Only the symmetric part is relevant due to the commuting nature of Φ – they satisfy

$$\Phi_{ijk}\Phi_{pqr} = \Phi_{pqr}\Phi_{ijk}.$$

To find the multiplicity of the trivial representation V_0 in the decomposition of (2.10), we first note that all irreducible representations of S_D can be chosen to be real and unitary, or equivalently orthogonal. Therefore, Schur's lemma implies that the trivial representation V_0 appears in the tensor product $V_R \otimes V_S$ of two irreducible representations V_R, V_S, if and only if the irreducible representations are isomorphic. If they are isomorphic, the multiplicity of the trivial representation in the tensor product decomposition is one. That is, we have

$$\dim \mathrm{Hom}_{S_D}(V_R \otimes V_S, V_0) = \delta_{RS},$$

where δ_{RS} is one if $V_R \cong V_S$ and zero otherwise.

From this observation, the construction of quadratic invariants is simple in the representation basis. In general, they take the form

$$\sum_a S_a^{\Lambda,\alpha} S_a^{\Lambda,\beta} = \sum_{\substack{i,j,k,\\p,q,r}} \sum_a C_{a,ijk}^{\Lambda,\alpha} \Phi^{ijk} C_{a,pqr}^{\Lambda,\beta} \Phi^{pqr}.$$

This follows from the fact that the representation of S_D acts with an orthogonal matrix in this basis. Therefore, summing over the index a gives the unique invariant vector in $V_\Lambda \otimes V_\Lambda$, up to normalisation.

Defining the invariant endomorphism tensors Q as

$$Q_{ijk;pqr}^{\Lambda,\alpha\beta} \equiv \sum_a C_{a,ijk}^{\Lambda,\alpha} C_{a,pqr}^{\Lambda,\beta}, \tag{2.11}$$

we have

$$\sum_a S_a^{\Lambda,\alpha} S_a^{\Lambda,\beta} = \sum_{\substack{i,j,k,\\p,q,r}} \Phi^{ijk} Q_{ijk;pqr}^{\Lambda,\alpha\beta} \Phi^{pqr}.$$

It follows from the definition (2.11) that they satisfy

$$Q_{\sigma(i)\sigma(j)\sigma(k);\sigma(p)\sigma(q)\sigma(r)}^{\Lambda,\alpha\beta} = Q_{ijk;pqr}^{\Lambda,\alpha\beta}, \tag{2.12}$$

and the transposition condition

$$Q_{ijk;pqr}^{\Lambda,\alpha\beta} = Q_{pqr;ijk}^{\Lambda,\beta\alpha},$$

and therefore span the S_D invariants of (2.10).

Counting the number of such combinations, we find 117 quadratic terms

$$\frac{5\cdot 6}{2} + \frac{10\cdot 11}{2} + \frac{6\cdot 7}{2} + \frac{6\cdot 7}{2} + \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{1\cdot 2}{2} = 117,$$

where the numbers 5, 10, 6, 6, 1, 2, 1 correspond to the multiplicities in (2.7).

To summarise, we can write the quadratic action, in the representation basis, as

$$\mathcal{S}_{\text{quadratic}}(\Phi) = \sum_{\Lambda,\alpha,\beta} g_{\alpha\beta}^{\Lambda} Q_{ijk;pqr}^{\Lambda,\alpha\beta} \Phi^{ijk} \Phi^{pqr},$$

where g^Λ are symmetric matrices of parameters labelled by irreps Λ and indexed by the multiplicity indices α, β. The matrices $g_{\alpha\beta}^{\Lambda}$ must have non-negative eigenvalues to define a convergent integral. Including the linear terms, we can write the full partition function

$$\begin{aligned} Z &= \int \mathrm{d}\Phi \exp(-\mathcal{S}(\Phi)) \\ &= \int \mathrm{d}\Phi \exp\left(\sum_{\alpha=1}^{5} \sum_{i,j,k} \mu_{[D],\alpha} C_{ijk}^{[D],\alpha} \Phi^{ijk} - \sum_{\Lambda,\alpha,\beta} \sum_{\substack{i,j,k\\p,q,r}} g_{\alpha\beta}^{\Lambda} Q_{ijk;pqr}^{\Lambda,\alpha\beta} \Phi^{ijk} \Phi^{pqr} \right). \end{aligned} \tag{2.13}$$

2.4. One-point and two-point functions in representation basis

Given the above expression for the partition function, we can now derive the one-point and two-point functions using standard techniques from quantum field theory. To derive the one- and two-point functions of Φ, we first compute

$$\langle S_a^{\Lambda,\alpha}\rangle, \quad \langle S_a^{\Lambda,\alpha} S_b^{\Lambda',\beta}\rangle,$$

defined in (2.3). To this end, it is useful to write the partition function entirely in terms of the representation basis elements. Further, we introduce auxiliary terms in the linear part of the action, multiplying non-invariant representation variables, with the understanding that the coupling of these auxiliary terms should be set to zero in order to recover the permutation invariant model. This object,

$$Z[\mu] = \int \mathrm{d}S \exp\Bigg(\sum_{\Lambda,\alpha,a} \mu_{\Lambda,\alpha}^a S_a^{\Lambda,\alpha} - \frac{1}{2} \sum_{\Lambda,\alpha,\beta,a} S_a^{\Lambda,\alpha} g_{\alpha\beta}^{\Lambda} S_a^{\Lambda,\beta} \Bigg), \tag{2.14}$$

with

$$\mathrm{d}S = \prod_{\Lambda,\alpha,a} \mathrm{d}S_a^{\Lambda,\alpha},$$

defines a generating function of expectation values in the representation basis. In this form, the integral can be computed using standard Gaussian integration techniques. For notational convenience, we define

$$\tilde{g}_\Lambda = g_\Lambda^{-1}.$$

Performing the integral on the right-hand side of (2.14) gives

$$Z[\mu] = \frac{(2\pi)^{\frac{D^3}{2}}}{(\det g)^{\frac{1}{2}}} \exp\Bigg(\frac{1}{2} \sum_{\Lambda,\alpha,\beta,a} \mu_{\Lambda,\alpha}^a \tilde{g}_\Lambda^{\alpha\beta} \mu_{\Lambda,\beta}^a \Bigg), \tag{2.15}$$

In order to calculate expectation values of the S variables, we differentiate $Z[\mu]$ with respect to μ. For example,

$$\begin{aligned}
\langle S_a^{\Lambda,\alpha}\rangle &= \frac{1}{Z} \int \mathrm{d}S\, S_a^{\Lambda,\alpha} e^{-\mathcal{S}} = \frac{1}{Z} \frac{\partial Z}{\partial \mu_{\Lambda,\alpha}^a}\bigg|_{\mu_\Lambda \neq 0 \text{ iff } \Lambda=[D]} \\
&= \frac{1}{Z} \tilde{g}_\Lambda^{\alpha\beta} \mu_{\Lambda,\beta} \delta(\Lambda,[D]) Z \\
&= \tilde{g}_\Lambda^{\alpha\beta} \mu_{\Lambda,\beta} \delta(\Lambda,[D]).
\end{aligned}$$

In going to the second line, we have differentiated the expression for $Z[\mu]$ given on the right-hand side of (2.15). The δ-function in the last line follows due to the fact that $\mu_{\Lambda,\beta} = 0$ unless $\Lambda = [D]$. We have also dropped the state index on the linear couplings from the second line as they do not depend on the state index – the only non-zero linear couplings are those for the trivial irrep which has dimension one. Writing this out explicitly, the non-zero linear expectation values are given by

$$\left\langle S^{[D],\alpha}\right\rangle = \sum_{\beta=1}^{5} \tilde{g}^{\alpha\beta}_{[D]}\mu_{[D],\beta},$$

with all other linear expectation values equal to zero.

In order to calculate quadratic expectation values of the S variables, we differentiate the partition function twice, giving

$$\begin{aligned}\left\langle S_a^{\Lambda_1,\alpha} S_b^{\Lambda_2,\beta}\right\rangle &= \frac{1}{Z}\int \mathrm{d}S\, S_a^{\Lambda_1,\alpha} S_b^{\Lambda_2,\beta} e^{-\mathcal{S}} = \frac{1}{Z}\frac{\partial}{\partial\mu^b_{\Lambda_2,\beta}}\frac{\partial Z}{\partial\mu^a_{\Lambda_1,\alpha}}\bigg|_{\mu_\Lambda\neq 0 \text{ iff } \Lambda=[D]}\\ &= \tilde{g}^{\alpha\beta}_{\Lambda_1}\delta(\Lambda_1,\Lambda_2)\delta_{ab} + \langle S_a^{\Lambda_1,\alpha}\rangle\left\langle S_b^{\Lambda_2,\beta}\right\rangle\\ &= \tilde{g}^{\alpha\beta}_{\Lambda_1}\delta(\Lambda_1,\Lambda_2)\delta_{ab}\\ &\quad+\sum_{\gamma,\rho}\tilde{g}^{\alpha\gamma}_{\Lambda_1}\mu_{\Lambda_1,\gamma}\tilde{g}^{\alpha\rho}_{\Lambda_2}\mu_{\Lambda_2,\rho}\delta(\Lambda_1,[D])\delta(\Lambda_2,[D]).\end{aligned} \tag{2.16}$$

For simplicity, we define the connected piece of the two-point function as

$$\left\langle S_a^{\Lambda_1,\alpha} S_b^{\Lambda_2,\beta}\right\rangle_{\text{conn}} \equiv \left\langle S_a^{\Lambda_1,\alpha} S_b^{\Lambda_2,\beta}\right\rangle - \langle S_a^{\Lambda_1,\alpha}\rangle\left\langle S_b^{\Lambda_2,\beta}\right\rangle = \tilde{g}^{\alpha\beta}_{\Lambda_1}\delta(\Lambda_1,\Lambda_2)\delta_{ab}. \tag{2.17}$$

We will now use these results to compute expressions for the one- and two-point functions in the tensor basis.

2.5. One-point and two-point functions in tensor basis

The one-point function in the tensor basis is given by

$$\begin{aligned}\left\langle\Phi_{ijk}\right\rangle &= \sum_{\Lambda}\sum_{\alpha}\sum_{a=1}^{\dim\Lambda} C^{\Lambda,\alpha}_{a,ijk}\langle S_a^{\Lambda,\alpha}\rangle\\ &= \sum_{\Lambda}\sum_{\alpha}\sum_{a=1}^{\dim\Lambda} C^{\Lambda,\alpha}_{a,ijk}\tilde{g}^{\alpha\beta}_{\Lambda}\mu_{\Lambda,\beta}\delta(\Lambda,[D])\\ &= \sum_{\alpha} C^{V_{[D]},\alpha}_{ijk}\tilde{g}^{\alpha\beta}_{[D]}\mu_{[D],\beta}.\end{aligned} \tag{2.18}$$

Expressions for the above Clebsch–Gordan coefficients are found in Appendix A. We can write the S_D invariant two-point function of two tensors as a sum over the S-variable two-point functions. In turn, this can be written as a sum of invariant endomorphism tensors Q:

$$
\begin{aligned}
&\langle \Phi_{ijk} \Phi_{pqr} \rangle \\
&= \sum_{\Lambda_1,\Lambda_2} \sum_{\alpha,\beta} \sum_{a=1}^{\dim \Lambda_1} \sum_{b=1}^{\dim \Lambda_2} C^{\Lambda_1,\alpha}_{a,ijk} C^{\Lambda_2,\beta}_{b,pqr} \langle S^{\Lambda_1,\alpha}_a S^{\Lambda_2,\beta}_b \rangle \\
&= \sum_{\Lambda_1,\Lambda_2} \sum_{\alpha,\beta} \sum_{a=1}^{\dim \Lambda_1} \sum_{b=1}^{\dim \Lambda_2} C^{\Lambda_1,\alpha}_{a,ijk} C^{\Lambda_2,\beta}_{b,pqr} \left(\tilde{g}^{\alpha\beta}_{\Lambda_1} \delta(\Lambda_1, \Lambda_2) \delta_{ab} + \langle S^{\Lambda_1 \alpha}_a \rangle \langle S^{\Lambda_2,\beta}_b \rangle \right) \\
&= \sum_{\Lambda_1} \sum_{\alpha,\beta} \sum_{a=1}^{\dim \Lambda_1} C^{\Lambda_1,\alpha}_{a,ijk} C^{\Lambda_1,\beta}_{a,pqr} \tilde{g}^{\alpha\beta}_{\Lambda_1} \\
&\quad + \sum_{\Lambda_1,\Lambda_2} \sum_{\alpha,\beta,\gamma,\rho} \sum_{a,b=1}^{\dim \Lambda_1} C^{\Lambda_1,\alpha}_{a,ijk} C^{\Lambda_2,\beta}_{b,pqr} \tilde{g}^{\alpha\gamma}_{\Lambda_1} \mu_{\Lambda_1,\gamma} \delta(\Lambda_1, [D]) \tilde{g}^{\beta\rho}_{\Lambda_2} \mu_{\Lambda_2,\rho} \delta(\Lambda_2, [D]) \\
&= \sum_{\Lambda_1} \sum_{\alpha,\beta} Q^{\Lambda_1,\alpha\beta}_{ijk;pqr} \tilde{g}^{\alpha\beta}_{\Lambda_1} + \sum_{\alpha,\beta,\gamma,\rho} Q^{[D],\alpha\beta}_{ijk;pqr} \tilde{g}^{\alpha\gamma}_{[D]} \mu_{[D],\gamma} \tilde{g}^{\beta\rho}_{[D]} \mu_{[D],\rho}. \qquad (2.19)
\end{aligned}
$$

In the second line, we have used the result (2.16), and in the final line, we have used the definition (2.11). Akin to (2.17), for simplicity, we again write the connected piece of the two-point function in the tensor basis

$$
\langle \Phi_{ijk} \Phi_{pqr} \rangle_{\text{conn}} \equiv \langle \Phi_{ijk} \Phi_{pqr} \rangle - \langle \Phi_{ijk} \rangle \langle \Phi_{pqr} \rangle = \sum_{\Lambda_1} \sum_{\alpha,\beta} Q^{\Lambda_1,\alpha\beta}_{ijk;pqr} \tilde{g}^{\alpha\beta}_{\Lambda_1}. \qquad (2.20)
$$

Given (2.20), our task in the next section is to find explicit expressions for the invariant endomorphism tensors $Q^{\Lambda,\alpha\beta}_{ijk;pqr}$ that determine the tensor two-point function.

3. Explicit computation of the two-point functions in the tensor basis

The aim of this section is to develop an algorithm for constructing the invariant endomorphism tensors (2.11), appearing in the two-point function in the tensor basis (2.19), for all $D \geq 6$. The algorithm involves resolving the labels Λ, α, β on the tensors $Q^{\Lambda,\alpha\beta}_{ijk;pqr}$. We show that the labels are described by a pair of graphs labelled by irreducible representations of S_D. Each pair of graphs is uniquely determined by a set of eigenvalue equations related to central elements in the $\mathbb{C}(S_D)$ group algebras. We solve the eigenvalue equations using a correspondence between central elements in $\mathbb{C}(S_D)$ and elements of partition algebras. This approach bypasses the explicit construction of the Clebsch–Gordan coefficients appearing in the definition of $Q^{\Lambda,\alpha\beta}_{ijk;pqr}$ (2.11). Specifying bases in irreps of S_N at general N, working out the corresponding Clebsch's, and summing them are the highly non-trivial steps which are bypassed. We find that the $Q^{\Lambda,\alpha\beta}_{ijk;pqr}$ themselves are elements of the partition algebra and the eigenvalue equations involve matrices whose elements are structure constants of the partition algebra. Combinatorially constructed eigenvalue systems for central elements in symmetric group algebras have also been used in the identification of representation-theoretic labels with motivations coming from holography, quantum information, and combinatorial representation theory [10, 19, 31].

In the construction, it is useful to view the invariant tensors $Q^{\Lambda,\alpha\beta}_{ijk;pqr}$ as equivariant maps

$$Q^{\Lambda,\alpha\beta} : V_D^{\otimes 3} \to V_D^{\otimes 3}, \tag{3.1}$$

where

$$Q^{\Lambda,\alpha\beta}(\Phi_{ijk}) = \sum_{p,q,r=1}^{D} Q^{\Lambda,\alpha\beta}_{pqr;ijk}\Phi_{pqr},$$

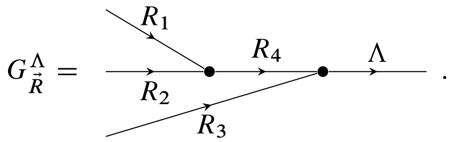
and from equation (2.12),

$$Q^{\Lambda,\alpha\beta}\rho(\sigma) = \rho(\sigma)Q^{\Lambda,\alpha\beta}.$$

They form a basis for the vector space of equivariant maps of the kind in (3.1), commonly denoted by

$$\mathrm{End}_{S_D}(V_D^{\otimes 3}) = \mathrm{Span}(Q^{\Lambda,\alpha\beta}).$$

This vector space is an algebra, with multiplication given by composition of maps. In Section 3.1, we will describe how the multiplicity labels α, β can be understood as graphs $G^{\Lambda}_{\vec{R}}$ decorated with irreducible representations $\vec{R} = (R_1, R_2, R_3, R_4)$ and Λ



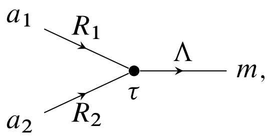
Sections 3.2–3.4 build towards the construction of a set of commuting operators whose eigenvalues distinguish the pairs of graphs, and consequently, have simultaneous eigenvectors $Q^{\Lambda,\alpha\beta}$. The last subsection gives an algorithm for solving these eigenvector equations as analytic functions of D.

3.1. Resolving multiplicity labels using graphs

As we now explain, the multiplicity indices α, β are in correspondence with decorated graphs. It will be useful to introduce some diagrammatic notation for Clebsch–Gordan coefficients used in [39]. Clebsch–Gordan coefficients for the decomposition

$V_{R_1} \otimes V_{R_2}$ can be represented by a graph

$$C^{R_1 R_2, \Lambda\, \tau}_{a_1\, a_2, m} =$$

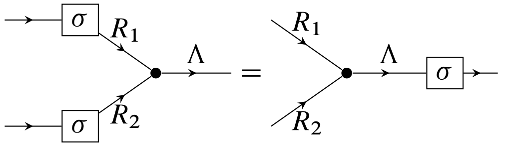


(3.2)

where a_1, a_2, m label orthonormal bases for V_{R_1}, V_{R_2}, V_Λ, respectively, and τ is a further index ranging over the multiplicity of Λ in the decomposition of $V_{R_1} \otimes V_{R_2}$. The equivariance of Clebsch–Gordan coefficients

$$\sum_{b_1, b_2} D^{R_1}_{a_1 b_1}(\sigma) D^{R_2}_{a_2 b_2}(\sigma) C^{R_1 R_2, \Lambda\, \tau}_{b_1\, b_2, m} = \sum_l C^{R_1 R_2, \Lambda\, \tau}_{a_1\, a_2, l} D^{\Lambda}_{lm}(\sigma) \quad \forall \sigma \in S_D$$

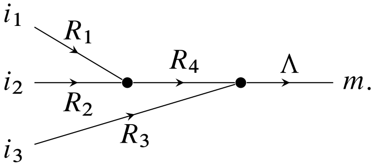
can be written diagrammatically as



$\forall \sigma \in S_D.$ (3.3)

The coefficients relevant to the decomposition of $V_D \otimes V_D \otimes V_D$ into irreducible representations are composed of two Clebsch–Gordan coefficients

$$C^{G^{\Lambda}_{\vec{R}}}_{i_1 i_2 i_3 \to m} = \sum_{m'} C^{R_1 R_2, R_4}_{i_1\, i_2, m'} C^{R_4 R_3, \Lambda}_{m'\, i_3, m} =$$

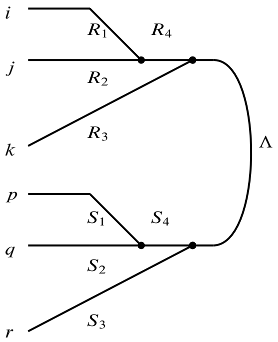


(3.4)

An important feature of the relevant Clebsch–Gordan coefficients appearing above is that they are all multiplicity free; that is, the τ appearing in (3.2) is 1 for any given $\vec{R}$ and Λ. We can therefore drop the multiplicity labels. We refer to the content of this graph as $G^{\Lambda}_{\vec{R}}$, Λ being the final irreducible representation and $\vec{R} = (R_1, R_2, R_3, R_4)$ specifying the intermediate irreducible representations. It follows that the multiplicities in (2.7) are uniquely specified by the content of such graphs.

Given the form of $Q^{\Lambda, \alpha\beta}_{ijk,pqr}$ in (2.11), we have the following diagrammatic expression:

$$Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr} =$$



where we have used a pair of graphs to label the equivariant map. We will now see how central elements are used to determine the graphs $G^{\Lambda}_{\vec{R}}$, $G^{\Lambda}_{\vec{R}}$, and subsequently the entire invariant endomorphism tensor $Q^{\Lambda,\alpha\beta}_{ijk,pqr}$.

3.2. Using central elements in $\mathbb{C}(S_D)$ to detect graphs

The centre $\mathcal{Z}(\mathbb{C}(S_D))$ of $\mathbb{C}(S_D)$, the symmetric group algebra, consists of elements

$$\mathcal{Z}(\mathbb{C}(S_D)) = \{z \in \mathbb{C}(S_D) : z\sigma = \sigma z \ \forall \sigma \in \mathbb{C}(S_D)\}.$$

Elements in the centre are called central elements. Central elements play a special role in representation theory because Schur's lemma implies that an irreducible matrix representation of a central element is proportional to the identity matrix. The proportionality constant is a normalised character. In particular, we have

$$D^{\Lambda_1}_{ab}(z) = \frac{\chi^{\Lambda_1}(z)}{\operatorname{Dim} V^{S_D}_{\Lambda_1}} \equiv \widehat{\chi}^{\Lambda_1}(z)\delta_{ab},$$

where $\chi^{\Lambda_1}(z)$ is the character of z in the irreducible representation Λ_1, and we have defined the hatted shorthand for normalised characters. Central elements act by constants on irreducible subspaces, and the constants can be used to determine the particular representation.

The element of $\mathbb{C}(S_D)$ formed by summing over all elements in a distinct conjugacy class of S_D is central. For example, we define the element $T_2 \in \mathcal{Z}(\mathbb{C}(S_D))$ as follows:

$$T_2 = \sum_{1 \leq i < j \leq D} (ij).$$

T_2 is the sum over all transpositions. Normalised characters of T_2 can be expressed in terms of combinatorial quantities (known as the contents) of boxes of Young diagrams (see [34, Example 7 in Section I.7]). Let Y_{Λ_1} be the Young diagram corresponding to the integer partition $\Lambda_1 \in \mathcal{Y}_S(k)$, the set of valid Young diagrams with k boxes. Then,

$$\widehat{\chi}^{\Lambda_1}(T_2) = \sum_{(i,j)\in Y_{\Lambda_1}} (j - i),$$

where (i, j) corresponds to the cell in the ith row and jth column of the Young diagram. (The top left box has coordinate $(1, 1)$.)

All of the irreducible representations we need to identify appear on the right-hand side of (2.7). The normalised characters of T_2 distinguish these representations; they are

$$\widehat{\chi}^{V_0}(T_2) = \frac{D(D-1)}{2},$$

$$\hat{\chi}^{V_H}(T_2) = \frac{D(D-3)}{2},$$
$$\hat{\chi}^{V_2}(T_2) = \frac{(D-1)(D-4)}{2},$$
$$\hat{\chi}^{V_3}(T_2) = \frac{D(D-5)}{2},$$
$$\hat{\chi}^{V_4}(T_2) = \frac{(D-4)(D-3)}{2},$$
$$\hat{\chi}^{V_5}(T_2) = \frac{(D-6)(D-1)}{2},$$
$$\hat{\chi}^{V_6}(T_2) = \frac{D(D-7)}{2}.$$

We define the following set of elements in $\mathbb{C}(S_D)^{\otimes 3}$:

$$T_2^{(111)} = \sum_{i<j} (ij) \otimes (ij) \otimes (ij), \tag{3.5}$$

$$T_2^{(110)} = \sum_{i<j} (ij) \otimes (ij) \otimes \mathbb{1}, \tag{3.6}$$

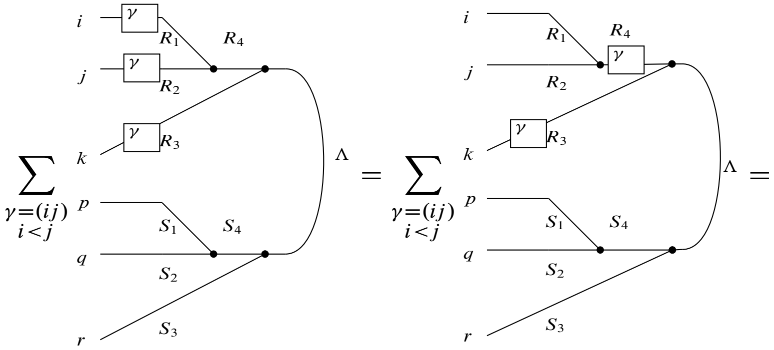
$$T_2^{(100)} = \sum_{i<j} (ij) \otimes \mathbb{1} \otimes \mathbb{1}, \tag{3.7}$$

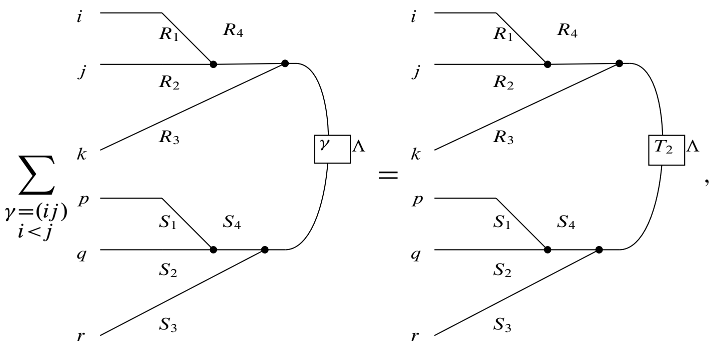
$$T_2^{(010)} = \sum_{i<j} \mathbb{1} \otimes (ij) \otimes \mathbb{1}, \tag{3.8}$$

$$T_2^{(001)} = \sum_{i<j} \mathbb{1} \otimes \mathbb{1} \otimes (ij) \tag{3.9}$$

and refer to them collectively as $T_2^{(b)}$, where b is one of the above binary strings. Notably, as operators on $V_D^{\otimes 3}$, they commute with the action of S_D; that is, they are elements of $\mathrm{End}_{S_D}(V_D^{\otimes 3})$. Furthermore, they commute among themselves. As we now show, the invariant endomorphism tensors $Q_{ijk,pqr}^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$ are simultaneous eigenvectors of the operators (3.5)–(3.9).

Consider the composition $T_2^{(111)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$ acting on $V_D^{\otimes 3}$. As a diagram equation, we have





where we have made repeated use of the equivariance property (3.3). Because the last equation involves an irreducible representation of a central element, it evaluates to multiplication of a normalised character and we get

$$\left(T_2^{(111)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}\right)_{ijk,pqr} = \widehat{\chi}^{\Lambda}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}. \tag{3.10}$$

Similar diagrammatic manipulations show that

$$\left(T_2^{(110)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}\right)_{ijk,pqr} = \widehat{\chi}^{R_4}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.11}$$

$$\left(T_2^{(100)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}\right)_{ijk,pqr} = \widehat{\chi}^{R_1}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.12}$$

$$\left(T_2^{(010)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}\right)_{ijk,pqr} = \widehat{\chi}^{R_2}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.13}$$

$$\left(T_2^{(001)} Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}\right)_{ijk,pqr} = \widehat{\chi}^{R_3}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}. \tag{3.14}$$

The five eigenvalue equations (3.10)–(3.14) are sufficient to determine the right eigenspaces in $\mathrm{End}(V_D^{\otimes 3})$ labelled by the graph $G^{\Lambda}_{\vec{R}}$. The left eigenspace labelled by graph $G^{\Lambda}_{\vec{S}}$ is determined through right action,

$$\left(Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}} T_2^{(110)}\right)_{ijk,pqr} = \widehat{\chi}^{S_4}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.15}$$

$$\left(Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}} T_2^{(100)}\right)_{ijk,pqr} = \widehat{\chi}^{S_1}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.16}$$

$$\left(Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}} T_2^{(010)}\right)_{ijk,pqr} = \widehat{\chi}^{S_2}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}, \tag{3.17}$$

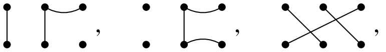
$$\left(Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}} T_2^{(001)}\right)_{ijk,pqr} = \widehat{\chi}^{S_3}(T_2) Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}_{ijk,pqr}. \tag{3.18}$$

These are eigenvector equations in a D^6-dimensional vector space $\mathrm{End}(V_D^{\otimes 3})$, and solving them directly for large D would be hopeless. Luckily, we know that $Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$, as well as the $T_2^{(b)}$ operators (equations (3.5)–(3.9)), lie in a much smaller subspace $\mathrm{End}_{S_D}(V_D^{\otimes 3})$. Further, its dimension is independent of D for $D \geq 6$. $\mathrm{End}_{S_D}(V_D^{\otimes 3})$

is a subalgebra. The composition of operators is closed in $\mathrm{End}_{S_D}(V_D^{\otimes 3})$. This greatly reduces the dimensionality of the problem, what remains is to give an explicit basis for $\mathrm{End}_{S_D}(V_D^{\otimes 3})$ in which the above eigenvector problem can be solved. This basis is given by the partition algebra, which we will now briefly introduce. For a more detailed exposition, the reader should consult [25].

3.3. Review: Partition algebra

The partition algebras $P_m(D)$ are a family of finite-dimensional diagram algebras, meaning they have a basis labelled by diagrams, where multiplication is defined combinatorially in terms of diagram concatenation. In particular, Dim $P_m(D) = B(2m)$ are the Bell numbers, which count the number of set partitions of the set $\{1, \ldots, 2m\}$. For example,

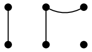


form a subset of the $B(6) = 203$ basis elements in $P_3(D)$.

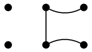
It is known [29, 36] that

$$\mathrm{End}_{S_D}(V_D^{\otimes 3}) \cong P_3(D),$$

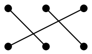
for $D \geq 2k$, such that every diagram corresponds to an element in $\mathrm{End}_{S_D}(V_D^{\otimes 3})$. The correspondence is simple; an edge connecting a vertex i to a vertex j is mapped to δ_{ij}. For example,



$$(\Phi_{ijk}) = \sum_{p,q,r} \delta_{ip}\delta_{jq}\delta_{qr}\Phi_{pqr} = \Phi_{ijj},$$



$$(\Phi_{ijk}) = \sum_{p,q,r} \delta_{jq}\delta_{qr}\delta_{jk}\Phi_{pqr} = D\delta_{jk}\Phi_{ijj},$$



$$(\Phi_{ijk}) = \sum_{p,q,r} \delta_{ir}\delta_{jp}\delta_{kq}\Phi_{pqr} = \Phi_{jki}.$$

This isomorphism implies that the maps $Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$ can be thought of as linear combinations of diagrams. In fact, the set of all Q's form a basis of $P_3(D)$ as

$$\mathrm{Span}(Q^{\Lambda,\alpha\beta}) = \mathrm{End}_{S_D}(V_D^{\otimes 3}) \cong P_3(D).$$

Indeed, given the Schur–Weyl result [25, 35, 36]

$$V_D^{\otimes 3} \cong \bigoplus_{\Lambda} V^{\Lambda}_{S_D} \otimes V^{\Lambda}_{P_3(D)}$$

and the S_D irreducible decomposition of $V_D^{\otimes 3}$ in (2.7), there are

$$5^2 + 10^2 + 6^2 + 6^2 + 1^2 + 2^2 + 1^2 = 203 = B(6) = \dim(P_3(D))$$

Q's, i.e., 203 choices of the set of labels Λ, α, β.

As we will now see, the operators $T_2^{(b)}$ also correspond to elements in $P_3(D)$. Therefore, the eigenvector equations (3.10)–(3.18) can be understood completely in this greatly reduced space $P_3(D)$.

3.4. $\mathbb{C}(S_D)$ central elements as partition algebra elements

Since the operators $T_2^{(b)}$ commute with the action of S_D on $V_D^{\otimes 3}$, there exist elements $\bar{T}_2^{(b)} \in P_3(D)$ such that

$$\bar{T}_2^{(b)}(\Phi_{ijk}) = T_2^{(b)}(\Phi_{ijk}).$$

The expansion of $\bar{T}_2^{(b)}$ in terms of diagrams can be found using Jucys–Murphy elements [16, 25] as we now explain. For every $P_m(D)$, there exists a family of Jucys–Murphy elements

$$L_i \in P_m(D), \quad i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots,$$

which is recursively defined in [16, Section 3]. Theorem 3.10 defines the following central element in $P_m(D)$:

$$z_m = L_{\frac{1}{2}} + L_1 + L_{\frac{3}{2}} + \cdots + L_m.$$

According to Proposition 5.4,

$$z_m = \sum_{i<j} (ij)^{\otimes m} - \left(\binom{D}{2} - mD\right)\mathbb{1} \tag{3.19}$$

as operators on $V_D^{\otimes m}$. In particular, rearranging (3.19) and taking $m = 1$, we have

$$\sum_{i<j}(ij) = L_{\frac{1}{2}} + L_1 + \binom{D}{2}\mathbb{1} - D\mathbb{1} \equiv \bar{T}_2^{(1)} \in P_1(D)$$

as operators on V_D,

$$\sum_{i<j}(ij)^{\otimes 2} = L_{\frac{1}{2}} + L_1 + L_{\frac{3}{2}} + L_2 + \binom{D}{2}\mathbb{1} - 2D\mathbb{1} \equiv \bar{T}_2^{(11)} \in P_2(D)$$

as operators on $V_D^{\otimes 2}$, and

$$\sum_{i<j}(ij)^{\otimes 3} = L_{\frac{1}{2}} + L_1 + L_{\frac{3}{2}} + L_2 + L_{\frac{5}{2}} + L_3 + \binom{D}{2}\mathbb{1} - 3D\mathbb{1} \equiv \bar{T}_2^{(111)} \in P_3(D)$$

as operators on $V_D^{\otimes 3}$. The partition algebras $P_1(D)$, $P_2(D)$ are embeddable into $P_3(D)$ by adding identity strands; i.e., they act trivially on all but one and two factors of $V_D^{\otimes 3}$, respectively. With the definitions

$$\begin{aligned}
\overline{T}_2^{(100)} &= \overline{T}_2^{(1)} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\overline{T}_2^{(010)} &= \mathbb{1} \otimes \overline{T}_2^{(1)} \otimes \mathbb{1}, \\
\overline{T}_2^{(001)} &= \mathbb{1} \otimes \mathbb{1} \otimes \overline{T}_2^{(1)}, \\
\overline{T}_2^{(110)} &= \overline{T}_2^{(11)} \otimes \mathbb{1},
\end{aligned}$$

we find that

$$\begin{aligned}
\overline{T}_2^{(100)} &= T_2^{(100)}, \\
\overline{T}_2^{(010)} &= T_2^{(010)}, \\
\overline{T}_2^{(001)} &= T_2^{(001)}, \\
\overline{T}_2^{(110)} &= T_2^{(110)}, \\
\overline{T}_2^{(111)} &= T_2^{(111)}
\end{aligned}$$

as operators on $V_D^{\otimes 3}$. For fixed D, it is straightforward to solve the eigenvector equations in (3.10)–(3.17) using partition algebras. But we are interested in all D constructions and this will be the subject of the next section.

3.5. All *D* constructions of invariant endomorphism tensors

In this section, we describe an all $D \geq 6$ construction of the $B(6) = 203$ Q's. In other words, we give functions

$$Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}(D) \in P_3(D)$$

which correspond to the invariant endomorphism tensors appearing in the two-point function (2.19) for all D.

It will be useful to define the following set of idempotents in $P_3(D)$:

$$\begin{aligned}
P_{R_1} &= \prod_{R \neq R_1} \frac{(\overline{T}_2^{(100)} - \widehat{\chi}^R(T_2^{(100)}))}{(\widehat{\chi}^{R_1}(T_2^{(100)}) - \widehat{\chi}^R(T_2^{(100)}))}, \\
P_{R_2} &= \prod_{R \neq R_2} \frac{(\overline{T}_2^{(010)} - \widehat{\chi}^R(T_2^{(010)}))}{(\widehat{\chi}^{R_2}(T_2^{(010)}) - \widehat{\chi}^R(T_2^{(010)}))}, \\
P_{R_3} &= \prod_{R \neq R_3} \frac{(\overline{T}_2^{(001)} - \widehat{\chi}^R(T_2^{(001)}))}{(\widehat{\chi}^{R_3}(T_2^{(001)}) - \widehat{\chi}^R(T_2^{(001)}))}, \qquad (3.20) \\
P_{R_4} &= \prod_{R \neq R_4} \frac{(\overline{T}_2^{(110)} - \widehat{\chi}^R(T_2^{(110)}))}{(\widehat{\chi}^{R_4}(T_2^{(110)}) - \widehat{\chi}^R(T_2^{(110)}))}, \\
P_{\Lambda} &= \prod_{R \neq \Lambda} \frac{(\overline{T}_2^{(111)} - \widehat{\chi}^R(T_2^{(111)}))}{(\widehat{\chi}^{\Lambda}(T_2^{(111)}) - \widehat{\chi}^R(T_2^{(111)}))},
\end{aligned}$$

where

$$
\begin{aligned}
&R_1, R_2, R_3 \in \{[D], [D-1,1]\},\\
&R_4 \in \{[D], [D-1,1], [D-2,2], [D-2,1,1]\},\\
&\Lambda \in \{[D], [D-1,1], [D-2,2], [D-2,1,1],\\
&\quad [D-3,3], [D-3,2,1], [D-3,1,1,1]\}.
\end{aligned}
$$

Given a graph $G^{\Lambda}_{\vec{R}}$, we construct the element

$$P_{G^{\Lambda}_{\vec{R}}} = P_{\Lambda} P_{R_4} P_{R_3} P_{R_2} P_{R_1} \in P_3(D), \tag{3.21}$$

and for a pair of graphs $(G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}})$ define the following projector on $P_3(D)$:

$$P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}(d) = P_{G^{\Lambda}_{\vec{S}}} d P_{G^{\Lambda}_{\vec{S}}} \quad \forall d \in P_3(D).$$

It projects onto the simultaneous left and right eigenspaces of $\overline{T}_2^{(b)}$ in $P_3(D)$, determined by the pair of graphs. Since the eigenspace is one-dimensional, we have

$$\operatorname{Im} P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}} = \operatorname{Span}\big(Q^{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}\big).$$

In general, a basis for the image of a matrix can be determined from its row echelon form [47, **2O** in Section 2.4]. This is reviewed in Appendix C for the convenience of the reader. We will use this to find $Q^{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$. Let $\mathcal{B}$ be a basis for $P_3(D)$ (e.g., the diagram basis). We define the matrix $(P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{d'd}$ in this basis by

$$P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}(d) = \sum_{d' \in \mathcal{B}} (P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{d'd}\, d'.$$

Since $\operatorname{Im} P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$ is one-dimensional and a basis for the image is determined by the pivot column of the row echelon form, we just have to find the first non-zero column in the first row of the echelon form. That is, suppose that

$$P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}} = E \widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}},$$

where $\widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$ is in row echelon form and E is an invertible matrix encoding the Gauss elimination steps. Let $(\widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{1d}$ be the pivot element; then,

$$\sum_{d' \in \mathcal{B}} (P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{d'd}\, d' \propto Q^{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}.$$

Because the matrix $P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$ has rational functions as entries, it is non-trivial to find the row echelon form $\widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$ and subsequent pivot column. Instead, to find a basis for general D, we will use the following method. Fix D to be any integer $n \geq 6$ and find the pivot column of

$$P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}\Big|_{D=n}.$$

Let d be the pivot column of the echelon form for $D = n$; then,

$$\sum_{d' \in \mathcal{B}} (P_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{d'd}\, d' = Q^{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}.$$

The relevant Sage code for implementing this algorithm can be found together with the arXiv version of this paper.

In practice, we worked with

$$D = 7$$

and verified that the resulting $Q^{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$ satisfies the correct eigenvalue equations (e.g., (3.5)–(3.9)) for all D. Theoretically, its validity can be argued for as follows. Suppose that we were able to find the row echelon form

$$\widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}}$$

for general D. This matrix will have a single non-zero row, and we call the pivot column d. The pivot entry is a rational function

$$f(D) = (\widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}})_{1d}.$$

Because it is a rational function, it has a finite number of zeros and poles. Away from these points, we can consider the reduced row echelon form

$$\frac{1}{f(D)} \widetilde{P}_{G^{\Lambda}_{\vec{R}}, G^{\Lambda}_{\vec{S}}},$$

whose pivot element is 1, and in particular independent of D. Therefore, away from the poles and zeros of $f(D)$, we can argue that the pivot column is independent of D.

3.6. Examples of invariant endomorphism tensors

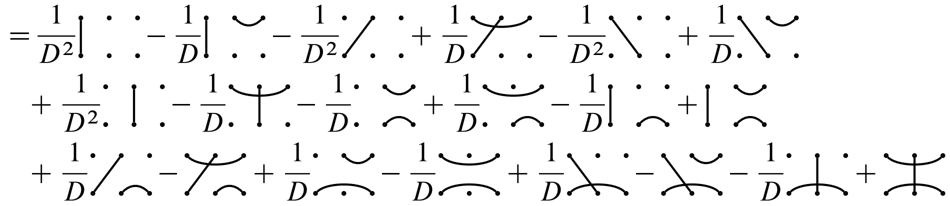
In this subsection, we give some examples of invariant tensors, represented by linear combinations of partition diagrams. More examples can be computed using the SageMath code accompanying the arXiv version of this paper.

For example, with

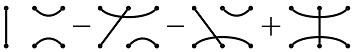
$$\vec{R} = \vec{S} = (V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-2,1,1]}), \quad \Lambda = V_{[D-1,1]},$$

the tensor $(Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}})_{i_1 i_2 i_3, i_4 i_5 i_6}$ has the following expansion in terms of diagrams:

$$Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$$



For the remaining examples, we will only give the dominant diagrams in the limit $D \to \infty$. For instance, the dominant diagrams in the above equation are



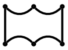
Note that we simply extracted the diagrams whose coefficients dominate at large D compared to the rest of the coefficients. This does not imply that they correspond to the diagrams which give dominant contributions to expectation values. We leave investigations of that question for future work.

For

$$\vec{R} = \vec{S} = (V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-2,2]}), \quad \Lambda = V_{[D-3,3]},$$

the dominant diagram is

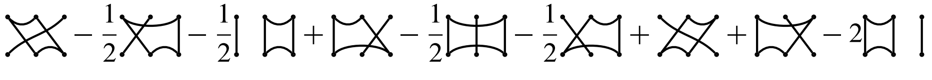
$$Q^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}} =$$



For

$$\vec{R} = \vec{S} = (V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-2,2]}), \quad \Lambda = V_{[D-3,2,1]},$$

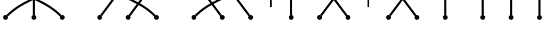
the dominant diagrams are



For

$$\vec{R} = \vec{S} = (V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-1,1]}, V_{[D-2,1,1]}), \quad \Lambda = V_{[D-3,1,1,1]},$$

the dominant diagrams are



4. Counting S_D invariant tensor observables

In this section, we derive a formula for the counting of invariant tensor observables using characters of permutations in the natural representation V_D. We then prove that a basis for the space of degree m observables of matrices of size D is in bijection with bipartite 3-coloured graphs with m white vertices and up to D black vertices.

4.1. Representation-theoretic counting

The dimension of the space of degree m observables is the same as the multiplicity of the trivial representation of S_D in the decomposition into irreducibles of

$$\mathrm{Sym}^m(V_D \otimes V_D \otimes V_D).$$

We call this

$$\mathrm{Dim}(D,m) = \text{Multiplicity of } V_0 \text{ in } \mathrm{Sym}^m(V_D \otimes V_D \otimes V_D).$$

Writing the linear operator for σ in V_D as $\rho(\sigma)$, the linear operator in $V_D^{\otimes 3}$ can be written as

$$\rho_{V_D^{\otimes 3}}(\sigma) = \rho(\sigma) \otimes \rho(\sigma) \otimes \rho(\sigma).$$

The tensor product $(V_D^{\otimes 3})^{\otimes m}$ has an action of

$$\rho_{V_D^{\otimes 3m}}(\sigma) = \underbrace{\rho_{V_D^{\otimes 3}}(\sigma) \otimes \cdots \otimes \rho_{V_D^{\otimes 3}}(\sigma)}_{m \text{ tensor factors}}.$$

The symmetric group S_m acts on $(V_D^{\otimes 3})^{\otimes m}$ by permuting tensor factors. For $\tau \in S_m$, the action is given by

$$\tau(\Phi_{i_1 j_1 k_1} \otimes \cdots \otimes \Phi_{i_m j_m k_m}) = \Phi_{i_{\tau(1)} j_{\tau(1)} k_{\tau(1)}} \otimes \cdots \otimes \Phi_{i_{\tau(m)} j_{\tau(m)} k_{\tau(m)}}.$$

We are interested in the symmetric subspace of $V_D^{\otimes 3m}$, which corresponds to the trivial representation of S_m. Define the projector to the trivial representation of S_m

$$P_0^{S_m} = \frac{1}{m!} \sum_{\tau \in S_m} \tau$$

and the corresponding projector for S_D

$$P_0^{S_D} = \frac{1}{D!} \sum_{\sigma \in S_D} \rho_{V_D^{\otimes 3m}}(\sigma).$$

The dimension of the space of degree m observables is

$$\begin{aligned}\mathrm{Dim}(D,m) &= \mathrm{tr}_{V_D^{\otimes 3m}}\left(P_0^{S_D} P_0^{S_k}\right) = \frac{1}{D!m!}\sum_{\sigma\in S_D}\sum_{\tau\in S_m}\mathrm{tr}_{V_D^{\otimes 3m}}\left(\rho_{V_D^{\otimes 3m}}(\sigma)\tau\right)\\ &= \frac{1}{D!m!}\sum_{\sigma\in S_D}\sum_{\tau\in S_m}\prod_{i=1}^{m}\mathrm{tr}_{V_D^{\otimes 3}}\left(\rho_{V_D^{\otimes 3}}(\sigma^i)\right)^{C_i(\tau)}. \qquad (4.1)\end{aligned}$$

Using

$$\mathrm{tr}_{V_D^{\otimes 3}}\left(\rho_{V_D^{\otimes 3}}(\sigma)\right) = \left(\mathrm{tr}_{V_D}(\rho(\sigma))\right)^3$$

and

$$\mathrm{tr}_{V_D}(\rho(\sigma^i)) = \sum_{l|i} l C_l(\sigma),$$

where the sum is over the divisors of i, we can rewrite (4.1) as

$$\mathrm{Dim}(D,m) = \frac{1}{D!m!}\sum_{\sigma\in S_D}\sum_{\tau\in S_m}\prod_{i=1}^{m}\Big(\sum_{l|i} l C_l(\sigma)\Big)^{3C_i(\tau)}.$$

We collapse the sums over permutations into sums over conjugacy classes to get

$$\mathrm{Dim}(D,m) = \sum_{p\vdash D}\sum_{q\vdash m}\frac{1}{\prod_{i=1} i^{p_i+q_i} p_i! q_i!}\prod_{i=1}^{m}\Big(\sum_{l|i} l p_l\Big)^{3q_i}, \qquad (4.2)$$

with p and q partitions obeying $\sum_i i p_i = D$ and $\sum_i i q_i = m$, respectively. For $m = 1,2,3,4$ and $D = 3m$, this gives

$$\mathrm{Dim}(D,m) = 5, 117, 3813, 187584.$$

4.2. Graph counting

Consider a bipartite 3-coloured graph with m (labelled) white vertices and k (labelled) black vertices and demand that the white vertices have exactly one red, one blue, and one green edge. Because the graph is bipartite, all edges coming out of a white vertex end on a black vertex. We encode the incidence of edges coming out of the ith white vertex using a triplet (r_i, b_i, g_i), where $r_i, b_i, g_i \in \{1,\dots,k\}$. Given this data – a list of m elements in $\{1,\dots,k\}^{\times 3}$ – we have a labelled graph. For an example of this correspondence, see Figure 1.

Let $N(k,m)$ be the number of unlabelled bipartite 3-coloured graphs with m white vertices (with one red, one blue, and one green edge) and up to k black vertices. As

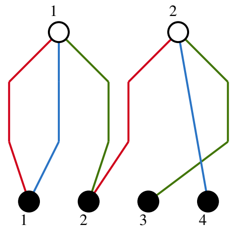


Figure 1. An example of a bipartite 3-coloured labelled graph with two white vertices and three black vertices. It corresponds to the pair of triplets $((1,1,2),(2,4,3))$, which in turn corresponds to the tensor invariant $\sum_{i,j,k,l}\Phi_{iij}\Phi_{jkl}$.

we will now show,

$$N(k,m)=\mathrm{Dim}(k,m).$$

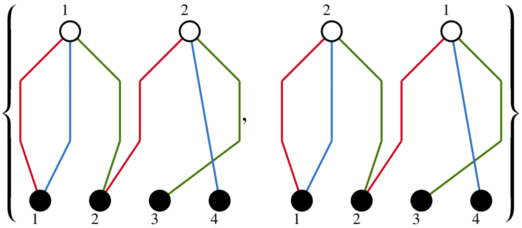
To count unlabelled graphs, we have to forget (form orbits under relabelling) the labels of white as well as black vertices. We start with the white vertex labels. An element $\tau\in S_m$ acts on lists of the above type as follows:

$$(r_1,b_1,g_1),\dots,(r_m,b_m,g_m)\mapsto(r_{\tau(1)},b_{\tau(1)},g_{\tau(1)}),\dots,(r_{\tau(m)},b_{\tau(m)},g_{\tau(m)}). \quad (4.3)$$

Orbits under this action correspond to graphs with unlabelled white vertices but labelled black vertices. For the example given in Figure 1, the orbit under S_2 acting on the white vertices is given in terms of triplets by

$$\{((1,1,2),(2,4,3)),((2,4,3),(1,1,2))\}.$$

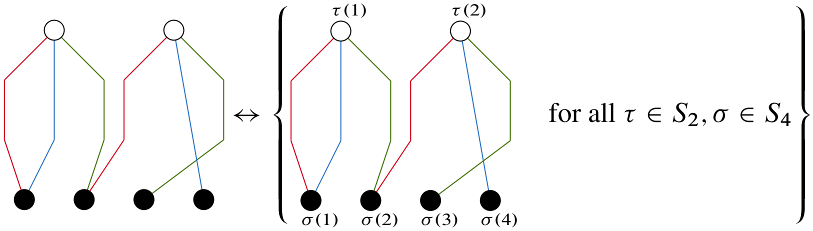
The corresponding set of graphs is



To forget the labels on black vertices, we take orbits under S_k, where $\sigma\in S_k$ acts on the list as

$$(r_1,b_1,g_1),\dots,(r_m,b_m,g_m)\mapsto(\sigma(r_1),\sigma(b_1),\sigma(g_1)),\dots,(\sigma(r_m),\sigma(b_m),\sigma(g_m)). \quad (4.4)$$

Combining the two actions gives an orbit that we identify with the unlabelled graph,



Burnside's lemma allows us to count the orbits of the above actions on the set of labelled graphs with m white and k black vertices,

$$\begin{aligned} N(k,m) &= \text{Average number of elements fixed by the above actions} \\ &= \frac{1}{k!m!} \sum_{\sigma \in S_k, \tau \in S_m} \text{Number of elements fixed by } \sigma \text{ and } \tau. \end{aligned}$$

This has an interpretation in terms of permutation representations. Let

$$U_{k,m} \cong (V_k \otimes V_k \otimes V_k)^{\otimes m}$$

be the vector space with basis

$$|r_1, b_1, g_1, \ldots, r_m, b_m, g_m\rangle \,,$$

where $\tau \in S_m$ and $\sigma \in S_k$ act as in equations (4.3) and (4.4), respectively. It follows that

$$\text{number of elements fixed by } \sigma \text{ and } \tau = \mathrm{Tr}_{U_{k,m}}(\sigma\tau)$$

and therefore

$$N(k,m) = \frac{1}{k!m!} \sum_{\sigma \in S_k, \tau \in S_m} \mathrm{Tr}_{U_{k,m}}(\sigma\tau) = \mathrm{Tr}_{U_{k,m}}\big(P_0^{S_k} P_0^{S_m}\big).$$

We have

$$\mathrm{Tr}_{U_{k,m}}\big(P_0^{S_k} P_0^{S_m}\big) = \sum_{p \vdash k, q \vdash m} \frac{1}{\prod_{i=1} i^{p_i + q_i} p_i! q_i!} \prod_{i=1}^{m} \Big(\sum_{r|i} r p_r\Big)^{3q_i},$$

and from equation (4.2),

$$N(k,m) = \mathrm{Dim}(k,m), \tag{4.5}$$

which proves the correspondence between unlabelled bipartite 3-coloured graphs and observables.

We emphasise that (4.5) holds for all k. Consequently, bipartite 3-coloured graphs with up to D vertices count tensor observables in the unstable limit $D < 3m$ and the stable limit $D \geq 3m$. This is analogous to the correspondence between directed graphs and matrix observables given in [5].

5. Summary and outlook

In this paper, we developed a permutation invariant statistical model of D-dimensional 3-index tensors Φ_{ijk}. The most general Gaussian model contains $5 + 117$ parameters. The basic structure of the model was given in Section 2, where it was solved using representation-theoretic variables. These variables give the most efficient descriptions of the one-point and two-point functions of Φ_{ijk} in terms of invariant tensors $C_{ijk}^{G^{[D]}_{\vec{R}}}$ and $Q_{ijk;pqr}^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$ which in general depend on D. Section 3 was devoted to developing techniques for determining the invariant tensors $Q_{ijk;pqr}^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$. These techniques used a combination of representation theory of S_D and partition algebras to set up a system of eigenvalue equations whose solutions determine the invariant tensors $Q_{ijk;pqr}^{G^{\Lambda}_{\vec{R}} G^{\Lambda}_{\vec{S}}}$. Importantly, the resulting algorithm is able to find the invariant tensors as exact functions of D. This is important for large D studies of the model. Section 4 contains a study of observables in this model. Observables are permutation invariant polynomials in the tensor. Observables were counted using representation theory, and by counting bipartite 3-coloured graphs, we proved a one-to-one correspondence between permutation invariant tensor observables and 3-coloured bipartite graphs.

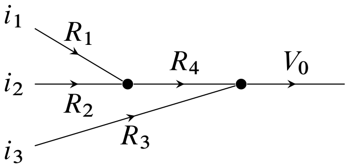
This work opens up several avenues of future research. A combinatorial algorithm for computing expectation values of permutation invariant 2-matrix observables, including computer code implementing the algorithm, was developed in [5]. The algorithm is based on Wick's theorem which holds for Gaussian tensor models as well. Developing analogous computer code for the tensor model studied here will be a useful project. Applications to computational linguistics along the lines of [28,43] should be possible given such an algorithm. In these approaches, the meanings of words are modelled by vector and tensor objects in a vector space; the changes of meaning under composition of words are given by the composition of these objects. The type of structure used to model the meaning of each word is dictated by the word's grammatical role in the sentence. For example, nouns correspond to vectors, adjectives to matrices, and transitive verbs to three index tensors. The statistics of the objects in this final grammatical category could be fruitfully studied with the model presented in this paper, and in fact the tensorial data has already been produced [50] for such a study. Two point functions of permutation invariant matrix observables were shown to exhibit large N factorisation in [4]. The proof relied on the close connection between permutation invariant matrix observables and set partitions, which naturally form a partially ordered set. The ordering and corresponding Hasse diagram was used to determine the powers of $1/N$ in the expansion of two-point functions of permutation invariant matrix observables and was therefore a crucial ingredient in the proof. Since tensor model observables also correspond to set partitions, we expect similar factorisation results to hold for permutation invariant tensor observables. In [6], a representation basis for the space of permutation invariant observables was developed in the context of quantum mechanical models of matrices. The representation basis is an eigenbasis for a set of commuting operators with known eigenvalues. The operators were used to define algebraic Hamiltonians with specified degeneracy patterns, including Hamiltonians with permutation-invariant ground states with large degeneracy. It would be interesting to explore quantum mechanical models of tensors with permutation symmetry using similar techniques.

A. Clebsch–Gordan coefficients for the trivial representation

In this appendix, we derive explicit expressions for the five Clebsch–Gordan coefficients for the trivial representation that appear in the linear part of the action (2.13) and subsequently in the formulae for one-point functions of tensors (2.18).

To compute expectation values, we need formulas for the five Clebsch–Gordan coefficients

$$C^{V_D^{\otimes 3}\to V_0;M}_{i_1 i_2 i_3} =$$



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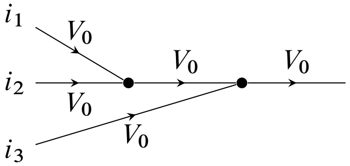
where the multiplicity label

$$M = (R_1, R_2, R_3, R_4, V_0).$$

The non-zero Clebsch–Gordan coefficients are associated with the graphs



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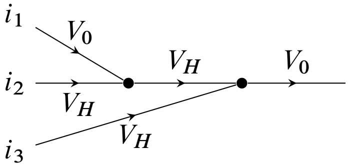

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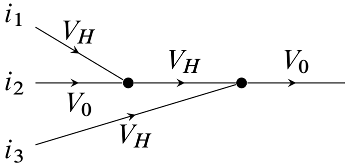

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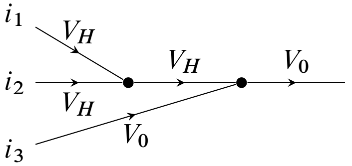

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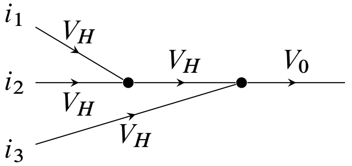

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From equation (3.4), these only involve known (see [42]) Clebsch–Gordan coefficients.

We now describe these in detail. The matrix elements of the maps $V_D \to V_0$ and $V_D \to V_H$ are

$$C_{0,i} = \frac{1}{\sqrt{D}}, \quad C_{m,i} = \frac{1}{\sqrt{m(m+1)}}\left(-m\delta_{i,m+1} + \sum_{j=1}^{m} \delta_{ji}\right),$$

respectively. It will be useful to define

$$F_{ij} = \sum_{a=1}^{D-1} C_{a,i} C_{a,j} = \delta_{ij} - \frac{1}{D}.$$

The Clebsch–Gordan coefficient for $V_H \otimes V_H \to V_0$ is given by

$$C^{V_H \otimes V_H \to V_0}_{mm'} = \frac{\delta_{mm'}}{\sqrt{D-1}},$$

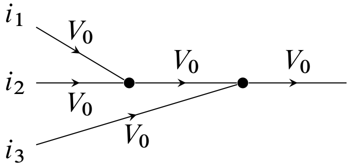
and for $V_H \otimes V_H \to V_H$,

$$C^{V_H \otimes V_H \to V_H}_{mm',m''} = \sqrt{\frac{D}{D-2}} \sum_{i=1}^{D} C_{m,i} C_{m',i} C_{m'',i}.$$

We will also use

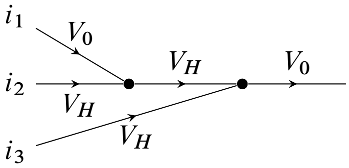
$$C^{V_0 \otimes V_0 \to V_0} = 1, \quad C^{V_0 \otimes V_H \to V_H}_{m,m'} = \delta_{mm'}.$$

The simplest Clebsch–Gordan coefficient to evaluate is



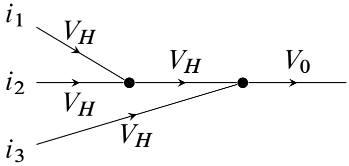
$$= C_{0,i_1}C_{0,i_2}C_{0,i_3} = \frac{1}{\sqrt{D^3}},$$

followed by



$$\begin{aligned}
&= C_{0,i_1}C_{m,i_2}C_{m',i_3}C^{V_0\otimes V_H\to V_H}_{m,m''}C^{V_H\otimes V_H\to V_0}_{m'm''}\\
&= \frac{1}{\sqrt{D}}C_{m,i_2}C_{m',i_3}\delta_{mm''}\frac{\delta_{m'm''}}{\sqrt{D-1}}\\
&= \frac{1}{\sqrt{D(D-1)}}F(i_2,i_3)\\
&= \frac{\delta_{i_2i_3}}{\sqrt{D(D-1)}} - \frac{1}{\sqrt{D^3(D-1)}}
\end{aligned}$$

and its variations given by cyclic permutations of i_1, i_2, i_3. The last coefficient is



$$\begin{aligned}
&= \sum_{m,m',m'',n} C_{m,i_1}C_{m',i_2}C_{m'',i_3}C^{V_H\otimes V_H\to V_H}_{mm',n}C^{V_H\otimes V_H\to V_0}_{nm''}\\
&= \sum_{m,m',m'',n=1}^{D-1} C_{m,i_1}C_{m',i_2}C_{m'',i_3}\sqrt{\frac{D}{D-2}}\sum_{l=1}^{D}C_{m,l}C_{m',l}C_{n,l}\frac{\delta_{nm''}}{\sqrt{D-1}}\\
&= \sqrt{\frac{D}{(D-1)(D-2)}}\sum_{l=1}^{D}F_{i_1l}F_{i_2l}F_{i_3l}.
\end{aligned}$$

We compute

$$\sum_{l=1}^{D}F_{i_1l}F_{i_2l}F_{i_3l} = \delta_{i_1i_2}\delta_{i_2i_3} - \frac{1}{D}(\delta_{i_1i_2}+\delta_{i_1i_3}+\delta_{i_2i_3}) + \frac{2}{D^2}.$$

B. Irreducible decomposition of $V_D \otimes V_D \otimes V_D$

The aim of this appendix is to explain the origin of each of the terms appearing in the irreducible decomposition of $V_D \otimes V_D \otimes V_D$:

$$\begin{aligned}\mathrm{Span}(\Phi_{ijk}) &\cong V_D \otimes V_D \otimes V_D \\ &\cong 5V_0 \oplus 10V_H \oplus 6V_2 \oplus 6V_3 \oplus V_4 \oplus 2V_5 \oplus V_6.\end{aligned} \tag{B.1}$$

Recall the notation defined earlier for Young diagrams in equation (2.8):

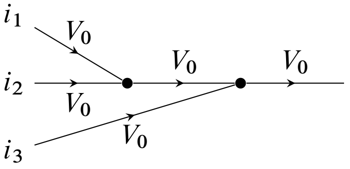
$$\begin{gathered}V_{[D]} \equiv V_0, \quad V_{[D-1,1]} \equiv V_H, \quad V_{[D-2,2]} \equiv V_2, \quad V_{[D-2,1,1]} \equiv V_3, \\ V_{[D-3,3]} \equiv V_4, \quad V_{[D-3,2,1]} \equiv V_5, \quad V_{[D-3,1,1,1]} \equiv V_6.\end{gathered}$$

As mentioned in the main text, this can be found with the rule [26, Section 7.13] for decomposing tensor products of the form $V_R \otimes V_{[D-1,1]}$ (also see [13]). This rule give the following multiplicity-free decompositions:

$$\begin{aligned}V_H \otimes V_H &\cong V_0 \oplus V_H \oplus V_2 \oplus V_3, \\ V_2 \otimes V_H &\cong V_H \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5, \\ V_3 \otimes V_H &\cong V_H \oplus V_2 \oplus V_3 \oplus V_5 \oplus V_6,\end{aligned}$$

which allows for a graphical description of the resulting multiplicities appearing on the right-hand side of (B.1) which we give here. Each of the following graphs corresponds to a single irreducible representation appearing in the decomposition

$\Lambda = V_0$:



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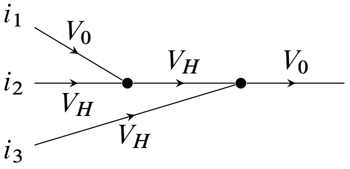

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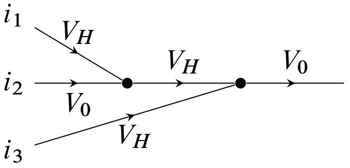

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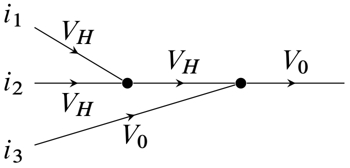

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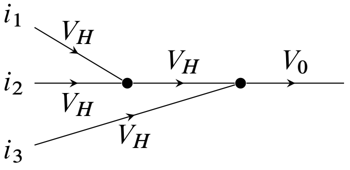

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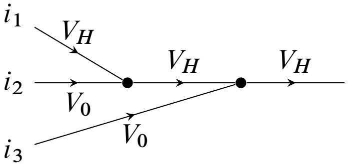
$\Lambda = V_H$:



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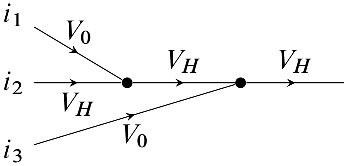

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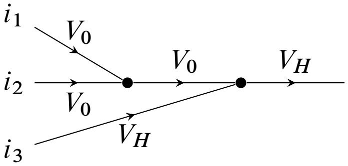

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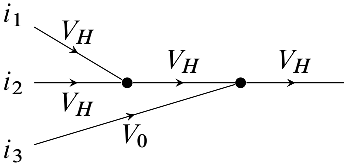

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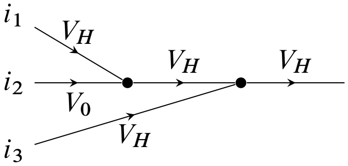

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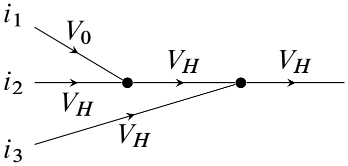

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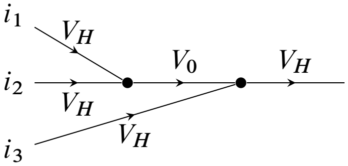

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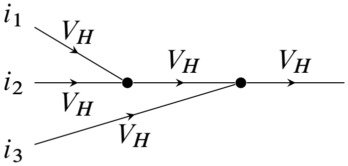

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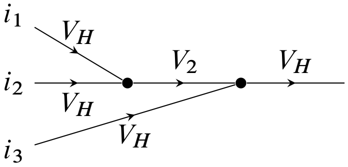

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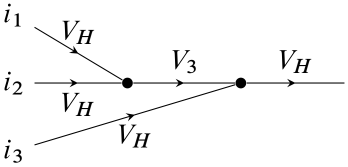

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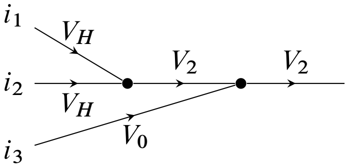
$\Lambda = V_2$:



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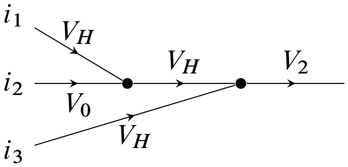

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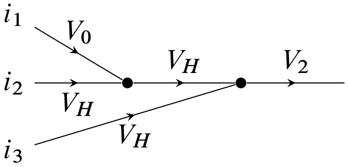

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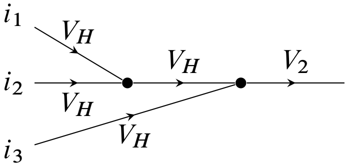

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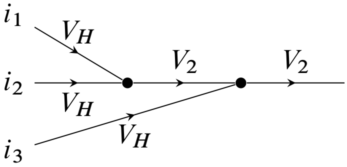

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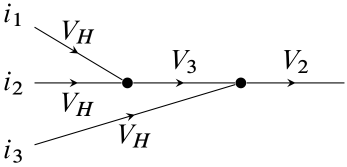

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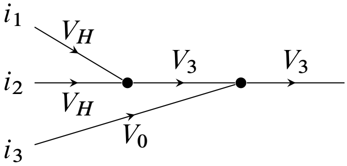
$\Lambda = V_3$:



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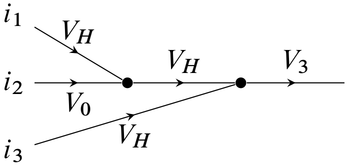

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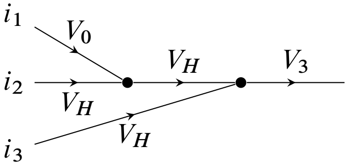

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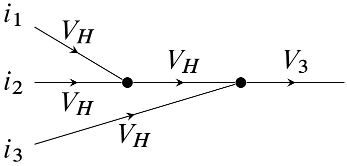

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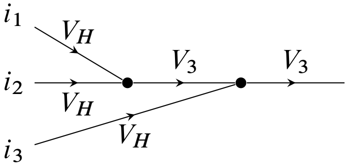

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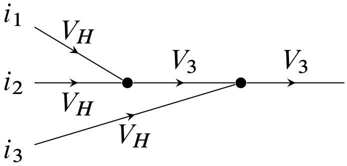

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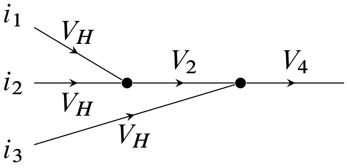
$\Lambda = V_4$:



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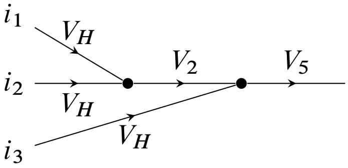
$\Lambda = V_5$:



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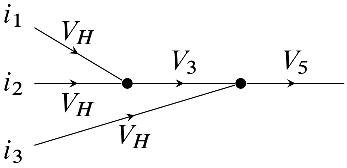

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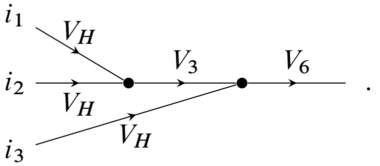
$\Lambda = V_6$:



C. Pivot columns and bases of image

In this appendix, we review pivot columns of matrices and how they are used to find bases for the image of a matrix (see [47, **2O** in Section 2.4]). This is used in Section 3.5 to give an all D construction of invariant endomorphism tensors.

Given a matrix B, it is said to be in row echelon form if the following hold.

- All rows consisting of only zeros are at the bottom.
- Reading the rows from the left, the first non-zero element in every row is to the right of the first non-zero element in every row above it.

For a matrix in row echelon form, the first non-zero element of a row is called the pivot. For example,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in row echelon form and the pivots are b_{11}, b_{23}. Consider B as list of column vectors

$$B = [b_1\, b_2\, b_3\, b_4].$$

The vectors

$$b_1 = \begin{pmatrix} b_{11} \\ 0 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} b_{13} \\ b_{23} \\ 0 \end{pmatrix}$$

form a basis for the image of B. In general, the columns containing pivots form a basis for the image of B. That is, all y such that

$$y = Bx = \sum_i b_i x_i$$

can be expanded in terms of pivot columns. This follows since every non-pivot column is a linear combination of the set of pivot columns before it.

Let A, B, E be $n \times n$ matrices, where

$$A = EB, \tag{C.1}$$

such that B is in row echelon form and E is invertible. Consider A, B as lists of column vectors

$$A = [a_1 \cdots a_n], \quad B = [b_1 \cdots b_n].$$

Equation (C.1) reads

$$a_i = Eb_i \quad \forall i = 1, \ldots, n. \tag{C.2}$$

As we will now show, if $\{b_{i_1}, \ldots, b_{i_k}\}$ is the set of pivot columns of B, the set of vectors $\{a_{i_1}, \ldots, a_{i_k}\}$ form a basis for the image of A. First, we show that they constitute a set of linearly independent vectors. That is, $x_p = 0$ is the only solution to

$$\sum_{p=1}^{k} x_p a_{i_p} = 0.$$

Using (C.2), we have

$$\sum_{p=1}^{k} x_p E b_{i_p} = 0,$$

and because E is invertible, this is equivalent to solving

$$\sum_{p=1}^{k} x_p b_{i_p} = 0.$$

But the pivot columns of B are linearly independent, therefore $x_p = 0$. Next, we show that all y such that

$$y = Ax$$

can be expanded in terms of $\{a_{i_1}, \ldots, a_{i_k}\}$. Using (C.1), we have

$$y = EBx,$$

or

$$E^{-1}y = Bx.$$

Define $y' = E^{-1}y$, y' is in the image of B, which has a basis of pivot columns. That is, there exists a unique set of numbers $y'_1, \ldots, y'_p$ such that

$$y' = \sum_{p=1}^{k} y'_p b_{i_p}.$$

Using the inverse of (C.2), we have

$$E^{-1}y = \sum_{p=1}^{k} y'_p E^{-1} a_{i_p},$$

or

$$y = \sum_{p=1}^{k} y'_p a_{i_p}.$$

We have proven that the set $\{a_{i_1}, \ldots, a_{i_k}\}$ of columns corresponding to pivot columns of B form a linearly independent set of vectors that span the image of A. That is, they form a basis for the image of A.

D. Finding invariant endomorphism tensors (code)

This appendix explains the code implementing the algorithm in Section 3.5 for constructing invariant endomorphism tensors. The code is implemented in SageMath [48].

The first step in computing the invariant endomorphism tensors, using partition algebras in SageMath, is to define the partition algebras $P_1(D)$, $P_2(D)$, $P_3(D)$ over the polynomial ring $\mathbb{Q}[D]$. This is done in the first cell of the Jupyter Notebook.

```
[1]: ## Define the partition algebras P_1(D) P_2(D) P_3(D) over the␣
      ↪polynomial ring QQ[D]
     R.<D> = QQ[]
     PA3 = PartitionAlgebra(3,D,R)
     PA2 = PartitionAlgebra(2,D,R)
     PA1 = PartitionAlgebra(1,D,R)
```

The second step is to compute the projectors in (3.20). Each projector is defined through a function that takes an irreducible representation R of S_D as input and returns the corresponding projector. The irreducible representations are to be given as integer partitions of D. In particular, for an integer partition $R = (D - k, \lambda)$ of D, where λ is an integer partition of k, only λ should be given as input. For example, for $R = [D]$, the corresponding input partition is $Partition([])$ – the empty partition. For $R = [D - 1, 1]$, one should input $Partition([1])$ and so on. Note that, in the current implementation, the projectors do not include the denominator in (3.20). This allows us to do the computations using partition algebras over polynomial rings, which is faster than computations using partition algebras over symbolic rings. Excluding the denominators does not change the image of the projectors but affects the overall normalisation of the output vector in the algorithm. If desired, the normalisation can be adjusted as a final step in the algorithm.

```
[2]: ## Define the "projector" P_{R_1} on the first representation␣
      ↪in G_A^\Lambda, see equation (3.52) in associated paper
     def P_R1(R1):
```

```
R.<D> = QQ[]
## Construct T2 \otimes \idn \otimes \idn defined in equation␣
↪(3.42)
T2_id_id = PA3(sum(PA1.jucys_murphy_element(i/2) for i in [1..
↪2*1])+(D*(D-1)/2-D*1)*PA1.one())
## Define the set of irreps and corresponding normalised␣
↪characters to take a product over in equation (3.52)
IrrepsEigenvaluesDictionary = {Partition([]): R(1/2*(D-1)*D),␣
↪Partition([1]): R((D-3)*D/2)}
proj = prod((T2_id_id-ev2*PA3.one()) for (rep, ev2) in␣
↪IrrepsEigenvaluesDictionary.items() if rep != R1)
return proj
```

[3]:

```
## Define the "projector" P_{R_2} on the second representation␣
↪in G_A^\Lambda, see equation (3.53) in associated paper
def P_R2(R2):
R.<D> = QQ[]
## Construct \idn \otimes T2 \otimes \idn defined in equation␣
↪(3.43)
id_T2_id = PA3([[-1,2],[-2,1],[-3,3]])*PA3(sum(PA1.
↪jucys_murphy_element(i/2) for i in [1..2*1])+(D*(D-1)/
↪2-D*1)*PA1.one())*PA3([[-1,2],[-2,1],[-3,3]])
## Define the set of irreps and corresponding normalised␣
↪characters to take a product over in equation (3.53)
IrrepsEigenvaluesDictionary = {Partition([]): R(1/2*(D-1)*D),␣
↪Partition([1]): R((D-3)*D/2)}
proj = prod((id_T2_id-ev2*PA3.one()) for (rep, ev2) in␣
↪IrrepsEigenvaluesDictionary.items() if rep != R2)
return proj
```

[4]:

```
## Define the "projector" P_{R_3} on the third representation␣
↪in G_A^\Lambda, see equation (3.54) in associated paper
def P_R3(R3):
R.<D> = QQ[]
## Construct \idn \otimes \idn \otimes T2 defined in equation␣
↪(3.44)
```

```
id_T2_id = PA3([[-1,3],[-3,1],[-2,2]])*PA3(sum(PA1.
↪jucys_murphy_element(i/2) for i in [1..2*1])+(D*(D-1)/
↪2-D*1)*PA1.one())*PA3([[-1,3],[-3,1],[-2,2]])
## Define the set of irreps and corresponding normalised␣
↪characters to take a product over in equation (3.54)
IrrepsEigenvaluesDictionary = {Partition([]): R(1/2*(D-1)*D),␣
↪Partition([1]): R((D-3)*D/2)}
proj = prod((id_T2_id-ev2*PA3.one()) for (rep, ev2) in␣
↪IrrepsEigenvaluesDictionary.items() if rep != R3)
return proj
```

[5]:
```
## Define the "projector" P_{R_4} on the fourth representation␣
↪in G_A^\Lambda, see equation (3.55) in associated paper
def P_R4(R4):
R.<D> = QQ[]
## Construct T2 \otimes \idn defined in equation (3.45)
T2_id = PA3(sum(PA2.jucys_murphy_element(i/2) for i in [1..
↪2*2])+(D*(D-1)/2-D*2)*PA2.one())
## Define the set of irreps and corresponding normalised␣
↪characters to take a product over in equation (3.55)
IrrepsEigenvaluesDictionary = {Partition([]): R(1/2*(D-1)*D),␣
↪Partition([1]): R((D-3)*D/2), Partition([2]): R(1/2*(D -␣
↪1)*(D - 4)), Partition([1,1]): R(1/2*(D - 5)*D)}
proj = prod((T2_id-ev2*PA3.one()) for (rep, ev2) in␣
↪IrrepsEigenvaluesDictionary.items() if rep != R4)
return proj
```

[6]:
```
## Define the "projector" P_{\Lambda} on the last␣
↪representation in G_A^\Lambda, see equation (3.55) in␣
↪associated paper
def P_Lambda(Lambda):
R.<D> = QQ[]
## Construct T2 defined in equation (3.41)
T2 = sum(PA3.jucys_murphy_element(i/2) for i in [1..
↪2*3])+(D*(D-1)/2-D*3)*PA3.one()
## Define the set of irreps and corresponding normalised␣
↪characters to take a product over in equation (3.55)
```

```
IrrepsEigenvaluesDictionary = {Partition([]): R(1/2*(D-1)*D),␣
 ↪Partition([1]): R((D-3)*D/2), Partition([2]): R(1/2*(D -␣
 ↪1)*(D - 4)), Partition([1,1]): R(1/2*(D - 5)*D),␣
 ↪Partition([3]): R(1/2*(D-3)*(D-4)), Partition([2,1]): R(1/
 ↪2*(D-1)*(D-6)), Partition([1,1,1]): R(1/2*D*(D-7))}
proj = prod((T2-ev2*PA3.one()) for (rep, ev2) in␣
 ↪IrrepsEigenvaluesDictionary.items() if rep != Lambda)
return proj
```

With all the projectors defined, we can compute the elements (3.21) for a pair of multiplicity graphs by taking products of several projectors. The graphs are specified by two lists of partitions, LeftReps and RightReps, respectively. The corresponding left and right projectors are computed as LeftProj and RightProj.

[7]:

```
## Define the set of irreducible representations R_1, R_2,␣
 ↪R_3, R_4, \Lambda appearing in G_A^\Lambda
LeftReps = (Partition([1]), Partition([1]), Partition([1]),␣
 ↪Partition([1,1]), Partition([1]))
## and the total "projector" P_{\Lambda} P_{R_4} P_{R_3}␣
 ↪P_{R_2} P_{R_1}
LeftProj =␣
 ↪P_Lambda(LeftReps[4])*P_R4(LeftReps[3])*P_R3(LeftReps[2])*
P_R2(LeftReps[1])*P_R1(LeftReps[0])
## Define the set of irreducible representations R_1, R_2,␣
 ↪R_3, R_4, \Lambda appearing in G_B^\Lambda
RightReps = (Partition([1]), Partition([1]), Partition([1]),␣
 ↪Partition([1,1]), Partition([1]))
## and the total "projector" P_{\Lambda} P_{R_4} P_{R_3}␣
 ↪P_{R_2} P_{R_1}
RightProj =␣
 ↪P_Lambda(RightReps[4])*P_R4(RightReps[3])*P_R3(RightReps[2])*
P_R2(RightReps[1])*P_R1(RightReps[0])
```

To find the image of the left and right multiplication of LeftProj and RightProj, we compute their corresponding matrix representations. SageMath has a built-in method called to_matrix() for this purpose. To get the left and right action, respectively, we give this method the input “side=left” and “side=right”. The matrix of the combined action is stored in ProjMatrix. The last step is to find the pivot column of ProjMatrix for a fixed value of D and extract the corresponding column of ProjMatrix. This element, called Q in the code, is proportional to the invariant tensor we were looking for.

```
[8]: ## Now produce the matrix corresponding to left action of␣
      ↪LeftProj and right action of RightProj
     ProjMatrix = LeftProj.to_matrix(side='left')*RightProj.
      ↪to_matrix(side='right')
     ## Compute the pivot columns of ProjMatrix for D=10
     pivot = ProjMatrix.subs(D=10).pivots()
     ## As long as ProjMatrix has rank 1, extract the pivot column,
     ## otherwise print a warning that G_A^\Lambda G_B^\Lambda do␣
      ↪not define a non-zero Q
     if len(pivot) > 0:
     Q = ProjMatrix.column(pivot[0])
     else:
     print('Q does not exist for the chosen G_A^\\Lambda and␣
      ↪G_B^\\Lambda')
```

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