

# Left orderability and taut foliations with one-sided branching

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**Abstract.** For a closed, orientable, irreducible 3-manifold  $M$  that admits a co-orientable taut foliation with one-sided branching, we show that  $\pi_1(M)$  is left orderable.

## 1. Introduction

The  $L$ -space conjecture is proposed by Boyer–Gordon–Watson [3] and Juhász [13], which states that, for every closed, orientable, irreducible 3-manifold  $M$ ,  $M$  is a non- $L$ -space if and only if  $\pi_1(M)$  is left orderable, and if and only if  $M$  admits a co-orientable taut foliation.

In [17], Ozsváth and Szabó prove that  $M$  is a non- $L$ -space if  $M$  admits a co-orientable taut foliation (see also [1, 14]). In [9], Gabai proves that  $M$  admits taut foliations if  $M$  has positive first Betti number. In [4], Boyer, Rolfsen, and Wiest prove that  $\pi_1(M)$  is left orderable if  $b_1(M) > 0$ . It is known that the  $L$ -space conjecture holds for every graph manifold, by the works of Boyer–Clay [2], Rasmussen [19], and Hanselman–Rasmussen–Rasmussen–Watson [12].

Taut foliations of 3-manifolds can be distinguished into three types according to the branching behavior of the leaf spaces (of the pull-back foliations in their universal covers):  $\mathbb{R}$ -covered, one-sided branching, and two-sided branching. The generic case for taut foliations is two-sided branching. The 3-manifolds that admit co-orientable  $\mathbb{R}$ -covered foliations are known to have left orderable fundamental group. In this paper, we consider taut foliations with one-sided branching and restrict the unknown case for the direction “co-orientable taut foliation  $\Rightarrow$  left orderability of  $\pi_1$ ” to two-sided branching.

**Theorem 1.1.** *Let  $M$  be a connected, closed, orientable, irreducible 3-manifold that admits a co-orientable taut foliation with one-sided branching. Then,  $\pi_1(M)$  is left orderable.*

In the proof of Theorem 1.1, we provide an injective homomorphism of  $\pi_1(M)$  to the group  $\mathcal{E}_\infty$  (see Definition 2.10 for the definition), which is shown to be left orderable by Navas and is shown to have no nontrivial homomorphism to  $\text{Homeo}_+(\mathbb{R})$  by Mann [15].

## 2. Preliminaries

Throughout this section,  $M$  will always be a connected, closed, orientable, irreducible 3-manifold and  $p : \tilde{M} \rightarrow M$  is the universal covering of  $M$ .

### 2.1. The leaf space of a taut foliation

Suppose that  $\mathcal{F}$  is a taut foliation of  $M$ . We will always adopt the following notations associated to  $\mathcal{F}$ .

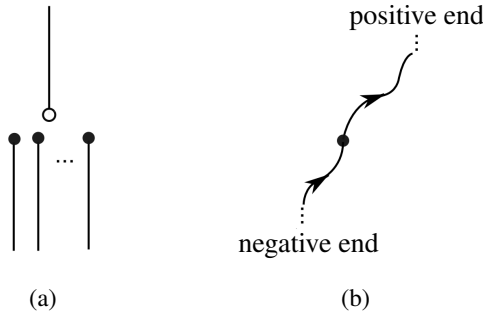
**Notation 2.1.** Let  $\tilde{\mathcal{F}}$  be the pull-back foliation of  $\mathcal{F}$  in  $\tilde{M}$ . Let  $L(\mathcal{F})$  be the leaf space of  $\mathcal{F}$ . The action of  $\pi_1(M)$  on  $\tilde{M}$  by deck transformations induces an action of  $\pi_1(M)$  on  $L(\mathcal{F})$ , which we refer to as the  $\pi_1$ -action on  $L(\mathcal{F})$ .

Note that  $L(\mathcal{F})$  is an orientable, connected, simply connected 1-manifold, which is second countable but possibly non-Hausdorff [11, 18]. In addition,  $L(\mathcal{F})$  is Hausdorff if and only if it is homeomorphic to  $\mathbb{R}$ . Whenever  $L(\mathcal{F})$  is not homeomorphic to  $\mathbb{R}$ ,  $L(\mathcal{F})$  has some “non-Hausdorff” places described as follows.

Consider two distinct points  $x, y \in L(\mathcal{F})$ . We say that  $x, y$  are *non-separated* if, for any point  $t \in L(\mathcal{F}) - \{x, y\}$ ,  $x, y$  are contained in the same component of  $L(\mathcal{F}) - \{t\}$ . Equivalently,  $x, y$  are non-separated if  $x, y$  have no disjoint neighborhoods. See Figure 1 (a) for a collection of points in  $L(\mathcal{F})$  which are pairwise non-separated.

We note that, for three points  $x, y, z \in L(\mathcal{F})$  such that  $x, y$  are non-separated and  $y, z$  are non-separated, it does not necessarily follow that  $x, z$  are non-separated. As discussed in [20, Section 4], we can blow-up countably many points in  $L(\mathcal{F})$  as in [20, Appendix A], to obtain a 1-manifold  $L'$  so that the relation “ $x \sim y$  if  $x, y \in L'$  are non-separated” is transitive on  $L'$ , and thus, forms an equivalence relation on  $L'$ . The quotient space  $L'/\sim$  is Hausdorff and remains simply connected (though  $L'/\sim$  is no longer a 1-manifold), which is referred to as the *Hausdorff tree associated to  $L(\mathcal{F})$*  [20, Definition 4.6].

At last, we explain the behavior of the  $\pi_1$ -action on  $L(\mathcal{F})$  in the case where  $\mathcal{F}$  is co-orientable. We choose a co-orientation on  $\mathcal{F}$ , which then induces a co-orientation on  $\tilde{\mathcal{F}}$  and, therefore, induces an orientation on  $L(\mathcal{F})$ . In this case,  $\pi_1(M)$  acts on  $L(\mathcal{F})$  via orientation-preserving homeomorphisms.



**Figure 1.** (a) describes a collection of points in  $L(\mathcal{F})$  which are pairwise non-separated. (b) describes the ends of  $L(\mathcal{F})$ : every positive end of  $L(\mathcal{F})$  can be represented by some positively oriented rays in  $L(\mathcal{F})$ , and every negative end of  $L(\mathcal{F})$  can be represented by some negatively oriented rays in  $L(\mathcal{F})$ .

**2.2. 1-manifolds with one-sided branching**

Let  $L$  be an oriented, connected, simply connected 1-manifold which is second countable but possibly non-Hausdorff.

By a ray of  $L$ , we mean an embedding  $r : [0, +\infty) \rightarrow L$  such that there is no embedding  $f : [0, +\infty) \rightarrow L$  with  $f(0) = r(0)$  and  $r([0, +\infty)) \subsetneq f([0, +\infty))$ .

Let  $\mathcal{R} = \{\text{rays of } L\}$ . And let  $\sim$  be the relation on  $\mathcal{R}$  such that, for arbitrary two elements  $r_1, r_2 : [0, +\infty) \rightarrow L$  of  $\mathcal{R}$ ,  $r_1 \sim r_2$  if there are  $t_1, t_2 \in [0, +\infty)$  with  $r_1([t_1, +\infty)) = r_2([t_2, +\infty))$ . It is not hard to see that “ $\sim$ ” is transitive on  $\mathcal{R}$ , and so, “ $\sim$ ” is an equivalence relation on  $\mathcal{R}$ . We define

$$\text{End}(L) = \mathcal{R} / \sim,$$

and we call each element of  $\text{End}(L)$  an *end* of  $L$ . For any  $r \in \mathcal{R}$ , we will always denote by  $[r]$  the end represented by  $r$ .

**Definition 2.2.** Let  $e : \mathbb{R} \rightarrow L$  be an embedding. We will always denote by  $e_+ : [0, +\infty) \rightarrow L$  the restriction of  $e$  to  $[0, +\infty)$ , and we will always denote by  $e_- : [0, +\infty) \rightarrow L$  the map defined by  $e_-(t) = e(-t)$  for any  $t \in [0, +\infty)$ . We call  $e$  a *proper embedding* if both of  $e_+, e_-$  are rays of  $L$ .

Given an embedded path  $j : [0, 1] \rightarrow L$ , a ray  $r : [0, +\infty) \rightarrow L$ , or a proper embedding  $e : \mathbb{R} \rightarrow L$ , each of  $j, r, e$  is said to be *positively oriented* (resp., *negatively oriented*) if the increasing orientation on its domain is consistent with (resp., *opposite to*) the orientation on  $L$ .

**Definition 2.3.** An end  $t$  of  $L$  is *positive* (resp., *negative*) if there is a positively oriented ray (resp., negatively oriented ray)  $r : [0, +\infty) \rightarrow L$  with  $[r] = t$ . See Figure 1 (b).

Let  $G$  be a group acting on  $L$  by homeomorphisms. We note that this action induces an action of  $G$  on  $\text{End}(L)$ . In addition, for any  $g \in G$ , if  $g : L \rightarrow L$  is an orientation-preserving homeomorphism, then the endomorphism on  $\text{End}(L)$  induced from  $g$  takes positive ends to positive ends and takes negative ends to negative ends. Thus, in the case that  $G$  acts on  $L$  via orientation-preserving homeomorphisms, under the induced action of  $G$  on  $\text{End}(L)$ , the set of positive ends and the set of negative ends are both  $G$ -invariant subsets of  $\text{End}(L)$ .

Below, we describe the branching behavior of  $L$ , which basically follows from [6, Chapter 4.7, page 169]. There is a strict partial order “ $\overset{L}{>}$ ” on  $L$  defined as follows. Let  $u, v \in L(\mathcal{F})$ . Then,  $u \overset{L}{>} v$  if  $u \neq v$  and there is a positively oriented embedded path in  $L$  from  $v$  to  $u$ . In addition,  $u, v$  are said to be *comparable* if one of these three cases holds:  $u = v, u \overset{L}{>} v, u \overset{L}{<} v$ . And  $u, v$  are said to be *incomparable* if  $u, v$  are not comparable.

Compare the following definition with [6, Chapter 4.7, Definition 4.41 (2)].

**Definition 2.4.**  $L$  is said to have *one-sided branching* if either of the following statements holds.

- $L$  is not homeomorphic to  $\mathbb{R}$ , and moreover, for any  $u, v \in L(\mathcal{F})$ , there exists  $s \in L(\mathcal{F})$  such that  $s \overset{L}{>} u, v$ . In this case,  $L$  is said to have *(one-sided) branching in the negative direction*.
- $L$  is not homeomorphic to  $\mathbb{R}$ , and moreover, for any  $u, v \in L(\mathcal{F})$ , there exists  $s \in L(\mathcal{F})$  such that  $s \overset{L}{<} u, v$ . In this case,  $L$  is said to have *(one-sided) branching in the positive direction*.

Now, we provide a description for Definition 2.4 from the ends of  $L$ . We consider  $L$  as an order tree; then, by [21, Definitions 3.2, 3.5, and Theorem 3.6], if  $u \overset{L}{>} v$  for  $u, v \in L$ , then there is a unique embedded path in  $L$  from  $v$  to  $u$ . We denote this embedded path by  $\gamma_L(v, u)$ .

**Lemma 2.5.** (a)  $L$  has one-sided branching in the negative direction if and only if  $L$  has exactly one positive end and more than one negative end.

(b)  $L$  has one-sided branching in the positive direction if and only if  $L$  has exactly one negative end and more than one positive end.

*Proof.* To be convenient, we only prove (a).

Suppose that  $L$  has exactly one positive end and more than one negative end. Let  $u, v \in L$ . We choose two positively oriented rays  $r_u, r_v : [0, +\infty) \rightarrow L$  with

$r_u(0) = u, r_v(0) = v$ . Because  $L$  has exactly one positive end,  $[r_u] = [r_v]$ . So, there exists  $s \in r_u([0, +\infty)) \cap r_v([0, +\infty))$ , and we have  $s \stackrel{L}{>} u, v$ . Note that  $L$  is not homeomorphic to  $\mathbb{R}$  since it has more than one negative end. Hence,  $L$  has one-sided branching in the negative direction.

Now, suppose that  $L$  has one-sided branching in the negative direction. We first claim that, for any two positively oriented rays  $r_1, r_2 : [0, +\infty) \rightarrow L$  with  $r_1(0) = r_2(0)$ , we have  $[r_1] = [r_2]$ . Let  $a = r_1(0) = r_2(0)$ . We assume otherwise  $[r_1] \neq [r_2]$ . Then, there is  $b \in r_1([0, +\infty)) - r_2([0, +\infty))$ . If  $b \stackrel{L}{>} t$  for all  $t \in r_2([0, +\infty))$ , then  $r_2([0, +\infty)) \subsetneq \gamma_L(a, b)$ , which is impossible. So, there is  $c \in r_2([0, +\infty))$  with  $b \not\stackrel{L}{>} c$  (here,  $b \not\stackrel{L}{>} c$  means that  $b \stackrel{L}{>} c$  does not hold). By Definition 2.4, there exists  $d \in L$  such that  $d \stackrel{L}{>} b, c$ . Note that  $a \stackrel{L}{<} b, c$  since  $b \in r_1([0, +\infty)), c \in r_2([0, +\infty))$ . Hence,  $a \stackrel{L}{<} d$  and  $b, c \in \gamma_L(a, d)$ . It follows that either  $b \in \gamma_L(a, c)$  or  $b \in \gamma_L(c, d) - \{c\}$ . As  $b \not\stackrel{L}{>} c$ , we have  $b \notin \gamma_L(c, d) - \{c\}$ . So,  $b \in \gamma_L(a, c)$ . Note that  $c \in r_2([0, +\infty))$  implies  $\gamma_L(a, c) \subseteq r_2([0, +\infty))$ , and therefore,  $b \in r_2([0, +\infty))$ . This contradicts the condition  $b \in r_1([0, +\infty)) - r_2([0, +\infty))$  given above. Thus,  $[r_1] = [r_2]$ .

Let  $x, y \in L$  and let  $r_x, r_y : [0, +\infty) \rightarrow L$  be two positively oriented rays of  $L$  with  $x = r_x(0), y = r_y(0)$ . By Definition 2.4, there exists  $t \in L$  such that  $t \stackrel{L}{>} x, y$ . Let  $r_t : [0, +\infty) \rightarrow L$  be a positively oriented ray of  $L$  with  $r_t(0) = t$ . Then,  $\gamma_L(x, t) * r_t$  is a positively oriented ray of  $L$  that starts at  $x$ , where  $\gamma_L(x, t) * r_t$  denotes the concatenation of  $\gamma_L(x, t)$  and  $r_t$ . Also,  $\gamma_L(y, t) * r_t$  is a positively oriented ray of  $L$  that starts at  $y$ . By the claim as above, we have

$$[r_x] = [\gamma_L(x, t) * r_t], \quad [r_y] = [\gamma_L(y, t) * r_t].$$

Then,

$$[r_x] = [\gamma_L(x, t) * r_t] = [r_t] = [\gamma_L(y, t) * r_t] = [r_y].$$

Thus, any two positively oriented rays in  $L$  represent the same positive end of  $L$ , and therefore,  $L$  has a unique positive end.

It remains to show that  $L$  has more than one negative end. Suppose otherwise that  $L$  has exactly one negative end. Since  $L$  is not homeomorphic to  $\mathbb{R}$ , there exist  $m, n \in L$  which are incomparable. Let  $r_m^+, r_n^+ : [0, +\infty) \rightarrow L$  be two positively oriented rays of  $L$  with  $r_m^+(0) = m, r_n^+(0) = n$ , and let  $r_m^-, r_n^- : [0, +\infty) \rightarrow L$  be two negatively oriented rays of  $L$  with  $r_m^-(0) = m, r_n^-(0) = n$ . Then,

$$[r_m^+] = [r_n^+], \quad [r_m^-] = [r_n^-].$$

So, there exist  $q \in r_m^+([0, +\infty)) \cap r_n^+([0, +\infty)), s \in r_m^-([0, +\infty)) \cap r_n^-([0, +\infty))$ , and we have  $q \stackrel{L}{>} m, n, s \stackrel{L}{<} m, n$ . Thus,  $s \stackrel{L}{<} q$  and  $m, n \in \gamma_L(s, q)$ . It follows that  $m, n$  are comparable, which is a contradiction. Therefore,  $L$  has more than one negative end. ■

### 2.3. The three types of foliations

Suppose that  $M$  admits a taut foliation  $\mathcal{F}$ . We fix a co-orientation on  $\tilde{\mathcal{F}}$ , which induces an orientation on  $L(\mathcal{F})$ , as described in [6, Definition 4.41].

**Definition 2.6.**  $\mathcal{F}$  has exactly one of the following three types:

- (a)  $\mathcal{F}$  is  $\mathbb{R}$ -covered if  $L(\mathcal{F})$  is homeomorphic to  $\mathbb{R}$ ,
- (b)  $\mathcal{F}$  has one-sided branching if  $L(\mathcal{F})$  has one-sided branching,
- (c)  $\mathcal{F}$  has two-sided branching if  $\mathcal{F}$  is not  $\mathbb{R}$ -covered and does not have one-sided branching.

For concreteness, we describe Definition 2.6 (b), (c) by using the ends of  $L(\mathcal{F})$ . We note that having more than one positive (or negative) end in  $L(\mathcal{F})$  implies the existence of infinitely many positive (or negative) ends in  $L(\mathcal{F})$ . For the reader's convenience, we include a proof of this property of  $L(\mathcal{F})$ .

**Lemma 2.7.** (a) *If  $L(\mathcal{F})$  has more than one positive end, then  $L(\mathcal{F})$  has infinitely many positive ends.*

(b) *If  $L(\mathcal{F})$  has more than one negative end, then  $L(\mathcal{F})$  has infinitely many negative ends.*

*Proof.* We only prove (a); the proof of (b) is entirely similar. Let  $G = \pi_1(M)$ , and we may consider  $G$  as the group of deck transformations of  $\tilde{M}$ . We first assume further that  $\mathcal{F}$  is co-orientable and prove (a) under this assumption.

For any  $s \in L(\mathcal{F})$ , we define

$$E_+(s) = \{[r] \mid r : [0, +\infty) \rightarrow L(\mathcal{F}) \text{ is a positively oriented ray with } r(0) = s\}.$$

We claim that there exists  $s \in L(\mathcal{F})$  with  $|E_+(s)| > 1$ , where  $|E_+(s)|$  denotes the cardinality of  $E_+(s)$ . Now, suppose otherwise that  $|E_+(s)| = 1$  for all  $s \in L(\mathcal{F})$ . Note that for any  $t_1, t_2 \in L(\mathcal{F})$  with  $t_1 \overset{L(\mathcal{F})}{>} t_2$ , we always have  $E_+(t_1) = E_+(t_2)$  since  $E_+(t_1) \subseteq E_+(t_2)$  and  $|E_+(t_1)| = |E_+(t_2)| = 1$ . Because  $L(\mathcal{F})$  has more than one positive end, there are  $u, v \in L(\mathcal{F})$  with  $E_+(u) \neq E_+(v)$ . As described in [10, Definition 6.9 (3), Proposition 6.10], there is a finite sequence of points  $u_0 = u, u_1, \dots, u_{n-1}, u_n = v$  such that  $u_i, u_{i+1}$  are comparable for every  $0 \leq i \leq n - 1$ . It follows that  $E_+(u_0), E_+(u_1), \dots, E_+(u_n)$  are all equal, which is a contradiction.

Thus, there exists  $s \in L(\mathcal{F})$  with  $|E_+(s)| > 1$ . Let  $r_1, r_2 : [0, +\infty) \rightarrow L(\mathcal{F})$  be two positively oriented rays such that  $r_1(0) = r_2(0) = s$  but  $[r_1] \neq [r_2]$ . We choose  $t \in r_1([0, +\infty)) - r_2([0, +\infty))$ . Then,  $[r_2] \notin E_+(t)$ . Because  $\mathcal{F}$  is a taut foliation, there is a simple closed curve in  $M$  transverse to  $\mathcal{F}$  that intersects all leaves of  $\mathcal{F}$ . This condition implies that, there is a positively oriented path in  $L(\mathcal{F})$  that starts at  $t$ , ends at certain point in the orbit of  $t$  (under the  $\pi_1$ -action on  $L(\mathcal{F})$ ), such that its

interior contains certain point  $s'$  in the orbit of  $s$  (under the  $\pi_1$ -action on  $L(\mathcal{F})$ ). Let  $g \in G$  with  $s' = g(s)$ . We have  $g(s) = s' \stackrel{L(\mathcal{F})}{>} t \stackrel{L(\mathcal{F})}{>} s$ . Since  $\mathcal{F}$  is co-orientable,  $g : L(\mathcal{F}) \rightarrow L(\mathcal{F})$  is an orientation-preserving homeomorphism, which implies that  $g(a) \stackrel{L(\mathcal{F})}{<} g(b)$  for any  $a, b \in L(\mathcal{F})$  with  $a \stackrel{L(\mathcal{F})}{<} b$ . So,  $s \stackrel{L(\mathcal{F})}{<} t \stackrel{L(\mathcal{F})}{<} g(s) \stackrel{L(\mathcal{F})}{<} g^2(s) \stackrel{L(\mathcal{F})}{<} g^3(s) \stackrel{L(\mathcal{F})}{<} \dots$ . Let  $n \in \mathbb{N}$ . Then, we have the following.

- (1) Since  $[r_2] \notin E_+(t)$  and  $g^n(s) \stackrel{L(\mathcal{F})}{>} t$ ,  $[r_2] \notin E_+(g^n(s))$ .
- (2) For simplicity, we denote by  $g^n([r_2])$  the image of  $[r_2]$  under the endomorphism on  $\text{End}(L(\mathcal{F}))$  induced from  $g^n : L(\mathcal{F}) \rightarrow L(\mathcal{F})$ . Since  $g^n$  is orientation-preserving,  $g^n([r_2])$  is a positive end of  $L(\mathcal{F})$ . Because  $[r_2] \in E_+(s)$ , we have  $g^n([r_2]) \in E_+(g^n(s))$ .

It follows that  $[r_2], g^n([r_2])$  are distinct positive ends of  $L(\mathcal{F})$ . Consequently,  $[r_2], g([r_2]), g^2([r_2]), \dots$  are positive ends of  $L(\mathcal{F})$  that are distinct from each other. So,  $L(\mathcal{F})$  has infinitely many positive ends.

Now, assume that  $\mathcal{F}$  is not co-orientable. Then,  $M$  has a double cover  $M'$  such that  $\mathcal{F}$  pulls-back to a co-orientable taut foliation of  $M'$  (denoted by  $\mathcal{F}'$ ). Note that  $\tilde{M}$  is also a universal cover of  $M'$  and  $\tilde{\mathcal{F}}$  is still the pull-back foliation of  $\mathcal{F}'$  in  $\tilde{M}$ . Thus,  $L(\mathcal{F})$  is homeomorphic to  $L(\mathcal{F}')$ . We assign  $L(\mathcal{F}')$  an orientation induced from the orientation on  $L(\mathcal{F})$ . Similar to the above discussions, we can ensure that  $L(\mathcal{F}')$  has infinitely many positive ends. Thus,  $L(\mathcal{F})$  also has infinitely many positive ends. ■

Combined with Lemma 2.5, we get the following corollary.

**Corollary 2.8.** (a)  $\mathcal{F}$  has one-sided branching if and only if  $L(\mathcal{F})$  either has exactly one positive end and infinitely many negative ends or has exactly one negative end and infinitely many positive ends.

(b)  $\mathcal{F}$  has two-sided branching if and only if  $L(\mathcal{F})$  has infinitely many positive ends and infinitely many negative ends.

For taut foliations,  $\mathbb{R}$ -covered is well-studied, two-sided branching is the generic case, and one-sided branching has an intermediate role between them.

As noted in [5, Subsection 2.1], we have the following fact.

**Fact 2.9.** If  $\mathcal{F}$  has one-sided branching, then  $\mathcal{F}$  is co-orientable.

We refer the reader to [5, 8] for more properties of taut foliations with one-sided branching.

There are many important examples of taut foliations with one-sided branching. In [16], Meigniez provides infinitely many taut foliations with one-sided branching in hyperbolic 3-manifolds. In [5, Example 5.02] and [6, Example 4.43], Calegari describes the construction of taut foliations with one-sided branching in some closed

3-manifolds obtained from taking finite branched covers over certain knots in some closed 3-manifolds with  $\mathbb{R}$ -covered foliations.

**2.4. The blowing-up operations**

For a set  $C \subseteq \mathbb{R}$  consisting of countably many points, Denjoy [7] introduces the *blowing-up* operation for  $C$  in  $\mathbb{R}$ . As a result, each point of  $C$  is “replaced” by a new closed subinterval of  $\mathbb{R}$  through the blowing-up operation. Similarly, the blowing-up operation can be defined for a leaf of a taut foliation and for a countable set of points in a possibly non-Hausdorff 1-manifold. Below, we provide a brief description of these aspects along with some references.

Suppose that  $M$  admits a taut foliation  $\mathcal{F}$ , and let  $\lambda$  be a leaf of  $\mathcal{F}$ . We refer the reader to [6, Example 4.14] for the blowing-up operation for  $\lambda$  in  $\mathcal{F}$ . In this operation, the leaf  $\lambda$  is replaced by a product subspace  $\lambda \times I$  of  $M$  foliated with leaves  $\{\lambda \times \{t\} \mid t \in I\}$ . It is not hard to see that the resulting foliation is still taut.

Let  $\mathcal{F}_0$  denote the taut foliation obtained from blowing-up  $\lambda$  in  $\mathcal{F}$ . Let  $\tilde{\lambda}$  be a component of  $p^{-1}(\lambda)$ , which is also a leaf of  $\tilde{\mathcal{F}}$ . Let  $G = \pi_1(M)$ , and we may consider  $G$  as the group of deck transformations of  $\tilde{M}$ . Now, consider the foliation  $\tilde{\mathcal{F}}_0$  in  $\tilde{M}$ . For each  $g \in G$ , there is a product subspace  $g(\tilde{\lambda}) \times I$  of  $\tilde{M}$  foliated with leaves  $\{g(\tilde{\lambda}) \times \{t\} \mid t \in I\}$  of  $\tilde{\mathcal{F}}_0$ , which is also a component of  $p^{-1}(\lambda \times I)$ .

We refer the reader to [20, 9. Appendix: Denjoy blow-ups] for the blowing-up operation for a countable set of points in a possibly non-Hausdorff 1-manifold. Now, we consider every leaf of  $\tilde{\mathcal{F}}$  as a point of  $L(\mathcal{F})$  and consider every leaf of  $\tilde{\mathcal{F}}_0$  as a point of  $L(\mathcal{F}_0)$ . Then,  $L(\mathcal{F}_0)$  is obtained from blowing-up the countable set  $\{g(\tilde{\lambda}) \mid g \in G\}$  of  $L(\mathcal{F})$ , where every point  $g(\tilde{\lambda}) \in L(\mathcal{F})$  is replaced by the interval  $g(\tilde{\lambda}) \times I \subseteq L(\mathcal{F}_0)$ .

**2.5. The group  $\mathcal{G}_\infty$**

We recall the definition of the group  $\mathcal{G}_\infty$  below (see, for example, [15, Definition 1.3]).

**Definition 2.10.** Let  $\mathcal{G}_\infty = \text{Homeo}_+(\mathbb{R}) / \sim$ , where  $\sim$  is the equivalence relation on  $\text{Homeo}_+(\mathbb{R})$  defined by  $g \sim f$  if there is  $n \in \mathbb{R}$  such that the restrictions of  $g, f$  to  $[n, +\infty)$  are equal. Henceforth, for every  $g \in \text{Homeo}_+(\mathbb{R})$ , we will always denote by  $[g]$  the image of  $g$  under the quotient map  $\text{Homeo}_+(\mathbb{R}) \rightarrow \mathcal{G}_\infty$ . We assume that the multiplication on  $\text{Homeo}_+(\mathbb{R})$  is given by the left group action, i.e.,  $fg = f \circ g$ . Define  $[f] \cdot [g] = [fg] = [f \circ g]$  for all  $f, g \in \text{Homeo}_+(\mathbb{R})$ . This multiplication is well-defined on  $\mathcal{G}_\infty$  and makes  $\mathcal{G}_\infty$  a group.

Navas proves the following theorem, by applying the criterion that a group  $G$  is left orderable if and only if, for any finite subset  $\{g_1, \dots, g_n\}$  of  $G$  that does not

contain the identity, there is  $\varepsilon_i \in \{-1, 1\}$  ( $1 \leq i \leq n$ ) such that the semigroup generated by  $g_1^{\varepsilon_1}, \dots, g_n^{\varepsilon_n}$  does not contain the identity. For a proof, see [15, Proposition 2.2].

**Theorem 2.11** (Navas).  *$\mathcal{G}_\infty$  is left orderable.*

**Remark 2.12.** Recall that a countable group is left orderable if and only if it acts effectively on  $\mathbb{R}$  via orientation-preserving homeomorphism, which implies that every countable subgroup of  $\mathcal{G}_\infty$  is a subgroup of  $\text{Homeo}_+(\mathbb{R})$ . However, Mann [15] proves that the cardinality of  $\mathcal{G}_\infty$  is equal to the cardinality of  $\text{Homeo}_+(\mathbb{R})$ , but there exists no nontrivial homomorphism  $\mathcal{G}_\infty \rightarrow \text{Homeo}_+(\mathbb{R})$ . See Mann [15] for more information about  $\mathcal{G}_\infty$ .

### 3. The proof of the main theorem

#### 3.1. The group action on a 1-manifold with one-sided branching

Let  $L$  be an oriented, connected, simply connected 1-manifold which is second countable but possibly non-Hausdorff, and we assume that  $L$  has one-sided branching. Without loss of generality, we may assume that  $L$  has branching in the negative direction. Let  $G$  be a group acting on  $L$  via orientation-preserving homeomorphisms. Let  $e : \mathbb{R} \rightarrow L$  be a positively oriented proper embedding. Then,

$$\begin{aligned} [e_+] &= \text{the positive end of } L, \\ [e_-] &\in \{\text{negative ends of } L\}. \end{aligned}$$

We fix this proper embedding  $e$  throughout this subsection.

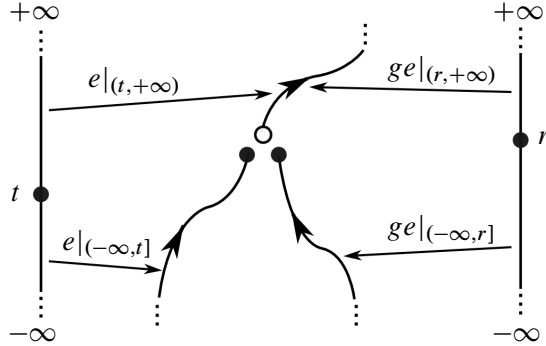
In this subsection, we prove the following proposition.

**Proposition 3.1.** *There is a homomorphism  $d : G \rightarrow \mathcal{G}_\infty$ , satisfying that  $d(h) \neq 1$  for all  $h \in G$  with the following property: for arbitrary  $n \in \mathbb{R}$ , there is  $m \in (n, +\infty)$  with  $h(e(m)) \neq e(m)$ .*

**Notation 3.2.** We assume that the action of  $G$  on  $L$  is a left group action, i.e., for arbitrary  $f, g \in G$ , we have  $fg(x) = f(g(x))$  for every  $x \in L$ . Henceforth, for arbitrary functions  $u : Y \rightarrow Z, v : X \rightarrow Y$ , we will always denote by  $uv$  the composition  $u \circ v$ . For example, given  $f, g \in G$ ,  $fge$  will always denote the function  $f \circ g \circ e : \mathbb{R} \xrightarrow{e} L \xrightarrow{g} L \xrightarrow{f} L$ .

Let  $g \in G$ . Since  $g : L \rightarrow L$  is an orientation-preserving homeomorphism,  $ge : \mathbb{R} \rightarrow L$  is also a positively oriented proper embedding. Thus,

$$[e_+] = [(ge)_+] = \text{the positive end of } L.$$



**Figure 2.** For  $g \in G$  with  $ge(\mathbb{R}) \neq e(\mathbb{R})$ , there are  $t, r \in \mathbb{R}$  such that the image of  $(t, +\infty)$  under  $e$  is same as the image of  $(r, +\infty)$  under  $ge$ , but  $e(t) \neq ge(r)$ .

We can observe that one of the following two possibilities happens:

- $e(\mathbb{R}) = ge(\mathbb{R})$ ,
- $e(\mathbb{R}) \neq ge(\mathbb{R})$ . Then, there are  $J = (t, +\infty), K = (r, +\infty)$  for some  $t, r \in \mathbb{R}$  such that

$$e(J) = ge(K) = e(\mathbb{R}) \cap ge(\mathbb{R}).$$

See Figure 2 for an example.

Thus, we have the following fact.

**Fact 3.3.** Let  $H = \{g_1, \dots, g_n\} \subseteq G$  be a finite subset of  $G$ . Let

$$\mu = \bigcap_{1 \leq i \leq n} g_i e(\mathbb{R}).$$

Then,  $\mu \neq \emptyset$ . Moreover, for each  $1 \leq i \leq n$ , there is  $J_i = [t_i, +\infty)$  for some  $t_i \in \mathbb{R}$  such that  $g_i e(J_i) \subseteq \mu$ .

**Definition 3.4.** Let  $g \in G$ , and let  $\mu = e(\mathbb{R}) \cap ge(\mathbb{R})$ . Then, we have the following.

- We fix a homeomorphism  $g_0 : e(\mathbb{R}) \rightarrow ge(\mathbb{R})$  such that the restriction of  $g_0$  to  $\mu$  is the identity.
- Let

$$g_* = e^{-1} g_0^{-1} g e : \mathbb{R} \xrightarrow{e} e(\mathbb{R}) \xrightarrow{g} ge(\mathbb{R}) \xrightarrow{g_0^{-1}} e(\mathbb{R}) \xrightarrow{e^{-1}} \mathbb{R}.$$

We define  $d(g) = [g_*] \in \mathcal{G}_\infty$ .

**Lemma 3.5.** For each  $g \in G$ ,  $d(g)$  is independent of the choice of  $g_0$ .

*Proof.* Let  $\mu = e(\mathbb{R}) \cap ge(\mathbb{R})$ . By Fact 3.3, there is  $J = [n, +\infty)$  for some  $n \in \mathbb{R}$  such that  $e(J), ge(J) \subseteq \mu$ . Then, the restriction of  $g_0^{-1}$  to  $ge(J)$  is the identity. Thus,

$$g_*|_J = e^{-1}g_0^{-1}(ge|_J) = e^{-1} \cdot 1 \cdot (ge|_J) = (e^{-1}ge)|_J.$$

So,  $g_*|_J$  is independent of the choice of  $g_0$ . ■

In the following, we prove that  $d : G \rightarrow \mathcal{G}_\infty$  is a homomorphism.

**Lemma 3.6.** *Let  $f, g \in G$ . Then,  $d(fg) = d(f)d(g)$ .*

*Proof.* We have

$$\begin{aligned} f_*g_* &= (e^{-1}f_0^{-1}fe)(e^{-1}g_0^{-1}ge) \\ &= e^{-1}f_0^{-1}fg_0^{-1}ge \end{aligned}$$

and

$$(fg)_* = e^{-1}((fg)_0)^{-1}fge.$$

Let  $\mu = e(\mathbb{R}) \cap fe(\mathbb{R}) \cap ge(\mathbb{R}) \cap fge(\mathbb{R})$ . By Fact 3.3, there is  $J = [t, +\infty)$  for some sufficiently large  $t \in \mathbb{R}$  such that

$$e(J), fe(J), ge(J), fge(J) \subseteq \mu.$$

Notice that the restriction of  $g_0^{-1}$  to  $ge(J)$  is the identity, since

$$ge(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap ge(\mathbb{R}).$$

And the restriction of  $f_0^{-1}$  to  $fge(J)$  is the identity, since

$$fge(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap fe(\mathbb{R}).$$

So,

$$\begin{aligned} (f_*g_*)|_J &= e^{-1}f_0^{-1}fg_0^{-1}(ge|_J) \\ &= e^{-1}f_0^{-1}f \cdot 1 \cdot (ge|_J) \\ &= e^{-1}f_0^{-1}(fge|_J) \\ &= e^{-1} \cdot 1 \cdot (fge|_J) \\ &= (e^{-1}fge)|_J. \end{aligned}$$

Also, the restriction of  $((fg)_0)^{-1}$  to  $fge(J) \subseteq \mu$  is the identity, since

$$fge(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap fge(\mathbb{R}).$$

So,

$$\begin{aligned} (fg)_*|_J &= e^{-1}((fg)_0)^{-1}(fge|_J) \\ &= e^{-1} \cdot 1 \cdot (fge|_J) \\ &= (e^{-1}fge)|_J. \end{aligned}$$

It follows that

$$(f_*g_*)|_J = (e^{-1}fge)|_J = (fg)_*|_J.$$

By Definition 2.10, we have

$$d(fg) = [(fg)_*] = [f_*][g_*] = d(f)d(g).$$

■

To complete the proof of Proposition 3.1, it remains to show the following lemma.

**Lemma 3.7.** *Let  $h \in G$ . Assume that for arbitrary  $n \in \mathbb{R}$ , there is  $m \in (n, +\infty)$  with  $he(m) \neq e(m)$ . Then,  $d(h) \neq 1$ .*

*Proof.* Let  $\mu = e(\mathbb{R}) \cap he(\mathbb{R})$ . By Fact 3.3, there is  $J = [t, +\infty)$  for some  $t \in \mathbb{R}$  such that  $e(J), he(J) \subseteq \mu$ . By Definition 3.4,  $d(h) = [e^{-1}h_0^{-1}he]$ . Let  $K = [r, +\infty)$  for some  $r \in (t, +\infty)$ . By our assumption, there is  $m \in K$  such that  $he(m) \neq e(m)$ . Since  $m \geq r > t$ ,  $he(m)$  is contained in  $\mu$ , and thus,  $h_0^{-1}he(m) = he(m)$ . Therefore,

$$e^{-1}h_0^{-1}he(m) = e^{-1}he(m) \neq e^{-1}(e(m)) = m.$$

So,  $e^{-1}h_0^{-1}he \sim id$  (where  $\sim$  denotes the equivalence relation defined in Definition 2.10). Therefore,  $d(h) \neq 1$ . ■

Proposition 3.1 now follows from Lemmas 3.6 and 3.7.

### 3.2. Proof of Theorem 1.1

Let  $M$  be a connected, orientable, irreducible 3-manifold. Suppose that  $M$  admits a co-oriented taut foliation  $\mathcal{F}$  which has one-sided branching. Let  $G = \pi_1(M)$ , and let  $p : \tilde{M} \rightarrow M$  be the universal covering of  $M$ .

In this subsection, we prove Theorem 1.1.

**Theorem 1.1.**  *$G$  is left orderable.*

Since  $\mathcal{F}$  has one-sided branching,  $L(\mathcal{F})$  is a non-Hausdorff 1-manifold with one-sided branching. We may assume that  $L(\mathcal{F})$  has an orientation induced from the co-orientation on  $\mathcal{F}$ , and that  $L(\mathcal{F})$  has branching in the negative direction. In the

following, for every  $g \in G$  and  $t \in L(\mathcal{F})$ ,  $g(t)$  will always denote the image of  $t$  under the transformation  $g : L(\mathcal{F}) \rightarrow L(\mathcal{F})$  given by the  $\pi_1$ -action on  $L(\mathcal{F})$ . And we will not distinguish the leaves of  $\tilde{\mathcal{F}}$  and the points in  $L(\mathcal{F})$ .

We first give a quick sketch of the proof of Theorem 1.1 in this paragraph. We blow-up some leaf  $\lambda$  of  $\mathcal{F}$  to obtain a new foliation  $\mathcal{F}_0$  with one-sided branching. Then, we construct an action  $\{\alpha_g : L(\mathcal{F}_0) \rightarrow L(\mathcal{F}_0) \mid g \in G\}$  of  $G$  on  $L(\mathcal{F}_0)$  such that some points in  $L(\mathcal{F}_0)$  have trivial stabilizer. Considering the action  $\{\alpha_g \mid g \in G\}$  on  $L(\mathcal{F}_0)$  and choosing some positively oriented proper embedding  $e : \mathbb{R} \rightarrow L(\mathcal{F}_0)$  as in Subsection 3.1, we can obtain an injective homomorphism  $G \rightarrow \mathcal{G}_\infty$  by Proposition 3.1.

Now, we give the details of the proof. Let  $\lambda$  be a leaf of  $\mathcal{F}$ , and let  $\tilde{\lambda}$  be a component of  $p^{-1}(\lambda)$ .

**Fact 3.8.** Let  $\rho : \mathbb{R} \rightarrow L$  be an arbitrary positively oriented proper embedding (then  $[\rho_+]$  is the positive end of  $L(\mathcal{F})$ ). Let  $J = (n, +\infty)$  for some  $n \in \mathbb{R}$ . Then, there is  $g \in G$  such that  $g(\tilde{\lambda}) \in \rho(J)$ .

*Proof.* Let  $\tilde{l} = \rho(m)$  for some  $m > n$ , and let  $l = p(\tilde{l})$ . Since  $\mathcal{F}$  is taut, there is a path  $\gamma$  in  $M$  such that  $\gamma$  starts and ends at the same point in the leaf  $l$ ,  $\gamma$  intersects every leaf of  $\mathcal{F}$ , and  $Int(\gamma)$  is a positively oriented transversal of  $\mathcal{F}$ . Let  $\tilde{\gamma}$  be a lift of  $\gamma$  in  $\tilde{M}$  that starts at some point in the leaf  $\tilde{l}$ , and we may consider  $\tilde{\gamma}$  as a path in  $L$  that starts at the point  $\tilde{l}$ . Notice that  $\tilde{\gamma}$  is positively oriented in  $L$  and  $L$  has branching in the negative direction, we have

$$\tilde{\gamma} \subseteq \rho([m, +\infty)) \subseteq \rho(J).$$

Since  $\gamma$  has nonempty intersection with  $\lambda$ , there is  $g \in G$  such that  $g(\tilde{\lambda}) \in \tilde{\gamma}$ . It follows that  $g(\tilde{\lambda}) \in \rho(J)$ . ■

We blow-up the leaf  $\lambda$  of  $\mathcal{F}$  to obtain a new foliation  $\mathcal{F}_0$  of  $M$ . Then,  $L(\mathcal{F}_0)$  is obtained from blowing-up  $\{g(\tilde{\lambda}) \mid g \in G\}$  in  $L(\mathcal{F})$ , and thus,  $L(\mathcal{F}_0)$  is still a non-Hausdorff 1-manifold with one-sided branching. We assume that  $L(\mathcal{F}_0)$  has an orientation induced from the orientation on  $L(\mathcal{F})$ . Now, for every  $g \in G$ , the point  $g(\tilde{\lambda}) \in L(\mathcal{F})$  is replaced by an interval  $g(\tilde{\lambda}) \times I$ .

Let  $K = \{g \in G \mid g(\tilde{\lambda}) = \tilde{\lambda}\}$ . By Novikov’s theorem (see, for example, [6, Theorem 4.35]),  $\lambda$  is incompressible in  $M$ , and thus, the restriction of  $p$  to  $\tilde{\lambda}$ ,  $p|_{\tilde{\lambda}} : \tilde{\lambda} \rightarrow \lambda$ , is a universal covering of  $\lambda$ . As  $K$  is the deck transformation group of the universal covering  $p|_{\tilde{\lambda}} : \tilde{\lambda} \rightarrow \lambda$ ,  $K$  is isomorphic to  $\pi_1(\lambda)$ . Since  $\lambda$  is an orientable surface,  $K$  is a countable left orderable group or a trivial group (when  $\lambda$  is a 2-plane). So, there is an action  $\phi : K \rightarrow \text{Homeo}_+(I)$  of  $K$  on  $I$  such that  $\phi(g)(\frac{1}{2}) \neq \frac{1}{2}$  for every  $g \in K - \{1\}$  (see [6, Lemma 2.43, Remark]). Here, we set  $\phi$  to be the trivial homomorphism when

$K$  is a trivial group. For each left coset  $gK$  ( $g \in G$ ) of  $K$  in  $G$ , we fix an element  $x_{gK} \in gK$ . And we set  $x_K = 1 \in K$ .

**Construction 3.9.** For each  $h \in G$ , we define a map  $\alpha_h : L(\mathcal{F}_0) \rightarrow L(\mathcal{F}_0)$  as follows.

- (1) Suppose that  $q \in L(\mathcal{F}_0) - \bigcup_{g \in G} (g(\tilde{\lambda}) \times I)$ . Then,  $q$  can be canonically identified with a point of  $L(\mathcal{F})$  (which we also denote by  $q$ ). We define  $\alpha_h(q) = h(q)$ .
- (2) Suppose that  $q \in \bigcup_{g \in G} (g(\tilde{\lambda}) \times I)$ . Then, there are  $g \in G, t \in I$  such that  $q = g(\tilde{\lambda}) \times \{t\}$ . We define

$$\alpha_h(g(\tilde{\lambda}) \times \{t\}) = hg(\tilde{\lambda}) \times \{\phi(x_{hgK}^{-1}hx_{gK})(t)\}.$$

Since  $x_{hgK}^{-1}hx_{gK} \in K$ , the map  $\alpha_h$  is well-defined. Notice that  $\alpha_h$  takes  $g(\tilde{\lambda}) \times I$  to  $hg(\tilde{\lambda}) \times I$  for every  $g \in G$ . So,  $\alpha_h$  is an orientation-preserving homeomorphism. Furthermore, we have the following lemma.

**Lemma 3.10.**  $\{\alpha_g : L(\mathcal{F}_0) \rightarrow L(\mathcal{F}_0) \mid g \in G\}$  is an action of  $G$  on  $L(\mathcal{F}_0)$ .

*Proof.* Let  $h, r \in G$ . Its clear that

$$\alpha_{hr}(q) = \alpha_h\alpha_r(q)$$

for every  $q \in L(\mathcal{F}_0) - \bigcup_{g \in G} (g(\tilde{\lambda}) \times I)$ . Now, we choose  $q \in \bigcup_{g \in G} (g(\tilde{\lambda}) \times I)$ . Let  $g \in G, t \in I$  for which  $q = g(\tilde{\lambda}) \times \{t\}$ . We have

$$\begin{aligned} \alpha_{hr}(q) &= \alpha_{hr}(g(\tilde{\lambda}) \times \{t\}) \\ &= hrg(\tilde{\lambda}) \times \{\phi(x_{hrgK}^{-1}hrx_{gK})(t)\} \end{aligned}$$

and

$$\begin{aligned} \alpha_h\alpha_r(q) &= \alpha_h\alpha_r(g(\tilde{\lambda}) \times \{t\}) \\ &= \alpha_h(rg(\tilde{\lambda}) \times \{\phi(x_{rgK}^{-1}rx_{gK})(t)\}) \\ &= hrg(\tilde{\lambda}) \times \{\phi(x_{hrgK}^{-1}hx_{rgK}x_{rgK}^{-1}rx_{gK})(t)\} \\ &= hrg(\tilde{\lambda}) \times \{\phi(x_{hrgK}^{-1}hrx_{gK})(t)\}. \end{aligned}$$

Thus,  $\alpha_{hr}(q) = \alpha_h\alpha_r(q)$ . Also, we have  $\alpha_1(q) = q$  for every  $q \in L(\mathcal{F}_0)$ . Therefore,  $\{\alpha_g : L(\mathcal{F}_0) \rightarrow L(\mathcal{F}_0) \mid g \in G\}$  is an action of  $G$  on  $L(\mathcal{F}_0)$ . ■

**Lemma 3.11.** The point  $\tilde{\lambda} \times \{\frac{1}{2}\}$  has trivial stabilizer under  $\{\alpha_g : L(\mathcal{F}_0) \rightarrow L(\mathcal{F}_0) \mid g \in G\}$ .

*Proof.* Let  $g \in G - \{1\}$ . We have

$$\alpha_g\left(\tilde{\lambda} \times \left\{\frac{1}{2}\right\}\right) = g(\tilde{\lambda}) \times \left\{\phi(x_{gK}^{-1}gx_K)\left(\frac{1}{2}\right)\right\} = g(\tilde{\lambda}) \times \left\{\phi(x_{gK}^{-1}g)\left(\frac{1}{2}\right)\right\}.$$

If  $g \notin K$ , then  $g(\tilde{\lambda}) \neq \tilde{\lambda}$ . If  $g \in K - \{1\}$ , then  $\phi(x_{gK}^{-1}g)\left(\frac{1}{2}\right) = \phi(g)\left(\frac{1}{2}\right) \neq \frac{1}{2}$ . So,  $\alpha_g(\tilde{\lambda} \times \{\frac{1}{2}\}) \neq \tilde{\lambda} \times \{\frac{1}{2}\}$  for every  $g \in G - \{1\}$ . ■

$$\text{Let } \tilde{\lambda}_0 = \tilde{\lambda} \times \left\{\frac{1}{2}\right\} \in L(\mathcal{F}_0).$$

**Definition 3.12.** Let  $e : \mathbb{R} \rightarrow L(\mathcal{F}_0)$  be an arbitrary positively oriented proper embedding.

**Lemma 3.13.** Let  $J = (n, +\infty)$  for some  $n \in \mathbb{R}$ . Then, there is  $h \in G$  such that  $\alpha_h(\tilde{\lambda}_0) \in e(J)$ .

*Proof.* This follows from Fact 3.8 and the fact that  $L(\mathcal{F}_0)$  is obtained from blowing-up  $\{g(\tilde{\lambda}) \mid g \in G\}$  in  $L(\mathcal{F})$  (and every interval  $g(\tilde{\lambda}) \times I$  ( $g \in G$ ) contains some images of  $\lambda_0$  under  $\{\alpha_h \mid h \in G\}$ ). ■

We have the following corollary.

**Corollary 3.14.** For every  $n \in \mathbb{R}$ , there is  $m \in (n, +\infty)$  such that  $\alpha_g e(m) \neq e(m)$  for all  $g \in G - \{1\}$ .

Now,  $(L(\mathcal{F}_0), \{\alpha_g \mid g \in G\}, e)$  can be considered as the triple  $(L, G, e)$  as given in Subsection 3.1. Let  $d : G \rightarrow \mathcal{G}_\infty$  be the homomorphism obtained by Proposition 3.1. Combined with Corollary 3.14, we have the following corollary.

**Corollary 3.15.** For every  $g \in G - \{1\}$ ,  $d(g) \neq 1$ .

So,  $d$  is injective, and thus,  $G$  is isomorphic to a subgroup of  $\mathcal{G}_\infty$ . By Theorem 2.11,  $\mathcal{G}_\infty$  is a left orderable group. This completes the proof of Theorem 1.1.

**Remark 3.16.** Since  $G$  is a countable left orderable group,  $G$  acts nontrivially on  $\mathbb{R}$ . However, our approach does not give such an action.

Theorem 2.11 only ensures that a left-invariant order of  $\mathcal{G}_\infty$  exists. There is no constructive proof for Theorem 2.11 as far as we know (see some remarks in [15, Section 1: Extension vs. realization]). So, we do not know a left-invariant order of  $\mathcal{G}_\infty$  (or its dynamic realization), and thus, the homomorphism  $d : G \rightarrow \mathcal{G}_\infty$  does not give a left-invariant order of  $G$  or a nontrivial action of  $G$  on  $\mathbb{R}$ .

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