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Boundary representations of mapping class groups

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Abstract. Let $S = S_g$ be a closed orientable surface of genus $g \geq 2$, and let $\mathrm{Mod}(S)$ be the mapping class group of S. In this paper, we initiate the study of the boundary representation of $\mathrm{Mod}(S)$ and show that this representation is ergodic using statistical hyperbolicity, thereby generalizing the classical result of Masur on the ergodicity of the action of $\mathrm{Mod}(S)$ on the projective measured foliation space $\mathcal{PMF}(S)$. As a corollary, we prove that the boundary representation of $\mathrm{Mod}(S)$ is irreducible.

1. Introduction

Let $S = S_g$ be a closed, connected, orientable surface of genus g. Recall that the mapping class group $\mathrm{Mod}(S)$ of S is defined as the group of isotopy classes of orientation-preserving homeomorphisms of S. Throughout this paper, we assume that the genus g is at least 2. (When $g = 2$, instead of using $\mathrm{Mod}(S)$, we will use $\widetilde{\mathrm{Mod}(S)} = \mathrm{Mod}(S)/Z(\mathrm{Mod}(S))$, the quotient of $\mathrm{Mod}(S)$ by its center, in the sequel. The main result holds for $\mathrm{Mod}(S)$ by considering the quotient map $\mathrm{Mod}(S) \to \widetilde{\mathrm{Mod}(S)}$.) *The space of measured foliations* $\mathcal{MF}(S)$ is the set of equivalence classes of nonzero measured foliations on S. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{MF}(S)$ and preserves a Radon measure ν_{Th}, called the *Thurston measure* on $\mathcal{MF}(S)$. Moreover, the space $\mathcal{MF}(S)$ is equipped with an $\mathbb{R}_+$-action that commutes with the $\mathrm{Mod}(S)$-action. Therefore, $\mathrm{Mod}(S)$ acts on the quotient $\mathcal{PMF}(S)$, called the *projective measured foliation space*, which is the quotient of $\mathcal{MF}(S)$ by $\mathbb{R}_+$, preserving the measure class $[\nu]$. The measure $\nu = \nu_o$, with respect to a hyperbolic structure o, is the probability measure induced by the Thurston measure ν_{Th} on $\mathcal{MF}(S)$. The reader is referred to Section 2.2 or to [1] for the construction of ν.

One motivation for this paper is to use geometric objects, such as $\mathcal{MF}(S)$ and $\mathcal{PMF}(S)$, to understand unitary representations of $\mathrm{Mod}(S)$ (see, for instance, [8, 28] for related topics).

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Main results. Recall that, for an action of a discrete group G on a Borel probability space (X, μ) that preserves the measure class $[\mu]$, the associated *quasi-regular representation* π_μ is defined as the unitary representation of G on $L^2(X, \mu)$. For a probability measure class-preserving ergodic action, a natural question is whether the quasi-regular representation is irreducible. Recall that a unitary representation is called *irreducible* if it has no nontrivial closed invariant subspaces. Note that this is not the case for a measure-preserving ergodic action, since such an action always has the space $\mathbb{C}\mathbb{1}_X$ of constant functions on X as a nontrivial closed invariant subspace. For the ergodic action of $\mathrm{Mod}(S)$ on $\mathcal{PMF}(S)$ with respect to ν, we will prove the following theorem.

Theorem 1.1 (See Corollary 2.1). *Let $S = S_g$ be a closed surface of genus $g \geq 2$. The quasi-regular unitary representation of the mapping class group* $\mathrm{Mod}(S)$ *on* $L^2(\mathcal{PMF}(S), \nu)$*, the space of square-integrable functions on* $\mathcal{PMF}(S)$ *with respect to ν, is irreducible.*

The quasi-regular unitary representation of the mapping class group $\mathrm{Mod}(S)$ on $L^2(\mathcal{PMF}(S), \nu)$ is called the *boundary representation* of $\mathrm{Mod}(S)$ in this paper. This theorem is a corollary of an ergodic-type theorem for quasi-regular representations, and it is the ergodic-type theorem itself that is much more interesting and has many corollaries.

Given an action of a discrete group G on a Borel probability space (X, μ) preserving the measure class $[\mu]$, denote the projection onto the subspace of constant functions on X by $P_{\mathbb{1}_X}$. The quasi-regular representation π_μ is said to be *ergodic with respect to* (K_n, e_n, f), where $K_n \subset G$ is a finite subset with $|K_n| \to \infty$, $e_n : K_n \to X$ is a map, and f is a bounded Borel function on X, if we have the following convergence in the weak operator topology, as $n \to \infty$:

$$\frac{1}{|K_n|} \sum_{g \in K_n} f(e_n(g)) \frac{\pi_\mu(g)}{\langle \pi_\mu(g)\mathbb{1}_X, \mathbb{1}_X \rangle} \to f P_{\mathbb{1}_X}.$$

The ergodicity of representations generalizes the ergodicity of group actions and implies the irreducibility of the associated unitary representations [6, Proposition 2.5]. Our main result is as follows.

Theorem 1.2 (See Theorem 2.17). *There exists a sequence of finite subsets $\{E_n\}$ of* $\mathrm{Mod}(S)$ *and maps $\{e_n : E_n \to \mathcal{PMF}(S)\}$ such that, for every bounded Borel function f, the quasi-regular representation π_ν, with respect to the Thurston measure ν, is ergodic with respect to (E_n, e_n, f).*

Thus, we obtain a generalization of Masur's classical result on the ergodicity of the action $\mathrm{Mod}(S) \curvearrowright \mathcal{PMF}(S)$ [21]. One of the key steps in our proof is the following result concerning matrix coefficients of representations, which may be of independent interest.

Theorem 1.3 (See Theorem 4.1). *There exists a sequence of finite subsets* $\{E_n\}$ *of* $\mathrm{Mod}(S)$ *for* $n \gg 0$ *with exponential growth, and constants* $a_1 > 0, a_2 > 0, b_1, b_2$ *such that for every* $g \in E_n$*,*

$$(a_1 n + b_1)e^{-(3g-3)n} \leq \langle \pi_\nu(g)\mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\mathcal{PMF}(S)}\rangle \leq (a_2 n + b_2)e^{-(3g-3)n}.$$

This theorem should be compared with [5, Proposition 3.2] for a related result. Finally, we remark that for this quasi-regular representation of $\mathrm{Mod}(S)$, we can additionally show that it is *tempered*, meaning it is weakly contained in the regular representation $\ell^2(\mathrm{Mod}(S))$ of $\mathrm{Mod}(S)$ (see Proposition 2.2).

Historical remarks. The main theorem is related to a question raised by Bader and Muchnik in the context of random walks on groups. Namely, let G be a discrete group and μ a probability measure on G. Let $(\partial G, \nu)$ be the Poisson boundary of G associated with the μ-random walk on G. Then, the measure class $[\nu]$ is G-invariant and, thus, defines a quasi-regular representation of G on $L^2(\partial G, \nu)$. In [2], inspired by the cases of free groups and lattices in Lie groups, Bader and Muchnik proposed the following conjecture.

Conjecture 1.4 ([2]). For a locally compact group G and a spread-out probability measure μ on G, the quasi-regular representation associated with the μ-Poisson boundary of G is irreducible.

We now briefly mention some progress toward this conjecture. As noted earlier, this conjecture holds for certain random walks on free groups and lattices in Lie groups (see [2] and references therein). Hence, it is true for the mapping class group $\mathrm{Mod}(S) = \mathrm{SL}(2, \mathbb{Z})$ of the closed surface of genus one, acting on $\mathcal{PMF}(S) = S^1$, with respect to the Lebesgue measure. This Lebesgue measure is identified with ν on $\mathcal{PMF}(S)$. Notice that all identifications are $\mathrm{Mod}(S)$-equivariant. We remark that for lattices in Lie groups, the irreducibility can also be deduced from the ergodicity of the associated quasi-regular representation (see [6]). The conjecture is verified in [2] for the fundamental groups of compact negatively curved manifolds, with respect to the Patterson–Sullivan measure, by Bader–Muchnik. Their result has been further generalized to non-elementary hyperbolic groups [11] with respect to the Patterson–Sullivan measure by Garncarek, and some discrete subgroups of the group of isometries of a CAT(-1) space with non-arithmetic spectrum by Boyer [5]. Note that in all the cases above, the Patterson–Sullivan measure on the Gromov boundary coincides with the Poisson boundary of (G, μ) for some probability measure μ on G [7]. However, Björklund–Hartman–Oppelmayer recently showed that there are random walks on some Lamplighter groups and solvable Baumslag–Solitar groups that provide counterexamples to this conjecture [4].

The relationship between the main theorem and the above progress is as follows. On the one hand, there is a long history of exploiting the similarities between mapping class groups and hyperbolic groups, which has proven to be quite fruitful. To name just a few among the vast literature, we mention [13,23,24]. On the other hand, by [1], the Thurston measure on $\mathcal{PMF}(S)$ is the Patterson–Sullivan measure on the Teichmüller boundary (more precisely, Gardiner–Masur boundary, though this is irrelevant for our purposes) of the Teichmüller space of S, which is in a similar situation to previously known cases.

Comments on the proofs. The proof of Theorem 1.2 highlights hyperbolic features of Teichmüller spaces. We make extensive use of the concept of statistical hyperbolicity, as defined by Dowdall–Duchin–Masur [9]. The proof consists of three main steps.

The first step is to construct the desired finite subsets of $\mathrm{Mod}(S)$. This is achieved by carefully selecting elements in $\mathrm{Mod}(S)$ with sufficient hyperbolicity so that the cardinality of these subsets goes to infinity (in fact, we require the growth to be exponential). The subsets are described in the paragraph preceding Lemma 2.12, based on the framework proposed in [9].

The second step is the core of this paper: the Harish-Chandra estimates (Theorem 4.1). Unlike the approaches in [2, 5], we derive the Harish-Chandra estimates using Teichmüller theory, particularly through extremal length functions and intersection number functions. The key idea is that, instead of directly performing estimations, we first relate the problem to finding the integrals of intersection number functions. We then use the map introduced by Masur–Minsky [23], which connects $\mathrm{Teich}(S)$ to the pants curves of S, to simplify these integrations. This approach is one of the novelties of this paper and directly yields condition (3) in Theorem 2.14.

The final step is to establish uniform boundness for the operators. We again leverage statistical hyperbolicity here. Since we do not have a metric structure on $\mathcal{PMF}(S)$ with required regularities, we utilize intersection number functions once more. Unlike Boyer [5], we complete the proof by counting lattice points in balls. Finally, we note that the idea of using intersection number functions as a type of conformal metric has already been employed by Rees in [31].

2. Quasi-regular unitary representations

2.1. Quasi-regular representations of discrete groups

In this section, we recall the concept of ergodic quasi-regular representations and present a criterion for proving ergodicity of representations. The reader is referred to [2,5,6] for more details.

Quasi-regular unitary representations. Let G be a locally compact second-countable group and X be a second-countable Hausdorff topological space. Let ν be a Borel probability measure on X. Assume that G acts on X as homeomorphisms and G preserves the measure class $[\nu]$ of ν, i.e., G preserves the null sets of ν. For every $\gamma \in G$, the measure $\gamma_*\nu$ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to $\gamma_*\nu$. Denote the corresponding Radon–Nikodym derivative by $c(\gamma,\nu) = \frac{d\gamma_*\nu}{d\nu}$. A unitary representation π_ν of G on $L^2(X,\nu)$ can be constructed as follows. For every $f \in L^2(X,\nu)$, every $x \in X$, and every $\gamma \in G$, $\pi_\nu(\gamma)f(x)$ is defined by $\pi_\nu(\gamma)f(x) = f(\gamma^{-1}x)c(\gamma,\nu)^{\frac{1}{2}}(x)$. This representation π_ν is called a *quasi-regular (unitary) representation* of G. We note that if ν and μ are in the same measure class, then π_ν and π_μ are unitarily equivalent. Assume that $c(\gamma,\nu)^{\frac{1}{2}}$ is integrable for each $\gamma \in G$ with respect to ν. The *Harish-Chandra function* Φ associated with π_ν is then defined as the integral

$$\Phi(\gamma) = \langle \pi_\nu(\gamma)\mathbb{1}_X, \mathbb{1}_X\rangle_{L^2(X,\nu)} = \int_X c(\gamma,\nu)^{\frac{1}{2}}(x)d\nu(x).$$

Ergodic quasi-regular representations. From now on, we assume that G is a discrete group. Let $(X,\nu), \pi_\nu$ be as above, and let $\mathcal{B}(L^2(X,\nu))$ denote the Banach space of bounded operators on $L^2(X,\nu)$. Let $e_K : K \to X$ be a map from a finite subset K of G to X and $f : X \to \mathbb{C}$ be a bounded Borel function. We use $|K|$ to denote the cardinality of K. Consider the following elements in $\mathcal{B}(L^2(X,\nu))$:

$$\begin{aligned} M^f_{(K,e_K)} &: L^2(X,\nu) \to L^2(X,\nu), \quad \phi \mapsto \frac{1}{|K|}\sum_{\gamma\in K} f(e_K(\gamma))\frac{\pi_\nu(\gamma)\phi}{\Phi(\gamma)}, \\ P_{\mathbb{1}_X} &: L^2(X,\nu) \to L^2(X,\nu), \quad \phi \mapsto \int_X \phi d\nu \mathbb{1}_X, \\ m(f) &: L^2(X,\nu) \to L^2(X,\nu), \quad \phi \mapsto f\phi. \end{aligned}$$

We now introduce a concept of ergodicity for quasi-regular representations, which generalizes the usual notion of ergodicity for measure class-preserving group actions. Recall that a sequence $F_n \in \mathcal{B}(L^2(X,\nu))$ *converges* to $F \in \mathcal{B}(L^2(X,\nu))$ as $n \to \infty$, written as $F_n \to F$, in the weak operator topology if, for every $\phi, \psi \in L^2(X,\nu)$, we have $\lim_{n\to\infty}\langle F_n(\phi),\psi\rangle_{L^2} = \langle F(\phi),\psi\rangle_{L^2}$.

Definition 2.1 ([6]). Let $G, (X,\nu), \pi_\nu, f$ be as above. Suppose that for every $n \in \mathbb{N}$, there is a pair $(K_n, e_n : K_n \to X)$ such that K_n is a finite subset of G, and $|K_n| \to \infty$ as $n \to \infty$. The representation π_ν is said to be ergodic with respect to (K_n, e_n) and f if we have the following convergence in the weak operator topology:

$$M^f_{(K_n,e_n)} \to m(f)P_{\mathbb{1}_X}.$$

Remark 2.2. It is easy to see that the ergodicity of a measure class-preserving group action is a weaker property than that of the associated quasi-regular representation. The proof can be found in [6, Proposition 2.5].

The following criterion for the ergodicity of a quasi-regular representation is essentially contained in [2] and summarized in [6]. For the definition of length functions on groups, we refer to [6]. The relevant fact here is that if a finitely generated group G acts isometrically and properly on a metric space Y, and if $o \in Y$ has a trivial stabilizer, then the map $g \mapsto d(g \cdot o, o)$ defines a length function on G.

Theorem 2.3 ([6, Theorem 2.2]). *Let G, (X, ν) be as above, and let π_ν be the associated quasi-regular representation of G on $L^2(X, \nu)$. Let L be a length function on G, and let (X, d) be a metric space that induces the topology of X. For each $n \in \mathbb{N}$, let E_n be a finite symmetric subset of G, that is, $E_n = E_n^{-1}$, and let $e_n : E_n \to X$ be a map. Assume that the following conditions hold:*

(1) *for every $g \in G$, $\|\pi_\nu(g)\mathbb{1}_X\|_{L^\infty(X,\nu)} < \infty$;*

(2) $\lim_{n\to\infty} |E_n| = \infty$*;*

(3) *for all Borel subsets $W, V \subset X$ such that $\nu(\partial W) = \nu(\partial V) = 0$,*
$$\limsup_{n\to\infty} \frac{1}{|E_n|} \left|\left\{\gamma \in E_n : e_n(\gamma^{-1}) \in W \text{ and } e_n(\gamma) \in V\right\}\right| \le \nu(W)\nu(V);$$

(4) *for every $r \ge 0$, there is a non-increasing function $h_r : [0, \infty) \to [0, \infty)$ such that*
$$\lim_{s\to\infty} h_r(s) = 0,$$
and
$$\forall n \in \mathbb{N}, \forall \gamma \in E_n, \qquad \frac{\langle \pi_\nu(\gamma)\mathbb{1}_X, \mathbb{1}_{\{x\in X: d(x,e_n(\gamma))\ge r\}}\rangle_{L^2}}{\Phi(\gamma)} \le h_r(L(\gamma));$$

(5) *we have*
$$\sup_n \|M_{E_n}^{\mathbb{1}_X} \mathbb{1}_X\|_{L^\infty(X,\nu)} < \infty.$$

Then, the quasi-regular representation π_ν is ergodic with respect to (E_n, e_n) and any $f \in \overline{H}^{L^\infty(X,\nu)}$, where H is the vector space generated by
$$\left\{\mathbb{1}_U : \nu(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } X\right\}.$$

Remark 2.4. Thanks to condition (1), the Harish-Chandra function is defined for every element γ in G.

Proposition 2.1 ([2]). *Under the assumptions in the above theorem, if ν is additionally a Radon measure, then π_ν is irreducible.*

2.2. Boundary representations of mapping class groups

We now consider quasi-regular representations of mapping class groups and, after the necessary preparations, state in detail the main theorem that we aim to prove. For further details on mapping class groups and Teichmüller theory, we refer the reader to [1, 10, 15].

Mapping class groups and Teichmüller spaces. Let $S = S_g$ be a closed, connected, orientable surface of genus g, where we always assume $g \geq 2$. The arguments here also apply to punctured hyperbolic surfaces with essential simple closed curves. The *mapping class group* $\operatorname{Mod}(S)$ of S is the group of isotopy classes of orientation-preserving homeomorphisms of S. That is, if the group of orientation-preserving homeomorphisms of S is denoted by $\operatorname{Homeo}^+(S)$ and the group of homeomorphisms of S that are isotopic to the identity is denoted by $\operatorname{Homeo}_0(S)$, then

$$\operatorname{Mod}(S) = \operatorname{Homeo}^+(S)/\operatorname{Homeo}_0(S).$$

We remark that mapping class groups of surfaces are finitely presented and are considered to be discrete groups. The *Teichmüller space* $\operatorname{Teich}(S)$ of S is the space of homotopy classes of hyperbolic structures. The Teichmüller space $\operatorname{Teich}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$, and the mapping class group $\operatorname{Mod}(S)$ acts on $\operatorname{Teich}(S)$. The quotient $\mathcal{M}(S) = \operatorname{Teich}(S)/\operatorname{Mod}(S)$ is the *moduli space* of S. Note that neither the Teichmüller space $\operatorname{Teich}(S)$ nor $\mathcal{M}(S)$ is compact.

There are several distance functions on $\operatorname{Teich}(S)$ under which $\operatorname{Mod}(S)$ acts as isometries, and the one we will use is the *Teichmüller distance* d_T. It is defined as follows: for $\mathcal{X} = [(X, \phi)]$, $\mathcal{Y} = [(Y, \psi)] \in \operatorname{Teich}(S)$, the distance $d_T(\mathcal{X}, \mathcal{Y})$ is given by $d_T(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \log K_f$, where $f : X \to Y$ is the Teichmüller mapping, locally of the form $x + iy \mapsto e^t x + i e^{-t} y$, in the isotopy class of $\psi \circ \phi^{-1}$, i.e., the quasi-conformal homeomorphism with minimal dilatation in the isotopy class of $\psi \circ \phi^{-1}$, and K_f is the dilatation of f. It is evident that $\operatorname{Mod}(S) \subset \operatorname{Isom}(\operatorname{Teich}(S), d_T)$.

Measured foliations. The Teichmüller space can be compactified in several ways. The compactification used in this paper is the Teichmüller compactification. Fix a point $o \in \operatorname{Teich}(S)$, which is represented as a Riemann surface X via uniformization. A *holomorphic quadratic differential* $q \in H^0(X, \Omega_X^{\otimes 2})$ on X is locally of the form $q(z)dz^2$, where $q(z)$ is a holomorphic function. Define a *norm* on q by

$$\|q\| = \int_X |q(z)| dx dy,$$

and consider the unit open ball $B^1(X)$ with respect to this norm. The set $\operatorname{QD}(X)$ of holomorphic quadratic differentials forms a vector space and can be identified with the cotangent space of $\operatorname{Teich}(S)$ at o. There is a homeomorphism $\pi : B^1(X) \to \operatorname{Teich}(S)$, which maps each open unit ray in $\mathrm{QD}(X)$ starting at the origin to a Teichmüller geodesic ray starting at o. The *Teichmüller compactification* is then the visual compactification obtained by adding the endpoints in the unit sphere of $\mathrm{QD}(X)$ to each ray. The Teichmüller compactification is denoted by $\overline{\mathrm{Teich}(S)}$, and its boundary $\partial\overline{\mathrm{Teich}(S)}$ is identified with the unit sphere $\mathrm{QD}^1(X)$.

One could give a geometric description of $\partial\overline{\mathrm{Teich}(S)}$ via projective measured foliations. A *measured foliation* on S is a singular foliation of S endowed with a transverse measure. The space $\mathcal{MF}(S)$ of nonzero measured foliations is the set of equivalent classes of nonzero measured foliations, where the equivalence is defined by Whitehead moves and isotopy. The space $\mathcal{MF}(S)$, endowed with the weak topology on measures, is homeomorphic to $\mathbb{R}^{6g-6} \setminus \{0\}$. The quotient of $\mathcal{MF}(S)$ by the natural action of $\mathbb{R}_+$, called the *projective measured foliation* space $\mathcal{PMF}(S)$ of S, is homeomorphic to the $6g-7$-sphere S^{6g-7}. Both $\mathcal{MF}(S)$ and $\mathcal{PMF}(S)$ carry a $\mathrm{Mod}(S)$-action.

There is a deep relation between $\mathcal{MF}(S)$ and $\mathrm{QD}(X)$. Specifically, for each holomorphic quadratic differential q, the *vertical measured foliation* $\mathcal{V}(q)$ associated with $q = q(z)dz^2$ is the foliation formed by the integral curves of the holomorphic tangent vector field on S, where each tangent vector has a value in negative real numbers under q. The transverse measure of $\mathcal{V}(q)$ is given by the integration of $|\mathrm{Re}\sqrt{q}|$. By a theorem of Hubbard and Masur, the map $\mathcal{V}$ that assigns to each nonzero holomorphic quadratic differential q on X its vertical measured foliation $\mathcal{V}(q)$ is a homeomorphism from $\mathrm{QD}(X) \setminus \{0\}$ onto $\mathcal{MF}(S)$.

The composition $\pi \circ \mathcal{V}$, where $\mathcal{V} : \mathrm{QD}(X) \setminus \{0\} \to \mathcal{MF}(S)$ is the map and $\pi : \mathcal{MF}(S) \to \mathcal{PMF}(S)$ is the quotient map, provides an identification of $\mathrm{QD}^1(X)$ with $\mathcal{PMF}(S)$. Thus, we will regard $\mathcal{PMF}(S)$ as the boundary of the Teichmüller compactification of $\mathrm{Teich}(S)$.

The equivalent class of $\xi \in \mathcal{MF}(S)$ in $\mathcal{PMF}(S)$ will be denoted by $[\xi]$. Any $q \in \mathrm{QD}^1(X)$ (and hence $[\mathcal{V}(q)] \in \mathcal{PMF}(S)$) determines a Teichmüller geodesic ray g_t starting from o, therefore, by abuse of terminology, we will refer to q and $[\mathcal{V}(q)]$ the *direction* of g_t, and sometimes write g_t as g_t^q or $\mathcal{V}(q)(t)$.

Any isotopy class α of essential simple closed curves on S defines a measured foliation λ_α. For $c \in \mathbb{R}_+$, the c-weighted isotopy class $c\alpha$ defines a measured foliation $\lambda_{c\alpha} \in \mathcal{MF}(S)$. As topological foliations, $\lambda_{c\alpha}$ is identical to λ_α, but its transverse measure is scaled by c. Let $\mathcal{C}(S)$ denote the set of isotopy classes of essential simple closed curves. There is an embedding of $\mathcal{C}(S) \times \mathbb{R}_+$ into $\mathcal{MF}(S)$ and the image is dense [33]. This embedding enables us to define three types of functions that will be useful in our analysis: intersection number functions, extremal length functions, and hyperbolic length functions. We will now define each of them.

The *intersection number function* $i : \mathcal{MF}(S) \times \mathcal{MF}(S) \to \mathbb{R}_+$ is the unique continuous function on $\mathcal{MF}(S) \times \mathcal{MF}(S)$ that extends the geometric intersection number of two essential simple closed curves. It satisfies the property that for any $c > 0$, $i(c\lambda, \xi) = ci(\lambda, \xi)$ for all $\lambda, \xi \in \mathcal{MF}(S)$ (see [31, Corollary 1.11]).

Let $o = [X] \in \mathrm{Teich}(S)$, where X is a Riemann surface. For an isotopy class γ of essential simple closed curves, the *extremal length* $\mathrm{Ext}_X(\gamma)$ of γ in X is defined by

$$\mathrm{Ext}_X(\gamma) = \sup_\rho \ell_\rho(\gamma)^2,$$

where ρ runs over all metrics with unit area in the conformal class of X, and $\ell_\rho(\gamma)$ denotes the infimum of the ρ-lengths of simple closed curves in γ. The extremal length $\mathrm{Ext}_X : \mathcal{MF}(S) \to \mathbb{R}_+$ is the unique continuous function on $\mathcal{MF}(S)$ that extends the extremal length of $\mathcal{C}(S)$ and satisfies

$$\mathrm{Ext}_X(c\lambda) = c^2\,\mathrm{Ext}_X(\lambda)$$

for $c \in \mathbb{R}_+$ (see [15, Proposition 3]). Note that the extremal length function is defined on $\mathrm{Teich}(S) \times \mathcal{MF}(S)$; that is, if $[X] = [Y] \in \mathrm{Teich}(S)$, then $\mathrm{Ext}_X(\cdot) = \mathrm{Ext}_Y(\cdot)$. Therefore, we will write $\mathrm{Ext}_o(\cdot)$ instead of $\mathrm{Ext}_X(\cdot)$ for $o = [X]$.

The *hyperbolic length* $\ell_o(\gamma)$, for $o = [X]$, is defined as the X-length of the unique X-hyperbolic geodesic $\widetilde{\gamma}$ in the isotopy class γ. The function $\ell_o(\cdot)$ can also be uniquely extended to $\mathcal{MF}(S)$ to obtain a continuous function ℓ_o on $\mathcal{MF}(S)$ [17].

We will use the following relation (see [1]): given a point $o \in \mathrm{Teich}(S)$, there exists a constant $C = C(o)$, depending on o such that for all $\xi \in \mathcal{MF}(S)$, we have

$$\frac{1}{C}\ell_o(\xi) \le \sqrt{\mathrm{Ext}_o(\xi)} \le C\ell_o(\xi).$$

Recall that a measured foliation λ is called *minimal* if it has no simple closed leaves. Two measured foliations are said to be *topologically equivalent* if, as topological foliations, they differ by isotopies and Whitehead moves. A measured foliation ξ is called *uniquely ergodic* if it is minimal and any measured foliation ζ that is topologically equivalent to ξ is measure equivalent to ξ, i.e., $[\xi] = [\zeta]$. When ξ is uniquely ergodic, we will refer to $[\xi]$ as (a) uniquely ergodic (point).

It is well known that the subset of uniquely ergodic measured foliations in $\mathcal{PMF}(S)$ has full measure with respect to the Thurston measure, which will be defined later. The following two lemmas are essential to our approach using intersection number functions.

Lemma 2.5 ([31, Theorem 1.12] and [20]). *Let λ be a uniquely ergodic measured foliation and η any measured foliation. Then, $i(\lambda, \eta) = 0$ if and only if $[\lambda] = [\eta]$.*

Lemma 2.6 (Masur's criterion [22]). *Consider $\varepsilon > 0$. If a Teichmüller geodesic ray g_t starting from o does not leave $\mathrm{Teich}_\varepsilon(S)$ eventually, then the direction of g_t is uniquely ergodic.*

One feature of the Teichmüller compactification is that the action of $\mathrm{Mod}(S)$ cannot be extended continuously to $\overline{\mathrm{Teich}(S)}$ [15]. However, uniquely ergodic measured foliations are "nice" points with respect to the $\mathrm{Mod}(S)$-action in the following sense.

Lemma 2.7 ([20]). *The mapping class group acts continuously on the Teichmüller compactification* $\overline{\mathrm{Teich}(S)}$ *at uniquely ergodic points on the boundary.*

The following formula, proved by Kerckhoff, for calculating Teichmüller distances will be used frequently.

Lemma 2.8 (See [15], Kerckhoff's formula). *We have*

$$\forall x, y \in \mathrm{Teich}(S), \quad d_T(x,y) = \frac{1}{2} \sup_{[\xi]\in\mathcal{PMF}(S)} \ln\left(\frac{\mathrm{Ext}_x(\xi)}{\mathrm{Ext}_y(\xi)}\right).$$

Hyperbolicity. It was first proved in [24] that the Teichmüller space $(\mathrm{Teich}(S), d_T)$ is not hyperbolic in the sense of Gromov. However, some triangles in $(\mathrm{Teich}(S), d_T)$ are thin. We now present two related results that will be useful for comparing neighborhoods in $\mathcal{PMF}(S)$, defined by the radial projections of balls in $\mathrm{Teich}(S)$, with those defined by intersection numbers.

Recall that, for $\varepsilon > 0$, the *ε-thick part* $\mathrm{Teich}_\varepsilon(S)$ of the Teichmüller space $\mathrm{Teich}(S)$ is defined as

$$\mathrm{Teich}_\varepsilon(S) = \{y \in \mathrm{Teich}(S) : \forall c \in \mathcal{C}(S), \mathrm{Ext}_y(c) \geq \varepsilon\}.$$

The following result, which generalizes a theorem of Rafi [30], characterizes when certain triangles are thin. For a subset A of $\mathrm{Teich}(S)$, by $\mathcal{N}_D(A)$ we denote the D-neighborhood of A. Recall that *a geodesic segment $I : [a,b] \to$* $\mathrm{Teich}(S)$ *has at least proportion θ in* $\mathrm{Teich}_\varepsilon(S)$ if

$$\mathrm{Thk}^{\%}_{\varepsilon}[I] \doteq \frac{|\{a \leq s \leq b : I(s) \in \mathrm{Teich}_\varepsilon(S)\}|}{b-a} \geq \theta.$$

Theorem 2.9 ([9]). *Given $\varepsilon > 0$ and $0 < \theta \leq 1$, there exist constants $D = D(\varepsilon,\theta)$, $L_0 = L_0(\varepsilon,\theta)$ such that if $I \subset [x,y]$ is a geodesic subinterval in* $\mathrm{Teich}(S)$ *of length at least L_0, and at least proportion θ of I is in* $\mathrm{Teich}_\varepsilon(S)$*, then for every $z \in$* $\mathrm{Teich}(S)$ *we have*

$$I \cap \mathcal{N}_D([x,z] \cup [y,z]) \neq \emptyset.$$

The next result will also be used later. Recall that two parametrized geodesic segments $\delta(t)$ and $\delta'(t)$ defined on $[a,b]$ are said to *P-fellow travel* in a parametrized fashion for $P > 0$ if, for every $t \in [a,b]$,

$$d_T(\delta(t), \delta'(t)) \leq P.$$

Theorem 2.10 ([30]). *Let $\varepsilon > 0$ and $R > 0$. Then, there exists $P = P(\varepsilon, R) > 0$ such that, whenever $x_1, x_2, y_1, y_2 \in \mathrm{Teich}_\varepsilon(S)$ with*

$$d_T(x_1, x_2) \le R, d_T(y_1, y_2) \le R,$$

the geodesic segment $[x_1, y_1]$ and $[x_2, y_2]$ are P-fellow traveling.

Boundary representations of mapping class groups. We are in a position to discuss a special type of quasi-regular unitary representation of mapping class groups. Fixing $o \in \mathrm{Teich}(S)$, we define a Radon measure ν_o on $\mathcal{PMF}(S)$. Let ν_{Th} be the Thurston measure on $\mathcal{MF}(S)$. For any open subset $U \subset \mathcal{PMF}(S)$, we define $\nu_o(U)$ by

$$\nu_o(U) = \nu_{Th}(\{\xi : [\xi] \in U, \mathrm{Ext}_o(\xi) \le 1\}).$$

One can verify that for all $\gamma \in \mathrm{Mod}(S)$, $\gamma_* \nu_o = \nu_{\gamma . o}$, and for all $x, y \in \mathrm{Teich}(S)$, $[\nu_x] = [\nu_y]$. Therefore, for all $x, y \in \mathrm{Teich}(S)$ and $[\xi] \in \mathcal{PMF}(S)$, we have

$$\frac{d\nu_x}{d\nu_y}([\xi]) = \left(\frac{\mathrm{Ext}_y(\xi)}{\mathrm{Ext}_x(\xi)}\right)^{\frac{6g-6}{2}}.$$

By the definition of extremal length, the function $[\xi] \mapsto \left(\frac{\mathrm{Ext}_y(\xi)}{\mathrm{Ext}_x(\xi)}\right)^{\frac{6g-6}{2}}$ is well defined on $\mathcal{PMF}(S)$. In particular, we have

$$\frac{d\gamma_* \nu_o}{d\nu_o}([\xi]) = \left(\frac{\mathrm{Ext}_o(\xi)}{\mathrm{Ext}_{\gamma . o}(\xi)}\right)^{\frac{6g-6}{2}}$$

for all $\gamma \in \mathrm{Mod}(S)$ and $[\xi] \in \mathcal{PMF}(S)$.

Hence, there exists a quasi-regular unitary representation π_{ν_o} of $\mathrm{Mod}(S)$ on the Hilbert space $L^2(\mathcal{PMF}(S), \nu_o)$. The quasi-regular representation π_{ν_o} of $\mathrm{Mod}(S)$ is called the *boundary representation* of $\mathrm{Mod}(S)$ (with respect to o).

As intersection numbers will be the main tool, we embed $\mathcal{PMF}(S)$ into $\mathcal{MF}(S)$. For each $[\xi]$ define $\tau(\xi) \in \mathcal{MF}(S)$ to be unique element in $[\xi]$ such that $\mathrm{Ext}_o(\tau(\xi)) = 1$. Hence, the map $\tau : \mathcal{PMF}(S) \to \mathcal{MF}(S)$ is a section of the projection $\pi : \mathcal{MF}(S) \to \mathcal{PMF}(S)$. When discussing intersection numbers for two points in $\mathcal{PMF}(S)$, except in Section 4.4, we will use the image of τ.

Ergodic boundary representation. From now on, let $S = S_g$ be a closed, orientable surface of genus g ($g \ge 2$), and fix a point $o = [X] \in \mathrm{Teich}(S)$. Normalize ν_o to be a probability measure. Denote $h = 6g - 6$. Let $\varepsilon > 0$ and $\theta > 0$. Let L be the length function on G induced by the Teichmüller distance d_T, namely, $L(g) = d_T(o, g \cdot o)$.

Inspired by [9, 12], we first describe our choice of E_n, which fits into Theorem 2.14. Assume that g_t^q is a Teichmüller geodesic ray starting from o with direction

$q \in \mathrm{QD}^1(X)$. For every $m > 0$, define

$$\mathrm{Thk}_{\varepsilon}^{\%}[o, g_m^q] \doteq \frac{|\{0 \le s \le m : g_s^q \in \mathrm{Teich}_{\varepsilon}(S)\}|}{d_T(o, g_m^q)}.$$

Theorem 2.11 ([9, Proposition 5.5]). *For all* $0 < \theta < 1$*, there exists* $\varepsilon > 0$ *such that for all* $o = [X] \in \mathrm{Teich}(S)$*, we have*

$$\lim_{R_0 \to \infty} \nu_o\big(\{q \in \mathrm{QD}^1(X) : \mathrm{Thk}_{\varepsilon}^{\%}[o, g_m^q] \ge \theta, \forall m > R_0\}\big) = 1.$$

We fix any $\theta \in (0,1)$ and choose $\varepsilon > 0$ smaller than the one in Theorem 2.11. We identify $\mathrm{QD}^1(X)$ with $\mathcal{PMF}(S)$ and g_t^q with $\mathcal{V}(q)(t)$. For each $R > 0$, we define

$$U(R, \theta, \varepsilon) = \big\{\xi \in \mathcal{PMF}(S) : \mathrm{Thk}_{\varepsilon}^{\%}[o, \xi(m)] \ge \theta, \forall m > R\big\}.$$

Then, if $R_2 \ge R_1 > 0$, we have

$$U(R_1, \theta, \varepsilon) \subset U(R_2, \theta, \varepsilon).$$

Next, we define

$$U(\theta, \varepsilon) = \bigcup_{R > 0} U(R, \theta, \varepsilon),$$

then by Theorem 2.11, we have

$$\nu_o(U(\theta, \varepsilon)) = 1.$$

Furthermore, after a suitable choice of θ, one has $\nu_o(\partial(U(\theta, \varepsilon))) = 0$ and by Masur's criterion (Lemma 2.6), the set $U(\varepsilon, \theta)$ consists of uniquely ergodic directions.

Now, we fix the choice of ε and θ and, for $\gamma \in \mathrm{Mod}(S)$, denote the direction determined by the oriented geodesic $[o, \gamma \cdot o]$ by ξ_γ. We are in a position to describe E_n.

Fix $\rho > 0$ and let $L_0 = L_0(\theta, \varepsilon)$ be the constant as in Theorem 2.9. For $\frac{1}{3h} \ln \ln n > \max\{L_0, \rho\}$, define *the set* $\mathcal{E}(\theta, \varepsilon, n, o, \rho)$, which will be the desired E_n, to be the set of all elements γ in $\mathrm{Mod}(S)$ satisfying the following:

(a) $d_T(\gamma \cdot o, o) \in (n - \rho, n + \rho)$;

(b) both ξ_γ and $\xi_{\gamma^{-1}}$ are in $U(\theta, \varepsilon)$;

(c) if $g(t)$ is either the geodesic ray $\xi_\gamma(t)$ or $\xi_{\gamma^{-1}}(t)$, then the segment $[o, g(\frac{1}{3h} \ln \ln n)]$ has at least proportion θ in $\mathrm{Teich}_{\varepsilon}(S)$.

Lemma 2.12. *Let* n *be large enough. Then, for* $\gamma \in \mathcal{E}(\theta, \varepsilon, n, o, \rho)$*, there exists a geodesic segment* I_γ *of length* $\frac{1}{3h} \ln \ln n$ *in the geodesic* $[o, \gamma \cdot o]$ *that has at least proportion* θ *in* $\mathrm{Teich}_{\varepsilon}(S)$ *and contains* $\gamma \cdot o$*.*

Proof. Let $\gamma \in \mathcal{E}(\theta, \varepsilon, n, o, \rho)$. Since the geodesic ray $\xi_{\gamma^{-1}}(t)$ satisfies (c) in the definition of $\mathcal{E}(\theta, \varepsilon, n, o, \rho)$, the first segment $I_{\gamma^{-1}}$ of $[o, \gamma^{-1} \cdot o]$ of length $\frac{1}{3h} \ln \ln n$ has at least proportion θ in $\mathrm{Teich}_\varepsilon(S)$. As $\gamma \cdot [o, \gamma^{-1} \cdot o] = [\gamma \cdot o, o]$, the geodesic $[o, \gamma \cdot o]$ contains a subinterval $\gamma \cdot I_{\gamma^{-1}}$ of length $\frac{1}{3h} \ln \ln n$, which has at least proportion θ in $\mathrm{Teich}_\varepsilon(S)$ and contains the point $\gamma \cdot o$. ∎

In the next section, we will prove that $E_n = \mathcal{E}(\theta, \varepsilon, n, o, \rho)$ has exponential growth. We state an obvious property of the boundary representation.

Lemma 2.13. *Let π_ν be the boundary representation of* $\mathrm{Mod}(S)$*. For every $g \in$* $\mathrm{Mod}(S)$*,*

$$\|\pi_{\nu_o}(g)\mathbb{1}_{\mathcal{PMF}(S)}\|_{L^\infty(\mathcal{PMF}(S),\nu_o)} < \infty.$$

Proof. The lemma is an easy consequence of Kerckhoff's formula, namely Lemma 2.8, on calculating Teichmüller distances. By Lemma 2.8, one has

$$\forall x, y \in \mathrm{Teich}(S), \forall [\xi] \in \mathcal{PMF}(S), \quad \left(\frac{\mathrm{Ext}_x(\xi)}{\mathrm{Ext}_y(\xi)}\right)^{\frac{1}{2}} \le e^{d_T(x,y)}.$$

Since

$$\pi_{\nu_o}(g)\mathbb{1}_{\mathcal{PMF}(S)} = \left(\frac{\mathrm{Ext}_o(\xi)}{\mathrm{Ext}_{g\cdot o}(\xi)}\right)^{\frac{6g-6}{4}},$$

one has

$$\|\pi_{\nu_o}(g)\mathbb{1}_{\mathcal{PMF}(S)}\|_{L^\infty(\mathcal{PMF}(S),\nu_o)} \le e^{\frac{6g-6}{2} d_T(o, g\cdot o)} < \infty. \qquad ∎$$

The following theorem is a slight variant of Theorem 2.3, and its proof is similar to the original one. For any set $E_n \subset \{g \in \mathrm{Mod}(S) : d_T(o, g \cdot o) \in [n-\rho, n+\rho]\}$ with $n \gg \rho$, *the radial projection* $\mathrm{Pr} : E_n \to \mathcal{PMF}(S)$ from o is the map that assigns to each element $\gamma \in E_n$ the projective vertical foliation of the unit quadratic differential defined by the oriented geodesic $[o, \gamma \cdot o]$.

Theorem 2.14. *Let π_{ν_o} be the associated quasi-regular representation of* $\mathrm{Mod}(S)$ *on $L^2(\mathcal{PMF}(S), \nu_o)$. Let $i(\cdot,\cdot)$ be the intersection number function on $\mathcal{PMF}(S)$. Let $n \gg \rho$ and $E_n = E_n(\rho) \subset \{g \in \mathrm{Mod}(S) : d_T(o, g \cdot o) \in [n-\rho, n+\rho]\}$ be a symmetric subset. Let $e_n = \mathrm{Pr} : E_n \to \mathcal{PMF}(S)$ be the radial projection from o. Assume that the following conditions hold:*

(1) $\lim_{n\to\infty} |E_n| = \infty$*;*

(2) *for all Borel subsets $W, V \subset \mathcal{PMF}(S)$ such that $\nu_o(\partial W) = \nu_o(\partial V) = 0$,*

$$\limsup_{n\to\infty} \frac{1}{|E_n|}\left|\{\gamma \in E_n : e_n(\gamma^{-1}) \in W \text{ and } e_n(\gamma) \in V\}\right| \le \nu_o(W)\nu_o(V);$$

(3) *there are two sequences of reals* $\{h_{r_n}(n,\rho)\}$, $\{r_n\}$ *such that* $\lim_{n\to\infty} h_{r_n}(n,\rho) = \lim_{n\to\infty} r_n = 0$ *and such that* $\forall n \gg \rho, \forall \gamma \in E_n$,

$$\frac{\langle \pi_{\nu_o}(\gamma)\mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{\xi\in\mathcal{PMF}(S):\, i(\xi,e_n(\gamma))\geq r_n\}}\rangle}{\Phi(\gamma)} \leq h_{r_n}(n,\rho);$$

(4) *we have*

$$\sup_n \left\| M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)} \right\|_{L^\infty(\mathcal{PMF}(S),\nu_o)} < \infty.$$

Then, the quasi-regular representation π_{ν_o} *is ergodic with respect to* (E_n, e_n) *and any* $f \in \overline{H}^{L^\infty(\mathcal{PMF}(S),\nu_o)}$, *where* H *is the vector space generated by*

$$\left\{\mathbb{1}_U : \nu_o(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } \mathcal{PMF}(S)\right\}.$$

Remark 2.15. Since Theorem 2.14 is slightly different from Theorem 2.3, we should make a few comments. It is easy to observe that the only difference is condition (3), as condition (1) in Theorem 2.3 is automatically verified by Lemma 2.13. Condition (3) in Theorem 2.14 is slightly weaker than condition (4) in Theorem 2.3, as we do not require h_r to be independent of $L(g)$. The cost of this weaker assumption is that we require the additional condition that the elements in E_n have almost the same induced distance n from id $\in \mathrm{Mod}(S)$.

Proof. As the proof of [6, Theorem 2.2] works quite well in our situation with only some modifications, we will only point out the necessary adjustments here. The main issue is that the intersection number function $i(\cdot,\cdot)$ does not define a metric structure, nor does it even define the usual topological structure on $\mathcal{PMF}(S)$, since there exist points $\xi \neq \eta$ with $i(\eta,\xi) = 0$. However, there is a full ν_o-measure subset $\mathcal{UF} \subset \mathcal{PMF}(S)$ consisting of uniquely ergodic projective measured foliations such that, for every nice metric on $\mathcal{PMF}(S)$, e.g., the round metric, the induced topology coincides with the topology induced by the intersection number functions when restricted to $\mathcal{UF}$, according to Lemma 2.5.

The proof of [6, Theorem 2.2, page 2037] relies on [6, Lemma 2.20] and [6, Proposition 2.21]. We will point out modifications to these two results, and the rest of the proof follows as in [6, Theorem 2.2].

The proof of [6, Theorem 2.2, page 2037] is ultimately based on [6, Lemma 2.19], and our argument follows the same structure. The following are our assumptions. The set $B = \mathcal{PMF}(S)$ is equipped with any metric d which is compatible with the quotient topology. When we refer to Borel subsets U, W, we mean compact subsets with null ν_o-zero boundaries. These assumptions are reasonable since we can take closures in all statements in [6, Lemma 2.19]. Note that, although we are working with metric measure spaces, metrics are only used in the following way: for subsets U, W, $d(U,W) > 0$ iff $\overline{U} \cap \overline{W} = \emptyset$.

We now define modified open sets $W(r)$ for $r > 0$ as follows:

$$\begin{aligned} W(r) &= \{\eta \in \mathcal{PMF}(S) : \exists \xi \in \mathcal{UF} \cap W, i(\eta, \xi) < r\} \\ &= \bigcup_{\xi \in \mathcal{UF} \cap W} \{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) < r\}. \end{aligned}$$

For the arguments in [6, Lemma 2.20], we modify the reasoning as follows: we replace $W(r)$ with $W(r_n)$, where r_n is the sequence given in condition (3), and then follow the proof therein, taking the limit $\lim_{n\to\infty} h_{\tau_n}(n, \rho)$, instead of $\lim_{s\to\infty} h_r(s)$ as in the original proof.

For the arguments in [6, Proposition 2.21], we modify the reasoning as follows: first, we replace $W(r)$ and $V(r)$ with $W(r_n)$ and $V(r_n)$, respectively, in the proof of [6, Proposition 2.21, page 2037]. Notice that, by the density of $\mathcal{UF}$ and the fact that W is closed, for $r \geq 0$,

$$\overline{W(r)} = \bigcup_{\xi \in W} \{\eta : i(\eta, \xi) \leq r\}.$$

Then, using the same notation,

$$\begin{aligned} &\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n, e_n)} \mathbb{1}_B, \mathbb{1}_W \rangle \\ &\quad = \limsup_{n\to\infty} \frac{1}{|E_n|} \int_{E_n} \mathbb{1}_U(e_n(g)) \mathbb{1}_{W(r_n)}(e_n(g)) dg \\ &\quad \leq \limsup_{n\to\infty} \frac{1}{|E_n|} \int_{E_n} \mathbb{1}_U(e_n(g)) \mathbb{1}_{\overline{W(r_n)}}(e_n(g)) dg. \end{aligned}$$

From

$$\forall r \leq r', \quad \overline{W(r)} \subset \overline{W(r')},$$

we deduce that, if $\nu_o(\partial \overline{W(r_k)}) = 0$ for all k, by condition (2),

$$\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n, e_n)} \mathbb{1}_B, \mathbb{1}_W \rangle \leq \nu_o(U \cap \overline{W(r_k)}) \quad \forall k.$$

Hence, if

$$\lim_{k\to\infty} \nu_o(\overline{W(r_k)} \cap U) \leq \nu_o(\overline{W(0)} \cap U),$$

then one can follow the same arguments as in [6, Proposition 2.21], since $U \cap \overline{W(0)}$ contains only non-uniquely ergodic points (by the assumption that $U \cap W = \emptyset$), and ν_o is supported on uniquely ergodic points.

Note that we may not have the property that $\partial W(r_k)$ has ν_o-null measure. We are therefore left to prove the following lemma, which serves as an analog of [6, Proposition 2.18]. Afterward, we slightly increase r_k to obtain r'_k such that $\nu_o(\partial W(r'_k)) = 0$, with $r'_k \to 0$ as $k \to \infty$. Replacing r_k with r'_k in the arguments in [6, Lemma 2.20 and Proposition 2.21] will yield the same modified conclusion as stated before. Finally, we follow the arguments of Boyer, Link, and Pittet [6, Proof of Theorem 2.2], ultimately leading to the desired conclusion. ■

Lemma 2.16. *Use the notation introduced in the proof above. We have the following:*

(a) *the set* $\{r \in \mathbb{R}_+ : \nu_o(\partial \overline{W(r)}) = 0\}$ *is dense;*

(b) *we have*

$$\lim_{k\to\infty} \nu_o(\overline{W(r_k)} \cap U) \leq \nu_o(\overline{W(0)} \cap U).$$

Proof. Assume that W is compact; if not, we replace it with its closure. Part (b) follows directly from the Monotone Convergence Theorem, since the sequence $\{r_k\}$ is decreasing, and hence,

$$\overline{W(r_{k+1})} \cap U \subset \overline{W(r_k)} \cap U.$$

For part (a), we proceed as in the proof of [6, Proposition 2.18]. Since W is compact, for $\xi \in \mathcal{PMF}(S)$, define $s(\xi, W) = \min\{i(\xi, w) : w \in W\}$. Thus, for any $r > 0$, we have

$$\partial \overline{W(r)} \subset \{\eta \in \mathcal{PMF}(S) : s(\eta, W) = r\}.$$

This implies that if $r \neq r'$, then $\partial \overline{W(r)} \cap \partial \overline{W(r')} = \emptyset$. Hence, if part (a) is incorrect, it would contradict the fact that the full ν_o-measure is finite. ■

Our main result is the following theorem.

Theorem 2.17. *There exist* $\theta > 0$ *and* $\varepsilon > 0$ *such that if* $E_n = \mathcal{E}(\theta, \varepsilon, n, o, \rho)$ *as described in the paragraph preceding Lemma 2.12, then, up to passing to a subsequence, the boundary representation* π_{ν_o} *is ergodic with respect to* $(E_n, \Pr)$ *and any* $f \in \overline{H}^{L^\infty}$. *In other words,* $\{(E_n = \mathcal{E}(\theta, \varepsilon, n, o, \rho), \Pr)\}$ *satisfies all the conditions listed in Theorem 2.14.*

As ν_o is a Radon measure, immediately the following two corollaries are obtained by Proposition 2.1 and Remark 2.2.

Corollary 2.1. *The boundary representation* π_{ν_o} *of* $\mathrm{Mod}(S)$ *is irreducible.*

Corollary 2.2. *The mapping class group* $\mathrm{Mod}(S)$ *acts ergodically on* $\mathcal{PMF}(S)$ *with respect to the measure class* $[\nu_o]$.

We now mention a property of the boundary representation π_{ν_o}. Recall that a unitary representation of a group G is called *tempered* if it is weakly contained in the regular representation $L^2(G)$ (cf. [3]).

Proposition 2.2. *The boundary representation* π_{ν_o} *of* $\mathrm{Mod}(S)$ *is tempered.*

Proof. We argue as in [11, Proposition 6.3]. By the main theorem in [19], we need to verify that the action of $\mathrm{Mod}(S)$ on $\mathcal{PMF}(S)$ is amenable. This is established in [13, Proposition 8.1] as a corollary of the topological amenability of the action of $\mathrm{Mod}(S)$ on $\mathcal{PMF}(S)$. ∎

Notations. We introduce the following notations and conventions, which will be used throughout the sequel.

- $S = S_g$: a genus $g \geq 2$, closed, oriented, connected surface.
- $h = 6g - 6$.
- o: the base point in $\mathrm{Teich}(S)$, chosen to be generic, in the sense $\mathrm{Stab}_o(\mathrm{Mod}(S)) = \mathrm{id}$.[1] Denote $\nu = \nu_o$, and the measure is normalized so that $\nu(\mathcal{PMF}(S)) = 1$
- The projective measured foliation space $\mathcal{PMF}(S)$ is regarded as a subset of $\mathcal{MF}(S)$ via the map τ. An element $[\xi]$ in $\mathcal{PMF}(S)$ is written as ξ, so both $[\xi]$ and ξ will be called directions when there are no ambiguity.
- $\mathrm{Pr}_y : \mathrm{Teich}(S) \setminus \{y\} \to \mathcal{PMF}(S)$: the radial projection from $\mathrm{Teich}(S)$ to the projective measured foliation space $\mathcal{PMF}(S)$, which assigns to every point $z \in \mathrm{Teich}(S) \setminus \{y\}$ the projective vertical measured foliation of the unit holomorphic quadratic differential defined by the oriented geodesic $[y, z]$. For $y = o$, we simply denote Pr_o by Pr. We will regard Pr as a map defined on $\mathrm{Mod}(S) \setminus \{\mathrm{id}\}$ via the map $\gamma \mapsto \gamma \cdot o$.
- $d(\cdot,\cdot)$: the Teichmüller distance $d_T(\cdot,\cdot)$.
- $B(y, R)$: the closed ball in $\mathrm{Teich}(S)$ of radius R at y with respect to the Teichmüller distance d.
- $\asymp$: if $A(t), B(t)$ are two functions, we use $A \asymp B$ when $\lim_{t\to\infty} \frac{A(t)}{B(t)} \to 1$, and $A \precsim B$ when $\lim_{t\to\infty} \frac{A(t)}{B(t)} \leq 1$. The notation $A \succsim B$ is defined similarly.
- $A \sim_\theta B$: there are multiplicative constants $C_1 > 0, C_2 > 0$, depending on θ such that
$$C_1 A \leq B \leq C_2 A.$$
$A \prec_\theta B$: there is a multiplicative constant $D = D(\theta) > 0$ so that
$$A \leq DB.$$
The notation $A \succ_\theta B$ is defined similarly.

[1]For $g = 2$, instead of using $\mathrm{Mod}(S)$, we will use $\widetilde{\mathrm{Mod}(S)} = \mathrm{Mod}(S)/Z(\mathrm{Mod}(S))$, the quotient of $\mathrm{Mod}(S)$ by its center, in the sequel. The main result holds for $\mathrm{Mod}(S)$ by considering the quotient map $\mathrm{Mod}(S) \to \widetilde{\mathrm{Mod}(S)}$.

- Fix arbitrary $\rho > 0$ and assume that $n \gg \rho$. It is always evident in the context to choose a lower bound for n that is much larger than ρ.
- Use the notation $U(\theta, \varepsilon)$ and $E_n = \mathcal{E}(\theta, \varepsilon, n, o, \rho)$ in the sequel, as described in the paragraph preceding Lemma 2.12, where we choose $\theta > 0.999$.
- $\xi_\gamma \in \mathcal{PMF}(S)$ (for $\gamma \in \mathrm{Mod}(S) \setminus \{\mathrm{id}\}$): the direction of the oriented geodesic segment $[o, \gamma \cdot o]$.

3. Exponential growth and shadow lemma

3.1. Exponential growth

In this subsection, we will demonstrate that $|E_n|$ tends to infinity. More specifically, we will prove that $|E_n|$ grows exponentially. For any Borel subset W of $\mathcal{PMF}(S)$, let Sect_W denote the union of geodesics starting from o and ending at points in W. We first recall the following theorem in [1], adapted to our setting. Define

$$C(n, \rho) = \big\{\gamma \in \mathrm{Mod}(S) : d(\gamma \cdot o, o) \in (n - \rho, n + \rho)\big\}.$$

Theorem 3.1 ([1, Theorem 2.10]). *Let W and V be two Borel subsets of $\mathcal{PMF}(S)$ with null measure boundaries. Then, as n tends to ∞,*

$$\Big|\big\{\gamma \in C(n, \rho) : \gamma \cdot o \in \mathrm{Sect}_W \text{ and } \gamma^{-1} \cdot o \in \mathrm{Sect}_V\big\}\Big| \asymp K e^{hn} \nu(W)\nu(V),$$

where K is a constant depending on g, ρ and o. In fact, using the notation from [1], one has $K = \frac{2\sinh(h\rho)\|\nu(\mathcal{PMF}(S))\|^2}{hm(\mathcal{M}_g)}$, where the measure m is the push-forward measure of the Masur–Veech volume.

Corollary 3.1. *Let $n \gg 0$, and let K be the constant in Theorem 3.1. Then, we have $|E_n| \asymp K e^{hn}$ (up to passing to a subsequence). In particular, this implies that $\lim_{n\to\infty} |E_n| = \infty$.*

Proof. As $E_n \subset C(n, \rho)$ and, by Theorem 3.1, we have $|C(n, \rho)| \asymp K e^{hn}$, it is obvious that

$$|E_n| \precsim K e^{hn}.$$

We now show that $|E_n| \succsim K e^{hn}$. Recall that

$$U(\theta, \varepsilon) = \bigcup_{R>0} U(R, \theta, \varepsilon),$$

where $\nu(U(\theta, \varepsilon)) = 1$, and $U(S, \theta, \varepsilon) \subset U(T, \theta, \varepsilon)$ for $T > S$. Let $\delta_1 > 0$ be sufficiently small and choose $R \gg 0$ such that

$$1 - \delta_1 \le \nu(U(R, \theta, \varepsilon)) \le 1, \nu(\partial U(R, \theta, \varepsilon)) = 0.$$

By Theorem 3.1 again, for any $\delta_2 > 0$ small enough, there exists $N(\delta_2)$ such that, whenever $n \geq N(\delta_2)$ and $\frac{1}{3h} \ln\ln n > R$,

$$\begin{aligned}
&\left|\left\{\gamma \in C(n,\rho) : \gamma \cdot o \in \mathrm{Sect}_{U(R,\theta,\varepsilon)} \text{ and } \gamma^{-1} \cdot o \in \mathrm{Sect}_{U(R,\theta,\varepsilon)}\right\}\right| \\
&\quad \geq K e^{-\delta_2} e^{hn} (\nu(U(R,\theta,\varepsilon)))^2 \\
&\quad \geq K e^{-\delta_2} (1-\delta_1)^2 e^{hn}.
\end{aligned}$$

By the choice of n and the definition of $U(R,\theta,\varepsilon)$, we have

$$\left\{\gamma \in C(n,\rho) : \gamma \cdot o \in \mathrm{Sect}_{U(R,\theta,\varepsilon)} \text{ and } \gamma^{-1} \cdot o \in \mathrm{Sect}_{U(R,\theta,\varepsilon)}\right\} \subset E_n.$$

This implies that $|E_n| \geq K e^{-\delta_2}(1-\delta_1)^2 e^{hn}$. As δ_1 and δ_2 can be arbitrary small, one has $|E_n| \gtrsim K e^{hn}$. ■

Corollary 3.2. *For all Borel subsets $W, V \subset \mathcal{PMF}(S)$ such that $\nu(\partial W) = \nu(\partial V) = 0$,*

$$\limsup_{n\to\infty} \frac{1}{|E_n|} \left|\left\{\gamma \in E_n : \mathrm{Pr}(\gamma^{-1}) \in W \text{ and } \mathrm{Pr}(\gamma) \in V\right\}\right| \leq \nu(W)\nu(V).$$

Proof. By using Corollary 3.1 and Theorem 3.1, we have $|E_n| \asymp |C(n,\rho)|$. Note that $\mathrm{Pr}(\gamma) \in V$ if and only if $\gamma \cdot o \in \mathrm{Sect}_V$. Hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|E_n|} \left|\left\{\gamma \in E_n : \mathrm{Pr}(\gamma^{-1}) \in W \text{ and } \mathrm{Pr}(\gamma) \in V\right\}\right| \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{|E_n|} \left|\left\{\gamma \in C(n,\rho) : \mathrm{Pr}(\gamma^{-1}) \in W \text{ and } \mathrm{Pr}(\gamma) \in V\right\}\right| \\
&\quad = \limsup_{n\to\infty} \frac{|C(n,\rho)|}{|E_n|} \frac{1}{|C(n,\rho)|} \left|\left\{\gamma \in C(n,\rho) : \mathrm{Pr}(\gamma^{-1}) \in W \text{ and } \mathrm{Pr}(\gamma) \in V\right\}\right| \\
&\quad \leq \nu(W)\nu(V).
\end{aligned}$$

■

3.2. Shadow lemma

The following shadow lemma will be used in the proof of uniform boundedness (Section 5.2).

Lemma 3.2 ([35, Lemma 6.3]). *There exists $R_0 > 0$ such that for every $R \geq R_0$, there is a real number $C \geq 1$, depending on R such that for all $n \gg \rho$ and $g \in E_n$,*

$$\frac{1}{C} \exp(-hd(o, g \cdot o)) \leq \nu(\mathrm{Pr}(B(g \cdot o, R))) \leq C \exp(-hd(o, g \cdot o)).$$

Recall that $U(\theta,\varepsilon)$ has full measure. We now need a lemma that relates Busemann functions to extremal lengths. Recall that if (X, d_X) is a metric space and ξ is a geodesic ray starting from a point $x_0 \in X$, then the *Busemann function* associated to the geodesic ray ξ is the function b_ξ on X defined by

$$b_\xi : x \mapsto \lim_{t\to\infty} (d_X(x, \xi(t)) - t).$$

For $(X = \mathrm{Teich}(S), d_X = d)$ and a geodesic ray ξ starting from o, we have the following lemma.

Lemma 3.3 ([32,34]). *If* $[\xi] \in \mathcal{PMF}(S)$ *is uniquely ergodic point, then the Busemann function* b_ξ *associated with the geodesic ray* ξ *in the direction* $[\xi]$ *is given by*

$$\forall x \in \mathrm{Teich}(S), \quad b_\xi(x) = \frac{1}{2} \ln \left(\frac{\mathrm{Ext}_x(\xi)}{\mathrm{Ext}_o(\xi)} \right).$$

4. Harish-Chandra estimates

This section is devoted to proving the following Harish-Chandra estimates, which involve some technical details.

Theorem 4.1. *Consider* $n \gg \rho$. *There exist constants* $a_1 > 0, a_2 > 0, b_1, b_2$, *depending on* $\varepsilon, o, g, \theta, \rho$ *such that*

$$\forall \gamma \in E_n, \quad (a_1 n + b_1) e^{-\frac{h}{2} n} \le \Phi(\gamma) \le (a_2 n + b_2) e^{-\frac{h}{2} n}.$$

Recall that we have

$$\begin{aligned}\Phi(\gamma) &= \langle \pi_\nu(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\mathcal{PMF}(S)} \rangle_{L^2(\mathcal{PMF}(S),\nu)} \\ &= \int_{\mathcal{PMF}(S)} \left(\frac{\mathrm{Ext}_o(\xi)}{\mathrm{Ext}_{\gamma\cdot o}(\xi)} \right)^{\frac{h}{4}} d\nu([\xi]).\end{aligned}$$

The proof is divided into several lemmas, and it will be given at the end of this section.

4.1. Reduction to intersection numbers

By our convention, for every $\xi \in \mathcal{PMF}(S)$, $\mathrm{Ext}_o(\xi) = 1$, we have

$$\Phi(\gamma) = \int_{\mathcal{PMF}(S)} \left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\zeta)} \right)^{\frac{h}{4}} d\nu(\zeta).$$

Let ξ_γ be the direction of the oriented geodesic $[o, \gamma \cdot o]$. In order to estimate $\Phi(\gamma)$, we will relate it to the following integral of the intersection number function:

$$\Psi(\gamma) = \int_{\mathcal{PMF}(S)} \left(\frac{1}{i(\xi_\gamma, \eta)} \right)^{\frac{h}{2}} d\nu(\eta).$$

Define

$$\Psi(\gamma)_{\geq A} = \int_{\{\eta\in\mathcal{PMF}(S):i(\xi_\gamma,\eta)\geq A\}} \left(\frac{1}{i(\xi_\gamma,\eta)}\right)^{\frac{h}{2}} d\nu(\eta).$$

For $\gamma \in E_n$, the first step is to bound $\Phi(\gamma)$ from above. This can be easily done by the inequality in the following lemma.

Lemma 4.2 ([25]). *Let ξ and η be two measured foliations on S and $x = [X] \in$ Teich(S), then*

$$i^2(\xi,\eta) \leq \mathrm{Ext}_x(\xi)\,\mathrm{Ext}_x(\eta),$$

where the equality holds if and only if there is a quadratic differential q so that the vertical measured foliation of q on X is ξ and the horizontal measured foliation is η.

Corollary 4.1. *There exist constants $C_3 = C_3(g,\rho) > 0$ and $C_4 = C_4(g,\rho) > 0$ such that, for every $M \in (0,1)$ and every $\gamma \in E_n (n \gg \rho)$,*

$$\Phi(\gamma) \leq C_3 e^{-\frac{h}{2}n}\Psi(\gamma)_{\geq M} + C_4 e^{\frac{h}{2}n}\nu(\{\eta\in\mathcal{PMF}(S) : i(\eta,\xi_\gamma)\leq M\}).$$

Proof. Decompose $\mathcal{PMF}(S)$ into two subsets: $A = \{\eta \in \mathcal{PMF}(S) : i(\eta,\xi_\gamma) \leq M\}$ and $B = \{\eta\in\mathcal{PMF}(S) : i(\eta,\xi_\gamma) \geq M\}$. Then, we have

$$\Phi(\gamma) = \underbrace{\int_B \left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta)}_{\mathrm{I}} + \underbrace{\int_A \left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\eta)}\right)^{\frac{h}{4}} d\nu(\eta)}_{\mathrm{II}} = \mathrm{I}+\mathrm{II}.$$

By Kerckhoff's formula, II $\prec_{g,\rho} e^{\frac{h}{2}n}\nu(\{\eta\in\mathcal{PMF}(S) : i(\eta,\xi_\gamma)\leq M\})$. Since $\gamma\in E_n$, by the construction of E_n, ξ_γ is uniquely ergodic. Thanks to Lemma 3.3, we can replace $\mathrm{Ext}_{\gamma\cdot o}(\xi_\gamma)$ in Lemma 4.2 by e^{-2n}, since in this case $b_{\xi_\gamma}(\gamma\cdot o) = -n$ and by our convention, $\mathrm{Ext}_o(\xi_\gamma) = 1$. Hence, we have

$$\frac{1}{i^2(\xi_\gamma,\eta)e^{2n}} \succ_{g,\rho} \frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\eta)},$$

which gives the bound for the term I. ■

Lemma 4.3. *Let $n \gg \rho$. There exists a positive constant F, depending on $g, o, \varepsilon, \theta, \rho$ such that if $i(\xi_\gamma,\eta) \geq F \ln n e^{-2n}$, where $\eta\in U(\varepsilon,\theta)$ and $\gamma\in E_n$ then $i^2(\xi_\gamma,\eta) \succ_{g,o,\varepsilon,\theta,\rho}$ $\mathrm{Ext}_{\gamma\cdot o}(\eta)e^{-2n}$.*

Proof. First, we remark that since both η and ξ_γ are uniquely ergodic, by [18, Proposition 5.1], there is a geodesic whose horizontal and vertical measured foliations are in the projective classes ξ_γ and η, respectively. Hence, we have a geodesic triangle $\triangle(o,\xi_\gamma,\eta)$. Since $\gamma\in E_n$, Lemma 2.12 implies that there is a geodesic segment I of length $\ell = \frac{1}{3h} \ln \ln n$ in $[o, \gamma \cdot o]$, ending at $\gamma \cdot o$, which has at least proportion θ in $\mathrm{Teich}_{\varepsilon}(S)$. Since $n \gg \rho$, by Theorem 2.9 (though the theorem is for triangles of finite lengths, it is easy to see that this theorem can be extended to include $\triangle(o, \xi_\gamma, \eta)$),

$$I \cap \mathcal{N}_D([\eta, \xi_\gamma] \cap [o, \eta]) \neq \emptyset,$$

where D comes from Theorem 2.9. Choose $q \in I \cap \mathcal{N}_D([\eta, \xi_\gamma] \cap [o, \eta])$. Then, there are two possibilities.

Case 1. $d(q, y) \leq D$ with $y \in [\xi_\gamma, \eta]$.

In this case, we have, by Kerckhoff's formula and Lemma 4.2,

$$\begin{aligned} i^2(\xi_\gamma, \eta) &= \mathrm{Ext}_y(\eta)\,\mathrm{Ext}_y(\xi_\gamma) \\ &\succ_{g,o,\theta,\varepsilon} \mathrm{Ext}_q(\eta)\,\mathrm{Ext}_q(\xi_\gamma) \\ &= \mathrm{Ext}_q(\eta) e^{-2d(o,q)} \\ &\geq \mathrm{Ext}_{\gamma \cdot o}(\eta) e^{-2n}. \end{aligned}$$

This implies that, in this case, we always have $i^2(\xi_\gamma, \eta) \succ_{g,o,\varepsilon,\theta} \mathrm{Ext}_{\gamma \cdot o}(\eta) e^{-2n}$.

Case 2. $d(q, y) \leq D$ with $y \in [o, \eta]$.

In this case, we have the following:

$$\begin{aligned} i^2(\xi_\gamma, \eta) &\leq \mathrm{Ext}_y(\eta)\,\mathrm{Ext}_y(\xi_\gamma) \\ &\sim_{g,o,\theta,\varepsilon} \mathrm{Ext}_y(\eta)\,\mathrm{Ext}_q(\xi_\gamma) \\ &\sim_{g,o,\theta,\varepsilon,\rho} e^{-4d(o,q)} = e^{-4(d(o,\gamma \cdot o) - d(q, \gamma \cdot o))} \leq e^{-4n} e^{4\ell}. \end{aligned}$$

Therefore, we have a positive constant F_1, depending on $g, o, \varepsilon, \theta, \rho$ such that

$$i(\xi_\gamma, \eta) \leq F_1 e^{-2n} e^{2\ell} \leq F_1 e^{-2n} e^{\ln \ln n} = F_1 \ln n e^{-2n}.$$

If we take $F \gg F_1$ and set $i(\xi_\gamma, \eta) \geq F \ln n e^{-2n}$, then Case 2 will never occur, which implies the conclusion that

$$i^2(\xi_\gamma, \eta) \succ_{g,o,\varepsilon,\theta} \mathrm{Ext}_{\gamma \cdot o}(\eta) e^{-2n}.$$

■

Corollary 4.2. *For every $\gamma \in E_n$, take $\bar{M} = F \ln n e^{-2n}$, where F is the constant in Lemma 4.3. Then, $\Phi(\gamma) \succ_{g,o,\varepsilon,\theta,\rho} e^{-\frac{h}{2} n} \Psi(\gamma)_{\geq \bar{M}}$.*

Proof. Note that $U(\varepsilon, \theta)$ has full measure. Hence, by Lemma 4.3, one has

$$\begin{aligned} \Phi(\gamma) &= \int_{U(\varepsilon,\theta)} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\eta)} \right)^{\frac{h}{4}} d\nu(\eta) \\ &\geq \int_{\{\eta \in U(\varepsilon,\theta) : i(\eta, \xi_\gamma) \geq \bar{M}\}} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\eta)} \right)^{\frac{h}{4}} d\nu(\eta) \succ_{g,o,\varepsilon,\theta} e^{-\frac{hn}{2}} \Psi(\gamma)_{\geq \bar{M}}. \end{aligned}$$

■

The following lemma allows us to reduce the estimate of $\Phi(\gamma)$ to the estimates of measures.

Lemma 4.4. *For any $\gamma \in E_n$, if there exist a sequence $\{\xi_k \in \mathcal{PMF}(S) : k \in \mathbb{N}\} \subset \mathcal{PMF}(S)$ that converges to ξ_γ, and constants $\delta_0 > 0, a > 0, b > 0$ such that, for all $k \in \mathbb{N}$ and for all $0 < N \le \delta_0$,*

$$aN^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_k) \le N\}) \le bN^{\frac{h}{2}},$$

then, there exist constants $\delta_0 > 0$, $A > 0$, $B > 0$, D_1, D_2 such that

$$\forall\, 0 < N \le \delta_0, \quad -A \ln N + D_1 \le \Psi(\gamma)_{\ge N} \le -B \ln N + D_2.$$

Proof. Fix $0 < N \le \delta_0$, then the set $\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_\gamma) \ge N\}$ is compact. Since $i(\xi_k, \cdot)$ converges to $i(\xi_\gamma, \cdot)$ uniformly on compact sets outside $\{\xi_\gamma\}$, there exists $K_1 > 0$ such that, for $k \ge K_1$,

$$\big\{\eta : i(\xi_k, \eta) \ge 2N\big\} \subset \big\{\eta : i(\xi_\gamma, \eta) \ge N\big\}.$$

Hence, for all $k \ge K_1$, we have

$$\int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \ge 2N\}} \left(\frac{1}{i(\xi_\gamma, \eta)}\right)^{\frac{h}{2}} d\nu \le \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_\gamma,\eta) \ge N\}} \left(\frac{1}{i(\xi_\gamma, \eta)}\right)^{\frac{h}{2}} d\nu.$$

By uniform convergence on compact sets again, there exists $K_2 > 0$ such that, for $k \ge K_2$, and for $\eta \in \{\eta : i(\eta, \xi_\gamma) \ge N\}$,

$$\frac{1}{2} \le \left(\frac{i(\xi_k, \eta)}{i(\xi_\gamma, \eta)}\right)^{\frac{h}{2}} \le 2.$$

Take $k > \max\{K_1, K_2\}$; then,

$$\begin{aligned}\frac{1}{2} \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \ge 2N\}} &\left(\frac{1}{i(\xi_k, \eta)}\right)^{\frac{h}{2}} d\nu \\ &\le \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \ge 2N\}} \left(\frac{1}{i(\xi_\gamma, \eta)}\right)^{\frac{h}{2}} d\nu \le \Psi(\gamma)_{\ge N}.\end{aligned}$$

Since, for $\eta \in \mathcal{PMF}(S)$,

$$\begin{aligned}\lim_{k\to\infty} \mathbb{1}_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \ge N\}}(\eta) &\left(\frac{1}{i(\xi_k, \eta)}\right)^{\frac{h}{2}} \\ &= \mathbb{1}_{\{\eta \in \mathcal{PMF}(S): i(\xi_\gamma,\eta) \ge N\}}(\eta) \left(\frac{1}{i(\xi_\gamma, \eta)}\right)^{\frac{h}{2}},\end{aligned}$$

by Fatou's lemma, there exists $K_3 > 0$ such that, for $k \geq K_3$,

$$\Psi(\gamma)_{\geq N} \leq \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \geq N\}} \left(\frac{1}{i(\xi_k,\eta)}\right)^{\frac{h}{2}} d\nu + 1.$$

Define

$$\Psi(\xi_k)_{\geq N} = \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \geq N\}} \left(\frac{1}{i(\xi_k,\eta)}\right)^{\frac{h}{2}} d\nu(\eta).$$

We have, for $k \gg 0$,

$$\frac{1}{2}\Psi(\xi_k)_{\geq 2N} \leq \Psi(\gamma)_{\geq N} \leq \Psi(\xi_k)_{\geq N} + 1.$$

Hence, it is sufficient to show that there exist A, B, D_1, D_2 such that

$$-A \ln N + D_1 \leq \Psi(\xi_k)_{\geq N} \leq -B \ln N + D_2.$$

The proof in the sequel is similar to part of [5, Proposition 3.2], and we repeat it here for completeness. Namely,

$$\begin{aligned}
\Psi(\xi_k)_{\geq N} &= \int_{\{\eta \in \mathcal{PMF}(S): i(\xi_k,\eta) \geq N\}} \left(\frac{1}{i(\xi_k,\eta)}\right)^{\frac{h}{2}} d\nu(\eta) \\
&= \int \nu\left(\left\{\eta \in \mathcal{PMF}(S) : \left(\frac{1}{i(\xi_k,\eta)}\right)^{\frac{h}{2}} \geq t\right\}\right) dt \\
&= \int_1^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{PMF}(S) : i(\xi_k,\eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt \\
&= \int_1^{(\frac{1}{\delta_0})^{\frac{h}{2}}} \nu\left(\left\{\eta \in \mathcal{PMF}(S) : i(\xi_k,\eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt \\
&\quad + \int_{(\frac{1}{\delta_0})^{\frac{h}{2}}}^{\frac{1}{N^{\frac{h}{2}}}} \nu\left(\left\{\eta \in \mathcal{PMF}(S) : i(\xi_k,\eta) \leq \frac{1}{t^{\frac{2}{h}}}\right\}\right) dt.
\end{aligned}$$

By the assumption and the fact that ν is a probability measure, the conclusion follows. ■

We include the following observation.

Lemma 4.5. *If there exist $\alpha_0 > 0$ small enough and a sequence of points $\{\xi_k \in \mathcal{PMF}(S) : k \in \mathbb{N}\}$ that converges to $\beta \in \mathcal{PMF}(S)$ such that, for all $k \in \mathbb{N}$, and for all $0 < N \leq \alpha_0$,*

$$\nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi_k) \leq N\}) \leq bN^{\frac{h}{2}},$$

then there exist constants $N_0 > 0, a' > 0$ *such that*

$$\forall 0 < N \leq \alpha_0, \quad \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \beta) \leq N\}) \leq a' N^{\frac{h}{2}}.$$

Proof. Denote

$$A_k(N) = \{\eta : i(\eta, \xi_k) \leq N\} \quad \text{and} \quad B(N) = \{\eta : i(\eta, \beta) \leq N\}.$$

Since $i(\cdot, \cdot)$ is continuous, we have ν-almost surely,

$$\lim_{k \to \infty} \mathbb{1}_{A_k(N)}(x) = \mathbb{1}_{B(N)}(x).$$

By Fatou's lemma,

$$\nu(B(N)) \leq \liminf_k \nu(A_k(N)) \leq bN^{\frac{h}{2}}.$$

■

4.2. A basic example

Before continuing our discussion, we specialize in the case of the once-punctured torus $S_{1,1}$, which will help the reader understand the ideas behind the estimates in the sequel. Some standard facts are taken from [27, 7.2 Examples].

Let $S = S_{1,1}$. Then, $\mathrm{Mod}(S) = \mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{Teich}(S) = \mathbb{H}^2$, the upper half-plane. Take o to be $i \in \mathbb{H}^2$. The space $\mathcal{MF}(S)$ of nonzero measured foliations can be identified with the real plane module the inversion, i.e., $\{\mathbb{R}^2 - (0,0)\}/\{I, -I\}$. By the ergodicity of the Thurston measure ν_{Th}, up to a constant multiple, the measure ν_{Th}, which is defined as the weak limit of counting measures on $\mathcal{MF}(S)$, can be identified with the Lebesgue measure on $\mathbb{R}^2 - (0,0)$. The rays in $\{\mathbb{R}^2 - (0,0)\}/\{I, -I\}$ are identified with the points in $\mathcal{PMF}(S)$. This implies that $\mathcal{PMF}(S)$ can be identified with $\mathbb{R}P^1$. Notice that all identifications here are $\mathrm{Mod}(S)$-equivariant. Hence, $\mathcal{PMF}(S)$ can be represented as

$$\{[x : y] : x^2 + y^2 \neq 0, x, y \in \mathbb{R}\},$$

or equivalently, $\mathbb{R} \cup \{\infty\}$. The compactification $\overline{\mathrm{Teich}(S)}$ is then the usual compactification of $\mathbb{H}^2$. In this case, $\mathrm{Mod}(S)$ acts on $\overline{\mathrm{Teich}(S)}$ via linear fractional transformations.

For $(x, y) \in \mathbb{R}^2$, its extremal length at o is

$$\mathrm{Ext}_o((x, y)) = x^2 + y^2.$$

Hence, the image of $\mathcal{PMF}(S)$ under the map τ is the circle. We will ignore the difference between $\mathbb{R}^2$ and $\mathbb{R}^2/\{I, -I\}$. For two points $(x, y), (p, q) \in \mathcal{MF}(S)$, their intersection number is $|qx - py|$.

Now, we write the image of $\mathcal{PMF}(S)$ in the form of $(\sin(\theta), \cos(\theta))$, and fix any $\xi = (\sin(\theta_0), \cos(\theta_0)) \in \mathcal{PMF}(S)$. Let M be small enough; then,

$$\begin{aligned}
&\{\eta \in \mathcal{PMF}(S) : i(\xi, \eta) \leq M\} \\
&\quad = \{\theta \in [0, 2\pi] : |\sin(\theta)\cos(\theta_0) - \cos(\theta)\sin(\theta_0)| \leq M\} \\
&\quad = \{\theta \in [0, 2\pi] : |\sin(\theta - \theta_0)| \leq M\} \\
&\quad = \{\theta \in [0, 2\pi] : -M \leq \sin(\theta - \theta_0) \leq M\}.
\end{aligned}$$

As M is small enough, $\sin(\theta)$ is almost the same as θ, so there exist constants A and B such that

$$AM \leq \nu(\{\eta \in \mathcal{PMF}(S) : i(\xi, \eta) \leq M\}) \leq BM,$$

Notice that when S is $S_{1,1}$, we have $h = 6g - 6 + 2n = 6 \times 1 - 6 + 2 \times 1 = 2$; hence, $\frac{h}{2} = 1$.

4.3. Approximations by pant curves

In this subsection, we will prove the following proposition, which is analogous to Diophantine approximation.

Proposition 4.1. *There exist constants $A > 0, B > 0$, depending on ε and o such that, for every $\eta \in U(\theta, \varepsilon)$, there exists a sequence $\{\xi_k = \xi_k(\eta)\}_{k=1}^{\infty} \subset \mathcal{PMF}(S)$ satisfying the following conditions.*

- $(\star_1)$ *For each k, $\xi_k = [x_k = \sum_{i=1}^{3g-3} \alpha_i^k]$, where $\{\alpha_i^k\}_{i=1}^{3g-3}$ is a pants decomposition of S.*
- $(\star_2)$ *For each k and $1 \leq i \leq 3g-3$, there is a sequence $\{t_k\}$ of positive reals such that $Ae^{2t_k} \leq \mathrm{Ext}_o(\alpha_i^k) \leq Be^{2t_k}$, $Ae^{2t_k} \leq \mathrm{Ext}_o(x_k) \leq Be^{2t_k}$, $\lim_{k\to\infty} t_k = \infty$.*
- $(\star_3)$ *The limit of $\left\{\xi_k = \frac{x_k}{\sqrt{\mathrm{Ext}_o(x_k)}}\right\}$ in $\mathcal{PMF}(S)$ is η.*

Any pants curve x_k in Proposition 4.1 will be called *a thick pants curve*, and the sequence $\{\xi_k(\eta)\}$ will be called the *pants curves associated with η*. This proposition, together with the next subsection, shows that the assumption in Lemma 4.4 holds. We prove this proposition via the systole map, which has been studied, for instance, in [23].

Recall that, thanks to Bers' theorem, there exists a constant $C_1 = C(g)$, depending only on the genus g such that for every point $x \in \mathrm{Teich}(S)$, there exists a pants decomposition, i.e., a collection of $3g-3$ disjoint essential simple closed curves $\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g-3}\}$ such that

$$\forall 1 \leq i \leq 3g-3, \quad \mathrm{Ext}_x(\alpha_i) \leq C_1^2.$$

The existence of such a pants decomposition may not be unique, but we can choose any of them. If x is further assumed to lie in $\mathrm{Teich}_\varepsilon(S)$, one can choose a collection of $3g-3$ essential simple closed curves $\{\alpha_1(x),\ldots,\alpha_{3g-3}(x)\}$ on S such that

$$\forall 1\le i\le 3g-3,\quad \varepsilon\le \mathrm{Ext}_x(\alpha_i(x))\le C_1^2.$$

Let β_x be the measured foliation $\beta_x=\sum_{i=1}^{3g-3}\alpha_i(x)$, and let $[\beta_x]$ be its projective class in $\mathcal{PMF}(S)$. By the aforementioned result of Hubbard–Masur, there is a unique unit holomorphic quadratic differential $q=q(\beta_x)$ on $o\in\mathrm{Teich}(S)$ whose projective class of the vertical measured foliation is $[\beta_x]$.

Given $\eta\in U(\theta,\varepsilon)$, by the construction of $U(\theta,\varepsilon)$ (cf. the discussion after Theorem 2.11), the geodesic ray $g_t=[o,\eta)$ cannot leave $\mathrm{Teich}_\varepsilon(S)$ eventually. Hence, there is a sequence of points $\{y_k=y_k(\eta)\in\mathrm{Teich}_\varepsilon(S)\}$ in g_t that converges to η in $\overline{\mathrm{Teich}(S)}$. By the above discussion, we obtain a sequence of pants curves $\{\beta_{y_k}\}$ and a sequence of points $\{[\beta_{y_k}]\}$ in $\mathcal{PMF}(S)$.

Let y be any $y_k(\eta)$ and denote $\beta_y=\sum\alpha_i(y)$. Let g_t be the Teichmüller geodesic ray starting from o in the direction $q(\beta_y)$. Along the geodesic ray $[o,[\beta_y])$, the modulus M_i of the core curve $\alpha_i(y)$, with respect to the flat metric given by $q(\beta_y)$, is multiplied by e^{2t} for each i, so by the geometric definition of the extremal length [15, page 32],

$$\mathrm{Ext}_{g_t}(\alpha_i(y))\le e^{-2t}\frac{1}{M_i}.$$

Hence, along the geodesic ray in the direction $[\beta_y]$, there is a point t_y in the geodesic ray such that t_y has maximal distance with o and

$$\forall 1\le i\le 3g-3,\quad \varepsilon\le \mathrm{Ext}_{t_y}(\alpha_i(y))\le C_1^2.$$

The proof of Proposition 4.1 is based on the following theorem, which is used in the form of [1, Theorem 5.3]. For the definition of the twist number $tw(\alpha,\beta)$, the reader is referred to [29].

Theorem 4.6 ([26]). *Let $x\in\mathrm{Teich}(S)$ and $\mathcal{P}=\{\alpha_1,\ldots,\alpha_{3g-3}\}$ be a pants decomposition produced by the Bers' theorem mentioned above. Then, for any simple closed curve β,*

$$\mathrm{Ext}_x(\beta)\sim_g \max_{1\le i\le 3g-3}\left(\frac{i^2(\beta,\alpha_i)}{\mathrm{Ext}_x(\alpha_i)}+tw^2(\beta,\alpha_i)\,\mathrm{Ext}_x(\alpha_i)\right).$$

In light of the above discussion, for each $\eta\in U(\theta,\varepsilon)$, we have a sequence $\{y_k=y_k(\eta)\}$ in the geodesic ray η, and for each y_k, we have $\beta_{y_k}=\sum_i\alpha_i(y_k)$. Notice that, for each k, $\{\alpha_i(y_k)\}_{i=1}^{3g-3}$ is a pants decomposition of S. We want to show that, up to a subsequence and certain adjustments, the sequence $\{\beta_{y_k}\}$ satisfies the conditions $(\star_2)$ and $(\star_3)$; therefore, it will be the desired $\{x_k\}$ in the proposition.

Proof of Proposition 4.1. The entire argument uses the notation introduced above. First, we prove that $\mathrm{Ext}_o(\beta_{y_k}) \sim_{g,\varepsilon} e^{2d(o,y_k)}$. Indeed, for each $y = y_k$, since η is uniquely ergodic and $\mathrm{Ext}_o(\eta) = 1$, $\mathrm{Ext}_y(\eta) = e^{-2d(y,o)}$ according to Lemma 3.3. Notice that Theorem 4.6 also applies for η, since the set of simple closed curves is dense in $\mathcal{PMF}(S)$. By the construction of $\alpha_i(y)$, $\mathrm{Ext}_y(\alpha_i(y)) \sim 1$ for each i. We can apply Theorem 4.6 to calculate $\mathrm{Ext}_y(\eta)$ using the pants decomposition $\{\alpha_i(y)\}_{i=1}^{3g-3}$, which implies that $\sum_i (i(\eta, \alpha_i(y)) + tw(\eta, \alpha_i(y)))$ is coarsely $e^{-d(y,o)}$. However, we know that $\mathrm{Ext}_o(\eta) = 1$. When we apply Theorem 4.6 to calculate $\mathrm{Ext}_o(\eta)$, with respect to the pants decomposition $\{\alpha_1(o), \dots, \alpha_{3g-3}(o)\}$, the sum of extremal lengths $\sum_i \mathrm{Ext}_o(\alpha_i(y))$ is coarsely $e^{2d(y,o)}$. This implies that $\mathrm{Ext}_o(\beta_{y_k}) \sim_{g,\varepsilon} e^{2d(o,y_k)}$ for each k. See [30] for the variation of twist numbers along geodesics.

Then, we show that the sequence $\{\frac{\beta_{y_k}}{\mathrm{Ext}_o(\beta_{y_k})}\}$ converges to η. Indeed, since η is uniquely ergodic, hence by [14, Proposition 2.4], the projection of $[o, \eta)$ to the curve graph is unbounded and the projection of the sequence y_k, up to a subsequence, converges to η. By [18, Theorem 1.4], $\{\frac{\beta_{y_k}}{\mathrm{Ext}_o(\beta_{y_k})}\}$, up to a subsequence, converges in $\mathcal{PMF}(S)$ to η. Notice that, the fact that $\{\frac{\beta_{y_k}}{\mathrm{Ext}_o(\beta_{y_k})}\}$ converges to η also implies that for each i, $\{\frac{\alpha_i(y)}{\mathrm{Ext}_o \alpha_i(y)}\}$ converges to η, since η is uniquely ergodic.

Finally, we show that $\{\beta_{y_k}\}$, after some modifications, satisfies the remaining part of $(\star_2)$ in Proposition 4.1, with t_k taken as $d(y_k, o)$. If there is a subsequence, still denote by $\{y_k\}$, of $\{y_k\}$ such that for each k and each $1 \le i \le 3g-3$, $\mathrm{Ext}_o(\alpha_i(y_k)) \sim_{g,\varepsilon} e^{2d_T(o,y_k)}$, then we are done. Otherwise, since $\mathrm{Ext}_o(\beta_{y_k}) \sim_{g,\varepsilon} e^{2d(o,y_k)}$ and

$$\mathrm{Ext}_o(\beta_{y_k}) = \sum_i \mathrm{Ext}_o(\alpha_i(y_k))$$

for each k, there is $i_k \in \{1, 2, \dots, 3g-3\}$ such that $\mathrm{Ext}_o(\alpha_{i_k}(y_k)) \sim_{g,\varepsilon} e^{2d(o,y_k)}$. For each k, we are now going to modify the pants decomposition $\{\alpha_j(y_k)\}$ as follows. Write $\beta_{y_k} = \alpha_1(y_k) + \alpha_2(y_k) + \cdots + \alpha_m(y_k) + \cdots + \alpha_{3g-3}(y_k)$ such that the first m terms have extremal lengths that are smaller than (and incomparable to) $e^{2d(o,y_k)}$, while the remaining $3g-3-m$ terms have extremal lengths that are comparable to $e^{2d(o,y_k)}$, where $A(t)$ is comparable to $B(t)$ if $\lim_{t\to\infty} \frac{A(t)}{B(t)}$ is a positive number. So, $i_k > m$. For $j = 1$, construct another curve $\alpha'_j(y_k)$ so that $\alpha'_j(y_k)$ intersects only with $\alpha_1(y_k)$ among the pants curve $\{\alpha_i(y_k)\}$, and has extremal length $\mathrm{Ext}_o(\alpha'_j(y_k))$ that is comparable to $e^{2d(o,y_k)}$. The simple closed curve $\alpha'_j(y_k)$ can be constructed by taking a certain number of Dehn twists along $\alpha_1(y_k)$, since $\mathrm{Ext}_o(\alpha_1(y_k))$ is less than $e^{2d(o,y_k)}$ and greater than ε, as assumed. Then, for $j = 2$, construct another simple closed curve $\alpha'_j(y_k)$ such that $\alpha'_j(y_k)$ intersects only $\alpha_2(y_k)$ among the pants curves $\{\alpha'_1(y_k), \alpha_2(y_k), \dots, \alpha_{3g-3}(y_k)\}$, and has extremal length $\mathrm{Ext}_o(\alpha'_j(y_k))$ that is comparable to $e^{2d(o,y_k)}$. We continue this process for m terms. We obtain, for each k, a pants decomposition $\{\alpha'_1(y_k), \dots, \alpha'_m(y_k), \alpha_{m+1}(y_k), \dots, \alpha_{3g-3}(y_k)\}$. Denote this pants decomposition by $\{\delta_i(y_k)\}$ and $\delta(y_k) = \sum_i \delta_i(y_k)$. This will be our desired $\{x_k : k \in \mathbb{N}\}$. On the one hand, by the construction and the fact that $\mathrm{Ext}_o(\delta(y_k)) \sim \sum_i \mathrm{Ext}_o(\delta_i(y_k))$, the condition $(\star_2)$ in Proposition 4.1 holds. On the other hand, since $\alpha_{i_k}(y_k)$ lies in $\{\delta_i(y_k)\}$, up to a subsequence, the sequence $\{\frac{\delta(y_k)}{\mathrm{Ext}_o(\delta(y_k))}\}$ has a limit in $\mathcal{PMF}(S)$, which is the same as the limit of $\{\frac{\alpha_{i_k}(y_k)}{\mathrm{Ext}_o(\alpha_{i_k}(y_k))}\}$. Hence, the sequence $\{\frac{\delta(y_k)}{\mathrm{Ext}_o(\delta(y_k))}\}$ converges to η. This completes the proof of Proposition 4.1. ■

4.4. Regularity at pants curves

We are now in a position to prove that the assumptions in Lemmas 4.5 and 4.4 hold.

More conventions. In this section, we will use the hyperbolic length function $\ell_o(\cdot)$. Since $\ell_o^2(\cdot) \sim_o \mathrm{Ext}_o(\cdot)$, we replace the function $\mathrm{Ext}_o(\cdot)$ with $\ell_o(\cdot)$ without affecting the results when we define the measure ν_o, the embedding $\tau : \mathcal{PMF}(S) \to \mathcal{MF}(S)$, and ξ_k. For example, for a measurable subset $U \in \mathcal{PMF}(S)$, we use

$$\nu_0(U) = \mu(\{\eta : [\eta] \in U, \ell_o(\eta) \le 1\}).$$

Setup 0. Let $\alpha = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of S, and consider it as a measured foliation, still denoted by α. Then, $[\alpha]$ defines a unit holomorphic quadratic differential q on o, namely, the unique q such that $[\mathcal{V}(q)] = [\alpha]$. Let $\xi = \frac{\alpha}{\ell_o(\alpha)}$, so ξ is the image of $[\alpha]$ under τ. We denote g_t as the Teichmüller geodesic defined by q. We assume that α is a thick pants curve, i.e., for all $i \in \{1, \dots, 3g-3\}$, both $\ell_o(\alpha_i)$ and $\ell_o(\alpha)$ are bounded below and above, up to multiplicative constants depending only on g, o, ε, by e^T for some T (cf. Proposition 4.1).

Theorem 4.7. *Under the above Setup 0, there exist constants $M_0 > 0, C > 0$ and $D > 0$, depending on g, o, ε such that when $0 < M < M_0$, we have*

$$CM^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \le M\}) \le DM^{\frac{h}{2}}.$$

The main tool used to prove the above theorem is the following Dehn–Thurston theorem. Let $P = \{\alpha_k\}$ be a pants decomposition. For each α_k, let $m_k : \mathcal{MF}(S) \to \mathbb{R}_{\ge 0}, \xi \mapsto i(\alpha_k, \xi)$ be the intersection function defined by α_k and $t_k = tw_k$ be the twist function associated with α_k.

Theorem 4.8 (The Dehn–Thurston theorem [29, Theorem 3.1.1]). *Let $S = S_g$ and $\alpha = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of S. Then, the map*

$$\varpi : \mathcal{MF}(S) \to \mathbb{R}^{6g-6}, \quad \mathcal{F} \mapsto (m_1(\mathcal{F}), \dots, m_{3g-3}(\mathcal{F}), t_1(\mathcal{F}), \dots, t_{3g-3}(\mathcal{F}))$$

gives a global coordinate for $\mathcal{MF}(S)$.

There are various (equivalent) definitions for the Thurston measure ν_{Th} on $\mathcal{MF}(S)$. For later computations, the Dehn–Thurston theorem allows us to define it as the measure $\frac{1}{(3g-3)!}\omega^{3g-3}$, where $\omega = dm_1 \wedge dt_1 + \cdots + dm_{3g-3} \wedge dt_{3g-3}$ is the symplectic form. Notice that although the symplectic form ω depends on α, the measure does not depend on the choice of α, since $\mathcal{MF}(S)$ has a piecewise integral linear structure [29, Section 3.1], which means that different pants decompositions give the same measure. Hence, the constants in Theorem 4.7 do not depend on ξ. We now denote $\mu = \nu_{Th}$ and α is fixed to be the one given in Setup 0. Note that $\frac{h}{2} = 3g - 3$.

Proof of Theorem 4.7. By Lemma 4.2, for every two elements ξ and η in $\mathcal{PMF}(S)$, the intersection number $i(\eta, \xi) \leq 1$, equality being achievable. The proof is then divided into two parts. In the sequel, denote $a = \frac{1}{\sum_i \ell_o(\alpha_i)}$ and let ℓ be given by

$$\ell = \frac{1}{a\prod(\ell_o(\alpha_i))^{\frac{2}{h}}} = \frac{\sum_i \ell_o(\alpha_i)}{\prod(\ell_o(\alpha_i))^{\frac{2}{h}}}.$$

By assumption (or see ($\star_2$) in Proposition 4.1), there exist constants $A_1 > 0$ and $B_1 > 0$, depending on g, o, ε such that $B_1 \leq \ell \leq A_1$.

First, we deduce the upper bound: $\nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M\}) \leq DM^{\frac{h}{2}}$.

Let $M \leq M_0$ and $M_0 = \frac{1}{4}$. By the definition of ν and the relation $i(\eta, \xi) = a\sum_{k=1}^{3g-3} i(\eta, \alpha_k)$, we have

$$\begin{aligned}
&\nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M\})\\
&\quad = \mu(\{t\eta \in \mathcal{MF}(S) : i(\eta, \xi) \leq M, \ell_o(\eta) = 1, 0 \leq t \leq 1\}) \quad \text{(by definition)}\\
&\quad = \mu\left(\left\{t\eta \in \mathcal{MF}(S) : a\sum_k m_k(\eta) \leq M, \ell_o(\eta) = 1, 0 \leq t \leq 1\right\}\right)\\
&\quad \leq \mu\left(\left\{\eta : \forall k, m_k(\eta) \leq \frac{M}{a}, t_i(\eta)\ell_o(\alpha_i) \leq A_2\right\}\right),
\end{aligned} \tag{4.1}$$

where A_2 is a constant that depends only on o. The last step follows from the fact that sufficiently many Dehn twists will result in large lengths (cf. [16, Lemma 3.2] for an explicit formula for computing hyperbolic lengths). Thus, there exists a constant D, depending only on g, o, and ε such that

$$\begin{aligned}
&\nu(\{\eta \in \mathcal{PMF}(S) : i(\eta, \xi) \leq M\})\\
&\leq \mu\left(\left\{(m_1, \ldots, m_{3g-3}, t_1, \ldots, t_{3g-3}) : \forall k, m_k \leq \frac{M}{a}, t_k \leq \frac{A_2}{\ell_o(\alpha_k)}\right\}\right) \quad \text{(by (4.1))}\\
&\leq A_3 M^{3g-3} \frac{1}{a^{3g-3}\prod_{k=1}^{3g-3}\ell_0(\alpha_k)} \quad \text{(by the definition of } \mu)\\
&= A_3 M^{3g-3}\ell^{3g-3} \leq A_3 M^{3g-3} A_1^{3g-3} \quad \text{(since } B_1 \leq \ell \leq A_1)\\
&\leq DM^{\frac{h}{2}}.
\end{aligned}$$

We now deduce the lower bound: $CM^{\frac{h}{2}} \le \nu(\{\eta \in \mathcal{PMF}(S) : i(\eta,\xi) \le M\})$. To bound the measure from below, we will construct a subset contained within this set and compute the measure of the subset.

First, fix a positive orientation for α_i for each i. Define

$$V(M) = \{t\eta \in \mathcal{MF}(S) : i(\eta,\xi) \le M, \ell_o(\eta) = 1, 0 \le t \le 1\}.$$

Then, $\eta^0 = \frac{1}{3g-3}\xi$ lies in $V(M)$. Let a and M be as above, and $\delta > 0$ be a positive number. Let $\kappa(g)$ be a sufficiently small positive number depending only on o. Define a set of $6g-6$-tuples $W_0(a, M, \delta)$ by

$$\{(x_1,\dots,x_{3g-3}, y_1,\dots,y_{3g-3}) : \forall i, 0 \le ax_i \le \kappa(g)M, 0 \le y_i\ell_o(\alpha_i) \le \delta\}.$$

Let ϖ be the coordinate map in Theorem 4.8. Then, on the one hand, by hyperbolic geometry (cf. [16, Lemma 3.2]), there exist constants $M_0 > 0$ and $\delta_0 = \delta_0(M_0)$, depending on o such that for all positive $M \le M_0$, the following holds:

$$\varpi^{-1}(\varpi(\eta^0) + W_0(a, M, \delta_0)) \subset V(M).$$

Notice that one can choose the constant a, depending on o and M_0 such that ϖ^{-1} is a homeomorphism on $\varpi(\eta^0) + W_0(a, M, \delta_0)$. Therefore, we have $\mu(V(M)) \ge \nu(\varpi^{-1}(\varpi(\eta^0) + W_0(a, M, \delta_0)))$. On the other hand,

$$\nu(\varpi^{-1}(\varpi(\eta^0) + W_0(a, M, \delta_0))) = \nu(\varpi^{-1}(W_0(a, M, \delta_0))).$$

This measure is clear to be at least $CM^{\frac{h}{2}}$, where C depends on M_0 and o. Hence, the proof is complete. ■

Proof of Theorem 4.1. Let $\gamma \in E_n$ and take $M = e^{-2n}$ in Corollary 4.1 and $\bar{M} = F \ln ne^{-2n}$ in Corollary 4.2, where F is the constant in Lemma 4.3. Then, Theorem 4.7 and Proposition 4.1 imply that the condition in Lemma 4.4 is satisfied, and hence, $\Psi(\gamma)_{\ge M} \sim_{g,o,\varepsilon} n$ and $\Psi(\gamma)_{\ge \bar{M}} \sim_{g,o,\varepsilon} a_1' n - c_1 \ln\ln n \sim a_1 n$. Finally, by Corollaries 4.1, 4.2, and Lemma 4.5, the proof is complete. ■

We can now prove the following theorem which will be used to demonstrate uniform boundedness. Let $\zeta \in U(\theta,\varepsilon)$. Define, for N sufficiently small,

$$\Psi(\zeta)_{\ge N} = \int_{\{\eta\in\mathcal{PMF}(S): i(\eta,\zeta)>N\}} \left(\frac{1}{i(\eta,\zeta)}\right)^{\frac{h}{2}} d\nu(\eta).$$

Theorem 4.9. *There exist constants $\delta_0 > 0$, $A > 0$ and $B > 0$, depending only on g,θ,o such that, for $0 < N < \delta_0$,*

$$\Psi(\zeta)_{\ge N} \le -A \ln N + B.$$

Proof. The proof follows by combining the last part of the proof of Lemma 4.4, with Lemma 4.5, Proposition 4.1, and Theorem 4.7. ■

5. Ergodicity of boundary representations

5.1. Main theorem

In this section, we will prove the main result namely, Theorem 2.17 of this paper.

Theorem 5.1. *Let $S = S_g (g \geq 2)$ and π_ν be the associated quasi-regular representation of the mapping class group* $\mathrm{Mod}(S)$ *on $L^2(\mathcal{PMF}(S), \nu)$. Let $n \gg \rho$ and $E_n = \mathcal{E}(\theta, \varepsilon, n, o, \rho)$ (up to a subsequence). Let $e_n = \mathrm{Pr} : E_n \to \mathcal{PMF}(S)$ be the radial projection, which maps $g \in E_n$ to the direction ξ_g of the oriented geodesic $[o, g \cdot o]$. Then, the quasi-regular representation π_ν is ergodic with respect to (E_n, e_n) and any $f \in \bar{H}^{L^\infty(\mathcal{PMF}(S),\nu)}$, where H is the vector space generated by*

$$\left\{\mathbb{1}_U : \nu(\partial U) = 0 \text{ and } U \text{ is a Borel subset of } \mathcal{PMF}(S)\right\}.$$

Proof. The proof consists of verifying all four conditions listed in Theorem 2.14 for E_n. The first two are derived from the fact that E_n has exponential growth (namely, Corollaries 3.1 and 3.2). The third one follows from Proposition 5.1. The last one is Theorem 5.4 in the next section. ∎

Proposition 5.1. *There exist two sequences of real numbers $\{h_{r_n}(n, \rho)\}$ and $\{r_n\}$ such that* $\lim_{n\to\infty} h_{r_n}(n, \rho) = \lim_{n\to\infty} r_n = 0$ *and $\forall n \gg \rho$, $\forall \gamma \in E_n$,*

$$\frac{\langle \pi_\nu(\gamma)\mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{\xi\in\mathcal{PMF}(S): i(\xi, e_n(\gamma))\geq r_n\}}\rangle}{\Phi(\gamma)} \leq h_{r_n}(n, \rho).$$

Proof. Now, let $n \gg \rho$ and $\gamma \in E_n$. Let ξ_γ be as before. Take $r_n = \frac{1}{n}$. By the Harish-Chandra estimates (Theorem 4.1), Corollary 4.1, Lemma 4.4, and the proof of its assumption (Theorem 4.7), there exist constants $c(g, o, \rho), a_1, b_1, D$ such that

$$\frac{\langle \pi_\nu(\gamma)\mathbb{1}_{\mathcal{PMF}(S)}, \mathbb{1}_{\{\xi\in\mathcal{PMF}(S): i(\xi, \xi_\gamma)\geq \frac{1}{n}\}}\rangle}{\Phi(\gamma)} \leq c(g, o, \rho)\frac{\ln n - D}{a_1 n + b_1}.$$

Take $h_{r_n}(n, \rho) = c(g, o, \rho)\frac{\ln n - D}{a_1 n + b_1}$, and we complete the proof. ∎

5.2. Uniform boundedness

In this section, we complete the proof of the main theorem by establishing the desired uniform boundedness property. We start with a few lemmas that compare two types of neighborhoods.

Lemma 5.2. *Using the notation introduced in the paragraph preceding Theorem 4.6. Suppose that $\gamma \in E_n$. Let $\xi_\gamma \in \mathcal{PMF}(S)$ be the direction of $[o, y = \gamma \cdot o]$ and $\xi^\gamma \in \mathcal{PMF}(S)$ the direction of $[o, t_y]$. Then, there exists a constant C such that $i(\xi_\gamma, \xi^\gamma) \leq Ce^{-2d(\gamma\cdot o, o)}$.*

Proof. By Lemma 4.2, one has

$$i^2(\xi_\gamma, \xi^\gamma) \le \mathrm{Ext}_{\gamma\cdot o}(\xi_\gamma)\,\mathrm{Ext}_{\gamma\cdot o}(\xi^\gamma).$$

Let $\alpha = \beta_\gamma$ be as defined previously (cf. the proof of Proposition 4.1). On the one hand, as shown in the proof of Proposition 4.1, we have $\xi^\gamma = \frac{\alpha}{\sqrt{\mathrm{Ext}_o(\alpha)}} \sim_{g,o} e^{-d(\gamma\cdot o,o)}\alpha$. Since, by construction, $\mathrm{Ext}_{\gamma\cdot o}(\alpha) \sim_{g,o} 1$, it follows that

$$\mathrm{Ext}_{\gamma\cdot o}(\xi^\gamma) = \frac{1}{\mathrm{Ext}_o(\alpha)}\,\mathrm{Ext}_{\gamma\cdot o}(\alpha) \prec_{g,o} e^{-2d(\gamma\cdot o,o)}.$$

On the other hand, $\mathrm{Ext}_{\gamma\cdot o}(\xi_\gamma) = e^{-2d(\gamma\cdot o,o)}$. By collecting all the terms, we complete the proof. ∎

Let $\eta \in \mathcal{PMF}(S)$ and $x \in [o,\eta]$, define

$$\mathcal{I}_C(\eta, x) = \left\{\xi \in \mathcal{PMF}(S) : i(\xi,\eta) \le Ce^{-2d(x,o)}\right\}.$$

Fix θ, ε as chosen according to our convention, and let $L_0 = L_0(\theta,\varepsilon)$ be as in Theorem 2.9.

Lemma 5.3. *Suppose that the geodesic ray $\eta \in \mathcal{PMF}(S)$ from o does not leave* $\mathrm{Teich}_\varepsilon(S)$ *eventually. Let $C \ge 1$ and $n \in \mathbb{N}$ be large enough such that $\frac{1}{3h}\ln\ln n \gg L_0$. Let $x \in [o,\eta]$ such that $d(x,o) = n$. If $\gamma \in E_n$ satisfies $\xi_\gamma \in \mathcal{I}_C(\eta,x)$, then $d(x,\gamma\cdot o) \le \frac{1}{h}\ln\ln n$.*

Proof. We argue as in Lemma 4.3. Denote $\xi = \xi_\gamma$. By assumption, $i(\xi,\eta) \le Ce^{-2n}$. Since both η and ξ are uniquely ergodic, we have a geodesic triangle $\triangle(o,\xi,\eta)$. As $\gamma \in E_n$, there is a geodesic segment I of length $\ell = \frac{1}{3h}\ln\ln n$ in $[o,\gamma\cdot o]$ ending at $p = \gamma\cdot o$ that has at least proportion θ in $\mathrm{Teich}_\varepsilon(S)$. By Theorem 2.9,

$$I \cap \mathcal{N}_D([o,\eta]\cap[\xi,\eta]) \ne \emptyset,$$

where D is the constant given in Theorem 2.9. Choose $q \in I \cap \mathcal{N}_D([o,\eta]\cap[\xi,\eta])$. Then, there are two possibilities.

Case 1. $d(q,y) \le D$ with $y \in [o,\eta]$.

In this case,

$$d(q,o) - D \le d(o,y) \le d(q,o) + D.$$

Since

$$n - \ell - \rho \le d(q,o) \le n + \rho,$$

we have

$$0 \le d(x,y) \le \ell + D + \rho.$$

Hence,

$$\begin{aligned} d(x, \gamma \cdot o) &\le d(x, y) + d(y, q) + d(q, p) \\ &\le \ell + D + D + \ell + \rho \\ &\le 2(\ell + D + \rho) \\ &\le 3\ell. \end{aligned}$$

Case 2. $d(q, y) \le D$ with $y \in [\xi, \eta]$.

By Lemma 4.2, one has

$$i^2(\eta, \xi) = \mathrm{Ext}_y(\xi)\,\mathrm{Ext}_y(\eta).$$

Now, since $d(q, y) \le D$, by Kerckhoff's formula, we have

$$e^{-2D}\,\mathrm{Ext}_q(\xi) \le \mathrm{Ext}_y(\xi), \quad e^{-2D}\,\mathrm{Ext}_q(\eta) \le \mathrm{Ext}_y(\eta).$$

Therefore,

$$e^{-4D}\,\mathrm{Ext}_q(\xi)\,\mathrm{Ext}_q(\eta) \le i^2(\xi, \eta).$$

On the other hand, we have

$$\mathrm{Ext}_q(\xi) = e^{-2d(o,q)}, \quad i(\xi, \eta) \le Ce^{-2n}.$$

We can deduce that

$$e^{-4D}\,\mathrm{Ext}_q(\eta)e^{-2d(o,q)} \le C^2 e^{-4n}.$$

That is,

$$e^{-4D}\,\mathrm{Ext}_q(\eta)e^{2(n-d(o,q))} \le C^2 e^{-2n}.$$

By Kerckhoff's formula again,

$$\mathrm{Ext}_p(\eta) \le C^2 e^{2\rho+4D} e^{-2n},$$

i.e.,

$$\frac{1}{2}\ln \mathrm{Ext}_p(\eta) \le \ln(Ce^{\rho+2D}) - n.$$

Applying Lemma 3.3, one could choose $z \in [o, \eta] \cap \mathrm{Teich}_\varepsilon(S)$ such that, if we denote $d(o, p) = t$, $d(p, z) = a$ and $d(z, o) = b$, $a - b \le -n + \ln(Ce^{2\rho+4D}) + 1$. Therefore, we have

$$0 \le t + a - b \le \ln(Ce^{2\rho+4D}) + 1 + \rho = c_1.$$

Note that c_1 is a constant depending on $C, \varepsilon, \theta, \rho$.

Now, consider the geodesic triangle $\Delta(o, p, z)$. Since the side $[o, p]$ has a segment $I = [s, p]$ ending at p which has at least proportion θ in the thick part, take the midpoint m of I, then, as $\theta = 0.999$, the subsegment $[s, m]$ has at least 0.1 in the thick part. By Theorem 2.9 again, there exist $g \in [s, m]$ and a constant D' when $n \gg 0$ such that

$$d(g, [o, z] \cup [p, z]) \leq D'.$$

If there exists $h_1 \in [o, z]$ such that $d(h_1, g) \leq D'$, the proof can be complete as in Case 1. Otherwise, there exists $h_2 \in [p, z]$ such that $d(h_2, g) \leq D'$. Then,

$$\begin{aligned} t + a - b &= d(o, g) + d(g, p) + d(p, h_2) + d(h_2, z) - d(o, z) \\ &= d(o, g) + D' + d(h_2, z) - d(o, z) + d(g, p) + d(p, h_2) - D' \\ &\geq \ell - 2D' \to \infty, \end{aligned}$$

which is impossible, since $t + a - b$ is bounded above by c_1. ■

Corollary 5.1. *Suppose that the geodesic ray $\eta \in \mathcal{PMF}(S)$ from o does not leave* $\mathrm{Teich}_\varepsilon(S)$ *eventually. Let $C > 0$ and $n \in \mathbb{N}$ large enough. Let $x \in [o, \eta]$ such that $d(x, o) = n$. Then,*

$$|\{\gamma \in E_n : \gamma \cdot o \in \mathrm{Sec}_{\mathcal{I}_C(\eta, x)}\}| \prec_{g,o,\rho} n.$$

Proof. By [1, Theorem 1.2], when n is large enough, there exists a constant $N_0 > 0$ such that $|B(x, R) \cap \mathrm{Mod}(S) \cdot o| \leq N_0 e^{hR}$. Applying Lemma 5.3, we obtain the conclusion. ■

Theorem 5.4. *Under the notation used in Theorem 5.1, we have*

$$\sup_n \left\| M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)} \right\|_{L^\infty(\mathcal{PMF}(S), \nu)} < \infty.$$

Recall that

$$\begin{aligned} &M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}([\xi]) \\ &\quad = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \frac{\pi_\nu(\gamma) \mathbb{1}_{\mathcal{PMF}(S)}([\xi])}{\Phi(\gamma)} \\ &\quad = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{\mathrm{Ext}_o(\xi)}{\mathrm{Ext}_{\gamma \cdot o}(\xi)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}. \end{aligned}$$

By using the embedding map τ of $\mathcal{PMF}(S)$ into $\mathcal{MF}(S)$, one can rewrite the above formula as

$$\begin{aligned} &M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}([\xi]) \\ &\quad = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{1}{\mathrm{Ext}_{\gamma . o}(\xi)} \right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}. \end{aligned}$$

First, we introduce a type of open set $\mathcal{IN}$ in $\mathcal{PMF}(S)$ defined by intersection numbers. For every $\eta \in \mathcal{PMF}(S)$, $C > 0$, and $t > 0$, define

$$\mathcal{IN}(\eta, t, C) = \left\{\xi \in \mathcal{PMF}(S) : i(\xi, \eta) \leq Ce^{-2t}\right\}.$$

Proof of Theorem 5.4. Fix $R > R_0$, where R_0 is the constant from Lemma 3.2. Let ε sufficiently small so that, for all $g \in \mathrm{Mod}(S)$, the open ball $B(g \cdot o, R) \subset \mathrm{Teich}_\varepsilon(S)$, and Theorem 2.11 holds. We know that $U(\varepsilon, \theta)$ is a subset of $\mathcal{PMF}(S)$ of full measure. We shall give a bound, independent of $n \gg \rho$, for $M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}(\zeta)$ for every point $\zeta \in U(\varepsilon, \theta)$. As usual, for $\gamma \in E_n$, denote ξ_γ to be the direction corresponding to $[o, \gamma \cdot o]$, hence a point in $\mathcal{PMF}(S)$. For each point $\gamma \cdot o$, consider the open ball $B(\gamma, R)$ of radius R centered at $\gamma \cdot o$. Denote the projection of $B(\gamma, R)$ to $\mathcal{PMF}(S)$ by $\mathcal{O}(\gamma \cdot o, R)$. Then, by Lemma 3.2, the measure $\nu(\mathcal{O}(\gamma \cdot o, R)) \sim_{g,R,\rho} e^{-hn}$.

Fix any $C \geq 1$, for instance, $C = 1$. Divide E_n into two sets E_n^1 and $E_n^2 = E_n - E_n^1$, where E_n^1 consists of $\gamma \in E_n$ such that $\xi_\gamma \notin \mathcal{IN}(\zeta, n, C)$. We then have, for each $\zeta \in U(\varepsilon, \theta)$,

$$\begin{aligned}
&M_{E_n}^{\mathbb{1}_{\mathcal{PMF}(S)}} \mathbb{1}_{\mathcal{PMF}(S)}(\zeta) \\
&\quad = \frac{1}{|E_n|} \sum_{\gamma \in E_n} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
&\quad = \underbrace{\frac{1}{|E_n|} \sum_{\gamma \in E_n^1} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}}_{\mathrm{I}} + \underbrace{\frac{1}{|E_n|} \sum_{\gamma \in E_n^2} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}}_{\mathrm{II}} \\
&\quad = \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{5.1}$$

We want to bound term I in equation (5.1). The set E_n^1 can be further decomposed into two sets: F_n^1 and $F_n^2 = E_n^1 - F_n^1$, where F_n^1 is defined as follows:

$$F_n^1 = \left\{\gamma \in E_n^1 : \mathcal{O}(\gamma \cdot o, R) \cap \mathcal{IN}(\zeta, n, C) = \emptyset\right\}.$$

One obtains

$$\begin{aligned}
\mathrm{I} &= \underbrace{\frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}}_{\mathrm{III}} + \underbrace{\frac{1}{|E_n|} \sum_{\gamma \in F_n^2} \left(\frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)}}_{\mathrm{IV}} \\
&= \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

For term III, notice that,

$$\forall y \in B(\gamma \cdot o, R), \quad \frac{1}{\mathrm{Ext}_{\gamma \cdot o}(\zeta)} \sim_R \frac{1}{\mathrm{Ext}_y(\zeta)}.$$

By Lemma 4.2, for ν-almost every endpoint $\xi_y \in \mathcal{O}(\gamma \cdot o, R)$,

$$\left(\frac{1}{\mathrm{Ext}_y(\zeta)}\right)^{\frac{h}{4}} \prec_{\rho,R} \frac{1}{e^{\frac{hn}{2}}(i(\xi_y,\zeta))^{\frac{h}{2}}}.$$

Hence, for ν-almost every $\xi \in \mathcal{O}(\gamma \cdot o, R)$,

$$\left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\zeta)}\right)^{\frac{h}{4}} \prec_{\rho,R} e^{-\frac{hn}{2}} \frac{1}{(i(\xi,\zeta))^{\frac{h}{2}}}.$$

Therefore,

$$\begin{aligned}
\mathrm{III} &= \frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
&\prec_R \frac{1}{|E_n|} \sum_{\gamma \in F_n^1} \frac{e^{-\frac{hn}{2}}}{\nu(\mathcal{O}(\gamma\cdot o, R))} \int_{\mathcal{O}(\gamma\cdot o,R)} \frac{1}{(i(\eta,\zeta))^{\frac{h}{2}}} d\nu(\eta) \frac{1}{\Phi(\gamma)}.
\end{aligned}$$

Note that there is a bounded number of intersections of open sets of the form $\mathcal{O}(\gamma \cdot o, R)$, and the bound depends on R and ρ. Thus, since $|E_n| \asymp e^{hn}$ (Corollary 3.1) and $\Phi(\gamma) \succ_{g,o,\rho} (a_1 n + b_1) e^{-\frac{hn}{2}}$ (Theorem 4.1), by combining all these estimates, one obtains

$$\mathrm{III} \prec_{g,o,\rho,R} \frac{1}{a_1 n + b_1} \int_{\{\eta \in \mathcal{PMF}(S): i(\eta,\zeta) > Ce^{-2n}\}} \left(\frac{1}{i(\eta,\zeta)}\right)^{\frac{h}{2}} d\nu(\eta) \prec_{g,o,\rho,R} 1.$$

The last inequality follows from the fact that $\zeta \in U(\varepsilon,\theta)$ and Theorem 4.9.

For terms II and IV, define $H_n = E_n^2 \cup F_n^2$, and hence,

$$\begin{aligned}
\mathrm{II} + \mathrm{IV} &= \frac{1}{|E_n|} \sum_{\gamma \in H_n} \left(\frac{1}{\mathrm{Ext}_{\gamma\cdot o}(\zeta)}\right)^{\frac{h}{4}} \frac{1}{\Phi(\gamma)} \\
&\le \frac{1}{|E_n|} \sum_{\gamma \in H_n} \frac{e^{\frac{hd(o,\gamma\cdot o)}{2}}}{\Phi(\gamma)} \\
&\sim_{g,\rho,o} e^{-hn} \sum_{\gamma \in H_n} \frac{e^{\frac{hn}{2}}}{(a_1 n + b_1) e^{\frac{-hn}{2}}} \\
&= \frac{1}{a_1 n + b_1} |H_n|.
\end{aligned}$$

We complete the proof by showing that $|H_n| \prec_{g,o,\rho,\varepsilon,\theta,R} n$, which implies that the sum II + IV is uniformly bounded.

By Corollary 5.1, $|E_n^2| \prec n$. We now show that so is $|F_n^2|$. The idea is to bound the distance between a nearby point to $\gamma \cdot o$ and the unique point $x \in [o, \zeta]$ with $d(x, o) = n$, which allows us to bound $|F_n^2|$ by n.

To this end, by the choice of R and ε, for every $\gamma \in \mathrm{Mod}(S)$ and every point $w \in B(\gamma \cdot o, R)$, we have $w \in \mathrm{Teich}_\varepsilon(S)$ and $d_T(\gamma \cdot o, w) \prec_R 1$. Assume now that $\gamma \in F_n^2$, meaning that $\mathcal{O}(\gamma \cdot o, R) \cap \mathcal{IN}(\zeta, n, C) \neq \emptyset$. Since $U(\varepsilon, \theta)$ has full measure, and, in particular, it is dense in $\mathcal{PMF}(S)$, one can choose $w \in B(\gamma \cdot o, R)$ such that the direction ξ_w of $[o, w]$ lies in $U(\varepsilon, \theta) \cap \mathcal{IN}(\zeta, n, C)$. By Theorem 2.10, there exists a constant $P = P(\varepsilon, R)$ such that the two geodesics $[o, \gamma \cdot o]$ and $[o, w]$ are P-fellow traveling in a parametrized fashion. Now, consider the P-neighborhood $\mathcal{N}_P$ of $\mathrm{Teich}_\varepsilon(S)$, namely, the union of points in $\mathrm{Teich}(S)$ that has distance at most P from a point in $\mathrm{Teich}_\varepsilon(S)$. As $\mathrm{Mod}(S)$ acts as isometries on $\mathrm{Teich}(S)$ and $\mathrm{Teich}_\varepsilon(S)$ is $\mathrm{Mod}(S)$-invariant and co-compact, the neighborhood $\mathcal{N}_P$ is $\mathrm{Mod}(S)$-invariant and co-compact. By Mumford's compactness theorem, there exists a small $\varepsilon' > 0$ such that $\mathcal{N}_P \subset \mathrm{Teich}_{\varepsilon'}(S)$.

It follows that since $\gamma \in E_n$, the geodesic segment $[o, w]$ contains a subsegment $I = [a, w]$ of length $\frac{1}{3h} \ln \ln n$ such that I has at least θ in $\mathrm{Teich}_{\varepsilon'}(S)$. Theorem 2.9 implies that there exist constants $D' = D'(\varepsilon', \theta)$ and $L'_0 = L'_0(\varepsilon', \theta)$ that satisfy the thin triangle property in Theorem 2.9. Take n large enough. We then follow the entire argument in the proof of Lemma 5.3, replacing $p = \gamma \cdot o$ therein with w, to bound the distance between w and $x \in [o, \zeta]$ with $d(x, o) = n$ by $\ln \ln n$. Since $d_T(\gamma \cdot o, w) \prec_R 1$, this gives a bound between $\gamma \cdot o$ and x. Since x is a fixed point, arguing as in the proof of Corollary 5.1, one has $|F_n^2| \prec n$. ∎

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