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Weak and strong solutions for polymeric fluid-structure interaction of Oldroyd-B type

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Abstract. We prove the existence of weak solutions and a unique strong solution to the Oldroyd-B dumbbell model describing the evolution of a two-dimensional dilute polymer fluid interacting with a one-dimensional viscoelastic shell. The polymer fluid consists of a mixture of an incompressible viscous solvent and a solute comprising two massless beads connected by a Hookean spring with center-of-mass diffusion. This solute-solvent mixture then interacts with a flexible structure that evolves in time. An arbitrary nondegenerate reference domain for the polymer fluid is allowed and both solutions exist globally in time provided no future degeneracies occur with the structure deformation. Furthermore, weak-strong uniqueness holds unconditionally.

1. Introduction

A polymer fluid is a complex fluid with a high molecular weight consisting of a mixture of a solvent and a solute. In this work, the solvent is a viscous fluid modeled by the incompressible Navier–Stokes equation and the solute is described by the macroscopic average of the probability distribution function of two monomers connected by a Hookean spring and modeled by the Oldroyd-B system (also referred to as the *convected Jeffreys model* by some authors [13]). We refer to [2] for the modeling of the generalized state-of-the-art where the fluid's density is also allowed to vary.

We initiate work on the rigorous analysis of such a polymer fluid within a flexible structure and the interaction between the evolution of the structure and the polymer fluid. The polymer fluid is two-dimensional and contained in a domain whose boundary is a flexible structure in 1D modeled by a viscoelastic shell equation. We deviate from the usual practice in the literature where one considers a simple flat reference domain for fluids and allows for a generalized reference domain for the solute-solvent mixture.

1.1. Geometric setup and equations of motion

We consider a fluid domain whose reference configuration is $\Omega \subset \mathbb{R}^2$. The boundary of this reference domain $\partial\Omega$ may consist of a flexible part $\omega \subset \mathbb{R}$ and a rigid part $\Gamma \subset \mathbb{R}$.

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However, because the analysis at the rigid part is significantly simpler, we shall identify the whole of $\partial\Omega$ with ω. Let $I := (0, T)$ represent a time interval for a given constant $T > 0$. We represent the time-dependent displacement of the structure by $\eta : \overline{I} \times \omega \to (-L, L)$ where $L > 0$ is a fixed length of the tubular neighborhood of $\partial\Omega$ given by

$$S_L := \big\{\mathbf{x} \in \mathbb{R}^2 : \mathrm{dist}(\mathbf{x}, \partial\Omega) < L\big\}.$$

Now, for some $k \in \mathbb{N}$ large enough, we assume that $\partial\Omega$ is parametrized by an injective mapping $\boldsymbol{\varphi} \in C^k(\omega; \mathbb{R}^2)$ with $\partial_y \boldsymbol{\varphi} \neq 0$ such that

$$\partial\Omega_{\eta(t)} = \big\{\boldsymbol{\varphi}_{\eta(t)} := \boldsymbol{\varphi}(y) + \mathbf{n}(y)\eta(t, y) : t \in I, y \in \omega\big\}.$$

The set $\partial\Omega_{\eta(t)}$ represents the boundary of the flexible domain at any instant of time $t \in I$ and the vector $\mathbf{n}(y)$ is a unit normal at the point $y \in \omega$. We also let $\mathbf{n}_{\eta(t)}(y)$ be the corresponding normal of $\partial\Omega_{\eta(t)}$ at the space-time point $y \in \omega$ and $t \in I$. Then for $L > 0$ sufficiently small, $\mathbf{n}_{\eta(t)}(y)$ is close to $\mathbf{n}(y)$ and $\boldsymbol{\varphi}_{\eta(t)}$ is close to $\boldsymbol{\varphi}$. Since $\partial_y \boldsymbol{\varphi} \neq 0$, it will follow that

$$\partial_y \boldsymbol{\varphi}_{\eta(t)} \neq 0 \quad \text{and} \quad \mathbf{n}(y) \cdot \mathbf{n}_{\eta(t)}(y) \neq 0$$

for $y \in \omega$ and $t \in I$. Thus, in particular, there is no loss of strict positivity of the Jacobian determinant, provided that $\|\eta\|_{L^\infty(I\times\omega)} < L$.

For the interior points, we transform the reference domain Ω into a time-dependent moving domain $\Omega_{\eta(t)}$ whose state at time $t \in \overline{I}$ is given by

$$\Omega_{\eta(t)} = \big\{\boldsymbol{\Psi}_{\eta(t)}(\mathbf{x}) : \mathbf{x} \in \Omega\big\}.$$

Here,

$$\boldsymbol{\Psi}_{\eta(t)}(\mathbf{x}) = \begin{cases} \mathbf{x} + \mathbf{n}(y(\mathbf{x}))\eta(t, y(\mathbf{x}))\phi(s(\mathbf{x})) & \text{if } \mathrm{dist}(\mathbf{x}, \partial\Omega) < L, \\ \mathbf{x} & \text{elsewhere} \end{cases}$$

is the Hanzawa transform with inverse $\boldsymbol{\Psi}_{-\eta(t)}$ and where, for a point $\mathbf{x}$ in the neighborhood of $\partial\Omega$, the vector $\mathbf{n}(y(\mathbf{x}))$ is the unit normal at the point $y(\mathbf{x}) = \mathrm{argmin}_{y\in\omega}|\mathbf{x} - \boldsymbol{\varphi}(y)|$. Also, $s(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\varphi}(y(\mathbf{x}))) \cdot \mathbf{n}(y(\mathbf{x}))$ and $\phi \in C^\infty(\mathbb{R})$ is a cut-off function, that is, $\phi \equiv 0$ in the neighborhood of $-L$ and $\phi \equiv 1$ in the neighborhood of 0. Note that $\boldsymbol{\Psi}_{\eta(t)}(\mathbf{x})$ can be rewritten as

$$\boldsymbol{\Psi}_{\eta(t)}(\mathbf{x}) = \begin{cases} \boldsymbol{\varphi}(y(\mathbf{x})) + \mathbf{n}(y(\mathbf{x}))[s(\mathbf{x}) + \eta(t, y(\mathbf{x}))\phi(s(\mathbf{x}))] & \text{if } \mathrm{dist}(\mathbf{x}, \partial\Omega) < L, \\ \mathbf{x} & \text{elsewhere.} \end{cases}$$

With the above preparation in hand, we consider the Oldroyd-B model for the flow of a dilute polymeric fluid interacting with a flexible structure in the closure of the deformed space-time cylinder

$$I \times \Omega_\eta := \bigcup_{t\in I}\{t\} \times \Omega_\eta$$

with $\Omega_\eta := \Omega_{\eta(t)}$. Our goal is to find a structure displacement function $\eta : (t, y) \in I \times \omega \mapsto \eta(t, y) \in \mathbb{R}$, a fluid velocity field $\mathbf{u} : (t, \mathbf{x}) \in I \times \Omega_\eta \mapsto \mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^2$, a pressure function

$p : (t,\mathbf{x}) \in I \times \Omega_\eta \mapsto p(t,\mathbf{x}) \in \mathbb{R}$, a polymer number density $\rho : (t,\mathbf{x}) \in I \times \Omega_\eta \mapsto \rho(t,\mathbf{x}) \in \mathbb{R}$ and an extra stress tensor $\mathbb{T} : (t,\mathbf{x}) \in I \times \Omega_\eta \mapsto \mathbb{T}(t,\mathbf{x}) \in \mathbb{R}^{2\times 2}$ such that the system of equations

$$\mathrm{div}_{\mathbf{x}}\mathbf{u} = 0, \tag{1.1}$$

$$\partial_t \rho + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\rho = \varepsilon\Delta_{\mathbf{x}}\rho, \tag{1.2}$$

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \nu\Delta_{\mathbf{x}}\mathbf{u} - \nabla_{\mathbf{x}} p + \mathbf{f} + K\mathrm{div}_{\mathbf{x}}\mathbb{T}, \tag{1.3}$$

$$\varrho_s\partial_t^2\eta - \gamma\partial_t\partial_y^2\eta + \alpha\partial_y^4\eta = g - (\mathbb{S}\mathbf{n}_\eta)\circ\boldsymbol{\varphi}_\eta\cdot\mathbf{n}\,\det(\partial_y\boldsymbol{\varphi}_\eta), \tag{1.4}$$

$$\partial_t\mathbb{T} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbb{T} = (\nabla_{\mathbf{x}}\mathbf{u})\mathbb{T} + \mathbb{T}(\nabla_{\mathbf{x}}\mathbf{u})^\top - 2k(\mathbb{T} - \rho\mathbb{I}) + \varepsilon\Delta_{\mathbf{x}}\mathbb{T} \tag{1.5}$$

holds on $I \times \Omega_\eta \subset \mathbb{R}^{1+2}$ where

$$\mathbb{S} = \nu(\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^\top) - p\mathbb{I} + K\mathbb{T},$$

the parameters $\varepsilon, K, \gamma, k, \nu, \varrho_s, \alpha$ are all positive constants, $\mathbf{n}_\eta$ is the normal at $\partial\Omega_\eta$, and $\mathbb{I}$ is the identity matrix. We complement (1.1)–(1.5) with the following initial and boundary conditions:

$$\eta(0,\cdot) = \eta_0(\cdot), \quad \partial_t\eta(0,\cdot) = \eta_\star(\cdot) \quad \text{in } \omega, \tag{1.6}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega_{\eta_0}, \tag{1.7}$$

$$\rho(0,\cdot) = \rho_0(\cdot), \quad \mathbb{T}(0,\cdot) = \mathbb{T}_0(\cdot) \quad \text{in } \Omega_{\eta_0}, \tag{1.8}$$

$$\mathbf{n}_\eta\cdot\nabla_{\mathbf{x}}\rho = 0, \quad \mathbf{n}_\eta\cdot\nabla_{\mathbf{x}}\mathbb{T} = 0 \quad \text{on } I\times\partial\Omega_\eta. \tag{1.9}$$

Furthermore, for simplicity, we impose periodicity on the boundary of ω and the following interface condition:

$$\mathbf{u}\circ\boldsymbol{\varphi}_\eta = (\partial_t\eta)\mathbf{n} \quad \text{on } I\times\omega \tag{1.10}$$

between the polymeric fluid and the flexible part of the boundary with normal $\mathbf{n}$.

The two unknowns ρ and $\mathbb{T}$ for the solute component of the polymer fluid are related via the identities

$$\mathbb{T}(t,\mathbf{x}) = k\int_B f(t,\mathbf{x},\mathbf{q})\mathbf{q}\otimes\mathbf{q}\,\mathrm{d}\mathbf{q}, \quad \rho(t,\mathbf{x}) = \int_B f(t,\mathbf{x},\mathbf{q})\,\mathrm{d}\mathbf{q}$$

where for $B = \mathbb{R}^2$ with elements $\mathbf{q}\in B$, the function f is the probability density function ($f \geq 0$ a.e. on $\overline{I}\times\Omega_\eta\times B$) satisfying the Fokker–Planck equation

$$\partial_t f + \mathrm{div}_{\mathbf{x}}(\mathbf{u}f) + \mathrm{div}_{\mathbf{q}}((\nabla_{\mathbf{x}}\mathbf{u})\mathbf{q}f) = \varepsilon\Delta_{\mathbf{x}} f + k\mathrm{div}_{\mathbf{q}}(M\nabla_{\mathbf{q}}(f/M)) \tag{1.11}$$

in $I\times\Omega_\eta\times B$ for a Hookean dumbbell spring potential and Maxwellian

$$U\Big(\frac{1}{2}|\mathbf{q}|^2\Big) = \frac{1}{2}|\mathbf{q}|^2, \quad M = \frac{\exp(-U(\frac{1}{2}|\mathbf{q}|^2))}{\int_B\exp(-U(\frac{1}{2}|\mathbf{q}|^2))\,\mathrm{d}\mathbf{q}},$$

respectively. Whether the center-of-mass diffusion parameter ε in (1.11) is zero or positive has been a source of many debates over the years. In this work, we follow the school of thought such as [4,25,49] that gives justifications for the inclusion of this term. As shown in [2], this choice leads to the elegant parabolized macroscopic closure equation for the polymer number density ρ and the elastic extra stress tensor $\mathbb{T}$ satisfying (1.2) and (1.5), respectively.

The existence of smooth solutions for the coupling of the Fokker–Planck equation (1.11) with a generalized viscous compressible fluid in a fixed domain has been shown in [17], whereas weak solutions have earlier been constructed in [9, 10, 29]. More work has been done in the incompressible case in fixed domains. Weak solutions are constructed in [3–8] and in [44] when $\varepsilon = 0$ in (1.11), whereas results on the existence of strong solutions are shown in [21,22,34,35,38,42,43,47,53].

There are several works on the rigorous analysis of Oldroyd-B-type models in a fixed spatial domain or for domains without boundaries. Strong solutions have been constructed in [32], where the authors also show the existence of stable time-periodic solutions when the forcing term is small and time periodic. For small data, the existence of global-in-time classical solutions has been shown in [36] using an incompressible limit argument. Small data global solutions living in Hölder spaces and Besov spaces are also shown in [24] and [28, 52, 55], respectively. See also [54] for the three-dimensional case. The authors in [24] also cover large data for a model in which the potential responds to high rates of strain in the fluid. When stress diffusion is taken into account, the authors in [23] study the global existence and regularity of strong solutions. See also [26,27,51], which cover the case where the solvent is inviscid and [20] where there is no damping or dissipation in the equation for the extra stress tensor. Results on weak solutions are unusually fewer and we refer to [11, 33, 39] where global weak solutions are constructed and to [12], which combines the Oldroyd-B and the Giesekus models. Related results also include the Peterlin model [41] and the compressible Oldroyd-B model without stress diffusion [40].

There are only a couple of results on polymer fluids evolving and interacting with a flexible structure. The existence of weak solutions to a finitely extensible dilute polymer (see (1.11)) of Warner type immersed in a viscous incompressible fluid (1.1)–(1.3) and with this 3D mixture interacting with a 2D elastic structure (1.4) ($\gamma = 0$) has been shown in [16]. These weak solutions are global provided the shell does not self-intersect and there are no degeneracies in the system. This was recently followed up with the local-in-time existence of a unique strong solution [15] for a viscoelastic structure with $\gamma > 0$. The mathematical analysis of an Oldroyd-B model interacting with a flexible structure is completely open. Related works, however, include [30,48] where the authors study strong solutions of the immersion of solid objects in viscoelastic fluids.

1.2. Work plan

The paper is structured in the following way: in Section 2, we introduce some notations, give the precise definition of the types of solutions we are investigating, and state our main results. To make the construction of solutions as concise as possible, we present in Section 3 the a priori estimates leading to global bounds. These bounds will be referred to several times in later sections. We then devote the entirety of Section 4 to the proof of the existence of weak solutions. This will come in three main steps. Firstly, we construct a finite-dimensional approximation of the linearized solvent-structure subproblem and of the solute subproblem independently of each other. Here, the linearized subproblem is advected by a given regularized vector field and evolves on a given regularized geometry. A fixed point argument is then used to obtain a finite-dimensional approximation of the fully coupled linearized system. After this, we pass to the limit in the discrete parameter to obtain a unique solution to the linear and regularized system. Our second step will be to remove the given advected vector field and the given spatial geometry to obtain a weak solution to the fully nonlinear regularized system posed on a regularized spatial geometry that is evolving according to the structure equation. Here, we use a Schauder-type fixed point argument whose key ingredient is an Aubin–Lions-type compactness result [46, Theorem 5.1] for establishing the compactness of the fluid's velocity fields. We lose uniqueness at this point due to the nonlinearity in the system and the fact that the regularity of weak solutions is low. The final step then involves the passage to the limit in the regularization parameter to obtain a weak solution to the original system.

We devote Section 5 to the construction of a unique strong solution via a fixed point argument. Strong solutions are only an instant of differentiability stronger than weak solutions. This low regularity of the strong solutions coupled with the boundary-valued effect of the moving domain (e.g., boundary terms from applying the Reynolds transport theorem) makes it very difficult to obtain useful estimates. Indeed, the most challenging part of Section 5 is the proof of contraction, which requires obtaining estimates for the difference of two solutions. Firstly, each solution is defined on a unique spatial domain evolving in time and as such, it is not clear what the spatial domain for the difference between the two solutions is. Secondly, the strong solutions are not very regular and are actually only an instant on integrability stronger than weak solutions. Consequently, the standard Hölder doubles or triples for estimating the product of functions leads one to require more regularity for some of the terms in the product than they actually possess.

There are at least two ways to remedy the first problem. We can either transform each of the solutions to the fixed reference domain and study the difference equation on this fixed domain or we transform one solution to the domain of the other and study the difference equation on this latter domain. The second approach is more difficult than the first. However, we opt for the second approach, since it allows us to take full advantage of the regularity properties of the latter geometry on which we study the difference equation. This approach has been used in [31] to show the continuous dependence on initial data in a fluid-structure problem, in [50] to obtain a weak-strong uniqueness result for a fluid interacting with elastic plates, and in [19] for elastic shells.

The key to solving the second problem is to find equivalent (in the sense that two spaces are continuously embedded in each other) square-integrable Sobolev spaces of fractional differentiability for non-square-integrable Sobolev spaces in order to obtain sharp interpolation between the weakest and strongest spaces of the functions being estimated.

Finally, we show in Section 6 that weak-strong uniqueness holds unconditionally, that is, weak solutions are always unique in the class of strong solutions.

2. Preliminaries and main results

Henceforth, without loss of generality, we set all the parameters $\{\varepsilon, K, \gamma, k, \nu, \varrho_s, \alpha\}$ in (1.1)–(1.9) to 1. For two non-negative quantities F and G, we write $F \lesssim G$ if there is a constant $c > 0$ such that $F \leq cG$. If $F \lesssim G$ and $G \lesssim F$ both hold, we use the notation $F \sim G$. The symbol $|\cdot|$ may be used in four different contexts. For a scalar function $f \in \mathbb{R}$, $|f|$ denotes the absolute value of f. For a vector $\mathbf{f} \in \mathbb{R}^d$, $|\mathbf{f}|$ denotes the Euclidean norm of $\mathbf{f}$. For a square matrix $\mathbb{F} \in \mathbb{R}^{d\times d}$, $|\mathbb{F}|$ shall denote the Frobenius norm $\sqrt{\mathrm{trace}(\mathbb{F}^T\mathbb{F})}$. Finally, if $S \subseteq \mathbb{R}^d$ is a (sub)set, then $|S|$ is the d-dimensional Lebesgue measure of S.

For $I = (0, T)$, $T > 0$, and $\eta \in C(\overline{I} \times \omega)$ with $\|\eta\|_{L^\infty(I\times\omega)} < L$, we define for $1 \leq p, r \leq \infty$,

$$\begin{aligned} L^p(I; L^r(\Omega_\eta)) &:= \big\{v \in L^1(I \times \Omega_\eta) : v(t,\cdot) \in L^r(\Omega_{\eta(t)}) \text{ for a.e. } t, \\ &\qquad \|v(t,\cdot)\|_{L^r(\Omega_{\eta(t)})} \in L^p(I)\big\}, \\ L^p(I; W^{1,r}(\Omega_\eta)) &:= \big\{v \in L^p(I; L^r(\Omega_\eta)) : \nabla_{\mathbf{x}} v \in L^p(I; L^r(\Omega_\eta))\big\}. \end{aligned}$$

Higher-order Sobolev spaces can be defined accordingly. For $k > 0$ with $k \notin \mathbb{N}$, we define the fractional Sobolev space $L^p(I; W^{k,r}(\Omega_\eta))$ as the class of $L^p(I; L^r(\Omega_\eta))$-functions v for which

$$\|v\|^p_{L^p(I;W^{k,r}(\Omega_\eta))} = \int_I \Big(\int_{\Omega_\eta} |v|^r \,\mathrm{d}\mathbf{x} + \int_{\Omega_\eta}\int_{\Omega_\eta} \frac{|v(\mathbf{x}) - v(\mathbf{x}')|^r}{|\mathbf{x} - \mathbf{x}'|^{d+kr}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \Big)^{\frac{p}{r}} \,\mathrm{d}t$$

is finite. Accordingly, we can also introduce fractional differentiability in time for the spaces on moving domains. Next, for $\eta \in C(\overline{I} \times \omega)$ satisfying

$$\|\eta\|_{L^\infty(I\times\omega)} \leq L \quad \text{and} \quad \|\partial_y \eta\|_{L^\infty(I\times\omega)} \leq c(L),$$

where $L > 0$ is a constant, we let

$$\mathrm{Bog}_\eta : C_0^\infty(\Omega_\eta) \to C_0^\infty(\Omega_\eta; \mathbb{R}^d) \quad \text{with} \quad \mathrm{div}_{\mathbf{x}} \mathrm{Bog}_\eta(f) = f - b \int_{\Omega_\eta} f \,\mathrm{d}\mathbf{x},$$

with $b \in C_0^\infty(\Omega_\eta \setminus S_L)$, be the time-dependent Bogovskij operator (see [19, Theorem 2.11 and Remark 2.12]). The operator Bog_η vanishes on the boundary $\partial\Omega_\eta$, it commutes with time derivatives, and it continuously maps scalar elements in $W^{s,p}(\Omega_\zeta)$ into vectors in

$W^{s+1,p}(\Omega_\zeta;\mathbb{R}^d)$ for any $p \in (1,\infty)$ and $s \geq 0$. Furthermore, the Hanzawa transform $\mathbf{\Psi}_\eta$ together with its inverse $\mathbf{\Psi}_\eta^{-1} : \Omega_\eta \to \Omega$ possesses the following properties; see [14, 19] for details: if for some $\ell, R > 0$, we assume that

$$\|\eta\|_{L^\infty(\omega)} + \|\zeta\|_{L^\infty(\omega)} < \ell < L \quad \text{and} \quad \|\partial_y\eta\|_{L^\infty(\omega)} + \|\partial_y\zeta\|_{L^\infty(\omega)} < R$$

holds, then for any $s > 0$, $\varrho, p \in [1,\infty]$ and for any $\eta, \zeta \in B^s_{\varrho,p}(\omega) \cap W^{1,\infty}(\omega)$ (where $B^s_{\varrho,p}$ is a Besov space), we have that the estimates

$$\begin{aligned}\|\mathbf{\Psi}_\eta\|_{B^s_{\varrho,p}(\Omega\cup S_\ell)} + \|\mathbf{\Psi}_\eta^{-1}\|_{B^s_{\varrho,p}(\Omega\cup S_\ell)} &\lesssim 1 + \|\eta\|_{B^s_{\varrho,p}(\omega)},\\ \|\mathbf{\Psi}_\eta - \mathbf{\Psi}_\zeta\|_{B^s_{\varrho,p}(\Omega\cup S_\ell)} + \|\mathbf{\Psi}_\eta^{-1} - \mathbf{\Psi}_\zeta^{-1}\|_{B^s_{\varrho,p}(\Omega\cup S_\ell)} &\lesssim \|\eta - \zeta\|_{B^s_{\varrho,p}(\omega)}\end{aligned}$$

and

$$\|\partial_t\mathbf{\Psi}_\eta\|_{B^s_{\varrho,p}(\Omega\cup S_\ell)} \lesssim \|\partial_t\eta\|_{B^s_{\varrho,p}(\omega)}, \qquad \eta \in W^{1,1}(I; B^s_{\varrho,p}(\omega))$$

holds uniformly in time with the hidden constants depending only on the reference geometry, $L - \ell$, and R.

2.1. The concept of solutions and the main results

Let us start with a precise definition of what we mean by a weak solution. We recall that $L > 0$ is a sufficiently small constant as introduced in Section 1.1.

Definition 2.1 (Weak solution). Let $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ be a dataset that satisfies

$$\begin{aligned}&\mathbf{f} \in L^2(I; L^2_{\mathrm{loc}}(\mathbb{R}^2)), \quad g \in L^2(I; L^2(\omega)),\\ &\eta_0 \in W^{2,2}(\omega) \text{ with } \|\eta_0\|_{L^\infty(\omega)} < L, \quad \eta_\star \in L^2(\omega),\\ &\mathbf{u}_0 \in L^2_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\eta_0}) \text{ is such that } \mathbf{u}_0 \circ \boldsymbol{\varphi}_{\eta_0} = \eta_\star\mathbf{n} \text{ on } \omega,\\ &\rho_0 \in L^2(\Omega_{\eta_0}), \quad \mathbb{T}_0 \in L^2(\Omega_{\eta_0}),\\ &\rho_0 \geq 0, \ \mathbb{T}_0 > 0, \quad \text{a.e. in } \Omega_{\eta_0}.\end{aligned} \tag{2.1}$$

We call $(\eta, \mathbf{u}, \rho, \mathbb{T})$ a *weak solution* of (1.1)–(1.10) with data $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ if:

(a) the properties

$$\begin{aligned}&\eta \in W^{1,\infty}(I; L^2(\omega)) \cap W^{1,2}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{2,2}(\omega)),\\ &\|\eta\|_{L^\infty(I\times\omega)} < L,\\ &\mathbf{u} \in L^\infty(I; L^2(\Omega_\eta)) \cap L^2(I; W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_\eta)),\\ &\rho \in L^\infty(I; L^2(\Omega_\eta)) \cap L^2(I; W^{1,2}(\Omega_\eta)),\\ &\mathbb{T} \in L^\infty(I; L^2(\Omega_\eta)) \cap L^2(I; W^{1,2}(\Omega_\eta))\end{aligned}$$

hold;

(b) for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \rho\psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_I \int_{\Omega_\eta} [\rho\partial_t\psi + (\rho\mathbf{u}\cdot\nabla_{\mathbf{x}})\psi]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \nabla_{\mathbf{x}}\rho\cdot\nabla_{\mathbf{x}}\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t; \tag{2.2}$$

(c) for all $\mathbb{Y} \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\begin{aligned}\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \mathbb{T} : \mathbb{Y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = &\int_I \int_{\Omega_\eta} [\mathbb{T} : \partial_t\mathbb{Y} + \mathbb{T} : (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbb{Y}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &+ \int_I \int_{\Omega_\eta} [(\nabla_{\mathbf{x}}\mathbf{u})\mathbb{T} + \mathbb{T}(\nabla_{\mathbf{x}}\mathbf{u})^\top] : \mathbb{Y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &- 2\int_I \int_{\Omega_\eta} (\mathbb{T} : \mathbb{Y} - \rho\mathrm{tr}(\mathbb{Y}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &- \int_I \int_{\Omega_\eta} \nabla_{\mathbf{x}}\mathbb{T} :: \nabla_{\mathbf{x}}\mathbb{Y}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\end{aligned} \tag{2.3}$$

where $\nabla_{\mathbf{x}}\mathbb{T} :: \nabla_{\mathbf{x}}\mathbb{Y} = \sum_{i=1}^2 \partial_{x_i}\mathbb{T} : \partial_{x_i}\mathbb{Y}$;

(d) for all $(\boldsymbol{\phi}, \phi) \in C^\infty_{\mathrm{div}_{\mathbf{x}}}(\overline{I} \times \mathbb{R}^2) \otimes C^\infty(\overline{I} \times \omega)$ with $\boldsymbol{\phi}(T, \cdot) = 0$, $\phi(T, \cdot) = 0$ and $\boldsymbol{\phi} \circ \boldsymbol{\varphi}_\eta = \phi\mathbf{n}$, we have

$$\begin{aligned}&\int_I \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\Omega_\eta} \mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x} + \int_\omega \partial_t\eta\phi\,\mathrm{d}y\Big)\,\mathrm{d}t \\ &\quad= \int_I \int_{\Omega_\eta} [\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \mathbf{u}\cdot(\mathbf{u}\cdot\nabla_{\mathbf{x}})\boldsymbol{\phi}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\qquad- \int_I \int_{\Omega_\eta} \big[\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\boldsymbol{\phi} - \mathbf{f}\cdot\boldsymbol{\phi} + \mathbb{T} : \nabla_{\mathbf{x}}\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\qquad+ \int_I \int_\omega \big[\partial_t\eta\partial_t\phi - \partial_t\partial_y\eta\partial_y\phi - \partial_y^2\eta\partial_y^2\phi + g\phi\big]\,\mathrm{d}y\,\mathrm{d}t;\end{aligned} \tag{2.4}$$

(e) the energy inequality

$$\begin{aligned}&\sup_{t\in I}\Big(\int_{\Omega_\eta} \mathrm{tr}(\mathbb{T}(t))\,\mathrm{d}\mathbf{x} + \|\mathbf{u}(t)\|^2_{L^2(\Omega_\eta)} + \|\partial_t\eta(t)\|^2_{L^2(\omega)} + \|\partial_y^2\eta(t)\|^2_{L^2(\omega)}\Big) \\ &\qquad+ \int_I \int_{\Omega_\eta} \mathrm{tr}(\mathbb{T})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t + \int_I \|\partial_t\partial_y\eta\|^2_{L^2(\omega)}\,\mathrm{d}t \\ &\quad\lesssim \int_{\Omega_{\eta_0}} \mathrm{tr}(\mathbb{T}_0)\,\mathrm{d}\mathbf{x} + \|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})} + \|\eta_\star\|^2_{L^2(\omega)} + \|\partial_y^2\eta_0\|^2_{L^2(\omega)} \\ &\qquad+ T\int_{\Omega_{\eta_0}} \rho_0\,\mathrm{d}\mathbf{x} + \int_I \|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t + \int_I \|g\|^2_{L^2(\omega)}\,\mathrm{d}t\end{aligned} \tag{2.5}$$

holds;

(f) in addition, we have

$$\begin{aligned}\sup_{t\in I}\big(\|\rho(t)\|^2_{L^2(\Omega_\eta)}+\|\mathbb{T}(t)\|^2_{L^2(\Omega_\eta)}\big)&+\int_I\big(\|\rho\|^2_{W^{1,2}(\Omega_\eta)}+\|\mathbb{T}\|^2_{W^{1,2}(\Omega_\eta)}\big)\,\mathrm{d}t\\&+\int_I\|\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\\&\lesssim\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}+\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})}+T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)\\&\qquad\times\exp\Big(c\int_I\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big).\end{aligned}\tag{2.6}$$

With this definition in hand, we now state our first main result.

Theorem 2.2. *For a dataset* $(\mathbf{f},g,\rho_0,\mathbb{T}_0,\mathbf{u}_0,\eta_0,\eta_\star)$ *that satisfies* (2.1), *there exists a weak solution* $(\eta,\mathbf{u},\rho,\mathbb{T})$ *of* (1.1)–(1.10).

Remark 2.3. As in [1] (see also [2, Section 8.2]), a solution $\mathbb{T}$ of (1.5) remains strictly positive if it were initially so. Furthermore, a solution ρ of (1.2) also remains non-negative if it were initially non-negative. Indeed, if we test (1.2) with the nonpositive part $\rho_-=\min\{0,\rho\}$ of ρ, integrate over Ω_η and use the boundary condition (1.9) together with the Reynolds transport theorem, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta}|\rho_-|^2\,\mathrm{d}\mathbf{x}+\int_{\Omega_\eta}|\nabla_{\mathbf{x}}\rho_-|^2\,\mathrm{d}\mathbf{x}=0.$$

Therefore, it follows that $\rho_-=0$ almost everywhere in Ω_η for any $t\in I$ and thus, $\rho=\rho_+=\max\{0,\rho\}$. Rigorously showing the strict positivity of ρ an $\mathbb{T}$, however, requires another approximating layer. For simplicity, however, we omit the requirement of preserving positivity in our notion of a weak solution.

We note that since the test function for the (weak) momentum equation in (2.4) is divergence free, our weak formulation is consequently pressure free. However, when the weak solution and the forcing $\mathbf{f}$ are sufficiently regular, the pressure can be recovered by solving an elliptic problem obtained by taking the divergence in (1.3). A first step to obtaining said regularity is by showing the existence of a strong solution, which is the subject of our second main result. Here, by a “strong solution”, we mean the following:

Definition 2.4 (Strong solution). Let $(\mathbf{f},g,\rho_0,\mathbb{T}_0,\mathbf{u}_0,\eta_0,\eta_\star)$ be a dataset that satisfies

$$\begin{aligned}&\mathbf{f}\in L^2(I;L^2_{\mathrm{loc}}(\mathbb{R}^2)),\quad g\in L^2(I;L^2(\omega)),\\&\eta_0\in W^{3,2}(\omega)\text{ with }\|\eta_0\|_{L^\infty(\omega)}<L,\quad \eta_\star\in W^{1,2}(\omega),\\&\mathbf{u}_0\in W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\eta_0})\text{ is such that }\mathbf{u}_0\circ\boldsymbol{\varphi}_{\eta_0}=\eta_\star\mathbf{n}\text{ on }\omega,\\&\rho_0\in W^{1,2}(\Omega_{\eta_0}),\quad \mathbb{T}_0\in W^{1,2}(\Omega_{\eta_0}),\\&\rho_0\geq 0,\ \mathbb{T}_0>0,\quad \text{a.e. in }\Omega_{\eta_0}.\end{aligned}\tag{2.7}$$

We call $(\eta, \mathbf{u}, p, \rho, \mathbb{T})$ a *strong solution* of (1.1)–(1.10) with data $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ if

(a) $(\eta, \mathbf{u}, \rho, \mathbb{T})$ is a weak solution of (1.1)–(1.10);

(b) the structure-function η is such that $\|\eta\|_{L^\infty(I\times\omega)} < L$ and

$$\eta \in W^{1,\infty}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{3,2}(\omega)) \cap W^{1,2}(I; W^{2,2}(\omega)) \\ \cap W^{2,2}(I; L^2(\omega)) \cap L^2(I; W^{4,2}(\omega));$$

(c) the velocity $\mathbf{u}$ is such that $\mathbf{u} \circ \boldsymbol{\varphi}_\eta = (\partial_t \eta)\mathbf{n}$ on $I \times \omega$ and

$$\mathbf{u} \in W^{1,2}(I; L^2_{\mathrm{div}_{\mathbf{x}}}(\Omega_\eta)) \cap L^2(I; W^{2,2}(\Omega_\eta));$$

(d) the pressure p is such that

$$p \in L^2(I; W^{1,2}(\Omega_\eta));$$

(e) the pair $(\rho, \mathbb{T})$ is such that

$$\rho, \mathbb{T} \in W^{1,2}(I; L^2(\Omega_\eta)) \cap L^\infty\big(I; W^{1,2}(\Omega_\eta)\big) \cap L^2(I; W^{2,2}(\Omega_\eta));$$

(f) equations (1.1)–(1.5) are satisfied almost everywhere in space-time with $\eta(0) = \eta_0$ and $\partial_t \eta(0) = \eta_\star$ almost everywhere in ω, as well as $\mathbf{u}(0) = \mathbf{u}_0$, $\rho(0) = \rho_0$ and $\mathbb{T}(0) = \mathbb{T}_0$ almost everywhere in Ω_{η_0}.

Our second main result concerning the existence of a strong solution is given as follows:

Theorem 2.5. *For a dataset* $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ *that satisfies* (2.7)*, there exists a strong solution* $(\eta, \mathbf{u}, p, \rho, \mathbb{T})$ *of* (1.1)–(1.10)*.*

Finally, a consequence of Theorem 2.5 and its proof is the following unconditional weak-strong uniqueness result:

Theorem 2.6. *Let* $(\eta_w, \mathbf{u}_w, \rho_w, \mathbb{T}_w)$ *be a weak solution of* (1.1)–(1.10) *with dataset* $(\mathbf{f}_w, g_w, \rho_{0,w}, \mathbb{T}_{0,w}, \mathbf{u}_{0,w}, \eta_{0,w}, \eta_{\star,w})$ *in the sense of Definition* 2.1 *and let* $(\eta_s, \mathbf{u}_s, \rho_s, \mathbb{T}_s)$ *be a strong solution of* (1.1)–(1.10) *with dataset* $(\mathbf{f}_s, g_s, \rho_{0,s}, \mathbb{T}_{0,s}, \mathbf{u}_{0,s}, \eta_{0,s}, \eta_{\star,s})$ *in the sense of Definition* 2.4*. Set*

$$\rho_{ws} := \rho_w - \overline{\rho}_s, \quad \mathbb{T}_{ws} := \mathbb{T}_w - \overline{\mathbb{T}}_s, \quad \mathbf{u}_{ws} = \mathbf{u}_w - \overline{\mathbf{u}}_s, \\ \eta_{ws} = \eta_w - \eta_s, \quad \mathbf{f}_{ws} := \mathbf{f}_w - \overline{\mathbf{f}}_s, \quad g_{ws} = g_w - g_s,$$

where $\overline{f_s} := f_s \circ \boldsymbol{\Psi}_{\eta_s - \eta_w}$*. Then the inequality*

$$\sup_{t\in I}\big(\|\rho_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\mathbb{T}_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\mathbf{u}_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\partial_t \eta_{ws}(t)\|^2_{L^2(\omega)}\big) \\ + \sup_{t\in I}\|\partial_y^2 \eta_{ws}(t)\|^2_{L^2(\omega)} + \int_I \big(\|\nabla_{\mathbf{x}} \rho_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\nabla_{\mathbf{x}} \mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta_w})}\big)\,\mathrm{d}t \\ + \int_I \big(\|\nabla_{\mathbf{x}} \mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\partial_t \partial_y \eta_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t$$

$$\lesssim \|\rho_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})}+\|\mathbb{T}_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})}+\|\mathbf{u}_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})}+\|\partial_t\eta_{ws}(0)\|^2_{L^2(\omega)}$$
$$+\|\partial_y^2\eta_{ws}(0)\|^2_{L^2(\omega)}+\int_I\big(\|\mathbf{f}_{ws}\|^2_{L^2(\Omega_{\eta_w})}+\|g_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t$$

hold.

3. A priori estimates

In this section, we derive formal estimates in energy norms satisfied by smooth solutions of (1.1)–(1.5). The first two estimates will be shown to be satisfied by weak solutions in subsequent sections and the last estimate will be satisfied by a strong solution.

Proposition 3.1. *Any smooth solution* $(\eta,\mathbf{u},p,\rho,\mathbb{T})$ *of* (1.1)–(1.10) *with smooth dataset* $(\mathbf{f},g,\rho_0,\mathbb{T}_0,\mathbf{u}_0,\eta_0,\eta_\star)$ *satisfies*

$$\begin{aligned}
&\sup_{t\in I}\Big(\int_{\Omega_\eta}\mathrm{tr}(\mathbb{T}(t))\,\mathrm{d}\mathbf{x}+\|\mathbf{u}(t)\|^2_{L^2(\Omega_\eta)}+\|\partial_t\eta(t)\|^2_{L^2(\omega)}+\|\partial_y^2\eta(t)\|^2_{L^2(\omega)}\Big)\\
&\qquad+\int_I\int_{\Omega_\eta}\mathrm{tr}(\mathbb{T})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|\partial_t\partial_y\eta\|^2_{L^2(\omega)}\,\mathrm{d}t\\
&\quad\lesssim\int_{\Omega_{\eta_0}}\mathrm{tr}(\mathbb{T}_0)\,\mathrm{d}\mathbf{x}+\|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})}+\|\eta_\star\|^2_{L^2(\omega)}+\|\partial_y^2\eta_0\|^2_{L^2(\omega)}\\
&\qquad+T\int_{\Omega_{\eta_0}}\rho_0\,\mathrm{d}\mathbf{x}+\int_I\|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|g\|^2_{L^2(\omega)}\,\mathrm{d}t.
\end{aligned}\tag{3.1}$$

Proof. Take $(\boldsymbol{\phi},\phi)=(\mathbf{u},\partial_t\eta)$ in (2.4) to obtain

$$\begin{aligned}
&\frac{1}{2}\int_I\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{u}\|^2_{L^2(\Omega_\eta)}+\|\partial_t\eta\|^2_{L^2(\omega)}+\|\partial_y^2\eta\|^2_{L^2(\omega)}\big)\,\mathrm{d}t\\
&\qquad+\int_I\big(\|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}+\|\partial_t\partial_y\eta\|^2_{L^2(\omega)}\big)\,\mathrm{d}t\\
&\quad=-\int_I\int_{\Omega_\eta}\mathbb{T}:\nabla_{\mathbf{x}}\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_{\Omega_\eta}\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\omega g\partial_t\eta\,\mathrm{d}y\,\mathrm{d}t
\end{aligned}\tag{3.2}$$

where

$$\int_I\int_{\Omega_\eta}\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\le c\int_I\|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\frac{1}{4}\sup_{t\in I}\|\mathbf{u}(t)\|^2_{L^2(\Omega_\eta)},$$
$$\int_I\int_\omega g\partial_t\eta\,\mathrm{d}y\,\mathrm{d}t\le c\int_I\|g\|^2_{L^2(\omega)}\,\mathrm{d}t+\frac{1}{4}\sup_{t\in I}\|\partial_t\eta(t)\|^2_{L^2(\omega)}.$$

On the other hand, if we take the trace in (1.5), integrate and use (1.9) and the relation $\mathrm{tr}(\mathbb{A}\mathbb{B}^\top) = \mathbb{A} : \mathbb{B}$ which holds for all $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}$, we obtain

$$\begin{aligned} \frac{1}{2}\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} \mathrm{tr}(\mathbb{T})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I\int_{\Omega_\eta} \mathrm{tr}(\mathbb{T})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ = \int_I\int_{\Omega_\eta} \mathbb{T} : \nabla_{\mathbf{x}}\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + 3\int_I\int_{\Omega_\eta} \rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

Now note that due to the Neumann boundary condition (1.9) for ρ, if we integrate (1.2) over space, and apply Gauss's theorem and the fact that the left-hand side is transported by divergence-free velocity, it follows that

$$\int_{\Omega_\eta} \rho\,\mathrm{d}\mathbf{x} = \int_{\Omega_{\eta_0}} \rho_0\,\mathrm{d}\mathbf{x}.$$

Combining everything finishes the proof. ∎

Proposition 3.2. *Any smooth solution* $(\eta, \mathbf{u}, p, \rho, \mathbb{T})$ *of* (1.1)–(1.10) *with smooth dataset* $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ *satisfies*

$$\begin{aligned} &\sup_{t\in I}\left(\|\rho(t)\|^2_{L^2(\Omega_\eta)} + \|\mathbb{T}(t)\|^2_{L^2(\Omega_\eta)}\right) + \int_I \left(\|\rho\|^2_{W^{1,2}(\Omega_\eta)} + \|\mathbb{T}\|^2_{W^{1,2}(\Omega_\eta)}\right)\mathrm{d}t \\ &\quad\lesssim \|\rho_0\|^2_{L^2(\Omega_{\eta_0})} + \left(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\right)\exp\left(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\right). \end{aligned} \tag{3.4}$$

Proof. If we set $\psi = \rho$ in (2.2), we obtain

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} |\rho|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_I\int_{\Omega_\eta} [\rho\partial_t\rho + (\rho\mathbf{u}\cdot\nabla_{\mathbf{x}})\rho]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_I\int_{\Omega_\eta} |\nabla_{\mathbf{x}}\rho|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Thus, by the Reynolds transport theorem, we obtain

$$\frac{1}{2}\sup_{t\in I}\|\rho(t)\|^2_{L^2(\Omega_\eta)} + \int_I \|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t = \frac{1}{2}\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}. \tag{3.5}$$

On the other hand, if we set $\mathbb{Y} = \mathbb{T}$ in (2.3), we obtain

$$\begin{aligned} \int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} |\mathbb{T}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_I\int_{\Omega_\eta} [\mathbb{T} : \partial_t\mathbb{T} + \mathbb{T} : (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbb{T}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \int_I\int_{\Omega_\eta} [(\nabla_{\mathbf{x}}\mathbf{u})\mathbb{T} + \mathbb{T}(\nabla_{\mathbf{x}}\mathbf{u})^\top] : \mathbb{T}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - 2\int_I\int_{\Omega_\eta} (|\mathbb{T}|^2 - \rho\mathrm{tr}(\mathbb{T}))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_I\int_{\Omega_\eta} |\nabla_{\mathbf{x}}\mathbb{T}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned}$$

If we now use the estimate

$$\int_{\Omega_\eta} |\mathrm{tr}(\mathbb{T})|^2\,\mathrm{d}\mathbf{x} \leq \int_{\Omega_\eta} |\mathbb{T}|^2\,\mathrm{d}\mathbf{x}$$

and the Reynolds transport theorem, we obtain

$$\begin{aligned}\frac{1}{2}\int_I \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{T}(t)\|_{L^2(\Omega_\eta)}^2\,\mathrm{d}t + \int_I \|\mathbb{T}\|_{W^{1,2}(\Omega_\eta)}^2\,\mathrm{d}t \\ \leq 2\int_I\int_{\Omega_\eta} |\nabla_{\mathbf{x}}\mathbf{u}||\mathbb{T}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \|\rho\|_{L^2(\Omega_\eta)}^2\,\mathrm{d}t \\ \leq 2\int_I\int_{\Omega_\eta} |\nabla_{\mathbf{x}}\mathbf{u}||\mathbb{T}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + T\|\rho_0\|_{L^2(\Omega_{\eta_0})}^2\end{aligned}$$

where the second inequality follows from (3.5). We now note that

$$\begin{aligned}2\int_{\Omega_\eta} |\nabla_{\mathbf{x}}\mathbf{u}||\mathbb{T}|^2\,\mathrm{d}\mathbf{x} &\lesssim \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}\|\mathbb{T}\|_{L^4(\Omega_\eta)}^2 \lesssim \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}\|\mathbb{T}\|_{L^2(\Omega_\eta)}\|\mathbb{T}\|_{W^{1,2}(\Omega_\eta)} \\ &\leq \delta\|\mathbb{T}\|_{W^{1,2}(\Omega_\eta)} + c\|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}^2\|\mathbb{T}\|_{L^2(\Omega_\eta)}^2\end{aligned}$$

holds for any $\delta > 0$. Consequently, it follows from Grönwall's lemma that

$$\begin{aligned}\sup_{t\in I}\|\mathbb{T}(t)\|_{L^2(\Omega_\eta)}^2 + \int_I \|\mathbb{T}\|_{W^{1,2}(\Omega_\eta)}^2\,\mathrm{d}t \\ \lesssim \big(\|\mathbb{T}_0\|_{L^2(\Omega_{\eta_0})}^2 + T\|\rho_0\|_{L^2(\Omega_{\eta_0})}^2\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}^2\,\mathrm{d}t\Big).\end{aligned} \tag{3.6}$$

Combining everything finishes the proof. ■

We now present a result that will be satisfied by strong solutions.

Proposition 3.3. *Any smooth solution* $(\eta, \mathbf{u}, p, \rho, \mathbb{T})$ *of* (1.1)–(1.10) *with smooth dataset* $(\mathbf{f}, g, \rho_0, \mathbb{T}_0, \mathbf{u}_0, \eta_0, \eta_\star)$ *satisfies*

$$\begin{aligned}&\int_I \big(\|\partial_t\rho\|_{L^2(\Omega_\eta)}^2 + \|\partial_t\mathbb{T}\|_{L^2(\Omega_\eta)}^2 + \|\partial_t\mathbf{u}\|_{L^2(\Omega_\eta)}^2 + \|\partial_t^2\eta\|_{L^2(\omega)}^2\big)\,\mathrm{d}t \\ &\quad + \sup_{t\in I}\big(\|\rho(t)\|_{W^{1,2}(\Omega_\eta)}^2 + \|\mathbb{T}(t)\|_{W^{1,2}(\Omega_\eta)}^2 + \|\mathbf{u}(t)\|_{W^{1,2}(\Omega_\eta)}^2 + \|\partial_t\eta(t)\|_{W^{1,2}(\omega)}^2\big) \\ &\quad + \sup_{t\in I}\|\eta(t)\|_{W^{3,2}(\omega)}^2 + \int_I \big(\|\rho\|_{W^{2,2}(\Omega_\eta)} + \|\mathbb{T}\|_{W^{2,2}(\Omega_\eta)} + \|\mathbf{u}\|_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t \\ &\quad + \int_I \big(\|\nabla_{\mathbf{x}}p\|_{L^2(\Omega_\eta)} + \|\partial_t\eta\|_{W^{2,2}(\omega)}^2 + \|\eta\|_{W^{4,2}(\omega)}^2\big)\,\mathrm{d}t \\ &\lesssim \|\rho_0\|_{W^{1,2}(\Omega_{\eta_0})} + \|\mathbb{T}_0\|_{W^{1,2}(\Omega_{\eta_0})} + \|\mathbf{u}_0\|_{W^{1,2}(\Omega_{\eta_0})} + \|\eta_\star\|_{W^{1,2}(\omega)}^2 + \|\eta_0\|_{W^{3,2}(\omega)}^2 \\ &\quad + \int_I \big(\|\mathbf{f}\|_{L^2(\Omega_\eta)}^2 + \|g\|_{L^2(\omega)}^2\big)\,\mathrm{d}t \\ &\quad + T\big(\|\mathbb{T}_0\|_{L^2(\Omega_{\eta_0})}^2 + T\|\rho_0\|_{L^2(\Omega_{\eta_0})}^2\big)\exp(ch_0),\end{aligned} \tag{3.7}$$

where h_0 *is the right-hand side of* (3.1).

Proof. We begin with the following acceleration estimate for the solvent-structure subsystem in (1.1) and (1.3)–(1.4) (see [14, (5.4)] and [19, (4.5)])

$$\begin{aligned}
&\sup_{t\in I}\big(\|\nabla_{\mathbf{x}}\mathbf{u}(t)\|^2_{L^2(\Omega_\eta)}+\|\partial_t\partial_y\eta(t)\|^2_{L^2(\omega)}+\|\partial_y^3\eta(t)\|^2_{L^2(\omega)}\big)\\
&\qquad+\int_I\big(\|\nabla^2_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}+\|\partial_t\mathbf{u}\|^2_{L^2(\Omega_\eta)}+\|\nabla_{\mathbf{x}}p\|^2_{L^2(\Omega_\eta)}\big)\,\mathrm{d}t\\
&\qquad+\int_I\big(\|\partial_t\partial_y^2\eta\|^2_{L^2(\omega)}+\|\partial_t^2\eta\|^2_{L^2(\omega)}\big)\,\mathrm{d}t\\
&\quad\lesssim\|\nabla_{\mathbf{x}}\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})}+\|\partial_y\eta_\star\|^2_{L^2(\omega)}+\|\partial_y^3\eta_0\|^2_{L^2(\omega)}\\
&\qquad+\int_I\|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|\nabla_{\mathbf{x}}\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|g\|^2_{L^2(\omega)}\,\mathrm{d}t.
\end{aligned}\tag{3.8}$$

The constant in the bound depends only on the right-hand side of (3.1) and the estimate is the key ingredient in showing the global-in-time existence of strong solutions for the solvent-structure subsystem in (1.1) and (1.3)–(1.4) satisfying $\|\eta\|_{L^\infty(I\times\omega)}<L$. The original estimate in [14, (5.4)] required the L^2-norm in time of $\|\partial_y g\|^2_{L^2(\omega)}$ on the right-hand side of (3.8). The fact that $\|g\|^2_{L^2(\omega)}$ (rather than $\|\partial_y g\|^2_{L^2(\omega)}$) is sufficient is shown in [19, (4.5)]. We also note that by (3.4),

$$\int_I\|\nabla_{\mathbf{x}}\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\lesssim\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}+\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})}+T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)e^{ch_0}\tag{3.9}$$

where

$$\begin{aligned}
h_0:=&\int_{\Omega_{\eta_0}}\mathrm{tr}(\mathbb{T}_0)\,\mathrm{d}\mathbf{x}+\|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})}+\|\eta_\star\|^2_{L^2(\omega)}+\|\partial_y^2\eta_0\|^2_{L^2(\omega)}\\
&+T\int_{\Omega_{\eta_0}}\rho_0\,\mathrm{d}\mathbf{x}+\int_I\|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|g\|^2_{L^2(\omega)}\,\mathrm{d}t.
\end{aligned}\tag{3.10}$$

We now test (1.2) with $\Delta_{\mathbf{x}}\rho$. This yields

$$\begin{aligned}
&\int_I\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+\int_I\|\Delta_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\\
&\quad=\int_I\int_{\Omega_\eta}((\mathbf{u}\cdot\nabla_{\mathbf{x}})\rho)\Delta_{\mathbf{x}}\rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\frac12\int_I\int_{\partial\Omega_\eta}(\partial_t\eta\mathbf{n})\circ\boldsymbol{\varphi}_\eta^{-1}\cdot\mathbf{n}_\eta|\nabla_{\mathbf{x}}\rho|^2\,\mathrm{d}\mathscr{H}^1\,\mathrm{d}t\\
&\quad=:I_1+I_2
\end{aligned}\tag{3.11}$$

where

$$I_1\le\delta\int_I\|\Delta_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t+c(\delta)\int_I\|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t$$

for any $\delta>0$, and by using $\eta\in L^\infty(I;W^{1,\infty}(\omega))$ and $\partial_t\eta\in L^\infty(I;W^{1,2}(\omega))$ (which follows from (3.8)), we also obtain

$$I_2\lesssim\int_I\|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\partial\Omega_\eta)}\|(\partial_t\eta\mathbf{n})\circ\boldsymbol{\varphi}_\eta^{-1}\cdot\mathbf{n}_\eta\|_{L^\infty(\partial\Omega_\eta)}\,\mathrm{d}t$$

$$\begin{aligned}
&\lesssim \int_I \|\rho\|_{W^{2,2}(\Omega_\eta)} \|\nabla_{\mathbf{x}}\rho\|_{W^{3/4,2}(\Omega_\eta)} \|\partial_y \eta\|_{L^\infty(\omega)} \|\partial_t \eta\|_{W^{5/4,2}(\omega)} \,\mathrm{d}t \\
&\lesssim \int_I \|\rho\|_{W^{2,2}(\Omega_\eta)}^{7/4} \|\nabla_{\mathbf{x}}\rho\|_{L^2(\Omega_\eta)}^{1/4} \|\partial_t \eta\|_{W^{1,2}(\omega)}^{3/4} \|\partial_t \eta\|_{W^{2,2}(\omega)}^{1/4} \,\mathrm{d}t \\
&\leq \delta \int_I \|\rho\|_{W^{2,2}(\Omega_\eta)}^2 + c(\delta) \int_I \|\nabla_{\mathbf{x}}\rho\|_{L^2(\Omega_\eta)}^2 \|\partial_t \eta\|_{W^{2,2}(\omega)}^2 \,\mathrm{d}t
\end{aligned}$$

or any $\delta > 0$. Consequently, it follows from Grönwall's lemma that

$$\begin{aligned}
&\sup_{t\in I} \|\nabla_{\mathbf{x}}\rho(t)\|_{L^2(\Omega_\eta)} + \int_I \|\Delta_{\mathbf{x}}\rho\|_{L^2(\Omega_\eta)} \,\mathrm{d}t \\
&\qquad \lesssim \|\nabla_{\mathbf{x}}\rho_0\|_{L^2(\Omega_{\eta_0})} + \int_I \big(\|\partial_t \eta\|_{W^{2,2}(\omega)}^2 + \|\mathbf{u}\|_{W^{2,2}(\Omega_\eta)}^2\big) \,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

If we also test (1.5) with $\Delta_{\mathbf{x}}\mathbb{T}$, we obtain

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t + \int_I \|\Delta_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t \\
&\quad = \int_I \int_{\Omega_\eta} ((\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbb{T})\Delta_{\mathbf{x}}\mathbb{T} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad + \frac{1}{2} \int_I \int_{\partial\Omega_\eta} (\partial_t \eta \mathbf{n}) \circ \boldsymbol{\varphi}_\eta^{-1} \cdot \mathbf{n}_\eta |\nabla_{\mathbf{x}}\mathbb{T}|^2 \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}t \\
&\qquad + 2 \int_I \int_{\Omega_\eta} (\mathbb{T} - \rho\mathbb{I})\Delta_{\mathbf{x}}\mathbb{T} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad - \int_I \int_{\Omega_\eta} ((\nabla_{\mathbf{x}}\mathbf{u})\mathbb{T} + \mathbb{T}(\nabla_{\mathbf{x}}\mathbf{u})^\top)\Delta_{\mathbf{x}}\mathbb{T} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad =: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.13}$$

The terms J_1 and J_2 can be treated as I_1 and I_2 above. By using (3.4), we obtain

$$\begin{aligned}
J_3 &\leq \delta \int_I \|\Delta_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t + cT\big(\|\mathbb{T}_0\|_{L^2(\Omega_{\eta_0})}^2 + T\|\rho_0\|_{L^2(\Omega_{\eta_0})}^2\big) \\
&\quad \times \exp\Big(c \int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t\Big)
\end{aligned}$$

for any $\delta > 0$ and by Ladyzhenskaya's inequality and (3.4),

$$\begin{aligned}
J_4 &\lesssim \int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}^{1/2} \|\mathbf{u}\|_{W^{2,2}(\Omega_\eta)}^{1/2} \|\mathbb{T}\|_{L^2(\Omega_\eta)}^{1/2} \|\nabla_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)}^{1/2} \|\Delta_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)} \,\mathrm{d}t \\
&\leq \delta \int_I \|\Delta_{\mathbf{x}}\mathbb{T}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t + c \int_I \|\mathbf{u}\|_{W^{2,2}(\Omega_\eta)}^2 \|\mathbb{T}\|_{W^{1,2}(\Omega_\eta)}^2 \,\mathrm{d}t \\
&\qquad + cT\big(\|\mathbb{T}_0\|_{L^2(\Omega_{\eta_0})}^2 + T\|\rho_0\|_{L^2(\Omega_{\eta_0})}^2\big) \exp\Big(c \int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|_{L^2(\Omega_\eta)}^2 \,\mathrm{d}t\Big).
\end{aligned}$$

Thus, it follows from Grönwall's lemma that

$$\begin{aligned}\sup_{t\in I}\|\mathbb{T}(t)\|^2_{W^{1,2}(\Omega_\eta)} &+ \int_I \|\Delta_{\mathbf{x}}\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t \\ &\lesssim \|\nabla_{\mathbf{x}}\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + \int_I \big(\|\partial_t\eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t \\ &\quad + T\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big). \qquad (3.14)\end{aligned}$$

To obtain regularity in time, we test (1.2) with $\partial_t\rho$. This yields

$$\begin{aligned}\int_I \|\partial_t\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t &+ \int_I \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t \\ &= \frac{1}{2}\int_I\int_{\partial\Omega_\eta} (\partial_t\eta\mathbf{n})\circ\boldsymbol{\varphi}_\eta^{-1}\cdot\mathbf{n}_\eta|\nabla_{\mathbf{x}}\rho|^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t - \int_I\int_{\Omega_\eta}(\mathbf{u}\cdot\nabla_{\mathbf{x}})\rho\partial_t\rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leq c\int_I \|\rho\|^2_{W^{2,2}(\Omega_\eta)} + c\int_I \|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\|\partial_t\eta\|^2_{W^{2,2}(\omega)}\,\mathrm{d}t \\ &\quad + c(\delta)\int_I \|\nabla_{\mathbf{x}}\rho\|^2_{L^2(\Omega_\eta)}\|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\,\mathrm{d}t + \delta\int_I \|\partial_t\rho\|^2_{L^2(\Omega_\eta)} \qquad (3.15)\end{aligned}$$

for any $\delta > 0$. Note the estimate for the boundary term done earlier in (3.11). By using estimate (3.12), it follows from (3.15) that

$$\begin{aligned}\int_I \|\partial_t\rho\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t &+ \sup_{t\in I}\|\nabla_{\mathbf{x}}\rho(t)\|^2_{L^2(\Omega_\eta)} \\ &\lesssim \|\nabla_{\mathbf{x}}\rho_0\|_{L^2(\Omega_{\eta_0})} + \int_I \big(\|\partial_t\eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t. \qquad (3.16)\end{aligned}$$

Now, we note that (compare with the estimate for J_3 in (3.13))

$$\begin{aligned}2k\int_I\int_{\Omega_\eta} &(\mathbb{T}-\rho\mathbb{I}):\partial_t\mathbb{T}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leq \delta\int_I \|\partial_t\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t + cT\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big) \\ &\quad \times\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big)\end{aligned}$$

and (compare with the estimate for J_4 in (3.13))

$$\begin{aligned}\int_I\int_{\Omega_\eta} &[(\nabla_{\mathbf{x}}\mathbf{u})\mathbb{T} + \mathbb{T}(\nabla_{\mathbf{x}}\mathbf{u})^\top]:\partial_t\mathbb{T}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leq \delta\int_I \|\partial_t\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t + c\int_I \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\|\nabla_{\mathbf{x}}\mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t \\ &\quad + cT\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big).\end{aligned}$$

Therefore, by testing (1.5) with $\partial_t \mathbb{T}$, we obtain

$$\begin{aligned}\int_I \|\partial_t \mathbb{T}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t &+ \sup_{t\in I}\|\nabla_{\mathbf{x}}\mathbb{T}(t)\|^2_{L^2(\Omega_\eta)}\\ &\lesssim \|\nabla_{\mathbf{x}}\mathbb{T}_0\|_{L^2(\Omega_{\eta_0})} + \int_I \big(\|\partial_t\eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t\\ &\qquad + cT\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big).\end{aligned} \tag{3.17}$$

If we now combine (3.12), (3.14), (3.16), and (3.17), we obtain

$$\begin{aligned}\int_I \big(\|\partial_t\rho\|^2_{L^2(\Omega_\eta)} + \|\partial_t\mathbb{T}\|^2_{L^2(\Omega_\eta)}\big)\,\mathrm{d}t &+ \sup_{t\in I}\big(\|\rho(t)\|^2_{W^{1,2}(\Omega_\eta)} + \|\mathbb{T}(t)\|^2_{W^{1,2}(\Omega_\eta)}\big)\\ &+ \int_I \big(\|\rho\|_{W^{2,2}(\Omega_\eta)} + \|\mathbb{T}\|_{W^{2,2}(\Omega_\eta)}\big)\\ &\lesssim \|\rho_0\|_{W^{1,2}(\Omega_{\eta_0})} + \|\mathbb{T}_0\|_{W^{1,2}(\Omega_{\eta_0})} + \int_I \big(\|\partial_t\eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t\\ &\qquad + T\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big).\end{aligned} \tag{3.18}$$

Further combining (3.8) and (3.18) yields the desired estimate. Note that we can use equation (1.4) to directly obtain a bound for $\int_I \|\eta\|^2_{W^{4,2}(\omega)}\,\mathrm{d}t$ in terms of the dataset by virtue of (3.8) and (3.18). ∎

4. Weak solutions

4.1. Galerkin approximation

Our goal in this section is to construct a finite-dimensional approximation of a linearized version of the polymeric fluid-structure system for a given geometric setup. The approach follows a fixed point argument where we decouple the equation for the solute from the solvent-structure system. In this case, the solute system becomes a bilinear system whose finite-dimensional approximation can be constructed using the classical Galerkin method. On the other hand, the problem for the solvent-structure system can follow the construction done in [37] where the Galerkin basis on the moving domain is constructed from the Piola transform of the basis of the fixed reference domain. For completeness, but to avoid repetition, we summarize the construction in what follows.

We consider a given structure displacement $\zeta \in C(\overline{I} \times \omega)$ with an initial state $\zeta(0,\cdot) = \eta_0$ and a given driving divergence-free velocity field $\mathbf{v} \in L^2_{\mathrm{div}_{\mathbf{x}}}(I \times \mathbb{R}^2)$. To ensure that the pair $(\zeta, \mathbf{v})$ are sufficiently smooth so that the subsequent analyses are well defined, we consider their spatial regularization $(\zeta_\kappa, \mathbf{v}_\kappa)$[1] and assume that they satisfy the interface

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[1] Here, $f_\kappa := \mathcal{R}_\kappa f$, where the regularizing kernels $(\mathcal{R}_\kappa)_{\kappa>0}$ commute with ∂_t. See [37] for more details.

condition $\mathbf{v}_\kappa \circ \boldsymbol{\varphi}_{\zeta_\kappa} = \mathbf{n}\partial_t \zeta_\kappa$ on $I \times \omega$. For $\kappa > 0$ fixed, we are going to use a Galerkin approximation to construct a weak solution $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ of the following linearized system:

$$\mathrm{div}_{\mathbf{x}} \mathbf{u}_\kappa = 0, \tag{4.1}$$

$$\partial_t \rho_\kappa + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\rho_\kappa = \Delta_{\mathbf{x}} \rho_\kappa, \tag{4.2}$$

$$\partial_t \mathbf{u}_\kappa + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbf{u}_\kappa = \Delta_{\mathbf{x}} \mathbf{u}_\kappa - \nabla_{\mathbf{x}} p_\kappa + \mathbf{f}_\kappa + \mathrm{div}_{\mathbf{x}} \mathbb{T}_\kappa, \tag{4.3}$$

$$\partial_t^2 \eta_\kappa - \partial_t \partial_y^2 \eta_\kappa + \partial_y^4 \eta_\kappa = g_\kappa - (\mathbb{S}_\kappa \mathbf{n}_{\zeta_\kappa}) \circ \boldsymbol{\varphi}_{\zeta_\kappa} \cdot \mathbf{n} \, \det(\partial_y \boldsymbol{\varphi}_{\zeta_\kappa}), \tag{4.4}$$

$$\partial_t \mathbb{T}_\kappa + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbb{T}_\kappa = (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)\mathbb{T}_\kappa + \mathbb{T}_\kappa (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)^\top - 2(\mathbb{T}_\kappa - \rho_\kappa \mathbb{I}) + \Delta_{\mathbf{x}} \mathbb{T}_\kappa \tag{4.5}$$

on $I \times \Omega_{\zeta_\kappa} \subset \mathbb{R}^{1+2}$ where

$$\mathbb{S}_\kappa = (\nabla_{\mathbf{x}} \mathbf{u}_\kappa + (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)^\top) - p_\kappa \mathbb{I} + \mathbb{T}_\kappa$$

and

$$\begin{aligned}
\eta_\kappa(0,\cdot) = \eta_{0,\kappa}(\cdot), \quad \partial_t \eta_\kappa(0,\cdot) &= \eta_{\star,\kappa}(\cdot) \quad \text{in } \omega, && (4.6)\\
\mathbf{u}_\kappa(0,\cdot) &= \mathbf{u}_{0,\kappa}(\cdot) \quad \text{in } \Omega_{\zeta_{0,\kappa}}, && (4.7)\\
\rho_\kappa(0,\cdot) = \rho_{0,\kappa}(\cdot), \quad \mathbb{T}_\kappa(0,\cdot) &= \mathbb{T}_{0,\kappa}(\cdot) \quad \text{in } \Omega_{\zeta_{0,\kappa}},\\
\mathbf{n}_{\zeta_\kappa} \cdot \nabla_{\mathbf{x}} \rho_\kappa = 0, \quad \mathbf{n}_{\zeta_\kappa} \cdot \nabla_{\mathbf{x}} \mathbb{T}_\kappa &= 0 \quad \text{on } I \times \partial\Omega_{\zeta_\kappa}.
\end{aligned}$$

We now make precise the notion of a weak solution $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ in this linearized setting.

Definition 4.1. Let $(\mathbf{f}_\kappa, g_\kappa, \rho_{0,\kappa}, \mathbb{T}_{0,\kappa}, \mathbf{u}_{0,\kappa}, \eta_{0,\kappa}, \eta_{\star,\kappa})$ be a dataset that satisfies

$$\begin{aligned}
&\mathbf{f}_\kappa \in L^2(I; L^2_{\mathrm{loc}}(\mathbb{R}^2)), \quad g \in L^2(I; L^2(\omega)),\\
&\eta_{0,\kappa} \in W^{2,2}(\omega) \text{ with } \|\eta_{0,\kappa}\|_{L^\infty(\omega)} < L, \quad \eta_{\star,\kappa} \in L^2(\omega),\\
&\mathbf{u}_{0,\kappa} \in L^2_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\eta_{0,\kappa}}) \text{ is such that } \mathbf{u}_{0,\kappa} \circ \boldsymbol{\varphi}_{\eta_{0,\kappa}} = \eta_{\star,\kappa}\mathbf{n} \text{ on } \omega,\\
&\rho_{0,\kappa} \in L^2(\Omega_{\eta_{0,\kappa}}), \quad \mathbb{T}_{0,\kappa} \in L^2(\Omega_{\eta_{0,\kappa}}),\\
&\rho_{0,\kappa} \geq 0, \ \mathbb{T}_{0,\kappa} > 0 \quad \text{a.e. in } \Omega_{\eta_0}.
\end{aligned} \tag{4.8}$$

We call $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ a *weak solution* of (4.1)–(4.5) if:

(a) the following properties

$$\begin{aligned}
&\eta_\kappa \in W^{1,\infty}(I; L^2(\omega)) \cap W^{1,2}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{2,2}(\omega)),\\
&\|\eta_\kappa\|_{L^\infty(I\times\omega)} < L,\\
&\mathbf{u}_\kappa \in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\zeta_\kappa})),\\
&\rho_\kappa \in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})),\\
&\mathbb{T}_\kappa \in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})),\\
&\rho_\kappa \geq 0, \ \mathbb{T}_\kappa > 0, \quad \text{a.e. in } I \times \Omega_{\zeta_\kappa}
\end{aligned}$$

holds;

(b) for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\zeta_\kappa}} \rho_\kappa \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_I \int_{\Omega_{\zeta_\kappa}} [\rho_\kappa \partial_t \psi + (\rho_\kappa \mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\psi] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_I \int_{\Omega_{\zeta_\kappa}} \nabla_{\mathbf{x}} \rho_\kappa \cdot \nabla_{\mathbf{x}} \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t;$$

(c) for all $\mathbb{Y} \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\zeta_\kappa}} \mathbb{T}_\kappa : \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_I \int_{\Omega_{\zeta_\kappa}} [\mathbb{T}_\kappa : \partial_t \mathbb{Y} + \mathbb{T}_\kappa : (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbb{Y}] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \int_I \int_{\Omega_{\zeta_\kappa}} [(\nabla_{\mathbf{x}} \mathbf{u}_\kappa)\mathbb{T}_\kappa + \mathbb{T}_\kappa (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)^\top] : \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - 2 \int_I \int_{\Omega_{\zeta_\kappa}} (\mathbb{T}_\kappa : \mathbb{Y} - \rho_\kappa \mathrm{tr}(\mathbb{Y})) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_I \int_{\Omega_{\zeta_\kappa}} \nabla_{\mathbf{x}} \mathbb{T}_\kappa :: \nabla_{\mathbf{x}} \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t;$$

(d) for all $(\boldsymbol{\phi}, \phi) \in C^\infty_{\mathrm{div}_{\mathbf{x}}}(\overline{I} \times \mathbb{R}^2) \otimes C^\infty(\overline{I} \times \omega)$ with $\boldsymbol{\phi}(T, \cdot) = 0$, $\phi(T, \cdot) = 0$ and $\boldsymbol{\phi} \circ \boldsymbol{\varphi}_{\zeta_\kappa} = \phi \mathbf{n}$, we have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\omega \partial_t \eta_\kappa \, \phi \,\mathrm{d}y + \int_{\Omega_{\zeta_\kappa}} \mathbf{u}_\kappa \cdot \boldsymbol{\phi} \,\mathrm{d}\mathbf{x} \Big) dt \\ = \int_I \int_\omega (\partial_t \eta_\kappa \partial_t \phi - \partial_t \partial_y \eta_\kappa \partial_y \phi + g_\kappa \phi - \Delta_{\mathrm{y}} \phi \Delta_{\mathrm{y}} \eta_\kappa) \,\mathrm{d}y\,\mathrm{d}t \\ + \int_I \int_\omega \Big(\frac{1}{2} \mathbf{n}_{\zeta_\kappa} \circ \boldsymbol{\varphi}_{\zeta_\kappa} \cdot \mathbf{n}^\top \phi \, \partial_t \zeta_\kappa \, \partial_t \eta_\kappa \, \det(\partial_y \boldsymbol{\varphi}_{\zeta_\kappa}) \Big) \,\mathrm{d}y\,\mathrm{d}t \\ + \int_I \int_{\Omega_{\zeta_\kappa}} \Big(\mathbf{u}_\kappa \cdot \partial_t \boldsymbol{\phi} - \frac{1}{2} ((\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}}) \mathbf{u}_\kappa) \cdot \boldsymbol{\phi} \Big) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \int_I \int_{\Omega_{\zeta_\kappa}} \Big(\frac{1}{2} ((\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}}) \boldsymbol{\phi}) \cdot \mathbf{u}_\kappa - \nabla_{\mathbf{x}} \mathbf{u}_\kappa : \nabla_{\mathbf{x}} \boldsymbol{\phi} + \mathbf{f}_\kappa \cdot \boldsymbol{\phi} \Big) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_I \int_{\Omega_{\zeta_\kappa}} \mathbb{T}_\kappa : \nabla_{\mathbf{x}} \boldsymbol{\phi} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t;$$

(e) the energy inequality

$$\sup_{t \in I} \Big(\int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}_\kappa(t)) \,\mathrm{d}\mathbf{x} + \|\mathbf{u}_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\partial_t \eta_\kappa(t)\|^2_{L^2(\omega)} + \|\partial_y^2 \eta_\kappa(t)\|^2_{L^2(\omega)} \Big) \\ + \int_I \int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}_\kappa) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \|\nabla_{\mathbf{x}} \mathbf{u}_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})} \,\mathrm{d}t + \int_I \|\partial_t \partial_y \eta_\kappa\|^2_{L^2(\omega)} \,\mathrm{d}t \\ \lesssim \int_{\Omega_{\eta_{0,\kappa}}} \mathrm{tr}(\mathbb{T}_{0,\kappa}) \,\mathrm{d}\mathbf{x} + \|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_{0,\kappa}})} + \|\eta_{\star,\kappa}\|^2_{L^2(\omega)} + \|\partial_y^2 \eta_{0,\kappa}\|^2_{L^2(\omega)}$$

$$+ T\int_{\Omega_{\eta_{0,\kappa}}} \rho_{0,\kappa}\,\mathrm{d}\mathbf{x} + \int_I \|\mathbf{f}_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t + \int_I \|g_\kappa\|^2_{L^2(\omega)}\,\mathrm{d}t \tag{4.9}$$

holds.

(f) In addition, we have

$$\begin{aligned}
&\sup_{t\in I}\big(\|\rho_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\mathbb{T}_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})}\big)\\
&\qquad + \int_I \big(\|\rho_\kappa\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})} + \|\mathbb{T}_\kappa\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\big)\,\mathrm{d}t\\
&\qquad \lesssim \|\rho_{0,\kappa}\|^2_{L^2(\Omega_{\eta_0})} + \|\mathbb{T}_{0,\kappa}\|^2_{L^2(\Omega_{\eta_0})}\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t\Big)\\
&\qquad + T\|\rho_{0,\kappa}\|^2_{L^2(\Omega_{\eta_0})}\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{u}_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t\Big).
\end{aligned}$$

Our main result in this section is now given as follows:

Theorem 4.2. *Let $\kappa > 0$ be fixed. For a dataset $(\mathbf{f}_\kappa, g_\kappa, \rho_{0,\kappa}, \mathbb{T}_{0,\kappa}, \mathbf{u}_{0,\kappa}, \eta_{0,\kappa}, \eta_{\star,\kappa})$ that satisfies (4.8), there exists a weak solution $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ of (4.1)–(4.5).*

Remark 4.3. We remark here that the subscript κ for the unknowns are meant to emphasize that these are the solutions to the regularized system constructed from a given regularized dataset.

We now devote the rest of this section to the proof of Theorem 4.2. To obtain $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$, we consider the basis $(\overline{X}_n)_{n\in\mathbb{N}}$ and $(\overline{Y}_n)_{n\in\mathbb{N}}$ of $W^{1,2}_{0,\mathrm{div}_{\mathbf{x}}}(\Omega)$ and $W^{2,2}(\omega)$ respectively. Then by [37, Theorem A.3], there exist solenoidal vector fields $(\overline{\boldsymbol{Y}}_n)_{n\in\mathbb{N}}$ that solve a Stokes system on the fixed reference domain with boundary data $(\mathbf{n}\overline{Y}_n)_{n\in\mathbb{N}}$. Now, for all $t \in \overline{I}$, we obtain from $(\overline{X}_n)_{n\in\mathbb{N}}$ the following basis:

$$X_n(t,\cdot) := \mathcal{J}_{\zeta_\kappa(t)}\overline{X}_n$$

for $W^{1,2}_{0,\mathrm{div}_{\mathbf{x}}}(\Omega_{\zeta_\kappa(t)})$ where, for a vector field $\mathbf{v}$, $\mathcal{J}_{\zeta_\kappa}\mathbf{v}$ defined by

$$\mathcal{J}_{\zeta_\kappa}\mathbf{v} = \big(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\kappa}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\kappa})^{-1}\mathbf{v}\big)\circ\boldsymbol{\Psi}^{-1}_{\zeta_\kappa}$$

is the Piola transform of $\mathbf{v}$ with respect to a mapping $\zeta : \omega \to \mathbb{R}$. The Piola transform is invertible with inverse

$$\mathcal{J}^{-1}_{\zeta_\kappa}\mathbf{v} = \big((\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\kappa})^{-1}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\kappa})\mathbf{v}\big)\circ\boldsymbol{\Psi}_{\zeta_\kappa}.$$

In order to ensure that the basis for the solvent system matches with the basis for the structure at the solvent-structure interface, additionally, we consider the Piola transform of the solenoidal vector fields $(\overline{\boldsymbol{Y}}_n)_{n\in\mathbb{N}}$ by setting

$$\boldsymbol{Y}_n(t,\cdot) := \mathcal{J}_{\zeta_\kappa(t)}\overline{\boldsymbol{Y}}_n.$$

Consequently, if we set

$$Y_n(t,\cdot) := (\det(\partial_y \boldsymbol{\varphi}_{\zeta_\kappa(t)}))^{-1}\overline{Y}_k$$

we obtain the interface condition

$$Y_n(t,\cdot)\mathbf{n} = \boldsymbol{Y}_n(t,\cdot)\circ \boldsymbol{\varphi}_{\zeta_\kappa(t)}.$$

Our extended basis for the moving domain will now consist of the pair $(\boldsymbol{\psi}_n, \psi_n)_{n\in\mathbb{N}}$ where

$$\boldsymbol{\psi}_n = \begin{cases} \boldsymbol{X}_n & n \text{ even}, \\ \boldsymbol{Y}_n & n \text{ odd} \end{cases} \qquad \text{and} \qquad \psi_n \mathbf{n} := \boldsymbol{\psi}_n \circ \boldsymbol{\varphi}_{\zeta_\kappa(t)}. \tag{4.10}$$

With basis (4.10) in hand, we can use the Picard–Lindelöf theorem to find functions $\alpha_n^N \in C^1(\overline{I})$, $n, N \in \mathbb{N}$ such that $\mathbf{u}_\kappa^N := \alpha_n^N \boldsymbol{\psi}_n$ and $\eta_\kappa^N = \int_0^t \alpha_n^N \psi_n \,\mathrm{d}s + \eta_{\kappa,0}^N$ solves[2]

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\omega \partial_t \eta_\kappa^N\, \psi_j \,\mathrm{d}y + \int_{\Omega_{\zeta_\kappa}} \mathbf{u}_\kappa^N \cdot \boldsymbol{\psi}_j \,\mathrm{d}\mathbf{x}\Big) \\
&= \int_\omega \big(\partial_t \eta_\kappa^N \partial_t \psi_j - \partial_t \partial_y \eta_\kappa^N \partial_y \psi_j + g_\kappa^N \psi_j - \Delta_{\mathbf{y}} \psi_j \Delta_{\mathbf{y}} \eta_\kappa^N\big)\,\mathrm{d}y \\
&\quad + \int_\omega \Big(\frac{1}{2}\mathbf{n}_{\zeta_\kappa} \circ \boldsymbol{\varphi}_{\zeta_\kappa} \cdot \mathbf{n}^\top \psi_j\, \partial_t \zeta_\kappa\, \partial_t \eta_\kappa^N\, \det(\partial_y \boldsymbol{\varphi}_{\zeta_\kappa})\Big)\,\mathrm{d}y \\
&\quad + \int_{\Omega_{\zeta_\kappa}} \Big(\mathbf{u}_\kappa^N \cdot \partial_t \boldsymbol{\psi}_j - \frac{1}{2}((\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbf{u}_\kappa^N) \cdot \boldsymbol{\psi}_j\Big)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_{\Omega_{\zeta_\kappa}} \Big(\frac{1}{2}((\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\boldsymbol{\psi}_j) \cdot \mathbf{u}_\kappa^N - \nabla_{\mathbf{x}} \mathbf{u}_\kappa^N : \nabla_{\mathbf{x}} \boldsymbol{\psi}_j + \mathbf{f}_\kappa^N \cdot \boldsymbol{\psi}_j - \mathbb{T}_\kappa^N : \nabla_{\mathbf{x}} \boldsymbol{\psi}_j\Big)\,\mathrm{d}\mathbf{x}
\end{aligned} \tag{4.11}$$

for all $1 \le j \le N$ with the pair $(\rho_\kappa^N, \mathbb{T}_\kappa^N)$ determined by

$$\partial_t \rho_\kappa^N + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\rho_\kappa^N = \Delta_{\mathbf{x}} \rho_\kappa^N, \tag{4.12}$$

$$\begin{aligned}\partial_t \mathbb{T}_\kappa^N + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbb{T}_\kappa^N &= (\nabla_{\mathbf{x}} \mathbf{u}_\kappa^N)\mathbb{T}_\kappa^N + \mathbb{T}_\kappa^N (\nabla_{\mathbf{x}} \mathbf{u}_\kappa^N)^\top - 2(\mathbb{T}_\kappa^N - \rho_\kappa^N \mathbb{I}) \\ &\quad + \Delta_{\mathbf{x}} \mathbb{T}_\kappa^N \end{aligned} \tag{4.13}$$

on $I \times \Omega_{\zeta_\kappa} \subset \mathbb{R}^{1+2}$. Here, we complement (4.12)–(4.13) with the following initial and boundary conditions:

$$\rho_{0,\kappa}^N \ge 0, \quad \mathbb{T}_{0,\kappa}^N > 0 \qquad \text{a.e in } I \times \Omega_{\zeta_\kappa}, \tag{4.14}$$

$$\rho_\kappa^N(0,\cdot) = \rho_{0,\kappa}^N(\cdot), \quad \mathbb{T}_\kappa^N(0,\cdot) = \mathbb{T}_{0,\kappa}^N(\cdot) \qquad \text{in } \Omega_{\zeta_{0,\kappa}}, \tag{4.15}$$

$$\mathbf{n}_{\zeta_\kappa} \cdot \nabla_{\mathbf{x}} \rho_\kappa^N = 0 \qquad \text{on } I \times \partial\Omega_{\zeta_\kappa}, \tag{4.16}$$

$$\mathbf{n}_{\zeta_\kappa} \cdot \nabla_{\mathbf{x}} \mathbb{T}_\kappa^N = 0 \qquad \text{on } I \times \partial\Omega_{\zeta_\kappa}. \tag{4.17}$$

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[2] The dependence of $\mathbf{u}_\kappa^N$ and η_κ^N on κ follows from the implicit dependence of $\boldsymbol{\psi}_n$ and ψ_n on κ

Indeed, for a fixed $N \in \mathbb{N}$, suppose that $(\overline{\eta^N_\kappa}, \overline{\mathbf{u}^N_\kappa})$ satisfying

$$\begin{aligned}
&\overline{\eta^N_\kappa} \in W^{1,\infty}(I; L^2(\omega)) \cap W^{1,2}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{2,2}(\omega)),\\
&\overline{\mathbf{u}^N_\kappa} \in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}_{\mathrm{div}_\mathbf{x}}(\Omega_{\zeta_\kappa})),\\
&\overline{\mathbf{u}^N_\kappa} \circ \boldsymbol{\varphi}_{\zeta_\kappa} = \mathbf{n}\partial_t \overline{\eta^N_\kappa} \qquad \text{on } I \times \omega
\end{aligned}$$

is given. Then due to the bilinearity of (4.12)–(4.13), a solution pair $(\rho^N_\kappa, \mathbb{T}^N_\kappa)$ of (4.12)–(4.17) satisfying

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\zeta_\kappa}} \rho^N_\kappa \phi_i \, \mathrm{d}\mathbf{x}\, \mathrm{d}t &= \int_I \int_{\Omega_{\zeta_\kappa}} [\rho^N_\kappa \partial_t \phi_i + (\rho^N_\kappa \mathbf{v}_\kappa \cdot \nabla_\mathbf{x})\phi_i] \, \mathrm{d}\mathbf{x}\, \mathrm{d}t\\
&\quad - \int_I \int_{\Omega_{\zeta_\kappa}} \nabla_\mathbf{x} \rho^N_\kappa \cdot \nabla_\mathbf{x} \phi_i \, \mathrm{d}\mathbf{x}\, \mathrm{d}t
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\zeta_\kappa}} \mathbb{T}^N_\kappa : \mathbb{Y}_i \, \mathrm{d}\mathbf{x}\, \mathrm{d}t &= \int_I \int_{\Omega_{\zeta_\kappa}} [\mathbb{T}^N_\kappa : \partial_t \mathbb{Y}_i + \mathbb{T}^N_\kappa : (\mathbf{v}_\kappa \cdot \nabla_\mathbf{x})\mathbb{Y}_i] \, \mathrm{d}\mathbf{x}\, \mathrm{d}t\\
&\quad + \int_I \int_{\Omega_{\zeta_\kappa}} [(\nabla_\mathbf{x} \overline{\mathbf{u}^N_\kappa})\mathbb{T}^N_\kappa + \mathbb{T}^N_\kappa (\nabla_\mathbf{x} \overline{\mathbf{u}^N_\kappa})^\top] : \mathbb{Y}_i \, \mathrm{d}\mathbf{x}\, \mathrm{d}t\\
&\quad - 2 \int_I \int_{\Omega_{\zeta_\kappa}} (\mathbb{T}^N_\kappa : \mathbb{Y}_i - \rho^N_\kappa \mathrm{tr}(\mathbb{Y}_i)) \, \mathrm{d}\mathbf{x}\, \mathrm{d}t\\
&\quad - \int_I \int_{\Omega_{\zeta_\kappa}} \nabla_\mathbf{x} \mathbb{T}^N_\kappa :: \nabla_\mathbf{x} \mathbb{Y}_i \, \mathrm{d}\mathbf{x}\, \mathrm{d}t
\end{aligned} \tag{4.19}$$

for all $1 \le i \le N$ is obtained as a limit $M \to \infty$ of, yet again, a Galerkin approximation $(\rho^{N,M}_\kappa, \mathbb{T}^{N,M}_\kappa)$ for a basis $(\phi_i, \mathbb{Y}_i)$ in $W^{1,2}_0(\Omega_{\zeta_\kappa}) \otimes W^{1,2}_0(\Omega_{\zeta_\kappa})$. Furthermore, if we take the trace in (4.13), then similar to (3.3), we obtain

$$\begin{aligned}
&\int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}^N_\kappa(t)) \, \mathrm{d}\mathbf{x} + 2 \int_0^t \int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}^N_\kappa) \, \mathrm{d}\mathbf{x}\, \mathrm{d}t'\\
&\quad \le \int_0^t \|\mathbb{T}^N_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})} \, \mathrm{d}t' + \int_0^t \|\nabla_\mathbf{x} \overline{\mathbf{u}^N_\kappa}\|^2_{L^2(\Omega_{\zeta_\kappa})} \, \mathrm{d}t'\\
&\qquad + 6 \int_{\Omega_{\eta_{0,\kappa}}} \rho^N_{\kappa,0} \, \mathrm{d}\mathbf{x} + \int_{\Omega_{\eta_{0,\kappa}}} \mathrm{tr}(\mathbb{T}^N_{\kappa,0}) \, \mathrm{d}\mathbf{x}
\end{aligned} \tag{4.20}$$

for all $t \in I$. On the other hand, if we take the inner product of (4.13) with $\mathbb{T}^N_\kappa$ and apply Grönwall's lemma, we obtain

$$\begin{aligned}
&\sup_{t \in I} \|\mathbb{T}^N_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \int_I \|\mathbb{T}^N_\kappa\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})} \, \mathrm{d}t\\
&\quad \lesssim \big(\|\mathbb{T}^N_{0,\kappa}\|^2_{L^2(\Omega_{\eta_{0,\kappa}})} + T\|\rho^N_{0,\kappa}\|^2_{L^2(\Omega_{\eta_{0,\kappa}})}\big) \exp\Big(c \int_I \|\nabla_\mathbf{x} \overline{\mathbf{u}^N_\kappa}\|^2_{L^2(\Omega_{\zeta_\kappa})} \, \mathrm{d}t\Big)
\end{aligned} \tag{4.21}$$

with a similar but simpler estimate holding for ρ_κ^N. If we now combine (4.20) and (4.21), we obtain

$$\begin{aligned}\sup_{t\in I}\int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}_\kappa^N(t))\,\mathrm{d}\mathbf{x} + \sup_{t\in I}\|\mathbb{T}_\kappa^N(t)\|_{L^2(\Omega_{\zeta_\kappa})}^2 + \int_I\int_{\Omega_{\zeta_\kappa}} \mathrm{tr}(\mathbb{T}_\kappa^N)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \int_I \|\mathbb{T}_\kappa^N\|_{W^{1,2}(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t \lesssim \mathcal{D}(\overline{\mathbf{u}_\kappa^N}, \rho_{0,\kappa}^N, \mathbb{T}_{0,\kappa}^N)\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}\mathcal{D}(\overline{\mathbf{u}_\kappa^N}, \rho_{0,\kappa}^N, \mathbb{T}_{0,\kappa}^N) := \int_I \|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^N}\|_{L^2(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t + \int_{\Omega_{\eta_{0,\kappa}}} \rho_{0,\kappa}^N\,\mathrm{d}\mathbf{x} + \int_{\Omega_{\eta_{0,\kappa}}} \mathrm{tr}(\mathbb{T}_{0,\kappa}^N)\,\mathrm{d}\mathbf{x} \\ + \big(\|\mathbb{T}_{0,\kappa}^N\|_{L^2(\Omega_{\eta_{0,\kappa}})}^2 + T\|\rho_{0,\kappa}^N\|_{L^2(\Omega_{\eta_{0,\kappa}})}^2\big) \\ \times \exp\Big(c\int_I \|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^N}\|_{L^2(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t\Big).\end{aligned}$$

The right-hand side of (4.22) is finite, and thus, we have constructed a solution $(\rho_\kappa^N, \mathbb{T}_\kappa^N)$ of (4.12)–(4.17) satisfying

$$\begin{aligned}\rho_\kappa^N &\in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})), \\ \mathbb{T}_\kappa^N &\in L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})).\end{aligned}$$

Now, with the constructed pair $(\rho_\kappa^N, \mathbb{T}_\kappa^N)$ in hand, we revisit (4.11). As stated earlier, its solution follows from Picard–Lindelöf theorem. Furthermore, by taking the pair $(\eta_\kappa^N, \mathbf{u}_\kappa^N)$ as test functions, we obtain the global bound

$$\begin{aligned}\sup_{t\in I}\big(\|\mathbf{u}_\kappa^N\|_{L^2(\Omega_{\zeta_\kappa})}^2 + \|\partial_t\eta_\kappa^N\|_{L^2(\omega)}^2 + \|\partial_y^2\eta_\kappa^N\|_{L^2(\omega)}^2\big) \\ + \int_I \big(\|\nabla_{\mathbf{x}}\mathbf{u}_\kappa^N\|_{L^2(\Omega_{\zeta_\kappa})}^2 + \|\partial_t\partial_y\eta_\kappa^N\|_{L^2(\omega)}^2\big)\,\mathrm{d}t \\ \lesssim \int_I \|\mathbb{T}_\kappa^N\|_{L^2(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t + \int_I \|\mathbf{f}_\kappa^N\|_{L^2(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t + \int_I \|g_\kappa^N\|_{L^2(\Omega_{\zeta_\kappa})}^2\,\mathrm{d}t \\ + \|\mathbf{u}_{0,\kappa}^N\|_{L^2(\Omega_{\eta_{0,\kappa}})}^2 + \|\eta_{\star,\kappa}^N\|_{L^2(\omega)}^2 + \|\partial_y^2\eta_{0,\kappa}^N\|_{L^2(\omega)}^2 \lesssim 1\end{aligned} \tag{4.23}$$

leading to $(\eta_\kappa^N, \mathbf{u}_\kappa^N) \in X^I$ where

$$\begin{aligned}X^I := \big(W^{1,\infty}(I; L^2(\omega)) \cap W^{1,2}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{2,2}(\omega))\big) \\ \otimes \big(L^\infty(I; L^2(\Omega_{\zeta_\kappa})) \cap L^2(I; W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\zeta_\kappa}))\big).\end{aligned}$$

At this point, on the one hand, we have obtained a solution $(\rho_\kappa^N, \mathbb{T}_\kappa^N)$ to the solute system (4.12)–(4.17) given a solvent-structure pair $(\overline{\eta_\kappa^N}, \overline{\mathbf{u}_\kappa^N})$. On the other hand, we have also constructed a solvent-structure pair $(\eta_\kappa^N, \mathbf{u}_\kappa^N)$ given a solute pair $(\rho_\kappa^N, \mathbb{T}_\kappa^N)$. We can now use a fixed point argument to get a local solution $(\eta_\kappa^N, \mathbf{u}_\kappa^N, \rho_\kappa^N, \mathbb{T}_\kappa^N)$ for the mutually coupled system. To do this, for a time $T_* > 0$, $I_* := (0, T_*)$ to be determined soon, we consider the solution map $\mathtt{T} = \mathtt{T}_1 \circ \mathtt{T}_2 : X^{I_*} \to X^{I_*}$ where

$$\mathtt{T}(\overline{\eta^N_\kappa}, \overline{\mathbf{u}^N_\kappa}) = (\eta^N_\kappa, \mathbf{u}^N_\kappa), \quad \mathtt{T}_2(\overline{\eta^N_\kappa}, \overline{\mathbf{u}^N_\kappa}) = (\rho^N_\kappa, \mathbb{T}^N_\kappa), \quad \mathtt{T}_1(\rho^N_\kappa, \mathbb{T}^N_\kappa) = (\eta^N_\kappa, \mathbf{u}^N_\kappa)$$

and the associated set

$$B^{I_*}_R := \Big\{(\overline{\eta^N_\kappa}, \overline{\mathbf{u}^N_\kappa}) \in X^{I_*} \text{ with } \overline{\mathbf{u}^N_\kappa} := \overline{\alpha}^N_n \boldsymbol{\psi}_n,\ \overline{\eta^N_\kappa} := \int_0^t \overline{\alpha}^N_n \psi_n \,\mathrm{d}s + \eta^N_{\kappa,0} \\ \text{such that } \|(\overline{\eta^N_\kappa}, \overline{\mathbf{u}^N_\kappa})\|^2_{X^{I_*}} \leq R^2 \text{ for } t \in \overline{I}_*\Big\}$$

for $R > 0$ large enough and for fixed $\kappa > 0$. From (4.22) and (4.23), it follows that $\mathtt{T} : B^{I_*}_R \to B^{I_*}_R$ for $R > 0$ large and $T_* > 0$ small, that is, the solution mapping $\mathtt{T}$ maps the ball $B^{I_*}_R$ onto itself. To show the contraction property leading to the existence of a unique local solution for the fully coupled system, we consider any two solution pairs $(\eta^{N,i}_\kappa, \mathbf{u}^{N,i}_\kappa)$, $i = 1, 2$ of the solvent-structure system with datasets $(\mathbf{f}^N_\kappa, g^N_\kappa, \mathbf{u}^N_{0,\kappa}, \eta^N_{0,\kappa}, \eta^N_{\star,\kappa}, \rho^{N,i}_\kappa, \mathbb{T}^{N,i}_\kappa)$, $i = 1, 2$, respectively. Thus, the difference

$$(\eta^{N,12}_\kappa,\ \mathbf{u}^{N,12}_\kappa) := (\eta^{N,1}_\kappa - \eta^{N,2}_\kappa,\ \mathbf{u}^{N,1}_\kappa - \mathbf{u}^{N,2}_\kappa)$$

solves

$$\partial^2_t \eta^{N,12}_\kappa - \partial_t \partial^2_y \eta^{N,12}_\kappa + \partial^4_y \eta^{N,12}_\kappa = -\mathbf{n}^\top \mathbb{S}^{N,12}_\kappa \circ \boldsymbol{\varphi}_{\zeta_\kappa} \mathbf{n}_{\zeta_\kappa} \det(\partial_y \boldsymbol{\varphi}_{\zeta_\kappa}) \tag{4.24}$$

in $I_* \times \omega$ and

$$\partial_t \mathbf{u}^{N,12}_\kappa + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{N,12}_\kappa = \Delta_{\mathbf{x}} \mathbf{u}^{N,12}_\kappa - \nabla_{\mathbf{x}} p^{N,12}_\kappa + \mathrm{div}_{\mathbf{x}} \mathbb{T}^{N,12}_\kappa \tag{4.25}$$

in $I_* \times \Omega_{\zeta_\kappa}$, where

$$\mathbb{S}^{N,12}_\kappa = (\nabla_{\mathbf{x}} \mathbf{u}^{N,12}_\kappa + \nabla_{\mathbf{x}} (\mathbf{u}^{N,12}_\kappa)^\top) - p^{N,12}_\kappa \mathbb{I} + \mathbb{T}^{N,12}_\kappa$$

with $p^{N,12}_\kappa := p^{N,1}_\kappa - p^{N,2}_\kappa$ and $\mathbb{T}^{N,12}_\kappa := \mathbb{T}^{N,1}_\kappa - \mathbb{T}^{N,2}_\kappa$. Similar to (3.2), we obtain

$$\begin{aligned} \sup_{t \in I_*} \big(\|\mathbf{u}^{N,12}_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\partial_t \eta^{N,12}_\kappa(t)\|^2_{L^2(\omega)} + \|\partial^2_y \eta^{N,12}_\kappa(t)\|^2_{L^2(\omega)}\big) \\ + \int_{I_*} \big(\|\nabla_{\mathbf{x}} \mathbf{u}^{N,12}_\kappa\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\partial_t \partial_y \eta^{N,12}_\kappa\|^2_{L^2(\omega)}\big) \,\mathrm{d}t \\ \lesssim T_* \sup_{t \in I_*} \|\mathbb{T}^{N,12}_\kappa(t)\|^2_{L^2(\Omega_{\zeta_\kappa})}. \end{aligned} \tag{4.26}$$

To obtain a contraction estimate for $\mathbb{T}^N_\kappa$, let us consider any two solution pairs $(\rho^{N,i}_\kappa, \mathbb{T}^{N,i}_\kappa)$, $i = 1, 2$ of (4.12)–(4.13) with datasets $(\rho^N_{0,\kappa}, \mathbb{T}^N_{0,\kappa}, \overline{\eta^{N,i}_\kappa}, \overline{\mathbf{u}^{N,i}_\kappa})$, $i = 1, 2$, respectively, so that the differences $\rho^{N,12}_\kappa := \rho^{N,1}_\kappa - \rho^{N,2}_\kappa$ and $\mathbb{T}^{N,12}_\kappa := \mathbb{T}^{N,1}_\kappa - \mathbb{T}^{N,2}_\kappa$ solve

$$\partial_t \rho^{N,12}_\kappa + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}}) \rho^{N,12}_\kappa = \Delta_{\mathbf{x}} \rho^{N,12}_\kappa \tag{4.27}$$

and

$$\begin{aligned}\partial_t \mathbb{T}_\kappa^{N,12} + (\mathbf{v}_\kappa \cdot \nabla_{\mathbf{x}})\mathbb{T}_\kappa^{N,12} = (\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}})\mathbb{T}_\kappa^{N,1} &+ \mathbb{T}_\kappa^{N,1}(\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}})^\top \\ &- 2(\mathbb{T}_\kappa^{N,12} - \rho_\kappa^{N,12}\mathbb{I}) + \Delta_{\mathbf{x}}\mathbb{T}_\kappa^{N,12} \\ &+ (\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}})\mathbb{T}_\kappa^{N,12} + \mathbb{T}_\kappa^{N,12}(\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}})^\top \end{aligned} \tag{4.28}$$

in $I_* \times \Omega_{\zeta_\kappa}$. We now test (4.27) with $\rho_\kappa^{N,12}$ and use boundary condition (4.16), which leads to

$$\sup_{t\in I_*} \|\rho_\kappa^{N,12}(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \int_{I_*} \|\nabla_{\mathbf{x}}\rho_\kappa^{N,12}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t = 0. \tag{4.29}$$

Next we test (4.28) with $\mathbb{T}_\kappa^{N,12}$. We obtain

$$\begin{aligned}&\|\mathbb{T}_\kappa^{N,12}(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \int_0^t \|\mathbb{T}_\kappa^{N,12}\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' \\ &\quad\lesssim T_*\|\rho_\kappa^{N,12}(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \int_0^t \|\mathbb{T}_\kappa^{N,12}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' \\ &\qquad + \int_0^t\int_{\Omega_{\zeta_\kappa}} \big(|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}}|\,|\mathbb{T}_\kappa^{N,1}| + |\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}}|\,|\mathbb{T}_\kappa^{N,12}|\big)\,|\mathbb{T}_\kappa^{N,12}|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'\end{aligned}$$

for all $t \in I_*$, whereas by using the Ladyzhenskaya inequality, we find that the estimate

$$\begin{aligned}&\int_0^t\int_{\Omega_{\zeta_\kappa}} |\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}}|\,|\mathbb{T}_\kappa^{N,1}|\,|\mathbb{T}_\kappa^{N,12}|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\ &\quad\lesssim \int_0^t \|\mathbb{T}_\kappa^{N,1}\|_{L^4(\Omega_{\zeta_\kappa})}\|\mathbb{T}_\kappa^{N,12}\|_{L^4(\Omega_{\zeta_\kappa})}\|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}}\|_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' \\ &\quad\le \delta\int_0^t \|\mathbb{T}_\kappa^{N,12}\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' + \delta\int_0^t \|\mathbb{T}_\kappa^{N,1}\|_{L^2(\Omega_{\zeta_\kappa})}\|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t \\ &\qquad + c\int_0^t \|\mathbb{T}_\kappa^{N,1}\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\|\mathbb{T}_\kappa^{N,12}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t'\end{aligned}$$

holds for any $\delta > 0$ and

$$\begin{aligned}\int_0^t\int_{\Omega_{\zeta_\kappa}} |\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}}|\,|\mathbb{T}_\kappa^{N,12}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' &\lesssim \int_0^t \|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}}\|_{L^2(\Omega_{\zeta_\kappa})}\|\mathbb{T}_\kappa^{N,12}\|^2_{L^4(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' \\ &\le \delta\int_0^t \|\mathbb{T}_\kappa^{N,12}\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\,\mathrm{d}t' \\ &\qquad + c\int_0^t \|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,2}}\|^2_{L^2(\Omega_{\zeta_\kappa})}\|\mathbb{T}_\kappa^{N,12}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t'.\end{aligned}$$

If we now use the regularity of $\mathbb{T}_\kappa^{N,1}, \overline{\mathbf{u}_\kappa^{N,2}}$ as well as (4.29), we obtain from Grönwall's lemma that

$$\begin{aligned}\sup_{t\in I_*}\|\mathbb{T}_\kappa^{N,12}(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \int_{I_*}\|\mathbb{T}_\kappa^{N,12}\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\,\mathrm{d}t \\ \leq \delta e^{cT_*}\int_{I_*}\|\nabla_{\mathbf{x}}\overline{\mathbf{u}_\kappa^{N,12}}\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t.\end{aligned} \tag{4.30}$$

By combining (4.26) and (4.30), we obtain

$$\|\mathtt{T}(\overline{\eta_\kappa^N}, \overline{\mathbf{u}_\kappa^N})\|^2_{X^{I_*}} = \|(\eta_\kappa^N, \mathbf{u}_\kappa^N)\|^2_{X^{I_*}} \leq c\delta T_* e^{cT_*}\|(\overline{\eta_\kappa^N}, \overline{\mathbf{u}_\kappa^N})\|^2_{X^{I_*}} \leq \tfrac{1}{2}\|(\overline{\eta_\kappa^N}, \overline{\mathbf{u}_\kappa^N})\|^2_{X^{I_*}}$$

for the choice of T_* such that $2T_*e^{cT_*} \leq (c\delta)^{-1}$. This completes the proof of the existence of a unique local-in-time weak solution for the fully coupled finite-dimensional system (4.11)–(4.13). The fact that this solution is global follows from the energy estimate. Similarly to (3.1), we obtain

$$\begin{aligned}&\sup_{t\in I}\Big(\int_{\Omega_{\zeta_\kappa}}\mathrm{tr}(\mathbb{T}_\kappa^N(t))\,\mathrm{d}\mathbf{x} + \|\mathbf{u}_\kappa^N(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\partial_t\eta_\kappa^N(t)\|^2_{L^2(\omega)} + \|\partial_y^2\eta_\kappa^N(t)\|^2_{L^2(\omega)}\Big) \\ &\qquad + \int_I\int_{\Omega_{\zeta_\kappa}}\mathrm{tr}(\mathbb{T}_\kappa^N)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I\|\nabla_{\mathbf{x}}\mathbf{u}_\kappa^N\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t + \int_I\|\partial_t\partial_y\eta_\kappa^N\|^2_{L^2(\omega)}\,\mathrm{d}t \\ &\lesssim \int_{\Omega_{\zeta_\kappa(0)}}\mathrm{tr}(\mathbb{T}_{0,\kappa}^N)\,\mathrm{d}\mathbf{x} + \|\mathbf{u}_{0,\kappa}^N\|^2_{L^2(\Omega_{\zeta_\kappa(0)})} + \|\eta_{\star,\kappa}^N\|^2_{L^2(\omega)} + \|\partial_y^2\eta_{0,\kappa}^N\|^2_{L^2(\omega)} \\ &\qquad + T\int_{\Omega_{\zeta_\kappa(0)}}\rho_{0,\kappa}^N\,\mathrm{d}\mathbf{x} + \int_I\|\mathbf{f}_\kappa^N\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t + \int_I\|g_\kappa^N\|^2_{L^2(\omega)}\,\mathrm{d}t\end{aligned} \tag{4.31}$$

and similar to (3.4), we obtain

$$\begin{aligned}&\sup_{t\in I}\big(\|\rho_\kappa^N(t)\|^2_{L^2(\Omega_{\zeta_\kappa})} + \|\mathbb{T}_\kappa^N(t)\|^2_{L^2(\Omega_{\zeta_\kappa})}\big) \\ &\qquad + \int_I\big(\|\rho_\kappa^N\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})} + \|\mathbb{T}_\kappa^N\|^2_{W^{1,2}(\Omega_{\zeta_\kappa})}\big)\,\mathrm{d}t \\ &\lesssim \|\rho_{0,\kappa}^N\|^2_{L^2(\Omega_{\zeta_\kappa(0)})} + \|\mathbb{T}_{0,\kappa}^N\|^2_{L^2(\Omega_{\zeta_\kappa(0)})}\exp\Big(c\int_I\|\nabla_{\mathbf{x}}\mathbf{u}_\kappa^N\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t\Big) \\ &\qquad + T\|\rho_{0,\kappa}^N\|^2_{L^2(\Omega_{\zeta_\kappa(0)})}\exp\Big(c\int_I\|\nabla_{\mathbf{x}}\mathbf{u}_\kappa^N\|^2_{L^2(\Omega_{\zeta_\kappa})}\,\mathrm{d}t\Big).\end{aligned} \tag{4.32}$$

By using the boundedness of the initial conditions, it follows from (4.31) and (4.32) that

$$\begin{aligned}\eta_\kappa^N &\rightarrow \eta_\kappa \quad \text{in } \big(L^\infty(I; W^{2,2}(\omega)), w^*\big), \\ \partial_t\eta_\kappa^N &\rightarrow \partial_t\eta_\kappa \quad \text{in } \big(L^\infty(I; L^2(\omega)), w^*\big) \cap \big(L^2(I; W^{1,2}(\omega)), w\big), \\ \mathbf{u}_\kappa^N &\rightarrow \mathbf{u}_\kappa \quad \text{in } \big(L^\infty(I; L^2(\Omega_{\zeta_\kappa})), w^*\big) \cap \big(L^2(I; W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\zeta_\kappa})), w\big), \\ \rho_\kappa^N &\rightarrow \rho_\kappa \quad \text{in } \big(L^\infty(I; L^2(\Omega_{\zeta_\kappa})), w^*\big) \cap \big(L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})), w\big), \\ \mathbb{T}_\kappa^N &\rightarrow \mathbb{T}_\kappa \quad \text{in } \big(L^\infty(I; L^2(\Omega_{\zeta_\kappa})), w^*\big) \cap \big(L^2(I; W^{1,2}(\Omega_{\zeta_\kappa})), w\big).\end{aligned}$$

Furthermore, by using a density argument and lower semi-continuity of norms, the convergences above offer all the ingredients to pass to the limit in (4.11), (4.18), (4.19), (4.31), and (4.32) to complete the proof of Theorem 4.2.

4.2. The regularized fully coupled system

In the previous section, we constructed a weak solution $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ to the linearized system posed on the given regularized space-time geometry $I \times \Omega_{\zeta_\kappa}$. In this section, we are going to use a fixed point argument to show that $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ actually solves the fully coupled nonlinear system posed on the unknown regularized geometry, that is,

$$\mathrm{div}_{\mathbf{x}} \mathbf{u}_\kappa = 0, \tag{4.33}$$

$$\partial_t \rho_\kappa + (\mathbf{u}_\kappa \cdot \nabla_{\mathbf{x}})\rho_\kappa = \Delta_{\mathbf{x}} \rho_\kappa, \tag{4.34}$$

$$\partial_t \mathbf{u}_\kappa + (\mathbf{u}_\kappa \cdot \nabla_{\mathbf{x}})\mathbf{u}_\kappa = \Delta_{\mathbf{x}} \mathbf{u}_\kappa - \nabla_{\mathbf{x}} p_\kappa + \mathbf{f}_\kappa + \mathrm{div}_{\mathbf{x}} \mathbb{T}_\kappa, \tag{4.35}$$

$$\partial_t^2 \eta_\kappa - \partial_t \partial_y^2 \eta_\kappa + \partial_y^4 \eta_\kappa = g_\kappa - (\mathbb{S}_\kappa \mathbf{n}_{\eta_\kappa}) \circ \boldsymbol{\varphi}_{\eta_\kappa} \cdot \mathbf{n} \det(\partial_y \boldsymbol{\varphi}_{\eta_\kappa}), \tag{4.36}$$

$$\partial_t \mathbb{T}_\kappa + (\mathbf{u}_\kappa \cdot \nabla_{\mathbf{x}})\mathbb{T}_\kappa = (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)\mathbb{T}_\kappa + \mathbb{T}_\kappa (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)^\top - 2(\mathbb{T}_\kappa - \rho_\kappa \mathbb{I}) + \Delta_{\mathbf{x}} \mathbb{T}_\kappa \tag{4.37}$$

on $I \times \Omega_{\eta_\kappa} \subset \mathbb{R}^{1+2}$ where

$$\mathbb{S}_\kappa = (\nabla_{\mathbf{x}} \mathbf{u}_\kappa + (\nabla_{\mathbf{x}} \mathbf{u}_\kappa)^\top) - p_\kappa \mathbb{I} + \mathbb{T}_\kappa.$$

A weak solution of (4.33)–(4.37) is defined in analogy to Definition 4.1. Our main result now reads

Theorem 4.4. *Let $\kappa > 0$ be arbitrary. For a dataset $(\mathbf{f}_\kappa, g_\kappa, \rho_{0,\kappa}, \mathbb{T}_{0,\kappa}, \mathbf{u}_{0,\kappa}, \eta_{0,\kappa}, \eta_{\star,\kappa})$ that satisfies* (4.8)*, there exists a weak solution $(\eta_\kappa, \mathbf{u}_\kappa, \rho_\kappa, \mathbb{T}_\kappa)$ of* (4.33)–(4.37).

Proof. We note that the only differences between (4.1)–(4.5) and the anticipated system (4.33)–(4.37) consists of the linearization by the given velocity $\mathbf{v}_\kappa$ (rather than $\mathbf{u}_\kappa$) in the advection term, the stress tensor term on the right-hand side of (4.4) being transformed by a coordinate transform with respect of ζ_κ (rather than η_κ), and finally the full system posed on Ω_{ζ_κ} (rather than Ω_{η_κ}). The proof of Theorem 4.4 therefore follows from the construction of a fixed point $\mathtt{T}(\zeta_\epsilon, \mathbf{v}_\epsilon) = (\eta_\epsilon, \mathbf{u}_\epsilon)$ of a certain solution map $\mathtt{T}$. Unfortunately, since we are dealing with weak solutions and the anticipated system given by (4.33)–(4.37) is nonlinear, we are unable to use a Banach fixed point-type argument as we did in the previous section. Consequently, we resort to a fixed point theorem for set-valued mappings that gives the existence (but not uniqueness) of a fixed point [37, Theorem A.4]. For this, with a slight abuse of notation, we (still) consider the interval $I_* := (0, T_*)$ where T_* is to be chosen later. For a tubular neighborhood S_α of $\partial\Omega$ with $\alpha \leq L$, we set

$$X := C(\overline{I}_* \times \omega) \otimes L^2(I_*; L^2(\Omega \cup S_\alpha))$$

and define the ball

$$B_R^X = \big\{(\zeta_\kappa, \mathbf{v}_\kappa) \in X : \zeta_\kappa(0) = \eta_{0,\kappa},\ \|(\zeta_\kappa, \mathbf{v}_\kappa)\|_X \leq R\big\}$$

for $R > 0$ large enough and for fixed $\kappa > 0$. Now let us consider the solution map $\mathtt{T} : B_R^X \subset X \to 2^{B_R^X}$ defined by $\mathtt{T}(\zeta_\kappa, \mathbf{v}_\kappa) = (\eta_\kappa, \mathbf{u}_\kappa)$. The critical requirement for a fixed point is to show compactness of the map $\mathtt{T}$. Thus, for a sequence $(\rho_\kappa^n, \mathbb{T}_\kappa^n)$ satisfying (4.12)–(4.13), we consider any sequence $(\zeta_\kappa^n, \mathbf{v}_\kappa^n)_{n\in\mathbb{N}} \in B_R^X$ with $\mathtt{T}(\zeta_\kappa^n, \mathbf{v}_\kappa^n) = (\eta_\kappa^n, \mathbf{u}_\kappa^n)$ (where the existence of such a solution map is guaranteed by (4.11) and (4.23)). Consequently, we have in particular that

$$\sup_{t\in I} \big(\|\partial_t \eta_\kappa^n\|_{L^2(\omega)}^2 + \|\partial_y^2 \eta_\kappa^n\|_{L^2(\omega)}^2\big) \lesssim 1$$

uniformly in $n \in \mathbb{N}$. Given that the embedding $W^{2,2}(\omega) \hookrightarrow C(\omega)$ is compact and the embedding $C(\omega) \hookrightarrow L^2(\omega)$ is continuous, it follows from Aubin–Lions lemma that

$$\eta_\kappa^n \to \eta_\kappa \quad \text{in } C(\overline{I}_* \times \omega).$$

Also, just as in [46, Lemma 6.3], we can use a reformulated Aubin–Lions lemma [46, Theorem 5.1] and the existence of a smooth solenoidal extension operator [46, Corollary 3.4] to also obtain

$$\partial_t \eta_\kappa^n \to \partial_t \eta_\kappa \quad \text{in } L^2(I_*; L^2(\omega)), \tag{4.38}$$

$$\mathbb{I}_{\Omega_{\zeta_\kappa^n}} \mathbf{u}_\kappa^n \to \mathbb{I}_{\Omega_{\zeta_\kappa}} \mathbf{u}_\kappa \quad \text{in } L^2(I_*; L^2(\Omega \cup S_\alpha)), \tag{4.39}$$

which finishes the proof of compactness of $\mathtt{T}$. Consequently, the map $\mathtt{T}$ posses a fixed point, that is, there exists $(\eta_\kappa, \mathbf{u}_\kappa) \in B_R^X$ with $\mathtt{T}(\eta_\kappa, \mathbf{u}_\kappa) = (\eta_\kappa, \mathbf{u}_\kappa)$. The fact that the solution is global follows from (3.1). ∎

Remark 4.5. We remark that the now standard method [46, Lemma 6.3] for obtaining compactness for the velocity sequence has been adapted to various settings, including a momentum equation with a forcing in divergence form [16, page 31] (just as we have in our present setting) and to stochastic models [18, page 24].

4.3. Limits of the regularized system

We are now going to pass to the limit $\kappa \to \infty$ in the regularization parameter to complete the proof of Theorem 2.2. Due to Theorem 4.4 (and (3.1)), it follows that

$$\begin{aligned}
\eta_\kappa &\to \eta && \text{in } (L^\infty(I; W^{2,2}(\omega)), w^*),\\
\partial_t \eta_\kappa &\to \partial_t \eta && \text{in } \big(L^\infty(I; L^2(\omega)), w^*\big) \cap \big(L^2(I; W^{1,2}(\omega)), w\big),\\
\mathbb{I}_{\Omega_{\eta_\kappa}} \mathbf{u}_\kappa &\to \mathbb{I}_{\Omega_\eta} \mathbf{u} && \text{in } \big(L^\infty(I; L^2(\Omega \cup S_\alpha)), w^*\big) \cap \big(L^2(I; W^{1,2}_{\mathrm{div}_x}(\Omega \cup S_\alpha)), w\big),\\
\mathbb{I}_{\Omega_{\eta_\kappa}} \rho_\kappa &\to \mathbb{I}_{\Omega_\eta} \rho && \text{in } \big(L^\infty(I; L^2(\Omega \cup S_\alpha)), w^*\big) \cap \big(L^2(I; W^{1,2}(\Omega \cup S_\alpha)), w\big),\\
\mathbb{I}_{\Omega_{\eta_\kappa}} \mathbb{T}_\kappa &\to \mathbb{I}_{\Omega_\eta} \mathbb{T} && \text{in } \big(L^\infty(I; L^2(\Omega \cup S_\alpha)), w^*\big) \cap \big(L^2(I; W^{1,2}(\Omega \cup S_\alpha)), w\big).
\end{aligned}$$

Furthermore, just as in (4.38)–(4.39), we also obtain

$$\begin{aligned}\partial_t\eta_\kappa &\rightarrow \partial_t\eta \quad \text{in } L^2(I_*;L^2(\omega)),\\ \mathbb{I}_{\Omega_{\eta_\kappa}}\mathbf{u}_\kappa &\rightarrow \mathbb{I}_{\Omega_\eta}\mathbf{u} \quad \text{in } L^2(I_*;L^2(\Omega\cup S_\alpha)).\end{aligned}$$

The above convergence results allow us to pass to the limit in the weak formulation and the energy inequality.

5. Strong solutions

We now attend to the proof of Theorem 2.5. Our strategy for constructing a solution also involves solving the solvent-structure subproblem and the solute subproblem independently of each other and using a fixed point argument to get a local solution to the fully coupled system. The extension to a global solution will then follow from estimate (3.7).

5.1. The solvent-structure subproblem

In the following, for a given pair $(\underline{\rho},\underline{\mathbb{T}})$, given body forces $\mathbf{f}$ and g, we wish to find a strong solution to the following system of equations:

$$\mathrm{div}_{\mathbf{x}}\mathbf{u} = 0, \tag{5.1}$$

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \Delta_{\mathbf{x}}\mathbf{u} - \nabla_{\mathbf{x}}p + \mathbf{f} + \mathrm{div}_{\mathbf{x}}\underline{\mathbb{T}}, \tag{5.2}$$

$$\partial_t^2\eta - \partial_t\partial_y^2\eta + \partial_y^4\eta = g - (\mathbb{S}\mathbf{n}_\eta)\circ\boldsymbol{\varphi}_\eta\cdot\mathbf{n}\,\det(\partial_y\boldsymbol{\varphi}_\eta), \tag{5.3}$$

defined on $I\times\Omega_\eta\subset\mathbb{R}^{1+2}$ where

$$\mathbb{S} = (\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^\top) - p\mathbb{I} + \underline{\mathbb{T}}.$$

We complement (5.1)–(5.3) with the following initial and interface conditions

$$\eta(0,\cdot) = \eta_0(\cdot), \quad \partial_t\eta(0,\cdot) = \eta_\star(\cdot) \quad \text{in } \omega, \tag{5.4}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \quad \text{in } \Omega_{\eta_0}. \tag{5.5}$$

$$\mathbf{u}\circ\boldsymbol{\varphi}_\eta = (\partial_t\eta)\mathbf{n} \quad \text{on } I\times\omega. \tag{5.6}$$

The precise definition of a strong solution is given as follows:

Definition 5.1 (Strong solution). Let $(\mathbf{f}, g, \eta_0, \eta_\star, \mathbf{u}_0, \underline{\mathbb{T}})$ be a dataset such that

$$\begin{aligned}&\mathbf{f}\in L^2(I;L^2_{\mathrm{loc}}(\mathbb{R}^2)),\ g\in L^2(I;L^2(\omega)),\ \eta_0\in W^{3,2}(\omega) \text{ with } \|\eta_0\|_{L^\infty(\omega)} < L,\\ &\eta_\star\in W^{1,2}(\omega),\ \underline{\mathbb{T}}\in L^2(I;W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)),\\ &\mathbf{u}_0\in W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\eta_0}) \text{ is such that } \mathbf{u}_0\circ\boldsymbol{\varphi}_{\eta_0} = \eta_\star\mathbf{n} \text{ on } \omega.\end{aligned}$$

We call $(\eta, \mathbf{u}, p)$ a *strong solution* of (5.1)–(5.6) with data $(\mathbf{f}, g, \eta_0, \eta_\star, \mathbf{u}_0, \mathbb{T})$ provided that the following hold:

(a) the structure-function η is such that $\|\eta\|_{L^\infty(I\times\omega)} < L$ and

$$\begin{aligned}\eta \in W^{1,\infty}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{3,2}(\omega)) \cap W^{1,2}(I; W^{2,2}(\omega))\\ \cap W^{2,2}(I; L^2(\omega)) \cap L^2(I; W^{4,2}(\omega));\end{aligned}$$

(b) the velocity $\mathbf{u}$ is such that $\mathbf{u} \circ \boldsymbol{\varphi}_\eta = (\partial_t \eta)\mathbf{n}$ on $I \times \omega$ and

$$\mathbf{u} \in W^{1,2}(I; L^2_{\mathrm{div}_\mathbf{x}}(\Omega_\eta)) \cap L^2(I; W^{2,2}(\Omega_\eta));$$

(c) the pressure p is such that

$$p \in L^2(I; W^{1,2}(\Omega_\eta));$$

(d) equations (5.1)–(5.3) are satisfied almost everywhere in space-time with $\eta(0) = \eta_0$ and $\partial_t \eta = \eta_\star$ almost everywhere in ω as well as $\mathbf{u}(0) = \mathbf{u}_0$ almost everywhere in Ω_{η_0}.

The existence of a unique global-in-time strong solution to (5.1)–(5.6) in the sense of Definition 5.1 is already shown in [14, Theorem 2.5], so we can proceed to the solute subproblem.

5.2. The solute subproblem

For a known flexible domain Ω_ζ and a known solenoidal vector field $\mathbf{v}$, we aim in this section to construct a strong solution of the following solute subproblem:

$$\partial_t \rho + (\mathbf{v} \cdot \nabla_\mathbf{x})\rho = \Delta_\mathbf{x} \rho, \tag{5.7}$$

$$\partial_t \mathbb{T} + (\mathbf{v} \cdot \nabla_\mathbf{x})\mathbb{T} = (\nabla_\mathbf{x} \mathbf{v})\mathbb{T} + \mathbb{T}(\nabla_\mathbf{x} \mathbf{v})^\top - 2(\mathbb{T} - \rho\mathbb{I}) + \Delta_\mathbf{x} \mathbb{T} \tag{5.8}$$

on $I \times \Omega_\zeta \subset \mathbb{R}^{1+2}$ subject to the initial and boundary conditions

$$\rho(0, \cdot) = \rho_0(\cdot), \quad \mathbb{T}(0, \cdot) = \mathbb{T}_0(\cdot) \quad \text{in } \Omega_{\zeta(0)}, \tag{5.9}$$

$$\mathbf{n}_\zeta \cdot \nabla_\mathbf{x} \rho = 0, \quad \mathbf{n}_\zeta \cdot \nabla_\mathbf{x} \mathbb{T} = 0 \quad \text{on } I \times \partial\Omega_\zeta. \tag{5.10}$$

The two unknowns ρ and $\mathbb{T}$ for the solute component of the polymer fluid are related via the identities

$$\mathbb{T}(t, \mathbf{x}) = \int_B f(t, \mathbf{x}, \mathbf{q})\mathbf{q} \otimes \mathbf{q}\, \mathrm{d}\mathbf{q}, \quad \rho(t, \mathbf{x}) = \int_B f(t, \mathbf{x}, \mathbf{q})\, \mathrm{d}\mathbf{q}$$

where f solves (1.11).

Let us start with a precise definition of what we mean by a strong solution.

Definition 5.2. Assume that $(\rho_0, \mathbb{T}_0, \mathbf{v}, \zeta)$ satisfies

$$\begin{aligned}
&\rho_0, \mathbb{T}_0 \in W^{1,2}(\Omega_{\zeta(0)}),\\
&\rho_0 \geq 0,\ \ \mathbb{T}_0 > 0 \quad \text{a.e. in } \Omega_{\zeta(0)},\\
&\mathbf{v} \in W^{1,2}(I; L^2_{\mathrm{div}_\mathbf{x}}(\Omega_\zeta)) \cap L^2(I; W^{2,2}(\Omega_\zeta)),\\
&\zeta \in W^{1,\infty}(I; W^{1,2}(\omega)) \cap L^\infty(I; W^{3,2}(\omega)) \cap W^{1,2}(I; W^{2,2}(\omega))\\
&\qquad \cap W^{2,2}(I; L^2(\omega)) \cap L^2(I; W^{4,2}(\omega)),\\
&\mathbf{v} \circ \boldsymbol{\varphi}_\zeta = (\partial_t \zeta)\mathbf{n} \,\text{on } I \times \omega, \quad \|\zeta\|_{L^\infty(I\times\omega)} < L.
\end{aligned} \tag{5.11}$$

We call $(\rho, \mathbb{T})$ a *strong solution* of (5.7)–(5.10) with dataset $(\rho_0, \mathbb{T}_0, \mathbf{v}, \zeta)$ if

(a) $(\rho, \mathbb{T})$ satisfies

$$\rho, \mathbb{T} \in W^{1,2}(I; L^2(\Omega_\zeta)) \cap L^2(I; W^{2,2}(\Omega_\zeta));$$

(b) for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\zeta} \rho\psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_I \int_{\Omega_\zeta} [\rho \partial_t \psi + (\rho \mathbf{v} \cdot \nabla_\mathbf{x})\psi] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad - \int_I \int_{\Omega_\zeta} \nabla_\mathbf{x} \rho \cdot \nabla_\mathbf{x} \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t;
\end{aligned}$$

(c) for all $\mathbb{Y} \in C^\infty(\overline{I} \times \mathbb{R}^2)$, we have

$$\begin{aligned}
\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\zeta} \mathbb{T} : \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_I \int_{\Omega_\zeta} [\mathbb{T} : \partial_t \mathbb{Y} + \mathbb{T} : (\mathbf{v} \cdot \nabla_\mathbf{x})\mathbb{Y}] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad + \int_I \int_{\Omega_\zeta} [(\nabla_\mathbf{x} \mathbf{v})\mathbb{T} + \mathbb{T}(\nabla_\mathbf{x} \mathbf{v})^\top] : \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad - 2\int_I \int_{\Omega_\zeta} (\mathbb{T} : \mathbb{Y} - \rho \mathrm{tr}(\mathbb{Y})) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad - \int_I \int_{\Omega_\zeta} \nabla_\mathbf{x} \mathbb{T} :: \nabla_\mathbf{x} \mathbb{Y} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

We now formulate our result on the existence of a unique strong solution of (5.7)–(5.10).

Theorem 5.3. *Let* $(\rho_0, \mathbb{T}_0, \mathbf{v}, \zeta)$ *satisfy* (5.11). *Then there is a unique strong solution* $(\rho, \mathbb{T})$ *of* (5.7)–(5.10)*, in the sense of Definition* 5.2*, such that*

$$\begin{aligned}
&\int_I \big(\|\partial_t \rho\|^2_{L^2(\Omega_\zeta)} + \|\partial_t \mathbb{T}\|^2_{L^2(\Omega_\zeta)}\big) \,\mathrm{d}t + \sup_{t\in I} \big(\|\rho(t)\|^2_{W^{1,2}(\Omega_\zeta)} + \|\mathbb{T}(t)\|^2_{W^{1,2}(\Omega_\zeta)}\big)\\
&\qquad + \int_I \big(\|\rho\|_{W^{2,2}(\Omega_\zeta)} + \|\mathbb{T}\|_{W^{2,2}(\Omega_\zeta)}\big)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\rho_0\|_{W^{1,2}(\Omega_{\zeta(0)})} + \|\mathbb{T}_0\|_{W^{1,2}(\Omega_{\zeta(0)})} + \int_I \big(\|\partial_t\zeta\|^2_{W^{2,2}(\omega)} + \|\mathbf{v}\|^2_{W^{2,2}(\Omega_\zeta)}\big)\,\mathrm{d}t \\
&\qquad + T\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\zeta(0)})} + T\|\rho_0\|^2_{L^2(\Omega_{\zeta(0)})}\big)\exp\Big(c\int_I \|\nabla_{\mathbf{x}}\mathbf{v}\|^2_{L^2(\Omega_\zeta)}\,\mathrm{d}t\Big). \qquad (5.12)
\end{aligned}
$$

holds with a constant depending on the $L^\infty(I;W^{1,2}(\omega))$*-norm of* $\partial_t\zeta$ *and the* $L^\infty(I;W^{1,\infty}(\omega))$*-norm of* ζ*, but otherwise being independent of the data.*

Since (5.7) and (5.8) are dissipative and bilinear, a strong solution of (5.7)–(5.10) is directly obtained by way of a limit to a Galerkin approximation. In particular, bound (5.12) for the finite-dimensional solution is obtained in the same manner as (3.18), after which one can pass to the limit.

5.3. The fully coupled system

In this section, we are going to use a fixed point argument to first establish the existence of a unique local strong solution to the fully coupled solute-solvent-structure system. This solution will hold globally in time because of Proposition 3.3. The fixed point argument requires showing the closedness of an anticipated solution in a ball and a contraction argument. These two properties will be shown in the following spaces:

$$
\begin{aligned}
X_\eta &:= W^{1,2}(I_*;L^2(\Omega_\eta)) \cap L^\infty(I_*;W^{1,2}(\Omega_\eta)) \cap L^2(I_*;W^{2,2}(\Omega_\eta)),\\
Y_\eta &:= L^\infty(I_*;L^2(\Omega_\eta)) \cap L^2(I_*;W^{1,2}(\Omega_\eta)),
\end{aligned}
$$

equipped with their canonical norms $\|\cdot\|_{X_\eta}$ and $\|\cdot\|_{Y_\eta}$, respectively. Here, I_* with $I_* = (0,T_*)$ is to be determined later.

Now, for $(\underline{\rho},\underline{\mathbb{T}}) \in Y_\eta \otimes Y_\eta$, let $(\eta,\mathbf{u},p)$ be a unique strong solution of (5.1)–(5.6) with data $(\mathbf{f},g,\eta_0,\eta_\star,\mathbf{u}_0,\underline{\mathbb{T}})$ as shown in [14, Theorem 2.5]. On the other hand, for

$$
\begin{aligned}
(\eta,\mathbf{u}) \in\ & W^{1,\infty}(I_*;W^{1,2}(\omega)) \cap L^\infty(I_*;W^{3,2}(\omega)) \cap W^{1,2}(I_*;W^{2,2}(\omega))\\
&\cap W^{2,2}(I_*;L^2(\omega)) \cap L^2(I;W^{4,2}(\omega))\\
&\otimes W^{1,2}(I_*;L^2_{\mathrm{div}_{\mathbf{x}}}(\Omega_\eta)) \cap L^2(I_*;W^{2,2}(\Omega_\eta)),
\end{aligned}
$$

let $(\rho,\mathbb{T})$ be the unique strong solution of (5.7)–(5.10) with dataset $(\rho_0,\mathbb{T}_0,\mathbf{u},\eta)$ as shown in Theorem 5.3. Now define the mapping $\mathtt{T} = \mathtt{T}_1 \circ \mathtt{T}_2$ where

$$
\mathtt{T}(\underline{\rho},\underline{\mathbb{T}}) = (\rho,\mathbb{T}), \quad \mathtt{T}_2(\underline{\rho},\underline{\mathbb{T}}) = (\eta,\mathbf{u},p), \quad \mathtt{T}_1(\eta,\mathbf{u},p) = (\rho,\mathbb{T})
$$

and let

$$
B_R := \big\{(\underline{\rho},\underline{\mathbb{T}}) \in X_\eta \otimes X_\eta : \|(\underline{\rho},\underline{\mathbb{T}})\|^2_{X_\eta\otimes X_\eta} \leq R^2\big\}.
$$

Let show that $\mathtt{T} : X_\eta \otimes X_\eta \to X_\eta \otimes X_\eta$ maps B_R into B_R, that is, for any $(\underline{\rho},\underline{\mathbb{T}}) \in B_R$, we have that

$$
\|(\rho,\mathbb{T})\|^2_{X_\eta\otimes X_\eta} = \|\mathtt{T}(\underline{\rho},\underline{\mathbb{T}})\|^2_{X_\eta\otimes X_\eta} = \|\mathtt{T}_1\circ\mathtt{T}_2(\underline{\rho},\underline{\mathbb{T}})\|^2_{X_\eta\otimes X_\eta} = \|\mathtt{T}_1(\eta,\mathbf{u},p)\|^2_{X_\eta\otimes X_\eta} \leq R^2.
$$

Indeed if we let $(\underline{\rho}, \underline{\mathbb{T}}) \in X_\eta \otimes X_\eta$ then by the a priori estimate in (5.12),

$$\begin{aligned}\|(\rho, \mathbb{T})\|^2_{X_\eta \otimes X_\eta} \lesssim \|\rho_0\|_{W^{1,2}(\Omega_{\eta_0})} + \|\mathbb{T}_0\|_{W^{1,2}(\Omega_{\eta_0})}&\\ + \int_{I_*} \big(\|\partial_t \eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t&\\ + cT_*\big(\|\mathbb{T}_0\|^2_{L^2(\Omega_{\eta_0})} + T_*\|\rho_0\|^2_{L^2(\Omega_{\eta_0})}\big)&\\ \times \exp\Big(c\int_{I_*} \|\nabla_{\mathbf{x}}\mathbf{u}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\Big).&\end{aligned} \tag{5.13}$$

However, by [14, (4.14) and Lemma 4.2], a unique strong solution $(\eta, \mathbf{u}, p)$ of (5.1)–(5.6) with data $(\mathbf{f}, g, \eta_0, \eta_\star, \mathbf{u}_0, \underline{\mathbb{T}})$ satisfies[3]

$$\begin{aligned}&\int_{I_*} \big(\|\partial_t \eta\|^2_{W^{2,2}(\omega)} + \|\mathbf{u}\|^2_{W^{2,2}(\Omega_\eta)}\big)\,\mathrm{d}t\\ &\quad\lesssim \|\mathbf{u}_0\|^2_{W^{1,2}(\eta_0)} + \|\eta_0\|^2_{W^{3,2}(\omega)} + \|\eta_\star\|^2_{W^{1,2}(\omega)}\\ &\qquad + \int_{I_*} \big(\|g\|^2_{L^2(\omega)} + \|\mathbf{f}\|^2_{L^2(\Omega_\eta)} + \|\underline{\mathbb{T}}\|^2_{W^{1,2}(\Omega_\eta)}\big)\,\mathrm{d}t.\end{aligned} \tag{5.14}$$

Given the regularity of the dataset and the fact that $\underline{\mathbb{T}} \in X_\eta$, for a large enough $R > 0$ and $T_* > 0$ small enough, we obtain

$$\|(\rho, \mathbb{T})\|^2_{X_\eta \otimes X_\eta} \leq R^2$$

by substituting (5.14) into (5.13). Thus, $\mathtt{T} : B_R \to B_R$.

To show the contraction property, we let $(\rho_i, \mathbb{T}_i)$, $i = 1, 2$ be two strong solutions of (5.7)–(5.10) with dataset $(\rho_0, \mathbb{T}_0, \mathbf{u}_i, \eta_i)$, $i = 1, 2$, respectively. Since the fluid domain depends on the deformation of the shell, we have to transform one solution, say $\mathbb{T}_2$, to the domain of $\mathbb{T}_1$ in order to get a difference estimate. To get the equation for the transformation of $\mathbb{T}_2$, we make use of the distributional formulation

$$\begin{aligned}&\int_{I_*}\int_{\Omega_{\eta_2}} [\partial_t \mathbb{T}_2 + (\mathbf{u}_2 \cdot \nabla_{\mathbf{x}})\mathbb{T}_2] : \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\\ &\quad= \int_{I_*}\int_{\Omega_{\eta_2}} [(\nabla_{\mathbf{x}}\mathbf{u}_2)\mathbb{T}_2 + \mathbb{T}_2(\nabla_{\mathbf{x}}\mathbf{u}_2)^\top] : \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &\qquad - 2\int_{I_*}\int_{\Omega_{\eta_2}} (\mathbb{T}_2 : \mathbb{Y}_2 - \rho_2 \mathrm{tr}(\mathbb{Y}_2))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{I_*}\int_{\Omega_{\eta_2}} \nabla_{\mathbf{x}}\mathbb{T}_2 :: \nabla_{\mathbf{x}}\mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\end{aligned}$$

for all $\mathbb{Y} \in C^\infty(\overline{I_*} \times \mathbb{R}^2)$. Let us set $\overline{\mathbb{T}}_2 = \mathbb{T}_2 \circ \boldsymbol{\Psi}_{\eta_2-\eta_1}$, $\overline{\mathbf{u}}_2 = \mathbf{u}_2 \circ \boldsymbol{\Psi}_{\eta_2-\eta_1}$, $\overline{\rho}_2 = \rho_2 \circ \boldsymbol{\Psi}_{\eta_2-\eta_1}$, and $\mathbb{Y}_2 = \overline{\mathbb{Y}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1}$, then

$$\int_{I_*}\int_{\Omega_{\eta_2}} \partial_t \mathbb{T}_2 : \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

---

[3] A careful analysis of [14, (4.14)] shows that $\|g\|^2_{L^2(\omega)}$ (rather than $\|g\|^2_{W^{1,2}(\omega)}$) is sufficient on the right-hand side of (5.14).

$$
\begin{aligned}
&= \int_{I_*}\int_{\Omega_{\eta_2}} (\partial_t \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} + \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \partial_t \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1}) : \overline{\mathbb{Y}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, (\partial_t \overline{\mathbb{T}}_2 + \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \cdot \partial_t \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1}) : \overline{\mathbb{Y}}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}
$$

where $J_{\eta_2-\eta_1} = \det(\nabla_{\mathbf{x}} \boldsymbol{\Psi}_{\eta_2-\eta_1})$. We also have

$$
\begin{aligned}
&\int_{I_*}\int_{\Omega_{\eta_2}} \mathbf{u}_2 \cdot \nabla_{\mathbf{x}} \mathbb{T}_2 : \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad = \int_{I_*}\int_{\Omega_{\eta_2}} \overline{\mathbf{u}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \cdot \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} : \overline{\mathbb{Y}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad = \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, \overline{\mathbf{u}}_2 \cdot \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1} : \overline{\mathbb{Y}}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\int_{I_*}\int_{\Omega_{\eta_2}} [(\nabla_{\mathbf{x}} \mathbf{u}_2)\mathbb{T}_2 + \mathbb{T}_2 (\nabla_{\mathbf{x}} \mathbf{u}_2)^\top] : \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{I_*}\int_{\Omega_{\eta_2}} [\nabla_{\mathbf{x}} \overline{\mathbf{u}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \\
&\qquad\quad + \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} (\nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1})^\top (\nabla_{\mathbf{x}} \overline{\mathbf{u}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1})^\top] : \overline{\mathbb{Y}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, \nabla_{\mathbf{x}} \overline{\mathbf{u}}_2 \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1} \overline{\mathbb{T}}_2 : \overline{\mathbb{Y}}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, \overline{\mathbb{T}}_2 (\nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1})^\top (\nabla_{\mathbf{x}} \overline{\mathbf{u}}_2)^\top : \overline{\mathbb{Y}}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
$$

Similarly,

$$
\int_{I_*}\int_{\Omega_{\eta_2}} (\mathbb{T}_2 : \mathbb{Y}_2 - \rho_2 \mathrm{tr}(\mathbb{Y}_2)) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, (\overline{\mathbb{T}}_2 : \overline{\mathbb{Y}}_2 - \overline{\rho}_2 \mathrm{tr}(\overline{\mathbb{Y}}_2)) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
$$

and

$$
\begin{aligned}
&\int_{I_*}\int_{\Omega_{\eta_2}} \nabla_{\mathbf{x}} \mathbb{T}_2 :: \nabla_{\mathbf{x}} \mathbb{Y}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{I_*}\int_{\Omega_{\eta_2}} \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} : \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \nabla_{\mathbf{x}} \overline{\mathbb{Y}}_2 \circ \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad = \int_{I_*}\int_{\Omega_{\eta_1}} J_{\eta_2-\eta_1} \, (\nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1})^\top \nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1} \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 :: \nabla_{\mathbf{x}} \overline{\mathbb{Y}}_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
$$

Now let

$$
\mathbb{A}_{\eta_2-\eta_1} := (\nabla_{\mathbf{x}} \boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1} \circ \boldsymbol{\Psi}_{\eta_2-\eta_1})^\top \mathbb{B}_{\eta_2-\eta_1},
$$

$$\text{where}\quad \mathbb{B}_{\eta_2-\eta_1} := J_{\eta_2-\eta_1}\nabla_{\mathbf{x}}\boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1}\circ\boldsymbol{\Psi}_{\eta_2-\eta_1};$$

then we obtain the equation

$$\begin{aligned}\partial_t\overline{\mathbb{T}}_2+\overline{\mathbf{u}}_2\cdot\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2 &= \nabla_{\mathbf{x}}\overline{\mathbf{u}}_2\overline{\mathbb{T}}_2+\overline{\mathbb{T}}_2(\nabla_{\mathbf{x}}\overline{\mathbf{u}}_2)^\top-2(\overline{\mathbb{T}}_2-\overline{\rho}_2\mathbb{I})+\Delta_{\mathbf{x}}\overline{\mathbb{T}}_2\\ &\quad-\mathrm{div}_{\mathbf{x}}((\mathbb{I}-\mathbb{A}_{\eta_2-\eta_1})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2)+\mathbb{H}_{\eta_2-\eta_1}(\overline{\rho}_2,\overline{\mathbf{u}}_2,\overline{\mathbb{T}}_2)\end{aligned}$$

defined on $I_*\times\Omega_{\eta_1}$ where

$$\begin{aligned}&\mathbb{H}_{\eta_2-\eta_1}(\overline{\rho}_2,\overline{\mathbf{u}}_2,\overline{\mathbb{T}}_2)\\ &\quad=(1-J_{\eta_2-\eta_1})\partial_t\overline{\mathbb{T}}_2-J_{\eta_2-\eta_1}\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2\cdot\partial_t\boldsymbol{\Psi}^{-1}_{\eta_2-\eta_1}\circ\boldsymbol{\Psi}_{\eta_2-\eta_1}+\nabla_{\mathbf{x}}\overline{\mathbf{u}}_2(\mathbb{B}_{\eta_2-\eta_1}-\mathbb{I})\overline{\mathbb{T}}_2\\ &\qquad+\overline{\mathbb{T}}_2(\mathbb{B}_{\eta_2-\eta_1}-\mathbb{I})^\top(\nabla_{\mathbf{x}}\overline{\mathbf{u}}_2)^\top+\overline{\mathbf{u}}_2\cdot\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2(\mathbb{I}-\mathbb{B}_{\eta_2-\eta_1})\\ &\qquad+2(1-J_{\eta_2-\eta_1})(\overline{\mathbb{T}}_2-\overline{\rho}_2\mathbb{I}).\end{aligned}$$

Now take the first solution

$$\partial_t\mathbb{T}_1+\mathbf{u}_1\cdot\nabla_{\mathbf{x}}\mathbb{T}_1=\nabla_{\mathbf{x}}\mathbf{u}_1\mathbb{T}_1+\mathbb{T}_1(\nabla_{\mathbf{x}}\mathbf{u}_1)^\top-2(\mathbb{T}_1-\rho_1\mathbb{I})+\Delta_{\mathbf{x}}\mathbb{T}_1$$

defined on $I_*\times\Omega_{\eta_1}$ and set

$$\mathbb{T}_{12}:=\mathbb{T}_1-\overline{\mathbb{T}}_2,\quad \mathbf{u}_{12}=\mathbf{u}_1-\overline{\mathbf{u}}_2,\quad \rho_{12}=\rho_1-\overline{\rho}_2,\quad \eta_{12}=\eta_1-\eta_2.$$

Then $\mathbb{T}_{12}$ solves

$$\begin{aligned}\partial_t\mathbb{T}_{12}+\mathbf{u}_1\cdot\nabla_{\mathbf{x}}\mathbb{T}_{12}&=\nabla_{\mathbf{x}}\mathbf{u}_1\mathbb{T}_{12}+\mathbb{T}_{12}(\nabla_{\mathbf{x}}\mathbf{u}_1)^\top-2(\mathbb{T}_{12}-\rho_{12}\mathbb{I})+\Delta_{\mathbf{x}}\mathbb{T}_{12}\\ &\quad-\mathbf{u}_{12}\cdot\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2+\nabla_{\mathbf{x}}\mathbf{u}_{12}\overline{\mathbb{T}}_2+\overline{\mathbb{T}}_2(\nabla_{\mathbf{x}}\mathbf{u}_{12})^\top\\ &\quad+\mathrm{div}_{\mathbf{x}}((\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2)-\mathbb{H}_{-\eta_{12}}(\overline{\rho}_2,\overline{\mathbf{u}}_2,\overline{\mathbb{T}}_2)\end{aligned}\tag{5.15}$$

on $I_*\times\Omega_{\eta_1}$ with identically zero initial condition. If we now test (5.15) with $\mathbb{T}_{12}$, then for $t\in I_*$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}+\|\mathbb{T}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}\\ &\quad\le\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}+\|\rho_{12}\|^2_{L^2(\Omega_{\eta_1})}+2\int_{\Omega_{\eta_1}}|\nabla_{\mathbf{x}}\mathbf{u}_1||\mathbb{T}_{12}|^2\,\mathrm{d}\mathbf{x}\\ &\qquad+\int_{\Omega_{\eta_1}}\big[\nabla_{\mathbf{x}}\mathbf{u}_{12}\overline{\mathbb{T}}_2+\overline{\mathbb{T}}_2(\nabla_{\mathbf{x}}\mathbf{u}_{12})^\top-\mathbf{u}_{12}\cdot\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2\big]:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\\ &\qquad+\int_{\Omega_{\eta_1}}\big[\mathrm{div}_{\mathbf{x}}((\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2)-\mathbb{H}_{-\eta_{12}}(\overline{\rho}_2,\overline{\mathbf{u}}_2,\overline{\mathbb{T}}_2)\big]:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{5.16}$$

Firstly, we obtain

$$2\int_{\Omega_{\eta_1}}|\nabla_{\mathbf{x}}\mathbf{u}_1||\mathbb{T}_{12}|^2\,\mathrm{d}\mathbf{x}\lesssim\|\nabla_{\mathbf{x}}\mathbf{u}_1\|_{L^2(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|^2_{L^4(\Omega_{\eta_1})}$$

$$\begin{aligned}&\lesssim \|\nabla_{\mathbf{x}}\mathbf{u}_1\|_{L^2(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}\\ &\leq c\|\nabla_{\mathbf{x}}\mathbf{u}_1\|^2_{L^2(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}+\delta\|\mathbb{T}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}\end{aligned}\tag{5.17}$$

for any $\delta > 0$. Next, by using the embedding $W^{2,2}(\Omega_{\eta_1}) \hookrightarrow L^\infty(\Omega_{\eta_1})$, we obtain

$$\begin{aligned}&\int_{\Omega_{\eta_1}}\left[\nabla_{\mathbf{x}}\mathbf{u}_{12}\overline{\mathbb{T}}_2+\overline{\mathbb{T}}_2(\nabla_{\mathbf{x}}\mathbf{u}_{12})^\top\right]:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\\ &\quad\lesssim \|\nabla_{\mathbf{x}}\mathbf{u}_{12}\|_{L^2(\Omega_{\eta_1})}\|\overline{\mathbb{T}}_2\|_{L^\infty(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}\\ &\quad\leq \delta\|\nabla_{\mathbf{x}}\mathbf{u}_{12}\|^2_{L^2(\Omega_{\eta_1})}+c\|\overline{\mathbb{T}}_2\|^2_{W^{2,2}(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}\end{aligned}\tag{5.18}$$

for any $\delta > 0$. Similarly,

$$\begin{aligned}\int_{\Omega_{\eta_1}}\mathbf{u}_{12}\cdot\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}&\lesssim\|\mathbf{u}_{12}\|_{L^4(\Omega_{\eta_1})}\|\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2\|_{L^4(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}\\ &\leq\delta\|\nabla_{\mathbf{x}}\mathbf{u}_{12}\|^2_{L^2(\Omega_{\eta_1})}+c\|\overline{\mathbb{T}}_2\|^2_{W^{2,2}(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}.\end{aligned}\tag{5.19}$$

Next, we write

$$\begin{aligned}\int_{\Omega_{\eta_1}}\mathrm{div}_{\mathbf{x}}((\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2):\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}&=\int_{\partial\Omega_{\eta_1}}\mathbf{n}_{\eta_1}\cdot(\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\\ &\quad-\int_{\Omega_{\eta_1}}(\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2::\nabla_{\mathbf{x}}\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\end{aligned}\tag{5.20}$$

where, by trace theorem and the fact that $\mathbb{I}-\mathbb{A}_{-\eta_{12}}\sim-\partial_y\eta_{12}$ holds in norm,

$$\begin{aligned}&\int_{\partial\Omega_{\eta_1}}\mathbf{n}_{\eta_1}\cdot(\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\\ &\quad\lesssim\|\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2\|_{L^4(\partial\Omega_{\eta_1})}\|\mathbb{I}-\mathbb{A}_{-\eta_{12}}\|_{L^4(\partial\Omega_{\eta_1})}\|\mathbb{T}_{12}\|_{L^2(\partial\Omega_{\eta_1})}\\ &\quad\lesssim\|\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2\|_{W^{3/4,2}(\Omega_{\eta_1})}\|\eta_{12}\|_{W^{1,4}(\omega)}\|\mathbb{T}_{12}\|_{W^{3/4,2}(\Omega_{\eta_1})}\\ &\quad\lesssim\|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}\|\eta_{12}\|_{W^{2,2}(\omega)}\|\mathbb{T}_{12}\|^{1/4}_{L^2(\Omega_{\eta_1})}\|\mathbb{T}_{12}\|^{3/4}_{W^{1,2}(\Omega_{\eta_1})}\\ &\quad\leq\delta\|\mathbb{T}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\delta\|\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}+c\|\overline{\mathbb{T}}_2\|^2_{W^{2,2}(\Omega_{\eta_1})}\|\eta_{12}\|^2_{W^{2,2}(\omega)}.\end{aligned}\tag{5.21}$$

We also obtain

$$\begin{aligned}&\int_{\Omega_{\eta_1}}(\mathbb{I}-\mathbb{A}_{-\eta_{12}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_2::\nabla_{\mathbf{x}}\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}\\ &\quad\leq\delta\|\nabla_{\mathbf{x}}\mathbb{T}_{12}\|^2_{L^2(\Omega_{\eta_1})}+c\|\overline{\mathbb{T}}_2\|^2_{W^{2,2}(\Omega_{\eta_1})}\|\eta_{12}\|^2_{W^{2,2}(\omega)}.\end{aligned}\tag{5.22}$$

Next, we rewrite

$$\int_{\Omega_{\eta_1}}\mathbb{H}_{-\eta_{12}}(\overline{\rho}_2,\overline{\mathbf{u}}_2,\overline{\mathbb{T}}_2):\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}=\int_{\Omega_{\eta_1}}(1-J_{-\eta_{12}})\partial_t\overline{\mathbb{T}}_2:\mathbb{T}_{12}\,\mathrm{d}\mathbf{x}$$

$$
\begin{aligned}
&- \int_{\Omega_{\eta_1}} J_{-\eta_{12}} \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 \cdot \partial_t \boldsymbol{\Psi}_{-\eta_{12}}^{-1} \circ \boldsymbol{\Psi}_{-\eta_{12}} : \mathbb{T}_{12} \, \mathrm{d}\mathbf{x} \\
&+ \int_{\Omega_{\eta_1}} \nabla_{\mathbf{x}} \overline{\mathbf{u}}_2 (\mathbb{B}_{-\eta_{12}} - \mathbb{I}) \overline{\mathbb{T}}_2 : \mathbb{T}_{12} \, \mathrm{d}\mathbf{x} \\
&+ \int_{\Omega_{\eta_1}} \overline{\mathbb{T}}_2 (\mathbb{B}_{-\eta_{12}} - \mathbb{I})^\top (\nabla_{\mathbf{x}} \overline{\mathbf{u}}_2)^\top : \mathbb{T}_{12} \, \mathrm{d}\mathbf{x} \\
&+ \int_{\Omega_{\eta_1}} \overline{\mathbf{u}}_2 \cdot \nabla_{\mathbf{x}} \overline{\mathbb{T}}_2 (\mathbb{I} - \mathbb{B}_{-\eta_{12}}) : \mathbb{T}_{12} \, \mathrm{d}\mathbf{x} \\
&+ 2 \int_{\Omega_{\eta_1}} (1 - J_{-\eta_{12}}) (\overline{\mathbb{T}}_2 - \overline{\rho}_2 \mathbb{I}) : \mathbb{T}_{12} \, \mathrm{d}\mathbf{x} \\
=:\ & I_1 + \ldots + I_6.
\end{aligned}
\tag{5.23}
$$

Then we have, by interpolation

$$
\begin{aligned}
I_1 &\lesssim \|\eta_{12}\|_{W^{1,4}(\omega)} \|\partial_t \overline{\mathbb{T}}_2\|_{L^2(\Omega_{\eta_1})} \|\mathbb{T}_{12}\|_{L^4(\Omega_{\eta_1})} \\
&\lesssim \|\eta_{12}\|_{W^{2,2}(\omega)} \|\partial_t \overline{\mathbb{T}}_2\|_{L^2(\Omega_{\eta_1})} \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^{1/2} \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^{1/2} \\
&\leq c \|\eta_{12}\|_{W^{2,2}(\omega)}^2 \|\partial_t \overline{\mathbb{T}}_2\|_{L^2(\Omega_{\eta_1})}^2 + 2\delta \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})} \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})} \\
&\leq c \|\eta_{12}\|_{W^{2,2}(\omega)}^2 \|\partial_t \overline{\mathbb{T}}_2\|_{L^2(\Omega_{\eta_1})}^2 + \delta \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^2 + \delta \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^2 .
\end{aligned}
$$

For I_2, we use the equivalence $W^{3/2,2}(\Omega_{\eta_1}) \equiv W^{1,4}(\Omega_{\eta_1})$ in 2D and interpolation to obtain

$$
\begin{aligned}
I_2 &\lesssim \|\partial_t \eta_{12}\|_{L^2(\omega)} \|\overline{\mathbb{T}}_2\|_{W^{3/2,2}(\Omega_{\eta_1})} \|\mathbb{T}_{12}\|_{L^4(\Omega_{\eta_1})} \\
&\lesssim \|\partial_t \eta_{12}\|_{L^2(\omega)} \|\overline{\mathbb{T}}_2\|_{W^{1,2}(\Omega_{\eta_1})}^{1/2} \|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^{1/2} \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^{1/2} \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^{1/2} \\
&\leq \delta \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^2 + c \|\partial_t \eta_{12}\|_{L^2(\omega)}^{4/3} \|\overline{\mathbb{T}}_2\|_{W^{1,2}(\Omega_{\eta_1})}^{2/3} \|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^{2/3} \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^{2/3} \\
&\leq \delta \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^2 + c \|\partial_t \eta_{12}\|_{L^2(\omega)}^2 \|\overline{\mathbb{T}}_2\|_{W^{1,2}(\Omega_{\eta_1})} + c \|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^2 \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^2 .
\end{aligned}
$$

Finally, we also obtain

$$
\begin{aligned}
I_3 + I_4 + I_5 &\lesssim \|\eta_{12}\|_{W^{2,2}(\omega)}^2 \|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^2 + \|\overline{\mathbf{u}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^2 \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^2 , \\
I_6 &\leq c \|\eta_{12}\|_{W^{2,2}(\omega)}^2 \big(\|\overline{\mathbb{T}}_2\|_{W^{1,2}(\Omega_{\eta_1})}^2 + \|\overline{\rho}_2\|_{W^{1,2}(\Omega_{\eta_1})}^2 \big) + \delta \|\mathbb{T}_{12}\|_{L^2(\Omega_{\eta_1})}^2 .
\end{aligned}
$$

We have shown that

$$
\begin{aligned}
&\sup_{t \in I_*} \|\mathbb{T}_{12}(t)\|_{L^2(\Omega_{\eta_1})}^2 + \int_{I_*} \|\mathbb{T}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^2 \, \mathrm{d}t \\
&\quad \lesssim \exp \Big(\int_{I_*} \big(1 + \|\overline{\mathbf{u}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^2 + \|\nabla_{\mathbf{x}} \mathbf{u}_1\|_{L^2(\Omega_{\eta_1})}^2 + \|\overline{\mathbb{T}}_2\|_{W^{2,2}(\Omega_{\eta_1})}^2 \big) \, \mathrm{d}t \Big) \\
&\qquad \times \Big[\int_{I_*} \|\rho_{12}\|_{L^2(\Omega_{\eta_1})}^2 \, \mathrm{d}t + \int_{I_*} \big(1 + \|\overline{\mathbb{T}}_2\|_{W^{1,2}(\Omega_{\eta_1})}^2 \big) \|\mathbf{u}_{12}\|_{W^{1,2}(\Omega_{\eta_1})}^2 \, \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{I_*}\big(1+\|\overline{\mathbb{T}}_2\|^2_{W^{1,2}(\Omega_{\eta_1})}\big)\|\partial_t\eta_{12}\|^2_{L^2(\omega)}\,\mathrm{d}t\\
&+\int_{I_*}\big(\|\partial_t\overline{\mathbb{T}}_2\|^2_{L^2(\Omega_{\eta_1})}+\|\overline{\mathbb{T}}_2\|^2_{W^{2,2}(\Omega_{\eta_1})}+\|\overline{\rho}_2\|^2_{W^{1,2}(\Omega_{\eta_1})}\big)\|\eta_{12}\|^2_{W^{2,2}(\omega)}\,\mathrm{d}t\Big],
\end{aligned}
\tag{5.24}
$$

which can be simplified to

$$
\begin{aligned}
&\sup_{t\in I_*}\|\mathbb{T}_{12}(t)\|^2_{L^2(\Omega_{\eta_1})}+\int_{I_*}\|\mathbb{T}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}\,\mathrm{d}t\\
&\qquad\lesssim\int_{I_*}\|\rho_{12}\|^2_{L^2(\Omega_{\eta_1})}\,\mathrm{d}t+\int_{I_*}\big(\|\mathbf{u}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\|\partial_t\eta_{12}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t\\
&\qquad\quad+\sup_{t\in I_*}\|\eta_{12}\|^2_{W^{2,2}(\omega)}
\end{aligned}
\tag{5.25}
$$

by using the regularity of the individual terms. Similarly, for the difference of two strong solutions of (5.7), we also obtain

$$
\begin{aligned}
&\sup_{t\in I_*}\|\rho_{12}(t)\|^2_{L^2(\Omega_{\eta_1})}+\int_{I_*}\|\rho_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}\,\mathrm{d}t\\
&\qquad\lesssim\int_{I_*}\big(\|\mathbf{u}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\|\partial_t\eta_{12}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t+\sup_{t\in I_*}\|\eta_{12}\|^2_{W^{2,2}(\omega)}.
\end{aligned}
\tag{5.26}
$$

Combining the two estimates above therefore yields

$$
\begin{aligned}
&\sup_{t\in I_*}\big(\|\rho_{12}(t)\|^2_{L^2(\Omega_{\eta_1})}+\|\mathbb{T}_{12}(t)\|^2_{L^2(\Omega_{\eta_1})}\big)+\int_{I_*}\big(\|\rho_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\|\mathbb{T}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}\big)\,\mathrm{d}t\\
&\quad\lesssim(1+T_*)\Big[\int_{I_*}\big(\|\mathbf{u}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\|\partial_t\eta_{12}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t+\sup_{t\in I_*}\|\eta_{12}\|^2_{W^{2,2}(\omega)}\Big].
\end{aligned}
\tag{5.27}
$$

Now, let us consider two strong solutions $(\eta_i,\mathbf{u}_i,p_i)$, $i=1,2$ of (5.1)–(5.6) with data $(\mathbf{f},g,\eta_0,\eta_\star,\mathbf{u}_0,\underline{\mathbb{T}}_i)$, respectively. The existence of these solutions is shown in [14, Theorem 2.5]. For

$$
\underline{\mathbb{T}}_{12}:=\underline{\mathbb{T}}_1-\overline{\underline{\mathbb{T}}}_2,\quad \mathbf{u}_{12}=\mathbf{u}_1-\overline{\mathbf{u}}_2,\quad \eta_{12}=\eta_1-\eta_2,
$$

where $\overline{\underline{\mathbb{T}}}_2:=\underline{\mathbb{T}}_2\circ\boldsymbol{\Psi}_{\eta_2-\eta_1}$, it follows from [19, Remark 5.2] that

$$
\begin{aligned}
\int_{I_*}\big(\|\mathbf{u}_{12}\|^2_{W^{1,2}(\Omega_{\eta_1})}+\|\partial_t\eta_{12}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t+\sup_{t\in I_*}\|\eta_{12}\|^2_{W^{2,2}(\omega)}&\lesssim\int_{I_*}\|\underline{\mathbb{T}}_{12}\|^2_{L^2(\Omega_{\eta_1})}\,\mathrm{d}t\\
&\lesssim T_*\|(\underline{\rho}_{12},\underline{\mathbb{T}}_{12})\|^2_{Y_{\eta_1}\otimes Y_{\eta_1}}.
\end{aligned}
$$

Inserting into (5.27) then yields

$$
\|(\rho_{12},\mathbb{T}_{12})\|^2_{Y_{\eta_1}\otimes Y_{\eta_1}}\lesssim(1+T_*)\big[T_*\|(\underline{\rho}_{12},\underline{\mathbb{T}}_{12})\|^2_{Y_{\eta_1}\otimes Y_{\eta_1}}\big].
$$

By choosing $T_*>0$ small enough, we obtain

$$
\|(\rho_{12},\mathbb{T}_{12})\|^2_{Y_{\eta_1}\otimes Y_{\eta_1}}\leq\tfrac{1}{2}\|(\underline{\rho}_{12},\underline{\mathbb{T}}_{12})\|^2_{Y_{\eta_1}\otimes Y_{\eta_1}}.
$$

The existence of the desired fixed point now follows.

The strong solution being global in time follows from Proposition 3.3.

6. Weak-strong uniqueness

Due to the fixed point argument shown above, we obtain a unique strong solution in the fixed ball. Nevertheless, one can extend uniqueness to the whole space using a continuity argument. In the following, however, we show a stronger weak-strong uniqueness result where one solution is allowed to be a generalized weak solution. Unlike the 3D/2D framework in [19] where a weak-strong uniqueness result is shown for the interaction of a 3D fluid with a 2D shell under the condition that

$$\eta \in L^\infty(I; C^1(\omega)), \tag{6.1}$$

$$\mathbf{u} \in L^r(I; L^s(\Omega_\eta)), \quad \tfrac{2}{r} + \tfrac{3}{s} \leq 1, \tag{6.2}$$

we can take advantage of our 2D/1D setup to obtain an *unconditional* weak-strong uniqueness result for our fully coupled solute-solvent-structure system.

To simplify our work, we first explain why (6.1)–(6.2) was needed in [19] and then justify why they are not needed in our setting.

Firstly, the requirement in (6.1) was needed in [19] for three reasons:

- The regularity for strong solutions consists of proving a so-called *acceleration estimate* where a key tool is to estimate $\nabla_{\mathbf{x}}^2 \mathbf{u}$ and $\nabla_{\mathbf{x}} p$ by means of $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u}$. This can be done by means of a steady Stokes theory for irregular domains [14] that strongly requires a boundary with a small local Lipschitz constant and thus, (6.1) is essentially needed. Note that since the Lipschitz constant needs to be small, it is not enough to have spatial regularity $C^{0,1}(\omega)$.
- To obtain weak-strong uniqueness, the authors in [19] required the introduction of the so-called *Universal Bogovskij operator* (see Section 2 and [19, Theorem 2.1]), which further requires that the structure or shell is Lipschitz continuous in space. For weak solutions, the highest spatial regularity for η is $W^{2,2}(\omega)$ and since the embedding $W^{2,2}(\omega) \hookrightarrow C^{0,1}(\omega)$ fails in 2D, one needs (6.1) to obtain weak-strong uniqueness. The aforementioned embedding is however true for our 1D shell, so (6.1) is not needed.
- The final reason for needing (6.1) has to do with Sobolev multipliers where one needs Lipschitz continuous shell displacements to get estimates of the form [19, (2.5)].

None of these issues arise for our 1D shell due to the validity of $W^{2,2}(\omega) \hookrightarrow C^{0,1}(\omega)$ in 1D with small Lipschitz constant.

The second requirement in (6.2), however, is crucial in the estimate of the convective term $(\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u}$ when one wants to improve weak solutions to strong ones in 3D. See the estimate for Roman number I [19, (4.8)] in the acceleration estimate. The corresponding 2D acceleration estimate, as shown in [14], does not require any additional condition such as (6.2). Also, even though (6.2) is used in constructing strong solution in [19], it plays no explicit role in the actual weak-strong uniqueness result. Consequently, we also do not need (6.2) in our current setting.

From the explanation above, it is clear that any potential conditional weak-strong uniqueness result can only depend on ρ or $\mathbb{T}$. A quick inspection of the contraction argument shown in the preceding section, however, shows that we do not need any extra assumption on ρ nor $\mathbb{T}$ to obtain our desired result. Indeed, if we let $(\eta_w, \mathbf{u}_w, \rho_w, \mathbb{T}_w)$ be a weak solution of (1.1)–(1.10) with dataset $(\mathbf{f}_w, g_w, \rho_{0,w}, \mathbb{T}_{0,w}, \mathbf{u}_{0,w}, \eta_{0,w}, \eta_{\star,w})$ and $(\eta_s, \mathbf{u}_s, \rho_s, \mathbb{T}_s)$ be a strong solution of (1.1)–(1.10) with dataset $(\mathbf{f}_s, g_s, \rho_{0,s}, \mathbb{T}_{0,s}, \mathbf{u}_{0,s}, \eta_{0,s}, \eta_{\star,s})$, then in particular,

$$\begin{aligned}\rho_w &\in L^\infty(I;L^2(\Omega_\eta)) \cap L^2(I;W^{1,2}(\Omega_\eta)),\\ \mathbb{T}_w &\in L^\infty(I;L^2(\Omega_\eta)) \cap L^2(I;W^{1,2}(\Omega_\eta)).\end{aligned}$$

This regularity (which should be compared with ρ_1 and $\mathbb{T}_1$ in the contraction argument earlier) is enough to make sense of all the terms in (5.16) and (5.26) without change. More importantly, the right-hand side of (5.16) (and of (5.26)) does not contain the individual weak solution terms ρ_w and $\mathbb{T}_w$.

With the explanation above in hand, we can now proceed to prove Theorem 2.6. For this, we set

$$\begin{aligned}&\rho_{ws} := \rho_w - \overline{\rho}_s, && \mathbb{T}_{ws} := \mathbb{T}_w - \overline{\mathbb{T}}_s, && \mathbf{u}_{ws} = \mathbf{u}_w - \overline{\mathbf{u}}_s,\\ &\eta_{ws} = \eta_w - \eta_s, && \mathbf{f}_{ws} := \mathbf{f}_w - \overline{\mathbf{f}}_s, && g_{ws} = g_w - g_s,\end{aligned}$$

and define the following:

$$\begin{aligned}\mathbb{B}_{-\eta_{ws}} &:= J_{-\eta_{ws}} \nabla_{\mathbf{x}} \mathbf{\Psi}_{-\eta_{ws}}^{-1} \circ \mathbf{\Psi}_{-\eta_{ws}},\\
\mathbb{A}_{-\eta_{ws}} &:= (\nabla_{\mathbf{x}} \mathbf{\Psi}_{-\eta_{ws}}^{-1} \circ \mathbf{\Psi}_{-\eta_{ws}})^\top \mathbb{B}_{-\eta_{ws}},\\
\mathbf{h}_{-\eta_{ws}}(\overline{\mathbf{u}}_s) &:= (1 - J_{-\eta_{ws}})\partial_t \overline{\mathbf{u}}_s - J_{-\eta_{ws}} \nabla_{\mathbf{x}} \overline{\mathbf{u}}_s \cdot \partial_t \mathbf{\Psi}_{-\eta_{ws}}^{-1} \circ \mathbf{\Psi}_{-\eta_{ws}}\\
&\quad + \overline{\mathbf{u}}_s \cdot \nabla_{\mathbf{x}} \overline{\mathbf{u}}_s (\mathbb{I} - \mathbb{B}_{-\eta_{ws}}),\\
h_{-\eta_{ws}}(\overline{\rho}_s, \overline{\mathbf{u}}_s) &:= (1 - J_{-\eta_{ws}})\partial_t \overline{\rho}_s - J_{-\eta_{ws}} \nabla_{\mathbf{x}} \overline{\rho}_s \cdot \partial_t \mathbf{\Psi}_{-\eta_{ws}}^{-1} \circ \mathbf{\Psi}_{-\eta_{ws}}\\
&\quad + \overline{\mathbf{u}}_s \cdot \nabla_{\mathbf{x}} \overline{\rho}_s (\mathbb{I} - \mathbb{B}_{-\eta_{ws}}),\\
\mathbb{H}_{-\eta_{ws}}(\overline{\rho}_s, \overline{\mathbf{u}}_s, \overline{\mathbb{T}}_s) &:= (1 - J_{-\eta_{ws}})\partial_t \overline{\mathbb{T}}_s - J_{-\eta_{ws}} \nabla_{\mathbf{x}} \overline{\mathbb{T}}_s \cdot \partial_t \mathbf{\Psi}_{-\eta_{ws}}^{-1} \circ \mathbf{\Psi}_{-\eta_{ws}}\\
&\quad + \nabla_{\mathbf{x}} \overline{\mathbf{u}}_s (\mathbb{B}_{-\eta_{ws}} - \mathbb{I}) \overline{\mathbb{T}}_s + \overline{\mathbb{T}}_s (\mathbb{B}_{-\eta_{ws}} - \mathbb{I})^\top (\nabla_{\mathbf{x}} \overline{\mathbf{u}}_s)^\top\\
&\quad + \overline{\mathbf{u}}_s \cdot \nabla_{\mathbf{x}} \overline{\mathbb{T}}_s (\mathbb{I} - \mathbb{B}_{-\eta_{ws}}) + 2(1 - J_{-\eta_{ws}})(\overline{\mathbb{T}}_s - \overline{\rho}_s \mathbb{I}).\end{aligned}$$

As in Section 5.3 (compare with [45, Section 4]), the difference equation satisfies

$$\begin{aligned}&\frac{1}{2}\big(\|\rho_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\mathbb{T}_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\mathbf{u}_{ws}(t)\|^2_{L^2(\Omega_{\eta_w(t)})} + \|\partial_t \eta_{ws}(t)\|^2_{L^2(\omega)}\big)\\
&\quad + \frac{1}{2}\|\partial_y^2 \eta_{ws}(t)\|^2_{L^2(\omega)} + \int_0^t \big(\|\nabla_{\mathbf{x}} \rho_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\nabla_{\mathbf{x}} \mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta_w})}\big)\,\mathrm{d}t'\\
&\quad + \int_0^t \big(\|\nabla_{\mathbf{x}} \mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\partial_t \partial_y \eta_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t'\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\big(\|\rho_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})} + \|\mathbb{T}_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})} + \|\mathbf{u}_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})}\big) \\
&\qquad + \frac{1}{2}\big(\|\partial_t \eta_{ws}(0)\|^2_{L^2(\omega)} + \|\partial_y^2 \eta_{ws}(0)\|^2_{L^2(\omega)}\big) + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4
\end{aligned}
\tag{6.3}
$$

for any $t \in I$ where

$$
\begin{aligned}
\mathfrak{R}_1 :=& \int_0^t \int_{\Omega_{\eta_w}} (1 - J_{-\eta_{ws}})\mathrm{div}_{\mathbf{x}}\overline{\mathbb{T}}_s \cdot (\mathbf{u}_{ws} + \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \mathrm{div}_{\mathbf{x}}\mathbb{T}_{ws} \cdot (\mathbf{u}_{ws} + \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t', \\
\mathfrak{R}_2 :=& \int_0^t \int_{\partial\Omega_{\eta_w}} (\mathbf{n}_{\eta_w} \cdot \nabla_{\mathbf{x}})\rho_{ws}\rho_{ws}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t' - \int_0^t \int_{\Omega_{\eta_w}} (\mathbf{u}_{ws} \cdot \nabla_{\mathbf{x}})\overline{\rho}_s\,\rho_{ws}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \big[\mathrm{div}_{\mathbf{x}}((\mathbb{I} - \mathbb{A}_{-\eta_{ws}})\nabla_{\mathbf{x}}\overline{\rho}_s) - h_{-\eta_{ws}}(\overline{\rho}_s, \overline{\mathbf{u}}_s)\big]\rho_{ws}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t', \\
\mathfrak{R}_3 :=& \int_0^t \big(\|\mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta_1})} + \|\rho_{ws}\|^2_{L^2(\Omega_{\eta_1})}\big)\,\mathrm{d}t' \\
&+ \int_0^t \int_{\partial\Omega_{\eta_w}} (\mathbf{n}_{\eta_w} \cdot \nabla_{\mathbf{x}})\mathbb{T}_{ws} : \mathbb{T}_{ws}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t' + 2\int_0^t \int_{\Omega_{\eta_w}} |\nabla_{\mathbf{x}}\mathbf{u}_w||\mathbb{T}_{ws}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \big[\nabla_{\mathbf{x}}\mathbf{u}_{ws}\overline{\mathbb{T}}_s + \overline{\mathbb{T}}_s(\nabla_{\mathbf{x}}\mathbf{u}_{ws})^\top - \mathbf{u}_{ws} \cdot \nabla_{\mathbf{x}}\overline{\mathbb{T}}_s\big] : \mathbb{T}_{ws}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \big[\mathrm{div}_{\mathbf{x}}((\mathbb{I} - \mathbb{A}_{-\eta_{ws}})\nabla_{\mathbf{x}}\overline{\mathbb{T}}_s) - \mathbb{H}_{-\eta_{ws}}(\overline{\rho}_s, \overline{\mathbf{u}}_s, \overline{\mathbb{T}}_s)\big] : \mathbb{T}_{ws}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{R}_4 =& \int_0^t \int_{\Omega_{\eta_w}} \big(\overline{\mathbf{u}}_s \cdot \nabla_{\mathbf{x}}\overline{\mathbf{u}}_s - \mathbf{h}_{-\eta_{ws}}(\overline{\mathbf{u}}_s)\big) \cdot \big(\mathbf{u}_{ws} + \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \frac{1}{2}\int_0^t \int_{\partial\Omega_{\eta_w}} \mathbf{n} \circ \boldsymbol{\varphi}_{\eta_w}^{-1} \cdot (\partial_t \eta_w \mathbf{n}_{\eta_w}) \circ \boldsymbol{\varphi}_{\eta_w}^{-1} |\overline{\mathbf{u}}_s|^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t' \\
&- \int_0^t \int_{\partial\Omega_{\eta_w}} \mathbf{n} \circ \boldsymbol{\varphi}_{\eta_w}^{-1} \cdot (\partial_t \eta_s \mathbf{n}_{\eta_w}) \circ \boldsymbol{\varphi}_{\eta_w}^{-1} |\mathbf{u}_w|^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \mathbf{u}_{ws} \cdot \partial_t \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' - \int_{\Omega_{\eta_w}} \mathbf{u}_{ws} \cdot \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\,\mathrm{d}\mathbf{x} \\
&+ \int_{\Omega_{\eta_w(0)}} \mathbf{u}_{ws}(0) \cdot \mathrm{Bog}_{\eta_w(0)}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s(0))\,\mathrm{d}\mathbf{x} \\
&- \int_0^t \int_{\Omega_{\eta_w}} \nabla_{\mathbf{x}}\mathbf{u}_{ws} : \nabla_{\mathbf{x}}\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+ \int_0^t \int_{\Omega_{\eta_w}} \big(\mathbf{A}_{-\eta_{ws}} - \mathbb{I}\big)\nabla_{\mathbf{x}}\overline{\mathbf{u}}_s : \nabla_{\mathbf{x}}(\mathbf{u}_{ws} + \mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'
\end{aligned}
$$

$$\begin{aligned}
&+\int_0^t\int_{\Omega_{\eta w}} \big(\mathbb{I}-\mathbf{B}_{-\eta_{ws}}\big)\overline{p}_s : \nabla_{\mathbf{x}}(\mathbf{u}_{ws}+\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+\int_0^t\int_{\omega} g_{ws}\,\partial_t\eta_{ws}\,\mathrm{d}y\,\mathrm{d}t' + \int_0^t\int_{\Omega_{\eta w}} \mathbf{f}_{ws}\cdot(\mathbf{u}_{ws}+\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+\int_0^t\int_{\Omega_{\eta w}} (1-J_{-\eta_{ws}})\overline{\mathbf{f}}_s\cdot(\mathbf{u}_{ws}+\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+\int_0^t\int_{\Omega_{\eta w}} \mathbf{u}_w\otimes\mathbf{u}_w : \nabla_{\mathbf{x}}\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+\int_0^t\int_{\Omega_{\eta w}} \mathbf{u}_w\cdot\nabla_{\mathbf{x}}\mathbf{u}_w\cdot\overline{\mathbf{u}}_s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' \\
&+\int_0^t\int_{\Omega_{\eta w}} \mathbf{f}_{ws}\cdot\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'.
\end{aligned}$$

To estimate $\mathfrak{R}_1$, we use the estimate

$$\|\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_s)\|^2_{W^{k,2}(\Omega_{\eta w})} = \|\mathrm{Bog}_{\eta_w}(\mathrm{div}_{\mathbf{x}}\overline{\mathbf{u}}_{ws})\|^2_{W^{k,2}(\Omega_{\eta w})} \lesssim \|\overline{\mathbf{u}}_{ws}\|^2_{W^{k,2}(\Omega_{\eta w})}$$

for $k\geq 0$ to obtain

$$\begin{aligned}
\mathfrak{R}_1 \leq \delta\int_0^t \|\nabla_{\mathbf{x}}\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta w})}\,\mathrm{d}t' + c(\delta)\int_0^t \|\nabla_{\mathbf{x}}\overline{\mathbb{T}}_s\|^2_{L^2(\Omega_{\eta w})}\|\partial_y^2\eta_{ws}\|^2_{L^2(\omega)}\,\mathrm{d}t' \\
+\delta\int_0^t \|\nabla_{\mathbf{x}}\mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta w})}\,\mathrm{d}t' + c(\delta)\int_0^t \|\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta w})}\,\mathrm{d}t'
\end{aligned}$$

for any $\delta>0$. Also, just as in (5.16) and (5.26),

$$\begin{aligned}
\mathfrak{R}_2+\mathfrak{R}_3 \leq \delta\int_0^t \Big(\|\nabla_{\mathbf{x}}\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta w})} + \|\nabla_{\mathbf{x}}\rho_{ws}\|^2_{L^2(\Omega_{\eta w})} + \|\nabla_{\mathbf{x}}\mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta w})}\Big)\,\mathrm{d}t' \\
+c(\delta)\int_0^t \Big(1+\|\nabla_{\mathbf{x}}\mathbf{u}_w\|^2_{L^2(\Omega_{\eta w})} + \|\overline{\mathbf{u}}_s\|^2_{W^{2,2}(\Omega_{\eta w})} \\
+\|\overline{\rho}_s\|^2_{W^{2,2}(\Omega_{\eta w})} + \|\overline{\mathbb{T}}_s\|^2_{W^{2,2}(\Omega_{\eta w})}\Big) \\
\times\Big(\|\rho_{ws}\|^2_{L^2(\Omega_{\eta w})} + \|\mathbb{T}_{ws}\|^2_{L^2(\Omega_{\eta w})}\Big)\,\mathrm{d}t'
\end{aligned}$$

$$\begin{aligned}
&+ c(\delta)\int_0^t \big(\|\overline{\rho}_s\|^2_{W^{2,2}(\Omega_{\eta_w})} + \|\overline{\mathbb{T}}_s\|^2_{W^{2,2}(\Omega_{\eta_w})} \\
&\qquad + \|\partial_t\overline{\rho}_s\|^2_{L^2(\Omega_{\eta_w})} + \|\partial_t\overline{\mathbb{T}}_s\|^2_{L^2(\Omega_{\eta_w})}\big) \\
&\qquad\qquad \times \big(\|\partial_y^2\eta_{ws}\|^2_{L^2(\omega)} + \|\partial_t\eta_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t'.
\end{aligned}$$

Finally, as shown in [19, (5.6)] (see also [19, Remark 5.2.]),

$$\begin{aligned}
\mathfrak{R}_4 \le{}& \delta\big(\|\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\partial_y^2\eta_{ws}\|^2_{L^2(\omega)}\big) \\
&+ \delta\int_0^t \big(\|\nabla_{\mathbf{x}}\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|\partial_t\partial_y\eta_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t' \\
&+ c(\delta)\int_0^t \big(1 + \|\overline{\mathbf{u}}_s\|^2_{W^{2,2}(\Omega_{\eta_w})} + \|\overline{\rho}_s\|^2_{W^{1,2}(\Omega_{\eta_w})}\big)\|\partial_y^2\eta_{ws}\|^2_{L^2(\omega)}\,\mathrm{d}t' \\
&+ c(\delta)\int_0^t \big(\|\partial_t\overline{\mathbf{u}}_s\|^2_{L^2(\Omega_{\eta_w})} + \|\overline{\mathbf{f}}_s\|^2_{L^2(\Omega_{\eta_w})}\big)\|\partial_y^2\eta_{ws}\|^2_{L^2(\omega)}\,\mathrm{d}t' \\
&+ c(\delta)\int_0^t \big(1 + \|\overline{\mathbf{u}}_s\|^2_{W^{2,2}(\Omega_{\eta_w})} + \|\partial_t\eta_s\|^2_{W^{2,2}(\omega)}\big)\|\mathbf{u}_{ws}\|^2_{L^2(\Omega_{\eta_w})}\,\mathrm{d}t' \\
&+ c(\delta)\int_0^t \big(1 + \|\overline{\mathbf{u}}_s\|^2_{W^{2,2}(\Omega_{\eta_w})} + \|\partial_t\eta_s\|^2_{W^{2,2}(\omega)}\big)\|\partial_t\eta_{ws}\|^2_{L^2(\omega)}\,\mathrm{d}t' \\
&+ c(\delta)\|\mathbf{u}_{ws}(0)\|^2_{L^2(\Omega_{\eta_w(0)})} + c(\delta)\int_0^t \big(\|\mathbf{f}_{ws}\|^2_{L^2(\Omega_{\eta_w})} + \|g_{ws}\|^2_{L^2(\omega)}\big)\,\mathrm{d}t'.
\end{aligned}$$

Substituting the estimates for the $\mathfrak{R}_i$s back into (6.3) and taking the supremum with respect to time, we obtain Theorem 2.6 by applying Grönwall's lemma. The proof is done!

7. Conclusion

In conclusion, we have presented in this work the Oldroyd-B dumbbell model describing the evolution of a two-dimensional dilute polymer fluid interacting with a one-dimensional viscoelastic shell. We have shown that if no degeneracies occur while the structure deforms, a weak and strong solution exist and the strong solution is unique. This result now opens the door to the study of further properties for this system including the qualitative and quantitative properties of their solutions.

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