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Complex embeddings, Toeplitz operators and transitivity of optimal holomorphic extensions

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Abstract. In a setting of a complex manifold with a positive line bundle and a submanifold, we consider the optimal Ohsawa–Takegoshi extension operator, sending a holomorphic section of the line bundle on the submanifold to the holomorphic extension of it on the ambient manifold with the minimal L^2-norm. We show that for a tower of submanifolds and large tensor powers of the line bundle, the extension operators act transitively modulo some small defect, which is a Toeplitz type operator. We calculate the first significant term in the asymptotic expansion of this "transitivity defect". As a byproduct, we deduce composition rules for Toeplitz type operators, the extension and restriction operators and calculate the second term in the asymptotic expansion of the optimal constant in the semi-classical version of the extension theorem.

Contents

1. Introduction

One of the main goals of this paper is to prove that for a tower of submanifolds, the transitivity property is satisfied for the Ohsawa–Takegoshi extension operator, sending holomorphic sections on the submanifold to their holomorphic extensions on the ambient manifold with the minimal L^2-norm, modulo some small error, which is a Toeplitz type operator.

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More precisely, we fix two (not necessarily compact) complex manifolds X, Y of dimensions n and m respectively. We fix also a complex embedding $\iota: Y \to X$, a positive line bundle (L, h^L) over X and an arbitrary Hermitian vector bundle (F, h^F) over X. In particular, we assume that for the curvature R^L of the Chern connection on (L, h^L), the closed real $(1,1)$-differential form

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L$$

is positive. We denote by g^{TX} the Riemannian metric on X induced by ω as follows

$$g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot), \tag{1.1}$$

where $J: TX \to TX$ is the complex structure on X. We denote by g^{TY} the induced metric on Y.

Note. *We assume throughout the whole article that the triple (X, Y, g^{TX}), and the Hermitian vector bundles (L, h^L), (F, h^F) are of bounded geometry in the sense of Definitions 2.3 and 2.5.*

This means that we assume uniform lower bounds $r_X, r_Y > 0$ on the injectivity radii of X, Y, the existence of the geodesic tubular neighborhood of Y of uniform size $r_\perp > 0$ in X, and some uniform bounds on related curvatures and the second fundamental form of the embedding.

Now, we fix some positive (with respect to the orientation given by the complex structure) volume forms dv_X, dv_Y on X and Y. For smooth sections f, f' of $L^p \otimes F$ over X, we define the L^2-scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_h$ induced by h^L and h^F as follows

$$\langle f, f' \rangle_{L^2(X)} := \int_X \langle f(x), f'(x) \rangle_h \, dv_X(x). \tag{1.2}$$

Similarly, using dv_Y, we introduce the L^2-scalar product for sections of $\iota^*(L^p \otimes F)$ over Y. We denote by $L^2(X, L^p \otimes F), L^2(Y, \iota^*(L^p \otimes F))$ the spaces of L^2-sections of $L^p \otimes F$ over X and Y.

Given a continuous smoothing linear operator

$$K: L^2(X, L^p \otimes F) \to L^2(X, L^p \otimes F),$$

the Schwartz kernel theorem guarantees the existence of the Schwartz kernel

$$K(x_1, x_2) \in (L^p \otimes F)_{x_1} \otimes (L^p \otimes F)^*_{x_2}, \quad x_1, x_2 \in X,$$

evaluated with respect to dv_X, i.e.

$$(Ks)(x_1) = \int_X K(x_1, x_2) \cdot s(x_2) \, dv_X(x_2), \quad s \in L^2(X, L^p \otimes F).$$

Similarly, we define the Schwartz kernels $K_1(y,x)$, $K_2(x,y)$, $x \in X$, $y \in Y$, for smoothing operators

$$K_1\colon L^2(X, L^p \otimes F) \to L^2\big(Y, \iota^*(L^p \otimes F)\big),$$
$$K_2\colon L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F)$$

with respect to the volume forms dv_X and dv_Y respectively.

For a Hermitian vector bundle (E, h^E) over X, we denote

$$\mathscr{C}^\infty_b(X,E) := \{f \in \mathscr{C}^\infty(X,E) : \text{for any } k \in \mathbb{N}, \text{ there is } C > 0 \text{ such that } |\nabla^k f| \leq C\},$$

where ∇ is the connection induced by the Chern connection on E and the Levi-Civita connection on TX, and $|\cdot|$ is the norm induced by the metrics g^{TX}, h^E. Assume that for the Riemannian volume forms $dv_{g^{TX}}$, $dv_{g^{TY}}$ of (X, g^{TX}), (Y, g^{TY}), we have

$$\frac{dv_{g^{TX}}}{dv_X}, \frac{dv_X}{dv_{g^{TX}}} \in \mathscr{C}^\infty_b(X), \qquad \frac{dv_{g^{TY}}}{dv_Y}, \frac{dv_Y}{dv_{g^{TY}}} \in \mathscr{C}^\infty_b(Y). \tag{1.3}$$

We denote by $H^0_{(2)}(X, L^p \otimes F)$ and $H^0_{(2)}(Y, \iota^*(L^p \otimes F))$ the vector spaces of holomorphic sections of $L^p \otimes F$ over X and Y respectively with bounded L^2-norm. In [9, §4], by relying on the bounded geometry assumption, we proved that the restriction to Y of any L^2-holomorphic section, defined on X, has finite L^2-norm. In other words, the operator

$$\mathrm{Res}^{Y|X}_p\colon H^0_{(2)}(X, L^p \otimes F) \to H^0_{(2)}\big(Y, \iota^*(L^p \otimes F)\big), \quad f \mapsto f|_Y, \tag{1.4}$$

is well defined. By extending the Ohsawa–Takegoshi theorem in [9, Theorem 4.1] (cf. [6, 20, 21], and [5, §13]), we established that there is $p_1 \in \mathbb{N}$ such that (1.4) is surjective for any $p \geq p_1$. The right inverse of this restriction, defined for $p \geq p_1$ by taking the holomorphic extension with the minimal L^2-norm, is called the (*Ohsawa–Takegoshi*) *extension* operator, and it is denoted by

$$\mathrm{E}^{X|Y}_p\colon H^0_{(2)}\big(Y, \iota^*(L^p \otimes F)\big) \to H^0_{(2)}(X, L^p \otimes F). \tag{1.5}$$

We identify the normal bundle $N^{X|Y}$ of Y in X as an orthogonal complement of TY in TX (with respect to g^{TX}), so that we have the following orthogonal decomposition

$$TX|_Y \to TY \oplus N^{X|Y}. \tag{1.6}$$

We denote by $g^{N^{X|Y}}$ the metric on $N^{X|Y}$ induced by g^{TX}, and by $P^{X|Y}_N$ the induced projection from $TX|_Y$ to $N^{X|Y}$. By an abuse of notation, we denote the induced projection from $(TX|_Y)^*$ to $(N^{X|Y})^*$ by the same symbol.

For $y \in Y$, $Z_N \in N_y^{X|Y}$, let $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$ be the geodesic in X in the direction Z_N. Bounded geometry condition means, in particular, that this map induces a diffeomorphism of $r_\perp$-neighborhood of the zero section in $N^{X|Y}$ with a tubular neighborhood U of Y in X.

Using this diffeomorphism, we define $\kappa_N^{X|Y} : U \to \mathbb{R}_+$ as the only function verifying

$$dv_X = \kappa_N^{X|Y} \cdot dv_Y \wedge dv_{N^{X|Y}}, \tag{1.7}$$

where $dv_{N^{X|Y}}$ is the relative Riemannian volume form on $(N^{X|Y}, g^{N^{X|Y}})$. We have $\kappa_N^{X|Y}|_Y = 1$ if

$$dv_X = dv_{g^{TX}}, \quad dv_Y = dv_{g^{TY}}. \tag{1.8}$$

Let us now fix a tower of submanifolds $Y \overset{\iota_1}{\hookrightarrow} W \overset{\iota_2}{\hookrightarrow} X$, $\iota := \iota_2 \circ \iota_1$ of dimensions m, l and n respectively. In addition to the volume forms dv_X, dv_Y, we fix a positive volume form dv_W on W, verifying assumptions similar to (1.3) with respect to the metric g^{TW} induced by g^{TX}. We assume, moreover, that the triples (X, W, g^{TX}), (W, Y, g^{TW}) are of bounded geometry in the sense of Definition 2.3. We denote by $\mathbf{r}^X$ (resp. $\mathbf{r}^W$) the scalar curvature of X (resp. W), and let $\Lambda_\omega[R^F] \in \mathrm{End}(F)$ be the contraction of the curvature of the Chern connection of (F, h^F) with the Kähler form ω. We denote by $\Lambda_{\iota_2^*\omega}[R^F] \in \mathrm{End}(\iota_2^* F)$ the analogous contraction defined on W.

We denote by $(N^{X|Y})^{(1,0)}$, $(N^{X|Y})^{(0,1)}$ the holomorphic, antiholomorphic components of $N^{X|Y} \otimes \mathbb{C}$, corresponding to $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of the induced complex structure action.

Theorem 1.1. *There are $p_1 \in \mathbb{N}^*$, $C > 0$, such that for any $p \geq p_1$, we have*

$$\| \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y} \| \leq C \cdot p^{-(n-m+1)/2}, \tag{1.9}$$

where $\| \cdot \|$ denotes the operator norm. Moreover, under the assumptions (1.8) and $dv_W = dv_{g^{TW}}$,

$$\| \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y} \| \sim C_0 \cdot p^{-(n-m+3)/2}. \tag{1.10}$$

The constant $C_0 \geq 0$ above is defined as follows:

$$C_0 := \frac{1}{\sqrt{\pi}} \sup_{y \in Y} \Big\| \frac{1}{8\pi} \partial_{N^{W|Y}} (\mathbf{r}_y^X - \mathbf{r}_y^W) \cdot \mathrm{Id}_{F_y} - \frac{1}{2\pi\sqrt{-1}} \nabla^{1,0}_{N^{W|Y}} \big(\Lambda_\omega[R_y^F] - \Lambda_{\iota_2^*\omega}[R_y^F] \big) \Big\|,$$

where the operator

$$\partial_{N^{W|Y}} : \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty\big(Y, (N^{W|Y})^{(1,0)*}\big)$$

is defined by the composition $P_N^{W|Y} \circ \partial$*, the operator*

$$\nabla^{1,0}_{N^{W|Y}} : \mathscr{C}^\infty\big(Y, \mathrm{End}(\iota^* F)\big) \to \mathscr{C}^\infty\big(Y, (N^{W|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big)$$

is similarly defined by the composition $P_N^{W|Y} \circ \nabla^{1,0}$ *for the* $(1,0)$*-component* $\nabla^{1,0}$ *of the Chern connection on* $\mathrm{End}(F)$*, endowed with the induced Hermitian metric, and the norm is considered as a norm of an element from* $(N^{W|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)$ *with the induced metric.*

Remark 1.2. (a) In [9, Theorem 1.1], we obtained that, as $p \to \infty$,

$$\| \mathrm{E}_p^{X|Y} \| \sim \sup_{y \in Y} \kappa_N^{X|Y}(y)^{1/2} \cdot p^{-(n-m)/2}. \tag{1.11}$$

Hence, (1.9) means that the "defect of transitivity" for the extension operator is of lower order of magnitude than the operator itself. We call this property the *asymptotic transitivity*.

(b) Bounded geometry condition implies that C_0 is a finite number.

(c) The estimate (1.9) alone can be obtained directly from [9, Theorem 1.1] and some local calculations, following from Section 2.4. Our motivation towards refining (1.9) to (1.10) was to understand the limits of transitivity. The stronger version (1.10) implies, in particular, that for generic submanifolds the defect of transitivity is not negligible.

The main goal of this paper is to develop the theory of Toeplitz operators in the setting of restriction and extension operators, implying a more precise asymptotic description of the sequence of operators

$$\mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y} : H^0_{(2)}\big(Y, \iota^*(L^p \otimes F)\big) \to H^0_{(2)}(X, L^p \otimes F), \quad p \geq p_1.$$

Remark that for different p, those operators act on different spaces, so the phrase "asymptotic description" itself has to be explained. For this, we introduce below *Toeplitz type operators*.

Let $H^{0,Y\perp}_{(2)}(X, L^p \otimes F)$ be the vector space of L^2-holomorphic sections which are orthogonal (with respect to the L^2-scalar product (1.2)) to L^2-holomorphic sections vanishing along Y. Denote by $B_p^{X|Y\perp}$, B_p^X the orthogonal projections from $L^2(X, L^p \otimes F)$ to $H^{0,Y\perp}_{(2)}(X, L^p \otimes F)$ and $H^0_{(2)}(X, L^p \otimes F)$, respectively. The operator B_p^X (resp. $B_p^{X|Y\perp}$) will be called the *Bergman projector* (resp. *orthogonal Bergman projector*). We extend $\mathrm{E}_p^{X|Y}$ to $L^2(Y, \iota^*(L^p \otimes F))$ as $f \mapsto (\mathrm{E}_p^{X|Y} \circ B_p^Y) f$.

Now, for a section $f \in \mathscr{C}^\infty_b(X, \mathrm{End}(F))$, we associate a sequence of linear operators $T_{f,p}^X \in \mathrm{End}(L^2(X, L^p \otimes F))$, $p \in \mathbb{N}$, called the *Berezin–Toeplitz operator*, by

$$T_{f,p}^X(g) := B_p^X(f \cdot B_p^X g).$$

We define the sequences of operators

$$T_{f,p}^{Y|X}: L^2(X, L^p \otimes F) \to L^2\big(Y, \iota^*(L^p \otimes F)\big),$$
$$T_{f,p}^{X|Y}: L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F), \quad p \in \mathbb{N},$$

by

$$\begin{aligned} T_{f,p}^{Y|X} &:= \mathrm{Res}_p^{Y|X} \circ T_{f,p}^X \circ (B_p^X - B_p^{X|Y\perp}), \\ T_{f,p}^{X|Y} &:= (B_p^X - B_p^{X|Y\perp}) \circ T_{f,p}^X \circ \mathrm{E}_p^{X|Y}. \end{aligned} \tag{1.12}$$

As we show in Proposition 4.8, the asymptotic study of operators $T_{f,p}^{Y|X}, T_{f,p}^{X|Y}$, fundamental to this paper, reduces to their study for some functions f, polynomial-like in the normal directions to Y. To describe these functions precisely, we fix a smooth function $\rho: \mathbb{R}_+ \to [0, 1]$, satisfying

$$\rho(x) = \begin{cases} 1 & \text{for } x < \frac{1}{4}, \\ 0 & \text{for } x > \frac{1}{2}. \end{cases} \tag{1.13}$$

Let $\pi: N^{X|Y} \to Y$ be the natural projection. We fix

$$g \in \mathscr{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big), \quad k \in \mathbb{N},$$

and construct a section $\{g\} \in \mathscr{C}^\infty(N^{X|Y}, \pi^* \mathrm{End}(\iota^* F))$, polynomial in the vertical directions, as follows $\{g\}(y, Z_N) := g(y) \cdot Z_N^{\otimes k}$, $y \in Y$, $Z_N \in N_y^{X|Y}$.

Recall that we introduced a diffeomorphism of $r_\perp$-neighborhood of the zero section in $N^{X|Y}$ with a tubular neighborhood U of Y in X after (1.6). By an abuse of notation, we denote by $\pi: U \to Y$ the projection $(y, Z_N) \mapsto y$ induced by π and the above diffeomorphism. Over U, we identify L, F to $\pi^*(\iota^* L), \pi^*(\iota^* F)$ by the parallel transport with respect to Chern connections along the geodesic

$$[0, 1] \ni t \mapsto (y, tZ_N) \in X, \quad |Z_N| < r_\perp.$$

From now on, we use these identifications implicitly. For fixed $p \in \mathbb{N}^*$, over U, we define the section $\langle\!\langle g \rangle\!\rangle \in \mathscr{C}_b^\infty(X, \mathrm{End}(F))$ as

$$\langle\!\langle g \rangle\!\rangle(y, Z_N) := p^{k/2} \cdot \rho\Big(\frac{|Z_N|}{r_\perp}\Big) \cdot \{g\}(y, Z_N), \tag{1.14}$$

where the norm $|Z_N|$, is taken with respect to $g^{N^{X|Y}}$. Away from U, we extend $\langle\!\langle g \rangle\!\rangle$ by zero. We extend the operator $\langle\!\langle \cdot \rangle\!\rangle$ linearly to

$$\bigoplus_{k=0}^{\infty} \mathscr{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big).$$

From (1.14), $\langle\!\langle\cdot\rangle\!\rangle$ clearly depends on p. We decided to neglect this in the notation because the symbol $\langle\!\langle\cdot\rangle\!\rangle$ will always appear in conjunction with another quantities, which depend explicitly on p.

Definition 1.3. A sequence of linear operators $T_p^Y \in \operatorname{End}\big(L^2(Y, \iota^*(L^p \otimes F))\big)$, $p \in \mathbb{N}$, respectively,

$$T_p^{Y|X} : L^2(X, L^p \otimes F) \to L^2\big(Y, \iota^*(L^p \otimes F)\big),$$
$$T_p^{X|Y} : L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F),$$

verifying $B_p^Y \circ T_p^Y \circ B_p^Y = T_p^Y$, respectively,

$$B_p^Y \circ T_p^{Y|X} \circ (B_p^X - B_p^{X|Y\perp}) = T_p^{Y|X}, \quad (B_p^X - B_p^{X|Y\perp}) \circ T_p^{X|Y} \circ B_p^Y = T_p^{X|Y},$$

is called a *Toeplitz operator with exponential decay* (resp. *of type* $Y|X$, $X|Y$) if there is a sequence $f_i \in \mathcal{C}_b^\infty(Y, \operatorname{End}(\iota^* F))$, respectively,

$$g_i^h \in \bigoplus_{k=0}^{\infty} \mathcal{C}_b^\infty\big(Y, \operatorname{Sym}^{2k+j}(N^{X|Y})^{(1,0)*} \otimes \operatorname{End}(\iota^* F)\big),$$
$$g_i^a \in \bigoplus_{k=0}^{\infty} \mathcal{C}_b^\infty\big(Y, \operatorname{Sym}^{2k+j}(N^{X|Y})^{(0,1)*} \otimes \operatorname{End}(\iota^* F)\big),$$

where $j \in \{1,2\}$ is of the same parity as i, and $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, the Schwartz kernels, evaluated with respect to dv_X, dv_Y for $y_1, y_2 \in Y$, $x \in X$, satisfy

$$\Big| T_p^Y(y_1, y_2) - \sum_{r=0}^{k} p^{-r} T_{f_r,p}^Y(y_1, y_2) \Big|_{\mathcal{C}^l} \leq C p^{m-k+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \operatorname{dist}_Y(y_1, y_2)\big),$$
$$\Big| T_p^{X|Y}(x, y_1) - \sum_{r=0}^{k} p^{-r/2} T^{X|Y}_{\langle\!\langle g_r^h \rangle\!\rangle, p}(x, y_1) \Big|_{\mathcal{C}^l} \leq C p^{m+(l-k)/2} \cdot \exp\big(-c\sqrt{p} \cdot \operatorname{dist}_X(x, y_1)\big),$$
$$\Big| T_p^{Y|X}(y_1, x) - \sum_{r=0}^{k} p^{-r/2} T^{Y|X}_{\langle\!\langle g_r^a \rangle\!\rangle, p}(y_1, x) \Big|_{\mathcal{C}^l} \leq C p^{n+(l-k)/2} \cdot \exp\big(-c\sqrt{p} \cdot \operatorname{dist}_X(y_1, x)\big), \tag{1.15}$$

where the pointwise $\mathcal{C}^l$-norm at a point $(y_1, y_2) \in Y \times Y$ is the sum of the norms induced by h^L, h^F and g^{TX}, evaluated at (y_1, y_2), of the derivatives up to order l with respect to the connection induced by the Chern connections on L, F and the Levi-Civita connection on TY, and similar notations are used for the other two norms at points $(y_1, x) \in Y \times X$ and $(x, y_1) \in X \times Y$. The sections f_i (resp. g_i^h, g_i^a) will later be denoted by $[T_p^X]_i$ (resp. $[T_p^{X|Y}]_i$, $[T_p^{Y|X}]_i$). We alternatively call the above operators *Toeplitz type operators* (*with exponential decay*).

Remark 1.4. (a) In Proposition 2.9, we show that (1.15) implies that for any $k \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, we have

$$\left\| T_p^Y - \sum_{r=0}^{k} p^{-r} T_{f_r,p}^Y \right\| \leq C p^{-k}, \quad \left\| T_p^{X|Y} - \sum_{r=0}^{k} p^{-r/2} T_{\langle\langle g_r^h \rangle\rangle,p}^{X|Y} \right\| \leq C p^{-(n-m+k)/2},$$
$$\left\| T_p^{Y|X} - \sum_{r=0}^{k} p^{-r/2} T_{\langle\langle g_r^a \rangle\rangle,p}^{Y|X} \right\| \leq C p^{(n-m-k)/2}.$$

In particular, for compact Y, the sequence of operators T_p^Y, $p \in \mathbb{N}$, forms a Toeplitz operator in the sense of Ma–Marinescu [14, §7].

(b) As we show in Corollary 3.6, our definition ultimately does not depend on the choice of ρ.

(c) In Corollary 3.13, we show that the sections f_i, $i \in \mathbb{N}$, (resp. g_i^h, g_i^a), verifying (1.15), are uniquely defined. Hence, the notation $[\cdot]_i$, $i \in \mathbb{N}$, from Definition 1.3 is well defined.

To state our main result, we place ourselves in the notations and assumptions of Theorem 1.1.

Theorem 1.5. *The sequence of operators*

$$D_p := \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y}, \quad p \in \mathbb{N},$$

forms a Toeplitz operator with exponential decay of type $X|Y$. Moreover, we have $[D_p]_0 = 0$. Also, under the assumptions (1.8) *and $dv_W = dv_{g^{TW}}$, we have*

$$[D_p]_1 = 0, \quad [D_p]_2 = 0, \quad [D_p]_3 \in \mathscr{C}_b^\infty\big(Y, (N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big)$$

for $n \in (N^{X|W})^{(1,0)}$, $[D_p]_3 \cdot n = 0$, and for $n \in (N^{W|Y})^{(1,0)}$:

$$[D_p]_3 \cdot n = \frac{1}{8\pi} \frac{\partial}{\partial n} \cdot (\mathbf{r}^X - \mathbf{r}^W) \cdot \mathrm{Id}_F$$
$$- \frac{1}{2\pi\sqrt{-1}} \nabla_n^{\mathrm{End}(E)} \big(\Lambda_\omega[R^F] - \Lambda_{\iota_2^*\omega}[R^F]\big).$$

Remark 1.6. Theorem 1.5 largely refines Theorem 1.1, see Remark 3.15 (b).

Now, in a slightly different direction, in Theorems 4.1 and 4.7, we show that for quasi-isometric embeddings, the set of Toeplitz type operators is closed under taking adjoints, restrictions, extensions and some products. This plays a crucial role in our approach to Theorem 1.5 and allows us to generalize Theorem 1.5 to towers of embeddings of arbitrary length, see Corollary 5.8 for a precise statement. As another direct consequence of our analysis, we obtain the following result.

Theorem 1.7. *As $p \to \infty$, the following asymptotic holds:*

$$\| \operatorname{Res}_p^{Y|X} \| \sim \sup_{y \in Y} \kappa_N^{X|Y}(y)^{-1/2} \cdot p^{(n-m)/2}. \tag{1.16}$$

Moreover, under assumption (1.8), as $p \to \infty$, we even have

$$\| \mathrm{E}_p^{X|Y} \| - \frac{1}{p^{(n-m)/2}} \sim \frac{C_3}{p^{(n-m+2)/2}}, \quad \| \operatorname{Res}_p^{Y|X} \| - p^{(n-m)/2} \sim C_4 \cdot p^{(n-m-2)/2}, \tag{1.17}$$

where the constants C_3, C_4 are defined as follows:

$$C_3 := -\frac{1}{2} \inf_{y \in Y} \left(\frac{\mathbf{r}_y^X - \mathbf{r}_y^Y}{8\pi} - \lambda_{\max} \left(\frac{\Lambda_\omega [R_y^F] - \Lambda_{\iota^* \omega}[R_y^F]}{2\pi\sqrt{-1}} \right) \right),$$
$$C_4 := \frac{1}{2} \sup_{y \in Y} \left(\frac{\mathbf{r}_y^X - \mathbf{r}_y^Y}{8\pi} - \lambda_{\min} \left(\frac{\Lambda_\omega [R_y^F] - \Lambda_{\iota^* \omega}[R_y^F]}{2\pi\sqrt{-1}} \right) \right),$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the values of the maximal and minimal eigenvalues.

Remark 1.8. (a) The first asymptotics (1.17) corresponds to the calculation of the optimal constant in Ohsawa–Takegoshi theorem. A less refined version was proved in [9, Theorem 1.1], see (1.11).

(b) In particular, from (1.11) and (1.16), we see that the sequence of operators $p^{(n-m)/2} \cdot \mathrm{E}_p^{X|Y}$, $p \in \mathbb{N}$, is an asymptotic isometry if and only if $\kappa_N^{X|Y}|_Y = 1$.

We say a few words about the proof of Theorem 1.5. The essential step consists in showing that the sequence D_p, $p \geq p_1$, from Theorem 1.5 satisfies the assumptions of the asymptotic characterizations of Toeplitz type operators similar to Ma–Marinescu [15]. For more amenable calculations of the higher order terms, in Section 4.2, for $p \geq p_1$, we introduce the sequence of operators $A_p^{X|Y}$, called *multiplicative defect*, relating extension and restriction maps. This sequence of operators has found further applications in the subsequent work of the author [8]. The general strategy for dealing with semi-classical limits here is inspired by Bismut [1] and Bismut–Vasserot [2].

This paper is organized as follows. In Section 2, we study the geometry of manifolds of bounded geometry. In Section 3, we study the asymptotics of Toeplitz type operators and derive asymptotic criteria for them. In Section 4, we study the algebraic properties of the set of Toeplitz type operators: we show that it is closed under taking adjoints, restrictions, extensions and some products. We study the adjoints of Toeplitz type operators and introduce multiplicative defect. Finally, in Section 5, using those preparations, we prove Theorems 1.1, 1.5 and 1.7, and generalize Theorem 1.5 to towers of submanifolds of arbitrary length.

Notations. We use notations X, Y for complex manifolds and M, H for real manifolds. The complex (resp. real) dimensions of X, Y (resp. M, H) are denoted here by n, m. An operator ι always means an embedding $\iota: Y \to X$ (resp. $\iota: H \to M$). We denote by Res_Y (resp. Res_H) the restriction operator from X to Y (resp. M to H).

For a Riemannian manifold (M, g^{TM}), we denote the Levi-Civita connection by ∇^{TM}, by R^{TM} the curvature of it, and by $dv_{g^{TM}}$ the Riemannian volume form. For a closed subset $W \subset M$, $r \geq 0$, let $B_W^M(r)$ be the ball of radius r around W.

For a fixed volume form dv_M on M, we denote by $L^2(dv_M, h^E)$ the space of L^2-sections of E with respect to dv_M and h^E. When $dv_M = dv_{g^{TM}}$, we also use the notation $L^2(g^{TM}, h^E)$. When there is no confusion about the data, we also use the simplified notation $L^2(M, E)$ or $L^2(M)$.

For $n \in \mathbb{N}^*$, we denote by $dv_{\mathbb{C}^n}$ the standard volume form on $\mathbb{C}^n$. We view $\mathbb{C}^m$ (resp. $\mathbb{R}^m$) embedded in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) by the first m coordinates. For $Z \in \mathbb{R}^k$, we denote by Z_l, $l = 1, \ldots, k$, the coordinates of Z. If $Z \in \mathbb{R}^{2n}$, we denote by z_i, $i = 1, \ldots, n$, the induced complex coordinates $z_i = Z_{2i-1} + \sqrt{-1} Z_{2i}$. We frequently use the decomposition $Z = (Z_Y, Z_N)$, where $Z_Y = (Z_1, \ldots, Z_{2m})$ and $Z_N = (Z_{2m+1}, \ldots, Z_{2n})$. For a fixed frame $(e_1, \ldots, e_{2n})$ in $T_x X$, $x \in X$, (resp. $y \in Y$) we implicitly identify Z (resp. Z_Y, Z_N) to an element in $T_x X$ (resp. $T_y Y$, $N_y^{X|Y}$) by

$$Z = \sum_{i=1}^{2n} Z_i e_i, \quad Z_Y = \sum_{i=1}^{2m} Z_i e_i, \quad Z_N = \sum_{i=2m+1}^{2n} Z_i e_i.$$

If the frame e_i satisfies the condition

$$J e_{2i-1} = e_{2i}, \tag{1.18}$$

we denote $\frac{\partial}{\partial z_i} := \frac{1}{2}(e_{2i-1} - \sqrt{-1} e_{2i})$, $\frac{\partial}{\partial \overline{z}_i} := \frac{1}{2}(e_{2i-1} + \sqrt{-1} e_{2i})$, and identify $z, \overline{z}$ to vectors in $T_x X \otimes_{\mathbb{R}} \mathbb{C}$ as follows

$$z = \sum_{i=1}^{n} z_i \cdot \frac{\partial}{\partial z_i}, \quad \overline{z} = \sum_{i=1}^{n} \overline{z}_i \cdot \frac{\partial}{\partial \overline{z}_i}.$$

Clearly, in this identification, $Z = z + \overline{z}$. We define

$$z_Y, \overline{z}_Y \in T_y Y \otimes_{\mathbb{R}} \mathbb{C}, \quad z_N, \overline{z}_N \in N_y^{X|Y} \otimes_{\mathbb{R}} \mathbb{C}$$

in a similar way. We sometimes further decompose $Z_N = (Z_{N^{W|Y}}, Z_{N^{X|W}})$ for $Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)}$, $Z_{N^{X|W}} \in \mathbb{R}^{2(n-l)}$, $m \leq l \leq n$, and use the analogous identifications. Sometimes, we use the notation $Z_W := (Z_Y, Z_{N^{W|Y}})$.

For a polynomial $P(Z, Z')$, $Z, Z' \in \mathbb{R}^n$, we denote by $\mathrm{Res}_m \circ P$ the polynomial in $Z_H \in \mathbb{R}^m$, $Z' \in \mathbb{R}^n$, defined as

$$\mathrm{Res}_m \circ P(Z_H, Z') = P\big((Z_H, 0), Z'\big).$$

Similarly, we define $P \circ \mathrm{Res}_m$.

For $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, $B = (B_1, \dots, B_k) \in \mathbb{C}^k$, we write by

$$|\alpha| = \sum_{i=1}^{k} \alpha_i, \quad \alpha! = \prod_{i=1}^{k} (\alpha_i)!, \quad B^\alpha = \prod_{i=1}^{k} B_i^{\alpha_i}.$$

2. Bounded geometry condition and local trivializations

The main goal of this section is to study the geometry of manifolds of bounded geometry. More precisely, in Section 2.1, we recall the definitions manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry and the quasi-isometry assumption. In Section 2.2, we study the convergence of exponential integrals on manifolds of bounded geometry. In Section 2.3, we recall some results comparing geodesic and Fermi coordinates and related trivializations of vector bundles. Finally, in Section 2.4, we extend those results to towers of submanifolds.

2.1. Introduction to bounded geometry condition

In this section, we recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. We also recall the basic facts about the second fundamental form and the quasi-isometry assumption. For a more detailed overview, we refer to [7, 11, 22], cf. [9].

Definition 2.1. We say that a Riemannian manifold (M, g^{TM}) is of bounded geometry if the following two conditions are satisfied:

(i) The injectivity radius of (M, g^{TM}) is bounded below by a positive constant r_M.

(ii) For the Riemann curvature tensor R^{TM} of M, we have

$$R^{TM} \in \mathcal{C}_b^\infty\big(M, \Lambda^2 T^*M \otimes \mathrm{End}(TM)\big).$$

Now, let (H, g^{TH}) be an embedded submanifold of (M, g^{TM}), $g^{TH} := g^{TM}|_H$. We identify the normal bundle $N^{M|H}$ of H in M to an orthogonal complement of TH in TM as in (1.6). We denote by $g^{N^{M|H}}$ the metric on $N^{M|H}$ induced by g^{TM}. We denote by $P_N^{M|H}: TM|_H \to N^{M|H}$, $P_H^{M|H}: TM|_H \to TH$, the projections induced by (1.6). Clearly, $\nabla^{N^{M|H}} := P_N^{M|H} \nabla^{TM}|_H$ defines a connection on $N^{M|H}$. We define the *second fundamental form* $A^{M|H} \in \mathscr{C}^\infty(H, T^*H \otimes \mathrm{End}(TM|_H))$ by

$$A^{M|H} := \nabla^{TM}|_H - \nabla^{TH} \oplus \nabla^{N^{M|H}}. \tag{2.1}$$

Recall that the *mean curvature* $\nu^{M|H} \in \mathscr{C}^\infty(H, N^{M|H})$ of ι is defined as follows

$$\nu^{M|H} := \frac{1}{m} \sum_{i=1}^{m} A^{M|H}(e_i) e_i, \tag{2.2}$$

where the sum runs over an orthonormal basis of (TH, g^{TH}).

Proposition 2.2. *The second fundamental form satisfies the following properties:*

(1) *It takes values in skew-symmetric endomorphisms of $TM|_H$, interchanging TH and $N^{M|H}$.*

(2) *For any $U, V \in TH$, we have $A^{M|H}(U)V = A^{M|H}(V)U$.*

Assume, moreover, that (M, g^{TM}) is Kähler. Then the following hold:

(3) *$A^{M|H}$ commutes with the action of the complex structure.*

(4) *For any $U \in TH$, $V \in TM$, $U = u + \overline{u}$, $V = v + \overline{v}$, $u, v \in T^{1,0}M$, we have*

$$A^{M|H}(U)v = A^{M|H}(\overline{u})v, \quad A^{M|H}(U)\overline{v} = A^{M|H}(u)\overline{v} \quad \text{if } V \in N^{M|H},$$
$$A^{M|H}(U)v = A^{M|H}(u)v, \quad A^{M|H}(U)\overline{v} = A^{M|H}(\overline{u})\overline{v} \quad \text{if } V \in TH.$$

(5) *We have $\nu^{M|H} = 0$.*

Proof. An elementary proof is omitted for brevity; the reader may refer to the arXiv version for a complete argument. ■

Definition 2.3. We say that the triple (M, H, g^{TM}) is of bounded geometry if the following conditions are fulfilled:

(i) The manifold (M, g^{TM}) is of bounded geometry.

(ii) The injectivity radius of (H, g^{TH}) is bounded below by a positive constant r_H.

(iii) There is a collar around H (a tubular neighborhood of fixed radius), i.e. there is $r_\perp > 0$ such that for any $x, y \in H$, the normal geodesic balls $B_{r_\perp}^{\perp}(x)$, $B_{r_\perp}^{\perp}(y)$, obtained by the application of the exponential mapping to vectors, orthogonal to H, of norm bounded by $r_\perp$, are disjoint.

(iv) The second fundamental form, $A^{M|H}$, satisfies

$$A^{M|H} \in \mathscr{C}_b^\infty\big(H, T^*M|_H \otimes \mathrm{End}(TM|_H)\big).$$

We will now introduce a coordinate system in M near a fixed point in H, which is particularly well adapted to the study of triples of bounded geometry. We fix a point $y_0 \in H$ and an orthonormal frame $(e_1, \dots, e_m)$ (resp. $(e_{m+1}, \dots, e_n)$) in $(T_{y_0}H, g^{TH}_{y_0})$ (resp. in $(N^{M|H}_{y_0}, g^{N^{M|H}}_{y_0})$). For

$$Z = (Z_H, Z_N), \quad Z_H \in \mathbb{R}^m, \quad Z_N \in \mathbb{R}^{n-m},$$
$$Z_H = (Z_1, \dots, Z_m), \quad Z_N = (Z_{m+1}, \dots, Z_n), \quad |Z_H| \leq r_H, \quad |Z_N| \leq r_\perp,$$

we define a coordinate system $\psi^{M|H}_{y_0} \colon B^{\mathbb{R}^m}_0(r_H) \times B^{\mathbb{R}^{n-m}}_0(r_\perp) \to M$ by

$$\psi^{M|H}_{y_0}(Z_H, Z_N) := \exp^M_{\exp^H_{y_0}(Z_H)}\big(Z_N(Z_H)\big), \tag{2.3}$$

where $Z_N(Z_H)$ is the parallel transport of $Z_N \in N^{M|H}_{y_0}$ along $\exp^H_{y_0}(tZ_H)$, $t = [0, 1]$, with respect to the connection $\nabla^{N^{M|H}}$ on $N^{M|H}$. The coordinates $\psi^{M|H}_{y_0}$ are called the *Fermi coordinates* at y_0. Their importance comes from the fact that the metric tensor has uniformly bounded coefficients in these coordinates, cf. [22, Lemma 3.9] and [11, Theorem 4.9].

Now, recall that an embedding $\iota \colon H \to M$ is called *quasi-isometry* if there are $A, B > 0$ such that for any $y_1, y_2 \in H$, we have

$$\operatorname{dist}_H(y_1, y_2) \leq A \operatorname{dist}_M\big(\iota(y_1), \iota(y_2)\big) + B. \tag{2.4}$$

In what follows, for brevity, we omit ι from the distance function. The following proposition will be useful in the study of restriction of Toeplitz operators.

Proposition 2.4. *Assume that a triple (M, H, g^{TM}) is of bounded geometry and the embedding ι is quasi-isometry. Then one can choose $B = 0$ in (2.4).*

Proof. An elementary proof is omitted for brevity; the reader may refer to the arXiv version for a complete argument. ■

Finally, we recall the last definition related to bounded geometry.

Definition 2.5. Let (E, ∇^E, h^E) be a Hermitian vector bundle with a fixed Hermitian connection over a manifold (M, g^{TM}) of bounded geometry. We say that (E, ∇^E, h^E) is of bounded geometry if $R^E \in \mathcal{C}^\infty_b(M, \Lambda^2 T^*M \otimes \operatorname{End}(E))$.

If (E, h^E) is a Hermitian vector bundle over a *complex manifold*, we say that it is of bounded geometry if (E, ∇^E, h^E) is of bounded geometry for the Chern connection ∇^E on (E, h^E).

2.2. Convergence of exponential integrals for triples of bounded geometry

The main goal of this section is to study convergence of exponential integrals for triples of bounded geometry. More precisely, fix a triple (M, H, g^{TM}) of bounded geometry. We conserve the notations from Definition 2.3.

Proposition 2.6 ([9, Corollary 3.3]). *There are $c, C' > 0$, which depend only on n, m, r_M, r_N, $r_\perp$ and* sup*-norm on R^{TM}, R^{TH}, $A^{M|H}$, such that for any $y_0 \in H$, $l > c$, the following bound holds:*

$$\int_H \exp\big(-l \operatorname{dist}_M(y_0, y)\big)\, dv_{g^{TH}}(y) < \frac{C'}{l^m}.$$

Let (E, h^E) be a Hermitian vector bundle over M and D is an operator acting on $L^2(H, \iota^* E)$. Assume that there are $c > 0$, l as in Proposition 2.6 and $C > 0$, such that for any $y_1, y_2 \in H$, the Schwartz kernel of D, evaluated with respect to $dv_{g^{TH}}$, satisfies the bound

$$|D(y_1, y_2)| \leq C l^m \exp\big(-l \operatorname{dist}_M(y_1, y_2)\big). \tag{2.5}$$

Corollary 2.7. *For $C' > 0$ as in Proposition 2.6, we have $\|D\| \leq CC'$.*

Proof. From Proposition 2.6 and (2.5), we deduce that there is $C' > 0$, as in Proposition 2.6, such that for any $y_0 \in H$, we have

$$\int_H |D(y_0, y)|\, dv_H(y) \leq CC', \qquad \int_H |D(y, y_0)|\, dv_H(y) \leq CC'. \tag{2.6}$$

We conclude directly from (2.6) and Young's inequality for integral operators, cf. [23, Theorem 0.3.1] applied for $p, q = 2$, $r = 1$ in the notations of [23]. ∎

Now, let $c > 0$ be as in Proposition 2.6. Let D_i, $i = 1, \dots, r$, be operators acting on $L^2(H, \iota^* E)$, such that for some $l \geq 2c$, $C > 0$, the bound (2.5) holds for $D := D_i$.

Corollary 2.8 ([9, Lemma 3.1]). *The Schwartz kernel $D_{r+1}(y_1, y_2)$ of the operator $D_{r+1} := D_1 \circ D_2 \circ \cdots \circ D_r$, evaluated with respect to $dv_{g^{TH}}$, is well defined for any $y_1, y_2 \in H$ and it satisfies the bound (2.5) for $D := D_{r+1}$, $l := l/2$, $C := (C')^r C^r$ for C' as in Proposition 2.6.*

Now, in addition to the triple (M, H, g^{TM}) of bounded geometry, we consider a Riemannian manifold (K, g^{TK}) with an embedding $\iota_1 \colon M \to K$, such that

$$\iota_1^* g^{TK} = g^{TM}.$$

We assume, moreover, that the triple (K, M, g^{TK}) is of bounded geometry. We let (E, h^E) be a Hermitian vector bundle over M and $D \colon L^2(H, \iota^* E) \to L^2(M, E)$ be a fixed linear operator. Assume that there is $c > 0$ as in Proposition 2.6 and $C > 0$, such that for some $l \geq c$ and any $y \in H$, $x \in M$, the Schwartz kernel of D, evaluated with respect to $dv_{g^{TH}}$, satisfies the bound

$$|D(x, y)| \leq C l^m \exp\big(-l \operatorname{dist}_K(x, y)\big).$$

Proposition 2.9. *There is $C' > 0$, which depends on the same data as constants from Proposition 2.6 and the analogous data on (K, M, g^{TK}), such that*

$$\|D\| \le \frac{C'C}{l^{(n-m)/2}}.$$

Remark 2.10. Clearly, Corollary 2.7 is a special case of Proposition 2.9 for $M := H$.

Proof. First of all, let us establish this result for $K := M$. We consider the exponential map $\exp_N^{M|H} \colon N^{M|H} \to M$. As M is complete, the map $\exp_N^{M|H}$ is surjective. We consider a subset V of $N^{M|H}$, consisting of points $u \in N^{M|H}$, such that

$$\operatorname{dist}\big(\exp_N^{M|H}(u), Y\big) = |u|,$$

where $|u|$ is the norm of u with respect to the induced metric on $N^{M|H}$.

We let $\delta = \min\{\inf_{x,T} \sec(x,T), -1\}$, where $\sec(x,T)$ is the sectional curvature of (M, g^{TM}), evaluated at $x \in M$ for the two-dimensional subspace $T \in T_x M$ of the tangent bundle, and the infimum is taken over all possible choices of x and T. Bounded geometry condition implies that δ is a finite constant. For $r > 0$, we denote

$$s_\delta(r) := \frac{1}{|\delta|^{1/2}} \sinh\big(|\delta|^{1/2} r\big) \quad \text{and} \quad c_\delta(r) := s'_\delta(r).$$

Then from the result of Heintze–Karcher [12, Corollary 3.3.1], for $u \in V$, $|u| = t$, $t > 0$, we obtain that

$$\big|\det\big(d \exp_N^{M|H}\big)_u\big| \le \Big(\frac{s_\delta(t)}{t}\Big)^{n-m-1} \cdot \big(c_\delta(t) - \langle \nu^{M|H}, u\rangle \cdot s_\delta(t)\big)^n,$$

where $\nu^{M|H}$ was defined in (2.2). In particular, due to bounded geometry assumption, we obtain that there are $c, C' > 0$, as in Proposition 2.6, such that

$$\big|\det\big(d \exp_N^{M|H}\big)_u\big| \le C' \exp(ct).$$

Using this fact, we deduce that there is $C > 0$ such that for any $f \in L^2(H, \iota^* E)$, we have

$$\|Df\|^2_{L^2} \le C \int_V \exp\big(c|u|\big) \tag{2.7}$$
$$\cdot \Big(l^m \int_H \exp\big(-l \operatorname{dist}_M\big(y_1, \exp^M_{y_2}(u)\big)\big) f(y)\, dv_{g^{TH}}(y_1)\Big)^2 dv_{N^{M|H}_{y_2}}(u)\, dv_{g^{TH}}(y_2).$$

However, clearly, we have $\operatorname{dist}_M(y_1, \exp^M_{y_2}(u)) \ge \operatorname{dist}_M(Y, \exp^M_{y_2}(u))$. But by the definition of the subset V, we have

$$\operatorname{dist}_M\big(Y, \exp^M_{y_2}(u)\big) = \operatorname{dist}_M\big(y_2, \exp^M_{y_2}(u)\big) = |u|.$$

From this and (2.7), for $l \geq 8c$, we obtain that

$$\|Df\|_{L^2}^2 \leq C \int_V \exp\Big(-\frac{l}{2}|u|\Big) \cdot \Big(l^m \int_H \exp\Big(-\frac{l}{8} \operatorname{dist}_M(y_1, y_2)\Big) f(y) \, dv_{g^{TH}}(y_1)\Big)^2 dv_{N_{y_2}^{M|H}}(u) \, dv_{g^{TH}}(y_2). \tag{2.8}$$

From the boundness of the exponential integral, for any $y \in Y$, we obtain that

$$\int_{V \cap N_y^{M|H}} \exp\Big(-\frac{l}{2}|u|\Big) \, dv_{N_y^{M|H}}(u) \leq \frac{C}{l^{n-m}}.$$

By combining with (2.8), it gives us

$$\|Df\|_{L^2}^2 \leq \frac{C}{l^{n-m}} \cdot \int_H \Big(l^m \int_H \exp\Big(-\frac{l}{4} \operatorname{dist}_M(y_1, y_2)\Big) f(y) \, dv_{g^{TH}}(y_1)\Big)^2 dv_{g^{TH}}(y_2). \tag{2.9}$$

Now, from Corollary 2.7, we obtain that there is $C > 0$, verifying

$$\int_H \Big(l^m \int_H \exp\Big(-\frac{l}{4} \operatorname{dist}_M(y_1, y_2)\Big) f(y) \, dv_{g^{TH}}(y_1)\Big)^2 dv_{g^{TH}}(y_2) \leq C \|f\|_{L^2}^2. \tag{2.10}$$

A combination of (2.9) and (2.10) finishes the proof for the case $K := M$. An easy verification shows that all the constants can be chosen as described in the statement of the proposition we are proving. The proof of the general case reduces to the case considered above through the use of the tubular neighborhood of M in K in the same way as in the proof of [9, Corollary 3.3]. ■

2.3. Fermi and geodesic coordinates; related trivializations of vector bundles

In this section, we recall some results comparing geodesic and Fermi coordinates and trivializations of vector bundles adapted to those coordinate systems. We place ourselves in the setting of a triple (M, H, g^{TM}) of bounded geometry.

Let us fix $x_0 \in M$ and an orthonormal frame $(e_1, \dots, e_n)$ of $(T_{x_0}M, g_{x_0}^{TM})$. We define the map $\phi_{x_0}^M : \mathbb{R}^n \to M$, $x_0 \in M$, as follows:

$$\phi_{x_0}^M(Z) := \exp_{x_0}^M(Z). \tag{2.11}$$

Define the constant $R > 0$ as follows:

$$R := \min\Big\{\frac{r_M}{2}, \frac{r_H}{4}, \frac{r_\perp}{4}\Big\}.$$

Assume now $x_0 = y_0$, where $y_0 \in H$ and let $(e_1, \dots, e_n)$ be as in (2.3). Recall that Fermi coordinates $\psi_{y_0}^{M|H}$ were defined in (2.3). Clearly, there is a (unique) smooth diffeomorphism $h_{y_0}^{M|H} : B_0^{\mathbb{R}^n}(R) \to \mathbb{R}^n$, $h_{y_0}^{M|H}(0) = 0$, such that the following identity holds

$$\psi_{y_0}^{M|H} = \phi_{y_0}^{M} \circ h_{y_0}^{M|H}. \tag{2.12}$$

We recall that $A^{M|H} \in \mathcal{C}^\infty(H, T^*H \otimes \mathrm{End}(TM|_H))$ was defined in (2.1). Let an auxiliary section $B^{M|H} \in \mathcal{C}^\infty(H, \mathrm{Sym}^2(T^*M|_H) \otimes TM|_H)$ for $Z \in TM|_H$, be defined as

$$B^{M|H}(Z) := B^{M|H}(Z, Z) := \frac{1}{2} A^{M|H}(Z_H) Z_H + A^{M|H}(Z_H) Z_N.$$

Proposition 2.11 ([9, Proposition 2.18]). *The diffeomorphism $h_{y_0}^{M|H}$ admits the Taylor expansion*

$$h_{y_0}^{M|H}(Z) = Z + B^{M|H}(Z) + O(|Z|^3).$$

In the second part of this section, we recall the comparison between two trivializations of vector bundles, done using parallel transport adapted to the above coordinate systems.

We fix an orthonormal frame $f_1, \dots, f_r \in (E_{x_0}, h_{x_0}^E)$. Let $\tilde{f}_1'^M, \dots, \tilde{f}_r'^M$ be a frame of E over $B_{x_0}^M(r_M)$, obtained by the parallel transport of $f_1, \dots, f_r$ along the curve $\phi_{x_0}^M(tZ)$, $t \in [0,1]$, $Z \in T_{x_0}M$, $|Z| < r_M$. Assume now $x_0 = y_0$, where $y_0 \in H$ and let $(e_1, \dots, e_n)$ be as in (2.3). We define $\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H}$ by the parallel transport of $f_1, \dots, f_r$ with respect to the connection ∇^E, first along the path $\psi_{y_0}^{M|H}(tZ_Y, 0)$, $t \in [0,1]$, and then along the path $\psi_{y_0}^{M|H}(Z_Y, tZ_N)$, $t \in [0,1]$, $Z_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z_Y| < r_Y$, $|Z_N| < r_\perp$.

Let $\xi_E^{M|H}$ be the unique smooth function over $B_{y_0}^M(R)$, with values in $\mathrm{End}(\mathbb{C}^r)$, such that $\xi_E^{M|H}(0) = 0$, and the following identity holds:

$$(\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H}) = \exp(\xi_E^{M|H}) \cdot (\tilde{f}_1'^M, \dots, \tilde{f}_r'^M), \tag{2.13}$$

where we view $(\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H})$ and $(\tilde{f}_1'^M, \dots, \tilde{f}_r'^M)$ as $r \times 1$ matrices.

Proposition 2.12 ([9, Proposition 2.22]). *The following asymptotic holds:*

$$\xi_E^{M|H}\big(\psi_{y_0}^{M|H}(Z)\big) = O\big(|Z|^2\big).$$

If, moreover, $(E, \nabla^E, h^E) := (L, \nabla^L, h^L)$ is a line bundle, and there is a skew-adjoint endomorphism Q of TM, which is parallel with respect to ∇^{TM} (i.e. $\nabla^{TM} Q = 0$), which commutes with $A^{M|H}$, the restriction of which to H respects the decomposition (1.6), and such that for the curvature R^L of ∇^L, and for any $u, v \in TM$, we have

$$\frac{\sqrt{-1}}{2\pi} R^L(u, v) = g^{TM}(Qu, v), \tag{2.14}$$

then the following more precise bound holds:

$$\xi_L^{M|H}\big(\psi^{M|H}(Z)\big) = -\frac{1}{4} R_{y_0}^L\big(Z_N, A^{M|H}(Z_H)Z_H\big) + O\big(|Z|^4\big). \tag{2.15}$$

Remark 2.13. Assume (M, g^{TM}) is endowed with a complex structure J, and g^{TM} is invariant under the action of it. Assume, moreover, that (2.14) holds for $Q := J$ as in (1.1). Then all the requirements are satisfied for $Q := J$, cf. [9, Remark 2.23].

2.4. Towers of embeddings, associated coordinates and holonomies

In this section, we compare natural coordinate systems and trivializations of vector bundles associated to towers of submanifolds. The proofs here are left to an interested reader.

We fix a tower of (real) manifolds $H \hookrightarrow K \hookrightarrow M$ of dimensions m, l and n, respectively. Endow M with the Riemannian metric g^{TM} and induce the metrics g^{TK} and g^{TH} on K and H. We fix $y_0 \in H$ and an orthonormal frame $(e_1, \dots, e_m)$ (resp. $(e_{m+1}, \dots, e_l)$ and $(e_{l+1}, \dots, e_n)$) of $(T_{y_0}H, g^{TH})$ (resp. $(N_{y_0}^{K|H}, g_{y_0}^{N^{K|H}})$ and $(N_{y_0}^{M|K}, g_{y_0}^{N^{M|K}})$). Recall that Fermi coordinates were defined in (2.3). Clearly, there is a (unique) smooth embedding $\sigma \colon \mathbb{R}^l \to \mathbb{R}^n$, $\sigma(0) = 0$, such that for any $Z \in \mathbb{R}^l$ small enough, we have

$$\psi_{y_0}^{M|H}(\sigma(Z)) = \psi_{y_0}^{K|H}(Z). \tag{2.16}$$

By comparing the Jacobians, we get the following result.

Proposition 2.14. *The function σ satisfies $\sigma(Z) = Z + O(|Z|^2)$.*

Next, we compare two natural trivializations of vector bundles for towers of submanifolds associated to Fermi coordinates: one for the pair (M, H), another one for (K, H).

Let (E, ∇^E, h^E) be a Hermitian vector bundle of bounded geometry and rank r over (M, g^{TM}). We fix an orthonormal frame $f_1, \dots, f_r \in (E_{y_0}, h_{y_0}^E)$. Recall that the frames $\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H}$; $\tilde{f}_1^{K|H}, \dots, \tilde{f}_r^{K|H}$ were defined before (2.13). Let τ_E be the (unique) smooth function, defined in $B_{y_0}^M(R)$, with values in $\mathrm{End}(\mathbb{C}^r)$, such that $\tau_E(0) = 0$, and

$$\mathrm{Res}_K\big(\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H}\big) = \exp(\tau_E) \cdot \big(\tilde{f}_1^{K|H}, \dots, \tilde{f}_r^{K|H}\big), \tag{2.17}$$

where we view $\mathrm{Res}_K(\tilde{f}_1^{M|H}, \dots, \tilde{f}_r^{M|H})$ and $(\tilde{f}_1^{K|H}, \dots, \tilde{f}_r^{K|H})$ as $r \times 1$ matrices.

Proposition 2.15. *In the notation of Proposition 2.12, the following asymptotic holds:*

$$\tau_E\big(\psi^{K|H}(Z)\big) = O\big(|Z|^2\big), \quad \tau_L\big(\psi^{K|H}(Z)\big) = O\big(|Z|^4\big). \tag{2.18}$$

Proof. The proof is done by a repetitive use of Propositions 2.11, 2.12 and 2.14. By (2.13), we have

$$\begin{aligned}\big(\widetilde{f}_1^{M|H},\dots,\widetilde{f}_r^{M|H}\big) &= \exp\big(\xi_E^{M|H}\big)\cdot\big(\widetilde{f}_1'^M,\dots,\widetilde{f}_r'^M\big),\\ \big(\widetilde{f}_1'^M,\dots,\widetilde{f}_r'^M\big) &= \exp\big(-\xi_E^{M|K}\big)\big(\widetilde{f}_1^{M|K},\dots,\widetilde{f}_r^{M|K}\big),\\ \mathrm{Res}_K\big(\widetilde{f}_1^{M|K},\dots,\widetilde{f}_r^{M|K}\big) &= \big(\widetilde{f}_1'^K,\dots,\widetilde{f}_r'^K\big),\\ \big(\widetilde{f}_1'^K,\dots,\widetilde{f}_r'^K\big) &= \exp\big(-\xi_E^{K|H}\big)\big(\widetilde{f}_1^{K|H},\dots,\widetilde{f}_r^{K|H}\big).\end{aligned} \tag{2.19}$$

Now, from (2.17) and (2.19), we obtain

$$\exp(\tau_E) = \mathrm{Res}_K\big(\exp\big(\xi_E^{M|H}\big)\cdot\exp\big(-\xi_E^{M|K}\big)\big)\cdot\exp\big(-\xi_E^{K|H}\big).$$

From Propositions 2.11, 2.12 and 2.14, we deduce the first part of (2.18). Remark the basic identity

$$A^{M|K}(U)V + A^{K|H}(U)V = A^{M|H}(U)V \quad \text{for } U,V\in TH. \tag{2.20}$$

The second part is now obtained by the same means, one only has to use (2.20) in addition to previous considerations. ■

We will now relate the Fermi coordinates for the pairs (M,K) and (M,H). Clearly, there is a unique diffeomorphism $\upsilon\colon\mathbb{R}^n\to\mathbb{R}^n$, $\upsilon(0)=0$, so that for any $Z\in\mathbb{R}^n$ small enough, we have

$$\psi_{y_0}^{M|K}(\upsilon(Z)) = \psi_{y_0}^{M|H}(Z). \tag{2.21}$$

By comparing the Jacobians, we get the following result.

Proposition 2.16. *The function υ has the following Taylor expansion:*

$$\upsilon(Z) = Z + O\big(|Z|^2\big).$$

Our next goal is to compare two trivializations of a vector bundle: one associated to the trivializations in Fermi coordinates for the pair (M,K), another one for (M,H). Let χ_E be the (unique) smooth function, defined over $B_{y_0}^M(R)$, with values in $\mathrm{End}(\mathbb{C}^r)$, such that $\chi_E(0)=0$ and

$$\big(\widetilde{f}_1^{M|K},\dots,\widetilde{f}_r^{M|K}\big) = \exp(\chi_E)\cdot\big(\widetilde{f}_1^{M|H},\dots,\widetilde{f}_r^{M|H}\big),$$

where we view $(\widetilde{f}_1^{M|K},\dots,\widetilde{f}_r^{M|K})$ and $(\widetilde{f}_1^{M|H},\dots,\widetilde{f}_r^{M|H})$ as $r\times 1$ matrices.

An elementary variation on the proof of Proposition 2.15 leads to the following result.

Proposition 2.17. *In the notation of Proposition 2.12, the following asymptotic holds:*

$$\chi_E\big(\psi^{M|H}(Z)\big) = O\big(|Z|^2\big),\quad \chi_L\big(\psi^{M|H}(Z)\big) = O\big(|Z|^3\big).$$

3. Asymptotics of Toeplitz type operators and kernel calculus

The main goal of this section is to study asymptotics of Schwartz kernels of Toeplitz type operators. More precisely, in Section 3.1, we consider the model situation, for which an explicit formula for the Schwartz kernels of Bergman projectors, the extension and restriction operators can be given. We then study the composition rules for the operators with related kernels. Those results play the foundational role in this article. In Section 3.2, we recall the asymptotics of Schwartz kernels of Bergman projectors and the extension operator. In Section 3.3, we study the asymptotic expansion of basic Toeplitz type operators, in particular, we establish Theorem 3.14. Finally, in Section 3.4, we establish asymptotic characterization of Toeplitz type operators with exponential decay.

3.1. Model operators on the complex vector space

In this section, we consider the model situation, for which an explicit formula for the Schwartz kernels of Bergman projectors, the extension and restriction operators can be given. We then use those explicit formulas to give a description for compositions of operators, the Schwartz kernels of which can be expressed using the above kernels. This section is motivated in many ways by the works of Ma–Marinescu [14, 16] and Dai–Liu–Ma [4].

Endow $\mathbb{C}^n$ with the standard Riemannian metric and consider a trivialized holomorphic line bundle L_0 on $\mathbb{C}^n$. We endow L_0 with the Hermitian metric h^{L_0}, given by

$$\|1\|_{h^{L_0}}(Z) = \exp\Big(-\frac{\pi}{2}|Z|^2\Big), \tag{3.1}$$

where Z is the natural real coordinate on $\mathbb{C}^n$, and 1 is the trivializing section of L_0. An easy verification shows that (3.1) implies that (1.1) holds in our setting. Recall that [14, §4.1.6] shows that the Kodaira Laplacian on $\mathscr{C}^\infty(X, L_0)$, multiplied by 2, which we denote here by $\mathscr{L}$, and view as an operator on $\mathscr{C}^\infty(X)$ using the orthonormal trivialization by $1\exp(\frac{\pi}{2}|Z|^2)$ of L_0, is given by

$$\mathscr{L} = \sum_{i=1}^n b_i b_i^+,$$

where b_i, b_i^+ are *creation* and *annihilation* operators, defined as

$$b_i = -2\frac{\partial}{\partial z_i} + \pi \overline{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \pi z_i.$$

We verify easily that

$$\big[g(z,\overline{z}), b_j\big] = 2\frac{\partial}{\partial z_j} g(z,\overline{z}). \tag{3.2}$$

From [14, Theorem 4.1.20], we know that for a multiindex $\alpha \in \mathbb{N}^n$, the functions

$$b^{\alpha}\Big(z^{\beta}\exp\Big(-\frac{\pi}{2}\sum_{i=1}^{n}|z_i|^2\Big)\Big), \tag{3.3}$$

form vector spaces of orthogonal eigenvectors of $\mathcal{L}$ (viewed as sections of $\mathcal{C}^{\infty}(X, L_0)$ using the above orthonormal trivialization) corresponding to the eigenvalues

$$4\pi\sum_{i=1}^{n}\alpha_i.$$

From this (cf. [15, Theorem 1.15] and [14, (4.1.84)]), the Bergman kernel $\mathcal{P}_n$ of $\mathbb{C}^n$ is given by

$$\mathcal{P}_n(Z, Z') = \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{n}\big(|z_i|^2 + |z'_i|^2 - 2z_i\overline{z}'_i\big)\Big) \quad \text{for } Z, Z' \in \mathbb{C}^n. \tag{3.4}$$

In particular, we deduce that

$$b_{j,z}\mathcal{P}_n(Z, Z') = 2\pi(\overline{z}_j - \overline{z}'_j)\mathcal{P}_n(Z, Z'). \tag{3.5}$$

Also, we see easily (cf. [9, (3.28), (3.29)]) that Schwartz kernels of the orthogonal Bergman kernel, $\mathcal{P}^{\perp}_{n,m}$, corresponding to the projection onto holomorphic sections orthogonal to the holomorphic sections which vanish along $\mathbb{C}^m$, and the L^2-extension operator $\mathcal{E}_{n,m}$, extending each element from $(\ker\mathcal{L})|_{\mathbb{C}^m}$ to an element from $\ker\mathcal{L}$ with the minimal L^2-norm, are given by

$$\begin{aligned}
\mathcal{P}^{\perp}_{n,m}(Z, Z') &= \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{m}\big(|z_i|^2 + |z'_i|^2 - 2z_i\overline{z}'_i\big) - \frac{\pi}{2}\sum_{i=m+1}^{n}\big(|z_i|^2 + |z'_i|^2\big)\Big),\\
\mathcal{E}_{n,m}(Z, Z'_Y) &= \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{m}\big(|z_i|^2 + |z'_i|^2 - 2z_i\overline{z}'_i\big) - \frac{\pi}{2}\sum_{i=m+1}^{n}|z_i|^2\Big).
\end{aligned} \tag{3.6}$$

We, finally, remark that the Schwartz kernel of the operator $\mathrm{Res}_{\mathbb{C}^m} \circ \mathcal{P}_n$ is given by

$$\mathcal{R}_{n,m}(Z_Y, Z') = \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{m}\big(|z_i|^2 + |z'_i|^2 - 2z_i\overline{z}'_i\big) - \frac{\pi}{2}\sum_{i=m+1}^{n}|z'_i|^2\Big). \tag{3.7}$$

Remark the trivial identity

$$\mathcal{R}_{n,m} = \mathcal{E}^{*}_{n,m}.$$

Now, a lot of calculations in this article will have something to do with compositions of operators having Schwartz kernels, given by the product of polynomials with the above kernels. For that reason, the following lemma will be of utmost importance in what follows.

Lemma 3.1. For any polynomials $A_1(Z, Z')$, $A_2(Z, Z')$, $Z, Z' \in \mathbb{R}^{2n}$, there is a polynomial $A_3 := \mathcal{K}_{n,m}[A_1, A_2]$, the coefficients of which are polynomials of the coefficients of A_1, A_2, such that

$$(A_1 \cdot \mathcal{P}_{n,m}^{\perp}) \circ (A_2 \cdot \mathcal{P}_{n,m}^{\perp}) = A_3 \cdot \mathcal{P}_{n,m}^{\perp}. \tag{3.8}$$

Moreover, $\deg A_3 \leq \deg A_1 + \deg A_2$. Also, if both polynomials A_1, A_2 are even or odd (resp. one is even, another is odd), then the polynomial A_3 is even (resp. odd). Similarly, there is a polynomial $A_3' := \mathcal{K}'_{n,m}[A_1, A_2]$ with the same properties as A_3, such that

$$(A_1 \cdot \mathcal{P}_n) \circ (A_2 \cdot \mathcal{P}_{n,m}^{\perp}) = A_3' \cdot \mathcal{P}_{n,m}^{\perp}. \tag{3.9}$$

Also, for any polynomials $A(Z, Z_Y')$, $D(Z_Y, Z_Y')$, $Z \in \mathbb{R}^{2n}$, $Z_Y, Z_Y' \in \mathbb{R}^{2m}$, there is a polynomial $A_3'' := \mathcal{K}_{n,m}^{EP}[A, D]$ with the same properties as A_3 such that

$$(A \cdot \mathcal{E}_{n,m}) \circ (D \cdot \mathcal{P}_m) = A_3'' \cdot \mathcal{E}_{n,m}. \tag{3.10}$$

For polynomials $B(Z, Z')$, $C(Z_Y, Z')$, $Z, Z' \in \mathbb{R}^{2n}$, $Z_Y \in \mathbb{R}^{2m}$, the following identities hold:

$$\begin{aligned}
(B \cdot \mathcal{P}_n) \circ (A \cdot \mathcal{E}_{n,m}) &= \big(\mathcal{K}_{n,n}[B, A] \circ \mathrm{Res}_m\big) \cdot \mathcal{E}_{n,m}, \\
(B \cdot \mathcal{P}_{n,m}^{\perp}) \circ (A \cdot \mathcal{E}_{n,m}) &= \big(\mathcal{K}_{n,m}[B, A] \circ \mathrm{Res}_m\big) \cdot \mathcal{E}_{n,m}, \\
(C \cdot \mathcal{R}_{n,m}) \circ (A \cdot \mathcal{E}_{n,m}) &= \big(\mathrm{Res}_m \circ \mathcal{K}_{n,m}[C, A] \circ \mathrm{Res}_m\big) \cdot \mathcal{P}_{m,m},
\end{aligned} \tag{3.11}$$

Finally, for any polynomials $A_4(Z, Z_W')$, $A_5(Z_W, Z_Y')$, $Z \in \mathbb{R}^{2n}$, $Z_Y' \in \mathbb{R}^{2m}$, $Z_W, Z_W' \in \mathbb{R}^{2l}$, there is a polynomial $A_3''' := \mathcal{K}_{n,m}^{E}[A_4, A_5]$, with the same properties as A_3, such that

$$(A_4 \cdot \mathcal{E}_{n,l}) \circ (A_5 \cdot \mathcal{E}_{l,m}) = A_3''' \cdot \mathcal{E}_{n,m}.$$

Remark 3.2. The statement (3.8) for $n = m$ is due to Ma–Marinescu [14, Lemma 7.1.1, (7.1.6)].

Proof. We decompose polynomials A_1, A_2 as follows

$$A_1(Z, Z') = \sum_{\alpha} Z_N^{\alpha} \cdot A_1^{\alpha}(Z_Y, Z'), \quad A_2(Z, Z') = \sum_{\alpha'} A_2^{\alpha'}(Z, Z_Y') Z_N'^{\alpha'},$$

where $\alpha, \alpha' \in \mathbb{N}^{2(n-m)}$ verify $|\alpha| \leq \deg A_1$, $|\alpha'| \leq \deg A_2$. In [9, (3.36)], we established the first statement and proved that

$$\mathcal{K}_{n,m}[A_1, A_2] = \sum_{\alpha} \sum_{\alpha'} Z_N^{\alpha} Z_N'^{\alpha'} \mathcal{K}_{n,n}[A_1^{\alpha}, A_2^{\alpha'}]. \tag{3.12}$$

Along the same lines, we obtained in [9, (3.38)] the second statement and

$$\mathcal{K}'_{n,m}[A_1, A_2](Z, Z') = \sum_{\alpha'} Z_N'^{\alpha'} \cdot \mathcal{K}_{n,n}[A_1, A_2^{\alpha'}](Z, Z_Y'). \tag{3.13}$$

To get the third statement, we represent $A(Z, Z'_Y) := \sum Z_N^\alpha \cdot A^\alpha(Z_Y, Z'_Y)$. Then, from (3.4) and (3.6), the following equation holds:

$$(A^\alpha \cdot \mathcal{E}_{n,m}) \circ (D \cdot \mathcal{P}_m) = \exp\Big(-\frac{\pi}{2}|Z_N|^2\Big) \cdot (A^\alpha \cdot \mathcal{P}_m) \circ (D \cdot \mathcal{P}_m).$$

By this and (3.8), we clearly have (3.10) for

$$\mathcal{K}^{EP}_{n,m}[A, D] = \sum_\alpha Z_N^\alpha \cdot \mathcal{K}_{m,m}[A^\alpha, D]. \tag{3.14}$$

Now, we note that an easy verification, based on (3.4), (3.6) and (3.7), shows that

$$\begin{aligned}\big((B \cdot \mathcal{P}_n) \circ (A \cdot \mathcal{E}_{n,m})\big)(Z, Z'_Y) &= \big((B \cdot \mathcal{P}_n) \circ (A \cdot \mathcal{P}_n)\big)(Z, Z'_Y),\\ \big((B \cdot \mathcal{P}^\perp_{n,m}) \circ (A \cdot \mathcal{E}_{n,m})\big)(Z, Z'_Y) &= \big((B \cdot \mathcal{P}^\perp_{n,m}) \circ (A \cdot \mathcal{P}^\perp_{n,m})\big)(Z, Z'_Y),\\ \big((C \cdot \mathcal{R}_{n,m}) \circ (A \cdot \mathcal{E}_{n,m})\big)(Z_Y, Z'_Y) &= \big((C \cdot \mathcal{P}^\perp_{n,m}) \circ (A \cdot \mathcal{P}^\perp_{n,m})\big)(Z_Y, Z'_Y).\end{aligned}$$

This clearly implies (3.11) by (3.8) and (3.9).

It is now only left to prove the fifth statement. For this, for

$$Z = (Z_Y, Z_{N^{W|Y}}, Z_{N^{X|W}}), \quad Z_Y, Z'_Y \in \mathbb{R}^{2m},$$
$$Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)}, \quad Z_{N^{X|W}} \in \mathbb{R}^{2(n-l)}, \quad Z'_W \in \mathbb{R}^{2l}, \quad Z_W := (Z_Y, Z_{N^{W|Y}}),$$

we decompose the polynomial A_4 as follows:

$$A_4(Z, Z'_W) = \sum_\alpha Z^\alpha_{N^{X|W}} \cdot A_4^\alpha(Z_W, Z'_W). \tag{3.15}$$

An easy verification shows that

$$\begin{aligned}&\big((A_4^\alpha \cdot \mathcal{E}_{n,l}) \circ (A_5 \cdot \mathcal{E}_{l,m})\big)(Z, Z'_Y)\\ &\qquad = \exp\Big(-\frac{\pi}{2}|Z_{N^{X|W}}|^2\Big) \cdot \big((A_4^\alpha \cdot \mathcal{P}_l) \circ (A_5 \cdot \mathcal{E}_{l,m})\big)(Z_W, Z'_Y).\end{aligned}$$

From (3.11), we obtain

$$\mathcal{K}^{E}_{n,m}[A_4, A_5] = \sum_\alpha Z_N^\alpha \cdot \big(\mathcal{K}_{l,l}[A_4^\alpha, A_5] \circ \mathrm{Res}_m\big), \tag{3.16}$$

which finishes the proof. The statements about the degrees of A_3, etc., follow from the validity of the corresponding statements for $\mathcal{K}_{n,n}$, proved by Ma–Marinescu in [16] and expressions (3.12), (3.13), (3.14), (3.15), (3.16). ■

From the above, we see that to compute the polynomials from Lemma 3.1, it suffices to give an algorithm for the calculation of $\mathcal{K}_{n,m}$. Below, we explain how to do this. Directly from the definitions, we see that

$$\mathcal{K}_{n,m}[1 \cdot P(Z'), A] = \mathcal{K}_{n,m}[1, P(Z) \cdot A]$$

for any polynomial A. Also, we trivially have

$$\mathcal{K}_{n,m}[P(Z) \cdot A(Z, Z'), A'(Z, Z')] = P(Z)\mathcal{K}_{n,m}[A(Z, Z'), A'(Z, Z')]$$

for any polynomials P, A, A'. Hence, it is enough to give an algorithm for the calculation of $\mathcal{K}_{n,m}$ where the first argument is given by 1. For this, remark that for any $i = 1, \dots, n$, $a, b \in \mathbb{N}$, we have

$$\mathcal{K}_{n,m}[1, P_i(Z) z_i^a \overline{z}_i^b] = \mathcal{K}_{n,m}[1, P_i(Z)] \cdot \mathcal{K}_{n,m}[1, z_i^a \overline{z}_i^b],$$

where the polynomial $P_i(Z)$ does not depend on z_i and $\overline{z}_i$. Hence, to understand $\mathcal{K}_{n,m}$, it suffices to know how to calculate it for polynomials $z_i^a \overline{z}_i^b$. We describe below the general formula.

Using (3.2) and (3.5), we see that for any $a, b \in \mathbb{N}$, $i \leq n$, we get

$$\mathcal{P}_n \circ (z_i^a \overline{z}_i^b \cdot \mathcal{P}_n) = \mathcal{P}_n \circ \Big(z_i^a \cdot \Big(\frac{b_i}{2\pi} + \overline{z}_i' \Big)^b \mathcal{P}_n \Big).$$

Remark that by (3.3), we have $\mathcal{P}_n \circ (b^\alpha z^\beta \cdot \mathcal{P}_n) = 0$ as long as $\alpha \neq 0$. Hence, from (3.2), we have

$$\mathcal{P}_n \circ (z_i^a \cdot b_i^k \mathcal{P}_n) = \begin{cases} \frac{a! 2^k}{(a-k)!} z_i^{a-k} \mathcal{P}_n & \text{for } a \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

From this, for $i \leq m$, we deduce directly the following identity

$$\mathcal{K}_{n,m}[1, z_i^a \overline{z}_i^b] = \sum_{l+k=b} \frac{1}{\pi^k} \frac{a! b!}{(a-k)! l! k!} z_i^{a-k} \overline{z}_i'^{\,l}. \tag{3.17}$$

Recall the following famous integral calculation: for $z = x + \sqrt{-1} y$, we have

$$\int_{\mathbb{C}} \exp(-\pi |z|^2) z^a \overline{z}^b \, dx \, dy = \delta_{ab} \frac{a!}{\pi^a}. \tag{3.18}$$

For $m + 1 \leq i \leq n$, (3.18) shows that

$$\mathcal{K}_{n,m}[1, z_i^a \overline{z}_i^b] = \delta_{ab} \frac{a!}{\pi^a}. \tag{3.19}$$

Corollary 3.3. *Assume that for a polynomial $A(Z, Z'_Y)$, $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$, the following equality holds:*

$$\mathcal{P}_n \circ (A \cdot \mathcal{E}_{n,m}) \circ \mathcal{P}_m = A \cdot \mathcal{E}_{n,m}.$$

Then A is a polynomial in $z, \overline{z}'_Y$.

Proof. First of all, our assumption clearly implies that $\mathcal{P}_n \circ (A \cdot \mathcal{E}_{n,m}) = A \cdot \mathcal{E}_{n,m}$. Hence, from (3.11), we deduce $\mathcal{K}_{n,n}[1, A] \circ \mathrm{Res}_m = A$. This along with (3.17) imply that A is a polynomial in z and Z'_Y. Now, again, our assumption implies that

$$(A \cdot \mathcal{E}_{n,m}) \circ \mathcal{P}_m = A \cdot \mathcal{E}_{n,m}.$$

Hence, $\mathcal{K}^{EP}_{n,n}[A, 1] = A$. This, in conjunction with (3.14) and (3.17), implies that A is a polynomial in z and $\overline{z}'_Y$, which concludes the proof. ■

3.2. Schwartz kernels of Bergman projectors and extension operator

The main goal of this section is to recall the results about the asymptotics of Schwartz kernels of Bergman projectors and the extension operator. We use notations from Section 1 and assume that the triple (X, Y, g^{TX}) is of bounded geometry. Let us recall first the results about the exponential decay of those Schwartz kernels.

Theorem 3.4 (Ma–Marinescu [19, Theorem 1]). *There are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, the following estimate holds:*

$$|B^X_p(x_1, x_2)|_{\mathcal{C}^k} \leq C p^{n+k/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x_1, x_2)\big),$$

where $\mathcal{C}^k$-norm here is interpreted as in Definition 1.3.

Theorem 3.5 ([9, Theorems 1.5 and 1.8]). *There are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, $y \in Y$, the following estimates hold:*

$$\begin{aligned} |\mathrm{E}^{X|Y}_p(x_1, y)|_{\mathcal{C}^k} &\leq C p^{m+k/2} \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x_1, y)\big), \\ |B^{X|Y\perp}_p(x_1, x_2)|_{\mathcal{C}^k} &\leq C p^{n+k/2} \\ &\qquad \cdot \exp\Big(-c\sqrt{p} \cdot \big(\mathrm{dist}(x_1, x_2) + \mathrm{dist}(x_1, Y) + \mathrm{dist}(x_2, Y)\big)\Big). \end{aligned}$$

Let us now give the first direct application of Theorems 3.4, 3.5. Assume that we start with two different choices of functions ρ_1, ρ_2 as in (1.13), and form two different brackets $\langle\!\langle \cdot \rangle\!\rangle_1$, $\langle\!\langle \cdot \rangle\!\rangle_2$, as in (1.14), corresponding to those choices.

Corollary 3.6. *There is $p_1 \in \mathbb{N}^*$ such that for any $g \in \mathcal{C}^\infty_b(Y, \mathrm{Sym}^k(N^{X|Y})^*)$, $k \in \mathbb{N}$, there are $c, C > 0$, such that for any $p \geq p_1$, $x \in X$, $y \in Y$, we have*

$$\begin{aligned} |T^{X|Y}_{\langle\!\langle g \rangle\!\rangle_1, p}(x, y) - T^{X|Y}_{\langle\!\langle g \rangle\!\rangle_2, p}(x, y)|_{\mathcal{C}^l} &\leq C \exp\big(-c\sqrt{p} \cdot \big(1 + \mathrm{dist}(x, y)\big)\big), \\ |T^{Y|X}_{\langle\!\langle g \rangle\!\rangle_1, p}(y, x) - T^{Y|X}_{\langle\!\langle g \rangle\!\rangle_2, p}(y, x)|_{\mathcal{C}^l} &\leq C \exp\big(-c\sqrt{p} \cdot \big(1 + \mathrm{dist}(x, y)\big)\big). \end{aligned}$$

In particular, Definition 1.3 ultimately does not depend on the choice of ρ.

Proof. The proof follows directly from Theorem 3.4 and Corollary 2.8. ∎

Theorem 3.5 shows that to understand fully the asymptotics of the Schwartz kernel of the Bergman projector (resp. orthogonal Bergman projector and the extension operator), it suffices to do so in a neighborhood of a fixed point on the diagonal of X (resp. Y), embedded in $X \times X$ (resp. $X \times X$ and $X \times Y$). Let us recall the results in this direction, showing that Schwartz kernels of our operators are essentially equal, up to a recalling, to Schwartz kernel of the model operators considered in Section 3.1. Before this, let us fix some notation.

We fix $x_0 \in X$ and an orthonormal frame $(e_1, \dots, e_{2n})$ of $(T_{x_0}X, g^{TX}_{x_0})$, verifying (1.18). Recall that geodesic coordinates were defined in (2.11). Define the function

$$\kappa^X_{\phi,x_0} : B_0^{\mathbb{R}^{2n}}(r_X) \to \mathbb{R}$$

by

$$\big((\phi^X_{x_0})^* \, dv_X\big)(Z) = \kappa^X_{\phi,x_0} \, dZ_1 \wedge \cdots \wedge dZ_{2n}. \tag{3.20}$$

Now, let $x_0 = y_0$, where $y_0 \in Y$, and $(e_1, \dots, e_{2n})$ be as (2.3). Recall that Fermi coordinates were defined in (2.3). Define the function

$$\kappa^{X|Y}_{\psi,y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \to \mathbb{R}$$

by

$$\big((\psi^{X|Y}_{y_0})^* \, dv_X\big)(Z) = \kappa^{X|Y}_{\psi,y_0} \, dZ_1 \wedge \cdots \wedge dZ_{2n}. \tag{3.21}$$

Recall that the function $\kappa^{X|Y}_N$ was defined in (1.7). Clearly, for $Z = (Z_Y, Z_N) \in \mathbb{R}^{2n}$, $Z_Y \in \mathbb{R}^{2m}$, we have the following relation between different κ-functions

$$\kappa^{X|Y}_{\psi,y_0}(Z) = \kappa^{X|Y}_N \big(\psi^{X|Y}_{y_0}(Z)\big) \cdot \kappa^Y_{\phi,y_0}(Z_Y). \tag{3.22}$$

Also, under assumptions (1.8), we have $\kappa^{X|Y}_{\psi,y_0}(0) = \kappa^Y_{\phi,y_0}(0) = 1$.

Recall that the second fundamental form $A^{X|Y} \in \mathscr{C}^\infty(Y, T^*Y \otimes \operatorname{End}(TX|_Y))$ was defined in (2.1). Recall that the functions $\mathcal{P}_n$, $\mathcal{P}^\perp_{n,m}$, $\mathcal{E}_{n,m}$, were defined in (3.4), (3.6).

We fix an orthonormal frame $(f_1, \dots, f_r)$ of $(F_{y_0}, h^F_{y_0})$ and define the orthonormal frames $(\tilde{f}_1^{X|Y}, \dots, \tilde{f}_r^{X|Y})$, $(\tilde{f}_1'^X, \dots, \tilde{f}_r'^X)$ of (F, h^F) in a neighborhood of y_0, as in Section 2.1.

Notation. For $g \in \mathscr{C}^\infty(X, F)$, by an abuse of notation, we write $g(\phi^X_{y_0}(Z)) \in \mathbb{R}^r$, $Z \in \mathbb{R}^{2n}$, $|Z| \leq R$, for coordinates of g in the frame $(\tilde{f}_1'^X, \dots, \tilde{f}_r'^X)$. We identify $g(\phi^X_{y_0}(Z))$ with an element in F_{y_0} using the frame $(f_1, \dots, f_r)$. Similarly, we denote by $g(\psi^{X|Y}_{y_0}(Z)) \in \mathbb{R}^r$ the coordinates in the frame $(\tilde{f}_1^{X|Y}, \dots, \tilde{f}_r^{X|Y})$ and identify them with an element from F_{y_0}. Similar notations are used for sections of F^*, $F \otimes L^p$, $(F \otimes L^p)^*$, $F \boxtimes F^*$, etc.

Theorem 3.7. *For any $r \in \mathbb{N}$, $y_0 \in Y$, there are $J_r^{X|Y}(Z,Z') \in \mathrm{End}(F_{y_0})$ polynomials in $Z, Z' \in \mathbb{R}^{2n}$, with the same parity as r and $\deg J_r^{X|Y} \leq 3r$, whose coefficients are polynomials in ω, R^{TX}, $A^{X|Y}$, R^F, $(dv_X/dv_{g^{TX}})^{\pm 1/2n}$, $(dv_Y/dv_{g^{TY}})^{\pm 1/2n}$, and their derivatives of order $\leq 2r$, all evaluated at y_0, such that for the functions $F_r^{X|Y} := J_r^{X|Y} \cdot \mathcal{P}_n$ over $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, the following holds: There are $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l, l' \in \mathbb{N}$, there exists $C > 0$, such that for any*

$$y_0 \in Y, \quad p \geq p_1, \quad Z, Z' \in \mathbb{R}^{2n}, \quad |Z|, |Z'| \leq \varepsilon,$$
$$\alpha, \alpha' \in \mathbb{N}^{2n}, \quad |\alpha| + |\alpha'| \leq l, \quad Q^1_{k,l,l'} := 3(n+k+l'+2) + l,$$

we have

$$\begin{aligned}\Bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Bigg(& \frac{1}{p^n} B_p^X\big(\psi_{y_0}^{X|Y}(Z), \psi_{y_0}^{X|Y}(Z')\big) \\ & - \sum_{r=0}^{k} p^{-r/2} F_r^{X|Y}(\sqrt{p}Z, \sqrt{p}Z') \kappa_\psi^{X|Y}(Z)^{-1/2} \kappa_\psi^{X|Y}(Z')^{-1/2} \Bigg) \Bigg|_{\mathcal{C}^{l'}} \\ & \leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|\big)^{Q^1_{k,l,l'}} \exp\big(-c\sqrt{p}|Z-Z'|\big),\end{aligned}$$

where the $\mathcal{C}^{l'}$-norm is taken with respect to y_0. Also, the following identity holds

$$J_0^{X|Y}(Z,Z') = \mathrm{Id}_{F_{y_0}}. \tag{3.23}$$

Moreover, under the assumption (1.8), we have

$$\begin{aligned} J_1^{X|Y}(Z,Z') = \mathrm{Id}_{F_{y_0}} \cdot \pi\big(& g\big(z_N, A^{X|Y}(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)\big) \\ & + g\big(\bar{z}'_N, A^{X|Y}(z_Y - z'_Y)(z_Y - z'_Y)\big)\big). \end{aligned} \tag{3.24}$$

Proof. For $X = Y$, the result is due to Dai–Liu–Ma [4] and the calculation of $J_1^{X|X}$ is due to Ma–Marinescu [14, Remark 4.1.26]. The proof of the general case is done in [9, Theorem 5.5] by relying on the result of [4] and some local calculations. ∎

Theorem 3.8 ([9, Theorem 1.6]). *For any $r \in \mathbb{N}$, $y_0 \in Y$, there are polynomials $J_r^{X|Y,E}(Z, Z'_Y) \in \mathrm{End}(F_{y_0})$ in $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$, with the same properties as in Theorem 3.7, such that for $F_r^{X|Y,E} := J_r^{X|Y,E} \cdot \mathcal{E}_{n,m}$, the following holds: There are $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l, l' \in \mathbb{N}$, there is $C > 0$, such that for any*

$$y_0 \in Y, \quad p \geq p_1, \quad Z = (Z_Y, Z_N), \quad Z_Y, Z'_Y \in \mathbb{R}^{2m}, \quad Z_N \in \mathbb{R}^{2(n-m)},$$
$$|Z|, |Z'_Y| \leq \varepsilon, \quad \alpha \in \mathbb{N}^{2n}, \quad \alpha' \in \mathbb{N}^{2m}, \quad |\alpha| + |\alpha'| \leq l$$

for $Q^2_{k,l,l'} := 6(16(n+2)(k+1)+l')+2l$, *the following bound holds:*

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial {Z'_Y}^{\alpha'}}\left(\frac{1}{p^m}\mathrm{E}_p^{X|Y}\left(\psi_{y_0}^{X|Y}(Z),\psi_{y_0}^{X|Y}(Z'_Y)\right)\right.\right.$$
$$\left.\left.-\sum_{r=0}^{k}p^{-r/2}F_r^{X|Y,E}(\sqrt{p}Z,\sqrt{p}Z'_Y)\kappa_{\psi}^{X|Y}(Z)^{-1/2}\kappa_{\phi}^{Y}(Z'_Y)^{-1/2}\right)\right|_{\mathcal{C}^{l'}}$$
$$\leq Cp^{-(k+1-l)/2}\left(1+\sqrt{p}|Z|+\sqrt{p}|Z'_Y|\right)^{Q^2_{k,l,l'}}$$
$$\cdot\exp\left(-c\sqrt{p}\left(|Z_Y-Z'_Y|+|Z_N|\right)\right),$$

where the $\mathcal{C}^{l'}$*-norm is taken with respect to* y_0. *Also, the following identity holds:*

$$J_0^{X|Y,E}(Z,Z'_Y)=\mathrm{Id}_{F_{y_0}}\cdot\kappa_N^{X|Y}(y_0)^{1/2}. \tag{3.25}$$

Theorem 3.9 ([9, Theorem 1.9]). *For any* $r\in\mathbb{N}$, $y_0\in Y$, *there are polynomials* $J_r^{X|Y,\perp}(Z,Z')\in\mathrm{End}(F_{y_0})$, $Z,Z'\in\mathbb{R}^{2n}$, *with the same properties as in Theorem 3.7 such that, for* $F_r^{X|Y,\perp}:=J_r^{X|Y,\perp}\cdot\mathcal{P}^{\perp}_{n,m}$, *the following holds: There are* $\varepsilon,c>0$ *and* $p_1\in\mathbb{N}^*$ *such that for any* $k,l,l'\in\mathbb{N}$, *there is* $C>0$, *such that for any*

$$y_0\in Y,\quad p\geq p_1,\quad Z=(Z_Y,Z_N),\quad Z'=(Z'_Y,Z'_N),\quad Z_Y,Z'_Y\in\mathbb{R}^{2m},$$
$$Z_N,Z'_N\in\mathbb{R}^{2(n-m)},\quad |Z|,|Z'|\leq\varepsilon,\quad \alpha,\alpha'\in\mathbb{N}^{2n},\quad |\alpha|+|\alpha'|\leq l,$$

for $Q^3_{k,l,l'} := 3(8(n+2)(k+1)+l')+l$, *we have*

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\left(\frac{1}{p^n}B_p^{X|Y\perp}\left(\psi_{y_0}^{X|Y}(Z),\psi_{y_0}^{X|Y}(Z')\right)\right.\right.$$
$$\left.\left.-\sum_{r=0}^{k}p^{-r/2}F_r^{X|Y,\perp}(\sqrt{p}Z,\sqrt{p}Z')\kappa_{\psi}^{X|Y}(Z)^{-1/2}\kappa_{\psi}^{X|Y}(Z')^{-1/2}\right)\right|_{\mathcal{C}^{l'}}$$
$$\leq Cp^{-(k+1-l)/2}\left(1+\sqrt{p}|Z|+\sqrt{p}|Z'|\right)^{Q^3_{k,l,l'}}$$
$$\cdot\exp\left(-c\sqrt{p}\left(|Z_Y-Z'_Y|+|Z_N|+|Z'_N|\right)\right),$$

where the $\mathcal{C}^{l'}$*-norm is taken with respect to* y_0. *Also, we have*

$$J_0^{X|Y,\perp}(Z,Z')=\mathrm{Id}_{F_{y_0}}. \tag{3.26}$$

Remark 3.10. In the paper [9, Theorems 1.6 and 1.9], we also explicitly calculated $J_1^{X|Y,E}$ and $J_1^{X|Y,\perp}$.

3.3. Basic Toeplitz type operators and their asymptotics

The main goal of this section is to study asymptotic expansions of basic Toeplitz type operators. In particular, we prove Theorem 3.14 and give precise formulas for the constants C_1, C_2 from (3.38). The following two lemmas will be crucial in what follows.

Lemma 3.11. Let (X, Y, g^{TX}) be of bounded geometry. There is $p_1 \in \mathbb{N}$, such that for any $f \in \mathscr{C}^\infty_b(Y, \mathrm{End}(\iota^* F))$, $g \in \bigoplus_{k=0}^\infty \mathscr{C}^\infty_b(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F))$, $l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, the Schwartz kernels

$$T^Y_{f,p}(y_1, y_2), \quad T^{X|Y}_{\langle\langle g\rangle\rangle,p}(x, y_1), \quad T^{Y|X}_{\langle\langle g\rangle\rangle,p}(y_1, x); \qquad x \in X, \ y_1, y_2 \in Y$$

of $T^Y_{f,p}$, $T^{X|Y}_{\langle\langle g\rangle\rangle,p}$, $T^{Y|X}_{\langle\langle g\rangle\rangle,p}$, evaluated with respect to dv_Y, dv_Y and dv_X respectively, satisfy

$$\begin{aligned} |T^Y_{f,p}(y_1, y_2)|_{\mathscr{C}^l} &\leq C p^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_Y(y_1, y_2)\big), \\ |T^{X|Y}_{\langle\langle g\rangle\rangle,p}(x, y_1)|_{\mathscr{C}^l} &\leq C p^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_X(x, y_1)\big), \\ |T^{Y|X}_{\langle\langle g\rangle\rangle,p}(y_1, x)|_{\mathscr{C}^l} &\leq C p^{n+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_X(x, y_1)\big). \end{aligned} \tag{3.27}$$

Proof. The first and third estimates of (3.27) follow trivially from Theorem 3.4 and Corollary 2.8. The second estimate is a consequence of Theorems 3.4, 3.5 and Corollary 2.8. ∎

We fix $y_0 \in Y$, a unitary frame $(f_1, \ldots, f_r)$ of $(F_{y_0}, h^F_{y_0})$ and use the notational conventions introduced before Theorem 3.7.

Lemma 3.12. For any $f \in \mathscr{C}^\infty_b(Y, \mathrm{End}(\iota^* F))$, respectively,

$$g \in \mathscr{C}^\infty_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big), \quad k \in \mathbb{N},$$

where $y_0 \in Y$, $r \in \mathbb{N}$, there are $J^Y_{r,f}(Z_Y, Z'_Y) \in \mathrm{End}(F_{y_0})$, respectively,

$$J^E_{r,g}(Z, Z'_Y) \in \mathrm{End}(F_{y_0}), \quad J^R_{r,g}(Z_Y, Z') \in \mathrm{End}(F_{y_0}),$$

polynomials in $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z, Z' \in \mathbb{R}^{2n}$ of the same parity as r (resp. $r + k$), such that the coefficients of $J^Y_{r,f}$ (resp. $J^E_{r,g}$, $J^R_{r,g}$) lie in $\mathscr{C}^\infty_b(Y, \mathrm{End}(\iota^* F))$, and for $F^Y_{r,f} := J^Y_{r,f} \cdot \mathscr{P}_m$ (resp. $F^E_{r,g} := J^E_{r,g} \cdot \mathscr{E}_{n,m}$, $F^R_{r,g} := J^R_{r,g} \cdot \mathscr{R}_{n,m}$), the following holds: There are $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any

$$\begin{gathered} y_0 \in Y, \quad p \geq p_1, \quad Z, Z' \in \mathbb{R}^{2n}, \quad Z = (Z_Y, Z_N), \quad Z' = (Z'_Y, Z'_N), \\ Z_Y, Z'_Y \in \mathbb{R}^{2m}, \quad |Z|, |Z'| \leq \varepsilon, \quad \alpha, \alpha' \in \mathbb{N}^{2n}, \quad \alpha_0, \alpha'_0 \in \mathbb{N}^{2m}, \\ |\alpha| + |\alpha'_0|, |\alpha_0| + |\alpha'|, |\alpha_0| + |\alpha'_0| \leq l, \end{gathered}$$

the following bounds hold:

$$\left| \frac{\partial^{|\alpha_0|+|\alpha_0'|}}{\partial Z_Y^{\alpha_0} \partial Z_Y'^{\alpha_0'}} \left(\frac{1}{p^m} T_{p,f}^Y \big(\phi_{y_0}^Y(Z_Y), \phi_{y_0}^Y(Z_Y')\big) \right.\right.$$
$$\left.\left. - \sum_{r=0}^{k} p^{-r/2} F_{r,f}^Y(\sqrt{p} Z_Y, \sqrt{p} Z_Y') \kappa_\phi^Y(Z_Y)^{-1/2} \kappa_\phi^Y(Z_Y')^{-1/2} \right) \right|_{\mathcal{C}^{l'}}$$
$$\leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z_Y'|\big)^Q$$
$$\cdot \exp\big(-c\sqrt{p}|Z_Y - Z_Y'|\big), \tag{3.28}$$

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z_Y'^{\alpha'}} \left(\frac{1}{p^m} T_{\langle\langle g \rangle\rangle,p}^{X|Y} \big(\psi_{y_0}^{X|Y}(Z), \phi_{y_0}^Y(Z_Y')\big) \right.\right.$$
$$\left.\left. - \sum_{r=0}^{k} p^{-r/2} F_{r,g}^E(\sqrt{p} Z, \sqrt{p} Z_Y') \kappa_\psi^{X|Y}(Z)^{-1/2} \kappa_\phi^Y(Z_Y')^{-1/2} \right) \right|_{\mathcal{C}^{l'}}$$
$$\leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z| + \sqrt{p}|Z_Y'|\big)^Q$$
$$\cdot \exp\big(-c\sqrt{p}\big(|Z_N| + |Z_Y - Z_Y'|\big)\big), \tag{3.29}$$

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^{\alpha} \partial Z'^{\alpha'}} \left(\frac{1}{p^n} T_{\langle\langle g \rangle\rangle,p}^{Y|X} \big(\phi_{y_0}^Y(Z_Y), \psi_{y_0}^{X|Y}(Z')\big) \right.\right.$$
$$\left.\left. - \sum_{r=0}^{k} p^{-r/2} F_{r,g}^R(\sqrt{p} Z_Y, \sqrt{p} Z') \kappa_\phi^Y(Z_Y)^{-1/2} \kappa_\psi^{X|Y}(Z')^{-1/2} \right) \right|_{\mathcal{C}^{l'}}$$
$$\leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'|\big)^Q$$
$$\cdot \exp\big(-c\sqrt{p}\big(|Z_N'| + |Z_Y - Z_Y'|\big)\big), \tag{3.30}$$

where the $\mathcal{C}^{l'}$-norm is taken with respect to y_0.

Moreover, we have $J_{0,f}(Z_Y, Z_Y') = f(y_0)$ and in the notations of (1.14), for $g^h \in \mathcal{C}_b^\infty(Y, \mathrm{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F))$, respectively,

$$g^a \in \mathcal{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(0,1)*} \otimes \mathrm{End}(\iota^* F)\big),$$

where $k \in \mathbb{N}$, the polynomial $J_{0,g^h}^E(Z, Z_Y')$ (resp. $J_{0,g^a}^R(Z_Y, Z')$) depends only on z_N (resp. $\overline{z}_N$), it has degree k, and as a section of $\mathrm{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)$, respectively,

$$\mathrm{Sym}^k(N^{X|Y})^{(0,1)*} \otimes \mathrm{End}(\iota^* F)$$

over Y, it coincides with $g^h \cdot \kappa_N^{X|Y}(y_0)^{1/2}$ (resp. $g^a \cdot \kappa_N^{X|Y}(y_0)^{-1/2}$) for $k \geq 1$ and equal to 0 for $k = 0$.

Proof. We establish each of the three statements one by one. Remark first that Ma–Marinescu in [16, Lemma 4.6] established the first part of this result for compact manifolds. We will now describe why essentially the same proof holds for the first statement of Lemma 3.12 for functions from $\mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$ on manifolds of bounded geometry (Y, g^{TY}). Trivially, we have

$$T_{f,p}^Y(y_1, y_2) = \int_Y B_p^Y(y_1, y_3) \cdot f(y_3) \cdot B_p^Y(y_3, y_2)\, dv_Y(y_3). \tag{3.31}$$

Now, let $\varepsilon > 0$ be as in Theorem 3.7. We put $\varepsilon_0 := \varepsilon/2$. Let $y_0 \in Y$ and $y_1, y_2 \in B_{y_0}^Y(\varepsilon_0)$. We decompose the integral in (3.31) into two parts: the first one over $B_{y_0}^Y(\varepsilon)$, and the second one is over its complement, which we denote by Q. Clearly, for $y_3 \in Q$, we get

$$\mathrm{dist}(y_1, y_3) + \mathrm{dist}(y_3, y_2) \geq \varepsilon.$$

Hence, from Theorem 3.4, Proposition 2.6 and (1.3), we see that the contribution from the integration over Q is smaller than $\exp(-c\sqrt{p}(1 + \mathrm{dist}(y_1, y_2)))$ for some constant $c > 0$. Consequently, only the integration over $B_{y_0}^Y(\varepsilon)$ is non-negligible. To evaluate it, we apply Theorem 3.4. We calculate the integral over the pull-back with respect to the exponential map of our differential forms. We use the notations introduced before Theorem 3.7. After the change of variables $Z \mapsto \sqrt{p}Z$, an estimate similar to the one which bounded the integral over Q, and the first part of Lemma 3.1, applied for $n := m$, we see that (3.28) holds for

$$J_{r,f}^Y := \sum_{a+b+|\alpha|=r} \sum_{\alpha} \mathcal{K}_{m,m}\Big[J_a^{Y|Y}, \frac{\partial^\alpha f(\phi_{y_0}^Y(Z_Y))}{\partial Z_Y^\alpha}(0) \cdot \frac{Z_Y^\alpha}{\alpha!} \cdot J_b^{Y|Y} \Big], \tag{3.32}$$

where α runs through the set of multiindices $\mathbb{N}^{2m}$. From (3.23) and (3.32), we deduce the statement about $J_{0,f}^Y$. From (3.32), our bounded geometry assumption, the fact that $f \in \mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$ and the fact that the coefficients of $J_r^{Y|Y}$ are polynomials in ω, R^{TY}, R^F, $(dv_Y/dv_{g^{TY}})^{\pm 1/2m}$, and their derivatives of order $\leq 2r$, we deduce that the coefficients of $J_{r,f}^Y$ lie in $\mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$. The statement about their parity follows from Lemma 3.1 and (3.32).

Now, let us establish the second part. The proof is absolutely analogous to the proof of the first part. One only has to, in addition to Theorems 3.4 and 3.7, use Theorems 3.5, 3.8 and 3.9. Again we use the notations introduced before Theorem 3.7. We view the section $\{g\}$, constructed before (1.14), as a function with values in $\mathrm{End}(F_{y_0})$. Clearly, $\{g\}(\psi_{y_0}^{X|Y}(Z_Y, Z_N))$ is polynomial in vertical directions; in other words

$$\{g\}\big(\psi_{y_0}^{X|Y}(Z_Y, Z_N)\big) = \sum_{\beta} g_\beta\big(\phi_{y_0}^Y(Z_Y)\big) \frac{Z_N^\beta}{\beta!}, \tag{3.33}$$

where $\beta \in \mathbb{N}^{2(n-m)}$ runs through all the multiindices and the sum is finite. From the first two equations of (3.11) and the reasoning similar to the one before (3.32), we obtain that the polynomials

$$J^E_{r,g}(Z,Z'_Y) := \sum_{\alpha}\sum_{\beta}\sum_{a+b+|\alpha|=r}$$
$$\Bigg(\mathcal{K}_{n,n}\Big[J^{X|Y}_a, \frac{\partial^\alpha g_\beta(\phi^Y_{y_0}(Z_Y))}{\partial Z^\alpha_Y}(0)\cdot\frac{Z^\beta_N}{\beta!}\cdot\frac{Z^\alpha_Y}{\alpha!}\cdot J^{X|Y,E}_b\Big]$$
$$-\mathcal{K}_{n,m}\Big[J^{X|Y,\perp}_a, \frac{\partial^\alpha g_\beta(\phi^Y_{y_0}(Z_Y))}{\partial Z^\alpha_Y}(0)\cdot\frac{Z^\beta_N}{\beta!}\cdot\frac{Z^\alpha_Y}{\alpha!}\cdot J^{X|Y,E}_b\Big]\Bigg)(Z,Z'_Y), \quad (3.34)$$

where α runs through the multiindices $\mathbb{N}^{2m}$, satisfy the second equation from (3.28). The statement about the parity of $J^E_{r,g}$ follows from (3.34) similarly to how it was done in (3.32). The calculation of $J^E_{0,g}$ follows from (3.17), (3.23), (3.25), (3.26) and (3.34).

Now, let us establish the third part. The proof is absolutely analogous to the proof of the first part. One only has to use Theorems 3.5 and 3.9, in addition to Theorems 3.4 and 3.7. More precisely, in the notations of (3.33), from (3.8) and (3.9), we obtain that the third equation from (3.28) holds for

$$J^R_{r,g}(Z_Y,Z') := \sum_{\alpha}\sum_{\beta}\sum_{a+b+|\alpha|=r}$$
$$\Bigg(\mathcal{K}_{n,n}\Big[J^{X|Y}_a, \frac{\partial^\alpha g_\beta(\phi^Y_{y_0}(Z_Y))}{\partial Z^\alpha_Y}(0)\cdot\frac{Z^\beta_N}{\beta!}\cdot\frac{Z^\alpha_Y}{\alpha!}\cdot J^{X|Y}_b\Big]$$
$$-\mathcal{K}'_{n,m}\Big[J^{X|Y}_a, \frac{\partial^\alpha g_\beta(\phi^Y_{y_0}(Z_Y))}{\partial Z^\alpha_Y}(0)\cdot\frac{Z^\beta_N}{\beta!}\cdot\frac{Z^\alpha_Y}{\alpha!}\cdot J^{X|Y,\perp}_b\Big]\Bigg)(Z_Y,Z'). \quad (3.35)$$

The statement about the parity of $J^R_{r,g}$ follows from (3.35), similarly to (3.32). The calculation of $J^R_{0,g}$ follows from (3.17), (3.23), (3.25), (3.26) and (3.35). ■

From Lemma 3.12, we obtain directly the following statement.

Corollary 3.13. *Assume that for*

$$f_1, f_2 \in \mathcal{C}^\infty_b\big(Y, \mathrm{End}(\iota^* F)\big),$$
$$g^h_1, g^h_2 \in \bigoplus_{k=1}^{\infty}\mathcal{C}^\infty_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(1,0)*}\otimes\mathrm{End}(\iota^* F)\big),$$
$$g^a_1, g^a_2 \in \bigoplus_{k=1}^{\infty}\mathcal{C}^\infty_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(0,1)*}\otimes\mathrm{End}(\iota^* F)\big),$$

there are $C>0$, $p_1\in\mathbb{N}^*$, such that for any $p\geq p_1$, in the notations of Lemma 3.11, we have

$$|T^Y_{f_1,p}(y_1,y_2) - T^Y_{f_2,p}(y_1,y_2)| \leq Cp^{m-1/2},$$

$$|T^{X|Y}_{\langle\!\langle g_1^h\rangle\!\rangle,p}(x,y_1) - T^{X|Y}_{\langle\!\langle g_2^h\rangle\!\rangle,p}(x,y_1)| \leq Cp^{m-1/2},$$
$$|T^{Y|X}_{\langle\!\langle g_1^a\rangle\!\rangle,p}(y_1,x) - T^{Y|X}_{\langle\!\langle g_2^a\rangle\!\rangle,p}(y_1,x)| \leq Cp^{n-1/2},$$

for any $x \in X$, $y_1, y_2 \in Y$. Then we have $f_1 = f_2$, $g_1^h = g_2^h$, $g_1^a = g_2^a$. In particular, the notation $[\cdot]_i$ from Definition 1.3 is well defined.

Proof. It follows directly from (3.28), (3.29), (3.30) and the precise descriptions of J^Y_{0,f_i}, J^E_{0,g_i^h}, J^R_{0,g_i^a} for $i = 1, 2$, given in the end of Lemma 3.12. ∎

We will now state a theorem which will help us to get a better understanding of the relation between Theorems 1.1 and 1.5. For

$$g \in \bigoplus_{k=0}^{\infty} \mathscr{C}^{\infty}_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big),$$

using the coordinate system as in (1.14), we define the sequence of operators

$$M^{X|Y}_{g,p} : L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F), \quad p \in \mathbb{N},$$

by

$$(M^{X|Y}_{g,p} f)(y, Z_N) = \langle\!\langle g\rangle\!\rangle(y, Z_N) \cdot \exp\Big(-p\frac{\pi}{2}|Z_N|^2\Big) \cdot (B^Y_p f)(y), \tag{3.36}$$

where $f \in L^2(Y, \iota^*(L^p \otimes F))$ and the norm $|Z_N|$, $Z_N \in N^{X|Y}$, is taken with respect to $g^{N^{X|Y}}$. We also define an operator

$$M^{Y|X,\dagger}_{g,p} : L^2(X, L^p \otimes F) \to L^2\big(Y, \iota^*(L^p \otimes F)\big), \quad p \in \mathbb{N},$$

as follows:

$$(M^{Y|X,\dagger}_{g,p} f)(y) = p^{n-m} \cdot B^Y_p \pi_* \Big(\langle\!\langle g\rangle\!\rangle(y, Z_N) \cdot \exp\Big(-p\frac{\pi}{2}|Z_N|^2\Big) \cdot f(y, Z_N) \cdot dv_{N^{X|Y}}(Z_N)\Big),$$

where we implicitly identified the restriction of $f \in L^2(X, L^p \otimes F)$ to U with an element from $L^2(U, \pi^*\iota^*(L^p \otimes F))$, and π_* is the integration over the fibers of $N^{X|Y}$. Remark that the integration is well defined because the function $\langle\!\langle g\rangle\!\rangle$ has support in a small tubular neighborhood of Y.

Theorem 3.14. *There is $p_1 \in \mathbb{N}^*$, such that for any*

$$g^h \in \bigoplus_{k=1}^{\infty} \mathscr{C}^{\infty}_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big),$$
$$g^a \in \bigoplus_{k=1}^{\infty} \mathscr{C}^{\infty}_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(0,1)*} \otimes \mathrm{End}(\iota^* F)\big),$$

there is $C > 0$, such that for any $p \geq p_1$, the following bounds hold:

$$\begin{aligned} \big\| T^{X|Y}_{\langle\langle g^h \rangle\rangle, p} - M^{X|Y}_{g^h, p} \big\| &\leq C p^{-(n-m+1)/2}, \\ \big\| T^{Y|X}_{\langle\langle g^a \rangle\rangle, p} - M^{Y|X,\dagger}_{g^a, p} \big\| &\leq C p^{(n-m-1)/2}. \end{aligned} \tag{3.37}$$

Remark 3.15. (a) In Proposition 3.18, we show that for non-zero

$$g \in \bigoplus_{k=0}^{\infty} \mathcal{C}^{\infty}_b \big(Y, \mathrm{Sym}^k (N^{X|Y})^* \otimes \mathrm{End}(\iota^* F) \big),$$

there are $C_1, C_2 > 0$, such that, as $p \to \infty$, we have

$$\big\| M^{X|Y}_{g,p} \big\| \sim C_1 p^{-(n-m)/2}, \quad \big\| M^{Y|X,\dagger}_{g,p} \big\| \sim C_2 p^{(n-m)/2}. \tag{3.38}$$

Hence, by (3.37), the operators $M^{X|Y}_{g^h,p}$, $M^{Y|X,\dagger}_{g^a,p}$, are asymptotic to $T^{X|Y}_{\langle\langle g^h \rangle\rangle, p}$, $T^{Y|X}_{\langle\langle g^a \rangle\rangle, p}$, respectively.

(b) From Theorem 3.14 and Remark 3.15 (a), we see that Theorem 1.5 refines Theorem 1.1.

Lemma 3.16. There are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $x \in X$, $y \in Y$, the following estimates hold:

$$\begin{aligned} \big| M^{X|Y}_{g,p}(x,y) \big| &\leq C p^m \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x,y) \big), \\ \big| M^{Y|X,\dagger}_{g,p}(y,x) \big| &\leq C p^n \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x,y) \big). \end{aligned}$$

Proof. It follows trivially from Theorem 3.4, (3.36) and the fact that the function $u^k \exp(-u)$ is bounded for $u \in \mathbb{R}_+$ for any $k \in \mathbb{N}$. ∎

Proof of Theorem 3.14. In order to prove the theorem, we consider the Schwartz kernel $M^{X|Y}_{g,p}(x,y)$ (resp. $M^{Y|X,\dagger}_{g,0,p}(y,x)$) of $M^{X|Y}_{g,p}$, (resp. $M^{Y|X,\dagger}_{g,p}$, viewed as an operator acting on the sections with support in a $r_\perp$-tubular neighborhood of Y) evaluated with respect to the volume form dv_Y (resp. $dv_Y \wedge dv_{N^{X|Y}}$). We use the notational convention introduced before Theorem 3.7. From Theorem 3.7 and (3.36), (3.4), and (3.6), we conclude that there are $\varepsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any

$$\begin{gathered} y_0 \in Y, \quad p \geq p_1, \quad Z, Z' \in \mathbb{R}^{2n}, \\ Z = (Z_Y, Z_N), \quad Z' = (Z'_Y, Z'_N), \quad |Z|, |Z'| \leq \varepsilon, \quad Z_Y, Z'_Y \in \mathbb{R}^{2m}, \end{gathered}$$

we have

$$\begin{aligned} &\Big| \frac{1}{p^m} M^{X|Y}_{g,p} \big(\psi^{X|Y}_{y_0}(Z), \phi^Y_{y_0}(Z'_Y) \big) \\ &\qquad - \{g\}(y_0, \sqrt{p} Z_N) \cdot \mathcal{E}_{n,m}(\sqrt{p} Z, \sqrt{p} Z'_Y) \cdot \kappa^Y_\phi(Z_Y)^{-1/2} \kappa^Y_\phi(Z'_Y)^{-1/2} \Big| \\ &\leq C p^{-1/2} \big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \big)^Q \exp\big(-c\sqrt{p} \big(|Z_Y - Z'_Y| + |Z_N| \big) \big), \end{aligned} \tag{3.39}$$

$$\begin{aligned}&\Big|\frac{1}{p^n}M^{Y|X,\dagger}_{g,0,p}\big(\phi^Y_{y_0}(Z_Y),\psi^{X|Y}_{y_0}(Z')\big)\\&\quad-\{g\}(y_0,\sqrt{p}Z'_N)\cdot\mathcal{R}_{n,m}(\sqrt{p}Z_Y,\sqrt{p}Z')\cdot\kappa^Y_\phi(Z_Y)^{-1/2}\kappa^Y_\phi(Z'_Y)^{-1/2}\Big|\\&\leq Cp^{-1/2}\big(1+\sqrt{p}|Z_Y|+\sqrt{p}|Z'|\big)^Q\exp\big(-c\sqrt{p}\big(|Z_Y-Z'_Y|+|Z'_N|\big)\big).\quad(3.40)\end{aligned}$$

We denote now by $M^{Y|X,\dagger}_{g,p}(y,x)$ the Schwartz kernel of $M^{Y|X,\dagger}_{g,p}$, evaluated with respect to dv_X. From (3.22), (3.39) and (3.40), for $Q_0:=\max\{Q,\deg g\}$, we then obtain that in the same notations (but, probably, for a different choice of C), we have

$$\begin{aligned}&\Big|\frac{1}{p^n}M^{Y|X,\dagger}_{g,p}\big(\phi^Y_{y_0}(Z_Y),\psi^{X|Y}_{y_0}(Z')\big)-\kappa^{X|Y}_N(y_0)^{-1/2}\\&\quad\cdot\{g\}(y_0,\sqrt{p}Z'_N)\cdot\mathcal{R}_{n,m}(\sqrt{p}Z_Y,\sqrt{p}Z')\cdot\kappa^Y_\phi(Z_Y)^{-1/2}\kappa^{X|Y}_\psi(Z')^{-1/2}\Big|\\&\leq Cp^{-1/2}\big(1+\sqrt{p}|Z_Y|+\sqrt{p}|Z'|\big)^{Q_0}\exp\big(-c\sqrt{p}\big(|Z_Y-Z'_Y|+|Z'_N|\big)\big).\quad(3.41)\end{aligned}$$

By comparing (3.39) and (3.41) with the expansions from Lemma 3.12, there is $Q_1\geq 0$, such that

$$\begin{aligned}&\big|M^{X|Y}_{g^h,p}\big(\psi^{X|Y}_{y_0}(Z),\phi^Y_{y_0}(Z'_Y)\big)-T^{X|Y}_{\langle\langle g^h\rangle\rangle,p}\big(\psi^{X|Y}_{y_0}(Z),\phi^Y_{y_0}(Z'_Y)\big)\big|\\&\leq Cp^{m-1/2}\big(1+\sqrt{p}|Z|+\sqrt{p}|Z'_Y|\big)^{Q_1}\exp\big(-c\sqrt{p}\big(|Z_Y-Z'_Y|+|Z_N|\big)\big),\quad(3.42)\\&\Big|M^{Y|X,\dagger}_{g^a,p}\big(\phi^Y_{y_0}(Z_Y),\psi^{X|Y}_{y_0}(Z')\big)-T^{Y|X}_{\langle\langle g^a\rangle\rangle,p}\big(\phi^Y_{y_0}(Z_Y),\psi^{X|Y}_{y_0}(Z')\big)\Big|\\&\leq Cp^{n-1/2}\big(1+\sqrt{p}|Z_Y|+\sqrt{p}|Z'|\big)^{Q_1}\exp\big(-c\sqrt{p}\big(|Z_Y-Z'_Y|+|Z'_N|\big)\big).\quad(3.43)\end{aligned}$$

From Proposition 2.9, Lemma 3.16, and (3.42) and (3.43), we finally deduce Theorem 3.14. ■

Let us now briefly describe how to calculate the asymptotics of the norms of operators $M^{X|Y}_{g,p}$ and $M^{Y|X,\dagger}_{g,p}$, $p\in\mathbb{N}$. For this, the following lemma will be of crucial importance.

Lemma 3.17 (Bordemann–Meinrenken–Schlichenmaier [3, Theorem 4.1], Finski [9, Theorem 1.1], Ma–Marinescu [16, Theorem 3.19, (3.91)]). *For $f\in\mathcal{C}^\infty_b(X,\mathrm{End}(F))$ non-zero, the following holds:*

$$\|T^X_{f,p}\|\sim\sup_{x\in X}\|f(x)\|,\qquad\|T^X_{f,p}\|\leq\sup_{x\in X}\|f(x)\|.$$

Moreover, if the above supremum is achieved in X, then there is $C>0$, such that for p big enough

$$\sup_{x\in X}\|f(x)\|-\frac{C}{\sqrt{p}}\leq\|T^X_{f,p}\|.$$

Now, let us introduce some further notations. We fix $y_0 \in Y$ and an orthogonal basis $w_1, \ldots, w_{n-m}$ of $(N^{X|Y}_{y_0})^{(1,0)}$, verifying $\|w_i\| = 1/\sqrt{2}$, $i = 1, \ldots, n-m$. For example, take $w_j = \partial/\partial z_{j+m}$, $j = 1, \ldots, n-m$, where z_i are the complex coordinates induced by Fermi coordinates at y_0. For any $i, j \in \mathbb{N}$, we define the operator

$$\Lambda_{\omega,=} \colon \mathrm{Sym}^i\big((N^{X|Y})^{(1,0)*}\big) \otimes \mathrm{Sym}^j\big((N^{X|Y})^{(0,1)*}\big) \to \mathbb{C}, \tag{3.44}$$

for multiindices $\alpha, \beta \in \mathbb{N}^{n-m}$, as follows:

$$\Lambda_{\omega,=}\big[w^\alpha \otimes \overline{w}^\beta\big] = \begin{cases} (1/\pi^{|\beta|})\beta! & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this operator does not depend on the choice of the basis $w_1, \ldots, w_{n-m}$. By linearity, we extend $\Lambda_{\omega,=}[\cdot]$ to $\mathrm{Sym}^i((N^{X|Y})^*) \otimes \mathrm{End}(\iota^* F) \otimes \mathbb{C}$.

Proposition 3.18. *The constants $C_1, C_2 > 0$ from (3.38) are given by*

$$C_1 = \sup_{y \in Y}\big(\kappa_N^{X|Y}(y)^{1/2} \cdot \big\|\Lambda_{\omega,=}[g_y^* \otimes g_y]\big\|^{1/2}\big),$$
$$C_2 = \sup_{y \in Y}\big(\kappa_N^{X|Y}(y)^{-1/2} \cdot \big\|\Lambda_{\omega,=}[g_y \otimes g_y^*]\big\|^{1/2}\big).$$

Proof. The main idea of the proof is to reduce the calculation of the norms from (3.38) to the calculation of the norms of some Toeplitz operators.

An easy calculation using (1.7) shows that for any $f \in L^2(Y, \iota^*(L^p \otimes F))$, we have

$$\|M_{g,p}^{X|Y} f\|_{L^2(dv_X)} = \|h_{g,p}(y) \cdot B_p^Y f\|_{L^2(dv_Y)}, \tag{3.45}$$

where the section $h_{g,p} \in \mathscr{C}^\infty(Y, \mathrm{End}(\iota^* F))$ satisfies

$$h_{g,p}(y)^2 := \int_{\mathbb{R}^{2(n-m)}} \kappa_N^{X|Y}(y, \sqrt{p} Z_N) \cdot \exp\big(-p\pi |Z_N|^2\big)$$
$$\cdot \{g\}(y, \sqrt{p}Z_N)^* \cdot \{g\}(y, \sqrt{p}Z_N) \cdot \rho\Big(\frac{|Z_N|}{r_\perp}\Big)^2 dZ_{2m+1} \wedge \cdots \wedge dZ_{2n}.$$

By an easy calculation using (3.18), there is $c > 0$, such that, as $p \to \infty$, we have

$$h_{g,p}(y)^2 = \frac{\kappa_N^{X|Y}(y) \cdot \Lambda_{\omega,=}[g(y)^* \otimes g(y)]}{p^{n-m}} + O\Big(\frac{1}{p^{n-m+1/2}}\Big). \tag{3.46}$$

Now, let $h \in \mathscr{C}^\infty(Y, \mathrm{End}(\iota^* F))$ verifies $h(y)^2 = \kappa_N^{X|Y}(y) \cdot \Lambda_{\omega,=}[g(y)^* \otimes g(y)]$. Clearly, the bounded geometry condition and assumption that

$$g \in \bigoplus_{k=0}^{\infty} \mathscr{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big)$$

imply that $h(y) \in \mathcal{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$. Trivially, Toeplitz operator $T_{h,p}^Y$ satisfies

$$\langle T_{h,p}^Y f, f \rangle_{L^2(dv_Y)} = \|h \cdot B_p^Y f\|_{L^2(dv_Y)}. \tag{3.47}$$

Thus, by (3.45), (3.46) and (3.47), we have

$$\|M_{g,p}^{X|Y}\| = \frac{1}{p^{(n-m)/2}} \|T_{h,p}^Y\| + O\Big(\frac{1}{p^{(n-m+1)/2}}\Big). \tag{3.48}$$

We deduce the first part of Proposition 3.18 by Lemma 3.17 and (3.48).

Now, to get the second part, let us first remark that the following formula holds

$$(M_{g,p}^{Y|X,\dagger})^* = (\kappa_N^{X|Y})^{-1} \cdot M_{g^*,p}^{X|Y}.$$

The proof now proceeds in the same way as the proof for the first part. ■

3.4. Asymptotic criteria for Toeplitz type operators

As we approach the study of Toeplitz operators with exponential decay through the asymptotic expansions of their Schwartz kernels, it is fundamental to find characterizations of the latter operators in terms of the former asymptotic expansions. This is the main goal of this section.

To state and prove those characterizations in the generality we need, we will introduce a weaker notion of Toeplitz operators, compared to the one from Definition 1.3. For this definition, we fix some Riemannian manifold (Z, g^{TZ}) and an embedding $\iota'\colon X \to Z$, such that $(\iota')^* g^{TZ} = g^{TX}$, and such that the triple (Z, X, g^{TZ}) is of bounded geometry.

Definition 3.19. A sequence of linear operators $T_p^Y \in \mathrm{End}(L^2(Y, \iota^*(L^p \otimes F)))$, respectively,

$$T_p^{Y|X}\colon L^2(X, L^p \otimes F) \to L^2\big(Y, \iota^*(L^p \otimes F)\big),$$
$$T_p^{X|Y}\colon L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F), \quad p \in \mathbb{N},$$

as in Definition 1.3 is called a *Toeplitz operator with <u>weak</u> exponential decay* (resp. *of type $Y|X$, $X|Y$) with respect to Z* if all the assumptions of Definition 1.3 hold, except that in the estimate (1.15) in the right-hand side instead of $\mathrm{dist}_Y(y_1, y_2)$ (resp. $\mathrm{dist}_X(x, y_1)$, $\mathrm{dist}_X(y_1, x)$), we have $\mathrm{dist}_Z(y_1, y_2)$ (resp. $\mathrm{dist}_Z(x, y_1)$, $\mathrm{dist}_Z(y_1, x)$). To shorten, we sometimes omit the reference to Z. The coefficients of the asymptotic expansions will still be denoted by $[T_p^Y]_i$, $[T_p^{Y|X}]_i$, $[T_p^{X|Y}]_i$.

Remark 3.20. (a) The analogue of Remark 1.4 (a) holds for this weaker notion of Toeplitz operators due to Propositions 2.6 and 2.9.

(b) From Proposition 2.4, we conclude that if ι' is quasi-isometry, the notions from Definition 3.19 coincide with those from Definition 1.3.

Theorem 3.21. *Let (Y, g^{TY}) be of bounded geometry. Then a family of linear operators $T_p^Y \in \operatorname{End}(L^2(Y, \iota^*(L^p \otimes F)))$, $p \in \mathbb{N}$, forms a Toeplitz operator with (resp. weak) exponential decay if and only if the following conditions hold:*

(1) *For any $p \in \mathbb{N}$, $T_p^Y = B_p^Y \circ T_p^Y \circ B_p^Y$.*

(2) *There is $p_1 \in \mathbb{N}$, such that for any $l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, the Schwartz kernel $T_p^Y(y_1, y_2)$; $y_1, y_2 \in Y$, of T_p^Y, evaluated with respect to dv_Y, satisfies*

$$|T_p^Y(y_1, y_2)|_{\mathscr{C}^l} \leq C p^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \operatorname{dist}_Y(y_1, y_2)\big), \tag{3.49}$$

respectively,

$$|T_p^Y(y_1, y_2)|_{\mathscr{C}^l} \leq C p^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \operatorname{dist}_Z(y_1, y_2)\big),$$

where in the last equation we used the notations from Definition 3.19.

(3) *For any $y_0 \in Y$, $r \in \mathbb{N}$, there are $I_r^Y(Z_Y, Z'_Y) \in \operatorname{End}(F_{y_0})$ polynomials in $Z_Y, Z'_Y \in \mathbb{R}^{2m}$ of the same parity as r, such that the coefficients of I_r^Y lie in $\mathscr{C}_b^\infty(Y, \operatorname{End}(\iota^* F))$, and for $F_r := I_r^Y \cdot \mathscr{P}_m$, the following holds: There are $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any*

$$y_0 \in Y, \quad p \geq p_1, \quad Z_Y, Z'_Y \in \mathbb{R}^{2m}, \quad |Z_Y|, |Z'_Y| \leq \varepsilon, \\ \alpha, \alpha' \in \mathbb{N}^{2m}, \quad |\alpha| + |\alpha'| \leq l,$$

the following bound holds:

$$\begin{aligned} &\bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'^{\alpha'}_Y} \bigg(\frac{1}{p^m} T_p^Y\big(\phi_{y_0}^Y(Z_Y), \phi_{y_0}^Y(Z'_Y)\big) \\ &\qquad - \sum_{r=0}^k p^{-r/2} F_r(\sqrt{p} Z_Y, \sqrt{p} Z'_Y) \kappa_\phi^Y(Z_Y)^{-1/2} \kappa_\phi^Y(Z'_Y)^{-1/2} \bigg) \bigg|_{\mathscr{C}^{l'}} \\ &\leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'_Y|\big)^Q \exp\big(-c\sqrt{p}|Z_Y - Z'_Y|\big), \end{aligned} \tag{3.50}$$

where the $\mathscr{C}^{l'}$-norm is taken with respect to y_0.

Moreover, (3.50) is related to the expansion from Definition 1.3 by $I_0^Y(0,0) = [T_p^Y]_0$.

Proof. The proof for Toeplitz operators with weak exponential decay is analogous to the proof for Toeplitz operators with exponential decay, so we only concentrate on the proof of the former case.

First of all, let us assume that the sequence of operators T_p^Y, $p \in \mathbb{N}$, forms a Toeplitz operator with exponential decay. Then the first condition of Theorem 3.21 holds by definition. The second holds due to Lemma 3.11. The third holds due to

Lemma 3.12. The identity $I_0^Y(0,0) = [T_p^Y]_0$ follows from Lemma 3.12. Overall, we obtain one direction of Theorem 3.21.

Let us now prove the opposite direction. Our proof is based on [15, Theorem 4.9], where the authors proved the analogous theorem for compact manifolds and Toeplitz operators in the sense of [14, §7], see Remark 1.4 (a). The first step of their proof shows that the polynomial $I_0^Y(Z_Y, Z'_Y)$ from Theorem 3.21 is constant, and hence equal to $I_0^Y(0,0)$. Their argument (which does not use the assumption on parity of $I_r^Y(Z_Y, Z'_Y)$, $r \in \mathbb{N}$) adapts line by line in our non-compact setting, except that the estimate [15, (4.47)] has to be replaced by Corollary 2.7.

Then Ma and Marinescu define a section $f_0 \in \mathcal{C}^\infty(Y, \mathrm{End}(\iota^* F))$ as

$$f_0(y_0) := I_0^Y(0,0).$$

Our assumption on the coefficients of I_r^Y implies that $f_0 \in \mathcal{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$. From Lemma 3.12, most notably the fact that

$$I_{0,f_0}^Y(Z_Y, Z'_Y) = f_0(y_0),$$

the fact that $I_0^Y(Z_Y, Z'_Y)$ is constant and the choice of f_0, we see that all the assumptions of Theorem 3.21 are satisfied for the sequence of operators $\sqrt{p} \cdot (T_p^Y - T_{f_0,p}^Y)$, $p \in \mathbb{N}$ (except for the parity of I_r^Y, which is now opposite to r).

By repeating this argument for $\sqrt{p} \cdot (T_p^Y - T_{f_0,p}^Y)$ instead of T_p^Y, we conclude that the first polynomial in Taylor-type expansion of $\sqrt{p} \cdot (T_p^Y - T_{f_0,p}^Y)$, as in (3.50), is constant. It is, however, of odd parity due to the parenthesized remark above. Hence, the first coefficient is equal to 0. Due to this, we see that all the assumptions of Theorem 3.21 (now, even for the parity of I_r^Y) are satisfied for the sequence of operators $p \cdot (T_p^Y - T_{f_0,p}^Y)$, $p \in \mathbb{N}$. In particular, the first equation of (1.15) holds for $k = 1$ by (3.49), applied for $p \cdot (T_p^Y - T_{f_0,p}^Y)$. We finish by induction. ∎

We will now describe the analogue of Theorem 3.21 for Toeplitz operators with exponential decay of type $X|Y$.

Theorem 3.22. *Let (X, Y, g^{TX}) be a triple of bounded geometry. Then a family*

$$T_p^{X|Y} : L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F), \quad p \in \mathbb{N},$$

of linear operators forms a Toeplitz operator with (resp. weak) exponential decay of type $X|Y$ if and only if the following conditions are satisfied:

(1) *For any $p \in \mathbb{N}$, $T_p^{X|Y} = (B_p^X - B_p^{X|Y\perp}) \circ T_p^{X|Y} \circ B_p^Y$.*

(2) *There is $p_1 \in \mathbb{N}^*$, such that for any $l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, the Schwartz kernel $T_p^{X|Y}(x, y)$; $x \in X$, $y \in Y$, of $T_p^{X|Y}$, evaluated with*

respect to dv_Y, satisfies

$$|T_p^{X|Y}(x,y)|_{\mathcal{C}^l} \leq Cp^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x,y)\big),$$

respectively,

$$|T_p^{X|Y}(x,y)|_{\mathcal{C}^l} \leq Cp^{m+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_Z(x,y)\big),$$

where in the last equation we used notations from Definition 3.19.

(3) *For any $y_0 \in Y$, $r \in \mathbb{N}$, there are $I_r^E(Z, Z'_Y) \in \mathrm{End}(F_{y_0})$ polynomials in $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$ of the same parity as r, such that the coefficients of I_r^E lie in $\mathcal{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$, and for $F_r^E := I_r^E \cdot \mathcal{E}_{n,m}$, the following holds. There are $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $k, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any*

$$y_0 \in Y, \quad p \geq p_1, \quad Z \in \mathbb{R}^{2n}, \quad Z'_Y \in \mathbb{R}^{2m}, \quad |Z|, |Z'_Y| \leq \varepsilon,$$
$$\alpha \in \mathbb{N}^{2n}, \quad \alpha' \in \mathbb{N}^{2m}, \quad |\alpha| + |\alpha'| \leq l,$$

we have

$$\begin{aligned} \bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial {Z'_Y}^{\alpha'}} \bigg(& \frac{1}{p^m} T_p^{X|Y}\big(\psi_{y_0}^{X|Y}(Z), \phi_{y_0}^Y(Z'_Y)\big) \\ & - \sum_{r=0}^k p^{-r/2} F_r^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_\psi^{X|Y}(Z)^{-1/2} \kappa_\phi^Y(Z'_Y)^{-1/2} \bigg) \bigg|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-(k+1-l)/2}\big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y|\big)^Q \\ & \quad \cdot \exp\big(-c\sqrt{p}\big(|Z_N| + |Z_Y - Z'_Y|\big)\big). \end{aligned} \tag{3.51}$$

Moreover, in the notations of (1.14), for any $y_0 \in Y$, the polynomial $I_0^E(Z, Z'_Y)$ depends only on z_N, and, as a section of $\bigoplus_{k=1}^\infty \mathrm{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)$ over Y, it coincides with $[T_p^{X|Y}]_0 \cdot \kappa_N^{X|Y}(y_0)^{1/2}$.

Proof. As the proof for the weak version of Toeplitz operators is completely analogous to the proof of non-weak version, we only concentrate on proving the former case.

First of all, let us assume that the sequence of operators $T_p^{X|Y}$, $p \in \mathbb{N}$, forms a Toeplitz operator with exponential decay of type $X|Y$. Then the first condition of Theorem 3.22 holds by definition. The second holds due to Lemma 3.11. The third holds due to Lemma 3.12. The relation between I_0^E and $[T_p^{X|Y}]_0$ follows from Lemma 3.12. Overall, we obtain one direction of Theorem 3.22.

Let us now prove the opposite direction. From the first condition of Theorem 3.22 and Corollary 3.3, we first deduce that in the notations of (3.51), $I_0^E(Z, Z'_Y)$ is a polynomial in $z, \overline{z}'_Y$. Let us show that it only depends on z_N.

Consider first the operator $T_p^Y := \mathrm{Res}_Y \circ T_p^{X|Y}$. Of course $T_p^Y = 0$ due to our first assumption on $T_p^{X|Y}$. From Lemma 3.12 and the fact that in Fermi coordinates, the submanifold Y corresponds to the embedding $\mathbb{R}^{2m} \hookrightarrow \mathbb{R}^{2n}$, we obtain as a consequence that $I_0^E(Z_Y, Z'_Y) = 0$.

Now, let U be a smooth vector field, such that

$$U|_Y \in N^{X|Y}, \quad U(y_0) = \frac{\partial}{\partial z_j}, \quad j = m+1, \dots, n,$$

where $z_1, \dots, z_n$ are the complex coordinates associated to Fermi coordinates. We consider the sequence of operators

$$T_{p,1}^Y := \frac{1}{\sqrt{p}} \mathrm{Res}_Y \circ B_p^X \circ \nabla_U T_p^{X|Y}, \quad p \in \mathbb{N}.$$

This sequence of operators is well defined due to (1.4). It follows from (1.4) that $T_{p,1}^Y$, $p \in \mathbb{N}^*$, satisfies the first condition from Theorem 3.21. The second condition associated to the weak notion also holds for $Z := X$ in the notations of Definition 3.19 (remark that the strong version holds only under additional quasi-isometry assumption, see Proposition 2.4. This technical caveat is one of the reasons why we need to consider weak version of Toeplitz type operators).

An easy verification using (3.9) shows that $T_{p,1}^Y$ satisfies the third assumption of Theorem 3.21 for

$$I_0^Y(Z_Y, Z'_Y) := \Big(\frac{\partial}{\partial z_j} I_0^E\Big)(Z_Y, Z'_Y).$$

Hence, by the results of Theorem 3.21 and its proof, we conclude that the sequence of operators $T_{p,1}^Y$, $p \in \mathbb{N}$, forms a Toeplitz operator with weak exponential decay associated to X, and $(\frac{\partial}{\partial z_j} I_0^E)(Z_Y, Z'_Y)$ a constant. Let

$$g'_1 \in \mathscr{C}_b^\infty\big(Y, (N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big)$$

be defined so that for any $n \in (N_{y_0}^{X|Y})^{(1,0)}$, we have

$$g'_1 \cdot n = \sum_{j=m+1}^{n} \Big(\frac{\partial}{\partial z_j} I_0^E\Big)(0,0) \cdot n_j,$$

where n_j are the coordinates of n in the basis $\frac{\partial}{\partial z_j}$. We define

$$T_{p,1}^{X|Y} := T_p^{X|Y} - \big(B_p^X - B_p^{X|Y\perp}\big) \circ \big(\langle\!\langle g'_1 \rangle\!\rangle \cdot \mathrm{E}_p^{X|Y}\big).$$

By Lemma 3.12, (3.51) and the remark after it, we deduce that the asymptotic expansion (3.50) holds for $T_p^{X|Y} := T_{p,1}^{X|Y}$ and $I_0^E := P_2 I_0^E$, where P_i, $i \in \mathbb{N}$, is the projection onto the vector space of polynomials of degree $\geq i$.

We then repeat the procedure for the pair of smooth vector fields U, V, verifying similar assumptions as above, and the sequence of operators

$$T^{Y}_{p,2} := \frac{1}{p} \operatorname{Res}_Y \circ B^X_p \circ \nabla_U \nabla_V T^{X|Y}_{p,1}, \quad p \in \mathbb{N},$$

to construct

$$g'_2 \in \mathcal{C}^\infty_b\big(Y, \operatorname{Sym}^2(N^{X|Y})^{(1,0)*} \otimes \operatorname{End}(\iota^* F)\big).$$

Then, as before, we form the sequence of operators

$$T^{X|Y}_{p,2} := T^{X|Y}_{p,1} - (B^X_p - B^{X|Y\perp}_p) \circ \big(\langle\!\langle g'_2 \rangle\!\rangle \cdot \mathrm{E}^{X|Y}_p\big), \quad p \in \mathbb{N}.$$

By continuing in the same fashion, we construct the sequence of elements

$$g'_k \in \mathcal{C}^\infty_b\big(Y, \operatorname{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \operatorname{End}(\iota^* F)\big), \quad k \in \mathbb{N}^*,$$

and operators $T^{X|Y}_{p,k}$, $k \in \mathbb{N}$, such that the asymptotic expansion (3.50) holds for

$$T^{X|Y}_p := T^{X|Y}_{p,k} \quad \text{and} \quad I^E_0 := P_{k+1} I^E_0.$$

Of course, since $I^E_0(Z, Z'_Y)$ is a polynomial, only a finite number of g'_k, $k \in \mathbb{N}^*$, is non-zero. We put $g_0 := \sum_{i=1}^\infty g'_i$. Clearly, g_0 has the same parity as I^E_0. By the above, we see that the asymptotic expansion (3.51) holds for

$$T^{X|Y}_p := T^{X|Y}_p - (B^X_p - B^{X|Y\perp}_p) \circ \big(\langle\!\langle g_0 \rangle\!\rangle \cdot \mathrm{E}^{X|Y}_p\big) \quad \text{and} \quad I^E_0 := 0.$$

Hence, the same asymptotic expansion (3.51) holds for

$$\sqrt{p}\big(T^{X|Y}_p - (B^X_p - B^{X|Y\perp}_p) \circ \big(\langle\!\langle g_0 \rangle\!\rangle \cdot \mathrm{E}^{X|Y}_p\big)\big).$$

We repeat the same procedure for the new sequence of operators and construct an element g_1. Clearly, by the assumptions on the parity of I^E_r, the parity of g_1 is different from g_0. By induction, we get a sequence of elements g_i, $i \in \mathbb{N}$, which satisfy the second equation from (1.15), and the parities of which are as we need. ■

We are finally ready to treat the last type of Toeplitz operators with exponential decay.

Theorem 3.23. *A family $T^{Y|X}_p \colon L^2(X, L^p \otimes F) \to L^2(Y, \iota^*(L^p \otimes F))$, $p \in \mathbb{N}$, of linear operators forms a Toeplitz operator with exponential decay of type $Y|X$ if and only if the following three conditions are satisfied:*

(1) *For any $p \in \mathbb{N}$, $T^{Y|X}_p = B^Y_p \circ T^{Y|X}_p \circ (B^X_p - B^{X|Y\perp}_p)$.*

(2) *There is* $p_1 \in \mathbb{N}^*$, *such that for any* $l \in \mathbb{N}$, *there is* $C > 0$, *such that for any* $p \geq p_1$, *the Schwartz kernel* $T_p^{Y|X}(y,x)$; $x \in X$, $y \in Y$, *of* $T_p^{Y|X}$, *evaluated with respect to* dv_X, *satisfies*

$$|T_p^{Y|X}(y,x)|_{\mathscr{C}^l} \leq C p^{n+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}(x,y)\big),$$

respectively,

$$|T_p^{Y|X}(y,x)|_{\mathscr{C}^l} \leq C p^{n+l/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_Z(x,y)\big),$$

where in the last equation we used the notations from Definition 3.19.

(3) *For any* $y_0 \in Y$, $r \in \mathbb{N}$, *there are* $I_r^R(Z_Y, Z') \in \mathrm{End}(F_{y_0})$ *polynomials in* $Z_Y \in \mathbb{R}^{2m}$, $Z' \in \mathbb{R}^{2n}$ *of the same parity as* r, *such that the coefficients of* I_r^R *lie in* $\mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$, *and for* $F_r^R := I_r^R \cdot \mathcal{R}_{n,m}$, *the following holds. There are* $\varepsilon, c > 0$, $p_1 \in \mathbb{N}^*$, *such that for any* $k, l, l' \in \mathbb{N}$, *there are* $C, Q > 0$, *such that for any*

$$y_0 \in Y, \quad p \geq p_1, \quad Z_Y \in \mathbb{R}^{2m}, \quad Z' \in \mathbb{R}^{2n}, \quad |Z_Y|, |Z'| \leq \varepsilon,$$
$$\alpha \in \mathbb{N}^{2m}, \quad \alpha' \in \mathbb{N}^{2n}, \quad |\alpha| + |\alpha'| \leq l,$$

we have

$$\begin{aligned}
&\bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^{\alpha} \partial Z'^{\alpha'}}\bigg(\frac{1}{p^n} T_p^{Y|X}\big(\phi_{y_0}^Y(Z_Y), \psi_{y_0}^{X|Y}(Z')\big) \\
&\quad - \sum_{r=0}^{k} p^{-r/2} F_r^R(\sqrt{p} Z_Y, \sqrt{p} Z') \kappa_\phi^Y(Z_Y)^{-1/2} \kappa_\psi^{X|Y}(Z')^{-1/2}\bigg)\bigg|_{\mathscr{C}^{l'}} \\
&\qquad \leq C p^{-(k+1-l)/2} \big(1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'|\big)^Q \\
&\qquad\quad \cdot \exp\big(-c\sqrt{p}\big(|Z'_N| + |Z_Y - Z'_Y|\big)\big).
\end{aligned} \tag{3.52}$$

Moreover, in the notations of (1.14) *and* (3.51), *the polynomial* $I_0^R(Z_Y, Z')$ *depends only on* $\overline{z}'_N$, *and as a section of* $\bigoplus_{k=1}^\infty \mathrm{Sym}^k(N^{X|Y})^{(0,1)*} \otimes \mathrm{End}(\iota^* F)$ *over* Y, *it coincides with* $[T_p^{Y|X}]_0 \cdot \kappa_N^{X|Y}(y_0)^{-1/2}$.

Clearly, Lemmas 3.11, 3.12 imply the first implication of Theorem 3.23. The proof of the second implication will be given in Section 4.2, where we study adjoints of Toeplitz type operators.

4. Toeplitz type operators: Algebraic properties and examples

The main goal of this section is to study algebraic properties of the set of Toeplitz type operators and to construct some examples of those operators. More precisely, in

Section 4.1, we show that the set of Toeplitz type operators is closed under taking restrictions, extensions and some products. In Section 4.2, we prove the analogous statement for the adjoints of Toeplitz type operators. To do this and out of independent interest, we introduce a sequence of operators, so-called multiplicative defect, which plays a crucial role in our approach to the main statements of this article. We also prove that the multiplicative defect is itself a Toeplitz type operator with weak exponential decay. Finally, in Section 4.3, we provide several examples of Toeplitz type operators.

4.1. Products, extensions and restrictions of Toeplitz type operators

The main goal of this section is to show that the set of Toeplitz type operators with (weak) exponential decay is closed under taking restrictions, extensions and some products.

To describe our main result, we fix some notations first. We have a natural isomorphism

$$N^{X|Y} \to \iota_1^* N^{X|W} \oplus N^{W|Y}.$$

We then extend the induced projection onto the $N^{W|Y}$ component to an operator on $\mathrm{Sym}^k(N^{X|Y})^{(1,0)*}$, and denote it by an abuse of notation $P_N^{W|Y}$. Recall that $\Lambda_{\omega,=}[\cdot]$ was defined in (3.44).

For $k, k' \in \mathbb{N}^*$, we fix

$$\begin{aligned}
g_1 &\in \mathcal{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big),\\
g_1' &\in \mathcal{C}_b^\infty\big(Y, \mathrm{Sym}^{k'}(N^{X|Y})^{(0,1)*} \otimes \mathrm{End}(\iota^* F)\big),\\
g_2 &\in \mathcal{C}_b^\infty\big(Y, \mathrm{Sym}^k(N^{W|Y})^{(1,0)*} \otimes \mathrm{End}(\iota_1^* F)\big),\\
g_3 &\in \mathcal{C}_b^\infty\big(W, \mathrm{Sym}^{k'}(N^{X|W})^{(1,0)*} \otimes \mathrm{End}(\iota_2^* F)\big).
\end{aligned}$$

The main result of this section goes as follows.

Theorem 4.1. *The sequences of operators*

(1) $\mathrm{Res}_W \circ T^{X|Y}_{\langle\langle g_1 \rangle\rangle, p}$, (2) $\mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{W|Y}$

for $p \in \mathbb{N}$, form a Toeplitz operator with weak exponential decay of type $W|Y$ with respect to X. The sequence of operators

(3) $T^{Y|X}_{\langle\langle g_1' \rangle\rangle, p} \circ T^{X|Y}_{\langle\langle g_1 \rangle\rangle, p}$

for $p \in \mathbb{N}$, forms a Toeplitz operator with weak exponential decay on Y with respect to X.

Finally, the sequences of operators

(4) $T^{X|W}_{\langle\langle g_3\rangle\rangle,p} \circ T^{W|Y}_{\langle\langle g_2\rangle\rangle,p}$, (5) $T^{X|W}_{\langle\langle g_3\rangle\rangle,p} \circ \mathrm{E}^{W|Y}_p$, (6) $\mathrm{E}^{X|W}_p \circ T^{W|Y}_{\langle\langle g_2\rangle\rangle,p}$,

(7) $T^{X|Y}_{\langle\langle g_1\rangle\rangle,p} \circ T^{Y}_{f,p}$,

for $p \in \mathbb{N}$, form Toeplitz operators with exponential decay of type $X|Y$. Moreover, we have

(1) $[\mathrm{Res}_W \circ T^{X|Y}_{\langle\langle g_1\rangle\rangle,p}]_0 = P^{W|Y}_N(g_1)$, (2) $[\mathrm{Res}_W \circ \mathrm{E}^{X|Y}_p - \mathrm{E}^{W|Y}_p]_0 = 0$,

(3) $[T^{Y|X}_{\langle\langle g_1'\rangle\rangle,p} \circ T^{X|Y}_{\langle\langle g_1\rangle\rangle,p}]_0 = \Lambda_{\omega,=}[g_1' \cdot g_1]$, (4) $[T^{X|W}_{\langle\langle g_3\rangle\rangle,p} \circ T^{W|Y}_{\langle\langle g_2\rangle\rangle,p}]_0 = \iota_1^*(g_3) \cdot g_2$,

(5) $[T^{X|W}_{\langle\langle g_3\rangle\rangle,p} \circ \mathrm{E}^{W|Y}_p]_0 = \iota_1^*(g_3)$, (6) $[\mathrm{E}^{X|W}_p \circ T^{W|Y}_{\langle\langle g_2\rangle\rangle,p}]_0 = g_2$,

(7) $[T^{X|Y}_{\langle\langle g_1\rangle\rangle,p} \circ T^{Y}_{f,p}]_0 = g_1 \cdot f$.

Remark 4.2. In particular, from Proposition 2.4, if the embedding $\iota_2 \colon W \to X$ is quasi-isometry then, in points (1) and (2), the related sequences of operators form Toeplitz type operator with exponential decay. The same holds for the point (3) if the embedding $\iota \colon Y \to X$ is quasi-isometry.

Proof. The proofs of all those statements proceed by the verification that relevant operators satisfy the assumptions of Theorems 3.21 and 3.22.

For statements (1) and (2), the validity of the first condition from Theorem 3.22 follows from (1.4). For statements (3), (4), (5), (6) and (7), the validity of the first condition from Theorems 3.21 and 3.22 is direct.

The weak version of the second condition for $Z := X$ for statement (1) (resp. (2)) follows trivially from Lemma 3.11 (resp. Theorem 3.5). For statements (4), (5), (6) and (7), (resp. (3)) the validity of the second condition (resp. weak version of the second condition for $Z := X$) from Theorem 3.22 follows from Corollary 2.8 and Lemma 3.11.

Hence, it is only left to verify the third statement for each of the operators. This is slightly more delicate, and will be done separately for each of the statements. By doing so, and employing the relationship between the polynomials from the third condition of Theorems 3.21 and 3.22 and the asymptotic expansions (1.15), we also establish the second part of Theorem 4.1.

Let us introduce the following notations first. For a function $f \colon \mathbb{R}^k \to \mathbb{R}$ and $s > 0$, we denote by $f_s(Z)$, the function given by $Z \mapsto \frac{1}{s} f(sZ)$. For functions $P(Z, Z')$, $R(Z, Z')$, $Z \in \mathbb{R}^r$, $Z' \in \mathbb{R}^e$; $r, e \in \mathbb{N}^*$, verifying $R(0,0) \neq 0$, and two functions $f \colon \mathbb{R}^r \to \mathbb{R}^r$, $g \colon \mathbb{R}^e \to \mathbb{R}^e$, verifying

$$f(Z) = Z + O\big(|Z|^2\big), \quad g(Z') = Z' + O\big(|Z'|^2\big),$$

we decompose $P(f_s(Z), g_s(Z'))$ as follows:

$$P(f_s(Z), g_s(Z')) = \sum_{i=0}^{k} P(f, g)_{[i]}(Z, Z')s^i + O(s^{k+1}),$$
$$\frac{R(f_s(Z), g_s(Z'))}{R(Z, Z')} = \sum_{i=0}^{k} R'(f, g)_{[i]}(Z, Z')s^i + O(s^{k+1}),$$

where $P(f, g)_{[i]}$, $R'(f, g)_{[i]}$ are functions, which do not depend on s. It is clear that $P(f, g)_{[i]}$, $R'(f, g)_{[i]}$ are polynomials if P is a polynomial and R is the exponential of a polynomial. When f (resp. g) is the identity map, we write $P(f, Z')_{[i]}$ (resp. $P(Z, g)_{[i]}$) for $P(f, g)_{[i]}$. When P or R depend only on Z or Z', we write $P(f)_{[i]}(Z)$, $P(g)_{[i]}(Z')$ and $R'(f)_{[i]}(Z)$, $R'(g)_{[i]}(Z')$ for the above polynomials.

We use the notations introduced before Theorem 3.7. From (2.16) and (2.17), we deduce that

$$\begin{aligned}\mathrm{Res}_W \circ T^{X|Y}_{\langle\langle g_1\rangle\rangle,p}\big(\psi^{W|Y}_{y_0}(Z_W), \phi^{Y}_{y_0}(Z'_Y)\big) \\ = T^{X|Y}_{\langle\langle g_1\rangle\rangle,p}\big(\psi^{X|Y}_{y_0}(\sigma(Z_W)), \phi^{Y}_{y_0}(Z'_Y)\big)\exp(p\tau_L + \tau_F)\big(\psi^{W|Y}_{y_0}(Z_W)\big).\end{aligned} \tag{4.1}$$

For $k \in \mathbb{N}$, we decompose $\exp(\tau_F)$, into power series expansion

$$\exp(\tau_F)(Z_W) = \sum_{i=0}^{k} \exp(\tau_F)_{[i]}(Z_W) + O\big(|Z_W|^{k+1}\big), \tag{4.2}$$

where $\exp(\tau_F)_{[i]}$ are homogeneous polynomials in Z_W of degree i. Using (2.15), we see that we can decompose $\exp(p\tau_L)$ as follows:

$$\exp(p\tau_L)(Z_W) = \sum_{i=0}^{2k} \sum_{j=0}^{\lfloor i/2\rfloor} \sqrt{p}^{\,j} \exp(p\tau_L)_{[i,j]}(Z_W) + O\big(\sqrt{p}^{\,k+1}|Z_W|^{2k+1}\big), \tag{4.3}$$

where $\exp(p\tau_L)_{[i,j]}$ are homogeneous polynomials in Z_W of degree i, independent of p.

Recall that κ-functions were defined in (1.7), (3.20), (3.21). Clearly, from (3.22), we have

$$\begin{aligned}\kappa^{X|Y}_{\psi}\big(\sigma(Z_W)\big) = \kappa^{W|Y}_{\psi}(Z_W) \cdot \kappa^{X|Y}_{N}\big(\psi^{X|Y}\big(\sigma(Z_W)\big)\big) \\ \cdot \kappa^{W|Y}_{N}\big(\psi^{W|Y}(Z_W)\big)^{-1} \cdot \frac{\kappa^{Y}_{\phi}(\sigma_Y(Z_W))}{\kappa^{Y}_{\phi}(Z_Y)},\end{aligned} \tag{4.4}$$

where σ_Y is the horizontal component of σ.

For $k \in \mathbb{N}$, let us expand in a neighborhood of y_0:

$$\begin{aligned} \kappa_N^{X|W}\big(\psi^{X|Y}\big(\sigma(Z_W)\big)\big)^{-1/2} &= \sum_{i=0}^{k} \kappa_{N,[i]}^{X|W}(\sigma)^{-1/2}(Z_W) + O\big(|Z_W|^{k+1}\big), \\ \kappa_N^{W|Y}\big(\psi^{W|Y}(Z_W)\big)^{1/2} &= \sum_{i=0}^{k} \big(\kappa_{N,[i]}^{W|Y}\big)^{1/2}(Z_W) + O\big(|Z_W|^{k+1}\big), \end{aligned} \tag{4.5}$$

where $\kappa_{N,[i]}^{X|W}(\sigma)^{-1/2}(Z_W)$ and $(\kappa_{N,[i]}^{W|Y})^{1/2}(Z_W)$ are homogeneous polynomials of degree i.

We also decompose

$$\left(\frac{\kappa_\phi^Y(\sigma_Y(Z_W))}{\kappa_\phi^Y(Z_Y)}\right)^{-1/2} = \sum_{i=0}^{k} \left(\frac{\kappa_\phi^Y(\sigma_Y)}{\kappa_\phi^Y}\right)_{[i]}^{-1/2}(Z_W) + O\big(|Z_W|^{k+1}\big),$$

where $(\kappa_\phi^Y(\sigma_Y)/\kappa_\phi^Y)_{[i]}^{-1/2}(Z_W)$ are homogeneous polynomials of degree i. Since

$$\sigma(Z_Y, 0) = Z_Y$$

for any $i \in \mathbb{N}^*$, the polynomials $(\kappa_\phi^Y(\sigma_Y)/\kappa_\phi^Y)_{[i]}^{-1/2}(Z_W)$ divide $Z_{N^{W|Y}}$, where $Z_W = (Z_Y, Z_{N^{W|Y}})$, and we have

$$\left(\frac{\kappa_\phi^Y(\sigma_Y)}{\kappa_\phi^Y}\right)_{[0]}^{-1/2} = 1.$$

For $r \in \mathbb{N}$, we now introduce

$$\begin{aligned} \kappa_{\mathrm{cor},[r]}^1(Z_W) := \sum_{a+b+c=r} & \kappa_{N,[a]}^{X|Y}(\sigma)^{-1/2}(Z_W) \\ & \cdot \big(\kappa_{N,[b]}^{W|Y}\big)^{1/2}(Z_W) \cdot \left(\frac{\kappa_\phi^Y(\sigma_Y)}{\kappa_\phi^Y}\right)_{[c]}^{-1/2}(Z_W). \end{aligned} \tag{4.6}$$

From (4.1)–(4.5) and (4.6), we deduce that the asymptotic expansion (3.51) holds for $X := W$ the operator $T_p^{X|Y} := \mathrm{Res}_W \circ T_{\langle\langle g_1 \rangle\rangle, p}^{X|Y}$ and the polynomials

$$\begin{aligned} I_r^E(Z_W, Z'_Y) := \sum_{a+b+c+d+e+f=r} & \big(\mathrm{Res}_l \circ J_{a,g_1}^E\big)(\sigma, Z'_Y)_{[b]} \cdot \big(\mathrm{Res}_l \circ \big(\mathcal{E}'_{n,m}(\sigma, Z'_Y)_{[c]}\big)\big) \\ & \cdot \exp(\tau_F)_{[d]}(Z_W) \cdot \sum_{i-j=e} \exp(p\tau_L)_{[i,j]}(Z_W) \cdot \kappa_{\mathrm{cor},[f]}^1(Z_W). \end{aligned} \tag{4.7}$$

From (4.3), we see that the second sum in (4.7) is finite. From this, we see that the first part of the first statement of Theorem 4.1 follows from Theorem 3.22. The fact that the coefficients of I_r^E are bounded with all their derivatives follows from Propositions 2.14, 2.15 and the corresponding statement for the polynomials J_{r,g_1}^E, $r \in \mathbb{N}$, from Lemma 3.12. The statement about the parity of I_r^E follows from the analogous statements for J_{b,g_1}^E from Lemma 3.12 and the fact that $\exp(p\tau_L)_{[i,j]}$ are non-zero only for even j. Now, from Propositions 2.14, 2.15, the expression for J_{0,g_1}^E from Lemma 3.12 and (4.7), we deduce that for any $Z_W = (Z_Y, Z_{N^{W|Y}}) \in \mathbb{R}^{2l}$; $Z_Y, Z'_Y \in \mathbb{R}^{2m}$:

$$I_0^E(Z_W, Z'_Y) = P_N^{W|Y}(g_1)(y_0) \cdot Z_{N^{W|Y}}^{\otimes k} \cdot \kappa_N^{W|Y}(y_0)^{1/2}.$$

From this, we deduce by the last remark from Theorem 3.22 the first statement of the second part of Theorem 4.1.

The proof for the second statement of the first part of Theorem 4.1 is completely analogous to the proof for the first statement. The only difference is that the asymptotic expansion (3.51) now holds for the operators $T_p^{X|Y} := \mathrm{Res}_W \circ E_p^{X|Y} - E_p^{W|Y}$ and the polynomials

$$\begin{aligned} I_r^E(Z_W, Z'_Y) := &\sum_{a+b+c+d+e+f=r} \big(\mathrm{Res}_l \circ J_a^{X|Y,E}\big)(\sigma, Z'_Y)_{[b]} \\ &\cdot \big(\mathrm{Res}_l \circ \big(\mathcal{E}'_{n,m}(\sigma, Z'_Y)_{[c]}\big)\big) \cdot \exp(\tau_F)_{[d]}(Z_W) \\ &\cdot \sum_{i-j=e} \exp(p\tau_L)_{[i,j]}(Z_W) \cdot \kappa^1_{\mathrm{cor},[f]}(Z_W) - J_r^{W|Y,E}(Z_W, Z'_Y). \end{aligned} \tag{4.8}$$

From (3.17), (4.8) and the expressions for $J_0^{X|Y,E}$, $J_0^{W|Y,E}$ from Theorem 3.8, we conclude that for $Z = (Z_Y, Z_N)$, we have

$$I_0^E(Z_W, Z'_Y) := 0. \tag{4.9}$$

This establishes the second statement from the second part of Theorem 4.1 by the remark in the end of Theorem 3.22.

We now treat the third statement from the first part of Theorem 4.1. Directly from Lemma 2.6, the last part of (3.11) and the analysis similar to the one before (3.32), we conclude that the expansion (3.50) holds for $T_p^Y := T_{\langle\langle g'_1\rangle\rangle,p}^{Y|X} \circ T_{\langle\langle g_1\rangle\rangle,p}^{X|Y}$ and

$$I_r^Y := \sum_{a+b=r} \mathrm{Res}_m \circ \mathcal{K}_{n,m}\big[J_{a,g'_1}^R, J_{b,g_1}^E\big] \circ \mathrm{Res}_m. \tag{4.10}$$

From (3.19), (4.10) and the expression for J_{0,g'_1}^R, J_{0,g_1}^E from Lemma 3.12, we conclude that for any $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, we have

$$J_0(Z_Y, Z'_Y) := \Lambda_{\omega,=}[g'_1 \cdot g_1].$$

The statement about the parity of I_r^Y follows from the corresponding statements for $J^R_{a,g_1'}$ and J^E_{b,g_1} from Lemma 3.12 and the parity statement from Lemma 3.1. This establishes the third statement from the first and the second parts of Theorem 4.1 by the last part of Theorem 3.21.

Let us now treat the fourth statement from the first part of Theorem 4.1. From (2.16) and (2.17), for $Z \in \mathbb{R}^{2n}$, $Z'_W \in \mathbb{R}^{2l}$, we deduce that

$$T^{X|W}_{\langle\langle g_3\rangle\rangle,p}\big(\psi^{X|Y}_{y_0}(Z), \psi^{W|Y}_{y_0}(Z'_W)\big) = \exp\big(p(\xi^{W|Y}_L)^* + (\xi^{W|Y}_F)^*\big)\big(\psi^{W|Y}_{y_0}(Z'_W)\big) \cdot T^{X|W}_{\langle\langle g_3\rangle\rangle,p}\big(\psi^{X|W}_{y_0}(\upsilon(Z)), \phi^W_{y_0}\big(h^{W|Y}(Z'_W)\big)\big) \cdot \exp(p\chi_L + \chi_F)\big(\psi^{X|Y}_{y_0}(Z)\big). \quad (4.11)$$

From (2.12), (3.20) and (3.21), we deduce that

$$\kappa^{W|Y}_\psi(Z_W) = \kappa^W_\phi\big(h^{W|Y}(Z_W)\big) \cdot \big(\det \mathrm{Jac}(h^{W|Y})\big)(Z_W). \quad (4.12)$$

From (2.12) and (2.21), we deduce that

$$h^{X|W}(\upsilon(Z)) = h^{X|Y}(Z).$$

Hence by (4.12), we obtain

$$\kappa^{X|W}_\psi(\upsilon(Z)) = \kappa^{X|Y}_\psi(Z) \cdot \frac{(\det \mathrm{Jac}(h^{X|W}))(\upsilon(Z))}{(\det \mathrm{Jac}(h^{X|Y}))(Z)}.$$

For $r \in \mathbb{N}$, we denote by

$$\big(\det \mathrm{Jac}(h^{X|W})\big)(\upsilon)^{-1/2}_{[r]}, \quad \big(\det \mathrm{Jac}(h^{X|Y})\big)^{1/2}_{[r]}, \quad \big(\det \mathrm{Jac}(h^{W|Y})\big)^{1/2}_{[r]}$$

the homogeneous polynomials of degree r, defined as in (4.5) from Taylor expansions of $(\det \mathrm{Jac}(h^{X|W}))(\upsilon)^{-1/2}$, $(\det \mathrm{Jac}(h^{X|Y}))^{1/2}$, and $(\det \mathrm{Jac}(h^{W|Y}))^{1/2}$. For $r \in \mathbb{N}$, we now introduce

$$\kappa^2_{\mathrm{cor},[r]}(Z, Z'_W) := \sum_{a+b+c=r} \big(\det \mathrm{Jac}(h^{X|W})\big)(\upsilon)^{-1/2}_{[a]}(Z) \cdot \big(\det \mathrm{Jac}(h^{X|Y})\big)^{1/2}_{[b]}(Z) \cdot \big(\det \mathrm{Jac}(h^{W|Y})\big)^{1/2}_{[c]}(Z'_W).$$

We use notations similar to (4.2) and (4.3) for

$$\exp\big((\xi^{W|Y}_F)^*\big), \quad \exp(\chi_F) \quad \text{and} \quad \exp\big(p(\xi^{W|Y}_L)^*\big), \quad \exp(p\chi_L).$$

From the analysis similar to the one before (3.32), (4.1), (4.2) and (4.11), we deduce that the asymptotic expansion (3.51) holds for the operator

$$T^{X|Y}_p := T^{X|W}_{\langle\langle g_3\rangle\rangle,p} \circ T^{W|Y}_{\langle\langle g_2\rangle\rangle,p}$$

and the polynomials

$$I_r^E := \sum_{a+b+c+d+e+f+g+h+k=r} \mathcal{K}_{n,m}^E \Big[\exp\big((\xi_F^{W|Y})^*\big)_{[a]}(Z'_W) \cdot \sum_{i-j=b} \exp\big(p(\xi_L^{W|Y})^*\big)_{[i,j]}(Z'_W) \cdot J_{c,g_3}^E(\upsilon, h^{W|Y})_{[d]} \cdot \mathcal{E}'_{n,l}(\upsilon, h^{W|Y})_{[e]} \cdot \exp(\chi_F)_{[f]}(Z) \cdot \sum_{i-j=g} \exp(p\chi_L)_{[i,j]}(Z) \cdot \kappa^2_{\mathrm{cor},[h]}(Z, Z'_W), J_{k,g_2}^E \Big], \tag{4.13}$$

where both sums run over a subset of natural numbers. From (2.15), similarly to the remark after (4.3), we see that the second and the third sums in (4.13) are actually finite. The statement about the parity of I_r^E follows from the analogous statements for J_{c,g_3}^E, J_{k,g_2}^E from Lemma 3.12 and the parity statement from Lemma 3.1. From this, we see that the fourth part of the first statement of Theorem 3.22 follows from Theorem 3.22. From (3.16), (3.17), (4.13), and the expressions for J_{r,g_3}^E, J_{r,g_2}^E, $r \in \mathbb{N}$ from Lemma 3.12, we conclude that for $Z = (Z_Y, Z_N)$, $Z_Y \in \mathbb{R}^{2m}$, we have

$$I_0^E(Z, Z'_Y) := \big(\iota_1^*(g_3) \cdot g_2\big)(y_0) \cdot Z_N^{\otimes(k+k')} \cdot \kappa_N^{X|Y}(y_0)^{1/2},$$

which establishes the fourth statement from the second part of Theorem 4.1 by the last part of Theorem 3.22.

The proofs of the fifth and sixth statements are completely analogous to the proof of the fourth one. The only difference is that the asymptotic expansion (3.51) holds for the operators

$$T_p^{X|Y} := T_{\langle\langle g_3\rangle\rangle,p}^{X|W} \circ E_p^{W|Y}, \quad T_p^{X|Y} := E_p^{X|W} \circ T_{\langle\langle g_2\rangle\rangle,p}^{W|Y}$$

and the polynomials

$$I_r^E := \sum_{a+b+c+d+e+f+g+h+k=r} \mathcal{K}_{n,m}^E \Big[\exp\big((\xi_F^{W|Y})^*\big)_{[a]}(Z'_W) \cdot \sum_{i-j=b} \exp\big(p(\xi_L^{W|Y})^*\big)_{[i,j]}(Z'_W) \cdot J_{c,g_3}^E(\upsilon, h^{W|Y})_{[d]} \cdot \mathcal{E}'_{n,l}(\upsilon, h^{W|Y})_{[e]} \cdot \exp(\chi_F)_{[f]}(Z) \cdot \sum_{i-j=g} \exp(p\chi_L)_{[i,j]}(Z) \cdot \kappa^2_{\mathrm{cor},[h]}(Z, Z'_W), J_k^{W|Y,E} \Big],$$

$$I_r^E := \sum_{a+b+c+d+e+f+g+h+k=r} \mathcal{K}_{n,m}^E \Big[\exp\big((\xi_F^{W|Y})^*\big)_{[a]}(Z'_W) \\ \cdot \sum_{i-j=b} \exp\big(p(\xi_L^{W|Y})^*\big)_{[i,j]}(Z'_W) \cdot J_c^{X|W,E}(\upsilon, h^{W|Y})_{[d]} \\ \cdot \mathcal{E}'_{n,l}(\upsilon, h^{W|Y})_{[e]} \cdot \exp(\chi_F)_{[f]}(Z) \\ \cdot \sum_{i-j=g} \exp(p\chi_L)_{[i,j]}(Z) \cdot \kappa^2_{\mathrm{cor},[h]}(Z, Z'_W), J_{k,g_2}^{W|Y,E} \Big], \tag{4.14}$$

respectively. The proofs of the parity statements are analogous. From (3.16), (3.17), (4.13), and the expressions for $J_0^{X|W,E}$, $J_0^{W|Y,E}$, J_{0,g_3}^E, J_{0,g_2}^E, from Theorem 3.8 and Lemma 3.12, we conclude that

$$I_0^E(Z, Z'_Y) := \iota_1^*(g_3)(y_0) \cdot Z_N^{\otimes k} \cdot \kappa_N^{X|Y}(y_0)^{1/2},$$
$$I_0^E(Z, Z'_Y) := g_2(y_0) \cdot Z_N^{\otimes k'} \cdot \kappa_N^{X|Y}(y_0)^{1/2},$$

respectively. This establishes fifth and sixth statements from the second part of Theorem 4.1 by the remark in the end of Theorem 3.22.

Let us now treat the seventh statement. Directly from Lemma 2.6, (3.10) and the analysis similar to the one before (3.32), we conclude that (3.51) holds for the operator $T_p^{X|Y} := T_{\langle\langle g_1 \rangle\rangle, p}^{X|Y} \circ T_{f,p}^Y$ and

$$I_r^E := \sum_{a+b=r} \mathcal{K}_{n,m}^{EP}[J_{a,g_1}^E, J_{b,f}^Y]. \tag{4.15}$$

The parity statement I_r^E holds by the same reasons as before. From (3.17), (4.15), and the expression for J_{0,g_1}^E, $J_{0,f}^Y$ from Lemma 3.12, we conclude that for $Z = (Z_Y, Z_N)$, we have

$$I_0^E(Z, Z'_Y) := g_1(y_0) \cdot Z_N^{\otimes k} \cdot f \cdot \kappa_N^{X|Y}(y_0)^{1/2},$$

which establishes the seventh statement from the second part of Theorem 4.1 by the last part of Theorem 3.21. ■

4.2. Multiplicative defect and adjoints of Toeplitz type operators

The main goal of this section is to study the adjoints of Toeplitz type operators. For this, we introduce the so-called multiplicative defect operator and study some of its properties. This operator will also play an important role in our calculations of the first significant term of the asymptotic expansion of the transitivity defect, D_p, from Theorem 1.5.

Theorem 4.3. *Assume* (X, Y, g^{TX}) *is of bounded geometry. Then there is* $p_1 \in \mathbb{N}^*$ *such that, for any* $p \geq p_1$*, there is a unique operator*

$$A_p^{X|Y} \in \mathrm{End}\big(H^0_{(2)}\big(Y, \iota^*(L^p \otimes F)\big)\big),$$

verifying

$$(\mathrm{Res}_Y \circ B_p^X)^* = \mathrm{E}_p^{X|Y} \circ A_p^{X|Y}. \tag{4.16}$$

Moreover, $(1/p^{n-m}) A_p^{X|Y}$, $p \geq p_1$*, when viewed as a sequence of elements from* $\mathrm{End}(L^2(Y, \iota^*(L^p \otimes F)))$ *by precomposing with* B_p^Y*, forms a Toeplitz operator with weak exponential decay with respect to* X*, and we have*

$$\Big[\frac{1}{p^{n-m}} A_p^{X|Y}\Big]_0 = \kappa_N^{X|Y}|_Y^{-1},$$

where $\kappa_N^{X|Y}$ *was defined in* (1.7).

Remark 4.4. (a) The sequence of operators $(1/p^{n-m}) A_p^{X|Y}$, $p \in \mathbb{N}^*$, will be later called "multiplicative defect".

(b) This theorem can be used to give an alternative proof of the main results from [9] bypassing some of the technical difficulties contained in [9, §§2.5 and 4], see [10] for details.

Proof. First of all, let us establish the existence and uniqueness of $A_p^{X|Y}$ for p big enough. Clearly, it suffices to prove that the kernels and the images of the operators $(\mathrm{Res}_Y \circ B_p^X)^*$ and $\mathrm{E}_p^{X|Y}$ coincide for p big enough. First of all, we have

$$\ker\big(\mathrm{Res}_Y \circ B_p^X\big)^* = \big(\mathrm{Im}\big(\mathrm{Res}_Y \circ B_p^X\big)\big)^{\perp}. \tag{4.17}$$

Now, in [9, (4.1)], we established that $\mathrm{Res}_Y \circ B_p^X$ has its image inside of

$$H^0_{(2)}\big(Y, \iota^*(L^p \otimes F)\big).$$

In [9, Theorem 4.4], by following the proof of Ohsawa–Takegoshi extension theorem, we proved that there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, the image of $\mathrm{Res}_Y \circ B_p^X$ coincides exactly with $H^0_{(2)}(Y, \iota^*(L^p \otimes F))$. From this, and (4.17), we see that the kernels of $(\mathrm{Res}_Y \circ B_p^X)^*$ and $\mathrm{E}_p^{X|Y}$ coincide. Similar reasoning shows that the images of those operators coincide as well. In particular, for $p \geq p_1$, there is a unique sequence of operators $A_p^{X|Y}$ as in (4.16).

Now, let us establish that the sequence of operators $(1/p^{n-m}) A_p^{X|Y}$, $p \geq p_1$, viewed as a sequence of elements from $\mathrm{End}(L^2(Y, \iota^*(L^p \otimes F)))$ by precomposing with B_p^Y, forms a Toeplitz operator with weak exponential decay. We do so by verifying that this sequence of operators satisfies all the properties of Theorem 3.21.

In fact, from (4.16) and the fact $\mathrm{Res}_Y \circ \mathrm{E}_p^{X|Y} = B_p^Y$, we obtain the explicit formula

$$A_p^{X|Y} = \mathrm{Res}_Y \circ \big(\mathrm{Res}_Y \circ B_p^X\big)^*. \tag{4.18}$$

Clearly, the first property from Theorem 3.21 follows from (1.4) and (4.18). The weak version of the second property with respect to X follows from Theorem 3.4 and (4.18).

We will now show that the third property is a direct consequence of Theorem 3.7. For the Taylor expansions of the κ-functions, we will use the same notation as in (4.5). From the fact that $\mathrm{Res}_Y(\widetilde{f}_1^{X|Y}, \dots, \widetilde{f}_r^{X|Y}) = \widetilde{f}_1'^Y, \dots, \widetilde{f}_r'^Y$ and (3.22), we see directly that the expansion (3.50) holds for $T_p^Y := (1/p^{n-m}) A_p^{X|Y}$ and for the polynomials $I_r^Y(Z_Y, Z'_Y)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, defined as follows:

$$I_r^Y(Z_Y, Z'_Y) := \sum_{a+b+c=r} J_a^{X|X}(Z_Y, Z'_Y) \cdot \kappa_{N,[b]}^{X|Y}(Z_Y)^{-1/2} \cdot \kappa_{N,[c]}^{X|Y}(Z'_Y)^{-1/2}. \tag{4.19}$$

From the parity properties of $J_a^{X|X}$ from Theorem 3.7 and the bounded geometry assumption, we see that the coefficients of I_r^Y are bounded with all their derivatives, and the parity of I_r^Y coincides with r. Hence, by Theorem 3.21, the sequence of operators $(1/p^{n-m}) A_p^{X|Y}$, $p \in \mathbb{N}^*$, forms a Toeplitz operator with weak exponential decay with respect to X. Moreover, from (3.23) and (4.19), we deduce

$$I_0^Y(Z_Y, Z'_Y) = \kappa_N^{X|Y}|_Y^{-1}.$$

From the last statement of Theorem 3.21, we deduce that

$$\Big[\frac{1}{p^{n-m}} A_p^{X|Y}\Big]_0 = \kappa_N^{X|Y}|_Y^{-1}. \qquad \blacksquare$$

For technical reasons, we will later need to consider the inverse of $(1/p^{n-m}) A_p^{X|Y}$. The following result gives a sufficient condition for inverting Toeplitz operators with weak exponential decay.

Lemma 4.5. Assume that a sequence of operators G_p, $p \in \mathbb{N}$, forms a Toeplitz operator with weak exponential decay with respect to a manifold Z in the notations from Definition 3.19. Assume that for $f := [G_p]_0$, we have $f \neq 0$ everywhere and $f^{-1} \in \mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$. Then there is $p_1 \in \mathbb{N}$, such that for $p \geq p_1$, the operators G_p are invertible. Moreover, the sequence of operators G_p^{-1}, $p \geq p_1$, forms a Toeplitz operator with weak exponential decay with respect to the same manifold Z and we have $[(G_p)^{-1}]_0 = f^{-1}$.

To prove this result, the following statement will be of utmost importance.

Lemma 4.6. For any $f_1, f_2 \in \mathscr{C}_b^\infty(Y, \mathrm{End}(\iota^* F))$, the sequence of operators

$$T_{f_1,p}^Y \circ T_{f_2,p}^Y, \quad p \in \mathbb{N},$$

forms a Toeplitz operator with exponential decay. Moreover, we have

$$\big[T^Y_{f_1,p} \circ T^Y_{f_2,p}\big]_0 = f_1 \cdot f_2.$$

In particular, a product of two Toeplitz type operators with weak exponential decay forms a Toeplitz type operator with weak exponential decay.

Proof. In the case of compact manifolds, the result is due to Bordemann–Meinrenken–Schlichenmaier [3] and Ma–Marinescu ([14, Theorem 7.4.1], [16]), who used the asymptotic characterization of Toeplitz operators as in Theorem 3.21. Since by Theorem 3.21, the analogous characterization holds in the setting of Toeplitz type operators, the same proof would give us the needed result. ∎

Proof of Lemma 4.5. First of all, let us consider a sequence of operators

$$K_p := G_p \circ T^Y_{f^{-1},p}, \quad p \in \mathbb{N}.$$

According to Lemma 4.6, K_p, $p \in \mathbb{N}$, form a Toeplitz operator with weak exponential decay with respect to Z and we can represent it in the form

$$K_p = 1 + \frac{Q_p}{p}, \tag{4.20}$$

where Q_p, $p \in \mathbb{N}$, is a Toeplitz operator with weak exponential decay with respect to Z. In particular, by Corollary 2.7, there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\|Q_p\| \leq C. \tag{4.21}$$

From (4.20) and (4.21), we deduce that there is $p_1 \in \mathbb{N}^*$, such that K_p is invertible for $p \geq p_1$, and

$$K_p^{-1} = \sum_{r=0}^{\infty} (-1)^r \frac{Q_p^r}{p^r}.$$

However, by Corollary 2.8, we infer that there are $C > 0$, $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, $r \in \mathbb{N}^*$, we have

$$\big|Q_p^r(y_1, y_2)\big|_{\mathscr{C}^k} \leq C^r p^{m+k/2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_X(y_1, y_2)\big). \tag{4.22}$$

We conclude by Lemma 4.6 and (4.22) that the sequence of operators K_p^{-1}, $p \geq p_1$, forms a Toeplitz type operator. But then again by Lemma 4.6, we obtain that the sequence of operators $T^Y_{f^{-1},p} \circ K_p^{-1}$, $p \geq p_1$, forms a Toeplitz operator with exponential decay. But trivially, we have

$$G_p \circ T^Y_{f^{-1},p} \circ K_p^{-1} = T^Y_{f^{-1},p} \circ K_p^{-1} \circ G_p = \mathrm{Id}.$$

Hence, G_p is invertible and $(G_p)^{-1} = T^Y_{f^{-1},p} \circ K_p^{-1}$, which finishes the proof. ∎

The following result will be useful in our further considerations.

Theorem 4.7. *A family* $T_p^{Y|X}: L^2(X, L^p \otimes F) \to L^2(Y, \iota^*(L^p \otimes F))$*,* $p \in \mathbb{N}$*, of linear operators forms a Toeplitz operator with exponential decay of type* $Y|X$ *if and only if the family of linear operators*

$$\frac{1}{p^{n-m}}(T_p^{Y|X})^*: L^2\big(Y, \iota^*(L^p \otimes F)\big) \to L^2(X, L^p \otimes F), \quad p \in \mathbb{N},$$

forms a Toeplitz operator with exponential decay of type $X|Y$*. Moreover, we have*

$$[T_p^{Y|X}]_0 = \left(\left[\frac{1}{p^{n-m}}(T_p^{Y|X})^*\right]_0\right)^* \cdot \kappa_N^{X|Y}|_Y. \tag{4.23}$$

Proof. Let us first assume that a sequence of operators $T_p^{Y|X}$, $p \in \mathbb{N}$, forms a Toeplitz operator with exponential decay of type $Y|X$. Clearly, it is enough to prove that for any $k \in \mathbb{N}$, $j = \{1, 2\}$,

$$g_j^a \in \mathscr{C}_b^\infty\big(Y, \operatorname{Sym}^{2k+j} N^{(0,1)*} \otimes \operatorname{End}(\iota^* F)\big),$$

for $T_p^{Y|X} = T_{\langle\langle g_j^a \rangle\rangle, p}^{Y|X}$, $j = 2$, and for $T_p^{Y|X} = (1/\sqrt{p}) T_{\langle\langle g_j^a \rangle\rangle, p}^{Y|X}$, $j = 1$, the sequence of operators $(1/p^{n-m})(T_p^{Y|X})^*$, $p \in \mathbb{N}^*$, forms a Toeplitz operator with exponential decay of type $X|Y$.

From Theorem 4.3 and (1.12), for $p \geq p_1$, where $p_1 \in \mathbb{N}^*$ is as in Theorem 4.3, we have

$$\big(T_{\langle\langle g_j^a \rangle\rangle, p}^{Y|X}\big)^* = T_{\langle\langle (g_j^a)^* \rangle\rangle, p}^{X|Y} \circ A_p^{X|Y}.$$

Hence, according to Theorems 4.1 (6) and 4.3, we see that $(1/p^{n-m})(T_{\langle\langle g_j^a \rangle\rangle, p}^{Y|X})^*$ forms a Toeplitz operator with exponential decay of type $X|Y$. The relation (4.23) follows from Theorems 4.1 (6) and 4.3. This proves the first direction of Theorem 4.7. The proof of the opposite direction is completely analogous and is left to the interested reader. ∎

Proof of Theorem 3.23. The proof of one implication of Theorem 3.23 was described in the end of Section 3.4. The inverse implication is a direct consequence of Theorems 3.22 and 4.7. ∎

4.3. Some examples of Toeplitz type operators

The main goal of this section is to give some examples of Toeplitz type operators. To state our results in this direction, we need to fix some notation first.

We fix $y_0 \in Y$, choose an orthogonal basis $w_1, \dots, w_{n-m}$ of $(N^{X|Y}_{y_0})^{(1,0)}$ as in (3.44). We define

$$\Lambda_{\omega,h}\colon \mathrm{Sym}^i\big((N^{X|Y})^{(1,0)*}\big) \otimes \mathrm{Sym}^j\big((N^{X|Y})^{(0,1)*}\big) \to \mathrm{Sym}^{\max\{i-j,0\}}\big((N^{X|Y})^{(1,0)*}\big),$$
$$\Lambda_{\omega,a}\colon \mathrm{Sym}^i\big((N^{X|Y})^{(1,0)*}\big) \otimes \mathrm{Sym}^j\big((N^{X|Y})^{(0,1)*}\big) \to \mathrm{Sym}^{\max\{j-i,0\}}\big((N^{X|Y})^{(0,1)*}\big),$$

for multiindices $\alpha, \beta \in \mathbb{N}^{n-m}$, as follows:

$$\Lambda_{\omega,h}(w^\alpha \otimes \overline{w}^\beta) = \begin{cases} \frac{1}{\pi^{|\alpha|}} \frac{\alpha!}{(\alpha-\beta)!} w^{\alpha-\beta} & \text{if } \alpha \geq \beta, \alpha \neq \beta, \\ 0 & \text{otherwise,} \end{cases}$$
$$\Lambda_{\omega,a}(w^\alpha \otimes \overline{w}^\beta) = \begin{cases} \frac{1}{\pi^{|\beta|}} \frac{\beta!}{(\beta-\alpha)!} \overline{w}^{\beta-\alpha} & \text{if } \alpha \geq \beta, \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, those operators do not depend on the choice of the basis. We extend $\Lambda_{\omega,h}[\cdot]$ and $\Lambda_{\omega,a}[\cdot]$ to $\mathrm{Sym}^k(N^{X|Y})^* \otimes \mathbb{C}$ linearly. For the next result, we will use the following notation. For $f \in \mathscr{C}^\infty_b(X, \mathrm{End}(F))$, we let $T^{Y|Y}_{f,p} := \mathrm{Res}_Y \circ T^X_{f,p} \circ \mathrm{E}^{X|Y}_p$.

Proposition 4.8. *For any*

$$f \in \mathscr{C}^\infty_b\big(X, \mathrm{End}(F)\big),$$
$$g \in \bigoplus_{k=0}^{\infty} \mathscr{C}^\infty_b\big(Y, \mathrm{Sym}^k(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big),$$
$$g_e \in \bigoplus_{k=0}^{\infty} \mathscr{C}^\infty_b\big(Y, \mathrm{Sym}^{2k}(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big),$$
$$g_o \in \bigoplus_{k=0}^{\infty} \mathscr{C}^\infty_b\big(Y, \mathrm{Sym}^{2k+1}(N^{X|Y})^* \otimes \mathrm{End}(\iota^* F)\big),$$

the sequences of operators

(1) $T^{Y|Y}_{f,p}$, (2) $T^{Y|Y}_{\langle\langle g \rangle\rangle,p}$

for $p \in \mathbb{N}$, form Toeplitz operators with exponential decay. Also, the sequences of operators

(1) $T^{X|Y}_{f,p}$, (2) $T^{X|Y}_{\langle\langle g_e \rangle\rangle,p}$, (3) $\frac{1}{\sqrt{p}} T^{X|Y}_{\langle\langle g_o \rangle\rangle,p}$

for $p \in \mathbb{N}$, form Toeplitz operators with exponential decay of type $X|Y$. The sequences of operators

(1) $T^{Y|X}_{f,p}$, (2) $T^{Y|X}_{\langle\langle g_e \rangle\rangle,p}$, (3) $\frac{1}{\sqrt{p}} T^{Y|X}_{\langle\langle g_o \rangle\rangle,p}$

for $p \in \mathbb{N}$, form Toeplitz operators with exponential decay of type $Y|X$.

Moreover, we have

(1) $[T^{Y|Y}_{f,p}]_0 = f$, (2) $[T^{X|Y}_{f,p}]_0 = 0$,

(3) $[T^{Y|Y}_{\langle\langle g\rangle\rangle,p}]_0 = \Lambda_{\omega,=}[g]$, (4) $[T^{X|Y}_{\langle\langle g_e\rangle\rangle,p}]_0 = \Lambda_{\omega,h}[g_e]$,

(5) $[\sqrt{p}T^{X|Y}_{\langle\langle g_o\rangle\rangle,p}]_0 = 0$, (6) $[T^{X|Y}_{\langle\langle g_o\rangle\rangle,p}]_1 = \Lambda_{\omega,h}[g_o]$,

(7) $[T^{Y|X}_{f,p}]_0 = 0$, (8) $[T^{Y|X}_{\langle\langle g_e\rangle\rangle,p}]_0 = \Lambda_{\omega,a}[g_e]$,

(9) $[\frac{1}{\sqrt{p}}T^{Y|X}_{\langle\langle g_o\rangle\rangle,p}]_0 = 0$, (10) $[\frac{1}{\sqrt{p}}T^{Y|X}_{\langle\langle g_o\rangle\rangle,p}]_1 = \Lambda_{\omega,a}[g_o]$.

Proof. The proofs of all the statements from the first part of Proposition 4.8 are very similar to the proofs from Theorem 4.1: they all proceed by the verification that the relevant operators satisfy the assumptions of Theorems 3.21, 3.22 and 3.23. The proofs of the second part are also analogous: we only need to calculate the first term of the asymptotic expansions as in (3.50), (3.51), and (3.52), and apply the last part of Theorems 3.21, 3.22, and 3.23. For brevity, we only present the proof for the fourth statement, which is slightly more complicated than the rest.

The validity of the first condition from Theorem 3.22 for $T^{X|Y}_{\langle\langle g_e\rangle\rangle,p}$ is direct. The second and the third conditions are proved in Lemma 3.12. Hence, by Theorem 3.22, the sequence of operators $T^{X|Y}_{\langle\langle g_e\rangle\rangle,p}$, $p \in \mathbb{N}$, form a Toeplitz operator with exponential decay of type $X|Y$. We now only need to calculate the first term of the asymptotic expansion of this sequence of operators to establish the second part of the theorem.

We decompose g_e as follows:

$$g_e = \sum_{i,j} g_{e,ij},$$

where $i, j \in \mathbb{N}$ and

$$g_{e,ij} \in \mathrm{Sym}^i(N^{X|Y})^{(1,0)*} \otimes \mathrm{Sym}^j(N^{X|Y})^{(0,1)*}.$$

From (3.17), (3.19) and (3.34), we deduce that for any $Z = (Z_Y, Z_N)$, $Z_N \in \mathbb{R}^{2(n-m)}$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, we have

$$J^E_{0,g_e}(Z, Z'_Y) = \sum_{i,j} \Lambda_{\omega,h}[g_{e,ij}] \cdot Z_N^{\otimes(i-j)} \cdot \kappa_N^{X|Y}(y_0)^{1/2}.$$

From this and the last part of Theorem 3.22, we conclude that the second part of Proposition 4.8 for the fourth point holds. ∎

5. Complex embeddings and associated Toeplitz type operators

The main goal of this section is to establish Theorems 1.1, 1.5 and 1.7. More precisely, in Section 5.1, we calculate the second term of the asymptotic expansion of the multiplicative defect introduced in Section 4.2 and, as a consequence, we prove Theorem 1.7. In Section 5.2, we establish Theorems 1.1 and 1.5, and the extension of Theorem 1.5 to towers of submanifolds of arbitrary length.

5.1. Optimal Ohsawa–Takegoshi theorem, multiplicative defect asymptotics

The main goal of this section is to calculate the second term of the asymptotic expansion of the multiplicative defect and to derive as a consequence the calculation of the asymptotics of the optimal constant in Ohsawa–Takegoshi theorem, i.e. to establish Theorem 1.7.

Theorem 5.1. *In the notations of Theorems 1.1 and 4.3, under assumption (1.8), we have*

$$\Big[\frac{1}{p^{n-m}}A_p^{X|Y}\Big]_0 = 1,$$
$$\Big[\frac{1}{p^{n-m}}A_p^{X|Y}\Big]_1 = \frac{1}{8\pi}(\mathbf{r}^X - \mathbf{r}^Y) - \frac{1}{2\pi\sqrt{-1}}\big(\Lambda_\omega[R^F] - \Lambda_{\iota^*\omega}[R^F]\big).$$

The proof of Theorem 5.1 will be based on the following result.

Theorem 5.2. *In the notations of Theorem 3.7, under assumptions (1.8), we have*

$$J_2^{X|X}(0,0) = \frac{1}{8\pi}\mathbf{r}^X_{x_0} - \frac{1}{2\pi\sqrt{-1}}\Lambda_\omega[R^F_{x_0}].$$

Proof. The proof is due to Lu [13] (for trivial (F, h^F)) and Wang [24] (for general (F, h^F)). See also Dai–Liu–Ma [14, Theorem 1.3] and [18, Proposition 4], where the authors calculate explicitly the polynomials $J_1^{X|X}$ and $J_2^{X|X}$. ■

Recall that in Lemma 3.12, for any $f \in \mathscr{C}^\infty_b(X, \mathrm{End}(F))$, $x_0 \in X$, $r \in \mathbb{N}$, we defined the polynomials $J^X_{r,f}(Z, Z') \in \mathrm{End}(F_{x_0})$, $Z, Z' \in \mathbb{R}^{2n}$.

Corollary 5.3. *Under the assumptions (1.8), we have*

$$J^X_{1,f}(Z, Z') = \nabla^{\mathrm{End}(F)}_{\frac{\partial}{\partial z}} f + \frac{\partial f}{\partial \overline{z}'}.$$

Proof. It follows directly from (3.17), (3.23), (3.32), (3.24) and the fact, following from Proposition 2.12, that a derivative of a sections of a vector bundle, written in the trivialization, considered in Theorem 3.7, corresponds to covariant derivatives. ■

Proof of Theorem 5.1. The first identity is a direct consequence of Theorem 4.3 and our assumption, see the remark before (1.8).

To establish the second identity, remark that from the first part and Theorem 4.3, the sequence of operators $p((1/p^{n-m})A_p^{X|Y} - B_p^Y)$, $p \geq p_1$ forms a Toeplitz operator with weak exponential decay with respect to X. Moreover, from (3.24) and (4.19), we see that the expansion (3.50) holds for $T_p^Y := p((1/p^{n-m})A_p^{X|Y} - B_p^Y)$ and for polynomials $I_r^Y(Z_Y, Z'_Y)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, verifying

$$I_0^Y(0,0) = J_2^{X|X}(0,0) - J_2^{Y|Y}(0,0).$$

From Theorem 5.2, we obtain that

$$I_0^Y(0,0) = \frac{1}{8\pi}(\mathbf{r}_{y_0}^X - \mathbf{r}_{y_0}^Y) - \frac{1}{2\pi\sqrt{-1}}\big(\Lambda_\omega[R_{y_0}^F] - \Lambda_{\iota^*\omega}[R_{y_0}^F]\big). \tag{5.1}$$

From the last part of Theorem 3.21 and (5.1), we obtain the needed result. ∎

Let us now give the first application of those calculations.

Proof of Theorem 1.7. From (4.16), remark that the following identities hold

$$(\mathrm{E}_p^{X|Y})^* \circ \mathrm{E}_p^{X|Y} = \big((A_p^{X|Y})^*\big)^{-1}, \quad \mathrm{Res}_p^{Y|X} \circ (\mathrm{Res}_p^{Y|X})^* = A_p^{X|Y}. \tag{5.2}$$

Clearly, we have

$$\|(\mathrm{E}_p^{X|Y})^* \circ \mathrm{E}_p^{X|Y}\| = \|\mathrm{E}_p^{X|Y}\|^2 \quad \text{and} \quad \|\mathrm{Res}_p^{Y|X} \circ (\mathrm{Res}_p^{Y|X})^*\| = \|\mathrm{Res}_p^{Y|X}\|^2.$$

The result now follows from this observation, Theorem 5.1, Lemma 3.17 and (5.2). ∎

Remark 5.4. From (5.2), we see that $A_p^{X|Y}$ is a self-adjoint operator.

5.2. Transitivity defect, proofs of Theorems 1.1 and 1.5

The main goal of this section is to study the asymptotic transitivity of the optimal holomorphic extension operator and to prove Theorems 1.1 and 1.5.

One possible way of proceeding would be to directly use the formula (4.8) to calculate the asymptotics of the sequence of operators

$$T_p^{W|Y} := \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{W|Y}, \quad p \geq p_1, \tag{5.3}$$

where $p_1 \in \mathbb{N}$ is as in (1.5) and study the first non-vanishing term of this asymptotics. Then, the needed result would follow by the use of the basic formula

$$\mathrm{E}_p^{X|W} \circ T_p^{W|Y} = D_p = \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y}, \tag{5.4}$$

and the subsequent use of the formula (4.14). This method is, although straightforward, computationally demanding. In fact, to calculate $[\mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y}]_3$ (which is the first significant term of our asymptotic expansion according to Theorem 1.5), we will need to calculate $J_3^{X|X}$, and the third terms of the Taylor expansions of τ_E, $p\tau_L$, σ, υ, etc. Although some information on $J_3^{X|X}$ is known, e.g. [17, Lemma 5.2], it seems that there is still no explicit formula.

Our approach here is different. We will still, however, base our consideration on the study of the sequence of operators $T_p^{W|Y}$, $p \in \mathbb{N}$, defined in (5.3), instead of D_p. But differently from the above approach, instead of using right away the explicit formula for the asymptotic expansion, we will first find an alternative expression for $T_p^{W|Y}$ in terms of the operators $A_p^{X|Y}$, $A_p^{X|W}$ and $A_p^{W|Y}$. Then the calculation of the asymptotic expansion for $T_p^{W|Y}$ will be essentially encapsulated in the calculations of the asymptotic expansions of $A_p^{X|Y}$, $A_p^{X|W}$ and $A_p^{W|Y}$, which was done in Theorem 5.1. More precisely, our formula looks as follows.

Lemma 5.5. There is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, the following expression for $T_p^{W|Y}$ holds:

$$\begin{aligned} T_p^{W|Y} &= \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \circ \Big[B_p^Y - \Big(\frac{1}{p^{n-m}} A_p^{X|Y} \Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y} \Big)^{-1} \Big] \\ &\quad + \Big[B_p^W - \Big(\frac{1}{p^{n-l}} A_p^{X|W} \Big)^{-1} \Big] \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \\ &\quad - \Big[B_p^W - \Big(\frac{1}{p^{n-l}} A_p^{X|W} \Big)^{-1} \Big] \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \\ &\quad \circ \Big[B_p^Y - \Big(\frac{1}{p^{n-m}} A_p^{X|Y} \Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y} \Big)^{-1} \Big]. \end{aligned} \tag{5.5}$$

Proof. First of all, recall that by Theorem 4.3 and Lemma 4.5, there is $p_1 \in \mathbb{N}$, such that for $p \geq p_1$, the operators $(1/p^{n-l}) A_p^{X|W}$, $(1/p^{n-m}) A_p^{X|Y}$, $(1/p^{l-m}) A_p^{W|Y}$ are invertible. In what follows, we work with such p with no further notice. From (4.16), we have

$$\mathrm{E}_p^{W|Y} = (\mathrm{Res}_Y \circ B_p^W)^* \circ (A_p^{W|Y})^{-1}. \tag{5.6}$$

Now, from the fact that $B_p^W = \mathrm{Res}_W \circ \mathrm{E}_p^{X|W}$ and (5.6), applied for $W := X$, $Y := W$, we obtain

$$B_p^W = \mathrm{Res}_W \circ (\mathrm{Res}_W \circ B_p^X)^* \circ (A_p^{X|W})^{-1}. \tag{5.7}$$

From (5.6), (5.7) and the trivial fact that $\mathrm{Res}_Y \circ \mathrm{Res}_W = \mathrm{Res}_Y$, we obtain

$$\mathrm{E}_p^{W|Y} = \big(\mathrm{Res}_Y \circ (\mathrm{Res}_W \circ B_p^X)^* \circ (A_p^{X|W})^{-1} \big)^* \circ (A_p^{W|Y})^{-1}. \tag{5.8}$$

We replace Res_Y in (5.8) by $\mathrm{Res}_Y \circ B_p^X$, open the brackets in (5.8), we then once again use (4.16) to give an alternative expression for $(\mathrm{Res}_Y \circ B_p^X)^*$ and use the trivial fact $B_p^X \circ \mathrm{E}_p^{X|Y} = \mathrm{E}_p^{X|Y}$, to obtain

$$\mathrm{E}_p^{W|Y} = \Big(\frac{1}{p^{n-l}} (A_p^{X|W})^* \Big)^{-1} \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \circ \Big(\frac{1}{p^{n-m}} A_p^{X|Y} \Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y} \Big)^{-1}. \tag{5.9}$$

The formula (5.5) is then a formal consequence of Remark 5.4, (5.9) and the fact that

$$\mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} = B_p^W \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \circ B_p^Y,$$

which follows from (1.4). ∎

To establish Theorem 1.5, we need two additional lemmas. To state the first, let us fix a function $f \in \mathscr{C}_b^\infty(X, \mathrm{End}(F))$. We use below the notational conventions introduced before Theorem 3.7. Compare the following result with the first part of Lemma 3.12.

Lemma 5.6. There are polynomials $J_{0,f}^{X|Y}(Z,Z')$, $J_{1,f}^{X|Y}(Z,Z')$ in $Z, Z' \in \mathbb{R}^{2n}$, such that for $F_{r,f}^{X|Y} := J_{r,f}^{X|Y} \cdot \mathscr{P}_n$, $r = 0, 1$, the following holds. There are $\varepsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $p \geq p_1$, $|Z|, |Z'| \leq \varepsilon$, the Schwartz kernel of $T_{f,p}^X$, evaluated with respect to the volume form dv_X, satisfies

$$\Big| \frac{1}{p^n} T_{f,p}^X\big(\psi_{y_0}^{X|Y}(Z), \psi_{y_0}^{X|Y}(Z')\big) \\ - \sum_{r=0}^{1} p^{-r/2} F_{r,f}^{X|Y}(\sqrt{p}Z, \sqrt{p}Z') \kappa_\psi^{X|Y}(Z)^{-1/2} \kappa_\psi^{X|Y}(Z')^{-1/2} \Big| \\ \leq Cp^{-1}\big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|\big)^Q \exp\big(-c\sqrt{p}|Z - Z'|\big).$$

Moreover, we have $J_{0,f}^{X|Y}(Z_Y, Z'_Y) = f(y_0)$, and for $Z = (0, Z_N)$, $Z_N \in \mathbb{R}^{2(n-m)}$, we have

$$J_{1,f}^{X|Y}(Z, 0) = \nabla_{\frac{\partial}{\partial z}}^{\mathrm{End}(E)} f.$$

Proof. First of all, recall that the diffeomorphism $h^{X|Y}$ was defined in (2.12), and the functions $\xi_L^{X|Y}, \xi_F^{X|Y}$ were defined in (2.13). Directly from the definitions, we obtain the following relation between the Schwartz kernels:

$$T_{f,p}^X\big(\psi_{y_0}^{X|Y}(Z), \psi_{y_0}^{X|Y}(Z')\big) = \exp\big(p(\xi_L^{X|Y})^* + (\xi_F^{X|Y})^*\big)\big(\psi_{y_0}^{X|Y}(Z')\big) \\ \cdot T_{f,p}^X\big(\phi_{y_0}^X(h^{X|Y}(Z)), \phi_{y_0}^X(h^{X|Y}(Z'))\big) \cdot \exp\big(p\xi_L^{X|Y} + \xi_F^{X|Y}\big)\big(\psi_{y_0}^{X|Y}(Z)\big). \tag{5.10}$$

Remark also that in the notations of (2.2), (3.20), (3.21), by [17, (3.26)] and [9, (5.35)], we have

$$\kappa_{\phi,y_0}^X(Z) = 1 + O\big(|Z|^2\big), \qquad \kappa_{\psi,y_0}^{X|Y}(Z) = 1 - g_{y_0}^{TX}(\nu^{X|Y}, Z) + O\big(|Z|^2\big). \tag{5.11}$$

By Proposition 2.2 (5) and (5.11), we deduce that

$$\frac{\kappa_{\phi,y_0}^X}{\kappa_{\psi,y_0}^{X|Y}} = 1 + O\big(|Z|^2\big). \tag{5.12}$$

From Lemma 3.12, Corollary 5.3, (5.10), (5.12) and the trivial fact that for $Z = (0, Z_N)$, $Z_N \in \mathbb{R}^{2(n-m)}$, we have

$$\xi_L^{X|Y}(Z) = \xi_F^{X|Y}(Z) = 0, \quad h^{X|Y}(Z) = Z,$$

we deduce the result. ■

Compare the following result with Theorem 3.8.

Lemma 5.7. There are polynomials $J_{0,\mathrm{Res}}^{W|Y}(Z_W, Z'_Y)$, $J_{1,\mathrm{Res}}^{W|Y}(Z_W, Z'_Y)$ in $Z_W \in \mathbb{R}^{2l}$, $Z'_Y \in \mathbb{R}^{2m}$, such that for

$$F_{r,\mathrm{Res}}^{W|Y} := J_{r,\mathrm{Res}}^{W|Y} \cdot \mathcal{E}_{l,m}, \quad r = 0, 1,$$

the following holds. There are $\varepsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z_W = (Z_Y, Z_{N^{W|Y}})$, $Z_Y \in \mathbb{R}^{2m}$, $|Z_W|, |Z'_Y| \leq \varepsilon$, the Schwartz kernel of $\mathrm{Res}_W \circ \mathrm{E}_p^{X|Y}$, evaluated with respect to dv_Y, satisfies the following bound:

$$\begin{aligned}&\Bigg| \frac{1}{p^m} \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \left(\psi_{y_0}^{W|Y}(Z_W), \phi_{y_0}^Y(Z'_Y)\right) \\ &- \sum_{r=0}^{1} p^{-r/2} F_{r,\mathrm{Res}}^{W|Y}(\sqrt{p}Z_W, \sqrt{p}Z'_Y)\kappa_\psi^{W|Y}(Z_W)^{-1/2}\kappa_\phi^W(Z'_Y)^{-1/2} \Bigg| \\ &\quad \leq Cp^{-1}\left(1 + \sqrt{p}|Z_W| + \sqrt{p}|Z'_Y|\right)^Q \\ &\qquad \cdot \exp\left(-c\sqrt{p}\left(|Z_Y - Z'_Y| + |Z_{N^{W|Y}}|\right)\right).\end{aligned}$$

Moreover, we have

$$J_{0,\mathrm{Res}}^{W|Y}(Z_W, Z'_Y) = 1,$$

and for $Z_W = (0, Z_{N^{W|Y}})$, $Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)}$, in the notations of Lemma 3.1, we have

$$\mathcal{K}_{l,l}[1, J_{1,\mathrm{Res}}^{W|Y}](Z_W, 0) = J_{1,\mathrm{Res}}^{W|Y}(Z_W, 0), \quad \mathcal{K}_{l,m}^{EP}[J_{1,\mathrm{Res}}^{W|Y}, 1] = J_{1,\mathrm{Res}}^{W|Y}. \tag{5.13}$$

Proof. The existence of polynomials was proved in (4.8). The calculation of $J_{0,\mathrm{Res}}^{W|Y}$ was included in (4.9). Now, to prove (5.13), we first remark that

$$B_p^W \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} = \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y}, \quad \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} = \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \circ B_p^Y. \tag{5.14}$$

Comparing the first order asymptotics of each side of (5.14), using Lemma 3.1 and the analysis similar to the one before (3.32), gives

$$\mathcal{K}_{l,l}[1, J_{1,\mathrm{Res}}^{W|Y}] + \mathcal{K}_{l,l}[J_1^{W|Y}, 1] = J_{1,\mathrm{Res}}^{W|Y},$$
$$\mathcal{K}_{l,m}^{EP}[J_{1,\mathrm{Res}}^{W|Y}, 1] + \mathcal{K}_{l,m}^{EP}[1, J_1^{Y|Y}] = J_{1,\mathrm{Res}}^{W|Y}.$$

However, an easy calculation, using (3.14), (3.17) and (3.24), shows that for $Z_W = (0, Z_{N^{W|Y}})$, $Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)}$, we have

$$\mathcal{K}_{l,l}[J_1^{W|Y}, 1](Z_W, 0) = 0, \quad \mathcal{K}_{l,m}^{EP}[1, J_1^{Y|Y}] = 0,$$

which obviously finishes the proof. ∎

Proof of Theorem 1.5. First of all, remark that in Theorem 4.1 (2), we already established that the sequence of operators $T_p^{W|Y}$, $p \geq p_1$, from (5.3), forms a Toeplitz operator with weak exponential decay with respect to X of type $W|Y$, and the identity $[T_p^{W|Y}]_0 = 0$ holds. From this, Theorem 4.1 (6) and (5.4), we obtain that the sequence of operators D_p, $p \in \mathbb{N}$, form a Toeplitz operator with exponential decay of type $X|Y$, and the identity $[D_p]_0 = 0$ holds. We see also that it suffices to prove that under the assumption (1.8) and $dv_W = dv_{g^{TW}}$, we have $[T_p^{W|Y}]_1 = 0$, $[T_p^{W|Y}]_2 = 0$, the polynomial $[T_p^{W|Y}]_3$ has degree 1, and for any $n \in (N^{W|Y})^{(1,0)}$, we have

$$[T_p^{W|Y}]_3 \cdot n = \frac{1}{8\pi} \frac{\partial}{\partial n} (\mathbf{r}^X - \mathbf{r}^W) \cdot \mathrm{Id}_F - \frac{1}{2\pi\sqrt{-1}} \nabla_n^{\mathrm{End}(F)} \big(\Lambda_\omega[R^F] - \Lambda_{\iota_2^*\omega}[R^F]\big). \tag{5.15}$$

Let us now establish all those statements. In what follows, we assume (1.8) as well as $dv_W = dv_{g^{TW}}$.

To simplify further presentation, we define $f \in \mathscr{C}^\infty(W, \iota_2^* F)$, $g \in \mathscr{C}^\infty(Y, \iota^* F)$, as follows

$$f := \Big[\frac{1}{p^{n-l}} A_p^{X|W}\Big]_1, \quad g := \Big[\frac{1}{p^{l-m}} A_p^{W|Y}\Big]_1 - \Big[\frac{1}{p^{n-m}} A_p^{X|Y}\Big]_1.$$

Remark that both f and g are self-adjoint due to Theorem 5.1, cf. Remark 5.4. From Theorem 5.1, we obtain the following identities

$$f = -g = \frac{1}{8\pi}(\mathbf{r}_{y_0}^X - \mathbf{r}_{y_0}^W) - \frac{1}{2\pi\sqrt{-1}} \big(\Lambda_\omega[R^F] - \Lambda_{\iota_2^*\omega}[R^F]\big). \tag{5.16}$$

Clearly, from Lemmas 4.5 and 5.5, the sequences of operators

$$T_{p,1}^W := B_p^W - \Big(\frac{1}{p^{n-l}} (A_p^{X|W})^*\Big)^{-1},$$
$$T_{p,2}^Y := B_p^Y - \Big(\frac{1}{p^{n-m}} A_p^{X|Y}\Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y}\Big)^{-1}, \quad p \in \mathbb{N}^*,$$

form Toeplitz operators with exponential decay, and we have

$$[T_{p,1}^W]_0 = 0, \quad [T_{p,1}^W]_1 = f, \quad [T_{p,2}^Y]_0 = 0, \quad [T_{p,2}^Y]_1 = g. \tag{5.17}$$

We now denote

$$T_{p,0}^{W|Y} := T_{f,p}^{W} \circ \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} + \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} \circ T_{g,p}^{Y}. \tag{5.18}$$

From Corollary 2.8, Lemma 5.5 and (5.17), we deduce that the Schwartz kernels $T_p^{W|Y}(x, y)$, $T_{p,0}^{W|Y}(x, y)$; $x \in W$, $y \in Y$, of $T_p^{W|Y}$, $T_{p,0}^{W|Y}$, evaluated with respect to dv_Y, are related by

$$\left| T_p^{W|Y}(x, y) - \frac{1}{p} T_{p,0}^{W|Y}(x, y) \right| \leq C p^{m-2} \cdot \exp\big(-c\sqrt{p} \cdot \mathrm{dist}_X(x, y)\big). \tag{5.19}$$

From Lemmas 3.1, 5.6 and 5.7, and (5.18), we see that there are polynomials

$$J_{0,0}^{W|Y}(Z_W, Z'_Y), \quad J_{0,1}^{X|Y}(Z_W, Z'_Y),$$
$$Z_W = (Z_Y, Z_{N^{W|Y}}), \quad Z_Y, Z'_Y \in \mathbb{R}^{2m}, \quad Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)},$$

verifying

$$\begin{aligned} J_{0,0}^{W|Y} &:= J_{0,f}^{W|Y} \cdot J_{0,\mathrm{Res}}^{W|Y} + J_{0,\mathrm{Res}}^{W|Y} \cdot J_{0,g}^{Y}, \\ J_{0,1}^{W|Y}(Z_W, 0) &:= \sum_{i=0}^{1} \mathcal{K}_{l,l}[J_{i,f}^{W|Y}, J_{1-i,\mathrm{Res}}^{W|Y}](Z_W, 0) \\ &\qquad + \sum_{i=0}^{1} \mathcal{K}_{l,m}^{EP}[J_{i,\mathrm{Res}}^{W|Y}, J_{1-i,g}^{Y}](Z_W, 0), \end{aligned} \tag{5.20}$$

such that for

$$F_{0,r}^{W|Y} := J_{0,r}^{W|Y} \cdot \mathcal{E}_{m,l}, \quad r = 0, 1,$$

the following holds. There are $\varepsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z_W \in \mathbb{R}^{2l}$, $|Z_W|, |Z'_Y| \leq \varepsilon$, the following bound holds

$$\begin{aligned} &\Bigg| \frac{1}{p^m} T_{p,0}^{W|Y}\big(\psi_{y_0}^{W|Y}(Z_W), \phi_{y_0}^{Y}(Z'_Y)\big) \\ &\quad - \sum_{r=0}^{1} p^{-r/2} F_{0,r}^{W|Y}(\sqrt{p}Z_W, \sqrt{p}Z'_Y) \kappa_{\psi}^{W|Y}(Z_W)^{-1/2} \kappa_{\phi}^{Y}(Z'_Y)^{-1/2} \Bigg| \\ &\qquad \leq C p^{-1}\big(1 + \sqrt{p}|Z_W| + \sqrt{p}|Z'_Y|\big)^Q \\ &\qquad\quad \cdot \exp\big(-c\sqrt{p}\big(|Z_Y - Z'_Y| + |Z_{N^{W|Y}}|\big)\big). \end{aligned} \tag{5.21}$$

From Lemmas 5.6, 5.7, (3.24) and (5.20), for $Z_W = (0, Z_{N^{W|Y}})$, $Z_{N^{W|Y}} \in \mathbb{R}^{2(l-m)}$, we deduce that

$$J_{0,0}^{W|Y} := f + g, \quad J_{0,1}^{W|Y}(Z_W, 0) := \nabla_{\frac{\partial}{\partial z_W}}^{\mathrm{End}(F)} f. \tag{5.22}$$

Now, from (5.19) and (5.21), we deduce that $[T_p^{W|Y}]_1 = 0$. Remark now that by (5.16), we have $f + g = 0$. From this, (5.19), (5.21) and (5.22), we deduce that

$$[T_p^{W|Y}]_2 = 0.$$

Finally, from (5.19), (5.21), (5.22) and the last part of Lemma 3.12, we deduce (5.15), which finishes the proof of Theorem 1.5, as we explained before (5.15). ■

Let us now generalize Theorem 1.5 to the tower of embeddings of an arbitrary length. We fix a tower of embeddings

$$Y \overset{\iota_1}{\hookrightarrow} W_1 \overset{\iota_2}{\hookrightarrow} \cdots \overset{\iota_r}{\hookrightarrow} W_r \overset{\iota_{r+1}}{\longhookrightarrow} X, \quad \iota := \iota_{r+1} \circ \cdots \circ \iota_1,$$

and volume forms dv_{W_i} on W_i, $i = 1, \ldots, r$, verifying assumptions similar to (1.3) with respect to the metric g^{TW_i} induced by g^{TX}. We assume that the triples

$$(X, W_r, g^{TX}), \ldots, (W_{i+1}, W_i, g^{TW_{i+1}}), (W_1, Y, g^{TW_1}), \quad i = 1, \ldots, r-1,$$

are of bounded geometry in the sense of Definition 2.3.

Corollary 5.8. *The sequence of operators*

$$D_{p,r} := \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W_r} \circ \mathrm{E}_p^{W_r|W_{r-1}} \circ \cdots \circ \mathrm{E}_p^{W_2|W_1} \circ \mathrm{E}_p^{W_1|Y}, \quad p \in \mathbb{N},$$

forms a Toeplitz operator with exponential decay of type $X|Y$. Moreover, we have

$$[D_{p,r}]_0 = 0.$$

Also, under assumptions (1.8) and $dv_{W_i} = dv_{g^{TW_i}}$, $i = 1, \ldots r$, we have

$$[D_{p,r}]_1 = 0, \quad [D_{p,r}]_2 = 0, \quad [D_{p,r}]_3 \in \mathscr{C}_b^\infty\big(Y, (N^{X|Y})^{(1,0)*} \otimes \mathrm{End}(\iota^* F)\big)$$

for $n \in (N^{X|Y})^{(1,0)}$, we have

$$[D_{p,r}]_3 \cdot n = \sum_{i=1}^{r} \Big\{ \frac{1}{8\pi} \frac{\partial}{\partial n_i} \cdot (\mathbf{r}^{W_{i+1}} - \mathbf{r}^{W_i}) + \frac{\sqrt{-1}}{2\pi} \nabla_{n_i}^{\mathrm{End}(F)} \big(\Lambda_{(\iota^{i+1})^*\omega}[R^F] - \Lambda_{(\iota^i)^*\omega}[R^F]\big) \Big\},$$

where we denoted $W_{r+1} := X$, $W_0 := Y$; $\iota^i : W_i \to X$ is defined as $\iota^i := \iota_r \circ \cdots \circ \iota_{i+1}$, and $n_i := P_N^{W_i|W_{i-1}} n$, $i = 1, \ldots, r$.

Proof. Let us rewrite $D_{p,r}$ in the following way:

$$D_{p,r} := \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W_r} \circ \mathrm{E}_p^{W_r|Y} + \mathrm{E}_p^{X|W_r} \circ \Big(\mathrm{E}_p^{W_r|Y} - \mathrm{E}_p^{W_r|W_{r-1}} \circ \cdots \circ \mathrm{E}_p^{W_2|W_1} \circ \mathrm{E}_p^{W_1|Y} \Big).$$

Now, the result follows directly from Theorems 1.5 and 4.1 (6) by induction. ∎

Proof of Theorem 1.1. It follows directly from Theorems 1.5, 3.14, Remark 3.15 (a) and Proposition 3.18. ∎

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