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Global results for the inhomogeneous Muskat problem

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Abstract. The inhomogeneous Muskat problem models the dynamics of an interface between two fluids of differing characteristics inside a nonuniform porous medium. We consider the case of a porous media with a permeability jump across a horizontal boundary away from an interface between two fluids of different viscosities and densities. For initial data of explicit medium size, depending on the characteristics of the fluids and porous media, we will prove the global existence and uniqueness of a solution that is instantly analytic and decays in time to the flat interface.

1. Introduction

The inhomogeneous Muskat problem models the dynamics of two incompressible, immiscible fluids in a nonuniform, porous medium. This scenario occurs naturally when, for example, oil and water flows meet in a sand and loam media. The physical principle governing the porous media flow is Darcy's law [17], given here in the two-dimensional setting:

$$\frac{\mu}{\kappa} u = -\nabla p - g \begin{bmatrix} 0 \\ \rho \end{bmatrix}, \tag{1.1}$$

where $u(x,t)$ is the fluid velocity, $\mu(x,t)$ is the fluid viscosity, $\kappa(x,t)$ is the permeability of the porous media, $p(x,t)$ is the pressure, g is the gravitational constant, and $\rho(x,t)$ is the fluid density. The incompressibility condition in each fluid domain is given by $\nabla \cdot u = 0$.

Given nonintersecting soil and fluid interfaces, we divide our domain into three time-dependent disjoint open regions $D_i(t)$ such that $D_3(t) = D_3$ is unchanging.

Denoting the smooth, simple boundaries as $\partial D_i(t)$ for $i = 1, 2, 3$,

$$D_1(t) \cup D_2(t) \cup D_3 \cup \partial D_1(t) \cup \partial D_3 = \mathbb{R}^2.$$

The two fluids occupy domains $D_1(t)$ and $D_2(t) \cup D_3$ respectively, and the two different soils occupy domains $D_1(t) \cup D_2(t)$ and D_3:

$$(\mu, \rho)(x,t) = \begin{cases} (\mu_1, \rho_1) & \text{if } x \in D_1(t), \\ (\mu_2, \rho_2) & \text{if } x \in D_2(t) \cup D_3 \end{cases}$$

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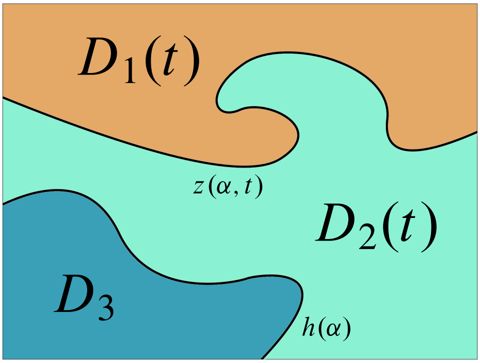

Figure 1. Inhomogeneous Muskat Problem.

and

$$\kappa(x) = \begin{cases} \kappa_1 & \text{if } x \in D_1(t) \cup D_2(t), \\ \kappa_2 & \text{if } x \in D_3. \end{cases}$$

We will study the evolution of the fluid interface $\partial D_1(t)$. This setting is a variation of the classical Muskat problem in which the porous medium is uniform and the permeability constant is often normalized to $\kappa = 1$. Note that in the case of uniform permeability, the distinction between the regions $D_2(t)$ and D_3 vanishes.

We obtain a contour equation for the fluid interface from (1.1) and the incompressibility condition on the fluid velocity by first parametrizing the fluid interface $\partial D_1(t)$ as

$$z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \quad \text{for } \alpha \in \mathbb{R},\ t \in \mathbb{R}_{\geq 0},$$

and the soil interface ∂D_3 as

$$h(\alpha) = (h_1(\alpha), h_2(\alpha)) \quad \text{for } \alpha \in \mathbb{R},$$

as depicted in Figure 1. We assume that $z(\alpha,t) \neq h(\alpha)$ for all $\alpha \in \mathbb{R}$.

Darcy's law implies the vorticity ($\omega = \nabla \times v$) can only be supported on the boundaries:

$$\omega(x,t) = \omega_1(\alpha,t)\delta(x - z(\alpha,t)) + \omega_2(\alpha,t)\delta(x - h(\alpha)).$$

The Biot–Savart law gives a solution for the fluid velocity in terms of the vorticity

$$\begin{aligned} u(x,t) &= BR(\omega_1, z)(x,t) + BR(\omega_2, h)(x,t) \\ &\stackrel{\text{def}}{=} \frac{1}{2\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2}\,\omega_1(\beta,t)\,d\beta \\ &\quad + \frac{1}{2\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{(x - h(\beta))^{\perp}}{|x - h(\beta)|^2}\,\omega_2(\beta,t)\,d\beta, \end{aligned}$$

where BR stands for the Birkoff–Rott integral defined above. Taking limits in the normal direction to the boundaries, one obtains

$$\partial_t z(\alpha,t) = BR(\omega_1,z)(z(\alpha,t),t) + BR(\omega_2,h)(z(\alpha,t),t) + c(\alpha,t)\,\partial_\alpha z(\alpha,t), \quad (1.2)$$

$$\omega_1(\alpha,t) = 2A_\mu(BR(\omega_1,z)(z(\alpha,t),t) + BR(\omega_2,h)(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t) - 2A_\rho\,\partial_\alpha z_2(\alpha,t), \quad (1.3)$$

$$\omega_2(\alpha,t) = -2A_\kappa(BR(\omega_1,z)(h(\alpha),t) + BR(\omega_2,h)(h(\alpha),t))\cdot\partial_\alpha h(\alpha), \quad (1.4)$$

where the constants are given by

$$A_\kappa = \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}, \quad A_\mu = \frac{\mu_1-\mu_2}{\mu_1+\mu_2}, \quad A_\rho = -\kappa_1\frac{\rho_1-\rho_2}{\mu_1+\mu_2}g. \quad (1.5)$$

In (1.2), the tangential term $c(\alpha,t)\partial_\alpha z(\alpha,t)$ is determined by the choice of parametrization of the curve and vanishes when considering the velocity normal to the interface. See [28] for more details. If there is no jump in permeability, the constant $A_\kappa = 0$ and the ω_2 terms disappear from the evolution equation for $z(\alpha,t)$.

The question of well-posedness of a fluid-fluid interface in porous media has been well studied. The local well-posedness depends on the system initially satisfying the Rayleigh–Taylor condition, which requires the jump of the gradient of the pressure in the normal direction to the interface to be strictly positive. A system satisfying the Rayleigh–Taylor condition is said to be in the stable regime (see, e.g., [5]). In the case that there is a density jump across the interface, $A_\rho \neq 0$, the Rayleigh–Taylor condition requires that the denser fluid lies below the interface, meaning $\rho_2 > \rho_1$. The scaling invariance of the Muskat problem gives the criticality of $H^{1+\frac{d}{2}}$ regularity, where d is the dimension of the interface. Similarly, it can be seen that $W^{1,\infty}$, C^1, and $\mathcal{F}_0^{1,1}$ are also all scale invariant. (For the definition of $\mathcal{F}_0^{1,1}$, see (2.1) below.)

The stable regime with uniform permeability $A_\kappa = 0$ has been extensively studied; see, for example, [1–3, 9, 10, 13–15, 21, 24–26], etc., and references therein. Of particular interest in this paper are results showing global well-posedness for initial data of *medium size* in the case of uniform permeability. Previously, global-in-time results are known in both two and three dimensions for systems with uniform permeability for medium size initial data without a viscosity jump [11, 12, 27] and with a viscosity jump [19] in the case of an infinite graph interface by using the norms (2.1). Under the effects of surface tension, the stability of medium size perturbations of a gravity unstable bubble interface has also been proven in [20]. A modulus of continuity approach in [7] and [8] gives a different medium size condition in both 2D and 3D without viscosity jump.

As discussed earlier, in this paper, we consider the nonuniform permeability setting, also called the inhomogeneous Muskat problem. In this setting, for a horizontal permeability jump boundary and without viscosity jump, graph interface solutions were shown to be locally well posed using energy estimates in Sobolev spaces and there exist interfaces starting in the stable regime that become unstable in finite time [6]. A class of graph solutions exhibiting this turning behavior was also demonstrated by a computer-assisted proof in [22]. In the same setting, [23] demonstrated global existence and decay to the flat fluid interface for small initial data in H^2. For a general interface curve and with a viscosity jump, local well-posedness was shown in [28] in Sobolev spaces. The existence of splash singularities was later shown in [16], although splat singularities were ruled out. This paper will address the problem of global well-posedness in the critical regularity $\mathcal{F}_0^{1,1}$ for medium-sized initial data in the regime that also has a viscosity jump. Moreover, the estimates proven in this paper will imply that the interface is instantly analytic and decays to the flat solution as a corollary. A result from [4] in the regime without a viscosity jump demonstrates precise conditions for H^3 solutions to show decay in the Lipschitz norm and further establishes a global existence and decay result for Lipschitz solutions with small initial data.

We will consider the case of a graphical fluid interface $z(\alpha,t) = (\alpha, f(\alpha,t))$ and a fixed horizontal permeability jump interface $h(\alpha) = (\alpha, -h_2)$ for $h_2 > 0$ as in [6], under the assumption $|f(\alpha,t)| < h_2$, and we will allow for a viscosity jump. Setting

$$\Delta_\beta f(\alpha) = \frac{f(\alpha) - f(\alpha-\beta)}{\beta},$$

the choice of a graph interface determines the tangential constant $c(\alpha,t)$ and turns the system (1.2)–(1.3)–(1.4) into

$$\partial_t f(\alpha) = \frac{1}{2\pi}(I_1(\alpha) + I_2(\alpha) + I_3(\alpha) + I_4(\alpha)) \tag{1.6}$$

for

$$I_1(\alpha) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{1+\Delta_\beta f(\alpha)^2} \frac{\omega_1(\alpha-\beta)}{\beta}\, d\beta, \tag{1.7}$$

$$I_2(\alpha) = \text{p.v.} \int_{\mathbb{R}} \frac{\partial_\alpha f(\alpha)\Delta_\beta f(\alpha)}{1+\Delta_\beta f(\alpha)^2} \frac{\omega_1(\alpha-\beta)}{\beta}\, d\beta, \tag{1.8}$$

$$I_3(\alpha) = \text{p.v.} \int_{\mathbb{R}} \frac{\beta}{\beta^2 + (f(\alpha)+h_2)^2}\, \omega_2(\alpha-\beta)\, d\beta, \tag{1.9}$$

$$I_4(\alpha) = \text{p.v.} \int_{\mathbb{R}} \frac{\partial_\alpha f(\alpha)(f(\alpha)+h_2)}{\beta^2 + (f(\alpha)+h_2)^2}\, \omega_2(\alpha-\beta)\, d\beta, \tag{1.10}$$

in which

$$\begin{aligned}\omega_1(\alpha) = {} & \frac{A_\mu}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\partial_\alpha f(\alpha) - \Delta_\beta f(\alpha)}{1+\Delta_\beta f(\alpha)^2} \frac{\omega_1(\alpha-\beta)}{\beta}\, d\beta \\ & + \frac{A_\mu}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\beta\partial_\alpha f(\alpha) - (f(\alpha)+h_2)}{\beta^2 + (f(\alpha)+h_2)^2} \omega_2(\alpha-\beta)\, d\beta \\ & - 2A_\rho \partial_\alpha f(\alpha),\end{aligned} \tag{1.11}$$

$$\omega_2(\alpha) = -\frac{A_\kappa}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(\alpha-\beta)+h_2}{\beta^2 + (f(\alpha-\beta)+h_2)^2}\, \omega_1(\alpha-\beta)\, d\beta. \tag{1.12}$$

Defining $\partial_\alpha \Omega_i = \omega_i$, it can be derived that

$$\Omega_1(\alpha) = -\frac{A_\mu}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{\Delta_\beta f(\alpha) - \partial_\alpha f(\alpha-\beta)}{1+\Delta_\beta f(\alpha)^2}\,\frac{\Omega_1(\alpha-\beta)}{\beta}\,d\beta$$

$$-\frac{A_\mu}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(\alpha)+h_2}{\beta^2+(f(\alpha)+h_2)^2}\Omega_2(\alpha-\beta)\,d\beta - 2A_\rho f(\alpha), \tag{1.13}$$

$$\Omega_2(\alpha) = -\frac{A_\kappa}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(\alpha-\beta)+h_2+\beta\partial_\alpha f(\alpha-\beta)}{\beta^2+(f(\alpha-\beta)+h_2)^2}\Omega_1(\alpha-\beta)\,d\beta. \tag{1.14}$$

Equations (1.6), (1.11), and (1.12) give a coupled system for the evolution of fluid-fluid graph interface $f(\alpha,t)$.

Outline of the paper. In Section 2, the main results of the paper are stated followed by some preliminary facts that will be used in the proof. In Section 3, we compute bounds on the vorticity terms by Taylor expanding the expressions of ω_i and Ω_i and then computing the Fourier transforms. Next, in Section 4, we decompose (1.6) into its linear and nonlinear parts and prove (2.2) for $s = 0, 1$. In Section 5, we prove (2.3) and then a higher order Sobolev estimate that is used for the existence argument. We conclude in Section 6 by proving Theorem 2.1.

2. Main Results

To study the evolution of the interface, we adopt the weighted Fourier norms defined as follows. For a function $g : \mathbb{R}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and for $s > -d$, define the norm

$$\|g\|_{\mathcal{F}^{s,p}_\nu}(t) = \|e^{\nu t|\xi|}\,|\xi|^s\,\widehat{g}(\xi,t)\|_{L^p} = \left(\int_{\mathbb{R}^d} e^{\nu t p|\xi|}|\xi|^{sp}|\widehat{g}(\xi,t)|^p\,d\xi\right)^{\frac{1}{p}} \tag{2.1}$$

where $\widehat{g}$ is the Fourier transform of g in the spatial variable

$$\widehat{g}(\xi,t) = \mathcal{F}(g(\cdot,t))(\xi) = \int_{\mathbb{R}^d} g(x,t)e^{-ix\cdot\xi}\,dx.$$

Let $\mathcal{F}^{s,p}_\nu$ be the space of all functions with finite $\|\cdot\|_{\mathcal{F}^{s,p}_\nu}$ norm.

Let θ be the constant defined in (4.21) and observe that $\theta > 0$ because $|A_\kappa| < 1$ in the setting of (1.1). Let $\sigma_s = \sigma_s(\|f_0\|_{\mathcal{F}^{0,1}_0}, \|f_0\|_{\mathcal{F}^{1,1}_0})$, $s = 0, 1, 2$ be continuous functions defined in (4.22), (4.23), and (5.13). Note that $\sigma_s(0,0) = 0$. Since the σ_s are continuous, we define the constants $k_0(|A_\mu|, |A_\kappa|) < h_2$ and $k_1(|A_\mu|, |A_\kappa|) < 1$ such that

$$\theta - \sigma_s(k_0(|A_\mu|, |A_\kappa|), k_1(|A_\mu|, |A_\kappa|)) > 0$$

for $s = 0, 1, 2$.

Theorem 2.1. *Suppose* $f_0 \in L^2 \cap \mathcal{F}_0^{1,1}$ *such that* $\|f_0\|_{\mathcal{F}_0^{0,1}} < k_0(|A_\mu|, |A_\kappa|)$ *and* $\|f_0\|_{\mathcal{F}_0^{1,1}} < k_1(|A_\mu|, |A_\kappa|)$ *hold. Then there exists a unique solution* $f \in L^\infty([0,T]; L^2 \cap \mathcal{F}_0^{1,1}) \cap L^1([0,T]; \mathcal{F}_0^{2,1})$ *to* (1.6) *for all* $T > 0$ *and a constant* $\nu > 0$ *satisfying for* $s = 0, 1$

$$\|f\|_{\mathcal{F}_\nu^{s,1}}(t) + (A_\rho\theta - A_\rho\sigma_s - \nu)\int_0^t \|f\|_{\mathcal{F}_\nu^{s+1,1}}(s)\, ds \le \|f_0\|_{\mathcal{F}_0^{s,1}} \tag{2.2}$$

and

$$\|f\|_{L_\nu^2}^2(t) \le \|f_0\|_{L^2}^2 \cdot \exp(R(\|f_0\|_{\mathcal{F}_0^{0,1}}, \|f_0\|_{\mathcal{F}_0^{1,1}}, t)) \tag{2.3}$$

for any $t \ge 0$ *for a positive function* R *that is bounded for* $\|f_0\|_{\mathcal{F}_0^{0,1}} < k_0(|A_\mu|, |A_\kappa|)$, $\|f_0\|_{\mathcal{F}_0^{1,1}} < k_1(|A_\mu|, |A_\kappa|)$, *and* $t \ge 0$.

Remark 2.2. By (2.2) and (2.3), it can be seen from the exponential weight that the solution gains analytic regularity instantly in time for all $t > 0$. Moreover, using the Hausdorff–Young inequality, since $k_0(|A_\mu|, |A_\kappa|) < h_2$, inequality (2.2) for $s = 0$ implies that $|f(\alpha, t)| < h_2$ for all $t \ge 0$.

Next, to show decay to the flat solution, we will need the Decay Lemma proved in [19], which we have restated for our setting below.

Lemma 2.3 (Decay Lemma). *Suppose* $\|g\|_{\mathcal{F}_0^{s_1,1}}(t) \le C_0$ *and*

$$\frac{d}{dt}\|g\|_{\mathcal{F}_0^{s_2,1}}(t) \le -C\|g\|_{\mathcal{F}_0^{s_2+1,1}}(t)$$

where $s_1 < s_2$. *Then*

$$\|g\|_{\mathcal{F}_0^{s_2,1}}(t) \lesssim (1+t)^{s_1 - s_2}.$$

The Decay Lemma along with (2.2) implies the large time decay of solutions to the inhomogeneous Muskat problem. Specifically, (2.2) implies uniform in time bounds of $\|f\|_{\mathcal{F}^{s,1}}$ for $s = 0, 1$ and then we use the Decay Lemma to obtain the following result.

Theorem 2.4. *Suppose* $f_0(\alpha)$ *is initial data satisfying the conditions of Theorem* 2.1. *Then the solution* $f(\alpha, t)$ *to* (1.6) *decays with the rate*

$$\|f\|_{\mathcal{F}^{1,1}}(t) \lesssim (1+t)^{-1}.$$

2.5. Preliminary Facts

To prove Theorem 2.1, we employ the following collection of useful facts. First, letting iterated convolutions be denoted as

$$(*^n g)(x) = (\underbrace{g * \cdots * g}_{n \text{ times}})(x)$$

we have the following product rule inequalities.

Lemma 2.6. *Given functions $f_k : \mathbb{R}^d \to \mathbb{R}$ for $1 \le k \le n$, we have*

$$e^{\nu t|\xi|}(|f_1| * |f_2| * \cdots * |f_n|) \le (e^{\nu t|\xi|}|f_1|) * (e^{\nu t|\xi|}|f_2|) * \cdots * (e^{\nu t|\xi|}|f_n|) \tag{2.4}$$

and for $0 < s \le 1$,

$$|\xi|^s(|f_1| * |f_2| * \cdots * |f_n|) \le \sum_{k=1}^{n}(|\xi|^s|f_k|) * (*_{j\ne k}|f_j|), \tag{2.5}$$

where

$$*_{j\ne k}|f_j|$$

indicates a convolution over the absolute values of all functions f_j except f_k.

Proof. By the triangle inequality, we have for $0 < s \le 1$,

$$|\xi_0|^s \le |\xi_0 - \xi_1|^s + |\xi_1 - \xi_2|^s + \cdots + |\xi_{n-2} - \xi_{n-1}|^s + |\xi_{n-1}|^s.$$

Applying this triangle inequality to the convolution gives us (2.5). Moreover,

$$e^{\nu t|\xi_0|} \le \prod_{j=0}^{n} e^{\nu t|\xi_j - \xi_{j+1}|}.$$

Plugging this inequality into the function convolution, we obtain (2.4). ■

We also have the interpolation inequality:

Proposition 2.7. *If $g \in \mathcal{F}_\nu^{s_1,1} \cap \mathcal{F}_\nu^{s_2,1}$ for $s_1 < s_2$, then $g \in \mathcal{F}_\nu^{s,1}$ for each $s \in [s_1, s_2]$ and g satisfies*

$$\|g\|_{\mathcal{F}_\nu^{s,1}} \le \|g\|^{\theta}_{\mathcal{F}_\nu^{s_1,1}} \|g\|^{1-\theta}_{\mathcal{F}_\nu^{s_2,1}} \tag{2.6}$$

for $\theta \in [0, 1]$ such that $s = \theta s_1 + (1-\theta)s_2$.

Finally, to compute the linearization of (1.6) and of the vorticity terms, the following Fourier transforms will be needed (see, e.g., [18, Chapter 3]):

Proposition 2.8. *For $a \in \mathbb{R}$, we have*

$$\mathcal{F}\left[\frac{a}{x^2 + a^2}\right](\xi) = \pi e^{-a|\xi|} \tag{2.7}$$

and

$$\mathcal{F}\left[\frac{x}{x^2 + a^2}\right](\xi) = -i\pi \,\mathrm{sgn}(\xi) e^{-a|\xi|}. \tag{2.8}$$

3. Potential jump and vorticity

In this section, we compute the Fourier transforms and bounds on the potential jump and vorticity terms.

First, we compute the Fourier transform of Ω_2. Write $\Omega_2 = \Omega_{21} + \Omega_{22}$, where

$$\Omega_{21}(\alpha) = -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} \frac{f(\alpha-\beta)+h_2}{\beta^2+(f(\alpha-\beta)+h_2)^2}\Omega_1(\alpha-\beta)\,d\beta,$$

and

$$\Omega_{22}(\alpha) = -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} \frac{\beta\partial_\alpha f(\alpha-\beta)}{\beta^2+(f(\alpha-\beta)+h_2)^2}\Omega_1(\alpha-\beta)\,d\beta.$$

Taking the Fourier transform of Ω_{21}, we obtain using (2.7)

$$\begin{aligned}
\widehat{\Omega}_{21}(\xi) &= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(\alpha-\beta)+h_2}{\beta^2+(f(\alpha-\beta)+h_2)^2}\ \Omega_1(\alpha-\beta)e^{-i\xi\alpha}\ d\beta d\alpha \\
&= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(y)+h_2}{\beta^2+(f(y)+h_2)^2}\ e^{-i\xi\beta}\ d\beta\ \Omega_1(y)e^{-i\xi y}\ dy \\
&= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} \pi e^{-(f(y)+h_2)|\xi|}\cdot\Omega_1(y)e^{-i\xi y}\ dy \\
&= -A_\kappa e^{-h_2|\xi|}\int_{\mathbb{R}} e^{-f(y)|\xi|}\cdot\Omega_1(y)e^{-i\xi y}\ dy \\
&= -A_\kappa\sum_{n=0}^{\infty} e^{-h_2|\xi|}\int_{\mathbb{R}} \frac{(-f(y)|\xi|)^n}{n!}\cdot\Omega_1(y)e^{-i\xi y}\ dy \\
&= -A_\kappa\sum_{n=0}^{\infty} e^{-h_2|\xi|}\frac{(-1)^n|\xi|^n}{n!}((*^n\widehat{f})*\widehat{\Omega}_1)(\xi).
\end{aligned}\tag{3.1}$$

For Ω_{22}, we have

$$\begin{aligned}
\widehat{\Omega}_{22}(\xi) &= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} d\beta\int_{\mathbb{R}} d\alpha\ e^{-i\xi\alpha}\frac{(\alpha-\beta)\partial_\alpha f(\beta)}{(\alpha-\beta)^2+(f(\beta)+h_2)^2}\Omega_1(\beta) \\
&= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} d\beta\ \partial_\alpha f(\beta)\Omega_1(\beta)\mathcal{F}\Big[\tau_\beta\Big(\frac{\alpha}{\alpha^2+(f(\beta)+h_2)^2}\Big)\Big](\xi) \\
&= -\frac{A_\kappa}{\pi}\int_{\mathbb{R}} d\beta\ e^{-i\xi\beta}\partial_\alpha f(\beta)\Omega_1(\beta)\cdot -i\pi\ \mathrm{sgn}(\xi)e^{-(f(\beta)+h_2)|\xi|} \\
&= A_\kappa\sum_{n=0}^{\infty} i\ \mathrm{sgn}(\xi)e^{-h_2|\xi|}\frac{(-1)^n|\xi|^n}{n!}((*^n\widehat{f})*\widehat{\partial_\alpha f}*\widehat{\Omega}_1)(\xi).
\end{aligned}\tag{3.2}$$

Next, we compute similarly for Ω_1. By (1.13), we write the term $\Omega_1 = \Omega_{11} + \Omega_{12} - 2A_\rho f(\alpha)$, where

$$\Omega_{11}(\alpha) = -\frac{A_\mu}{\pi}\int_{\mathbb{R}} \frac{\Delta_\beta f(\alpha)-\partial_\alpha f(\alpha-\beta)}{1+\Delta_\beta f(\alpha)^2}\,\frac{\Omega_1(\alpha-\beta)}{\beta}\,d\beta \tag{3.3}$$

and

$$\Omega_{12}(\alpha) = -\frac{A_\mu}{\pi}\int_{\mathbb{R}} \frac{f(\alpha)+h_2}{\beta^2+(f(\alpha)+h_2)^2}\Omega_2(\alpha-\beta)\,d\beta.$$

It can be seen that Ω_{11} has no linear part. For Ω_{12}, taking the Fourier transform, we obtain

$$\begin{aligned}
\widehat{\Omega}_{12}(\xi) &= -\frac{A_\mu}{\pi}\int_{\mathbb{R}} e^{-i\xi\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\Omega}_2(\xi_1)e^{i\xi_1(\alpha-\beta)}d\xi_1 \frac{f(\alpha)+h_2}{\beta^2+(f(\alpha)+h_2)^2}\,d\beta d\alpha \\
&= -\frac{A_\mu}{\pi}\int_{\mathbb{R}} e^{-i\xi\alpha}\int_{\mathbb{R}} \widehat{\Omega}_2(\xi_1)e^{i\xi_1\alpha}\Big(\int_{\mathbb{R}} e^{-i\xi_1\beta}\frac{f(\alpha)+h_2}{\beta^2+(f(\alpha)+h_2)^2}\,d\beta\Big)d\xi_1 d\alpha \\
&= -\frac{A_\mu}{\pi}\int_{\mathbb{R}} e^{-i\xi\alpha}\int_{\mathbb{R}} e^{i\xi_1\alpha}\,\widehat{\Omega}_2(\xi_1)\,\pi e^{-(f(\alpha)+h_2)|\xi_1|}\,d\xi_1 d\alpha \\
&= -A_\mu\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}} e^{-i\xi\alpha}f(\alpha)^n\int_{\mathbb{R}}\widehat{\Omega}_2(\xi_1)\,e^{-h_2|\xi_1|}\,|\xi_1|^n e^{i\xi_1\alpha}\,d\xi_1 d\alpha \\
&= -A_\mu\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}((\widehat{\Omega}_2\,e^{-h_2|\xi|}\,|\xi|^n) * (*^n\widehat{f}))(\xi).
\end{aligned} \tag{3.4}$$

The first term, Ω_{11}, satisfies the following bound (where the proof technique from [12] is used):

Lemma 3.1. *For* $\|f\|_{\mathcal{F}_0^{1,1}} < 1$, *we have the bound*

$$|\widehat{\Omega}_{11}(\xi)| \le 2|A_\mu|\sum_{n=0}^{\infty}(*^{2n+1}|\widehat{\partial_\alpha f}| * |\widehat{\Omega}_1|)(\xi). \tag{3.5}$$

Proof. We first consider the term

$$\begin{aligned}
\Omega_{11}(\alpha) &= -\frac{A_\mu}{\pi}\int_{\mathbb{R}}\frac{\Delta_\beta f(\alpha)-\partial_\alpha f(\alpha-\beta)}{1+\Delta_\beta f(\alpha)^2}\frac{\Omega_1(\alpha-\beta)}{\beta}\,d\beta \\
&\stackrel{\text{def}}{=} \Omega_{111}(\alpha)+\Omega_{112}(\alpha).
\end{aligned}$$

Taking the Fourier transform of the first term Ω_{111} and Taylor expanding the denominator for $|\Delta_\beta f(\alpha)| < 1$, we obtain

$$\begin{aligned}
\widehat{\Omega}_{111}(\xi) &= -\frac{A_\mu}{\pi}\int_{\mathbb{R}}\mathcal{F}\Big[\frac{\Delta_\beta f(\alpha)}{1+\Delta_\beta f(\alpha)^2}\frac{\Omega_1(\alpha-\beta)}{\beta}\Big](\xi)\,d\beta \\
&= -\frac{A_\mu}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_{\mathbb{R}}\mathcal{F}\Big[\Delta_\beta f(\alpha)^{2n+1}\frac{\Omega_1(\alpha-\beta)}{\beta}\Big](\xi)\,d\beta \\
&= -\frac{A_\mu}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_{\mathbb{R}}(*^{2n+1}(\widehat{f}m_\beta) * \widehat{\tau_\beta\Omega_1})(\xi)\,\frac{d\beta}{\beta},
\end{aligned}$$

in which

$$m_\beta(\xi) = \frac{1-e^{-i\xi\beta}}{\beta}.$$

Expanding the convolution of the n-th term in the sum, we have

$$\int_{\mathbb{R}} (*^{2n+1}(\hat{f} m_\beta) * \widehat{\tau_\beta \Omega_1})(\xi) \frac{d\beta}{\beta} = \int_{\mathbb{R}} d\xi_1 \ldots \int_{\mathbb{R}} d\xi_{2n+1} \hat{f}(\xi - \xi_1) \cdot \\ \ldots \cdot \hat{f}(\xi_{2n} - \xi_{2n+1}) \cdot \hat{\Omega}_1(\xi_{2n+1}) \cdot M_n,$$

where

$$M_n = M_n(\xi, \xi_1, \ldots, \xi_{2n+1}) = \int_{\mathbb{R}} d\beta \frac{e^{-i\xi_{2n+1}\beta}}{\beta} m_\beta(\xi - \xi_1) \cdot \ldots \cdot m_\beta(\xi_{2n} - \xi_{2n+1}).$$

Since

$$m_\beta(\xi) = i\xi \int_0^1 ds\, e^{i\beta(s-1)\xi},$$

we have

$$\begin{aligned} |M_n| &= \left| i^{2n}(\xi - \xi_1) \cdot \ldots \cdot (\xi_{2n} - \xi_{2n+1}) \int_0^1 ds_1 \cdots \int_0^1 ds_{2n+1} \int_{\mathbb{R}} d\beta \frac{e^{-i\xi_{2n+1}\beta}}{\beta} e^{i\beta A} \right| \\ &= \left| (\xi - \xi_1) \cdot \ldots \cdot (\xi_{2n} - \xi_{2n+1}) \int_0^1 ds_1 \cdots \int_0^1 ds_{2n+1}\, i\pi \operatorname{sgn}(A - \xi_{2n+1}) \right| \\ &\leq \pi \cdot |\xi - \xi_1| \cdot \ldots \cdot |\xi_{2n} - \xi_{2n+1}|, \end{aligned}$$

where

$$A = (s_1 - 1)(\xi - \xi_1) + \cdots + (s_{2n+1} - 1)(\xi_{2n} - \xi_{2n+1}).$$

Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}} (*^{2n+1}(\hat{f} m_\beta) * \widehat{\tau_\beta \omega_1})(\xi) \frac{d\beta}{\beta} \right| &\leq \int_{\mathbb{R}} d\xi_1 \cdots \int_{\mathbb{R}} d\xi_{2n+1} |\hat{f}(\xi - \xi_1)| \cdot \\ &\quad \ldots \cdot |\hat{f}(\xi_{2n} - \xi_{2n+1})| \cdot |\hat{\Omega}_1(\xi_{2n+1})| \cdot |M_n| \\ &\leq \pi(*^{2n+1}|\widehat{\partial_\alpha f}| * |\hat{\Omega}_1|)(\xi). \end{aligned}$$

The term Ω_{112} is bounded by the same quantity using a similar computation. This concludes the proof. ∎

Next, we consider the vorticity terms. The Fourier transform of ω_2 can be computed similarly to Ω_{21}

$$\hat{\omega}_2(\xi) = -A_\kappa \sum_{n=0}^{\infty} e^{-h_2|\xi|} \frac{(-1)^n |\xi|^n}{n!} ((*^n \hat{f}) * \hat{\omega}_1)(\xi). \tag{3.6}$$

For ω_1, similarly to Ω_1, we decompose $\omega_1 = \omega_{11} + \omega_{12} - 2A_\rho \partial_\alpha f$, where we have the analogous bound on $\hat{\omega}_{11}$

$$|\hat{\omega}_{11}(\xi)| \leq 2|A_\mu| \sum_{n=0}^{\infty} (*^{2n+1}|\widehat{\partial_\alpha f}| * |\hat{\omega}_1|)(\xi), \tag{3.7}$$

and, $\widehat{\omega}_{12}(\xi)$ is explicitly computed as $-i\xi\widehat{\Omega}_{12}(\xi)$ using (2.5):

$$\widehat{\omega}_{12}(\xi) = A_\mu \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!}\big((\, i\ \mathrm{sgn}(\cdot)e^{-h_2|\cdot|}\ |\cdot|^n \widehat{\omega}_2) * \widehat{\partial_\alpha f} * (*^n \widehat{f})\big)(\xi)$$
$$- A_\mu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\big((\widehat{\omega}_2\ e^{-h_2|\cdot|}\ |\cdot|^n) * (*^n \widehat{f})\big)(\xi) \tag{3.8}$$

Using the computations above, we can bound the vorticity terms in the frequency space norms $\mathcal{F}_\nu^{s,1}$ for $s = 0, 1$.

Proposition 3.2. *The term ω_2 satisfies*

$$\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \le |A_\kappa| C_0 \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \tag{3.9}$$

and

$$\|\omega_2\|_{\mathcal{F}_\nu^{1,1}} \le |A_\kappa| \frac{C_2}{\|f\|_{\mathcal{F}_\nu^{0,1}}} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \|f\|_{\mathcal{F}_\nu^{1,1}} + |A_\kappa| C_0 \|\omega_1\|_{\mathcal{F}_\nu^{1,1}} \tag{3.10}$$

where

$$C_0 = C_0(\|f\|_{\mathcal{F}_\nu^{0,1}}) = \sum_{n=0}^{\infty} \left\| e^{-h_2|\xi|} \frac{|\xi|^n}{n!} \right\|_{L^\infty} \|f\|^n_{\mathcal{F}_\nu^{0,1}} = \sum_{n=0}^{\infty} \frac{n^n}{e^n n!} \Big(\frac{\|f\|_{\mathcal{F}_\nu^{0,1}}}{h_2} \Big)^n \tag{3.11}$$

and

$$C_2 = \sum_{n=1}^{\infty} n \left\| e^{-h_2|\xi|} \frac{|\xi|^n}{n!} \right\|_{L^\infty} \|f\|^n_{\mathcal{F}_\nu^{0,1}} = \sum_{n=1}^{\infty} \frac{n^{n+1}}{e^n n!} \Big(\frac{\|f\|_{\mathcal{F}_\nu^{0,1}}}{h_2} \Big)^n, \tag{3.12}$$

which converge for $\|f\|_{\mathcal{F}_\nu^{0,1}} < h_2$. Note that $C_0 \to 1$ and $C_2 \to 0$ in the limit $\|f\|_{\mathcal{F}_\nu^{0,1}} \to 0$.

Proof. Using Young's inequalities for convolutions, (2.4), and (3.6),

$$\begin{aligned}
\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} &\le |A_\kappa| \sum_{n=0}^{\infty} \left\| e^{\nu t|\xi|} e^{-h_2|\xi|} \frac{(-1)^n |\xi|^n}{n!} \Big((*^n \widehat{f}) * \widehat{\omega}_1 \Big)(\xi) \right\|_{L^1} \\
&\le |A_\kappa| \sum_{n=0}^{\infty} \left\| e^{-h_2|\xi|} \frac{|\xi|^n}{n!} \right\|_{L^\infty} \left\| e^{\nu t|\xi|} (*^n \widehat{f}) * \widehat{\omega}_1 \right\|_{L^1} \\
&\le |A_\kappa| \Big(\sum_{n=0}^{\infty} \left\| e^{-h_2|\xi|} \frac{|\xi|^n}{n!} \right\|_{L^\infty} \|f\|^n_{\mathcal{F}_\nu^{0,1}} \Big) \|\omega_1\|_{\mathcal{F}_\nu^{0,1}}.
\end{aligned}$$

This computation yields (3.9). Applying (2.5), we obtain (3.10). ∎

We will use (3.9) and (3.10) implicitly to bound ω_1 in the same norms.

Lemma 3.3. *The vorticity term ω_1 satisfies the bounds*

$$\|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \leq 2A_\rho C_1 \|f\|_{\mathcal{F}_\nu^{1,1}} \tag{3.13}$$

and

$$\|\omega_1\|_{\mathcal{F}_\nu^{1,1}} \leq 2A_\rho C_1 C_3 \|f\|_{\mathcal{F}_\nu^{2,1}} \tag{3.14}$$

where

$$C_1 = \Big(1 - |A_\mu|\Big[\frac{2\|f\|_{\mathcal{F}_\nu^{1,1}}}{1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}}} + |A_\kappa| C_0^2 (1 + \|f\|_{\mathcal{F}_\nu^{1,1}})\Big]\Big)^{-1} \tag{3.15}$$

and

$$\begin{aligned} C_3 = 1 + 2|A_\mu| C_1 \Big(& \frac{\|f\|_{\mathcal{F}_\nu^{1,1}} (1 + \|f\|^2_{\mathcal{F}_\nu^{1,1}})}{(1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}})^2} \\ & + \frac{1}{2} |A_\kappa| C_0 \big[(C_0 + 2C_2)\|f\|_{\mathcal{F}_\nu^{1,1}} + C_2 (1 + \|f\|_{\mathcal{F}_\nu^{1,1}})\big]\Big) \end{aligned} \tag{3.16}$$

are defined for $\|f\|_{\mathcal{F}_\nu^{0,1}} < k_0(|A_\mu|, |A_\kappa|)$ and $\|f\|_{\mathcal{F}_\nu^{1,1}} < k_1(|A_\mu|, |A_\kappa|)$. Note that as $\|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0$, we have $C_1 \to (1 - |A_\kappa||A_\mu|)^{-1}$ and $C_3 \to 1$.

Proof. Similarly to the proof of (3.9), we use (2.4) and Young's inequality to obtain from (3.7) that

$$\|\omega_{11}\|_{\mathcal{F}_\nu^{0,1}} \leq 2|A_\mu| \sum_{n=0}^{\infty} \|f\|^{2n+1}_{\mathcal{F}_\nu^{1,1}} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \leq |A_\mu| \frac{2\|f\|_{\mathcal{F}_\nu^{1,1}}}{1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}}} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}}. \tag{3.17}$$

We can also bound ω_{12} from (3.8) by

$$\begin{aligned} \|\omega_{12}\|_{\mathcal{F}_\nu^{0,1}} &\leq |A_\mu| C_0 (1 + \|f\|_{\mathcal{F}_\nu^{1,1}}) \|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \\ &\leq |A_\kappa||A_\mu| C_0^2 (1 + \|f\|_{\mathcal{F}_\nu^{1,1}}) \|\omega_1\|_{\mathcal{F}_\nu^{0,1}}, \end{aligned} \tag{3.18}$$

where we used (3.9) in the second inequality. Hence, we now have that

$$\begin{aligned} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \leq{}& 2|A_\mu| \frac{\|f\|_{\mathcal{F}_\nu^{1,1}}}{1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}}} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} + |A_\kappa||A_\mu| C_0^2 (1 + \|f\|_{\mathcal{F}_\nu^{1,1}}) \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \\ & + 2A_\rho \|f\|_{\mathcal{F}_\nu^{1,1}}. \end{aligned}$$

Using (3.9) solving for $\|\omega_1\|_{\mathcal{F}_\nu^{0,1}}$ in the inequality implies (3.13). Estimate (3.14) follows similarly by applying (2.5). Next, via (2.6) we compute

$$\|\omega_{11}\|_{\mathcal{F}_\nu^{1,1}} \leq 2|A_\mu| \frac{1 + \|f\|^2_{\mathcal{F}_\nu^{1,1}}}{(1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}})^2} \|f\|_{\mathcal{F}_\nu^{2,1}} \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} + |A_\mu| \frac{2\|f\|_{\mathcal{F}_\nu^{1,1}}}{1 - \|f\|^2_{\mathcal{F}_\nu^{1,1}}} \|\omega_1\|_{\mathcal{F}_\nu^{1,1}},$$

$$\|\omega_{12}\|_{\mathcal{F}_\nu^{1,1}} \le |A_\mu|(C_0+C_2)\|\omega_2\|_{\mathcal{F}_\nu^{0,1}}\|f\|_{\mathcal{F}_\nu^{2,1}} + |A_\mu|C_0\|f\|_{\mathcal{F}_\nu^{1,1}}\|\omega_2\|_{\mathcal{F}_\nu^{1,1}}$$
$$+ |A_\mu|\Big(\frac{C_2}{\|f\|_{\mathcal{F}_\nu^{0,1}}}\|f\|_{\mathcal{F}_\nu^{1,1}}\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} + C_0\|\omega_2\|_{\mathcal{F}_\nu^{1,1}}\Big).$$

Now, using (3.13) and (3.9) gives (3.21) from the next proposition. Using (3.21), (3.10), and (3.13), and applying interpolation (2.6), we obtain

$$\|\omega_{11}\|_{\mathcal{F}_\nu^{1,1}} \le 4A_\rho|A_\mu|C_1\frac{\|f\|_{\mathcal{F}_\nu^{1,1}}(1+\|f\|^2_{\mathcal{F}_\nu^{1,1}})}{(1-\|f\|^2_{\mathcal{F}_\nu^{1,1}})^2}\|f\|_{\mathcal{F}_\nu^{2,1}}$$
$$+ 2|A_\mu|\frac{\|f\|_{\mathcal{F}_\nu^{1,1}}}{1-\|f\|^2_{\mathcal{F}_\nu^{1,1}}}\|\omega_1\|_{\mathcal{F}_\nu^{1,1}} \tag{3.19}$$
$$\|\omega_{12}\|_{\mathcal{F}_\nu^{1,1}} \le 2A_\rho|A_\kappa||A_\mu|C_0C_1(C_0+2C_2)\|f\|_{\mathcal{F}_\nu^{1,1}}\|f\|_{\mathcal{F}_\nu^{2,1}}$$
$$+ |A_\kappa||A_\mu|C_0^2(1+\|f\|_{\mathcal{F}_\nu^{1,1}})\|\omega_1\|_{\mathcal{F}_\nu^{1,1}}$$
$$+ 2A_\rho|A_\kappa||A_\mu|C_0C_1C_2(1+\|f\|_{\mathcal{F}_\nu^{1,1}})\|f\|_{\mathcal{F}_\nu^{2,1}}. \tag{3.20}$$

Computing implicitly as before yields the bound. ∎

Plugging estimates (3.13) and (3.14) into (3.9) and (3.10), and then using (2.6), we obtain

Proposition 3.4. *The term ω_2 is bounded as*

$$\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \le 2A_\rho|A_\kappa|C_0C_1\|f\|_{\mathcal{F}_\nu^{1,1}}, \tag{3.21}$$
$$\|\omega_2\|_{\mathcal{F}_\nu^{1,1}} \le 2A_\rho|A_\kappa|C_1C_4\|f\|_{\mathcal{F}_\nu^{2,1}} \tag{3.22}$$

where

$$C_4 = C_2 + C_0C_3 \to 1 \quad \textit{as} \quad \|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0. \tag{3.23}$$

4. Instant analyticity and decay inequality for the interface

In this section, we will prove inequality (2.2), which will imply the instantaneous gain of analytic regularity and the decay to the flat solution of the density jump interface for initial data of an explicitly calculable size. We perform these estimates in the spaces defined in (2.1) for $p = 1$, $\nu > 0$, and $0 \le s \le 1$. We first need to linearize the contour equation for the fluid-fluid interface $f(\alpha, t)$. To do so, we need to extract the linear part of ω_i. First, by (3.6), we can write the decomposition in of ω_2 in frequency space:

$$\widehat{\omega}_2(\xi) = L(\widehat{\omega}_2)(\xi) + N(\widehat{\omega}_2)(\xi), \tag{4.1}$$

where

$$L(\widehat{\omega}_2)(\xi) = -A_\kappa e^{-h_2|\xi|}\widehat{\omega}_1(\xi)$$

and

$$N(\widehat{\omega}_2)(\xi) = -A_\kappa \sum_{n=1}^{\infty} e^{-h_2|\xi|} \frac{(-1)^n |\xi|^n}{n!} \big((*^n \widehat{f}) * \widehat{\omega}_1\big)(\xi).$$

From (3.8), we can write

$$\widehat{\omega}_1(\xi) = L(\widehat{\omega}_1)(\xi) + N(\widehat{\omega}_1)(\xi) \tag{4.2}$$

where

$$L(\widehat{\omega}_1)(\xi) = -A_\mu e^{-h_2|\xi|} \widehat{\omega}_2(\xi) - 2A_\rho \widehat{\partial_\alpha f}(\xi)$$

and

$$\begin{aligned} N(\widehat{\omega}_1)(\xi) = \widehat{\omega}_{11}(\xi) - A_\mu \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \big((\widehat{\omega}_2\, e^{-h_2|\xi|}\, |\xi|^n) * (*^n \widehat{f})\big)(\xi) \\ + A_\mu \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \big((\, i \operatorname{sgn}(\xi) e^{-h_2|\xi|}\, |\xi|^n \widehat{\omega}_2) * \widehat{\partial_\alpha f} * (*^n \widehat{f})\big)(\xi). \end{aligned} \tag{4.3}$$

We can now use relations (4.1) and (4.2) to say

$$\begin{aligned} L(\widehat{\omega}_2) &= -A_\kappa e^{-h_2|\xi|} (L(\widehat{\omega}_1)(\xi) + N(\widehat{\omega}_1)(\xi)) \\ &= A_\kappa A_\mu e^{-2h_2|\xi|} (L(\widehat{\omega}_2)(\xi) + N(\widehat{\omega}_2)(\xi)) \\ &\quad + 2A_\kappa A_\rho e^{-h_2|\xi|} \widehat{\partial_\alpha f}(\xi) - A_\kappa e^{-h_2|\xi|} N(\widehat{\omega}_1). \end{aligned}$$

Combining terms and solving for $L(\widehat{\omega}_2)$, we find

$$\begin{aligned} L(\widehat{\omega}_2)(\xi) = 2A_\kappa A_\rho \widehat{\partial_\alpha f}(\xi) \frac{e^{h_2|\xi|}}{e^{2h_2|\xi|} - A_\kappa A_\mu} \\ - A_\kappa N(\widehat{\omega}_1)(\xi) \frac{e^{h_2|\xi|}}{e^{2h_2|\xi|} - A_\kappa A_\mu} + N(\widehat{\omega}_2)(\xi) \frac{A_\kappa A_\mu}{e^{2h_2|\xi|} - A_\kappa A_\mu}. \end{aligned} \tag{4.4}$$

With (4.4) in hand, we are ready to begin analyzing the interface decay. Differentiating in time, we obtain

$$\begin{aligned} \frac{d}{dt} \|f\|_{\mathcal{F}_\nu^{s,1}} &= \frac{d}{dt} \int_{\mathbb{R}} e^{\nu t|\xi|} |\xi|^s |\widehat{f}(\xi)|\, d\xi \\ &= \nu \int_{\mathbb{R}} e^{\nu t|\xi|} |\xi|^{s+1} |\widehat{f}(\xi)|\, d\xi \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{\nu t|\xi|} |\xi|^s \frac{\widehat{f}(\xi) \overline{\widehat{\partial_t f}(\xi)} + \widehat{\partial_t f}(\xi) \overline{\widehat{f}(\xi)}}{|\widehat{f}(\xi)|}\, d\xi. \end{aligned}$$

Next, to obtain the decay term in the expression above, we need to decompose the evolution equation for the interface into the linear and nonlinear terms

$$\widehat{\partial_t f}(\xi) = \frac{1}{2\pi} (\widehat{I}_1(\xi) + \widehat{I}_2(\xi) + \widehat{I}_3(\xi) + \widehat{I}_4(\xi)).$$

Defining N_0 as

$$I_1(\alpha) = \text{p.v.} \int_{\mathbb{R}} \frac{\omega_1(\alpha-\beta)}{\beta}\, d\beta - \int_{\mathbb{R}} \frac{\Delta_\beta f(\alpha)^2}{1+\Delta_\beta f(\alpha)^2} \frac{\omega_1(\alpha-\beta)}{\beta}\, d\beta$$
$$\stackrel{\text{def}}{=} \pi H(\omega_1)(\alpha) + 2\pi N_0(\alpha),$$

where H is the Hilbert transform, we use equation (4.2) to find

$$\begin{aligned}\frac{1}{2\pi}\widehat{I}_1(\xi) &= \frac{1}{2}\widehat{H(\omega_1)}(\xi) + \widehat{N}_0(\xi)\\ &= -A_\rho|\xi|\widehat{f}(\xi) + \frac{i}{2}\operatorname{sgn}(\xi)A_\mu e^{-h_2|\xi|}\widehat{\omega}_2(\xi) - \frac{i}{2}\operatorname{sgn}(\xi)N(\widehat{\omega}_1)(\xi)\\ &\quad + \widehat{N}_0(\xi).\end{aligned} \tag{4.5}$$

Expanding $\widehat{\omega}_2$ via (4.1) and (4.4) gives

$$\begin{aligned}\frac{i}{2}\operatorname{sgn}(\xi)A_\mu e^{-h_2|\xi|}\widehat{\omega}_2(\xi) &= -A_\rho|\xi|\widehat{f}(\xi)\frac{A_\kappa A_\mu}{e^{2h_2|\xi|}-A_\kappa A_\mu}\\ &\quad - A_\mu(\widehat{N}_1+\widehat{N}_2+\widehat{N}_3)(\xi),\end{aligned} \tag{4.6}$$

in which

$$\widehat{N}_1(\xi) = -\frac{i}{2}\operatorname{sgn}(\xi)e^{-h_2|\xi|}N(\widehat{\omega}_2)(\xi), \tag{4.7}$$

$$\widehat{N}_2(\xi) = \frac{i}{2}\operatorname{sgn}(\xi)\frac{A_\kappa}{e^{2h_2|\xi|}-A_\kappa A_\mu}N(\widehat{\omega}_1)(\xi), \tag{4.8}$$

$$\widehat{N}_3(\xi) = \frac{i}{2}\operatorname{sgn}(\xi)\frac{A_\kappa A_\mu e^{-h_2|\xi|}}{e^{2h_2|\xi|}-A_\kappa A_\mu}N(\widehat{\omega}_2)(\xi). \tag{4.9}$$

Combining (4.5) and (4.6) leads to

$$\begin{aligned}\frac{1}{2\pi}\widehat{I}_1(\xi) &= -A_\rho|\xi|\widehat{f}(\xi)\Big(1+\frac{A_\kappa A_\mu}{e^{2h_2|\xi|}-A_\kappa A_\mu}\Big)\\ &\quad + (\widehat{N}_0 + A_\mu(\widehat{N}_1+\widehat{N}_2+\widehat{N}_3))(\xi) - \frac{i}{2}\operatorname{sgn}(\xi)N(\widehat{\omega}_1)(\xi).\end{aligned}$$

Next,

$$\begin{aligned}I_3(\alpha) &= \int_{\mathbb{R}} \frac{\beta}{\beta^2+h_2^2}\,\omega_2(\alpha-\beta)\, d\beta\\ &\quad - \int_{\mathbb{R}} \frac{f(\alpha)+2h_2}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta f(\alpha)}{\beta^2+h_2^2}\,\omega_2(\alpha-\beta)\, d\beta,\end{aligned} \tag{4.10}$$

and denoting the nonlinear part as

$$N_4 = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{f(\alpha)+2h_2}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta f(\alpha)}{\beta^2+h_2^2}\,\omega_2(\alpha-\beta)\, d\beta,$$

we apply (2.8) and (4.4) once again to obtain

$$\begin{aligned}\frac{1}{2\pi}\widehat{I}_3(\xi) &= -\frac{i}{2}\,\mathrm{sgn}(\xi)e^{-h_2|\xi|}\widehat{\omega}_2(\xi) + \widehat{N}_4(\xi) \\ &= A_\rho|\xi|\widehat{f}(\xi)\frac{A_\kappa}{e^{2h_2|\xi|}-A_\kappa A_\mu} + (\widehat{N}_1+\widehat{N}_2+\widehat{N}_3+\widehat{N}_4)(\xi). \end{aligned} \tag{4.11}$$

Collecting terms, we have

$$\begin{aligned}\widehat{\partial_t f}(\xi) = &-A_\rho|\xi|\widehat{f}(\xi)\Big(1-\frac{A_\kappa(1-A_\mu)}{e^{2h_2|\xi|}-A_\kappa A_\mu}\Big) + \frac{1}{2\pi}(\widehat{I}_2+\widehat{I}_4)(\xi) \\ &+ \widehat{N}_0(\xi) + (1+A_\mu)(\widehat{N}_1+\widehat{N}_2+\widehat{N}_3)(\xi) + \widehat{N}_4(\xi) \\ &- \frac{i}{2}\,\mathrm{sgn}(\xi)N(\omega_1)(\xi). \end{aligned} \tag{4.12}$$

In (4.12), the linear terms will give the decay of the interface as long as the nonlinear terms are sufficiently bounded. So let us bound the nonlinear terms by following analogous computations to those in Section 3. The nonlinear bounds in this section grow arbitrarily small as $\|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0$.

Defining

$$C_5 = \Big\|\frac{A_\kappa A_\mu e^{-h_2|\xi|}}{e^{2h_2|\xi|}-A_\kappa A_\mu}\Big\|_{L^\infty}, \tag{4.13}$$

$$C_6 = \frac{\|f\|_{\mathcal{F}_\nu^{1,1}}}{1-\|f\|^2_{\mathcal{F}_\nu^{1,1}}} \to 0 \quad \text{as} \quad \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0, \tag{4.14}$$

we have the following estimates:

$$\frac{1}{2\pi}\|I_2\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1}{2}\frac{\|f\|^2_{\mathcal{F}_\nu^{1,1}}}{1-\|f\|^2_{\mathcal{F}_\nu^{1,1}}}\|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \le A_\rho C_1 C_6 \|f\|^2_{\mathcal{F}_\nu^{1,1}}, \tag{4.15}$$

$$\begin{aligned}\frac{1}{2\pi}\|I_2\|_{\mathcal{F}_\nu^{1,1}} &\le \frac{\|f\|_{\mathcal{F}_\nu^{1,1}}}{(1-\|f\|^2_{\mathcal{F}_\nu^{1,1}})^2}\Big(\|\omega_1\|_{\mathcal{F}_\nu^{0,1}}\|f\|_{\mathcal{F}_\nu^{2,1}} \\ &\qquad + \frac{1}{2}(1-\|f\|^2_{\mathcal{F}_\nu^{1,1}})\|f\|_{\mathcal{F}_\nu^{1,1}}\|\omega_1\|_{\mathcal{F}_\nu^{1,1}}\Big) \\ &\le 2A_\rho C_1\big(1+\frac{1}{2}C_3(1-\|f\|^2_{\mathcal{F}_\nu^{1,1}})\big)C_6^2\|f\|_{\mathcal{F}_\nu^{2,1}}. \end{aligned} \tag{4.16}$$

Above, the computation of $\widehat{I}_2$ is done similarly to (3.5) and the norms are computed using (2.4), (2.5), and (2.6). Next, similar to (3.4),

$$\widehat{I}_4(\xi) = \pi\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big((\widehat{\omega}_2\, e^{-h_2|\cdot|}\,|\cdot|^n) * (*^n\widehat{f}) * \widehat{\partial_\alpha f}\Big)(\xi),$$

and the bounds on I_4 are

$$\frac{1}{2\pi}\|I_4\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1}{2}C_0\|f\|_{\mathcal{F}_\nu^{1,1}}\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \le A_\rho|A_\kappa|C_0^2C_1\|f\|^2_{\mathcal{F}_\nu^{1,1}} \tag{4.17}$$

$$\begin{aligned}\frac{1}{2\pi}\|I_4\|_{\mathcal{F}_\nu^{1,1}} &\le A_\rho|A_\kappa|C_0C_1(C_0+C_2)\|f\|_{\mathcal{F}_\nu^{1,1}}\|f\|_{\mathcal{F}_\nu^{2,1}} + C_0\|f\|_{\mathcal{F}_\nu^{1,1}}\|\omega_2\|_{\mathcal{F}_\nu^{1,1}}\\ &\le A_\rho|A_\kappa|C_0C_1(C_0+C_2+C_4)\|f\|_{\mathcal{F}_\nu^{1,1}}\|f\|_{\mathcal{F}_\nu^{2,1}}\\ &\stackrel{\text{def}}{=} A_\rho|A_\kappa|C_1\lambda_0\|f\|_{\mathcal{F}_\nu^{2,1}}\end{aligned} \tag{4.18}$$

where

$$\lambda_0 = C_0(C_0+C_2+C_4)\|f\|_{\mathcal{F}_0^{1,1}} \to 0 \quad \text{as} \quad \|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0. \tag{4.19}$$

Next, like the term Ω_{111},

$$|\widehat{N}_0| \le \frac{1}{2}\sum_{n=0}^{\infty}((*^{2n+2}|\widehat{\partial_\alpha f}|) * |\widehat{\omega}_1|)(\xi),$$

which leads to

$$\begin{aligned}\|N_0\|_{\mathcal{F}_\nu^{0,1}} &\le A_\rho C_1C_6\|f\|^2_{\mathcal{F}_\nu^{1,1}},\\ \|N_0\|_{\mathcal{F}_\nu^{1,1}} &\le A_\rho C_1C_6\Big(C_3\|f\|_{\mathcal{F}_\nu^{1,1}} + \frac{2C_6}{\|f\|_{\mathcal{F}_\nu^{1,1}}}\Big)\|f\|_{\mathcal{F}_\nu^{2,1}}.\end{aligned}$$

Now, reusing techniques from the Fourier transforms in Section 3, we have

$$\begin{aligned}-2\pi\widehat{N}_4(\xi) &= \mathcal{F}\Big[\int_{\mathbb{R}} d\beta\ \frac{f(\alpha)+2h_2}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta f(\alpha)}{\beta^2+h_2^2}\omega_2(\alpha-\beta)\Big](\xi)\\ &= \int_{\mathbb{R}} d\beta\ \Big(\mathcal{F}\Big(\frac{f(\alpha)+2h_2}{\beta^2+(f+h_2)^2}\frac{\beta f(\alpha)}{\beta^2+h_2^2}\Big) * \widehat{\tau_\beta\omega_2}\Big)(\xi)\\ &= \int_{\mathbb{R}} d\beta\int_{\mathbb{R}} d\xi_1\ \mathcal{F}\Big(\frac{(f(\alpha)+2h_2)}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta f(\alpha)}{\beta^2+h_2^2}\Big)(\xi-\xi_1)\ \ e^{-i\beta\xi_1}\widehat{\omega}_2(\xi_1)\\ &= \int_{\mathbb{R}} d\xi_1\ \widehat{\omega}_2(\xi_1)\int_{\mathbb{R}} d\alpha\ e^{-i(\xi-\xi_1)\alpha}f(\alpha)\\ &\qquad \cdot\int_{\mathbb{R}} d\beta\ e^{-i\beta\xi_1}\frac{f(\alpha)+2h_2}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta}{\beta^2+h_2^2}.\end{aligned}$$

By (2.7) and (2.8), the integral in β is the convolution

$$\begin{aligned}T(\xi_1) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} d\beta\ e^{-i\beta\xi_1}\frac{f(\alpha)+2h_2}{\beta^2+(f(\alpha)+h_2)^2}\frac{\beta}{\beta^2+h_2^2}\\ &= \Big(\frac{\pi(f(\alpha)+2h_2)}{f(\alpha)+h_2}e^{-(f(\alpha)+h_2)|\cdot|} * -i\pi\ \mathrm{sgn}(\cdot)e^{-h_2|\cdot|}\Big)(\xi_1),\end{aligned}$$

which can be calculated via the identity

$$(e^{-a|\cdot|} * \operatorname{sgn}(\cdot)e^{-b|\cdot|})(x) = -\operatorname{sgn}(x)\frac{2a(e^{-a|x|} - e^{-b|x|})}{a^2 - b^2}$$

as

$$\begin{aligned} T(\xi_1) &= i\pi^2 \frac{f(\alpha) + 2h_2}{f(\alpha) + h_2} \cdot \operatorname{sgn}(\xi_1) \frac{2(f(\alpha) + h_2)(e^{-(f(\alpha)+h_2)|\xi_1|} - e^{-h_2|\xi_1|})}{(f(\alpha) + h_2)^2 - h_2^2} \\ &= 2i\pi^2 \operatorname{sgn}(\xi_1) e^{-h_2|\xi_1|} \frac{(e^{-f(\alpha)|\xi_1|} - 1)}{f(\alpha)} \\ &= 2i\pi^2 \operatorname{sgn}(\xi_1) e^{-h_2|\xi_1|} \sum_{n=1}^{\infty} \frac{(-f(\alpha))^{n-1}|\xi_1|^n}{n!}, \end{aligned}$$

and so

$$\widehat{N}_4(\xi) = -i\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{\mathbb{R}} d\xi_1\, \widehat{\omega}_2(\xi_1) \operatorname{sgn}(\xi_1)|\xi_1|^n e^{-h_2|\xi_1|} \int_{\mathbb{R}} d\alpha\, e^{-i(\xi-\xi_1)\alpha} f(\alpha)^n.$$

Hence,

$$\widehat{N}_4(\xi) = i\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \big((\widehat{\omega}_2 \ \operatorname{sgn}(\xi)\ |\xi|^n\ e^{-h_2|\xi|}) * (*^n \widehat{f})\big)(\xi). \tag{4.20}$$

Using (2.6) gives us the bounds

$$\begin{aligned} \|N_4\|_{\mathcal{F}_\nu^{0,1}} &\le \pi(C_0 - 1)\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \le 2\pi A_\rho |A_\kappa| C_0(C_0 - 1)C_1 \|f\|_{\mathcal{F}_\nu^{1,1}}, \\ \|N_4\|_{\mathcal{F}_\nu^{1,1}} &\le 2\pi A_\rho |A_\kappa| C_0 C_1 C_2 \|f\|_{\mathcal{F}_\nu^{2,1}} + \pi(C_0 - 1)\|\omega_2\|_{\mathcal{F}_\nu^{1,1}} \\ &\le 2\pi A_\rho |A_\kappa| C_1(C_0 C_2 + (C_0 - 1)C_4)\|f\|_{\mathcal{F}_\nu^{2,1}}. \end{aligned}$$

At this point, bounds on $N(\omega_1)$ and $N(\omega_2)$ will lead to estimates of all the remaining terms. In $\mathcal{F}^{0,1}$,

$$\|N(\omega_2)\|_{\mathcal{F}_\nu^{0,1}} \le |A_\kappa|(C_0 - 1)\|\omega_1\|_{\mathcal{F}_\nu^{0,1}} \le 2A_\rho |A_\kappa|(C_0 - 1)C_1 \|f\|_{\mathcal{F}_\nu^{1,1}}$$

and

$$\begin{aligned} \|N(\omega_1)\|_{\mathcal{F}_\nu^{0,1}} &\le 2|A_\mu| C_6 \|\omega_1\|_{\mathcal{F}_\nu^{0,1}} + |A_\mu|(C_0\|f\|_{\mathcal{F}_\nu^{1,1}} + (C_0 - 1))\|\omega_2\|_{\mathcal{F}_\nu^{0,1}} \\ &\le A_\rho |A_\mu| C_1 \lambda_1 \|f\|_{\mathcal{F}_\nu^{1,1}} \end{aligned}$$

where the second inequality follows from (3.13) and (3.21). Above,

$$\lambda_1 = 4C_6 + 2|A_\kappa| C_0(C_0\|f\|_{\mathcal{F}_\nu^{1,1}} + (C_0 - 1)) \to 0 \quad \text{as} \quad \|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0.$$

Next in $\mathcal{F}^{1,1}$

$$\|N(\omega_2)\|_{\mathcal{F}^{1,1}_\nu} \le |A_\kappa|\Big((C_0-1)\|\omega_1\|_{\mathcal{F}^{1,1}_\nu} + \frac{C_2}{\|f\|_{\mathcal{F}^{0,1}_\nu}}\|f\|_{\mathcal{F}^{1,1}_\nu}\|\omega_1\|_{\mathcal{F}^{0,1}_\nu}\Big) \\ \le A_\rho|A_\kappa|C_1\lambda_2\|f\|_{\mathcal{F}^{2,1}_\nu},$$

where λ_2 is defined by applying (3.13), (3.14), and then (2.6):

$$\lambda_2 = 2(C_0-1)C_3 + 2C_2 \to 0 \quad \text{as} \quad \|f\|_{\mathcal{F}^{0,1}_\nu} + \|f\|_{\mathcal{F}^{1,1}_\nu} \to 0.$$

Finally, calculating as in (3.20),

$$\|N(\omega_1)\|_{\mathcal{F}^{1,1}_\nu} \le \|\omega_{11}\|_{\mathcal{F}^{1,1}_\nu} + |A_\mu|\Big((C_0+C_2)\|\omega_2\|_{\mathcal{F}^{0,1}_\nu}\|f\|_{\mathcal{F}^{2,1}_\nu} + C_0\|f\|_{\mathcal{F}^{1,1}_\nu}\|\omega_2\|_{\mathcal{F}^{1,1}_\nu} \\ + \frac{C_2}{\|f\|_{\mathcal{F}^{0,1}_\nu}}\|f\|_{\mathcal{F}^{1,1}_\nu}\|\omega_2\|_{\mathcal{F}^{0,1}_\nu} + (C_0-1)\|\omega_2\|_{\mathcal{F}^{1,1}_\nu}\Big) \\ \le A_\rho|A_\mu|C_1\lambda_3\|f\|_{\mathcal{F}^{2,1}_\nu},$$

in which λ_3 is defined by using (3.13), (3.14), (3.19), (3.21), and (3.22):

$$\lambda_3 = 4\frac{1+\|f\|^2_{\mathcal{F}^{1,1}_\nu}}{(1-\|f\|^2_{\mathcal{F}^{1,1}_\nu})^2}\|f\|_{\mathcal{F}^{1,1}_\nu} + 4C_3C_6 + 2|A_\kappa|(C_0(C_0+C_2+C_4)\|f\|_{\mathcal{F}^{1,1}_\nu} \\ + 2|A_\kappa|(C_0C_2 + (C_0-1)C_4),$$

and $\lambda_3 \to 0$ as $\|f\|_{\mathcal{F}^{0,1}_\nu} + \|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$.

Now, fix

$$\theta = \inf_{\xi\in\mathbb{R}}\Big(1 - \frac{A_\kappa(1-A_\mu)}{e^{2h_2|\xi|} - A_\kappa A_\mu}\Big) \tag{4.21}$$

and note that $\theta > 0$, since $|A_\kappa| < 1$. Then by (4.12) and the above estimates,

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{0,1}_\nu} \le (-A_\rho\theta + \nu)\|f\|_{\mathcal{F}^{1,1}_\nu} + A_\rho\sigma_0\|f\|_{\mathcal{F}^{1,1}_\nu},$$

where

$$\sigma_0 \stackrel{\text{def}}{=} \sigma_0(\|f\|_{\mathcal{F}^{0,1}_\nu}, \|f\|_{\mathcal{F}^{1,1}_\nu}) \\ = C_1\Big[(C_6 + |A_\kappa|C_0^2)\|f\|_{\mathcal{F}^{1,1}_\nu} \\ + \frac{1}{2}|1+A_\mu|\Big(2|A_\kappa|(C_0-1) + C_5\Big(2|A_\kappa|(C_0-1)+\lambda_1\Big)\Big) \\ + 2\pi|A_\kappa|C_0(C_0-1) + |A_\mu|\lambda_1 + C_6\|f\|_{\mathcal{F}^{1,1}_\nu}\Big] \tag{4.22}$$

and

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{1,1}_\nu} \le (-A_\rho\theta + \nu)\|f\|_{\mathcal{F}^{2,1}_\nu} + A_\rho\sigma_1\|f\|_{\mathcal{F}^{2,1}_\nu}$$

where

$$\begin{aligned}\sigma_1 &\stackrel{\text{def}}{=} \sigma_1(\|f\|_{\mathcal{F}_\nu^{0,1}}, \|f\|_{\mathcal{F}_\nu^{1,1}})\\ &= C_1\Big[2(1+\frac{1}{2}C_3(1-\|f\|^2_{\mathcal{F}_\nu^{1,1}}))C_6^2 + |A_\kappa|\lambda_0\\ &\quad + \frac{1}{2}|1+A_\mu|(|A_\kappa|\lambda_2 + C_5(|A_\kappa|\lambda_2+\lambda_3))\\ &\quad + 2\pi|A_\kappa|(C_0C_2 + (C_0^2-1)C_4) + \frac{1}{2}|A_\mu|\lambda_3 + C_3C_6\|f\|_{\mathcal{F}_\nu^{1,1}}\\ &\quad + \frac{2C_6^2}{\|f\|_{\mathcal{F}_\nu^{1,1}}}\Big]. \end{aligned} \tag{4.23}$$

Note that $\sigma_i(\|f\|_{\mathcal{F}_\nu^{0,1}}, \|f\|_{\mathcal{F}_\nu^{1,1}})$, $i=1,2$, are continuous functions in $(\|f\|_{\mathcal{F}_\nu^{0,1}}, \|f\|_{\mathcal{F}_\nu^{1,1}})$ such that $\sigma_i(0,0)=0$.

5. L^2 estimates

5.1. Analytic estimates

In this section, we will prove the L^2 estimate (see (2.3)) of Theorem 2.1. We will introduce the notation $\lesssim$, which indicates a constant depending on $\|f\|_{\mathcal{F}_\nu^{0,1}}$ and $\|f\|_{\mathcal{F}_\nu^{1,1}}$. Let us begin by differentiating

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) &= \nu\|f\|^2_{\dot{H}^{1/2}_\nu}\\ &\quad - A_\rho\int_{\mathbb{R}} e^{2\nu t|\xi|}|\xi||\widehat{f}(\xi)|^2\left(1-\frac{A_\kappa(1-A_\mu)}{e^{2h_2|\xi|}-A_\kappa A_\mu}\right)d\xi\\ &\quad + \langle \widehat{f}, \text{h.o.t.}\rangle_{L^2_\nu} \end{aligned} \tag{5.1}$$

in which the higher order terms are defined as

$$\begin{aligned}\text{h.o.t.} = \frac{1}{2\pi}(\widehat{I}_2+\widehat{I}_4) + \widehat{N}_0(\xi) + (1+A_\mu)(\widehat{N}_1+\widehat{N}_2+\widehat{N}_3)(\xi) + \widehat{N}_4(\xi)\\ - \frac{i}{2}\,\text{sgn}(\xi)N(\widehat{\omega}_1)(\xi).\end{aligned}$$

In the following estimates, we will use the convention $\widetilde{g}(x) = g(-x)$ and the convolution identities

$$\begin{aligned}\int_{\mathbb{R}} g_1(x)(g_2*g_3)(x)dx &= \int_{\mathbb{R}} g_1(x)\int_{\mathbb{R}} g_2(x-y)g_3(y)dydx\\ &= \int_{\mathbb{R}} g_3(y)\int_{\mathbb{R}} g_1(x)g_2(x-y)dxdy\\ &= \int_{\mathbb{R}} g_3(y)(g_1*\widetilde{g}_2)(y)dy\end{aligned}$$

and

$$\widetilde{g_1 * g_2} = \widetilde{g_1} * \widetilde{g_2}.$$

Using the substitution $|\widehat{\omega}_1(\xi)| = |\xi||\widehat{\Omega}_1(\xi)|$, we obtain

$$\begin{aligned}
\langle \widehat{f}, \widehat{I}_2 \rangle_{L^2_\nu} &\le \int_{\mathbb{R}} e^{2\nu t|\xi|} |\widehat{f}(\xi)| \cdot \pi \sum_{n=0}^{\infty} \big((*^{2n+2} |\widehat{\partial_\alpha f}|) * |\cdot||\widehat{\Omega}_1| \big)(\xi)\, d\xi \\
&\le \pi \sum_{n=0}^{\infty} \int_{\mathbb{R}} e^{\nu t|\cdot|} |\widehat{f}(\xi)| \big((*^{2n+2} e^{\nu t|\cdot|} |\widehat{\partial_\alpha f}|) * e^{\nu t|\cdot|} |\cdot||\widehat{\Omega}_1| \big)(\xi)\, d\xi \\
&\le \pi \sum_{n=0}^{\infty} \int_{\mathbb{R}} e^{\nu t|\cdot|} |\xi||\widehat{\Omega}_1(\xi)| \big((*^{2n+2} e^{\nu t|\cdot|} |\widetilde{\widehat{\partial_\alpha f}}|) * e^{\nu t|\cdot|} |\widetilde{f}| \big)(\xi)\, d\xi.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \widehat{f}, \widehat{I}_2 \rangle_{L^2_\nu} &\le \pi \sum_{n=0}^{\infty} \|\Omega_1\|_{\dot{H}^{1/2}_\nu} \big\| |\xi|^{1/2} \big((*^{2n+2} e^{\nu t|\cdot|} |\widetilde{\widehat{\partial_\alpha f}}|) * e^{\nu t|\cdot|} |\widetilde{f}| \big) \big\|_{L^2} \\
&\le \pi \sum_{n=0}^{\infty} \|\Omega_1\|_{\dot{H}^{1/2}_\nu} \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu} \big((2n+2) \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|_{L^2_\nu} + \|f\|_{\mathcal{F}^{1,1}_\nu} \|f\|_{\dot{H}^{1/2}_\nu} \big) \\
&\le \pi \sum_{n=0}^{\infty} \frac{\epsilon_n}{2} \|\Omega_1\|^2_{\dot{H}^{1/2}_\nu} + \frac{1}{2\epsilon_n} (2n+2)^2 \|f\|^{4n+2}_{\mathcal{F}^{1,1}_\nu} \|f\|^2_{\mathcal{F}^{3/2,1}_\nu} \|f\|^2_{L^2_\nu} \\
&\quad + \|f\|^{2n+2}_{\mathcal{F}^{1,1}_\nu} \|\Omega_1\|_{\dot{H}^{1/2}_\nu} \|f\|_{\dot{H}^{1/2}_\nu}
\end{aligned} \tag{5.2}$$

where the last inequality is from Young's inequality for products where we choose $\epsilon_n = \epsilon/(1+n^2)$ for a small value $\epsilon > 0$. The other nonlinear terms can be estimated via similar methods

$$\begin{aligned}
\langle \widehat{f}, \widehat{I}_4 \rangle_{L^2_\nu} \le \pi \sum_{n=0}^{\infty} \Big\| e^{-h_2|\xi|} \frac{|\xi|^n}{n!} \Big\|_{L^\infty} \|\Omega_2\|_{\dot{H}^{1/2}_\nu} \|f\|^n_{\mathcal{F}^{0,1}_\nu} \\
\cdot \big(\|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|_{L^2_\nu} + (n+1) \|f\|_{\mathcal{F}^{1,1}_\nu} \|f\|_{\dot{H}^{1/2}_\nu} \big)
\end{aligned}$$

and

$$\begin{aligned}
\langle \widehat{f}, \widehat{N}_1 \rangle_{L^2_\nu} &\le \frac{1}{2} \int_{\mathbb{R}} d\xi\, e^{2\nu t|\xi|} e^{-h_2|\xi|} |\widehat{f}(\xi)| |N(\widehat{\partial_\alpha \Omega_2})(\xi)| \\
&\le \frac{|A_\kappa|}{2} \sum_{n=1}^{\infty} \int_{\mathbb{R}} d\xi\, e^{2\nu t|\xi|} e^{-2h_2|\xi|} \frac{|\xi|^n}{n!} \cdot |\widehat{f}(\xi)| \big((*^n |\widehat{f}|) * |\cdot||\widehat{\Omega}_1| \big)(\xi) \\
&\le \frac{|A_\kappa|}{2} \sum_{n=1}^{\infty} \Big\| e^{-2h_2|\xi|} \frac{|\xi|^n}{n!} \Big\|_{L^\infty} \int_{\mathbb{R}} d\xi\, e^{2\nu t|\xi|} |\xi||\widehat{\Omega}_1(\xi)| (*^n |\widetilde{\widehat{f}}| * |\widehat{f}|)(\xi)
\end{aligned}$$

$$\leq \frac{|A_\kappa|}{2}\sum_{n=1}^{\infty}\Big\|e^{-2h_2|\xi|}\frac{|\xi|^n}{n!}\Big\|_{L^\infty}(n+1)\|f\|^n_{\mathcal{F}^{0,1}_\nu}\|\Omega_1\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu}$$
$$\stackrel{\text{def}}{=} |A_\kappa|\lambda_4\|\Omega_1\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu}$$

where $\lambda_4 \to 0$ as $\|f\|_{\mathcal{F}^{0,1}_\nu} + \|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$. Next,

$$\begin{aligned}
&\langle \widehat{f}, \widehat{N}_2\rangle_{L^2_\nu}\\
&\quad\leq \frac{1}{2}C_5\Big(\|\Omega_1\|_{\dot H^{1/2}_\nu}\sum_{n=0}^{\infty}2(\|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu}\|f\|_{\dot H^{1/2}_\nu}+(2n+1)\|f\|^{2n}_{\mathcal{F}^{1,1}_\nu}\|f\|_{\mathcal{F}^{3/2,1}_\nu}\|f\|_{L^2_\nu})\\
&\qquad+\|\Omega_1\|_{\dot H^{1/2}_\nu}\sum_{n=0}^{\infty}\Big\|e^{-h_2|\xi|}\frac{|\xi|^n}{n!}\Big\|_{L^\infty}\|f\|^n_{\mathcal{F}^{0,1}_\nu}(\|f\|_{\mathcal{F}^{3/2,1}_\nu}\|f\|_{L^2_\nu}\\
&\qquad\qquad+(n+1)\|f\|_{\mathcal{F}^{1,1}_\nu}\|f\|_{\dot H^{1/2}_\nu})\\
&\qquad+\|\Omega_2\|_{\dot H^{1/2}_\nu}\sum_{n=1}^{\infty}\Big\|e^{-h_2|\xi|}\frac{|\xi|^n}{n!}\Big\|_{L^\infty}(n+1)\|f\|^n_{\mathcal{F}^{0,1}_\nu}\|f\|_{\dot H^{1/2}_\nu}\Big)\\
&\quad\stackrel{\text{def}}{=}\frac{1}{2}C_5\big(\|\Omega_1\|_{\dot H^{1/2}_\nu}(\lambda_5\|f\|_{\dot H^{1/2}_\nu}+\lambda_6\|f\|_{\mathcal{F}^{3/2,1}_\nu}\|f\|_{L^2_\nu})\\
&\qquad+\lambda_7\|\Omega_2\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu}\big),
\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}
\langle \widehat{f}, \widehat{N}_3\rangle_{L^2_\nu} &\leq \frac{1}{2}C_5\int_{\mathbb{R}}d\xi\; e^{2\nu t|\xi|}e^{-h_2|\xi|}|\widehat{f}(\xi)||N(\widehat{\Omega}_2)(\xi)|\\
&\leq \frac{1}{2}C_5|A_\kappa|\lambda_4\|\Omega_1\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu},\\
\langle \widehat{f}, \widehat{N}_4\rangle_{L^2_\nu} &\leq \pi\sum_{n=1}^{\infty}\Big\|e^{-h_2|\xi|}\frac{|\xi|^n}{n!}\Big\|_{L^\infty}(n+1)\|f\|^n_{\mathcal{F}^{0,1}_\nu}\|\Omega_2\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu},
\end{aligned}$$

in which

$$\lambda_5,\lambda_7\to 0 \quad\text{and}\quad \lambda_6\lesssim 1 \quad\text{as}\quad \|f\|_{\mathcal{F}^{0,1}_\nu}+\|f\|_{\mathcal{F}^{1,1}_\nu}\to 0. \tag{5.4}$$

Related to the computations in (5.3),

$$\begin{aligned}
\langle \widehat{f}, \widehat{N}_0\rangle_{L^2_\nu} &\leq \frac{1}{2}\|\Omega_1\|_{\dot H^{1/2}_\nu}\sum_{n=0}^{\infty}\|f\|^{2n+2}_{\mathcal{F}^{1,1}_\nu}\|f\|_{\dot H^{1/2}_\nu}\\
&\qquad+(2n+2)\|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu}\|f\|_{\mathcal{F}^{3/2,1}_\nu}\|f\|_{L^2_\nu},\\
\langle \widehat{f}, N(\widehat{\partial_\alpha\Omega_1})\rangle_{L^2_\nu} &\leq \|\Omega_1\|_{\dot H^{1/2}_\nu}(\lambda_5\|f\|_{\dot H^{1/2}_\nu}+\lambda_6\|f\|_{\mathcal{F}^{3/2,1}_\nu}\|f\|_{L^2_\nu})\\
&\qquad+\lambda_7\|\Omega_2\|_{\dot H^{1/2}_\nu}\|f\|_{\dot H^{1/2}_\nu}.
\end{aligned}$$

So, it is enough to control Ω_1 and Ω_2. We can write

$$\widehat{\Omega}_1 = \widehat{\Omega}_{11} + \widehat{\Omega}_{12} - 2A_\rho\widehat{f}.$$

Using (3.4) and (3.5), we have

$$\|\Omega_{11}\|_{\dot{H}_\nu^{1/2}} \le |A_\mu| \sum_{n=0}^{\infty} \|f\|_{\mathcal{F}_\nu^{1,1}}^{2n+1} \|\Omega_1\|_{\dot{H}_\nu^{1/2}} + (2n+1)\|f\|_{\mathcal{F}_\nu^{1,1}}^{2n} \|f\|_{\mathcal{F}_\nu^{3/2,1}} \|\Omega_1\|_{L_\nu^2}$$

$$\stackrel{\text{def}}{=} |A_\mu|(C_6\|\Omega_1\|_{\dot{H}_\nu^{1/2}} + C_7\|\Omega_1\|_{L_\nu^2}) \tag{5.5}$$

$$\|\Omega_{12}\|_{\dot{H}_\nu^{1/2}} \le |A_\mu|(C_0 + C_8)\|\Omega_2\|_{\dot{H}_\nu^{1/2}}, \tag{5.6}$$

where

$$C_7 \to \|f\|_{\mathcal{F}_\nu^{3/2,1}},$$

$$C_8 = \sum_{n=1}^{\infty} n \left\| e^{-h_2|\xi|} \frac{|\xi|^{n-1/2}}{n!} \right\|_{L^\infty} \|f\|_{\mathcal{F}_\nu^{0,1}}^{n-1} \|f\|_{\mathcal{F}_\nu^{1/2,1}} \to \frac{1}{\sqrt{2eh_2}} \|f\|_{\mathcal{F}_\nu^{1/2,1}} \to 0$$

as $\|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0$. With (3.1) and (3.2), we have the bounds

$$\|\Omega_2\|_{\dot{H}_\nu^{1/2}} \le |A_\kappa| C_0(1 + \|f\|_{\mathcal{F}_\nu^{1,1}})\|\Omega_1\|_{\dot{H}_\nu^{1/2}}$$

$$+ |A_\kappa| \Big(\frac{C_2}{\|f\|_{\mathcal{F}_\nu^{0,1}}} \|f\|_{\mathcal{F}_\nu^{1/2,1}} (1 + \|f\|_{\mathcal{F}_\nu^{1,1}}) + C_0\|f\|_{\mathcal{F}_\nu^{3/2,1}} \Big) \|\Omega_1\|_{L_\nu^2}$$

$$\stackrel{\text{def}}{=} |A_\kappa|(C_9\|\Omega_1\|_{\dot{H}_\nu^{1/2}} + C_{10}\|\Omega_1\|_{L_\nu^2}) \tag{5.7}$$

with $C_9 \to 1$ and $C_{10} \to |A_\kappa| \|f\|_{\mathcal{F}_\nu^{3/2,1}}$ as $\|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0$. Similarly to the above calculations, we derive

$$\|\Omega_2\|_{L_\nu^2} \le |A_\kappa| C_0(1 + \|f\|_{\mathcal{F}_\nu^{1,1}})\|\Omega_1\|_{L_\nu^2} = |A_\kappa| C_9 \|\Omega_1\|_{L_\nu^2}$$

and therefore, by (3.4) and (3.5), we have

$$(1 - |A_\mu|C_6)\|\Omega_1\|_{L_\nu^2} \le |A_\mu| C_0 \|\Omega_2\|_{L_\nu^2} + 2A_\rho \|f\|_{L_\nu^2}$$

$$\le |A_\kappa||A_\mu| C_0 C_9 \|\Omega_1\|_{L_\nu^2} + 2A_\rho \|f\|_{L_\nu^2}$$

which gives

$$\|\Omega_1\|_{L_\nu^2} \le 2A_\rho C_{11} \|f\|_{L_\nu^2}, \tag{5.8}$$

$$\|\Omega_2\|_{L_\nu^2} \le 2A_\rho |A_\kappa| C_9 C_{11} \|f\|_{L_\nu^2} \tag{5.9}$$

where

$$C_{11} = (1 - |A_\mu|C_6 - |A_\kappa||A_\mu|C_0C_9)^{-1}$$

$$\to (1 - |A_\kappa||A_\mu|)^{-1} \quad \text{as} \quad \|f\|_{\mathcal{F}_\nu^{0,1}} + \|f\|_{\mathcal{F}_\nu^{1,1}} \to 0. \tag{5.10}$$

Collecting terms from (5.5), (5.6), (5.7), (5.8), and (5.9) we find

$$\|\Omega_1\|_{\dot{H}_\nu^{1/2}} \le 2A_\rho C_{12}(|A_\mu| C_{11}(C_7 + |A_\kappa|(C_0 + C_8)C_{10})\|f\|_{L_\nu^2} + \|f\|_{\dot{H}_\nu^{1/2}})$$

$$\stackrel{\text{def}}{=} 2A_\rho C_{12}(C_{13}\|f\|_{L_\nu^2} + \|f\|_{\dot{H}_\nu^{1/2}}) \tag{5.11}$$

$$\begin{aligned}\|\Omega_2\|_{\dot{H}_\nu^{1/2}} &\leq 2A_\rho|A_\kappa|((C_9C_{12}C_{13}+C_{10}C_{11})\|f\|_{L_\nu^2}+C_9C_{12}\|f\|_{\dot{H}_\nu^{1/2}})\\ &\stackrel{\text{def}}{=} 2A_\rho|A_\kappa|(C_{14}\|f\|_{L_\nu^2}+C_9C_{12}\|f\|_{\dot{H}_\nu^{1/2}})\end{aligned} \tag{5.12}$$

for

$$\begin{aligned}&C_{12}=(1-|A_\mu|(C_6+|A_\kappa|(C_0+C_8)C_9))^{-1}\to(1-|A_\kappa||A_\mu|)^{-1},\\ &C_{13}\lesssim 1+\|f\|_{\mathcal{F}_\nu^{3/2,1}},\quad C_{14}\lesssim 1+\|f\|_{\mathcal{F}_\nu^{3/2,1}}.\end{aligned}$$

We see two expression types. For the first type of expression, of the form

$$\|\Omega_i\|_{\dot{H}_\nu^{1/2}}\|f\|_{L_\nu^2},$$

we can apply Young's inequality for products and control the $\dot{H}_\nu^{1/2}$ terms as in (5.2), for example, using (5.11):

$$\begin{aligned}\|\Omega_1\|_{\dot{H}_\nu^{1/2}}\|f\|_{L_\nu^2} &\leq 2A_\rho C_{12}(C_{13}\|f\|_{L_\nu^2}^2+\|f\|_{\dot{H}_\nu^{1/2}}\|f\|_{L_\nu^2})\\ &\leq \frac{C}{\varepsilon_n}\|f\|_{L_\nu^2}^2+\varepsilon_n\|f\|_{\dot{H}_\nu^{1/2}}^2.\end{aligned}$$

Here ε_n as earlier in (5.2) can always be chosen arbitrarily small. For the second type of expression $\|\Omega_i\|_{\dot{H}_\nu^{1/2}}\|f\|_{\dot{H}_\nu^{1/2}}$, after applying (5.11) or (5.12) we can control the resulting $\|f\|_{\dot{H}_\nu^{1/2}}^2$ term via the linear decay term in the interface equation; see σ_2 below. In terms with $\|f\|_{L_\nu^2}$ that contain a coefficient of $\|f\|_{\mathcal{F}_\nu^{3/2,1}}^2$, such as the middle term in (5.2), we use $\|f\|_{\mathcal{F}_\nu^{3/2,1}}^2\leq\|f\|_{\mathcal{F}_\nu^{1,1}}\|f\|_{\mathcal{F}_\nu^{2,1}}$ by (2.6). Collecting terms from above, (5.1) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|_{L_\nu^2}^2(t) &\leq(-A_\rho\theta+\nu+A_\rho\sigma_2+\varepsilon)\|f\|_{\dot{H}_\nu^{1/2}}^2\\ &\quad+(R_1+R_2\cdot(\|f\|_{\mathcal{F}_\nu^{3/2,1}}+\|f\|_{\mathcal{F}_\nu^{2,1}}))\|f\|_{L_\nu^2}^2,\end{aligned}$$

where

$$\begin{aligned}\sigma_2 &\stackrel{\text{def}}{=}\sigma_2(\|f\|_{\mathcal{F}_\nu^{0,1}},\|f\|_{\mathcal{F}_\nu^{1,1}})\\ &=C_{12}(C_6+C_0+C_2)\|f\|_{\mathcal{F}_\nu^{1,1}}\\ &\quad+|1+A_\mu|C_{12}(2|A_\kappa|\lambda_4+C_5(\lambda_5+|A_\kappa|(\lambda_7C_9+\lambda_4)))\\ &\quad+2\pi|A_\kappa|(C_0-1+C_2)C_9C_{12}+C_6C_{12}\|f\|_{\mathcal{F}_\nu^{1,1}}+2\lambda_5C_{12}\\ &\quad+2|A_\kappa|\lambda_7C_9C_{12},\end{aligned} \tag{5.13}$$

in which $\varepsilon=\varepsilon(\|f_0\|_{\mathcal{F}_0^{0,1}},\|f_0\|_{\mathcal{F}_0^{1,1}})$ can be chosen to be sufficiently small and $R_i=R_i(\|f_0\|_{\mathcal{F}_\nu^{0,1}},\|f_0\|_{\mathcal{F}_\nu^{1,1}})$ for $i=1,2$ is bounded for medium-sized initial data. Again, recall that

$$\|f\|_{\mathcal{F}_\nu^{3/2,1}}^2\leq\|f\|_{\mathcal{F}_\nu^{1,1}}\|f\|_{\mathcal{F}_\nu^{2,1}}$$

by (2.6). Hence, by (2.2), $\|f\|_{\mathcal{F}_\nu^{1,1}}(t)$ and $\|f\|_{\mathcal{F}_\nu^{2,1}}(t)$ are L^1 functions in time on $[0,T]$ for any $T>0$, and hence, so is $\|f\|_{\mathcal{F}_\nu^{3/2,1}}(t)$ with L^1 norm bounded by initial data. By Gronwall's inequality, we obtain (2.3).

5.2. Sobolev space estimates

In this section, we will prove an evolution estimate for a subcritical Sobolev norm $H^{\frac{3}{2}+\epsilon}$ of the interface. This estimate will be used in the local existence proof in the next section. We have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\dot{H}^{\frac{3}{2}+\epsilon}}(t) = -A_\rho \int_{\mathbb{R}} |\xi|^{4+2\epsilon}|\hat{f}(\xi)|^2\Big(1 - \frac{A_\kappa(1-A_\mu)}{e^{2h_2|\xi|} - A_\kappa A_\mu}\Big)\, d\xi + \langle \hat{f}, \text{h.o.t.}\rangle_{H^{\frac{3}{2}+\epsilon}}.$$

The weighted triangle inequality

$$\Big|\sum_{j=1}^{n} a_j\Big|^{1+\epsilon} \le n^\epsilon \sum_{j=1}^{n} |a_j|^{1+\epsilon}$$

allows us to use the constants from prior sections with the convention that for $0 \le k \le 14$, $C_{k,\epsilon}$ is C_k adjusted for the small ϵ weight from the weighted triangle inequality. All of the $C_{k,\epsilon}$ are bounded and $C_{k,\epsilon} \to C_k$ as $\epsilon \to 0$. As an example, we compute the constants $C_{0,\epsilon}$, $C_{1,\epsilon}$, and $C_{2,\epsilon}$ explicitly.

By (3.6), we have

$$\begin{aligned}\|\Omega_2\|_{\dot{H}^{2+\epsilon}} &= \|\omega_2\|_{\dot{H}^{1+\epsilon}} \\ &\le |A_\kappa|\Big(\frac{C_{2,\epsilon}}{\|f\|_{\mathcal{F}_0^{0,1}}}\|\omega_1\|_{\mathcal{F}_0^{0,1}}\|f\|_{\dot{H}^{1+\epsilon}} + C_{0,\epsilon}\|\Omega_1\|_{\dot{H}^{2+\epsilon}}\Big),\end{aligned}$$

where

$$C_{0,\epsilon} = \sum_{n=0}^{\infty} \frac{n^n(n+1)^\epsilon}{e^n n!}\Big(\frac{\|f\|_{\mathcal{F}_\nu^{0,1}}}{h_2}\Big)^n, \quad C_{2,\epsilon} = \sum_{n=1}^{\infty} \frac{n^{n+1}(n+1)^\epsilon}{e^n n!}\Big(\frac{\|f\|_{\mathcal{F}_\nu^{0,1}}}{h_2}\Big)^n.$$

Similarly, by (3.7) and (3.8),

$$\begin{aligned}&\|\Omega_1\|_{\dot{H}^{2+\epsilon}} \\ &\le |A_\mu|\Big(2\frac{1+\|f\|^2_{\mathcal{F}_0^{1,1}}}{(1-\|f\|^2_{\mathcal{F}_0^{1,1}})^2}\|\omega_1\|_{\mathcal{F}_0^{0,1}}\|f\|_{\dot{H}^{2+\epsilon}} + 2C_{6,\epsilon}\|\Omega_1\|_{\dot{H}^{2+\epsilon}} \\ &\quad + C_{0,\epsilon}(1+\|f\|_{\mathcal{F}_0^{1,1}})\|\Omega_2\|_{\dot{H}^{2+\epsilon}} + \frac{C_{2,\epsilon}}{\|f\|_{\mathcal{F}_0^{0,1}}}(1+\|f\|_{\mathcal{F}_0^{1,1}})\|\omega_2\|_{\mathcal{F}_0^{0,1}}\|f\|_{\dot{H}^{1+\epsilon}} \\ &\quad + C_{0,\epsilon}\|\omega_2\|_{\mathcal{F}_0^{0,1}}\|f\|_{\dot{H}^{2+\epsilon}}\Big) + 2A_\rho\|f\|_{\dot{H}^{2+\epsilon}}.\end{aligned}$$

From this, we derive

$$\|\Omega_1\|_{\dot{H}^{2+\epsilon}} \le 2A_\rho C_{1,\epsilon}(C_{15,\epsilon}\|f\|_{\dot{H}^{1+\epsilon}} + C_{16,\epsilon}\|f\|_{\dot{H}^{2+\epsilon}}) \tag{5.14}$$

where, for example, we recover the ϵ adjusted constant

$$C_{1,\epsilon} = \big(1 - 2|A_\mu|C_{6,\epsilon} - |A_\mu||A_\kappa|C^2_{0,\epsilon}(1+\|f\|_{\mathcal{F}_0^{1,1}})\big)^{-1}$$

and $\lim_{\epsilon\to 0} C_{16,\epsilon} < C_3$ and $C_{15,\epsilon}$ is a finite constant obtained using (3.21). This in turn implies

$$\|\Omega_2\|_{\dot{H}^{2+\epsilon}} \le 2A_\rho(C_{17,\epsilon}\|f\|_{\dot{H}^{1+\epsilon}} + C_{18,\epsilon}\|f\|_{\dot{H}^{2+\epsilon}}), \tag{5.15}$$

where $C_{17,\epsilon}$ is a finite constant computed similarly to $C_{15,\epsilon}$ and where

$$C_{18,\epsilon} = C_{0,\epsilon}C_{1,\epsilon}C_{16,\epsilon}$$

which, as $\epsilon \to 0$, approaches a value strictly less than C_1C_4.

Next, similarly computed to (4.15) and (4.16), we have that

$$\begin{aligned}\langle \hat{f}, \hat{I}_2\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \pi \int_{\mathbb{R}} |\xi|^{3+2\epsilon}|\hat{f}(\xi)| \sum_{n=0}^{\infty} (*^{2n+2}|\widehat{\partial_\alpha f}| * (|\cdot||\Omega_1|))(\xi)d\xi \\ &\le \pi\|f\|_{\dot{H}^{2+\epsilon}} \sum_{n=0}^{\infty} \left\| *^{2n+2}|\widehat{\partial_\alpha f}| * |\cdot||\Omega_1| \right\|_{\dot{H}^{1+\epsilon}}.\end{aligned}$$

Applying the weighted triangle inequality and then using (5.14) and (5.15), we obtain

$$\begin{aligned}&\langle \hat{f}, \hat{I}_2\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} \\ &\quad\le \pi\|f\|_{\dot{H}^{2+\epsilon}} \sum_{n=0}^{\infty}(2n+3)^\epsilon\Big((2n+2)\|f\|^{2n+1}_{\mathcal{F}^{1,1}_0}\|\Omega_1\|_{\mathcal{F}^{1,1}_0}\|f\|_{\dot{H}^{2+\epsilon}} \\ &\qquad + \|f\|^{2n+2}_{\mathcal{F}^{1,1}_0}\|\Omega_1\|_{\dot{H}^{2+\epsilon}}\Big).\end{aligned}$$

Similarly, making use of the fact that the $\|N(\Omega_i)\|_{H^{2+\epsilon}}$ can be bounded in the same way as in (5.14) or (5.15),

$$\begin{aligned}\langle \hat{f}, \hat{I}_4\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \pi\|f\|_{\dot{H}^{2+\epsilon}}\Big(C_{0,\epsilon}(\|f\|_{\mathcal{F}^{1,1}_\nu}\|\Omega_2\|_{\dot{H}^{2+\epsilon}} + \|\omega_2\|_{\mathcal{F}^{0,1}_0}\|f\|_{\dot{H}^{2+\epsilon}}) \\ &\qquad + \frac{C_{2,\epsilon}}{\|f\|_{\mathcal{F}^{0,1}_0}}\|f\|_{\mathcal{F}^{1,1}_0}\|\omega_2\|_{\mathcal{F}^{0,1}_0}\|f\|_{\dot{H}^{1+\epsilon}}\Big), \\ \langle \hat{f}, \hat{N}_0\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \frac{1}{2}\|f\|_{\dot{H}^{2+\epsilon}}(2n+3)^\epsilon \\ &\qquad \cdot\Big((2n+2)\|f\|^{2n+1}_{\mathcal{F}^{1,1}_0}\|f\|_{\mathcal{F}^{1,1}_0}\|\Omega_1\|_{\dot{H}^{1+\epsilon}} + \|f\|^{2n+2}_{\mathcal{F}^{1,1}_0}\|\Omega_1\|_{\dot{H}^{2+\epsilon}}\Big), \\ \langle \hat{f}, \hat{N}_1\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \frac{1}{2}\|f\|_{\dot{H}^{2+\epsilon}}\|N(\Omega_2)\|_{\dot{H}^{2+\epsilon}}, \\ \langle \hat{f}, \hat{N}_2\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \frac{1}{2}\left\|\frac{A_\kappa}{e^{2h_2|\xi|} - A_\kappa A_\mu}\right\|_{L^\infty}\|f\|_{\dot{H}^{2+\epsilon}}\|N(\Omega_1)\|_{\dot{H}^{2+\epsilon}}, \\ \langle \hat{f}, \hat{N}_3\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \frac{1}{2}C_5\|f\|_{\dot{H}^{2+\epsilon}}\|N(\Omega_2)\|_{\dot{H}^{2+\epsilon}}, \\ \langle \hat{f}, \hat{N}_4\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \pi\|f\|_{\dot{H}^{2+\epsilon}}\Big((C_{0,\epsilon}-1)\|\Omega_2\|_{\dot{H}^{2+\epsilon}} + \frac{C_{2,\epsilon}}{\|f\|_{\mathcal{F}^{0,1}_0}}\|f\|_{\mathcal{F}^{1,1}_0}\|\Omega_2\|_{\dot{H}^{1+\epsilon}}\Big), \\ \langle \hat{f}, \widehat{N(\partial_\alpha\Omega_1)}\rangle_{\dot{H}^{\frac{3}{2}+\epsilon}} &\le \|f\|_{\dot{H}^{2+\epsilon}}\|N(\Omega_1)\|_{\dot{H}^{2+\epsilon}}.\end{aligned}$$

Combining terms and using Young's inequality for products for terms involving $\|f\|_{\dot{H}^{1+\epsilon}}$, we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\dot{H}^{\frac{3}{2}+\epsilon}}(t) \le -A_\rho(\theta-\widetilde{\sigma}_\epsilon)\|f\|^2_{\dot{H}^{2+\epsilon}} + S(\|f\|_{\mathcal{F}_0^{0,1}}, \|f\|_{\mathcal{F}_0^{1,1}})\|f\|^2_{\dot{H}^{1+\epsilon}}, \quad (5.16)$$

in which, as $\epsilon \to 0$,

$$\widetilde{\sigma}_\epsilon < \sigma_1 \quad (5.17)$$

and S is a continuous function of $\|f\|_{\mathcal{F}_0^{0,1}}$ and $\|f\|_{\mathcal{F}_0^{1,1}}$ that vanishes at $(0, 0)$. So, for every finite $T > 0$ we can bound $\frac{d}{dt}\|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}}(t)$ on $[0, T]$ giving the desired derivative in time bound.

6. Proof of the theorem

We argue similarly to [19] for the proof of existence and uniqueness of solutions to (1.6). Uniqueness of solutions is proven at the level of $\mathcal{F}_0^{0,1}$ and follows exactly as in [19]. It yields an inequality of the type

$$\frac{d}{dt}\|f-g\|_{\mathcal{F}_0^{0,1}} \le -c_1\|f-g\|_{\mathcal{F}_0^{1,1}} + c_2\|f-g\|_{\mathcal{F}_0^{0,1}} \quad (6.1)$$

for two solutions f, g of (1.6) and where $c_i = c_i(\|f_0\|_{\mathcal{F}_0^{0,1}}, \|f_0\|_{\mathcal{F}_0^{1,1}}, \|g_0\|_{\mathcal{F}_0^{0,1}}, \|g_0\|_{\mathcal{F}_0^{1,1}}) > 0$ are positive constants. Note that the key difference with [19] is that for some terms, the difference of solutions occurs at the level of $\mathcal{F}_0^{0,1}$ which, unlike the Muskat problem without a permeability jump, cannot be absorbed into the decay term.

Consider the mollified system with initial data $f_0^\varepsilon = \varphi^\varepsilon * f_0 \in H^s$ for any $s > 0$ (since $f_0 \in L^2$) and the evolution equation

$$\partial_t f^\varepsilon = \mathcal{L}(\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon) + \varphi^\varepsilon * \mathcal{N}(\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon, \omega_1^\varepsilon, \omega_2^\varepsilon) \quad (6.2)$$

where

$$\widehat{\mathcal{L}g}(\xi) = -A_\rho|\xi|\hat{g}(\xi)\Big(1 - \frac{A_\kappa(1-A_\mu)}{e^{2h_2|\xi|} - A_\kappa A_\mu}\Big)$$

and $\mathcal{N}$ denotes the remaining nonlinear terms from (4.12). Here $\omega_i^\varepsilon = \partial_\alpha \Omega_i^\varepsilon$ are given by the mollified (1.11), (1.12), (1.13), and (1.14) where f is replaced by the mollified $\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon$.

For any $\delta > 0$, the mollified system satisfies the hypothesis to apply Picard's Theorem on the open set in $H^{\frac{3}{2}+\delta}$ given by the condition that

$$\|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon\|_{\mathcal{F}_0^{0,1}} + \|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon\|_{\mathcal{F}_0^{1,1}} < 1,$$

noting that $H^{\frac{3}{2}+\delta} \hookrightarrow \mathcal{F}_0^{0,1} \cap \mathcal{F}_0^{1,1}$. Hence, we have a local solution $f^\varepsilon \in C([0,T_\varepsilon); H^{\frac{3}{2}+\delta})$. Next, we can reproduce the analogous estimates for the medium-size condition on f_0 for $0 \le s \le 1$:

$$\|f^\varepsilon\|_{\mathcal{F}_\nu^{s,1}}(t) + (A_\rho\theta - A_\rho\sigma_s - \nu)\int_0^t \|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon\|_{\mathcal{F}_\nu^{s+1,1}}(s)\, ds \le \|f_0\|_{\mathcal{F}_0^{s,1}} \tag{6.3}$$

and

$$\|f^\varepsilon\|_{L_\nu^2}^2(t) \le \|f_0\|_{L^2}^2 \cdot \exp(R(\|f_0\|_{\mathcal{F}_0^{0,1}}, \|f_0\|_{\mathcal{F}_0^{1,1}})). \tag{6.4}$$

Due to the exponential weight in L_ν^2, estimate (6.4) implies that $\|f^\varepsilon\|_{H^{\frac{3}{2}+\delta}}(t) \le C_{\delta,\varepsilon}(t)\|f_0\|_{L^2}^2 \cdot \exp(R(\|f_0\|_{\mathcal{F}_0^{0,1}}, \|f_0\|_{\mathcal{F}_0^{1,1}}))$ for $t > 0$ where $C_{\delta,\varepsilon}(t)$ is a bounded decreasing constant for $t > 0$. Moreover, it can be seen by combining the proof of (2.3) and (5.16) that

$$\frac{d}{dt}\|f^\varepsilon\|_{H^{\frac{3}{2}+\delta}}^2 \le G(\|f^\varepsilon\|_{H^{\frac{3}{2}+\delta}}, \|f_0\|_{\mathcal{F}_0^{0,1}}, \|f_0\|_{\mathcal{F}_0^{1,1}})$$

for a continuous function G. Hence, we can take the limit continuously to the endpoint of $[0,T]$ and then continuously extend the local time of existence. Thus, the local solution can be extended to $C([0,T]; H^{\frac{3}{2}+\delta})$ for any $T > 0$.

By (6.1) and following the argument in [19], the sequence f^{ε_n} is shown to be Cauchy in $L^\infty([0,T]; \mathcal{F}_0^{0,1})$ for any $\varepsilon_n \to 0$. The main idea is that by the argument from uniqueness, we have

$$\begin{aligned}\frac{d}{dt}\|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}} \le &-c_1\|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon - \varphi^{\varepsilon'} * \varphi^{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}_0^{1,1}}\\ &+ c_2\|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon - \varphi^{\varepsilon'} * \varphi^{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}},\end{aligned}$$

and hence,

$$\begin{aligned}\|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(t) \le &\|\varphi^\varepsilon * f_0 - \varphi^{\varepsilon'} * f_0\|_{\mathcal{F}_0^{0,1}}\\ &+ c_2\int_0^t \|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon - \varphi^{\varepsilon'} * \varphi^{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(s)ds.\end{aligned}$$

Using the Mean Value Theorem in the mollifiers in the Fourier variables and assuming $\varepsilon \ge \varepsilon'$,

$$\|\varphi^\varepsilon * f_0 - \varphi^{\varepsilon'} * f_0\|_{\mathcal{F}_0^{0,1}} \le C\|f_0\|_{\mathcal{F}_0^{1,1}}\varepsilon^{\frac{1}{2}}$$

and

$$\begin{aligned}\|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon - \varphi^{\varepsilon'} * \varphi^{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}} &\le \|\varphi^\varepsilon * \varphi^\varepsilon * f^\varepsilon - \varphi^\varepsilon * \varphi^\varepsilon * f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}\\ &\quad + \|\varphi^\varepsilon * \varphi^\varepsilon * f^{\varepsilon'} - \varphi^{\varepsilon'} * \varphi^{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}\\ &\le \|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}} + C\|f^{\varepsilon'}\|_{\mathcal{F}_0^{1,1}}\varepsilon^{\frac{1}{2}}\\ &\le \|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}} + C\|f_0\|_{\mathcal{F}_0^{1,1}}\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(t) \leq C(1 + c_2 t)\|f_0\|_{\mathcal{F}_0^{1,1}} \varepsilon^{\frac{1}{2}} + c_2 \int_0^t \|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(s) ds. \quad (6.5)$$

Gronwall's inequality finally yields

$$\|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(t) \leq C(1 + c_2 t)e^{c_2 t}\|f_0\|_{\mathcal{F}_0^{1,1}} \varepsilon^{\frac{1}{2}}.$$

Hence, there exists a limit $f^{\varepsilon} \to f$ in $L^{\infty}([0,T]; \mathcal{F}_0^{0,1})$. Hence, we can obtain pointwise almost everywhere convergence of a subsequence $\widehat{f}^{\varepsilon_n}(\xi, t)$ and $\widehat{\varphi}^{\varepsilon_n}(\xi, t)^2 \widehat{f}^{\varepsilon_n}(\xi, t)$ to $\widehat{f}(\xi, t)$. Thus, Fatou's Lemma applied to (6.3) allows us to conclude that the limit f indeed satisfies inequality (2.2) for $s = 0$ and $s = 1$.

Interpolation (2.6) with (6.5) and the $s = 1$ case of (6.3) yields strong convergence of $\varphi^{\varepsilon} * \varphi^{\varepsilon} * f^{\varepsilon}$ to f in $L^2([0,T]; \mathcal{F}_0^{1,1})$, similar to the argument in [19]:

$$\begin{aligned}\int_0^t \|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{1,1}}(s)^2 ds &\lesssim \int_0^t \|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{0,1}}(s)\|f^{\varepsilon} - f^{\varepsilon'}\|_{\mathcal{F}_0^{2,1}}(s) ds \\ &\leq C(1 + c_2 t)e^{c_2 t}\|f_0\|^2_{\mathcal{F}_0^{1,1}} \varepsilon^{\frac{1}{2}}.\end{aligned}$$

Finally, we can now take weak limits in (6.2) and with the regularity of (2.2), we get the limiting function f as a solution to (1.6).

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