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The special Brauer group and twisted Picard varieties

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Abstract. We generalise the notion of the Tate–Šafarevič group $Ш(S/\mathbb{P}^1)$ of an elliptic K3 surface $S \to \mathbb{P}^1$ with a section to $Ш(S,h)$ of a K3 surface S endowed with a linear system $|h|$. The construction, which uses Grothendieck's special Brauer group, provides an efficient way to deal with moduli spaces of twisted sheaves supported on curves in $|h|$.

1. Introduction

Let $S_0 \to \mathbb{P}^1$ be an elliptic K3 surface with a section. Another elliptic K3 surface $S \to \mathbb{P}^1$ (without a section) is called a twist of S_0 if its Jacobian fibration is $S_0 \to \mathbb{P}^1$, i.e., $S_0 \simeq \overline{\mathrm{Pic}}^0(S/\mathbb{P}^1)$ relative over $\mathbb{P}^1$. The twists of a fixed $S_0 \to \mathbb{P}^1$ are parametrised by the Tate–Šafarevič group $Ш(S_0/\mathbb{P}^1)$. According to a result originally due to Grothendieck [6] and Artin–Tate [19], this group is naturally isomorphic to the Brauer group, so $Ш(S_0/\mathbb{P}^1) \simeq \mathrm{Br}(S_0)$.

Changing perspective, every twist $S \to \mathbb{P}^1$ of $S_0 \to \mathbb{P}^1$ can be viewed as a moduli space of rank one sheaves on the fibres of $S_0 \to \mathbb{P}^1$ twisted by some class $\alpha \in \mathrm{Br}(S_0)$. We will rephrase this by writing $S \to \mathbb{P}^1$ as $\overline{\mathrm{Pic}}^0_\alpha(S_0/\mathbb{P}^1)$ for some Brauer class $\alpha \in \mathrm{Br}(S_0)$.

1.1.

In this article, we generalise the classical picture and consider twisted Picard varieties for arbitrary base point free, complete linear systems $\mathcal{C} \to |h|$ of curves contained in a K3 surface S. More specifically, this will lead us to consider twisted relative Jacobians $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ of the family $\mathcal{C} \to |h|_{\mathrm{sm}}$ of all smooth curves in $|h|$, generalising the classical relative Picard varieties $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$. However, in general the twists are not indexed by elements in the Brauer group $\mathrm{Br}(S)$ but by elements in a certain extension of it.

Theorem 1.1. *Consider a base point free, complete linear system $\mathcal{C} \to |h|$ on a* K3 *surface S. Then there exists a natural finite cyclic extension $Ш(S,h)$ (the Tate–Šafarevič group):*

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow Ш(S,h) \longrightarrow \mathrm{Br}(S) \longrightarrow 0$$

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that parametrises (possibly non-effectively) all $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$*-torsors isomorphic to a twisted relative Jacobian* $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$.

Furthermore, the product in $Ш(S,h)$ *corresponds to the products of twisted Picard varieties viewed as torsors for* $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}}) \to |h|_{\mathrm{sm}}$*, see Remarks* 3.9 *and* 4.8.

The integer m in the theorem is the *divisibility* of h as an element in the lattice $\mathrm{NS}(S)$, so by definition m satisfies $m\mathbb{Z} = (\mathrm{NS}(S).h)$, and we will later see that also all $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ for non-zero d are taken care of by $Ш(S,h)$, see Proposition 4.9.

This new Tate–Šafarevič group $Ш(S,h)$ extends the notion of the classical Tate–Šafarevič group of an elliptic K3 surface $S_0 \to \mathbb{P}^1$ with a section to an arbitrary elliptic K3 surface $S \to \mathbb{P}^1$. It turns out that there exists a natural isomorphism

$$Ш(S,f) \simeq Ш(S_0/\mathbb{P}^1)$$

between the new Tate–Šafarevič group $Ш(S,f)$ and the classical one $Ш(S_0/\mathbb{P}^1)$ of its Jacobian fibration $S_0 = \overline{\mathrm{Pic}}^0(S/\mathbb{P}^1)$, see Section 5.1.

1.2.

Working with twisted sheaves poses a number of technical problems. Firstly, in order to talk about twisted sheaves it is not enough to just fix a Brauer class $\alpha \in \mathrm{Br}(S)$. A certain geometric realisation is needed. This could be an Azumaya algebra, a gerbe, a Brauer–Severi variety, or a Čech cocycle. Secondly, to deal with moduli spaces, certain numerical invariants, e.g., Chern classes or Mukai vectors, need to be fixed. For example, one cannot define, without introducing a certain ambiguity, the degree of an α-twisted sheaf on a fibre of $S \to \mathbb{P}^1$. As was shown by Lieblich [13, Ch. 5], the ambiguity can be lifted by passing to μ_n-gerbes, which is roughly the same as lifting a Brauer class to the special Brauer group. Working with the special Brauer group allows us to deal with all Brauer classes at the same time.

To address these issues we make use of Grothendieck's special Brauer group $\mathrm{SBr}(S)$ which is a certain extension of $\mathrm{Br}(S)$ by $\mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$, cf. [5]. However, it will turn out that it is more convenient to work with a smaller subgroup $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S)$, the restricted special Brauer group, which is a natural extension of $\mathrm{Br}(S)$ by the discriminant group of $\mathrm{NS}(S)$:

$$0 \longrightarrow \mathrm{NS}(S)^*/\,\mathrm{NS}(S) \longrightarrow \mathrm{SBr}^{\mathrm{o}}(S) \longrightarrow \mathrm{Br}(S) \longrightarrow 0.$$

The various Tate–Šafarevič groups $Ш(S,h)$ of curve classes on S are then constructed as certain quotients $\mathrm{SBr}^{\mathrm{o}}(S) \twoheadrightarrow Ш(S,h)$.

1.3.

To get an idea of the role of the special Brauer group, let us look at the case of a smooth projective integral curve C over an arbitrary field k of characteristic zero. In this case, there exists a short exact sequence of the form

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{SBr}(C) \longrightarrow \mathrm{Br}(C) \longrightarrow 0.$$

In addition to the classical Picard varieties $\mathrm{Pic}^d(C)$, $d \in \mathbb{Z}$, which are torsors for $\mathrm{Pic}^0(C)$, one can define varieties $\mathrm{Pic}^d_\alpha(C)$ for any class $\alpha \in \mathrm{SBr}(C)$ and any rational number $d \in \mathbb{Q}$. If not empty, they are again torsors for $\mathrm{Pic}^0(C)$. Furthermore, if $\alpha \in \mathrm{SBr}(C)$ is contained in the subgroup $\mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(C)$, then $\mathrm{Pic}^d_\alpha(C)$ is a trivial torsor. The converse holds if there exists a rational point $x \in C(k)$ such that the restriction $\alpha_x \in \mathrm{Br}(k(x) \simeq k)$ is trivial. See the discussion in Section 3.1, especially Remark 3.6 (i), for further details.

1.4.

Our discussion also sheds a new light on work of Markman [16]. Building upon his work, we will prove the following result in Section 5.3.

Theorem 1.2. *Let $X \to \mathbb{P}^n$ be Lagrangian fibration of a projective hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type and of Picard rank two. Then there exists a* K3 *surface S, a complete linear system $\mathcal{C} \to |h|$ on S, and a class $\alpha \in Ш(S,h)$ such that X is birational to* $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ *relative over $|h| \simeq \mathbb{P}^n$.*

The assumption on the Picard number can be weakened to a genericity assumption, cf. Section 5.3. If X is not assumed projective, the K3 surface S may be non-projective and for the class α one may have to use an analytic version of the special Brauer group.

Conventions. In most of this article, we deliberately restrict to complex projective K3 surfaces, but there are interesting questions to explore for more general types of varieties as well as in more arithmetic settings. In our discussion it will sometimes be convenient to deal with the scheme-theoretic generic curve in a linear system on a complex projective K3 surface, in which case we work with curves over function fields $\mathbb{C}(t_1, \ldots, t_n)$.

For simplicity we will always assume that the complete linear system $\mathcal{C} \to |h|$ parametrises at least one smooth curve or, equivalently, that the generic fibre is smooth. The cases of interest to us are ample complete linear systems and elliptic pencils.

For an Azumaya algebra $\mathcal{A}$ we call $d(\mathcal{A}) := \sqrt{\mathrm{rk}(\mathcal{A})}$ the degree of $\mathcal{A}$.

2. Grothendieck's special Brauer group

The special Brauer group introduced by Grothendieck in [5, Rem. 3.9] is an extension of the classical Brauer group. In this section, we recall its definition and its cohomological description and explain how to use it to define Chern characters of twisted sheaves. In the next section, we will then explain in what sense it is better suited to study moduli spaces of twisted sheaves.

2.1. Hodge theory

Assume X is a smooth complex projective variety. We are mainly interested in the case of K3 surfaces, but ultimately we will want to apply everything also to projective hyperkähler manifolds.

By $\mathrm{NS}(X)$ we denote the Néron–Severi group of X and by $T(X)$ its transcendental lattice, i.e., the smallest saturated sub-Hodge structure of $H^2(X,\mathbb{Z})$ with $H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$. The inclusion $\mathrm{NS}(X) \oplus T(X) \subset H^2(X,\mathbb{Z})$ is rarely an equality but always of finite index. For simplicity we will ignore any torsion in $H^2(X,\mathbb{Z})$. Next, we define $T'(X)$ by the short exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow T'(X) \longrightarrow 0. \tag{2.1}$$

Then the composition $T(X) \subset H^2(X,\mathbb{Z}) \twoheadrightarrow T'(X)$ realizes the transcendental lattice as a subgroup $T(X) \subset T'(X)$ of finite index. The quotient is a finite group, which we will denote

$$A(X) := T'(X)/T(X).$$

For a surface S, the unimodular intersection form provides a natural identification $T'(S) \simeq T(S)^*$ and $A(S)$ is nothing but the discriminant group of the transcendental lattice or, equivalently, of the Néron–Severi lattice:

$$A(S) \simeq T(S)^*/T(S) \simeq \mathrm{NS}(S)^*/\mathrm{NS}(S).$$

Tensoring the exact sequence $0 \to T(X) \to T'(X) \to A(X) \to 0$ with $\mathbb{Q}/\mathbb{Z}$ induces an exact sequence, cf. [4, Sec. 5.3] or [7, (9) and Cor. 1.5]:

$$0 \longrightarrow A(X) \longrightarrow T(X) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow T'(X) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{2.2}$$

where we identified $\mathrm{Tor}_1(A(X), \mathbb{Q}/\mathbb{Z})$ with $A(X)$. The inclusion is explicitly described by first lifting an element in $A(X)$ to a class in $T'(X)$, which is unique up to elements in $T(X)$, and then viewing it as an element in $T(X) \otimes \mathbb{Q} = T'(X) \otimes \mathbb{Q}$.

Remark 2.1. Under our assumptions, a result of Gabber and de Jong, cf. [4, Ch. 4], shows that the Brauer group equals the cohomological Brauer group, so $\mathrm{Br}(X) \simeq H^2_{\text{ét}}(X, \mathbb{G}_m)$. The latter group can be identified with the torsion subgroup $H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}} \subset H^2(X, \mathcal{O}_X^*)$, using the analytic topology. Furthermore, the exponential sequence provides an exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \longrightarrow H^3(X,\mathbb{Z})$$

and in particular both groups $T(X) \subset T'(X)$ can be viewed as subgroups of $H^2(X, \mathcal{O}_X)$.

If $H^3(X,\mathbb{Z}) = 0$, then

$$\mathrm{Br}(X) \simeq H^2(X,\mathcal{O}_X^*)_{\mathrm{tor}} \simeq \big(H^2(X,\mathcal{O}_X)/T'(X)\big)_{\mathrm{tor}} \simeq T'(X) \otimes \mathbb{Q}/\mathbb{Z}, \tag{2.3}$$

which for a surface S, using $T(S)^* \simeq T'(S)$, is often written as

$$\mathrm{Br}(S) \simeq \mathrm{Hom}\big(T(S), \mathbb{Q}/\mathbb{Z}\big).$$

Without any assumption on $H^3(X,\mathbb{Z})$, the isomorphism (2.3) only describes the divisible part of $\mathrm{Br}(X)$.

2.2. Special Brauer group

In the very last remark of [5], Grothendieck introduced the special Brauer group $\mathrm{SBr}(X)$ of a scheme X. It seems that this group has not been used much explicitly over the last fifty years, mainly because of its purely topological character. However, it reflects a point of view that is central for Lieblich's treatment of twisted sheaves, see e.g., [13], and is exactly what is needed for our purposes. The definition is similar to the definition of the Brauer group $\mathrm{Br}(X)$ as the group of Morita equivalence classes of Azumaya algebras.

Definition 2.2. The *special Brauer group* $\mathrm{SBr}(X)$ of a scheme X is the group of equivalence classes of Azumaya algebras $\mathcal{A}$ with respect to the equivalence relation generated by $\mathcal{A} \sim \mathcal{A} \otimes \mathcal{E}nd(F)$ with the locally free sheaf F required to have trivial determinant $\det(F) \simeq \mathcal{O}_X$.

Remark 2.3. As the special Brauer group is not often discussed in the literature, we add a few comments.

(i) In order to ensure that with this definition the tensor product $\mathcal{A} \otimes \mathcal{A}'$ defines a group structure on $\mathrm{SBr}(X)$ one needs to use the fact that any Azumaya algebra $\mathcal{A}$ has trivial determinant.

(ii) Variants of the above definition exist. For example, instead of requiring $\det(F)$ to be trivial one can ask it to be only torsion or algebraically (or cohomologically) trivial. It turns out that all these conditions eventually lead to the same group.

By construction, $\mathrm{SBr}(X)$ naturally surjects onto $\mathrm{Br}(X)$ and the kernel has been determined in [5, Rem. 3.9]: For X over a field of characteristic zero, there exists a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{SBr}(X) \longrightarrow \mathrm{Br}(X) \longrightarrow 0. \tag{2.4}$$

As explained in [5, Rem. 3.9], the cohomological version of (2.4) is obtained as the limit of the exact sequences

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \cdot \mathrm{Pic}(X) \longrightarrow H^2(X, \mu_{\ell^n}) \longrightarrow\!\!\!\!\rightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)[\ell^n] \longrightarrow 0$$

induced by the Kummer sequence. In particular, whenever $\mathrm{Br}(X) \xrightarrow{\sim} H^2_{\text{ét}}(X, \mathbb{G}_m)$, then

$$\mathrm{SBr}(X)[\ell^\infty] \simeq H^2\big(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\big) = \varinjlim H^2(X, \mu_{\ell^n}).$$

If we assume that X is a smooth complex projective variety, we can use singular cohomology to describe the situation. One finds

$$\mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{NS}(X) \otimes \mathbb{Q}/\mathbb{Z} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus \rho(X)}$$

and

$$\mathrm{SBr}(X) \simeq H^2(X, \mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus b_2(X)} \oplus H^3(X, \mathbb{Z})_{\text{tors}}. \tag{2.5}$$

If $H^3(X,\mathbb{Z})_{\text{tors}} = 0$, the sequence (2.4) is obtained from (2.1) by tensoring with $\mathbb{Q}/\mathbb{Z}$:

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}(X)\otimes\mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{SBr}(X) \longrightarrow \mathrm{Br}(X) \longrightarrow 0 \\ \simeq \mathrm{NS}(X)\otimes\mathbb{Q}/\mathbb{Z} \simeq H^2(X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} \simeq T'(X)\otimes\mathbb{Q}/\mathbb{Z}. \end{aligned} \tag{2.6}$$

Remark 2.4. The Brauer group $\mathrm{Br}(X) \simeq H^2(X,\mathbb{G}_m)$ can also be viewed as the set of isomorphism classes of $\mathbb{G}_m$-gerbes on X. Similarly, $H^2(X,\mu_n)$ is the set of isomorphism classes of μ_n-gerbes on X. From this perspective, $\mathrm{SBr}(X)$ is the set of all μ_n-gerbes on X, where a μ_n-gerbe is identified with the naturally associated μ_{nk}-gerbe under $\mu_n \subset \mu_{nk}$.

Remark 2.5. Here are a few further comments on the special Brauer group and its links to μ_n-gerbes and the Picard group.

(i) The injection in (2.6) is geometrically realised by sending $(1/r)\cdot L$ with $L \in \mathrm{Pic}(X)$ to the Azumaya algebra given by $\mathcal{E}nd(F)$, where F is any vector bundle of rank r and determinant L. Use $\mathcal{E}nd(F_1)\otimes\mathcal{E}nd(F_1^*\otimes F_2) \simeq \mathcal{E}nd(F_2)\otimes\mathcal{E}nd(F_1\otimes F_1^*)$ to see that this is independent of the choice of F.

(ii) This description fits with the interpretation of $\mathrm{SBr}(X)$ as parametrising μ_n-gerbes. The μ_n-gerbe associated with $\mathcal{E}nd(F)$, i.e., its image under $H^1(X,\mathrm{PGl}(r))\to H^2(X,\mu_n)$, is the image of $L \in \mathrm{Pic}(X) = H^1(X,\mathbb{G}_m)$ under the boundary map

$$\delta_r\colon H^1(X,\mathbb{G}_m) \longrightarrow H^2(X,\mu_r)$$

induced by the Kummer sequence.

(iii) As for the classical Brauer group, one checks that if $\alpha \in \mathrm{SBr}(X)$ is represented by an Azumaya algebra $\mathcal{A}$, then the order of α as an element in the group $\mathrm{SBr}(X)$ divides $d(\mathcal{A}) = \sqrt{\mathrm{rk}(\mathcal{A})}$. For example, if $\mathcal{A} = \mathcal{E}nd(F)$ with $\mathrm{rk}(F) = r$, then α^r is realised by the endomorphism bundle of $F^{\otimes r}\otimes\det(F)^*$ which has trivial determinant.

The cohomological description (2.5) of the special Brauer group reveals that $\mathrm{SBr}(X)$ is a purely topological invariant of X, unlike the standard version $\mathrm{Br}(X)$, which depends on the Picard number and thus may change under deformations. The exact sequence (2.6) provides a geometric interpretation for the link between jumps of the Picard number and drops of the rank of the Brauer group.

Note that both sequences, (2.4) and (2.6), actually split, but only non-canonically.

Remark 2.6. For a K3 surface S, it is common to call a lift of a class $\alpha \in \mathrm{Br}(S)$ to an element in $H^2(S,\mathbb{Q})$ a *B-field* lift of α. Such a B-field lift then induces a lift of α to a class in the special Brauer group $\mathrm{SBr}(S)$, see Section 4.6 for more details.

Example 2.7. For a smooth projective irreducible curve C over an arbitrary field k of characteristic zero, the sequence (2.4) becomes

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{SBr}(C) \longrightarrow \mathrm{Br}(C) \longrightarrow 0.$$

If k is algebraically closed, then $\mathrm{Br}(C)$ is trivial and $\mathbb{Q}/\mathbb{Z} \simeq \mathrm{SBr}(C)$. Although we mostly consider complex curves, the more general situation sheds light on the situation when applied to the scheme-theoretic generic fibre $\mathcal{C}_\eta$ of a linear system $\mathcal{C} \to |h|$ on a surface.

2.3. Chern character

Instead of looking at classes in the Brauer group $\mathrm{Br}(X)$ we now fix a class in the special Brauer group
$$\alpha \in \mathrm{SBr}(X).$$
Representing α by an Azumaya algebra $\mathcal{A}$, we can consider the abelian category $\mathrm{Coh}(X, \mathcal{A})$ (and its derived category). The same class α is also represented by any Azumaya algebra of the form $\mathcal{A}' := \mathcal{A} \otimes \mathcal{E}nd(F)$ with F locally free and such that $\det(F) \simeq \mathcal{O}_X$. The two categories are equivalent to each other via
$$\mathrm{Coh}(X, \mathcal{A}) \xrightarrow{\sim} \mathrm{Coh}(X, \mathcal{A}'), \quad E \longmapsto E \otimes F. \tag{2.7}$$
Since $\mathcal{E}nd(F') \simeq \mathcal{E}nd(F)$ if and only if $F' \simeq F \otimes L$ for some line bundle L which, moreover, is torsion if $\det(F') \simeq \det(F)$, the equivalence (2.7) is canonical up to equivalences given by tensoring with torsion line bundles. In particular, there is a distinguished equivalence (2.7) for K3 surfaces and hyperkähler manifolds.

Since X is assumed to be a smooth complex projective variety, we may use singular cohomology and Hodge theory to fix cohomological invariants, but the following definitions can be adapted to other situations. As in [9, 12], we consider the Chern character
$$\mathrm{ch}_{\mathcal{A}} \colon \mathrm{Coh}(X, \mathcal{A}) \longrightarrow H^*(X, \mathbb{Q}), \quad E \longmapsto \mathrm{ch}_{\mathcal{A}}(E) := \sqrt{\mathrm{ch}(\mathcal{A})}^{-1} \cdot \mathrm{ch}(E).$$
Since we will mostly work with sheaves on curves (contained in K3 surfaces), multiplication with $\sqrt{\mathrm{ch}(\mathcal{A})}^{-1}$ is just division by the integer $d(\mathcal{A})$. For K3 surfaces, it is often more convenient to work with the Mukai vector $v_{\mathcal{A}}(E) := \mathrm{ch}_{\mathcal{A}}(E) \cdot \mathrm{td}(S)^{1/2}$, which for sheaves concentrated on curves in the K3 surface is the same as $\mathrm{ch}_{\mathcal{A}}(E)$.

Note that on curves and surfaces $\mathrm{ch}_{\mathcal{A}}$ commutes with the equivalence (2.7). More precisely, if $\det(F) \simeq \mathcal{O}$ and $\mathcal{A}' = \mathcal{A} \otimes \mathcal{E}nd(F)$ then
$$\mathrm{ch}_{\mathcal{A}'}(E \otimes F) = \mathrm{ch}_{\mathcal{A}}(E),$$
for then $\mathrm{ch}(\mathcal{E}nd(F)) = \mathrm{ch}(F^*) \cdot \mathrm{ch}(F) = \mathrm{ch}(F)^2$.

Definition-Proposition 2.8. *Let $\alpha \in \mathrm{SBr}(X)$ be a class in the special Brauer group on a variety of dimension at most two. Then the* twisted Chern character
$$\mathrm{ch}_\alpha(E) := \mathrm{ch}_{\mathcal{A}}(E)$$
for $E \in \mathrm{Coh}(X, \alpha) := \mathrm{Coh}(X, \mathcal{A})$ is independent of the choice of the Azumaya algebra $\mathcal{A}$ representing the class $\alpha \in \mathrm{SBr}(X)$.

Remark 2.9. If $\alpha \in \mathrm{SBr}(X)$ is represented by an Azumaya algebra $\mathcal{A} \in H^1(X, \mathrm{PGl}(n))$, then $\mathrm{Coh}(X, \mathcal{A})$ is equivalent to the category $\mathrm{Coh}(\mathcal{M})_1$ of sheaves of weight one on the associated μ_n-gerbe $\mathcal{M}_{\mathcal{A}}$, cf. [14] and [9, Sec. 3] for further references. For two Azumaya algebras $\mathcal{A}, \mathcal{A}' \in H^1(X, \mathrm{PGl}(n))$ with isomorphic μ_n-gerbes $\mathcal{M}_{\mathcal{A}} \simeq \mathcal{M}_{\mathcal{A}'}$, i.e., inducing the same class in $H^2(X, \mu_n)$, one could work with the Chern character $\mathrm{ch}_{\mathcal{M}} = \mathrm{ch}_{\mathcal{M}'}$ on the DM stacks $\mathcal{M}_{\mathcal{A}} \simeq \mathcal{M}_{\mathcal{A}'}$ instead of ch_α.

However, when passing from the μ_n-gerbe $\mathcal{M}_{\mathcal{A}}$ associated with an Azumaya algebra $\mathcal{A}$ to the μ_{nr}-gerbe $\mathcal{M}_{\mathcal{A}'}$ associated with $\mathcal{A}' := \mathcal{A} \otimes \mathcal{E}nd(E)$ for some locally free sheaf E of rank r and trivial determinant, the situation becomes less clear. The induced map between the DM stacks $\mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}'}$ (over X), which is compatible with the natural inclusion $\mu_n \hookrightarrow \mu_{nr}$, will, by functoriality of the Chern character, pull-back $\mathrm{ch}_{\mathcal{M}_{\mathcal{A}'}}(F \otimes E)$ to $\mathrm{ch}_{\mathcal{M}_{\mathcal{A}}}(F)$. Here, $F \otimes E \in \mathrm{Coh}(X, \mathcal{A}') \simeq \mathrm{Coh}(\mathcal{M}_{\mathcal{A}'})_1$ is the image of $F \in \mathrm{Coh}(X, \mathcal{A}) \simeq \mathrm{Coh}(\mathcal{M}_{\mathcal{A}})_1$ under the equivalence

$$\mathrm{Coh}(X, \mathcal{A}) \xrightarrow{\sim} \mathrm{Coh}(X, \mathcal{A}'), \quad F \longmapsto F \otimes E.$$

Now, the fact that Proposition 2.8 only holds in dimension ≤ 2, shows that in general $\mathrm{ch}_{\mathcal{M}_{\mathcal{A}}}(F) \neq \mathrm{ch}_{\mathcal{A}}(F)$. Thus, expressing $\mathrm{ch}_{\mathcal{M}}(F)$ for $F \in \mathrm{Coh}(\mathcal{M})$ in terms of classical Chern classes of F viewed as coherent sheaf on X seems to involve first passing to the 'minimal' gerbe of $\mathcal{M}$. There one would expect $\mathrm{ch}_{\mathcal{M}}(F) = \mathrm{ch}_{\mathcal{A}}(F)$.

This has bearings on the comparison between moduli spaces of sheaves over Azumaya algebras and moduli spaces of sheaves on gerbes, see Remark 3.7.

3. Moduli spaces of twisted sheaves

Moduli spaces of (semi-)stable twisted sheaves have been constructed by Lieblich [14] and Yoshioka [20], for the construction from the point of view of modules over Azumaya algebras see [8, 17]. Most of the classical arguments carry over. However, it is important to emphasise, that the construction of the moduli space requires an additional choice, it is not enough to fix a Brauer class $\alpha \in \mathrm{Br}(S)$. For example, one possibility is to choose a Čech coycle $\{\alpha_{ijk}\}$ representing α, which allows one to use twisted sheaves. Another possibility is to fix a Brauer–Severi variety $P_\alpha \to S$ representing α, which is used in [20]. For us it will be more convenient to view these moduli spaces as moduli spaces of untwisted coherent sheaves E with a module structure over an Azumaya algebra $\mathcal{A}$ representing a given Brauer class α.

As we will only be interested in the birational type of moduli spaces, the difference between stability and semi-stability and the choice of a polarisation, already needed to define stability, are all irrelevant. However, the question whether a moduli space is a coarse or a fine moduli space is important. Over the stable locus, which is never empty in our setting, it is independent of the choice of a birational model.

3.1. Moduli on curves

Let us again first consider the case of a smooth projective irreducible curve C over an arbitrary field k of characteristic zero. The discussion should be compared to the one by Lieblich, e.g., [15, Sec. 3.2.2] using the language of sheaves on μ_n-gerbes, see also the introduction of Section 3.2.

Definition 3.1. For a class $\alpha \in \mathrm{SBr}(C)$ in the special Brauer group represented by an Azumaya algebra $\mathcal{A}$ and $r \in \mathbb{Z}$, $d \in \mathbb{Q}$ we denote by $M_\alpha(r,d)$ the moduli space of semi-stable vector bundles $E \in \mathrm{Coh}(C,\mathcal{A})$ with $\mathrm{ch}_{\mathcal{A}}(E) = (r,d)$. For the special case $r = 1$ we use the notation

$$\mathrm{Pic}^d_\alpha(C) := M_\alpha(1,d)$$

and call it the *twisted Picard group* of degree d.

There is no ambiguity in this definition, as for any other Azumaya algebra $\mathcal{A}'$ representing $\alpha \in \mathrm{SBr}(C)$ there exists an equivalence $\mathrm{Coh}(C,\mathcal{A}) \simeq \mathrm{Coh}(C,\mathcal{A}')$ unique up to tensoring with torsion line bundles which identifies semi-stable vector bundles of Mukai vector $\mathrm{ch}_{\mathcal{A}} = (r,d)$ in $\mathrm{Coh}(C,\mathcal{A})$ with those with $\mathrm{ch}_{\mathcal{A}'} = (r,d)$ in $\mathrm{Coh}(C,\mathcal{A}')$. However, the isomorphism between two moduli spaces defined by means of different Azumaya algebras representing the class in $\mathrm{SBr}(C)$ is natural only up to tensoring with torsion line bundles.

Note that the degree $d(\mathcal{A}) = \sqrt{\mathrm{rk}(\mathcal{A})}$ of $\mathcal{A}$ divides the rank $\mathrm{rk}(E)$ (as a coherent sheaf) of any locally free sheaf $E \in \mathrm{Coh}(C,\mathcal{A})$. Therefore, the constant coefficient of $\mathrm{ch}_{\mathcal{A}}(E)$ is indeed an integer, which explains why we fixed $r \in \mathbb{Z}$. However, this is not true for the degree d, which is usually only contained in $(1/d(\mathcal{A}))\mathbb{Z}$.

Remark 3.2. If $E \in \mathrm{Coh}(C,\mathcal{A})$ is (semi-)stable with $\mathrm{ch}_{\mathcal{A}}(E) = (r,d)$, then also any line bundle twist $E \otimes L \in \mathrm{Coh}(C,\mathcal{A})$ is (semi-)stable and $\mathrm{ch}_{\mathcal{A}}(E \otimes L) = (r, d + r \cdot \deg(L))$. So, if there exists a line bundle L of degree one on C, then for fixed r every moduli space $M_\alpha(r,d)$ is isomorphic to one with $d \in (1/d(\mathcal{A}))\mathbb{Z} \cap [0,r)$. If we allow ourselves to twist only with line bundles of degree in $m\mathbb{Z}$, then $[0,r)$ has to be replaced by $[0, m \cdot r)$.

Remark 3.3. Here are some comments on twisted Picard schemes.

(i) All non-empty $\mathrm{Pic}^d_\alpha(C)$, $d \in \mathbb{Q}$, $\alpha \in \mathrm{SBr}(C)$ are torsors for $\mathrm{Pic}^0(C)$ and this torsor structure does not depend on the choice of the Azumaya algebra either. If k is algebraically closed, all non-empty $\mathrm{Pic}^d_\alpha(C)$ are trivial torsors and, therefore, non-naturally isomorphic to $\mathrm{Pic}^0(C)$.

(ii) Assume that there exists an $E \in \mathrm{Coh}(C,\mathcal{A})$ defining a point in $\mathrm{Pic}^d_\alpha(C)$. Then $\mathrm{rk}(E) = d(\mathcal{A})$ and, therefore, the natural injection $\mathcal{A} \hookrightarrow \mathcal{E}nd(E)$ is an isomorphism. For the latter, use that both sheaves are locally free of the same rank and with trivial determinant. In particular,

$$\alpha \in \mathrm{Pic}(C) \otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(C).$$

If, moreover, there exists a line bundle L on C with $L^{\mathrm{rk}(E)} \simeq \det(E)$, then $\mathcal{A} \simeq \mathcal{E}nd(E \otimes L^*)$ with $\det(E \otimes L^*) \simeq \mathcal{O}_X$ and, hence, $\alpha \in \mathrm{SBr}(C)$ is trivial. In particular, for k algebraically closed, the non-emptyness of $\mathrm{Pic}^d_\alpha(C)$ with d an integer(!) implies that $\alpha \in \mathrm{SBr}(C) \simeq \mathbb{Q}/\mathbb{Z}$ is trivial, because then $\mathrm{rk}(E) \mid \deg(E)$.

This is no longer true for $d \notin \mathbb{Z}$. Indeed, any vector bundle F of rank r can be viewed as a stable sheaf over $\mathcal{A} = \mathcal{E}nd(F)$ with $\mathrm{ch}_{\mathcal{A}}(F) = 1 + \mu(F)$, i.e., $F \in \mathrm{Pic}^{\mu(F)}_\alpha(C)$ for $\alpha = [\mathcal{A}] \in \mathrm{Pic}(C) \otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(C)$.

Let us now only fix an ordinary Brauer class $\alpha \in \mathrm{Br}(C)$. Then the moduli space $M_\alpha(r)$ of all stable twisted sheaves of rank r is still well defined but only locally of finite type. For example,

$$\mathrm{Pic}_\alpha(C) = M_\alpha(1)$$

parametrises all locally free $E \in \mathrm{Coh}(C, \mathcal{A})$ of rank $\mathrm{rk}(E) = d(\mathcal{A})$. The definition does not depend on the choice of the Azumaya algebra $\mathcal{A}$ representing $\alpha \in \mathrm{Br}(C)$.

Note that $\mathrm{Pic}_\alpha(C)$ is a countable disjoint union of smooth projective varieties which are torsors for $\mathrm{Pic}^0(C)$. For later use we state this as the following.

Proposition 3.4. *The Picard scheme* $\mathrm{Pic}_\alpha(C)$ *is always non-empty and it is naturally a torsor for the group scheme* $\mathrm{Pic}(C) = \bigsqcup \mathrm{Pic}^d(C)$*, where the action is given by tensor product. If* $\tilde{\alpha} \in \mathrm{SBr}(C)$ *is a lift of* $\alpha \in \mathrm{Br}(C)$ *and* $\mathrm{Pic}^0_{\tilde{\alpha}}(C)$ *is not empty, then*

$$\mathrm{Pic}_\alpha(C) \simeq \big(\mathrm{Pic}(C) \times \mathrm{Pic}^0_{\tilde{\alpha}}(C)\big)/\mathrm{Pic}^0(C)$$

as torsors for $\mathrm{Pic}(C)$.

Proof. The twisted Picard scheme $\mathrm{Pic}_\alpha(C)$ is always non-empty, which can be verified after passing to the algebraic closure of the base field. Indeed, for k algebraically closed, $\mathrm{Br}(C)$ is trivial and for $\tilde{\alpha} = (1/r) \cdot L \in \mathrm{NS}(C) \otimes \mathbb{Q}/\mathbb{Z}$ the bundle $F = L \oplus \mathcal{O}_C^{\oplus r-1}$ defines a point in $\mathrm{Pic}^d_{\tilde{\alpha}}(C)$ with $d = \deg(L)/r$. ■

We shall need the following general fact, cf. [13, Sec. 5.1.3] or [15, Prop. 3.2.2.6] for the version for μ_n-gerbes. For completeness sake, we include a sketch of the proof. As is common, we will write $\bar{C}$ for the base change of C over k to some algebraic closure of k.

Proposition 3.5. *Let* C *be a smooth projective curve over a field* k*. Then the map*

$$\mathrm{Br}(C) \simeq H^2_{ét}(C, \mathbb{G}_m) \longrightarrow H^1\big(k, \mathrm{Pic}(\bar{C})\big) \tag{3.1}$$

induced by the Hochschild–Serre spectral sequence maps a Brauer class $\alpha \in \mathrm{Br}(C)$ *to the class of the* $\mathrm{Pic}(C)$*-torsor* $\mathrm{Pic}_\alpha(C)$.

There is a version of this result for arbitrary schemes X, but the map (3.1) is then only defined on the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(\bar{X})$.

Proof. The map (3.1) is part of the commutative diagram

$$\begin{array}{ccc}
\mathrm{Br}(C) \simeq H^2_{\text{ét}}(C, \mathbb{G}_m) & \longrightarrow & H^1\big(k, H^1_{\text{ét}}(\bar{C}, \mathbb{G}_m)\big) \\
\uparrow & & \uparrow \\
H^1\big(C, \mathrm{PGL}(n)\big) & \longrightarrow & H^0\big(k, H^1\big(\bar{C}, \mathrm{PGL}(n)\big)\big),
\end{array}$$

where the right vertical map is the boundary map induced by the short exact sequence

$$0 \longrightarrow \mathrm{Pic}(\bar{C}) \simeq H^1(\bar{C}, \mathbb{G}_m) \longrightarrow H^1\big(\bar{C}, \mathrm{GL}(n)\big) \longrightarrow H^1\big(\bar{C}, \mathrm{PGL}(n)\big) \longrightarrow 0.$$

Computing this boundary map explicitly shows that $\mathcal{A} \in H^1(C, \mathrm{PGL}(n))$ is mapped to the torsor which parametrises all $E \in \mathrm{Coh}(C, \mathcal{A})$ with $\mathcal{E}nd(E) \simeq \mathcal{A}$, which is nothing but $\mathrm{Pic}_\alpha(C)$. ∎

The reader will have noticed that we are using $\mathrm{Pic}(\bar{C})$ to denote the Picard group of $\bar{C}$, while at most other places, e.g., $\mathrm{Pic}(C)$ means the Picard scheme and $\mathrm{Pic}^0(C)$ the Jacobian variety. It should be clear from the context which is meant.

Remark 3.6. Let us elaborate on Remark 3.2.

(i) For simplicity we first assume that there exists a line bundle L_0 of degree one on C and so, in particular, $\mathrm{Pic}^1(C)(k)$ is not empty. Consider a class $\alpha = (p/r) \cdot L_0 \in \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(C)$ with $p, r \in \mathbb{Z}$ coprime, so that $|\alpha| = r$, and pick a locally free sheaf F on C with $\mathrm{rk}(F) = r$ and $\det(F) = L_0^p$. Then $\mathcal{A}_F := \mathcal{E}nd(F)$ represents the class $\alpha \in \mathrm{SBr}(C)$ and locally free sheaves $E \in \mathrm{Coh}(C, \mathcal{A}_F)$ with $\mathrm{ch}_{\mathcal{A}_F}(E) = (1, d)$ are all of the form $E \simeq F \otimes L$ for some line bundle L.

Since $(1/r)\deg(F \otimes L) = (p/r) + \deg(L)$, this shows that $d \equiv (p/r)$ modulo $\mathbb{Z}$, i.e., $\bar{d} = \alpha \in \mathbb{Q}/\mathbb{Z}$. In other words, up to isomorphisms induced by multiplication with line bundles there exists only one non-empty twisted Picard variety $\mathrm{Pic}_\alpha^\alpha(C)$ (admittedly, a somewhat confusing notation) which is furthermore non-naturally isomorphic to $\mathrm{Pic}^0(C)$.

(ii) Let us now consider the case that the minimal positive degree of a line bundle L_0 on C is m, also called the index of C. Similar arguments as above show the following: For a given $\alpha = (p/r) \cdot L_0 \in \mathrm{Pic}(C) \otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(C)$, there exist, up to tensor products with line bundles, at most m twisted Picard varieties $\mathrm{Pic}_\alpha^{d+i}(C)$, $i = 0, \ldots, m-1$, where $d = (m \cdot p/r)$, all torsors for $\mathrm{Pic}^0(C)$. In fact, there might be even fewer, as the index can be replaced by the period of the curve, i.e., the minimal d such that $\mathrm{Pic}^d(C)$ is a trivial torsor.

(iii) The two situations considered above will later be mixed as follows, cf. also [13, Sec. 5.1.2]. We will consider curves $C \subset S$ in a complex projective K3 surface. In this case, there clearly exists a line bundle of degree one on each individual C, so that we can consider $\alpha = (p/r) \cdot L_0 \in \mathrm{SBr}(C)$ with $\deg(L_0) = 1$. However, in order to let C vary in its linear system, we only allow twists by line bundles L of degree m, where m is determined by $(\mathrm{NS}(S).[C]) = m\mathbb{Z}$. Thus, one considers $\mathrm{Pic}_\alpha^{d+i}(C)$ with $d = (p/r)$ and $i = 0, \ldots, m-1$.

3.2. Moduli on surfaces

Let us now turn to sheaves on projective K3 surfaces over an algebraically closed field. Again, the discussion should be compared to the work of Lieblich, e.g., in [13, Sec. 5.1]. In particular, he already introduced twisted Picard schemes for fibred surfaces. Note that in our setting, the curves are not necessarily the fibres of a morphism but elements in a linear system.

According to Definition–Proposition 2.8, the twisted Chern character $\mathrm{ch}_\alpha(E)$ is independent of the choice of the Azumaya algebra, i.e., $\mathrm{ch}_{\mathcal{A}}(E) = \mathrm{ch}_{\mathcal{A}'}(E \otimes F)$ for $\mathcal{A}' =$

$\mathcal{A} \otimes \mathcal{E}nd(F)$ with $\det(F) \simeq \mathcal{O}_S$. Furthermore, the equivalence $\mathrm{Coh}(S, \mathcal{A}) \simeq \mathrm{Coh}(S, \mathcal{A}')$ preserves stability with respect to a polarisation $\mathcal{O}(1)$ which we will suppress in the notation.[1] In the case of K3 surfaces, it is more convenient to work with the (twisted) Mukai vector $v_\alpha(E) = \mathrm{ch}(E) \cdot \mathrm{td}(S)^{1/2}$, which is also well defined.

Hence, by [8, 17, 20] the moduli space $M_\alpha(v)$ of semi-stable α-twisted sheaves with twisted Mukai vector $v_\alpha(E) = v$ is well defined as long as $\alpha \in \mathrm{SBr}(S)$ is fixed as a class in the special Brauer group.

Remark 3.7. Once again, there should be a way to phrase everything in terms of Lieblich's moduli spaces of sheaves on μ_n-gerbes, cf. [14, 15]. In particular, if a class $\alpha \in \mathrm{SBr}(X)$ is represented by a μ_n-gerbe $\mathcal{M}$, one should be able to compare moduli spaces of sheaves on $\mathcal{M}$ with a certain moduli space of sheaves on the μ_{nk}-gerbe naturally associated with $\mathcal{M}$ via the inclusion $\mu_n \subset \mu_{nk}$. However, comparing Chern characters, and so Hilbert polynomials, and stability is tricky in general, see Remark 2.9 and [14, Lem. 2.3.2.8].

We will be mainly interested in the case of sheaves E on S supported on curves $C \subset S$ in a base point free linear system $|h|$ of dimension g, in which case there is no difference between the Mukai vector $v(E)$ and the Chern character $\mathrm{ch}(E)$. More precisely, the Mukai vector is in this case of the form $v(E) = (0, r \cdot h, s)$, where $s = \chi(E)$. If E is a line bundle on C, then $r = 1$. In general, if C is integral, r is the rank of E as a sheaf on C.

We can now twist the situation with respect to a class $\alpha \in \mathrm{SBr}(S)$ in the special Brauer group. Let us spell this out in detail. Choosing an Azumaya algebra $\mathcal{A}$ representing α, we consider all sheaves $E \in \mathrm{Coh}(S, \alpha) = \mathrm{Coh}(S, \mathcal{A})$ with $v_\alpha(E) = v_{\mathcal{A}}(E) = \mathrm{ch}_{\mathcal{A}}(E) = (0, r \cdot h, s)$, where $s = \chi(E)$.

Definition 3.8. Consider a K3 surface S with a base point free, complete linear system $\mathcal{C} \to |h|$ and a special Brauer class $\alpha \in \mathrm{SBr}(S)$. Then we denote the moduli space $M_\alpha(0, h, s)$ of all semi-stable sheaves $E \in \mathrm{Coh}(S, \alpha) = \mathrm{Coh}(S, \mathcal{A})$ with $v_\alpha(E) = v_{\mathcal{A}}(E) = (0, h, s)$ by

$$\overline{\mathrm{Pic}}^d_\alpha(\mathcal{C}/|h|) := M_\alpha(0, h, s),$$

with $d := s + (h.h)/2 = s + g - 1$, and call it the *compactified twisted relative Picard variety*.

We will abuse the notation slightly and denote by

$$\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \subset \overline{\mathrm{Pic}}^d_\alpha(\mathcal{C}/|h|)$$

the open subset $\pi^{-1}(|h|_{\mathrm{sm}})$ of twisted sheaves concentrated on smooth curves $C \in |h|$. Here,

$$\pi \colon \overline{\mathrm{Pic}}^d_\alpha(\mathcal{C}/|h|) \longrightarrow |h| \tag{3.2}$$

[1] Note that in general only μ-stability is preserved under this equivalence. However, on surfaces and under the assumption that F has trivial determinant, also Gieseker stability is preserved.

is the natural projection. Note that stability depends on the choice of a polarisation and so $\overline{\mathrm{Pic}}^d_\alpha(\mathcal{C}/|h|)$ depends on it as well. However, the open part $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ does not and as we are only interested in these moduli spaces up to birational isomorphism (over the linear system $|h|$), we can safely ignore the polarisation.

Remark 3.9. The fibre $\pi^{-1}(C)$ of (3.2) over a smooth curve $C \in |h|$ consists of all (stable) locally free $\mathcal{A}|_C$-sheaves of rank $d(\mathcal{A})$ and degree $d \cdot d(\mathcal{A})$. Thus, it is nothing but $\mathrm{Pic}^d_{\alpha|_C}(C)$ as introduced in the previous section. In particular, if not empty, the fibre is naturally a torsor for $\mathrm{Pic}^0(C)$. More globally, if not empty, $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|) \to |h|_{\mathrm{sm}}$ is a torsor for the abelian group scheme $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}}) \to |h|_{\mathrm{sm}}$.

Note that, according to Remark 3.3, if the fibre is non-empty and $d \in \mathbb{Z}$ is an integer, then $\alpha|_C = 1 \in \mathrm{SBr}(C)$.

Tensor product with a line bundle L on S with $(L.h) = m$ defines an isomorphism

$$\overline{\mathrm{Pic}}^d_\alpha(\mathcal{C}/|h|) \xrightarrow{\sim} \overline{\mathrm{Pic}}^{d+m}_\alpha(\mathcal{C}/|h|) \quad \text{and} \quad \mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \xrightarrow{\sim} \mathrm{Pic}^{d+m}_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}).$$

As in the case of curves, s and d need not be integers. But note that if for a fixed class $\alpha \in \mathrm{SBr}(S)$ and two $d, d' \in \mathbb{Q}$ the two relative twisted Picard varieties $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ and $\mathrm{Pic}^{d'}_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ are both not empty, then $d' = d + i$ for some integer i, see Remarks 3.2 and 3.6.

Remark 3.10. Using Remark 3.6, we find that up to tensoring with line bundles on S there are at most m α-twisted relative Picard schemes $\mathrm{Pic}^{d+i}_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ with $i = 0, \ldots, m-1$. Here, as before, m satisfies $m\mathbb{Z} = (\mathrm{NS}(S).h)$, i.e., it is the divisibility of h as an element of the lattices $\mathrm{NS}(S)$, and the rational number $d = p/r$ is determined by writing $\alpha|_C = (p/r) \cdot L_0$ for some line bundle L_0 of degree one on C and with coprime p and r.

In particular, unless $d = 0$, there is no preferred choice for the degree $d + i$ that would work well with the group structure of $\mathrm{SBr}(S)$. This observation will give rise to introducing the restricted special Brauer group in the next section.

Let us conclude this section by explaining a relative version of Remark 3.3.

Remark 3.11. If instead of a class in $\mathrm{SBr}(S)$ we only fix a Brauer class $\alpha \in \mathrm{Br}(S)$, then we define

$$\overline{\mathrm{Pic}}_\alpha(\mathcal{C}/|h|) \longrightarrow |h|$$

as the moduli space of all (semi-)stable sheaves $E \in \mathrm{Coh}(S, \mathcal{A})$ with Mukai vector

$$\mathrm{ch}_{\mathcal{A}}(E) = (0, h, *).$$

This is a countable disjoint union of projective schemes over $|h|$. The scheme-theoretic generic fibre is $\mathrm{Pic}_{\alpha|_{\mathcal{C}_\eta}}(\mathcal{C}_\eta)$, where $\mathcal{C}_\eta$ is the generic fibre of $\mathcal{C} \to |h|$.

Restricting to smooth curves in $|h|$, the scheme $\mathrm{Pic}_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \to |h|_{\mathrm{sm}}$ is a torsor for the countable union of projective group schemes

$$\mathrm{Pic}(\mathcal{C}/|h|_{\mathrm{sm}}) = \bigsqcup \mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}}) \longrightarrow |h|_{\mathrm{sm}}.$$

4. The restricted special Brauer group

From the perspective of moduli spaces of twisted sheaves (or, rather, of modules over Azumaya algebras) on curves contained in K3 surfaces, not all classes $\alpha \in \mathrm{SBr}(S)$ are relevant. Only classes in a drastically smaller group give naturally rise to non-empty moduli spaces, cf. Remarks 3.3 and 3.10. This leads to the notion of the restricted special Brauer group.

From now on, S will always be a complex projective K3 surface.

4.1. Restricted special Brauer group

Using the dual of the inclusion $\mathrm{NS}(S) \subset H^2(S,\mathbb{Z})$ and the unimodularity of $H^2(S,\mathbb{Z})$, one defines a natural map

$$\mathrm{SBr}(S) \simeq H^2(S,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{NS}(S)^* \otimes \mathbb{Q}/\mathbb{Z}. \tag{4.1}$$

The *restricted special Brauer group* $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S)$ is then defined as the kernel of this map. Equivalently, $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S)$ is the subgroup that annihilates $\mathrm{NS}(S)$ under the natural pairing

$$\mathrm{SBr}(S) \times \mathrm{NS}(S) \simeq H^2(S,\mathbb{Q}/\mathbb{Z}) \times \mathrm{NS}(S) \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{4.2}$$

The pairing can equivalently be viewed as induced by the intersection pairing on the second cohomology or by the restriction $\mathrm{SBr}(S) \times \mathrm{NS}(S) \to \mathbb{Q}/\mathbb{Z}$, $(\alpha, [C]) \mapsto \alpha|_C \in \mathrm{SBr}(C) \simeq \mathbb{Q}/\mathbb{Z}$.

From the definition of the restricted special Brauer group $\mathrm{SBr}^{\mathrm{o}}(S)$ we deduce the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(S) & \longrightarrow & \mathrm{SBr}^{\mathrm{o}}(S) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{SBr}(S) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & & & \\
 & & \mathrm{NS}(S)^* \otimes \mathbb{Q}/\mathbb{Z} & = & \mathrm{NS}(S)^* \otimes \mathbb{Q}/\mathbb{Z}. & & & &
\end{array} \tag{4.3}$$

It is not difficult to check that $\mathrm{SBr}^{\mathrm{o}}(S)$ still surjects onto $\mathrm{Br}(S)$.

The subgroup $\mathrm{SBr}^{\mathrm{o}}(S)$ parametrises all Brauer classes with respect to which every complete linear system can be twisted. In other words, $\mathrm{Pic}^0_\alpha(C)$ is non-empty for all smooth $C \subset S$ if and only if $\alpha \in \mathrm{SBr}^{\mathrm{o}}(S)$, use the arguments in the proof of Proposition 4.2 to see this. However, once a base point free, complete linear system $|h|$ on S is fixed, we replace $\mathrm{SBr}^{\mathrm{o}}(S)$ by a larger subgroup

$$\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S,h) \subset \mathrm{SBr}(S).$$

To define the *primitive special Brauer group* $\mathrm{SBr}(S,h)$, we consider again the pairing (4.2) and let $\mathrm{SBr}(S,h) \subset \mathrm{SBr}(S)$ be the annihilator of $h \in \mathrm{NS}(S)$, i.e., the set of all classes $\alpha \in \mathrm{SBr}(S) \simeq H^2(S,\mathbb{Q}/\mathbb{Z})$ with $(\alpha.h) = 0$ in $\mathbb{Q}/\mathbb{Z}$. (The analogy with the primitive cohomology $H^2(S,\mathbb{Z})_{\mathrm{pr}}$ was pointed out to us by E. Brakkee and A. Varilly-Alvarado.) Analogously, we define $A(S,h) \subset \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$ as the annihilator of $h \in \mathrm{NS}(S)$ with respect to the pairing

$$\big(\mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}\big) \times \mathrm{NS}(S) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Altogether, we have the commutative diagram of short exact sequences that completes (4.3):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(S) & \longrightarrow & \mathrm{SBr}^{\mathrm{o}}(S) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & A(S,h) & \longrightarrow & \mathrm{SBr}(S,h) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{SBr}(S) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0.
\end{array}$$

Again, the surjectivities on the right are straightforward to verify.

Remark 4.1. Viewing $\mathrm{SBr}(S)$ as the group that parametrises all μ_n-gerbes, $n \in \mathbb{Z}$, on S, see Remark 2.4, we find that $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S)$ is the subgroup of all μ_n-gerbes that become trivial on all smooth, integral curves $C \subset S$.

Indeed, $\mathrm{NS}(S)$ is generated by classes of smooth integral curves and for a class $\gamma \in H^2(S,\mathbb{Q})$ the condition $(\gamma.[C]) \in \mathbb{Z}$ is equivalent to $\gamma|_C \in H^2(C,\mathbb{Q})$ being contained in $H^2(C,\mathbb{Z})$. In other words, the special Brauer class induced by γ is contained in the kernel of

$$\mathrm{SBr}(S) \longrightarrow \mathrm{SBr}(C) \simeq H^2(C,\mathbb{Q}/\mathbb{Z}).$$

Similarly, $\mathrm{SBr}(S,h)$ is the set of classes $\alpha \in \mathrm{SBr}(S)$ with $\alpha|_C \in \mathrm{SBr}(C) \simeq H^2(C,\mathbb{Q}/\mathbb{Z})$ being trivial for all smooth curves $C \in |h|$.

4.2. Restricted Brauer group via moduli spaces

The restricted special Brauer group can be alternatively characterised by the non-emptiness of moduli spaces.

Proposition 4.2. *We assume again that $\mathcal{C} \to |h|$ is a base point free, complete linear system. Then $\mathrm{SBr}(S,h) \subset \mathrm{SBr}(S)$ is the subgroup of all special Brauer classes such that $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ is not empty.*

Proof. Let $\alpha \in \mathrm{SBr}(S,h)$ and pick any smooth curve $C \in |h|$. By Remark 4.1, the class $\alpha|_C \in \mathrm{SBr}(C)$ is trivial. Hence, $\alpha|_C$ can be represented by $\mathcal{O}_C$ and any degree zero line bundle, e.g., $\mathcal{O}_C$ itself, defines a point in $\mathrm{Pic}^0_{\alpha|_C}(C)$. In particular, $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \neq \emptyset$.

Conversely, if $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \neq \emptyset$ for a fixed class $\alpha \in \mathrm{SBr}(S)$, then according to Remark 3.3,

$$\alpha|_C \in \mathrm{SBr}(C)$$

is trivial for all smooth curves $C \in |h|$ and hence $\alpha \in \mathrm{SBr}(S, h)$. ∎

It is important to emphasise that although $\alpha|_C \in \mathrm{SBr}(C)$ is trivial for a class $\alpha \in \mathrm{SBr}(S, h)$ and any smooth curve $C \in |h|$, the restriction $\alpha|_{\mathcal{C}_\eta} \in \mathrm{SBr}(\mathcal{C}_\eta)$ to the generic fibre of the complete linear system $\mathcal{C} \to |h|$ is trivial if and only if α is of the form

$$(1/r) \cdot L \in \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$$

with $\deg(L|_C) = 0$. So, in particular, the associated class $\bar{\alpha} \in \mathrm{Br}(S)$ would be trivial in this case, for the restriction map $\mathrm{Br}(S) \hookrightarrow \mathrm{Br}(\mathcal{C}_\eta)$ is injective. For the latter use that $\mathcal{C} \to S$ is a Zariski locally trivial bundle and $\mathrm{Br}(\mathcal{C}_\eta)$ can understood as the limit of the Brauer groups $\mathrm{Br}(\mathcal{C}_U)$ over all open subsets $\mathcal{C}_U$, with $\emptyset \neq U \subset |h|$.

4.3. Generalised Tate–Šafarevič group

It turns out that for a given complete linear system $\mathcal{C} \to |h|$ the restricted special Brauer group $\mathrm{SBr}(S, h)$ does not parametrise the moduli spaces $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ effectively. We will show that the map that associates with a class $\alpha \in \mathrm{SBr}(S, h)$ the moduli space $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ factorises via a certain quotient of $\mathrm{SBr}(S) \twoheadrightarrow Ш(S, h)$.

For this purpose, we consider for any $h \in \mathrm{NS}(S)$ the natural map

$$\zeta_h\colon A(S, h) \twoheadrightarrow \mathbb{Z}/\big(\mathrm{NS}(S).h\big), \quad \varphi \longmapsto \varphi(h).$$

By the very definition of $A(S, h)$, this map is indeed surjective.

Definition 4.3. Let $\mathcal{C} \to |h|$ be a base point free, complete linear system on a K3 surface S. Then the *Tate–Šafarevič group* of (S, h) is defined as

$$Ш(S, h) := \mathrm{SBr}(S, h)/\ker(\zeta_h)$$

and we denote the projection by

$$\xi_h\colon \mathrm{SBr}(S, h) \twoheadrightarrow Ш(S, h). \tag{4.4}$$

Remark 4.4. Note that for h primitive also the restriction of ζ_h to $A(S) \subset A(S, h)$ is surjective, which has the consequence that in the definition of the generalised Tate–Šafarevič group $Ш(S, h)$, one could as well use the smaller $\mathrm{SBr}^{\mathrm{o}}(S)$. However, for $h = kh_0$ with h_0 primitive, the subgroup $\zeta_h(A(S))$ is of index k.

Thus, there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}/\big(\mathrm{NS}(S).h\big) \longrightarrow Ш(S, h) \longrightarrow \mathrm{Br}(S) \longrightarrow 0, \tag{4.5}$$

which together with (4.3) is part of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{NS}(S)\otimes\mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{SBr}(S) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & A(S,h) & \longrightarrow & \mathrm{SBr}(S,h) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0\\
 & & {\scriptstyle \xi_h}\downarrow & & {\scriptstyle \xi_h}\downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/\big(\mathrm{NS}(S).h\big) & \longrightarrow & Ш(S,h) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0.
\end{array} \tag{4.6}$$

Example 4.5. The natural projection is an isomorphism $Ш(S,h) \xrightarrow{\sim} \mathrm{Br}(S)$ if and only if $h \in \mathrm{NS}(S)$ has divisibility one, i.e., $(\mathrm{NS}(S).h) = \mathbb{Z}$. This is reminiscent of the case of elliptic K3 surfaces and will be discussed in detail further below.

Proposition 4.6. *Consider a base point free, complete linear system $\mathcal{C} \to |h|$. Then, for any two classes $\alpha_1, \alpha_2 \in \mathrm{SBr}(S,h)$ with $\xi_h(\alpha_1) = \xi_h(\alpha_2)$, the two moduli spaces $\mathrm{Pic}^0_{\alpha_1}(\mathcal{C}/|h|_{\mathrm{sm}})$ and $\mathrm{Pic}^0_{\alpha_2}(\mathcal{C}/|h|_{\mathrm{sm}})$ are naturally isomorphic torsors for $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$.*

Proof. Write $\alpha_2 = \alpha_1 \cdot \gamma$ with $\gamma = (1/r)\cdot L \in \mathrm{NS}(S)\otimes\mathbb{Q}/\mathbb{Z}$. If $\gamma \in A(S,h) \subset \mathrm{SBr}(S,h)$, then $r \mid (L.h)$, and under the stronger assumption $\gamma \in \ker(\zeta_h) \subset A(S,h)$, we find in addition a line bundle $L' \in \mathrm{NS}(S)$ such that $(L'.h) + (1/r)(L.h) = 0$. Thus, we can assume that γ is of the form $(1/r)\cdot L$ with $(L.h) = 0$.

Let now F be a locally free sheaf on S with $\det(F) \simeq L$ and $\mathrm{rk}(F) = r$, e.g., $F = L \oplus \mathcal{O}_S^{\oplus r-1}$. Then

$$\mathrm{Pic}^0_{\alpha_1}(\mathcal{C}/|h|_{\mathrm{sm}}) \xrightarrow{\sim} \mathrm{Pic}^0_{\alpha_2}(\mathcal{C}/|h|_{\mathrm{sm}}), \quad E \longmapsto E \otimes F$$

is an isomorphism of torsors for $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$. ∎

Remark 4.7. Consider a base point free, complete linear system $\mathcal{C} \to |h|$. Then, without any ambiguity, the proposition allows us to speak of the moduli spaces $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ for any class $\alpha \in Ш(S,h)$ in the Tate–Šafarevič group of (S,h).

Remark 4.8. As already hinted at in the last proof, the group structure of $Ш(S,h)$ as a quotient of $\mathrm{SBr}(S,h)$ is compatible with taking twisted Picard varieties. More precisely, for two classes α_1, α_2 the torsor $\mathrm{Pic}^0_{\alpha_1\alpha_2}(\mathcal{C}/|h|_{\mathrm{sm}})$ for $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$, is the quotient

$$\mathrm{Pic}^0_{\alpha_1\alpha_2}(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \big(\mathrm{Pic}^0_{\alpha_1}(\mathcal{C}/|h|_{\mathrm{sm}}) \times_{|h|} \mathrm{Pic}^0_{\alpha_2}(\mathcal{C}/|h|_{\mathrm{sm}})\big)/\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}}),$$

where the isomorphism is given by tensor product.

4.4. All twisted Picard varieties

Although we only consider classes in the restricted special Brauer group $\mathrm{SBr}(S,h) \subset \mathrm{SBr}(S)$ and only consider $d = 0$, we still get all twisted Picard varieties of arbitrary degree for $\mathcal{C} \to |h|_{\mathrm{sm}}$. This is the next result which morally is a consequence of the surjectivity $\mathrm{SBr}(S,h) \twoheadrightarrow \mathrm{Br}(S)$.

Proposition 4.9. *Consider a base point free, complete linear system* $\mathcal{C} \to |h|$. *Fix* $d \in \mathbb{Q}$ *and* $\alpha \in \mathrm{SBr}(S)$, *such that* $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ *is non-empty. Then there exists a class* $\alpha_0 \in \mathrm{SBr}(S,h)$ *(or* $\alpha_0 \in \text{Ш}(S,h)$*) and a natural isomorphism of* $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$*-torsors*

$$\mathrm{Pic}^0_{\alpha_0}(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}).$$

Proof. To ease the notation, we will simply write Pic^d_α instead of $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$, etc.

As a warmup, let us first discuss the special case that $\alpha = (1/r)\cdot L \in \mathrm{NS}(S)\otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(S)$ and $d = (L.h)/r$. Then we let α_0 be the trivial class and define

$$\mathrm{Pic}^0_{\alpha_0} = \mathrm{Pic}^0 \xrightarrow{\sim} \mathrm{Pic}^d_\alpha, \quad E \longmapsto E \otimes F.$$

Here, F is a locally free sheaf on S of rank r and determinant L, hence $\alpha = [\mathcal{E}nd(F)]$, and E is a degree zero line bundle on some smooth $C \in |h|$. In particular, the tensor product with F is actually the tensor product with $F|_C$ on C, cf. Remarks 3.3 and 3.6. The existence of the above isomorphism is of course equivalent to saying that the Pic^0-torsor Pic^d_α is trivial, as it admits the section provided by the restriction of F.

This proves the result for $\alpha \in \mathrm{NS}(S)\otimes\mathbb{Q}/\mathbb{Z}$, but only for one particular d. To obtain all $d+i$, one has to work with non-trivial classes $\alpha_0 \in A(S,h)$. To be precise, fix a line bundle L_0 on S with $(L_0.h)\mathbb{Z} = (\mathrm{NS}(S).h)$ and let $\alpha_0 = (i/(L_0.h))\cdot L_0 \in A(S,h)$. Pick a locally free sheaf F_0 with $\mathrm{rk}(F_0) = (L_0.h)$ and $\det(F_0) = L_0^i$. Then for $\alpha_0 := [\mathcal{E}nd(F_0)]$ one has

$$\mathrm{Pic}^0_{\alpha_0} \xrightarrow{\sim} \mathrm{Pic}^{d+i}_\alpha \quad \text{via } E \longmapsto E \otimes F_0.$$

Note that in this case, the torsor $\mathrm{Pic}^{d+i}_\alpha$ is not necessarily trivial anymore.

For the general case, write any $\alpha \in \mathrm{SBr}(S)$ as a product $\alpha = \alpha_0\cdot\alpha_1$ with $\alpha_0 \in \mathrm{SBr}(S,h)$ and $\alpha_1 \in \mathrm{NS}(S)\otimes\mathbb{Q}/\mathbb{Z}$. Once the decomposition is picked, we first deal with $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ for one choice of d and then show how to modify the decomposition $\alpha = \alpha_0\cdot\alpha_1$ to $\alpha = (\alpha_0\cdot\gamma^{-1})\cdot(\gamma\cdot\alpha_1)$ by some $\gamma \in A(S,h)$ to obtain all $d+i$, cf. Remark 3.9.

For the fixed choice of $\alpha_0 = [\mathcal{A}_0]$ and $\alpha_1 = [\mathcal{E}nd(F)]$, one has

$$\mathrm{Pic}^0_{\alpha_0} \xrightarrow{\sim} \mathrm{Pic}^d_\alpha, \quad E \longmapsto E \otimes F$$

with $d = \mu(F|_C)$. Now let $\gamma \in A(S,h)$ be the class $(i/(L_0.h))\cdot L_0$ and choose F_0 as above. Then $\mathrm{Pic}^0_{\alpha_0\cdot\gamma^{-1}} \xrightarrow{\sim} \mathrm{Pic}^{d+i}_\alpha$ via $E \mapsto E\otimes(F\otimes F_0)$. ∎

Remark 4.10. We reiterate, see beginning Section 4.3, that for h primitive $\text{Ш}(S,h)$ can also be defined as the quotient of the smaller subgroup $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}(S,h)$ by the kernel of the restriction $\zeta_h\colon A(S) \to \mathbb{Z}/(\mathrm{NS}(S).h)$, cf. Remark 4.4. In particular, as can also be seen by going through the above proof, the α_0 in the proposition can be chosen to be contained in $\mathrm{SBr}^{\mathrm{o}}(S)$.

Note that for $h = k\cdot h_0$, one can view $\text{Ш}(S,h_0)$ as a subgroup of $\text{Ш}(S,h)$ with a cyclic quotient of order k. This inclusion maps $\mathrm{Pic}^1(\mathcal{C}/|h_0|_{\mathrm{sm}})$ to $\mathrm{Pic}^k(\mathcal{C}/|h|_{\mathrm{sm}})$.

Example 4.11. In particular, all the untwisted relative Picard varieties $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$ are still accounted for, namely

$$\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}}).$$

Here, $\bar{d} \in \mathbb{Z}/(\mathrm{NS}(S).h)$ is defined as the image of $(d/(L_0.h))L_0 \in A(S,h) \subset \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$, where $L_0 \in \mathrm{NS}(S)$ satisfies $(L_0.h)\mathbb{Z} = (\mathrm{NS}(S).h)$. Hence, $\mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}})$ is a moduli space in $\mathrm{Coh}(S, \mathcal{E}nd(F))$ with F any locally free sheaf of rank $(L_0.h)$ and determinant L_0^{-d}. To conclude, use that $\deg(F|_C \otimes L) = -(L_0^d.C) + d \cdot \mathrm{rk}(F) = 0$ which leads to an isomorphism

$$\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}}) \xrightarrow{\sim} \mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}}), \quad L \longmapsto F|_C \otimes L.$$

The discussion so far says that

$$\begin{aligned} Ш(S,h) &\longrightarrow H^1\big(|h|_{\mathrm{sm}}, \mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})\big) \simeq \{\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})\text{-torsors}\}/_{\simeq} \\ \alpha &\longmapsto \mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}}) \end{aligned} \tag{4.7}$$

is a group homomorphism whose image consists of all torsors of the form $\mathrm{Pic}^d_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$.

Ideally, one would like to prove the injectivity of (4.7).[2] For this, one would need to show that if for a class $\alpha \in \mathrm{SBr}(S,h)$ the torsor $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ is trivial, then α is contained in the subgroup $\ker(\xi_h) = \ker(\zeta_h)$. Let us first explain the two problems one has to overcome and then explain in Proposition 4.12 how to deal with them under the assumption that the order of $\alpha \in \mathrm{SBr}(S,h)$ is sufficiently large.

(i) Assume $\alpha \in \mathrm{SBr}(S,h)$ such that $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ is a trivial torsor. Then the pull-back $\bar{\alpha}|_{\mathcal{C}} \in \mathrm{Br}(\mathcal{C})$ of $\bar{\alpha} \in \mathrm{Br}(S)$ to the smooth part $\pi\colon \mathcal{C} \to |h|_{\mathrm{sm}}$ is contained in the kernel of the natural map $\mathrm{Br}(\mathcal{C}) \to H^1(|h|_{\mathrm{sm}}, R^1\pi_*\mathbb{G}_m)$. This kernel is a quotient of $\mathrm{Br}(|h|_{\mathrm{sm}})$ which might be non-trivial. If $\bar{\alpha}|_{\mathcal{C}} \in \mathrm{Br}(\mathcal{C})$ could be shown to be trivial, then the injectivity of $\mathrm{Br}(S) \hookrightarrow \mathrm{Br}(\mathcal{C})$ would prove that also $\bar{\alpha} \in \mathrm{Br}(S)$ is trivial.

(ii) Assuming that (i) has been carried out, then we know already that α is contained in

$$A(S,h) = \ker\big(\mathrm{SBr}(S,h) \longrightarrow \mathrm{Br}(S)\big).$$

As we are only interested in the image of α in the group $\mathbb{Z}/(\mathrm{NS}(S).h) \subset Ш(S,h)$, we may assume that α is of the form $(d/(L_0.h))L_0$, where $L_0 \in \mathrm{NS}(S)$ with $(L_0.h)\mathbb{Z} = (\mathrm{NS}(S).h)$. Then, $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ is isomorphic to the untwisted torsor $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$, see Example 4.11 below for details. Since we are assuming that $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ is trivial, both torsors, $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ and, hence also, $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$, admit sections.

Suppose now that the section of $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$ corresponds to a line bundle $\mathcal{L}$ on $\mathcal{C} \to |h|_{\mathrm{sm}}$ (this is a priori obstructed by a class in $\mathrm{Br}(|h|_{\mathrm{sm}})$.) Then extending $\mathcal{L}$ to the

[2] The fact that we prove in the proposition below the injectivity only for classes of sufficiently high order is the reason for the 'possibly non effectively' in Theorem 1.1.

complete family $\overline{\mathcal{C}} \to |h|$ and using that $\overline{\mathcal{C}} \to S$ is a projective bundle, we may write $\mathcal{L}$ as a pull-back of a line bundle L on S up to twists by the tautological bundle on $|h|$. But then $(L.h) = d$ is a multiple of $(L_0.h)$ and, therefore, the class of α in $\mathbb{Z}/(L_0.h)\mathbb{Z}$ is trivial.

Summarising, in both steps, (i) and (ii), there is potentially an obstruction in $\mathrm{Br}(|h|_{\mathrm{sm}})$ to carry out the program and prove the injectivity of (4.7). Note that for elliptic K3 surfaces, i.e., $|h|$ is of dimension one, the Brauer group $\mathrm{Br}(|h|_{\mathrm{sm}})$ is trivial and hence (4.7) injective, see Proposition 5.4. But we expect that both obstructions vanish also for higher-dimensional linear systems, for which one would need to extend them to $|h|$ and then use that $\mathrm{Br}(|h|)$ is trivial.

Proposition 4.12. *The order of every element in the kernel of* (4.7) *divides m^2, where m is the positive generator of* $(\mathrm{NS}(S).h)$.

Proof. Assume $\alpha \in \mathrm{SBr}(S,h)$ is an element in the kernel of (4.7). The first part of the following argument works for any $\alpha \in \mathrm{SBr}(S,h)$

The relative moduli space $\mathrm{Pic}^0_\alpha := \mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ over $P = |h|_{\mathrm{sm}}$ is not necessarily fine. So, a universal bundle $\mathcal{P} \to \mathrm{Pic}^0_\alpha \times_P \mathcal{C}$ only exists as a $(\beta,\bar{\alpha})$-twisted sheaf for some $\beta \in \mathrm{Br}(\mathrm{Pic}^0_\alpha)$. Here, $(\beta,\bar{\alpha})$ denotes the product of β and $\bar{\alpha}$ pulled back to $\mathrm{Pic}^0_\alpha \times_P \mathcal{C}$ via the two projections. Note that the order of β divides m. Indeed, any divisor D on S with $(D.h) = m$ defines a relative cycle $\widetilde{P}$ of degree m of $\pi\colon \mathcal{C} \to P$ on which every Brauer class on $\mathcal{C}$ coming from S vanishes. This applies to α and implies that the base change $\mathrm{Pic}^0_\alpha \times_P \widetilde{P}$ is a fine moduli space and hence $|\beta|$ divides m. In fact, the argument shows $\mathrm{ind}(\beta) \mid m$, but we will not need this.

On the other hand, we assume that $\mathrm{Pic}^0_\alpha \to P$ is a trivial torsor and, therefore, comes with a section $\sigma\colon P \to \mathrm{Pic}^0_\alpha$. The restriction of $\mathcal{P}$ to $\sigma(P) \times_P \mathcal{C}$ is then locally free of rank one twisted with respect to $(\beta,\bar{\alpha})|_{\sigma(P)\times_P\mathcal{C}} = \pi^*(\beta|_{\sigma(P)=P}) \cdot \bar{\alpha}$. In other words, $\bar{\alpha} = \pi^*(\beta|_{\sigma(P)})^{-1}$ and, therefore, the order of $\bar{\alpha} \in \mathrm{Br}(S)$ divides m. Hence, the order of $\alpha \in \mathrm{SBr}(S)$ divides m^2. ∎

Note that in the final part of the proof, proving injectivity under the assumption on the order of α, we only had to address (i). The hypothesis of Proposition 4.12 can be slightly weakened by also addressing (ii) while using that the untwisted Pic^d (as an open subset of the moduli space $M(0,h,d+(1-g))$ on S) is a fine moduli space if $d+(1-g)$ and $2g-2$ are coprime (or, slightly weaker, if $d+(1-g)$ and m are coprime). We leave this to the reader.

Proof of Theorem 1.1. We conclude the proof of Theorem 1.1 by combining Proposition 4.6 and Remarks 4.7 and 4.8. ∎

4.5. Analytic Tate–Šafarevič group

Before continuing our discussion we make a digression on the analytic version of our construction. We introduce the analytic Brauer of a complex projective K3 surface S and compare it to a construction of Markman [16], cf. [2]. This will not be used in the rest of the paper. We will restrict to the case that h is primitive, which allows us to view $\text{Ш}(S,h)$ as a quotient of $\mathrm{SBr}^{\mathrm{o}}(S)$, cf. Section 4.3 and Remark 4.10.

In our context, it seems natural to introduce the *analytic special Brauer group* as

$$\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} := H^2(S,\mathcal{O}_S)/T(S).$$

It comes with a surjection onto the analytic Brauer group

$$\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} = H^2(S,\mathcal{O}_S)/T(S) \twoheadrightarrow \mathrm{Br}(S)^{\mathrm{an}} := H^2(S,\mathcal{O}_S^*)$$

and naturally contains the (algebraic) special Brauer group as the subgroup of all torsion elements $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$. We have the natural commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & A(S) & \longrightarrow & \mathrm{SBr}^{\mathrm{o}}(S) \simeq T(S)\otimes\mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{Br}(S) \simeq T(S)^*\otimes\mathbb{Q}/\mathbb{Z} & \longrightarrow 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \longrightarrow & A(S) & \longrightarrow & \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} \simeq H^2(S,\mathcal{O}_S)/T(S) & \longrightarrow & \mathrm{Br}(S)^{\mathrm{an}} = H^2(S,\mathcal{O}_S^*) & \longrightarrow 0.
\end{array}$$

Also, analogously to (4.4), one can define the analytic version $\text{Ш}(S,h)^{\mathrm{an}}$ of $\text{Ш}(S,h)$ as a quotient $\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} \twoheadrightarrow \text{Ш}(S,h)^{\mathrm{an}}$. Then $\text{Ш}(S,h) \subset \text{Ш}(S,h)^{\mathrm{an}}$ is the torsion subgroup. The corresponding commutative diagram is

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathbb{Z}/\big(\mathrm{NS}(S).h\big) & \longrightarrow & \text{Ш}(S,h) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathbb{Z}/\big(\mathrm{NS}(S).h\big) & \longrightarrow & \text{Ш}(S,h)^{\mathrm{an}} & \longrightarrow & \mathrm{Br}(S)^{\mathrm{an}} & \longrightarrow 0.
\end{array}$$

It is possible to introduce $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$ for any class in the analytic special Brauer $\alpha \in \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$ or $\alpha \in \text{Ш}(S,h)^{\mathrm{an}}$. However, if α is not torsion, then it will be non-algebraic and, in particular, it is more difficult to talk about its generic fibre. In other words, only a bimeromorphic equivalence class of a complex manifold together with a holomorphic projection to $|h|$ will be defined.

Markman introduces a group called Ш^0, see [16, (7.7)]. He identifies it with a certain finite quotient of $\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} = H^2(S,\mathcal{O}_S)/T(S)$ by enlarging the transcendental lattice $T(S)$ by all classes in $T(S)\otimes\mathbb{Q}$ that can be completed by elements in $\mathrm{NS}(S)\otimes\mathbb{Q}$ to integral classes in $H^2(S,\mathbb{Z})$ that are orthogonal to all irreducible components of all curves $C \in |h|$. In particular, if S has Picard number one, then $\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} = \text{Ш}^0$, but in general Ш^0 depends on the choice of h and approximates our analytic Tate–Šafarevič group $\text{Ш}(S,h)^{\mathrm{an}}$. More precisely, the analytic version of the quotient (4.4) factors through Ш^0:

$$\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}} \twoheadrightarrow \text{Ш}^0 \twoheadrightarrow \text{Ш}(S,h)^{\mathrm{an}}.$$

For non-primitive complete linear system $h = k\cdot h_0$ the discussion shows that the torsion group of Markman's Ш^0 maps onto the subgroup $\text{Ш}(S,h_0) \subset \text{Ш}(S,h)$ of index k, cf. Remark 4.10.

4.6. Digression on B-fields

We briefly come back to Remark 2.6. Assume $\alpha \in \mathrm{Br}(S) \simeq H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}} \simeq T'(S) \otimes \mathbb{Q}/\mathbb{Z}$ and pick a B-field lift of α, i.e., a class $B \in H^2(S, \mathbb{Q})$ such that its image under the exponential map $H^2(S, \mathbb{Q}) \to H^2(S, \mathcal{O}_S) \to H^2(S, \mathcal{O}_S^*)$, or, equivalently, under the projection $H^2(S, \mathbb{Q}) \twoheadrightarrow T'(S) \otimes \mathbb{Q} \twoheadrightarrow T'(S) \otimes \mathbb{Q}/\mathbb{Z}$, gives back α.

Using the decomposition $H^2(S, \mathbb{Q}) = (\mathrm{NS}(S) \otimes \mathbb{Q}) \oplus (T(S) \otimes \mathbb{Q})$, the class B can also be projected onto a class in $T(S) \otimes \mathbb{Q}$ and then further onto a class $\tilde{\alpha} \in \mathrm{SBr}^{\mathrm{o}}(S) \simeq T(S) \otimes \mathbb{Q}/\mathbb{Z}$, which maps to α under $\mathrm{SBr}^{\mathrm{o}}(S) \twoheadrightarrow \mathrm{Br}(S)$. In general, picking a B-field lift for a class $\alpha \in \mathrm{Br}(S)$ is strictly more information than is actually needed. We will now explain why for most practical purposes, a lift to a class in $\mathrm{SBr}^{\mathrm{o}}(S)$ suffices.

(i) Via the exponential map, the class $\alpha \in \mathrm{SBr}(S)$ or rather the corresponding class in $H^2(S, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}$ maps to a class $\alpha^{0,2} \in H^{0,2}(S) = H^2(S, \mathcal{O}_S)$. This allows one to associate with a generator $\sigma \in H^{2,0}(S) = H^0(S, \omega_S)$ the class $\sigma_\alpha := \sigma + \sigma \wedge \alpha^{0,2} \in H^2(S, \mathbb{C}) \oplus H^4(S, \mathbb{C})$, so that we can define a Hodge structure,

$$\widetilde{H}(S, \alpha, \mathbb{Z})$$

of K3 type associated with $\alpha \in \mathrm{SBr}^{\mathrm{o}}(S)$ as follows: The underlying lattice is nothing but the extended Mukai lattice $\widetilde{H}(S, \mathbb{Z})$ and its $(2,0)$-part is spanned by σ_α. All other parts of the Hodge structure are then determined by the usual orthogonality requirements. See [9, 11].

(ii) Representing the class $\alpha \in \mathrm{SBr}(S)$ in the special Brauer group by an Azumaya algebra $\mathcal{A}$, allows us to realise the abelian category $\mathrm{Coh}(S, \alpha)$ as $\mathrm{Coh}(S, \mathcal{A})$. Then, for any $E \in \mathrm{Coh}(S, \alpha)$ we consider the Mukai vector $v_\alpha(E) = \mathrm{ch}_\alpha(E) \cdot \sqrt{\mathrm{td}(S)}$, well defined and independent of the choice of $\mathcal{A}$, see Definition–Proposition 2.8. But the Mukai vector is of type $(1,1)$ with respect to the untwisted Hodge structure and in general not with respect to the twisted Hodge structure. However, things are better in our situation, as indeed

$$v_\alpha(E) \in \widetilde{H}^{1,1}(S, \alpha, \mathbb{Q})$$

for sheaves $E \in \mathrm{Coh}(S, \alpha)$ supported on curves in S.

5. Elliptic and Lagrangian fibrations

We specialise to the case of elliptic K3 surfaces and link our theory to the classical Ogg–Tate–Šafarevič theory, see [10, Ch. 11] for comments and references. We wish to point out that moduli spaces of twisted line bundles on fibred surfaces have been studied by Lieblich in his thesis [13, Ch. 5].

For an elliptic K3 surface $S_0 \to \mathbb{P}^1$ with a section, the Tate–Šafarevič group $Ш(S_0/\mathbb{P}^1)$ parametrises elliptic K3 surfaces $S \to \mathbb{P}^1$ together with an isomorphism $S_0 \simeq \overline{\mathrm{Pic}}^0(S/\mathbb{P}^1)$ over $\mathbb{P}^1$. The surface S_0 is thus a moduli space of stable sheaves on S. However, usually it is only a coarse moduli space, i.e., a universal sheaf on $S_0 \times_{\mathbb{P}^1} S$ exists only as a twisted sheaf with respect to a Brauer class $\beta \in \mathrm{Br}(S_0)$. Turning this around, shows that the twists S parametrised by $\text{Ш}(S_0/\mathbb{P}^1)$ can be viewed as moduli spaces of twisted sheaves of rank one on the fibres of $S_0 \to \mathbb{P}^1$. One would like to write this last observation as $\overline{\mathrm{Pic}}^0_\beta(S_0) \simeq S$, although β seems to occur naturally as a class in $\mathrm{Br}(S_0)$ and not as a class in $\mathrm{SBr}^{\mathrm{o}}(S_0)$. We will now see how to resolve this.

5.1. Comparison with the classical Tate–Šafarevič group

We begin with an elliptic K3 surface $S \to \mathbb{P}^1$ without a section. We denote by f the class of a fibre and let m be such that $m\mathbb{Z} = (\mathrm{NS}(S).f)$. Then $S_0 := \overline{\mathrm{Pic}}^0(S/\mathbb{P}^1) = M(0, f, 0)$ is the relative Jacobian of S, which is an elliptic K3 surface $S_0 \to \mathbb{P}^1$ with a section. Here and in the following, we always tacitly choose a generic polarisation to define stability.

Denoting by f also the class of a fibre of $S_0 \to \mathbb{P}^1$, the existence of a section implies $(\mathrm{NS}(S_0).f) = \mathbb{Z}$. Then, by virtue of Remark 4.5, the natural projection in (4.6) is an isomorphism

$$\text{Ш}(S_0, f) \xrightarrow{\sim} \mathrm{Br}(S_0). \tag{5.1}$$

On the other hand, viewing $\mathrm{Pic}^0_\beta(S_0/\mathbb{P}^1)$ as a torsor for $\mathrm{Pic}^0(S_0/\mathbb{P}^1)$, which compactifies to S_0, defines a group homomorphism $\text{Ш}(S_0, f) \to \text{Ш}(S_0/\mathbb{P}^1)$, which is in fact an isomorphism.

In the following, $\overline{\mathrm{Pic}}^0_\beta(S_0/\mathbb{P}^1)$ is the smooth compactification provided by the moduli space of stable twisted sheaves, where stability is considered with respect to a generic polarisation.

Lemma 5.1. *Mapping $\beta \in \text{Ш}(S_0, f)$ to $\overline{\mathrm{Pic}}^0_\beta(S_0/\mathbb{P}^1)$ defines an isomorphism of groups*

$$\mathrm{Br}(S_0) \simeq \text{Ш}(S_0, f) \xrightarrow{\sim} \text{Ш}(S_0/\mathbb{P}^1). \tag{5.2}$$

Proof. Indeed, as recalled above, we know that every twist S is a moduli space of twisted sheaves of rank one on the fibres of $S_0 \to \mathbb{P}^1$. Due to the existence of the section, $\mathrm{Pic}^d_\beta(S_0/\mathbb{P}^1) \simeq \mathrm{Pic}^0_\beta(S_0/\mathbb{P}^1)$ for all d, so the map is surjective.

The injectivity is a consequence of the arguments before Proposition 4.12. ∎

Combining (5.1) and (5.2), one recovers directly the well-known isomorphism

$$\text{Ш}(S_0/\mathbb{P}^1) \simeq \mathrm{Br}(S_0). \tag{5.3}$$

However, at this point it is not evident that our isomorphism coincides with classical one by Artin and Tate [5, Sec. 4] that uses the Leray spectral sequence, cf. Remark 5.6.

The classical Tate–Šafarevič group $\text{Ш}(S_0/\mathbb{P}^1)$ is only defined for elliptic K3 surfaces with a section. But the description of it as $\text{Ш}(S_0, f)$ allows us to speak of the Tate–Šafarevič group $\text{Ш}(S, f)$ of an elliptic K3 surface $S \to \mathbb{P}^1$ now possibly without a section.

5.2. Tate–Šafarevič group without a section

As a next step we will explain the link between this new Tate–Šafarevič group $Ш(S, f)$ of an elliptic K3 surface $S \to \mathbb{P}^1$ and the classical Tate–Šafarevič group $Ш(S_0, f) \simeq Ш(S_0/\mathbb{P}^1)$ of its Jacobian fibration.

We begin by recalling the short exact sequence

$$0 \longrightarrow \langle\beta\rangle \longrightarrow \mathrm{Br}(S_0) \longrightarrow \mathrm{Br}(S) \longrightarrow 0, \tag{5.4}$$

where $\beta \in \mathrm{Br}(S_0) \simeq Ш(S_0/\mathbb{P}^1) \simeq Ш(S_0, f)$ is the class corresponding to $S \to \mathbb{P}^1$. The result goes back to Artin and Tate, cf. [5, Sec. 4]. First consider the scheme-theoretic generic fibre E of $S \to \mathbb{P}^1$ which is a smooth curve of genus one over $\mathbb{C}(t)$. Its Jacobian $E_0 := \mathrm{Pic}^0(E)$ is the identity component of $\mathrm{Pic}(E)$. More precisely, there exists a short exact sequence $0 \to E_0 = \mathrm{Pic}^0(E) \to \mathrm{Pic}(E) \to \mathbb{Z} \to 0$. The relative version of this is a short exact sequence

$$0 \longrightarrow \mathcal{S}_0 \longrightarrow R^1\pi_*\mathbb{G}_m/_{\mathrm{vert}} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

cf. [10, Rem. 11.5.9]. Here, $\pi\colon S \to \mathbb{P}^1$ is the projection and $\mathcal{S}_0$ is the sheaf of étale (or analytic) local section of the Jacobian fibration $S_0 = \mathrm{Pic}^0(S/\mathbb{P}^1) \to \mathbb{P}^1$ and $/_{\mathrm{vert}}$ denotes the quotient by all vertical divisors. Taking cohomology leads to the exact sequence

$$H^0(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m) \longrightarrow \mathbb{Z} \longrightarrow H^1(\mathbb{P}^1, \mathcal{S}_0) \longrightarrow H^1(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m) \longrightarrow 0. \tag{5.5}$$

(Use that $H^i(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m) \to H^i(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m/_{\mathrm{vert}})$ is surjective for $i = 0$ and an isomorphism for $i = 1$.) The Leray spectral sequence induces isomorphisms $\mathrm{Pic}(S) \simeq H^0(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m)$ and $\mathrm{Br}(S) \simeq H^2(S, \mathbb{G}_m) \simeq H^1(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m)$. Also, the first map is nothing but $L \mapsto (L.f)$ and, therefore, its cokernel is $\mathbb{Z}/(\mathrm{NS}(S).f)$. Applied to the Jacobian fibration $S_0 \to \mathbb{P}^1$ itself, (5.5) induces an isomorphism $H^1(\mathbb{P}^1, \mathcal{S}_0) \simeq \mathrm{Br}(S_0)$. In general, one obtains a short exact sequence

$$0 \longrightarrow \mathbb{Z}/\big(\mathrm{NS}(S).f\big) \longrightarrow \mathrm{Br}(S_0) \longrightarrow \mathrm{Br}(S) \longrightarrow 0$$

and the kernel can indeed be shown to be just the subgroup $\langle\beta\rangle$. For example, the order $|\beta|$ of the subgroup $\langle\beta\rangle$ equals the minimal fibre degree of any line bundle on S, i.e., $|\beta|\mathbb{Z} = (\mathrm{NS}(S).f)$.

Remark 5.2. Let us discuss the link between the Brauer group and twists of the generic fibre.

(i) Over the generic point of $\mathbb{P}^1$, the surjection $\mathrm{Br}(S_0) \twoheadrightarrow \mathrm{Br}(S)$, which we view as the map $H^1(\mathbb{P}^1, \mathcal{S}_0) \twoheadrightarrow H^1(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m)$, corresponds to

$$H^1\big(\mathbb{C}(t), \bar{E}_0\big) \longrightarrow H^1\big(\mathbb{C}(t), \mathrm{Pic}(\bar{E})\big)$$

which maps a torsor E' for E_0 to the torsor $(\mathrm{Pic}(E) \times E')/E_0$ for $\mathrm{Pic}(E)$.

(ii) The map

$$\mathrm{Br}(S) \xrightarrow{\sim} H^1(\mathbb{P}^1, R^1\pi_*\mathbb{G}_m) \hookrightarrow H^1\big(\mathbb{C}(t), \mathrm{Pic}(\bar{E})\big)$$

is realised by mapping a Brauer class $\alpha \in \mathrm{Br}(S)$ to the $\mathrm{Pic}(E)$-torsor $\mathrm{Pic}_{\alpha|_E}(E)$, see Remarks 3.3 and 3.11, and Proposition 3.5. Recall that the Leray spectral sequence for the projection $S \to \mathbb{P}^1$ restricts to the Hochschild–Serre spectral sequence on the generic fibre. The related Weil–Châtelet group $\mathrm{WC}(E_0/\mathbb{C}(t)) \simeq H^1(\mathbb{C}(t), \bar{E}_0)$ will come up in the proof of the next proposition.

Remark 5.3. An alternative way of constructing (5.4) uses Hodge theory. Indeed, the universal $(\beta, 1)$-twisted sheaf on $S_0 \times_{\mathbb{P}^1} S$ induces a Hodge isometry $\widetilde{H}(S,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S_0, \beta, \mathbb{Z})$ which restricts to the isometry $T(S) \xrightarrow{\sim} T(S_0, \beta) = \ker(\beta\colon T(S_0) \to \mathbb{Q}/\mathbb{Z})$, cf. Section 4.6, [10, Ch. 14.4.1] or [11, Sec. 4]. Applying $\mathrm{Hom}(\,, \mathbb{Q}/\mathbb{Z})$ and using $\mathrm{Br}(S) \simeq \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$ leads to an exact sequence $0 \to \langle\beta\rangle \to \mathrm{Br}(S_0) \to \mathrm{Br}(S) \to 0$. This is indeed nothing but (5.4), but as we will not use this fact here, we do not give a proof.

Comparing (5.4) with the bottom sequence in (4.6) suggests the next result.

Proposition 5.4. *Mapping a class $\alpha \in Ш(S, f)$ to the smooth compactification $\overline{\mathrm{Pic}}^0_\alpha(S/\mathbb{P}^1)$ defines an isomorphism $Ш(S, f) \xrightarrow{\sim} Ш(S_0/\mathbb{P}^1)$ which can be completed to a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & Ш(S, f) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0 \\ & & \downarrow\simeq & & \downarrow\simeq & & \| & & \\ 0 & \longrightarrow & \langle\beta\rangle & \longrightarrow & Ш(S_0/\mathbb{P}^1) & \longrightarrow & \mathrm{Br}(S) & \longrightarrow & 0. \end{array}$$

In particular, the generator of the subgroup $\mathbb{Z}/m\mathbb{Z} \subset Ш(S, f)$ is mapped to the class $\beta \in Ш(S_0/\mathbb{P}^1)$ corresponding to S.

Proof. The generic fibre $\mathrm{Pic}^0_{\alpha|E}(E)$ of the projection $\overline{\mathrm{Pic}}^0_\alpha(S/\mathbb{P}^1) \to \mathbb{P}^1$ is a torsor for E_0 and, therefore, defines an element in the Weil–Châtelet group

$$\mathrm{WC}\,\big(E_0/\mathbb{C}(t)\big) \simeq H^1\big(\mathbb{C}(t), \bar{E}_0\big).$$

The map $Ш(S, f) \to \mathrm{WC}(E_0/\mathbb{C}(t))$ defined in this way takes image in the Tate–Šafarevič group $Ш(S_0/\mathbb{P}^1) \subset \mathrm{WC}(E_0/\mathbb{C}(t))$.

The injectivity of the map is a consequence of the comments at the end of the proof of Proposition 4.12 and the surjectivity follows from both groups being divisible of the same rank.

Next we show that $\bar{1} \in \mathbb{Z}/m\mathbb{Z}$ is mapped to β. This is a consequence of the more general observation made in Example 4.11. In this particular case it says that for the class $\alpha \in \mathrm{SBr}(S)$ represented by $\mathcal{E}nd(F)$, where F is locally free of rank m with its determinant satisfying $(\det(F).f) = m$, there exists an isomorphism $\mathrm{Pic}^0_\alpha(S/\mathbb{P}^1) \simeq \mathrm{Pic}^1(S/\mathbb{P}^1) \subset S$.

To prove the commutativity of the diagram on the right, it suffices to control the generic fibre. According to Remark 5.2, the map

$$\text{Ш}(S,f)\longrightarrow \text{Ш}(S_0/\mathbb{P}^1)\simeq \mathrm{Br}(S_0)\longrightarrow \mathrm{Br}(S)\hookrightarrow H^1\big(\mathbb{C}(t),\mathrm{Pic}(\bar{E})\big)$$

is given by $\alpha\mapsto \mathrm{Pic}^0_{\alpha|_E}(E)\mapsto (\mathrm{Pic}(E)\times\mathrm{Pic}^0_{\alpha|E}(E))/E_0$ and the map

$$\mathrm{SBr}^{\mathrm{o}}(S)\twoheadrightarrow \mathrm{Br}(S)\hookrightarrow H^1\big(\mathbb{C}(t),\mathrm{Pic}(\bar{E})\big)$$

sends $\alpha\in\mathrm{SBr}^{\mathrm{o}}(S)$ first to $\bar{\alpha}\in\mathrm{Br}(S)$ and then to $\mathrm{Pic}_{\bar{\alpha}|_E}(E)$, which by Proposition 3.4 is the torsor $(\mathrm{Pic}(E)\times\mathrm{Pic}^0_{\alpha|_E}(E))/E_0$. ∎

Remark 5.5. Observe that the surjectivity says that for every $\gamma\in\text{Ш}(S_0/\mathbb{P}^1)\simeq\mathrm{Br}(S_0)$ there exists a class $\alpha\in\mathrm{SBr}^{\mathrm{o}}(S)$ such that

$$\mathrm{Pic}^0_\gamma(S_0/\mathbb{P}^1)\simeq\mathrm{Pic}^0_\alpha(S/\mathbb{P}^1).$$

It would be interesting to find a geometric proof for this. One idea could be to use the two twisted universal families on $S_0\times_{\mathbb{P}^1}\mathrm{Pic}^0_\gamma(S_0/\mathbb{P}^1)$ and on $S\times_{\mathbb{P}^1}S_0$ to produce a family on $S\times_{\mathbb{P}^1}\mathrm{Pic}^0_\gamma(S_0/\mathbb{P}^1)$ inducing the desired isomorphism by universality. However, as the first family is twisted by $(\gamma\times 1)$ and the second one by $(1\times\beta)$, they do not concatenate directly. One would first need to transform, e.g., the first one to a family twisted by some $(\beta^{-1}\times\delta)$.

Remark 5.6. Let us add a few more comments on the above proof.

(i) Observe that the above arguments in particular show that the isomorphism (5.3) coincides with the classical one constructed via the Leray spectral sequence.

(ii) There is yet another way of linking $\mathrm{Br}(S_0)$ and $\text{Ш}(S_0/\mathbb{P}^1)$, cf. [3, Sec. 5.4] or [10, Rem. 11.5.9]. If S is the twist associated with $\beta\in\text{Ш}(S_0/\mathbb{P}^1)$, then S_0 can be viewed as a moduli space of sheaves on S, namely $\mathrm{Pic}^0(S/\mathbb{P}^1)\simeq S_0$. However, S_0 is only a coarse moduli space and the obstruction to the existence of a universal family is a class $\gamma_\beta\in\mathrm{Br}(S_0)$. This defines a map

$$\text{Ш}(S_0/\mathbb{P}^1)\longrightarrow\mathrm{Br}(S_0),\quad \beta\longmapsto\gamma_\beta,$$

which again is nothing but (5.3).

5.3. Twisting Lagrangian fibrations

We conclude by linking our discussion with a result by Markman [16].

Among other things, he proves that every non-special hyperkähler manifold of $\mathrm{K3}^{[n]}$-type X together with a Lagrangian fibration $X\to\mathbb{P}^n$ is a certain 'twist' M_s of a Mukai system $M=\overline{\mathrm{Pic}}^d(\mathcal{C}/|h|)\to|h|$ associated with some K3 surface S and a complete linear system $\mathcal{C}\to|h|$ on it. His twists are parametrised by elements $s\in\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$ of the analytic special Brauer group $\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$ (or rather of Ш^0 used in [16], see Section 4.5), which contains the restricted special Brauer group $\mathrm{SBr}^{\mathrm{o}}(S)\subset\mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$ as its torsion group.

How do these twists compare to our twisted Picard varieties $\mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})$, where α is an element in the smaller group $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$? A priori, our setting seems at the same time more special and more general for the following reasons.

(i) Markman only considers relative Picard varieties $\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}})$ which can be compactfied to hyperkähler manifolds. In other words, only those moduli spaces

$$\overline{\mathrm{Pic}}^d(\mathcal{C}/|h|) \simeq M(0, h, s),$$

where $d = s + (1/2)(h.h)$, are considered for which the Mukai vector $(0, h, s)$ is primitive. As we are only concerned with the smooth curves $\mathcal{C} \to |h|_{\mathrm{sm}}$, this restriction is irrelevant for us.

(ii) On the other hand, Markman 'twists' an arbitrary smooth

$$\overline{\mathrm{Pic}}^d(\mathcal{C}/|h|) = M(0, h, s) \longrightarrow |h|,$$

and not only those with a section as our $\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}})$, to obtain the given Lagrangian fibration $X \to \mathbb{P}^n$. This can be remedied by applying Proposition 4.9 and Example 4.11.

Theorem 1.2 reinterprets [16, Thm. 1.5] in the projective setting.

Proof of Theorem 1.2. The first step consists of observing that for $M = \overline{\mathrm{Pic}}^d(\mathcal{C}/|h|)$ and $s \in \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$, Markman's twist M_s is algebraic if and only if s is contained in $\mathrm{SBr}^{\mathrm{o}}(S) \subset \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$. For this observe that $\alpha = s \in \mathrm{SBr}^{\mathrm{o}}(S)^{\mathrm{an}}$ is torsion, or equivalently contained in $\mathrm{SBr}^{\mathrm{o}}(S)$, if and only if its image $\bar{\alpha} \in \mathrm{Br}(S)^{\mathrm{an}}$ is torsion, i.e., contained in $\mathrm{Br}(S) \subset \mathrm{Br}(S)^{\mathrm{an}}$. The result was proved in broader generality by Abasheva and Rogov [2, Thm. 5.19] and [1, Thm. A].

Next, according to Example 4.11, we know that M is birational to the torsor

$$\mathrm{Pic}^d(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}})$$

and by virtue of Remark 4.8 we have

$$\left(\mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}}) \times_{|h|} \mathrm{Pic}^0_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})\right)/\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \mathrm{Pic}^0_{-\bar{d}\cdot\alpha}(\mathcal{C}/|h|_{\mathrm{sm}}). \tag{5.6}$$

To conclude one shows that for $\alpha = s \in Ш(S, h)$ the twist M_s is the left-hand side of (5.6), which is a direct consequence of the discussion in [16, Sec. 7.2]. This is proved analogously to the two-dimensional case, cf. Remark 5.6. Hence, M_s is birational to $\mathrm{Pic}^0_{-\bar{d}\cdot\alpha}(\mathcal{C}/|h|_{\mathrm{sm}})$. ∎

As an alternative for the last step, one could first prove a version of (5.6) for the total Picard varieties:

$$\left(\mathrm{Pic}^0_{-\bar{d}}(\mathcal{C}/|h|_{\mathrm{sm}}) \times_{|h|} \mathrm{Pic}_\alpha(\mathcal{C}/|h|_{\mathrm{sm}})\right)/\mathrm{Pic}^0(\mathcal{C}/|h|_{\mathrm{sm}}) \simeq \mathrm{Pic}_{-\bar{d}\cdot\alpha}(\mathcal{C}/|h|_{\mathrm{sm}})$$

and compare the left-hand side with M_s via Remark 5.2, see also Proposition 3.4.

In order to apply the results of Markman, we a priori have to assume in Theorem 1.2 that X is 'non-special'. This assumption has subsequently be removed by Abasheva [1, Thm. A] and Soldatenkov–Verbitsky [18, Thm. 4.7].

Our assumption $\rho(X) = 2$ implies that $\rho(S) = 1$ which ensures that Markman's results can be applied, cf. [16, Ass. 7.1 and Rem. 7.2]. While the condition $\rho(X) = 2$ determines a complement of a countable union of hypersurfaces in the space of all Lagrangian fibred X, Markman's condition [16, Ass. 7.1] is Zariski open. Hence, our Theorem 1.2 also holds for a Zariski open, dense subset of the space of all Lagrangian fibrations $X \to \mathbb{P}^n$ of $\mathrm{K3}^{[n]}$-type.

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