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A convergent augmented SAV scheme for stochastic Cahn–Hilliard equations with dynamic boundary conditions describing contact line tension

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Abstract. We augment a thermodynamically consistent diffuse interface model for the description of line tension phenomena by multiplicative stochastic noise to capture the effects of thermal fluctuations and establish the existence of pathwise-unique (stochastically) strong solutions. By starting from a fully discrete linear finite element scheme, we do not only prove the well-posedness of the model, but also provide a practicable and convergent scheme for its numerical treatment. Conceptually, our discrete scheme relies on a recently developed augmentation of the scalar auxiliary variable approach, which reduces the requirements on the time regularity of the solution. By showing that fully discrete solutions to this scheme satisfy an energy estimate, we obtain first uniform regularity results. Establishing Nikolskii estimates with respect to time, we are able to show convergence toward pathwise-unique martingale solutions by applying Jakubowski's generalization of Skorokhod's theorem. Finally, a generalization of the Gyöngy–Krylov characterization of convergence in probability provides convergence toward strong solutions and thereby completes the proof.

1. Introduction

The description of the evolution of two immiscible fluids in a confined domain $\mathcal{O}$ has been a significant research topic throughout the last centuries. Thereby, the evolution of the three-phase contact line between the two fluids and the solid wall was of particular interest. The investigation of mathematical formulas to predict the contact angles of a droplet wetting a solid substrate dates back to the beginning of the 19th century, when Young [66] proposed the following famous formula for the equilibrium contact angle Θ:

$$\widetilde{\sigma} \cos \Theta = \gamma_{fs,1} - \gamma_{fs,2}. \tag{1.1}$$

Here, $\gamma_{fs,2}$ denotes the contact energy density between the wetting fluid and the solid substrate, $\gamma_{fs,1}$ denotes the contact energy density between the surrounding fluid (e.g., air or vapor) and the substrate, and $\widetilde{\sigma}$ describes the interfacial tension between the two fluids.

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Relation (1.1) can be derived by minimizing the energy

$$\mathcal{E}_1 := \int_{I_f} \widetilde{\sigma} \, \mathrm{d}\Gamma + \int_A (\gamma_{fs,2} - \gamma_{fs,1}) \, \mathrm{d}\Gamma,$$

where I_f denotes the fluid-fluid interface and $A \subset \partial\mathcal{O}$ denotes the surface wetted by the droplet. It was, however, already noted by J. W. Gibbs toward the end of the 19th century that contact line effects should be included (cf. [22, Chapter III]), yet Gibbs did not provide any mathematical formulation. Since then, the influence of contact lines on the contact angle was investigated by many authors (see, e.g., [2, 6, 36, 61–63, 67], and the references therein). In order to include contact line effects in the description, the energy $\mathcal{E}_1$ has to be augmented by an additional contact line integral, that is,

$$\mathcal{E}_2 := \int_{I_f} \widetilde{\sigma} \, \mathrm{d}\Gamma + \int_A (\gamma_{fs,2} - \gamma_{fs,1}) \, \mathrm{d}\Gamma + \int_{I_f \cap \partial\mathcal{O}} \widetilde{\kappa} \, \mathrm{d}s.$$

Here, $I_f \cap \partial\mathcal{O}$ is the three-phase contact line on the boundary $\partial\mathcal{O}$ of the fluid domain and $\widetilde{\kappa}$ denotes the line tension. Although the line tension can be negative, we will restrict ourselves to the case $\widetilde{\kappa} > 0$. Assuming that the droplet is spherical (or a spherical cap when attached to the substrate) and minimizing $\mathcal{E}_2$ leads to the following formula for the stationary contact line (cf. [63]):

$$\widetilde{\sigma} \cos\Theta = (\gamma_{fs,1} - \gamma_{fs,2}) + \frac{\widetilde{\kappa}}{r}. \tag{1.2}$$

Formula (1.2) indicates that Young's original description (see (1.1)) predicts the stationary contact angle sufficiently well, if the radius r of the circular contact area A is sufficiently large. For smaller droplets with smaller contact areas, however, this effect is able to cause significant deviations from the results predicted by Young's formula (see, e.g., [63]). As it was noted in [61] these descriptions should also take thermally excited fluctuations into account, as they are able to change the contact line contour and thereby modify the contact angle.

In this publication, we analyze a diffuse interface model with dynamic boundary conditions describing contact line tension effects including thermal fluctuations. The basic idea of a diffuse interface model is to replace surface (or line) Dirac functions of I_f (or $I_f \cap \partial\mathcal{O}$, respectively) by smooth approximations of the form $\frac{1}{2}\varepsilon|\nabla\phi|^2 + \varepsilon^{-1}F(\phi)$, where ϕ is the phase-field parameter describing the two fluid phases, F is a double-well potential with minima in (or close to) the values ± 1 indicating the pure phases, and ε is a small parameter related to the width of the diffuse interface. Typical choices for this double-well potential are the logarithmic double-well potential

$$W_{\log}(\phi) := \frac{\vartheta}{2}(1+\phi)\log(1+\phi) + \frac{\vartheta}{2}(1-\phi)\log(1-\phi) - \frac{\vartheta_c}{2}\phi^2$$

with $0 < \vartheta < \vartheta_c$, the double obstacle potential

$$W_{\mathrm{obst}}(\phi) := \begin{cases} \vartheta(1-\phi^2) & \text{if } \phi \in [-1,+1], \\ \infty & \text{else} \end{cases}$$

with $\vartheta > 0$, and the polynomial double-well potential $W_{\text{pol}}(\phi) := \frac{1}{4}(\phi^2 - 1)^2$. The logarithmic and double obstacle potentials are of great analytical interest, as they allow us to confine the phase-field parameter to the physically meaningful interval $[-1, +1]$. The numerical treatment of these potentials, however, is rather intricate (cf. [4, 5, 7, 8, 16, 21]) and most numerical schemes are based on the polynomial double-well potential W_{pol}. Assuming $F(\phi) \equiv W_{\text{pol}}(\phi)$, we approximate $\mathcal{E}_2$ by

$$\widetilde{\mathcal{E}}(\phi) := \int_{\mathcal{O}} \sigma\Big(\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\Big)\,\mathrm{d}x + \int_{\partial\mathcal{O}} \gamma_{fs}(\phi)\,\mathrm{d}\Gamma + \int_{\partial\mathcal{O}} \kappa\Big(\frac{\varepsilon}{2}|\nabla_\Gamma\phi|^2 + \frac{1}{\varepsilon}F(\phi)\Big)\,\mathrm{d}\Gamma,$$

with γ_{fs} interpolating between $\gamma_{fs,1}$ and $\gamma_{fs,2}$, σ and κ being rescaled versions of $\widetilde{\sigma}$ and $\widetilde{\kappa}$, and ∇_Γ denoting the surface gradient. A deterministic evolution of the phase-field that minimizes $\widetilde{\mathcal{E}}$ and conserves $\int_{\mathcal{O}} \phi\,\mathrm{d}x$ can be described by

$$\partial_t\phi = \Delta\mu \quad \text{in } \mathcal{O}, \tag{1.3a}$$
$$\mu = -\sigma\varepsilon\Delta\phi + \sigma\varepsilon^{-1}F'(\phi) \quad \text{in } \mathcal{O}, \tag{1.3b}$$
$$\partial_t\phi = -\gamma_{fs}'(\phi) - \kappa(-\varepsilon\Delta_\Gamma\phi + \varepsilon^{-1}F'(\phi)) - \sigma\varepsilon\nabla\phi\cdot\boldsymbol{n} \quad \text{on } \partial\mathcal{O}, \tag{1.3c}$$
$$\nabla\mu\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathcal{O}, \tag{1.3d}$$

that is, by a Cahn–Hilliard equation with Allen–Cahn-type dynamic boundary conditions. For a more rigorous derivation of system (1.3) and a discussion of the involved parameters, we refer the reader to [67]. Similar equations were derived in [50], where system (1.3) with $\kappa = 0$ was coupled with suitable Navier–Stokes equations to describe moving contact lines. Such Allen–Cahn-type boundary conditions have been extensively studied. A by no means exhausting list of contributions includes, for example, [12–14, 19, 20, 23, 32, 45, 46, 51, 64]. As in this publication we are not interested in the sharp interface limit $\varepsilon \searrow 0$, we will simplify the representation of system (1.3) by setting $\varepsilon = \sigma = \kappa = 1$ and introduce $G(\phi) := F(\phi) + \gamma_{fs}(\phi)$.

To include thermal fluctuations in the model, we consider a $\mathcal{Q}$-Wiener process $W = (W_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$. The exact assumptions on the $\mathcal{Q}$-Wiener process are listed in Assumptions **(W1)** and **(W2)** in Section 2. They in particular imply that its trace $([W_t]|_\Gamma)_{t\in[0,T]}$ on $\Gamma := \partial\Omega$ is well defined. With this $\mathcal{Q}$-Wiener process we augment system (1.3) by multiplicative Itô noise:

$$\mathrm{d}\phi - \Delta\mu\,\mathrm{d}t = \Phi(\phi)\mathrm{d}W \quad \text{in } \mathcal{O}, \tag{1.4a}$$
$$\mu = -\Delta\phi + F'(\phi) \quad \text{in } \mathcal{O}, \tag{1.4b}$$
$$\mathrm{d}\,[\phi]|_\Gamma + (-\Delta_\Gamma[\phi]|_\Gamma + G'([\phi]|_\Gamma) + \nabla\phi\cdot\boldsymbol{n})\,\mathrm{d}t = [\Phi(\phi)\mathrm{d}W]|_\Gamma \quad \text{on } \partial\mathcal{O}, \tag{1.4c}$$
$$\nabla\mu\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathcal{O} \tag{1.4d}$$

with $[\cdot]|_\Gamma$ denoting the trace operator, that is, we model the noise on the boundary $\Gamma = \partial\mathcal{O}$ as the trace of the noise in the bulk. The operator Φ maps the stochastic process ϕ into the space of Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}L^2(\mathcal{O})$. For a detailed definition, we refer the reader to Section 2 below.

To show the existence of solutions to system (1.4), we will start from a fully discrete finite element scheme. Hence, we will not only prove the well-posedness of system (1.4) but also establish convergence for our numerical scheme. The considered scheme relies on an augmented version of the scalar auxiliary variable (SAV) approach. Originally, the SAV approach was introduced in [55] for deterministic PDEs describing gradient flows. This approach has been applied to various deterministic problems (see, e.g., [56, 57] and the references therein) and many variations of this approach have been developed and tested (see, e.g., [28, 29, 35, 38, 65, 68]). As this approach provides linear schemes, it allows for a significant reduction in computation time. Hence, an application to stochastic PDEs (SPDEs), where often multiple different paths need to be simulated, is tempting. Yet, a straightforward application of the standard SAV scheme to SPDEs is not always crowned with success. Although there are positive results for the stochastic wave equation (see, e.g., [17]), for most SPDEs the poor time regularity of the solutions impedes convergence results. As shown in [44] this is not only an artificial analytical problem that jeopardizes rigorous convergence proofs; it can also lead to wrong results in practical simulations. To overcome these difficulties, the author proposed an augmented version of the SAV scheme in [44], which extends the applicability of the SAV schemes to SPDEs with less-regular solutions.

The outline of this paper is as follows: In Section 2, we will introduce the relevant function spaces and interpolation operators. We will also collect our assumptions on the data and important auxiliary results. In Section 3, we present the numerical scheme. The main convergence results of this publication can be found in Section 4: Theorem 4.2 provides convergence toward pathwise-unique martingale solutions and Theorem 4.3 provides for a given $\mathcal{Q}$-Wiener process the convergence of discrete solutions toward (stochastically) strong solutions. The results are proven in the remaining sections: The existence of solutions to the numerical scheme is established in Section 5. Uniform regularity results for these solutions are established in Section 6. In Section 7, we apply Jakubowski's theorem to identify weakly and strongly converging subsequences based on the prior established regularity results. In Section 8, we pass to the limit $(h, \tau) \searrow 0$ and establish the convergence toward (and existence of) martingale solutions. Exploiting monotonicity arguments in Section 9, we show that these martingale solutions are pathwise unique, which concludes the proof of Theorem 4.2. The proof of Theorem 4.3 can be found in Section 10, where we apply a generalized version of the Gyöngy–Krylov characterization of convergence in probability to show that for a given $\mathcal{Q}$-Wiener process our discrete solutions converge toward strong solutions to system (1.4).

2. Notation and assumptions

The spatial domain $\mathcal{O} \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ is assumed to be bounded and convex with Lipschitzian boundary $\Gamma := \partial\mathcal{O}$. To avoid additional technicalities, we shall assume that $\mathcal{O}$ is polygonal (or polyhedral, respectively). We denote the space of k-times weakly differentiable functions with weak derivatives in $L^p(\mathcal{O})$ by $W^{k,p}(\mathcal{O})$. For $p = 2$, these spaces are Hilbert spaces, which we shall denote by $H^k(\mathcal{O}) := W^{k,2}(\mathcal{O})$. The dual space of $H^1(\mathcal{O})$ will be denoted by $(H^1(\mathcal{O}))'$. The symbol $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $(H^1(\mathcal{O}))'$ and $H^1(\mathcal{O})$, which satisfies $\langle u, v \rangle = (u, v)_{L^2(\mathcal{O})}$ for $u \in L^2(\mathcal{O})$ and $v \in H^1(\mathcal{O})$. For $\varphi \in (H^1(\mathcal{O}))'$ we define a generalized mean value by

$$(\varphi)_{\mathcal{O}} := |\mathcal{O}|^{-1} \langle \varphi, 1 \rangle.$$

We shall use a similar notation for the function spaces on Γ. As we always assume that the domain $\mathcal{O}$ has a Lipschitzian boundary, the spaces $L^p(\Gamma)$ and $W^{1,p}(\Gamma)$ are well defined for all $p \geq 1$ (cf. [34]) and the trace operator $[\cdot]|_\Gamma$ is uniquely defined as an element of $\mathcal{L}(W^{1,p}(\mathcal{O}); W^{1-1/p,p}(\Gamma))$ (cf. [47]).

We also define the space

$$\mathcal{H}^1 := H^1(\mathcal{O}) \times H^1(\Gamma),$$

which is a Hilbert space with respect to the inner product

$$((\phi, \psi), (\zeta, \xi))_{\mathcal{H}^1} := (\phi, \zeta)_{H^1(\mathcal{O})} + (\psi, \xi)_{H^1(\Gamma)}$$

for all $(\phi, \psi), (\zeta, \xi) \in \mathcal{H}^1$. Furthermore, we introduce the following subspace of $H^1(\mathcal{O})$:

$$\mathcal{V} := \left\{\varphi \in H^1(\mathcal{O}) : (\varphi, [\varphi]|_\Gamma) \in \mathcal{H}^1\right\}.$$

For a Banach space X and a set I, the symbol $L^p(I; X)$ ($p \in (1, \infty)$) stands for the Bochner space of strongly measurable L^p-integrable functions on I with values in X. If X is only separable (and not reflexive), we follow the notation used in [18, Chapter 0.3] and denote the dual space of $L^{p/(p-1)}(I; X)$ ($p \in (1, \infty)$) by $L^p_{\text{weak-}(*)}(I; X')$. A space X endowed with the weak topology is denoted by X_{weak}.

By $C^{k,\alpha}(I; X)$ with $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$, we denote the space of k-times continuously differentiable functions from I to X whose k-th derivatives are Hölder continuous with Hölder exponent α. If $I = X = \mathbb{R}$, we shall write $C^{k,\alpha}(\mathbb{R})$. We shall also introduce the Nikolskii spaces $N^{\alpha,\beta}$ ($\alpha \in (0, 1)$, $\beta \in [1, \infty]$), which are defined for a time interval $(0, T)$ and a Banach space X via

$$N^{\alpha,\beta}(0, T; X) := \left\{f \in L^\beta(0, T; X) : \sup_{k>0} k^{-\alpha} \|f(\cdot + k) - f(\cdot)\|_{L^\beta(-k,T-k;X)} < \infty\right\}.$$

Here, the standard convention $f \equiv 0$ outside of $(0, T)$ for $f \in L^\beta(0, T; X)$ is used.

Concerning the discretization with respect to time, we shall assume that

(T) the time interval $I := [0, T]$ is subdivided into N equidistant intervals I_n given by nodes $(t^n)_{n=0,\ldots,N}$ with $t^0 = 0$, $t^N = T$, and $t^n - t^{n-1} = \tau =: \frac{T}{N}$. Without loss of generality, we assume $\tau < 1$.

Concerning the spacial domain, we consider a bounded and convex domain $\mathcal{O} \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) with boundary $\Gamma = \partial\mathcal{O}$. Furthermore, we shall assume that $\mathcal{O}$ is polygonal (or polyhedral, respectively) to avoid additional technicalities. The spatial discretization is based on partitions $\mathcal{T}_h$ of $\mathcal{O}$ depending on a discretization parameter $h > 0$ satisfying the following assumption:

(S1) Let $\{\mathcal{T}_h\}_{h>0}$ be a quasiuniform family (in the sense of [10]) of partitions of $\mathcal{O}$ into disjoint, open simplices K, satisfying

$$\overline{\mathcal{O}} \equiv \bigcup_{K \in \mathcal{T}_h} \overline{K} \quad \text{with} \quad \max_{K \in \mathcal{T}_h} \operatorname{diam}(K) \leq h.$$

For the discretization of Γ, we follow the ideas employed in [33,41,43] and use a partition consisting of the edges (or faces, respectively) of elements of $\mathcal{T}_h$. In particular, we shall consider partitions $\mathcal{T}_h^\Gamma$ of Γ satisfying the following assumption:

(S2) Let $\{\mathcal{T}_h^\Gamma\}_{h>0}$ be a quasiuniform family of partitions of Γ into disjoint open simplices K^Γ, satisfying

$$\forall K^\Gamma \in \mathcal{T}_h^\Gamma, \quad \exists! K \in \mathcal{T}_h \text{ such that } \overline{K^\Gamma} = \overline{K} \cap \Gamma,$$

and

$$\Gamma \equiv \bigcup_{K^\Gamma \in \mathcal{T}_h^\Gamma} \overline{K^\Gamma} \quad \text{with} \quad \max_{K^\Gamma \in \mathcal{T}_h^\Gamma} \operatorname{diam}(K^\Gamma) \leq h.$$

For the approximation of the phase-field ϕ and the chemical potential μ, we use continuous, piecewise linear finite element functions on $\mathcal{T}_h$. This space will be denoted by $U_h^\mathcal{O}$ and is spanned by functions $\{\chi_{h,k}\}_{k=1,\ldots,\dim U_h^\mathcal{O}}$ forming a dual basis to the vertices $\{\boldsymbol{x}_{h,k}\}_{k=1,\ldots,\dim U_h^\mathcal{O}}$ of $\mathcal{T}_h$. For the discretization of $[\phi]|_\Gamma$ and θ, we use continuous, piecewise linear finite element functions on $\mathcal{T}_h^\Gamma$, which shall be denoted by U_h^Γ. Due to the assumptions on $\mathcal{T}_h$ and $\mathcal{T}_h^\Gamma$, this space is given by traces of functions in $U_h^\mathcal{O}$, that is,

$$U_h^\Gamma := \operatorname{span}\left\{[\psi_h]|_\Gamma : \psi_h \in U_h^\mathcal{O}\right\}. \tag{2.1}$$

This space can also be described as the span of functions $\{\chi_{h,k}^\Gamma\}_{k=1,\ldots,\dim U_h^\Gamma}$, which form a dual basis to the vertices

$$\left\{\boldsymbol{x}_{h,k}^\Gamma\right\}_{k=1,\ldots,\dim U_h^\Gamma} \subset \left\{\boldsymbol{x}_{h,j}\right\}_{j=1,\ldots,\dim U_h^\mathcal{O}}$$

of $\mathcal{T}_h^\Gamma$. For a different discretization approach that relaxes the connection between $U_h^\mathcal{O}$ and U_h^Γ by introducing Lagrange multipliers, we refer to [1] and the references therein.

We define the nodal interpolation operators $\mathcal{I}_h : C^0(\overline{\mathcal{O}}) \to U_h^{\mathcal{O}}$ and $\mathcal{I}_h^\Gamma : C^0(\Gamma) \to U_h^\Gamma$ by

$$\mathcal{I}_h\{a\} := \sum_{k=1}^{\dim U_h^{\mathcal{O}}} a(\boldsymbol{x}_{h,k})\chi_{h,k} \quad \text{and} \quad \mathcal{I}_h^\Gamma\{a\} := \sum_{k=1}^{\dim U_h^\Gamma} a(\boldsymbol{x}_{h,k}^\Gamma)\chi_{h,k}^\Gamma.$$

For future reference, we state the following norm equivalences for $p \in [1, \infty)$ and $f_h \in U_h^{\mathcal{O}}$, $g_h \in U_h^\Gamma$:

$$c\Big(\int_{\mathcal{O}} |f_h|^p \,\mathrm{d}x\Big)^{1/p} \le \Big(\int_{\mathcal{O}} \mathcal{I}_h\{|f_h|^p\} \,\mathrm{d}x\Big)^{1/p} \le C\Big(\int_{\mathcal{O}} |f_h|^p \,\mathrm{d}x\Big)^{1/p}, \tag{2.2a}$$

$$c\Big(\int_{\Gamma} |g_h|^p \,\mathrm{d}\Gamma\Big)^{1/p} \le \Big(\int_{\Gamma} \mathcal{I}_h^\Gamma\{|g_h|^p\} \,\mathrm{d}\Gamma\Big)^{1/p} \le C\Big(\int_{\Gamma} |g_h|^p \,\mathrm{d}\Gamma\Big)^{1/p} \tag{2.2b}$$

with $c, C > 0$ independent of h. The proof of this estimate relies on standard inverse estimates and can be found, for example, in [58, Lemma 3.2.11]. To simplify the notation, we introduce the discrete norms

$$\|\zeta_h\|_{h,\mathcal{O}} := \sqrt{\int_{\mathcal{O}} \mathcal{I}_h\{|\zeta_h|^2\} \,\mathrm{d}x} \;\text{ and }\; \|\zeta_h\|_{H_h^1(\mathcal{O})} := \sqrt{\|\zeta_h\|_{h,\mathcal{O}}^2 + \|\nabla\zeta_h\|_{L^2(\mathcal{O})}^2}, \tag{2.3a}$$

$$\|\widehat{\zeta}_h\|_{h,\Gamma} := \sqrt{\int_{\Gamma} \mathcal{I}_h^\Gamma\{|\widehat{\zeta}_h|^2\} \,\mathrm{d}x} \;\text{ and }\; \|\widehat{\zeta}_h\|_{H_h^1(\Gamma)} := \sqrt{\|\widehat{\zeta}_h\|_{h,\Gamma}^2 + \|\nabla_\Gamma\widehat{\zeta}_h\|_{L^2(\Gamma)}^2} \tag{2.3b}$$

on $U_h^{\mathcal{O}}$ and U_h^Γ. Due to estimates (2.2), these norms are equivalent to the usual L^2- and H^1-norms on Ω and Γ, respectively. Furthermore, we state the following lemma that was proven in [42]:

Lemma 2.1. *Let $\mathcal{T}_h$ and $\mathcal{T}_h^\Gamma$ satisfy assumptions* **(S1)** *and* **(S2)***, respectively. Furthermore, let $p \in [1, \infty)$, $1 \le q \le \infty$ and $q^* = \frac{q}{q-1}$ for $q < \infty$ or $q^* = 1$ for $q = \infty$. Then*

$$\|(1 - \mathcal{I}_h)\{f_h g_h\}\|_{L^p(\mathcal{O})} \le Ch^2 \|\nabla f_h\|_{L^{pq}(\mathcal{O})} \|\nabla g_h\|_{L^{pq^*}(\mathcal{O})}, \tag{2.4a}$$

$$\|(1 - \mathcal{I}_h)\{f_h g_h\}\|_{W^{1,p}(\mathcal{O})} \le Ch \|\nabla f_h\|_{L^{pq}(\mathcal{O})} \|\nabla g_h\|_{L^{pq^*}(\mathcal{O})}, \tag{2.4b}$$

$$\|(1 - \mathcal{I}_h^\Gamma)\{u_h v_h\}\|_{L^p(\Gamma)} \le Ch^2 \|\nabla_\Gamma u_h\|_{L^{pq}(\Gamma)} \|\nabla_\Gamma v_h\|_{L^{pq^*}(\Gamma)}, \tag{2.4c}$$

$$\|(1 - \mathcal{I}_h^\Gamma)\{u_h v_h\}\|_{W^{1,p}(\Gamma)} \le Ch \|\nabla_\Gamma u_h\|_{L^{pq}(\Gamma)} \|\nabla_\Gamma v_h\|_{L^{pq^*}(\Gamma)} \tag{2.4d}$$

hold true for all $f_h, g_h \in U_h^{\mathcal{O}}$ and $u_h, v_h \in U_h^\Gamma$

Using the nodal interpolation operator, we define the discrete Laplacian Δ_h on $\mathcal{O}$ via

$$\int_{\mathcal{O}} \mathcal{I}_h\{\Delta_h\zeta_h\psi_h\} \,\mathrm{d}x = -\int_{\mathcal{O}} \nabla\zeta_h \cdot \nabla\psi_h \,\mathrm{d}x \tag{2.5}$$

for $\zeta_h, \psi_h \in U_h^{\mathcal{O}}$.

Concerning the potentials F and G, we postulate assumptions similar to the ones used in [44]. In particular, we shall assume that

(P) $F, G \in C^2(\mathbb{R})$ are bounded from below by a positive constant $\gamma > 0$ and satisfy the growth estimates

$$c_\gamma(1+|\zeta|^4) \leq F(\zeta) \leq C(1+|\zeta|^4),\ |F'(\zeta)| \leq C(1+|\zeta|^3),\ |F''(\zeta)| \leq C(1+|\zeta|^2),$$
$$c_\gamma(1+|\zeta|^4) \leq G(\zeta) \leq C(1+|\zeta|^4),\ |G'(\zeta)| \leq C(1+|\zeta|^3),\ |G''(\zeta)| \leq C(1+|\zeta|^2)$$

for all $\zeta \in \mathbb{R}$ and a positive constant c_γ.
Furthermore, F and G can be decomposed into $F_1, G_1 \in C^{2,\nu}(\mathbb{R})$ with $\nu \in (0,1)$ and $F_2, G_2 \in C^3(\mathbb{R})$, $|F_2'''(\zeta)| \leq C(1+|\zeta|^2)$, and $|G_2'''(\zeta)| \leq C(1+|\zeta|^2)$.

Remark 2.2. Having a positive lower bound for F and G is a purely technical assumption that is required to state the SAV scheme, as we will need that $\left(\int_{\mathcal{O}} F(\phi)\,\mathrm{d}x\right)^{-1/2}$ and $\left(\int_\Gamma G(\phi)\,\mathrm{d}\Gamma\right)^{-1/2}$ are well defined and bounded. As system (1.4) only depends on the derivatives of F and G, it is always possible to add arbitrary constants without changing system (1.4). Hence, any lower bounds for F and G are sufficient. The necessary shift can then be interpreted as a numerical parameter.

The statements given in assumption (P) are satisfied for instance by a shifted polynomial double-well potential $\widetilde{W}_{\mathrm{pol}}(\phi) := \frac{1}{4}(\phi^2-1)^2 + \gamma$ with $\gamma > 0$. We are, however, not limited to this specific choice. As stated before in system (1.3), we also consider a wetting energy density γ_{fs}, which can be chosen, for example, as

$$\gamma_{fs}(\phi) := \begin{cases} \dfrac{\gamma_{fs,2}-\gamma_{fs,1}}{2}\left(\dfrac{3}{8}\phi^5 - \dfrac{5}{4}\phi^3 + \dfrac{15}{5}\phi\right) + \dfrac{\gamma_{fs,2}+\gamma_{fs,1}}{2} & \text{if } \phi \in [-1,+1], \\ \gamma_{fs,1} & \text{if } \phi < -1, \\ \gamma_{fs,2} & \text{if } \phi > 1 \end{cases}$$

to satisfy assumption (P). An admissible choice obtained by merging this wetting energy density with the polynomial double-well potential is, for example,

$$G(\phi) = \widetilde{W}_{\mathrm{pol}} + \gamma_{fs}(\phi) - \min\left\{\gamma_{fs,1}, \gamma_{fs,2}\right\}.$$

In comparison to the assumptions used in [43] for the numerical treatment of a deterministic Cahn–Hilliard-Cahn–Hilliard system, assumptions (P) are stricter. In particular, the decrease in regularity caused by the additional stochastic noise terms requires us to enhance our discrete scheme by additional terms involving higher derivatives of the potentials (cf. Section 3). Hence, the potentials used in the stochastic setting need to allow for one derivative more than their counterparts in the deterministic setting. The growth conditions imposed on F and G are necessary to control the source terms using a Gronwall argument. As discussed in [35, Remark 5], these growth conditions seem to be sharp even for deterministic source terms.

For simplicity, we consider deterministic initial data and assume that

(I) $\phi_0 \in H^2(\mathcal{O})$. The discrete initial data $\phi_h^0 \in U_h^{\mathcal{O}}$ is then obtained via $\phi_h^0 := \mathcal{I}_h\{\phi_0\}$.

Immediate consequences of assumption (I) are

$$\|\phi_h^0\|_{W^{1,6}(\mathcal{O})} + \|[\phi_h^0]|_\Gamma\|_{W^{1,4}(\Gamma)} \leq C(\phi_0), \tag{2.6a}$$

$$\text{and} \quad \|\phi_h^0 - \phi_0\|_{H^1(\mathcal{O})} + \|[\phi_h^0]|_\Gamma - [\phi_0]|_\Gamma\|_{L^4(\Gamma)} \leq C(\phi_0)h. \tag{2.6b}$$

The stochastic source terms in system (1.4) are governed by a $\mathcal{Q}$-Wiener process and an operator Φ. Throughout this paper, we shall assume that

(W1) the trace class operator $\mathcal{Q} : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ satisfies

$$\mathcal{Q}g := \sum_{k\in\mathbb{Z}} \lambda_k^2 (g, \mathfrak{g}_k)_{L^2(\mathcal{O})} \mathfrak{g}_k,$$

where $(\lambda_k)_{k\in\mathbb{Z}}$ are given real numbers, and $(\mathfrak{g}_k)_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathcal{O})$.

Hence, $\mathcal{Q}^{1/2}$ given by $\mathcal{Q}^{1/2}g := \sum_{k\in\mathbb{Z}} \lambda_k (g, \mathfrak{g}_k)_{L^2(\mathcal{O})} \mathfrak{g}_k$ is a Hilbert–Schmidt operator from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$. We shall denote its image of $L^2(\mathcal{O})$ by

$$\mathcal{Q}^{1/2}L^2(\mathcal{O}) := \left\{\mathcal{Q}^{1/2}g : g \in L^2(\mathcal{O})\right\}.$$

Hence, the $\mathcal{Q}$-Wiener process $(W_t)_{t\in[0,T]}$ has the representation

$$W_t := \sum_{k\in\mathbb{Z}} \lambda_k \mathfrak{g}_k \beta_k(t)$$

with mutually independent Brownian motions $(\beta_k)_{k\in\mathbb{Z}}$. Furthermore, we assume that the $\mathcal{Q}$-Wiener process is colored in the sense that

(W2) there exists a positive constant $\widetilde{C}$ such that

$$\sum_{k\in\mathbb{Z}} \lambda_k^2 \|\mathfrak{g}_k\|_{W^{2,\infty}(\mathcal{O})}^2 \leq \widetilde{C}. \tag{2.7}$$

The operator Φ mapping the stochastic process ϕ into the space of Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ is defined via

$$\Phi(\phi)g := \varrho(\phi) \sum_{k\in\mathbb{Z}} (g, \mathfrak{g}_k)_{L^2(\mathcal{O})} \mathfrak{g}_k \tag{2.8}$$

for all $g \in \mathcal{Q}^{1/2}L^2(\mathcal{O})$. Concerning the coefficient function $\varrho : \mathbb{R} \to \mathbb{R}$, we follow [40] and [44] and assume that

(C) $\varrho \in L^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$.

This assumption in particular guarantees that Φ as defined in (2.8) is indeed a mapping into the space of Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}L^2(\mathcal{O})$ to $L^2(\mathcal{O})$, since

$$\sum_{k\in\mathbb{Z}} \|\Phi(\phi)(\lambda_k)\mathfrak{g}_k\|_{L^2(\mathcal{O})}^2 = \sum_{k\in\mathbb{Z}} \|\varrho(\phi)\lambda_k \mathfrak{g}_k\|_{L^2(\mathcal{O})}^2 \leq \|\varrho\|_{L^\infty(\mathbb{R})}^2 \sum_{k\in\mathbb{Z}} \|\mathcal{Q}^{1/2}\mathfrak{g}_k\|_{L^2(\mathcal{O})}^2 \leq C.$$

Hence, the source terms on the right-hand side of system (1.4) are governed by

$$\Phi(\phi)\mathrm{d}W := \sum_{k\in\mathbb{Z}} \lambda_k \mathfrak{g}_k \varrho(\phi)\mathrm{d}\beta_k. \tag{2.9}$$

Another straightforward consequence of assumption **(C)** are the estimates

$$\|\nabla \mathcal{I}_h\{\varrho(\zeta_h)\}\|_{L^p(\mathcal{O})} \le \overline{C}\|\nabla\zeta_h\|_{L^p(\mathcal{O})},$$
$$\|\nabla_\Gamma \mathcal{I}_h^\Gamma\{\varrho(\widehat{\zeta}_h)\}\|_{L^p(\Gamma)} \le \overline{C}\|\nabla_\Gamma\widehat{\zeta}_h\|_{L^p(\Gamma)}$$

for any $\zeta_h \in U_h^{\mathcal{O}}$, $\widehat{\zeta}_h \in U_h^\Gamma$, and $p\in[1,\infty]$ with a constant $\overline{C}$ depending on the Lipschitz constant of ϱ.

Remark 2.3. As discussed in [54], the multiplicative noise can be chosen in the form

$$\Phi(\phi)\mathfrak{g}_k := \widetilde{\varrho}_k(\phi) - \int_{\mathcal{O}} \widetilde{\varrho}_k(\phi)\,\mathrm{d}x,$$

resulting in $\int_{\mathcal{O}}\phi\,\mathrm{d}x$ being conserved. Such a choice makes sense from the modeling point of view, but is not necessary for the results presented in this paper.

In our discrete scheme, we shall approximate the $\mathcal{Q}$-Wiener process W by a sequence of discrete stochastic increments $\{\blacktriangle^n\boldsymbol{\xi}^\tau\}_{n=1,\dots,N}$ that are supposed to have the following properties:

(D0) Let $\mathcal{F}^\tau = (\mathcal{F}_t^\tau)_{t\in[0,T]}$ be defined by $\mathcal{F}_t^\tau = \mathcal{F}_{t^{n-1}}$ for $t\in[t^{n-1},t^n)$.

(D1) $\blacktriangle^n\boldsymbol{\xi}^\tau$ is $\mathcal{F}_{t^n}^\tau$-measurable and independent of $\mathcal{F}_{t^m}^\tau$ for all $0\le m\le n-1$.

(D2) There exist mutually independent symmetric random variables $\xi_k^{n,\tau}$ such that

$$\blacktriangle^n\boldsymbol{\xi}^\tau = \sqrt{\tau}\sum_{k\in\mathbb{Z}}\lambda_k\mathfrak{g}_k\xi_k^{n,\tau},$$

where $\mathbb{E}[\xi_k^{n,\tau}]=0$, $\mathbb{E}[|\xi_k^{n,\tau}|^2]=1$, and $\mathbb{E}[|\xi_k^{n,\tau}|^p]\le C_p$ for all integers $p\ge 2$ with a constant C_p depending on the exponent p but not on h, τ, k, or n.

Here $(\mathfrak{g}_k)_{k\in\mathbb{Z}}$ is an orthogonal basis of $L^2(\mathcal{O})$ and $(\lambda_k)_{k\in\mathbb{Z}}$ are given real numbers such that

(D3) the noise $\blacktriangle^n\boldsymbol{\xi}^\tau$ is colored in the sense that estimate (2.7) in assumption **(W2)** is satisfied for a positive constant $\widetilde{C}$ that is independent of h and τ.

Due to the trace theorem (see, e.g., [34, 47]), assumption **(D3)** also implies

$$\sum_{k\in\mathbb{Z}}\lambda_k^2\|[\mathfrak{g}_k]|_\Gamma\|_{W^{1,\infty}(\Gamma)}^2 \le C. \tag{2.10}$$

Summing $\blacktriangle^n\boldsymbol{\xi}^\tau$ from $n=1$ to m provides the discrete approximation $\boldsymbol{\xi}^{m,\tau}$ of the $\mathcal{Q}$-Wiener process. We want to emphasize that assumption **(D2)** is more general than assuming that $\boldsymbol{\xi}^{m,\tau}$ is obtained by evaluating a $\mathcal{Q}$-Wiener process satisfying assumption **(W1)** at given points in time, as assumption (D2) does not require the independent random variables $\xi_k^{n,\tau}$ to be $\mathcal{N}(0,1)$ distributed. Further, we shall approximate Φ by Φ_h, which is defined via

$$\Phi_h(\zeta_h)f := \sum_{k\in\mathbb{Z}_h} \mathcal{I}_h\{\varrho(\zeta_h)(f,\mathfrak{g}_k)_{L^2(\mathcal{O})}\mathfrak{g}_k\} \tag{2.11}$$

for all $f \in \mathcal{Q}^{1/2}L^2(\mathcal{O})$ and $\zeta_h \in U_h^{\mathcal{O}}$. Here, $\mathbb{Z}_h$ is an h-dependent finite subset of $\mathbb{Z}$ satisfying $\bigcup_{h>0}\mathbb{Z}_h = \mathbb{Z}$ and $\mathbb{Z}_{h_1} \subset \mathbb{Z}_{h_2}$ for $h_1 \geq h_2$. As a consequence, only a finite number of modes will enter the discrete scheme for every given h. Hence, it suffices to approximate the $\mathcal{Q}$-Wiener process by

$$\boldsymbol{\xi}_h^{m,\tau} := \sum_{n=1}^{m} \sqrt{\tau} \sum_{k\in\mathbb{Z}_h} \lambda_k \mathfrak{g}_k \xi_k^{n,\tau}. \tag{2.12}$$

In order to control higher moments of the (discrete) stochastic integrals, we will need the following estimates:

Lemma 2.4. *Let $(\blacktriangle^n\boldsymbol{\xi}^\tau)_{n=1,\dots,N}$ be a sequence of discrete stochastic increments satisfying assumptions (D0)–(D3). Also let $(\Phi_h^{n-1})_{n=1,\dots,N}$ be a sequence of $(\mathcal{F}_{t^{n-1}}^\tau)_{n=1,\dots,N}$-measurable mappings from Ω to the set of Hilbert–Schmidt operators mapping $\mathcal{Q}^{1/2}L^2(\mathcal{O})$ to a separable Hilbert space H, that is, $\Phi_h^{n-1} \in L_2(\mathcal{Q}^{1/2}L^2(\mathcal{O}); H)$, satisfying $\Phi_h^{n-1}\mathfrak{g}_k = 0$ for all $k \notin \mathbb{Z}_h$ (cf. equation (2.11)). Then, for $p \in [1,\infty)$, the estimates*

$$\mathbb{E}\big[\|\Phi_h\blacktriangle^n\boldsymbol{\xi}^\tau\|_H^p\big] \leq C_p\tau^{p/2}\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\|\lambda_k\Phi_h^{n-1}\mathfrak{g}_k\|_H^2\Big)^{p/2}\Big], \tag{2.13a}$$

$$\mathbb{E}\Big[\max_{1\leq l\leq m}\Big\|\sum_{n=1}^{l}\Phi_h^{n-1}\blacktriangle^n\boldsymbol{\xi}^\tau\Big\|_H^p\Big] \leq C_p(m\tau)^{\frac{p-2}{2}}\sum_{n=1}^{m}\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\|\lambda_k\Phi_h^{n-1}\mathfrak{g}_k\|_H^2\Big)^{p/2}\Big] \tag{2.13b}$$

hold true with a constant $C_p > 0$ that depends on p but is independent of h and τ.

These results are an extension of the results established for $\mathbb{R}$-valued stochastic increments in [49, Lemma 2.8]. For a proof of Lemma 2.4, we refer to [44, Lemma 2.2].

When passing to the limit $(h,\tau) \to (0,0)$ with families of fully discrete random variables, we need time interpolants of our time-discrete random variables. For a time-discrete function a^n, $n = 0,\dots,N$, we introduce some time-index-free notation as follows:

$$a^\tau(\cdot,t) := \frac{t-t^{n-1}}{\tau}a^n(\cdot) + \frac{t^n-t}{\tau}a^{n-1}(\cdot) \quad t\in[t^{n-1},t^n],\ n\geq 1, \tag{2.14a}$$

$$a^{\tau,-}(\cdot,t) := a^{n-1}(\cdot) \quad t\in[t^{n-1},t^n),\ n\geq 1, \tag{2.14b}$$

$$a^{\tau,+}(\cdot,t) := a^n(\cdot) \quad t\in(t^{n-1},t^n],\ n\geq 1. \tag{2.14c}$$

We shall complete this definition by setting $a^{\tau,+}(\cdot,0) := a^0(\cdot)$. Obviously, we have

$$a^{\tau}(\cdot,t) - a^{\tau,-}(\cdot,t) = \frac{t-t^{n-1}}{\tau}(a^n - a^{n-1}) \quad \text{for all } t \in [t^{n-1}, t^n).$$

To indicate that a statement is valid for all three time interpolants defined in equation (2.14), we use the abbreviation $a^{\tau,(\pm)}$.

3. The discrete scheme

For the ease of representation, we define the abbreviations

$$E_h^{\mathcal{O}}(\zeta_h) := \int_{\mathcal{O}} \mathcal{I}_h\{F(\zeta_h)\}\,\mathrm{d}x, \tag{3.1a}$$

$$E_h^{\Gamma}(\widehat{\zeta}_h) := \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G(\widehat{\zeta}_h)\}\,\mathrm{d}\Gamma \tag{3.1b}$$

for $\zeta_h \in U_h^{\mathcal{O}}$ and $\widehat{\zeta}_h \in U_h^{\Gamma}$. To linearize the discrete system, we introduce the stochastic auxiliary variables $r,\ s\ :\ \Omega \times [0,T] \to \mathbb{R}$ with $r(\omega,t) := \sqrt{\int_{\mathcal{O}} F(\phi(\omega,t,x))\,\mathrm{d}x}$ and $s(\omega,t) := \sqrt{\int_{\Gamma} G([\phi(\omega,t,x)]|_{\Gamma})\,\mathrm{d}\Gamma}$ as the square roots of the nonquadratic parts of the energy. An application of the chain rule suggests that the evolution of these auxiliary variables is given by

$$\partial_t r = \frac{1}{2\sqrt{\int_{\mathcal{O}} F(\phi)\,\mathrm{d}x}} \int_{\mathcal{O}} F'(\phi)\partial_t\phi\,\mathrm{d}x, \tag{3.2a}$$

$$\partial_t s = \frac{1}{2\sqrt{\int_{\Gamma} G([\phi]|_{\Gamma})\,\mathrm{d}\Gamma}} \int_{\Gamma} G'([\phi]|_{\Gamma})\partial_t[\phi]|_{\Gamma}\,\mathrm{d}\Gamma. \tag{3.2b}$$

The original idea of the scalar auxiliary variable method, as presented in [55], is to approximate equation (3.2a) by

$$r(t^n) - r(t^{n-1}) = \frac{1}{2\sqrt{\int_{\mathcal{O}} F(\phi(t^{n-1}))\,\mathrm{d}x}} \int_{\mathcal{O}} F'(\phi(t^{n-1}))(\phi(t^n) - \phi(t^{n-1}))\,\mathrm{d}x, \tag{3.3}$$

which leads to a time-discrete scheme that is linear with respect to the unknown quantities, but also still stable with respect to the modified energy expressed in terms of $r(t^n)$ and $s(t^n)$ instead of $\int_{\mathcal{O}} F(\phi(t^n))\,\mathrm{d}x$ and $\int_{\Gamma} G([\phi(t^n)]|_{\Gamma})\,\mathrm{d}\Gamma$. When discussing the convergence behavior of the auxiliary variables, that is, the question whether the discrete approximations of $r(t^n)$ converge toward $\sqrt{\int_{\mathcal{O}} F(\phi(t^n))\,\mathrm{d}x}$ in a suitable sense, it is important to notice that the right-hand side of equation (3.3) is given as the first-order Taylor approximation of $\sqrt{\int_{\mathcal{O}} F(\phi(t^n))\,\mathrm{d}x}$ around $\phi(t^{n-1})$. Hence, as outlined in [44, Remark 3.1], we can anticipate that the additional approximation error per time step stemming from the time discretization of r will depend on the term $(\phi(t^n) - \phi(t^{n-1}))^2$, that is, the global approximation error will depend on $\sum_{k=1}^{n} (\phi(t^k) - \phi(t^{k-1}))^2$. Unfortunately, in the stochastic setting, the time regularity of ϕ is severely limited by the regularity of the Brownian motions on the right-hand side of system (1.4). As a consequence, this approximation error will not vanish for $\tau \searrow 0$. Hence, we follow the ideas presented in [44] and add suitable linear approximations of the second-order terms of the Taylor approximation that guarantee convergence of the scalar auxiliary variables. For the rigorous computations, we refer to Lemma 6.4 below.

In our discrete scheme, the scalar auxiliary variables are approximated using a sequence of random variables $(r_h^n)_{n=0,\dots,N}$ and $(s_h^n)_{n=0,\dots,N}$, which are supposed to approximate

$$\Big(\sqrt{E_h^{\mathcal{O}}(\phi_h^n)}\Big)_{n=0,\dots,N} \quad \text{and} \quad \Big(\sqrt{E_h^{\Gamma}([\phi_h^n]|_\Gamma)}\Big)_{n=0,\dots,N}.$$

Defining the discrete initial data via $\phi_h^0 := \mathcal{I}_h\{\phi_0\}$ (cf. Assumption (I)), $r_h^0 := \sqrt{E_h^{\mathcal{O}}(\phi_h^0)}$, and $s_h^0 := \sqrt{E_h^{\Gamma}(\phi_h^0)}$, we state the following discrete scheme:

For a given $U_h^{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}$-valued random variable $(\phi_h^{n-1}, r_h^{n-1}, s_h^{n-1})$, find a $U_h^{\mathcal{O}} \times U_h^{\mathcal{O}} \times U_h^{\Gamma} \times \mathbb{R} \times \mathbb{R}$-valued random variable $(\phi_h^n, \mu_h^n, \theta_h^n, r_h^n, s_h^n)$ such that pathwise

$$\begin{aligned}\int_{\mathcal{O}} \mathcal{I}_h\{(\phi_h^n - \phi_h^{n-1})\psi_h\}\,\mathrm{d}x + \tau \int_{\mathcal{O}} \nabla\mu_h^n \cdot \nabla\psi_h\,\mathrm{d}x \\ = \int_{\mathcal{O}} \mathcal{I}_h\{\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau \psi_h\}\,\mathrm{d}x,\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{[\phi_h^n - \phi_h^{n-1}]|_\Gamma \widehat{\psi}_h\}\,\mathrm{d}\Gamma + \tau \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{\theta_h^n \widehat{\psi}_h\}\,\mathrm{d}\Gamma \\ = \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \widehat{\psi}_h\}\,\mathrm{d}\Gamma,\end{aligned} \tag{3.4b}$$

$$\begin{aligned}&\int_{\mathcal{O}} \mathcal{I}_h\{\mu_h^n \eta_h\}\,\mathrm{d}x + \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{\theta_h^n [\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma \\ &= \int_{\mathcal{O}} \nabla\phi_h^n \cdot \nabla\eta_h\,\mathrm{d}x + \int_{\Gamma} \nabla_\Gamma[\phi_h^n]|_\Gamma \cdot \nabla_\Gamma[\eta_h]|_\Gamma\,\mathrm{d}\Gamma \\ &\quad + \Bigg[\frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}x \\ &\quad - \frac{r_h^n}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\}\,\mathrm{d}x \int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}x \\ &\quad + \frac{r_h^n}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau \eta_h\}\,\mathrm{d}x\Bigg] \\ &\quad + \Bigg[\frac{s_h^n}{\sqrt{E_h^{\Gamma}([\phi_h^{n-1}]|_\Gamma)}} \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G'([\phi_h^{n-1}]|_\Gamma)[\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma\end{aligned}$$

$$
\begin{aligned}
&- \frac{s_h^n}{4[E_h^\Gamma([\phi_h^{n-1}]|_\Gamma)]^{3/2}} \int_\Gamma \mathcal{I}_h^\Gamma\big\{G'([\phi_h^{n-1}]|_\Gamma)[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&\quad \times \int_\Gamma \mathcal{I}_h^\Gamma\big\{G'([\phi_h^{n-1}]|_\Gamma)[\eta_h]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&+ \frac{s_h^n}{2\sqrt{E_h^\Gamma([\phi_h^{n-1}]|_\Gamma)}} \int_\Gamma \mathcal{I}_h^\Gamma\big\{G''([\phi_h^{n-1}]|_\Gamma)[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma[\eta_h]|_\Gamma\big\}\,\mathrm{d}\Gamma\Big], \qquad (3.4c)
\end{aligned}
$$

for all $\psi_h, \eta_h \in U_h^{\mathcal{O}}$, and $\widehat{\psi}_h \in U_h^\Gamma$ together with

$$
\begin{aligned}
r_h^n - r_h^{n-1} &= \frac{1}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \\
&\quad - \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau\big\}\,\mathrm{d}x \\
&\quad \times \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \\
&\quad + \frac{1}{4\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \\
&\quad \times \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \qquad (3.4d)
\end{aligned}
$$

and

$$
\begin{aligned}
s_h^n - s_h^{n-1} &= \frac{1}{2\sqrt{E_h^\Gamma([\phi_h^{n-1}]|_\Gamma)}} \int_\Gamma \mathcal{I}_h^\Gamma\big\{G'([\phi_h^{n-1}]|_\Gamma)[\phi_h^n - \phi_h^{n-1}]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&\quad - \frac{1}{8[E_h^\Gamma([\phi_h^{n-1}]|_\Gamma)]^{3/2}} \int_\Gamma \mathcal{I}_h^\Gamma\big\{G'([\phi_h^{n-1}]|_\Gamma)[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&\quad \times \int_\Gamma \mathcal{I}_h^\Gamma\big\{G'([\phi_h^{n-1}]|_\Gamma)[\phi_h^n - \phi_h^{n-1}]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&\quad + \frac{1}{4\sqrt{E_h^\Gamma([\phi_h^{n-1}]|_\Gamma)}} \\
&\quad \times \int_\Gamma \mathcal{I}_h^\Gamma\big\{G''([\phi_h^{n-1}]|_\Gamma)[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma[\phi_h^n - \phi_h^{n-1}]|_\Gamma\big\}\,\mathrm{d}\Gamma. \qquad (3.4e)
\end{aligned}
$$

Here, the terms in the square brackets in equation (3.4c) are the discrete approximations of the nonlinear terms $\int_{\mathcal{O}} F'(\phi(t^n))\eta\,\mathrm{d}x$ and $\int_\Gamma G'([\phi(t^n)]|_\Gamma)[\eta]|_\Gamma\,\mathrm{d}\Gamma$. To simplify the notation, we shall suppress the trace operator $[\cdot]|_\Gamma$ whenever the notation is unambiguous. For future reference, we shall denote the additional terms added to equation (3.4c) by the more accurate Taylor expansion by $U_h^{\mathcal{O}}$- and U_h^Γ-valued random variables $\Xi_{h,\mathcal{O}}^n$ and $\Xi_{h,\Gamma}^n$, that is,

$$\Xi^n_{h,\mathcal{O}} := -\frac{r^n_h}{4[E^{\mathcal{O}}_h(\phi^{n-1}_h)]^{3/2}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi^{n-1}_h)\Phi_h(\phi^{n-1}_h)\blacktriangle^n \xi^\tau\big\}\,\mathrm{d}x\, \mathcal{I}_h\big\{F'(\phi^{n-1}_h)\big\} + \frac{r^n_h}{2\sqrt{E^{\mathcal{O}}_h(\phi^{n-1}_h)}} \mathcal{I}_h\big\{F''(\phi^{n-1}_h)\Phi_h(\phi^{n-1}_h)\blacktriangle^n \xi^\tau\big\}, \tag{3.5a}$$

$$\Xi^n_{h,\Gamma} := -\frac{s^n_h}{4[E^{\Gamma}_h(\phi^{n-1}_h)]^{3/2}} \int_{\Gamma} \mathcal{I}^\Gamma_h\big\{G'(\phi^{n-1}_h)[\Phi_h(\phi^{n-1}_h)\blacktriangle^n \xi^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma\, \mathcal{I}^\Gamma_h\big\{G'(\phi^{n-1}_h)\big\} + \frac{s^n_h}{2\sqrt{E^{\Gamma}_h(\phi^{n-1}_h)}} \mathcal{I}^\Gamma_h\big\{G''(\phi^{n-1}_h)[\Phi_h(\phi^{n-1}_h)\blacktriangle^n \xi^\tau]|_\Gamma\big\}. \tag{3.5b}$$

As we shall show in Lemma 6.2, these additional terms vanish for $\tau \searrow 0$. With these definitions, equation (3.4c) can be written as

$$\int_{\mathcal{O}} \mathcal{I}_h\{\mu^n_h \eta_h\}\,\mathrm{d}x + \int_\Gamma \mathcal{I}^\Gamma_h\{\theta^n_h[\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma = \int_{\mathcal{O}} \nabla\phi^n_h\cdot\nabla\eta_h\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\phi^n_h\cdot\nabla_\Gamma\eta_h\,\mathrm{d}\Gamma + \frac{r^n_h}{\sqrt{E^{\mathcal{O}}_h(\phi^{n-1}_h)}}\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi^{n-1}_h)\eta_h\big\}\,\mathrm{d}x + \int_{\mathcal{O}} \mathcal{I}_h\big\{\Xi^n_{h,\mathcal{O}}\eta_h\big\}\,\mathrm{d}x + \frac{s^n_h}{\sqrt{E^{\Gamma}_h(\phi^{n-1}_h)}}\int_\Gamma \mathcal{I}^\Gamma_h\big\{G'(\phi^{n-1}_h)\eta_h\big\}\,\mathrm{d}\Gamma + \int_\Gamma \mathcal{I}^\Gamma_h\big\{\Xi^n_{h,\Gamma}\eta_h\big\}\,\mathrm{d}\Gamma.$$

4. Main results

In this section, we shall state the main results, which will be proven in the subsequent sections. The first result provides the existence of pathwise-unique solutions to the discrete scheme (3.4).

Lemma 4.1. *Let the assumptions* **(T)**, **(S1)**, **(S2)**, *and* **(D0)** *hold true. Then, there exists a sequence* $(\phi^n_h, \mu^n_h, \theta^n_h, r^n_h, s^n_h)_{n\geq 1}$ *of* $U^{\mathcal{O}}_h \times U^{\mathcal{O}}_h \times U^\Gamma_h \times \mathbb{R}\times\mathbb{R}$*-valued random variables that solves scheme* (3.4) *for each* $\omega\in\Omega$*. Furthermore, the map* $(\phi^n_h, \mu^n_h, \theta^n_h, r^n_h, s^n_h) : \Omega \to U^{\mathcal{O}}_h \times U^{\mathcal{O}}_h \times U^\Gamma_h \times \mathbb{R}\times\mathbb{R}$ *is* $\mathcal{F}_{t^n}$*-measurable.*

The proof of this result can be found in Section 5. Starting from these fully discrete solutions, we can pass to the limit $(h,\tau)\to(0,0)$ to obtain the existence of pathwise-unique martingale solutions to system (1.4).

Theorem 4.2. *Let the assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)–(D3)**, *hold true. Then, there exists a filtered probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *and a sequence of random variables* $(\widetilde{\phi}^\tau_h, \widetilde{\phi_\Gamma}^\tau_h, \widetilde{\mu}^{\tau,+}_h, \widetilde{\theta}^{\tau,+}_h)_{h,\tau}$ *on* $\widetilde{\Omega}$ *whose laws coincide with the laws of the* *time interpolants* $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma, \mu_h^{\tau,+}, \theta_h^{\tau,+})_{h,\tau}$ *of the discrete solutions to system* (3.4) *established in Lemma* 4.1*. Furthermore, there exists an* $\widetilde{\mathcal{F}}$*-measurable* $\mathcal{Q}$*-Wiener process* $\widetilde{W} := \sum_{k\in\mathbb{Z}} \lambda_k \mathfrak{g}_k \widetilde{\beta}_k$ *on* $\widetilde{\Omega}$ *and progressively* $\mathcal{F}$*-measurable processes*

$$
\begin{aligned}
\widetilde{\phi} &\in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T;H^1(\mathcal{O}))) \cap L^{4\mathfrak{p}}(\widetilde{\Omega}; C^{0,(\mathfrak{p}-1)/(4\mathfrak{p})}([0,T];L^2(\mathcal{O}))),\\
\widetilde{\phi_\Gamma} &\in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T;H^1(\Gamma))) \cap L^{2\mathfrak{p}}(\widetilde{\Omega}; C^{0,(\mathfrak{p}-1)/(2\mathfrak{p})}([0,T];L^2(\Gamma))),\\
\widetilde{\mu} &\in L^{2\mathfrak{p}}(\widetilde{\Omega}; L^2(0,T;H^1(\mathcal{O}))),\\
\widetilde{\theta} &\in L^{2\mathfrak{p}}(\widetilde{\Omega}; L^2(0,T;L^2(\Gamma)))
\end{aligned}
$$

for $\mathfrak{p} \in (1,\infty)$ *such that* $\widetilde{\mathbb{P}}$*-almost surely*

$$
\begin{aligned}
\lim_{(h,\tau)\to(0,0)} \widetilde{\phi}_h^\tau &= \widetilde{\phi} && \text{in } C([0,T];L^s(\mathcal{O})),\\
\lim_{(h,\tau)\to(0,0)} \widetilde{\phi_\Gamma}_h^\tau &= \widetilde{\phi_\Gamma} && \text{in } C([0,T];L^r(\Gamma)),\\
\lim_{(h,\tau)\to(0,0)} \widetilde{\mu}_h^{\tau,+} &= \widetilde{\mu} && \text{in } L^2(0,T;H^1(\mathcal{O}))_{\text{weak}},\\
\lim_{(h,\tau)\to(0,0)} \widetilde{\theta}_h^{\tau,+} &= \widetilde{\theta} && \text{in } L^2(0,T;L^2(\Gamma))_{\text{weak}}
\end{aligned}
$$

with $s \in [1, \frac{2d}{d-2})$ *and* $r \in [1,\infty)$*. These processes are pathwise unique and satisfy*

$$
\begin{aligned}
&\int_{\mathcal{O}} (\widetilde{\phi}(t) - \phi(0))\psi \,\mathrm{d}x + \int_0^t \int_{\mathcal{O}} \nabla\widetilde{\mu}\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s\\
&\qquad = \sum_{k\in\mathbb{Z}} \int_0^t \int_{\mathcal{O}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k \psi \,\mathrm{d}x\,\mathrm{d}\widetilde{\beta}_k,
\end{aligned} \tag{4.1a}
$$

$$
\begin{aligned}
&\int_\Gamma (\widetilde{\phi_\Gamma}(t) - [\phi]|_\Gamma(0))\widehat{\psi} \,\mathrm{d}\Gamma + \int_0^t \int_\Gamma \widetilde{\theta}\widehat{\psi} \,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\qquad = \sum_{k\in\mathbb{Z}} \int_0^t \int_\Gamma [\varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k]\big|_\Gamma \widehat{\psi} \,\mathrm{d}\Gamma\,\mathrm{d}\widetilde{\beta}_k,
\end{aligned} \tag{4.1b}
$$

$\widetilde{\mathbb{P}}$*-almost surely for all* $t \in [0,T]$ *and test functions* $\psi \in H^1(\mathcal{O})$ *and* $\widehat{\psi} \in L^2(\Gamma)$*. Furthermore,*

$$
\begin{aligned}
\int_{\mathcal{O}} \widetilde{\mu}\eta \,\mathrm{d}x + \int_\Gamma \widetilde{\theta}[\eta]|_\Gamma \,\mathrm{d}\Gamma &= \int_{\mathcal{O}} \nabla\widetilde{\phi}\cdot\nabla\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\widetilde{\phi_\Gamma}\cdot\nabla_\Gamma[\eta]|_\Gamma \,\mathrm{d}\Gamma\\
&\quad + \int_{\mathcal{O}} F'(\widetilde{\phi})\eta \,\mathrm{d}x + \int_\Gamma G'(\widetilde{\phi_\Gamma})[\eta]|_\Gamma \,\mathrm{d}\Gamma
\end{aligned} \tag{4.1c}
$$

holds true $\widetilde{\mathbb{P}}$*-almost surely for almost all* $t \in [0,T]$ *and test functions* $\eta \in \mathcal{V}$*. In addition,* $[\widetilde{\phi}]|_\Gamma = \widetilde{\phi_\Gamma}$ $\widetilde{\mathbb{P}}$*-almost surely almost everywhere on* $(0,T)\times\Gamma$*.*

The above results use an arbitrary finite-dimensional random walk $\boldsymbol{\xi}_h^{m,\tau}$ (cf. equation (2.12)) satisfying assumptions **(D0)**–**(D3)** as starting point in the finite element scheme and provide the existence of pathwise-unique martingale solutions. Hence, the Yamada–Watanabe theorem provides the existence of strong solutions (cf. [52] or [39, Theorem E.0.8]). These strong solutions can also be obtained as the limit of a sequence of fully discrete solutions: if we approximate a $\mathcal{Q}$-Wiener process W satisfying assumptions **(W1)** and **(W2)** via

$$\boldsymbol{\xi}_h^{m,\tau} = \sum_{k\in\mathbb{Z}_h} (W(t^m), \mathfrak{g}_k)_{L^2(\mathcal{O})}\mathfrak{g}_k, \tag{4.2}$$

then $\boldsymbol{\xi}_h^{m,\tau}$ still satisfies assumptions **(D0)**–**(D3)** with Gaussian random variables $\xi_k^{n,\tau}$. For this choice, we can establish convergence toward strong solutions.

Theorem 4.3. *Let W be a Wiener process defined on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ satisfying assumptions* **(W1)** *and* **(W2)** *with finite-dimensional approximations given by equation* (4.2). *Furthermore, let assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)**–**(D2)** *hold true. Then, there exist pathwise-unique, progressively $\mathcal{F}$-measurable processes*

$$\begin{aligned}
\phi &\in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\Omega; L^\infty(0,T;H^1(\mathcal{O}))) \cap L^{4\mathfrak{p}}(\Omega; C^{0,(\mathfrak{p}-1)/(4\mathfrak{p})}([0,T];L^2(\mathcal{O}))),\\
\phi_\Gamma &\in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\Omega; L^\infty(0,T;H^1(\Gamma))) \cap L^{2\mathfrak{p}}(\Omega; C^{0,(\mathfrak{p}-1)/(2\mathfrak{p})}([0,T];L^2(\Gamma))),\\
\mu &\in L^{2\mathfrak{p}}(\Omega; L^2(0,T;H^1(\mathcal{O}))),\\
\theta &\in L^{2\mathfrak{p}}(\Omega; L^2(0,T;L^2(\Gamma)))
\end{aligned}$$

for $\mathfrak{p}\in(1,\infty)$, which are the limits of the time interpolants $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma, \mu_h^{\tau,+}, \theta_h^{\tau,+})_{h,\tau}$ of the discrete solutions to equation (3.4). *In particular, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}
\lim_{(h,\tau)\to(0,0)} \phi_h^\tau &= \phi && \text{in } C([0,T];L^s(\mathcal{O})),\\
\lim_{(h,\tau)\to(0,0)} [\phi_h^\tau]|_\Gamma &= \phi_\Gamma && \text{in } C([0,T];L^r(\Gamma)),\\
\lim_{(h,\tau)\to(0,0)} \mu_h^{\tau,+} &= \mu && \text{in } L^2(0,T;H^1(\mathcal{O}))_{\text{weak}},\\
\lim_{(h,\tau)\to(0,0)} \theta_h^{\tau,+} &= \theta && \text{in } L^2(0,T;L^2(\Gamma))_{\text{weak}}
\end{aligned}$$

for $s\in[1,\frac{2d}{d-2})$ and $r\in[1,\infty)$. These processes satisfy

$$\begin{aligned}
&\int_{\mathcal{O}} (\phi(t)-\phi(0))\psi \,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} \nabla\mu\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s = \sum_{k\in\mathbb{Z}}\int_0^t\int_{\mathcal{O}} \varrho(\phi)\lambda_k\mathfrak{g}_k\psi\,\mathrm{d}x\,\mathrm{d}\beta_k,\\
&\int_\Gamma (\phi_\Gamma(t)-[\phi]|_\Gamma(0))\widehat{\psi}\,\mathrm{d}\Gamma + \int_0^t\int_\Gamma \theta\widehat{\psi}\,\mathrm{d}\Gamma\,\mathrm{d}s = \sum_{k\in\mathbb{Z}}\int_0^t\int_\Gamma [\varrho(\phi)\lambda_k\mathfrak{g}_k]|_\Gamma\widehat{\psi}\,\mathrm{d}\Gamma\,\mathrm{d}\beta_k,
\end{aligned}$$

$\mathbb{P}$*-almost surely for all* $t \in [0,T]$ *and test functions* $\psi \in H^1(\mathcal{O})$ *and* $\widehat{\psi} \in L^2(\Gamma)$*. Furthermore,*

$$\int_{\mathcal{O}} \mu\eta \,\mathrm{d}x + \int_\Gamma \theta[\eta]|_\Gamma \,\mathrm{d}\Gamma = \int_{\mathcal{O}} \nabla\phi\cdot\nabla\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\phi_\Gamma\cdot\nabla_\Gamma[\eta]|_\Gamma \,\mathrm{d}\Gamma + \int_{\mathcal{O}} F'(\phi)\eta \,\mathrm{d}x + \int_\Gamma G'(\phi_\Gamma)[\eta]|_\Gamma \,\mathrm{d}\Gamma$$

$\mathbb{P}$*-almost surely for almost all* $t \in [0,T]$ *and test functions* $\eta \in \mathcal{V}$*. Further,* $[\phi]|_\Gamma = \phi_\Gamma$ $\mathbb{P}$*-almost surely almost everywhere on* $(0,T)\times\Gamma$*.*

Here, the term ‘strong solution’ refers to a solution concept, where the filtered probability space and the Wiener process are a priori specified and solution processes on this filtered probability space satisfying the SPDE with the given Wiener process must be found. This distinguishes the results in Theorem 4.3 from the concept of martingale solutions used in Theorem 4.2, where constructing a suitable filtered probability space is part of the solution concept (see, e.g., [39, 49]).

The remainder of the paper is structured as follows: in Section 5, we present the proof of Lemma 4.1. The proof of Theorem 4.2 can be found in Sections 6–9: in Section 6, we establish regularity results for discrete solutions to system (3.4) that are independent of the discretization parameters h and τ. These results will be used in Section 7 together with Jakubowski’s theorem (cf. [30]) to establish the existence of converging subsequences. In Section 8, we discuss the passage to the limit $(h,\tau)\searrow 0$, which provides the existence statement in Theorem 4.2. The pathwise uniqueness of these martingale solutions is then established in Section 9. Section 10 is devoted to the proof of Theorem 4.3.

5. Existence of discrete solutions

In this section we will prove Lemma 4.1 by establishing the existence of pathwise-unique solutions to system (3.4). We adapt the ideas of [44] and start by showing that for each fixed $\omega \in \Omega$, system (3.4) has a unique solution: Since $\blacktriangle^n\boldsymbol{\xi}^\tau$ is given for fixed ω, system (3.4) is linear with respect to the unknown quantities ϕ_h^n, μ_h^n, θ_h^n, r_h^n, and s_h^n. As for finite-dimensional linear problems, uniqueness of possible solutions guarantees the existence of solutions for arbitrary right-hand sides, we assume that for given ϕ_h^{n-1}, r_h^{n-1}, s_h^{n-1}, and $\blacktriangle^n\boldsymbol{\xi}^\tau$, there exist two solutions. We denote their difference by $\widehat{\phi}$, $\widehat{\mu}$, $\widehat{\theta}$, $\widehat{r}$, and $\widehat{s}$. Obviously, these differences satisfy

$$\int_{\mathcal{O}} \mathcal{I}_h\{\widehat{\phi}\psi_h\}\,\mathrm{d}x + \tau\int_{\mathcal{O}} \nabla\widehat{\mu}\cdot\nabla\psi_h\,\mathrm{d}x = 0, \tag{5.1a}$$

$$\int_\Gamma \mathcal{I}_h^\Gamma\{\widehat{\phi}\widehat{\psi}_h\}\,\mathrm{d}\Gamma + \tau\int_\Gamma \mathcal{I}_h^\Gamma\{\widehat{\theta}\widehat{\psi}_h\}\,\mathrm{d}\Gamma = 0, \tag{5.1b}$$

$$\int_{\mathcal{O}} \mathcal{I}_h\{\widehat{\mu}\eta_h\}\,\mathrm{d}x + \int_\Gamma \mathcal{I}_h^\Gamma\{\widehat{\theta}\eta_h\}\,\mathrm{d}\Gamma = \int_{\mathcal{O}} \nabla\widehat{\phi}\cdot\nabla\eta_h\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\widehat{\phi}\cdot\nabla_\Gamma\eta_h\,\mathrm{d}\Gamma$$
$$+ \frac{\widehat{r}}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}x + \frac{\widehat{s}}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}\Gamma$$
$$- \frac{\widehat{r}}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}x$$
$$+ \frac{\widehat{r}}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}} \mathcal{I}_h\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\eta_h\}\,\mathrm{d}x$$
$$- \frac{\widehat{s}}{4[E_h^{\Gamma}(\phi_h^{n-1})]^{3/2}}\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\eta_h\}\,\mathrm{d}\Gamma$$
$$+ \frac{\widehat{s}}{2\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_\Gamma \mathcal{I}_h^\Gamma\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\eta_h\}\,\mathrm{d}\Gamma, \tag{5.1c}$$

$$\widehat{r} = \frac{1}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\widehat{\phi}\}\,\mathrm{d}x$$
$$- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{n-1})\widehat{\phi}\}\,\mathrm{d}x$$
$$+ \frac{1}{4\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}} \mathcal{I}_h\{F''(\phi_h^{n-1})\widehat{\phi}\Phi(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x, \tag{5.1d}$$

$$\widehat{s} = \frac{1}{2\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\widehat{\phi}\}\,\mathrm{d}\Gamma$$
$$- \frac{1}{8[E_h^{\Gamma}(\phi_h^{n-1})]^{3/2}}\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\int_\Gamma \mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\widehat{\phi}\}\mathrm{d}\Gamma$$
$$+ \frac{1}{4\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_\Gamma \mathcal{I}_h^\Gamma\{G''(\phi_h^{n-1})\widehat{\phi}[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma. \tag{5.1e}$$

Choosing $\psi_h \equiv \widehat{\mu}$ in equation (5.1a), $\widehat{\psi}_h \equiv \tau^{-1}\widehat{\phi}$ in equation (5.1b), and $\eta_h \equiv \widehat{\phi}$ in equation (5.1c), we obtain after substituting equations (5.1d) and (5.1e)

$$\tau\int_{\mathcal{O}}|\nabla\widehat{\mu}|^2 dx + \tau^{-1}\int_\Gamma \mathcal{I}_h^\Gamma\{|\widehat{\phi}|^2\}\,\mathrm{d}\Gamma + \int_{\mathcal{O}}|\nabla\widehat{\phi}|^2\,\mathrm{d}x + \int_\Gamma|\nabla_\Gamma\widehat{\phi}|^2\,\mathrm{d}\Gamma$$
$$+ 2|\widehat{r}|^2 + 2|\widehat{s}|^2 = 0. \tag{5.2}$$

This immediately implies $\widehat{r} = \widehat{s} = 0$ and $\widehat{\phi} \equiv 0$, which entails $\widehat{\theta} \equiv 0$ by equation (5.1b). Finally, using $\eta_h \equiv 1$ in equation (5.1c), we obtain $\int_{\mathcal{O}} \widehat{\mu}\,\mathrm{d}x = 0$. Due to equation (5.2), we have $\widehat{\mu} \equiv 0$. This provides the uniqueness and therefore the existence of solutions. As shown in, for example, [24, Theorem 6.7] the uniqueness of the solution obtained for each $\omega \in \Omega$ also entails its measurability.

6. Regularity results

In this section, we shall establish uniform regularity results for the discrete solutions obtained in Lemma 4.1.

Lemma 6.1. *Let the assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)**–**(D3)** *hold true. Then, for every* $1 \leq \mathfrak{p} < \infty$, *there exists a constant* $C \equiv C(\mathfrak{p}, T) > 0$ *independent of* h *and* τ *such that*

$$\begin{aligned}
&\mathbb{E}\Big[\max_{0\leq m\leq N} \|\phi_h^m\|_{H^1(\mathcal{O})}^{2\mathfrak{p}}\Big] + \mathbb{E}\Big[\max_{0\leq m\leq N} \|\phi_h^m\|_{H^1(\Gamma)}^{2\mathfrak{p}}\Big] + \mathbb{E}\Big[\max_{0\leq m\leq N} |r_h^m|^{2\mathfrak{p}}\Big] + \mathbb{E}\Big[\max_{0\leq m\leq N} |s_h^m|^{2\mathfrak{p}}\Big] \\
&\quad + \mathbb{E}\Big[\Big(\sum_{n=1}^{N} \|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^2\Big)^{\mathfrak{p}}\Big] + \mathbb{E}\Big[\Big(\sum_{n=1}^{N} \|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^2\Big)^{\mathfrak{p}}\Big] \\
&\quad + \mathbb{E}\Big[\Big(\sum_{n=1}^{m} |r_h^n - r_h^{n-1}|^2\Big)^{\mathfrak{p}}\Big] + \mathbb{E}\Big[\Big(\sum_{n=1}^{m} |s_h^n - s_h^{n-1}|^2\Big)^{\mathfrak{p}}\Big] \\
&\quad + \mathbb{E}\Big[\Big(\sum_{n=1}^{N} \tau \|\nabla \mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^{\mathfrak{p}}\Big] + \mathbb{E}\Big[\Big(\sum_{n=1}^{N} \tau \|\theta_h^n\|_{L^2(\Gamma)}^2\Big)^{\mathfrak{p}}\Big] \leq C. \qquad (6.1)
\end{aligned}$$

Proof. In a first step, we will establish

$$\begin{aligned}
&\max_{0\leq n\leq N} \mathbb{E}[\|\phi_h^m\|_{H^1(\mathcal{O})}^{2\mathfrak{p}}] + \max_{0\leq n\leq N} \mathbb{E}[\|\phi_h^m\|_{H^1(\Gamma)}^{2\mathfrak{p}}] + \max_{0\leq n\leq N} \mathbb{E}[|r_h^m|^{2\mathfrak{p}}] \\
&\quad + \max_{0\leq n\leq N} \mathbb{E}[|s_h^m|^{2\mathfrak{p}}] \leq C \qquad (6.2)
\end{aligned}$$

before proving estimate (6.1). For fixed $\omega \in \Omega$, we test equation (3.4a) by $\psi_h \equiv \mu_h^n$, equation (3.4b) by $\widehat{\psi}_h \equiv \theta_h^n$, and equation (3.4c) by $\eta_h \equiv \phi_h^n - \phi_h^{n-1}$ and $\eta_h \equiv -\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau$. This provides

$$\begin{aligned}
0 &= \int_{\mathcal{O}} \nabla \phi_h^n \cdot \nabla(\phi_h^n - \phi_h^{n-1}) \,\mathrm{d}x + \int_{\Gamma} \nabla_\Gamma \phi_h^n \cdot \nabla_\Gamma(\phi_h^n - \phi_h^{n-1}) \,\mathrm{d}\Gamma \\
&\quad + \frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\} \,\mathrm{d}x \\
&\quad - \frac{r_h^n}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau\big\} \,\mathrm{d}x \\
&\qquad \times \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\} \,\mathrm{d}x \\
&\quad + \frac{r_h^n}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau(\phi_h^n - \phi_h^{n-1})\big\} \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{s_h^n}{\sqrt{E_h^\Gamma(\phi_h^{n-1})}} \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1}) \big\} \, \mathrm{d}\Gamma \\
&- \frac{s_h^n}{4[E_h^\Gamma(\phi_h^{n-1})]^{3/2}} \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \big\} \, \mathrm{d}\Gamma \\
&\quad \times \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1}) \big\} \, \mathrm{d}\Gamma \\
&+ \frac{s_h^n}{2\sqrt{E_h^\Gamma(\phi_h^{n-1})}} \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma (\phi_h^n - \phi_h^{n-1}) \big\} \, \mathrm{d}\Gamma \\
&- \int_{\mathcal{O}} \nabla \phi_h^n \cdot \nabla (\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau) \, \mathrm{d}x - \int_\Gamma \nabla_\Gamma \phi_h^n \cdot \nabla_\Gamma [\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \, \mathrm{d}\Gamma \\
&- \frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h \big\{ F'(\phi_h^{n-1}) \Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau \big\} \, \mathrm{d}x \\
&+ \frac{r_h^n}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big| \int_{\mathcal{O}} \mathcal{I}_h \big\{ F'(\phi_h^{n-1}) \Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau \big\} \, \mathrm{d}x \Big|^2 \\
&- \frac{r_h^n}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h \big\{ F''(\phi_h^{n-1}) |\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau|^2 \big\} \, \mathrm{d}x \\
&- \frac{s_h^n}{\sqrt{E_h^\Gamma(\phi_h^{n-1})}} \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \big\} \, \mathrm{d}\Gamma \\
&+ \frac{s_h^n}{4[E_h^\Gamma(\phi_h^{n-1})]^{3/2}} \Big| \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \big\} \, \mathrm{d}\Gamma \Big|^2 \\
&- \frac{s_h^n}{2\sqrt{E_h^\Gamma(\phi_h^{n-1})}} \int_\Gamma \mathcal{I}_h^\Gamma \big\{ G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma^2 \big\} \, \mathrm{d}\Gamma \\
&+ \tau \int_{\mathcal{O}} |\nabla \mu_h^n|^2 \, \mathrm{d}x + \tau \int_\Gamma \mathcal{I}_h^\Gamma \big\{ |\theta_h^n|^2 \big\} \, \mathrm{d}\Gamma.
\end{aligned}
$$

Using equations (5.1d) and (5.1e) multiplied by r_h^n and s_h^n, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\nabla \phi_h^n\|_{L^2(\mathcal{O})}^2 + \frac{1}{2}\|\nabla(\phi_h^n - \phi_h^{n-1})\|_{L^2(\mathcal{O})}^2 - \frac{1}{2}\|\nabla \phi_h^{n-1}\|_{L^2(\mathcal{O})}^2 \\
&\quad + \frac{1}{2}\|\nabla_\Gamma \phi_h^n\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|\nabla_\Gamma(\phi_h^n - \phi_h^{n-1})\|_{L^2(\Gamma)}^2 - \frac{1}{2}\|\nabla_\Gamma \phi_h^{n-1}\|_{L^2(\Gamma)}^2 \\
&\quad + |r_h^n|^2 + |r_h^n - r_h^{n-1}|^2 - |r_h^{n-1}|^2 + |s_h^n|^2 + |s_h^n - s_h^{n-1}|^2 - |s_h^{n-1}|^2 \\
&\quad + \tau \|\nabla \mu_h^n\|_{L^2(\mathcal{O})}^2 + \tau \|\theta_h^n\|_{h,\Gamma}^2 \\
&= \int_{\mathcal{O}} \nabla \phi_h^n \cdot \nabla (\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau) \, \mathrm{d}x + \int_\Gamma \nabla_\Gamma \phi_h^n \cdot \nabla_\Gamma [\Phi_h(\phi_h^{n-1}) \blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma \, \mathrm{d}\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x \\
&- \frac{r_h^n}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\Big|^2 \\
&+ \frac{r_h^n}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\phi_h^{n-1})|\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau|^2\big\}\,\mathrm{d}x \\
&+ \frac{s_h^n}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}} \int_{\Gamma} \mathcal{I}_h^\Gamma\big\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&- \frac{s_h^n}{4[E_h^{\Gamma}(\phi_h^{n-1})]^{3/2}} \Big|\int_{\Gamma} \mathcal{I}_h^\Gamma\big\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma\Big|^2 \\
&+ \frac{s_h^n}{2\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}} \int_{\Gamma} \mathcal{I}_h^\Gamma\big\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma^2\big\}\,\mathrm{d}\Gamma \\
&= S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8. \qquad (6.3)
\end{aligned}
$$

As the scalar auxiliary variables r_h^n and s_h^n are merely approximations of positive terms, there is no non-negativity result available. Therefore, the terms S_4 and S_7 can not simply be neglected. Similarly to [44], we apply Young's inequality to separate the implicit terms and the stochastic increments $\blacktriangle^n \boldsymbol{\xi}^\tau$. In particular, we obtain

$$
\begin{aligned}
S_1 \le\, & \frac{1}{4}\|\nabla\phi_h^n - \nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2 + C\|\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau)\|_{L^2(\mathcal{O})}^2 \\
&+ \int_{\mathcal{O}} \nabla\phi_h^{n-1}\cdot\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau)\,\mathrm{d}x, \\
S_2 \le\, & \frac{1}{4}\|\nabla_\Gamma\phi_h^n - \nabla_\Gamma\phi_h^{n-1}\|_{L^2(\Gamma)}^2 + C\|\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma\|_{L^2(\Gamma)} \\
&+ \int_{\Gamma} \nabla_\Gamma\phi_h^{n-1}\cdot\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau]|_\Gamma\,\mathrm{d}\Gamma, \\
S_3 \le\, & \frac{1}{4}|r_h^n - r_h^{n-1}|^2 + C\frac{1}{E_h^{\mathcal{O}}(\phi_h^{n-1})}\Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\Big|^2 \\
&+ \frac{r_h^{n-1}}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x, \\
S_4 \le\, & \frac{1}{4}|r_h^n - r_h^{n-1}|^2 + C\frac{1}{[E_h^{\mathcal{O}}(\phi_h^{n-1})]^3}\Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\Big|^4 \\
&- \frac{r_h^{n-1}}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n \boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\Big|^2,
\end{aligned}
$$

$$S_5 \leq \frac{1}{4}|r_h^n - r_h^{n-1}|^2 + C\frac{1}{E_h^{\mathcal{O}}(\phi_h^{n-1})}\left|\int_{\mathcal{O}} \mathcal{I}_h\{F''(\phi_h^{n-1})|\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau|^2\}\,\mathrm{d}x\right|^2 + \frac{r_h^{n-1}}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}} \mathcal{I}_h\{F''(\phi_h^{n-1})|\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau|^2\}\,\mathrm{d}x,$$

$$S_6 \leq \frac{1}{4}|s_h^n - s_h^{n-1}|^2 + C\frac{1}{E_h^{\Gamma}(\phi_h^{n-1})}\left|\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\right|^2 + \frac{s_h^{n-1}}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma,$$

$$S_7 \leq \frac{1}{4}|s_h^n - s_h^{n-1}|^2 + C\frac{1}{[E_h^{\Gamma}(\phi_h^{n-1})]^2}\left|\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\right|^4 - \frac{s_h^{n-1}}{4[E_h^{\Gamma}(\phi_h^{n-1})]^{3/2}}\left|\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\right|^2,$$

$$S_8 \leq \frac{1}{4}|s_h^n - s_h^{n-1}|^2 + C\frac{1}{E_h^{\Gamma}(\phi_h^{n-1})}\left|\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma^2\}\,\mathrm{d}\Gamma\right|^2 + \frac{s_h^{n-1}}{2\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma} \mathcal{I}_h^{\Gamma}\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma^2\}\,\mathrm{d}\Gamma.$$

As we allow the stochastic term in equation (3.4a) to act as a source or sink term, controlling the $H^1(\mathcal{O})$-seminorm of the phase-field parameter is insufficient. In order to obtain the full H^1-norms on the left-hand side of equation (6.3), we choose $\psi_h \equiv \phi_h^n$ in equation (3.4a) and $\widehat{\psi}_h \equiv [\phi_h^n]|_\Gamma$ in equation (3.4b). After applying Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\phi_h^n\|_{h,\mathcal{O}}^2 + \frac{1}{2}\|\phi_h^n - \phi_h^{n-1}\|_{h,\mathcal{O}}^2 - \frac{1}{2}\|\phi_h^{n-1}\|_{h,\mathcal{O}}^2 + \frac{1}{2}\|\phi_h^n\|_{h,\Gamma}^2 \\
&\quad + \frac{1}{2}\|\phi_h^n - \phi_h^{n-1}\|_{h,\Gamma}^2 - \frac{1}{2}\|\phi_h^{n-1}\|_{h,\Gamma}^2 \\
&\leq \tau\frac{3}{4}\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2 + \tau\frac{1}{3}\|\nabla\phi_h^n\|_{L^2(\mathcal{O})}^2 + \frac{1}{4}\|\phi_h^n - \phi_h^{n-1}\|_{h,\mathcal{O}}^2 + \|\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau\|_{h,\mathcal{O}}^2 \\
&\quad + \int_{\mathcal{O}} \mathcal{I}_h\{\phi_h^{n-1}\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau\}\,\mathrm{d}x + \tau\frac{3}{4}\|\theta_h^n\|_{h,\Gamma}^2 + \tau\frac{1}{3}\|\phi_h^n\|_{h,\Gamma}^2 \\
&\quad + \frac{1}{4}\|\phi_h^n - \phi_h^{n-1}\|_{h,\Gamma}^2 + \|[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\|_{h,\Gamma}^2 \\
&\quad + \int_{\Gamma} \mathcal{I}_h^{\Gamma}\{\phi_h^{n-1}[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma.
\end{aligned}$$

Combining the above equations and summing from $n = 1$ to m, we obtain

$$\frac{1}{2}\|\phi_h^m\|_{H_h^1(\mathcal{O})}^2 + \frac{1}{2}\|\phi_h^m\|_{H_h^1(\Gamma)}^2 + |r_h^m|^2 + |s_h^m|^2 + \frac{1}{4}\sum_{n=1}^{m}\|\phi_h^n - \phi_h^{n-1}\|_{H_h^1(\mathcal{O})}^2$$

$$
\begin{aligned}
&+ \frac{1}{4}\sum_{n=1}^{m}\|\phi_h^n - \phi_h^{n-1}\|_{H_h^1(\Gamma)}^2 + \frac{1}{4}\sum_{n=1}^{m}|r_h^n - r_h^{n-1}|^2 + \frac{1}{4}\sum_{n=1}^{m}|s_h^n - s_h^{n-1}|^2 \\
&+ \frac{1}{4}\tau\sum_{n=1}^{m}\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2 + \frac{1}{4}\tau\sum_{n=1}^{m}\|\theta_h^n\|_{h,\Gamma}^2 \\
\leq\ & \frac{1}{2}\|\phi_h^0\|_{H_h^1(\mathcal{O})}^2 + \frac{1}{2}\|\phi_h^0\|_{H_h^1(\Gamma)}^2 + |r_h^0|^2 + |s_h^0|^2 + \frac{1}{3}\tau\sum_{n=1}^{m}\|\nabla\phi_h^n\|_{L^2(\mathcal{O})}^2 + \frac{1}{3}\tau\sum_{n=1}^{m}\|\phi_h^n\|_{h,\Gamma}^2 \\
&+ C\sum_{n=1}^{m}\|\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau)\|_{L^2(\mathcal{O})}^2 + \sum_{n=1}^{m}\int_{\mathcal{O}}\nabla\phi_h^{n-1}\cdot\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau)\,\mathrm{d}x \\
&+ C\sum_{n=1}^{m}\frac{1}{E_h^{\mathcal{O}}(\phi_h^{n-1})}\left|\int_{\mathcal{O}}\mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\right|^2 \\
&+ \sum_{n=1}^{m}\frac{r_h^{n-1}}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x \\
&+ C\sum_{n=1}^{m}\frac{1}{[E_h^{\mathcal{O}}(\phi_h^{n-1})]^3}\left|\int_{\mathcal{O}}\mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\right|^4 \\
&- \sum_{n=1}^{m}\frac{r_h^{n-1}}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}\left|\int_{\mathcal{O}}\mathcal{I}_h\big\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau\big\}\,\mathrm{d}x\right|^2 \\
&+ C\sum_{n=1}^{m}\frac{1}{E_h^{\mathcal{O}}(\phi_h^{n-1})}\left|\int_{\mathcal{O}}\mathcal{I}_h\big\{F''(\phi_h^{n-1})|\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau|^2\big\}\,\mathrm{d}x\right|^2 \\
&+ \sum_{n=1}^{m}\frac{r_h^{n-1}}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\big\{F''(\phi_h^{n-1})|\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau|^2\big\}\,\mathrm{d}x \\
&+ C\sum_{n=1}^{m}\|\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau]|_\Gamma\|_{L^2(\Gamma)}^2 \\
&+ \sum_{n=1}^{m}\int_\Gamma\nabla_\Gamma\phi_h^{n-1}\cdot\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau]|_\Gamma\,\mathrm{d}\Gamma \\
&+ C\sum_{n=1}^{m}\frac{1}{E_h^\Gamma(\phi_h^{n-1})}\left|\int_\Gamma\mathcal{I}_h^\Gamma\big\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma\right|^2 \\
&+ \sum_{n=1}^{m}\frac{s_h^{n-1}}{\sqrt{E_h^\Gamma(\phi_h^{n-1})}}\int_\Gamma\mathcal{I}_h^\Gamma\big\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma \\
&+ C\sum_{n=1}^{m}\frac{1}{[E_h^\Gamma(\phi_h^{n-1})]^3}\left|\int_\Gamma\mathcal{I}_h^\Gamma\big\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau]|_\Gamma\big\}\,\mathrm{d}\Gamma\right|^4
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{m} \frac{s_h^{n-1}}{4[E_h^\Gamma(\phi_h^{n-1})]^{3/2}} \Big| \int_\Gamma \mathcal{I}_h^\Gamma \{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma\} \,\mathrm{d}\Gamma \Big|^2 \\
&+ C \sum_{n=1}^{m} \frac{1}{E_h^\Gamma(\phi_h^{n-1})} \Big| \int_\Gamma \mathcal{I}_h^\Gamma \{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma^2\} \,\mathrm{d}\Gamma \Big|^2 \\
&+ \sum_{n=1}^{m} \frac{s_h^{n-1}}{2\sqrt{E_h^\Gamma(\phi_h^{n-1})}} \int_\Gamma \mathcal{I}_h^\Gamma \{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau]|_\Gamma^2\} \,\mathrm{d}\Gamma \\
=: &\frac{1}{2}\|\phi_h^0\|_{H_h^1(\mathcal{O})}^2 + \frac{1}{2}\|\phi_h^0\|_{H_h^1(\Gamma)}^2 + |r_h^0|^2 + |s_h^0|^2 + \frac{1}{3}\tau\sum_{n=1}^{m}\|\nabla\phi_h^n\|_{L^2(\mathcal{O})}^2 + \frac{1}{3}\tau\sum_{n=1}^{m}\|\phi_h^n\|_{h,\Gamma}^2 \\
&+ \sum_{\alpha=1}^{16} R_{\alpha,m}.
\end{aligned}
$$

Absorbing $\tau\frac{1}{3}\|\nabla\phi_h^m\|_{L^2(\mathcal{O})}^2$ and $\tau\frac{1}{3}\|\phi_h^m\|_{h,\Gamma}^2$ on the left-hand side, taking the $\mathfrak{p}$-th power, and using the equivalence between the norms defined in equation (2.3) and their standard counterparts, we obtain for any $m \in \{0, \dots, N\}$ the estimate

$$
\begin{aligned}
&\|\phi_h^m\|_{H^1(\mathcal{O})}^{2\mathfrak{p}} + \|\phi_h^m\|_{H^1(\Gamma)}^{2\mathfrak{p}} + |r_h^m|^{2\mathfrak{p}} + |s_h^m|^{2\mathfrak{p}} + \Big(\sum_{n=1}^{m}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^2\Big)^{\mathfrak{p}} \\
&\quad + \Big(\sum_{n=1}^{m}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^2\Big)^{\mathfrak{p}} + \Big(\sum_{n=1}^{m}|r_h^n - r_h^{n-1}|^2\Big)^{\mathfrak{p}} + \Big(\sum_{n=1}^{m}|s_h^n - s_h^{n-1}|^2\Big)^{\mathfrak{p}} \\
&\quad + \Big(\tau\sum_{n=1}^{m}\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^{\mathfrak{p}} + \Big(\tau\sum_{n=1}^{m}\|\theta_h^n\|_{L^2(\Gamma)}^2\Big)^{\mathfrak{p}} \\
&\le C\|\phi_h^0\|_{H^1(\mathcal{O})}^{2\mathfrak{p}} + C\|\phi_h^0\|_{H^1(\Gamma)}^{2\mathfrak{p}} + C|r_h^0|^{2\mathfrak{p}} + C|s_h^0|^{2\mathfrak{p}} + C\sum_{n=1}^{m}\tau\|\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\mathfrak{p}} \\
&\quad + C\sum_{n=1}^{m}\tau\|\phi_h^{n-1}\|_{L^2(\Gamma)}^{2\mathfrak{p}} + C\sum_{\alpha=1}^{16}|R_{\alpha,m}|^{\mathfrak{p}},
\end{aligned}
\tag{6.4}
$$

where the constant C depends on $\mathfrak{p}$ but not on h or τ. We will now follow the lines of [44] to derive estimates for the expected values of the stochastic terms $|R_{1,m}|^{\mathfrak{p}}, \dots, |R_{16,m}|^{\mathfrak{p}}$. We start by deducing an estimate for $\mathbb{E}[|R_{1,m}|^{\mathfrak{p}}]$. Applying Hölder's inequality, Lemma 2.4, inequality (2.2), and assumptions **(D3)** and **(C)**, we compute

$$
\begin{aligned}
\mathbb{E}\big[|R_{1,m}|^{\mathfrak{p}}\big] &\le C m^{\mathfrak{p}-1}\sum_{n=1}^{m}\mathbb{E}\big[\|\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau)\|_{L^2(\mathcal{O})}^{2\mathfrak{p}}\big] \\
&\le C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\|\lambda_k\nabla(\Phi_h(\phi_h^{n-1})\mathfrak{g}_k)\|_{L^2(\mathcal{O})}^2\Big)^{\mathfrak{p}}\Big]
\end{aligned}
$$

$$\leq C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})\}\|_{H^1(\mathcal{O})}^{2\mathfrak{p}}\Big(\sum_{k\in\mathbb{Z}}\lambda_k^2\|\mathfrak{g}_k\|_{W^{1,\infty}(\mathcal{O})}^2\Big)^{\mathfrak{p}}\Big]$$
$$\leq C\sum_{n=1}^{m}\tau\mathbb{E}\big[(1+\|\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\mathfrak{p}})\big]. \tag{6.5}$$

For $R_{2,m}$, we obtain from Lemma 2.4

$$\begin{aligned}\mathbb{E}\big[|R_{2,m}|^{\mathfrak{p}}\big] &\leq \mathbb{E}\Big[\Big(\max_{1\leq l\leq m}\Big|\sum_{n=1}^{l}\int_{\mathcal{O}}\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau)\cdot\nabla\phi_h^{n-1}\,\mathrm{d}x\Big|\Big)^{\mathfrak{p}}\Big]\\
&\leq C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\lambda_k^2\Big|\int_{\mathcal{O}}\nabla\mathcal{I}_h\{\varrho(\phi_h^{n-1})\mathfrak{g}_k\}\cdot\nabla\phi_h^{n-1}\,\mathrm{d}x\Big|^2\Big)^{\mathfrak{p}/2}\Big]\\
&\leq C\sum_{n=1}^{m}\tau\mathbb{E}\big[\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})\}\|_{H^1(\mathcal{O})}^{\mathfrak{p}}\|\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{\mathfrak{p}}\big]\\
&\leq C+C\mathbb{E}\Big[\sum_{n=1}^{m}\tau\|\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\mathfrak{p}}\Big].\end{aligned}$$

Using similar arguments, we obtain for $R_{3,m}$

$$\mathbb{E}\big[|R_{3,m}|^{\mathfrak{p}}\big]\leq C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\frac{1}{[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{\mathfrak{p}}}\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})F'(\phi_h^{n-1})\}\|_{L^1(\mathcal{O})}^{2\mathfrak{p}}\Big].$$

As the left-hand side of estimate (6.4) only includes $2\mathfrak{p}$-th powers of ϕ_h^m, it will not be possible to control $\|F'(\phi_h^{n-1})\|_{L^1(\mathcal{O})}^{2\mathfrak{p}}$. We therefore follow the approach used in [44] (see also [35]) and use the negative powers of $E_h^{\mathcal{O}}(\phi_h^{n-1})$ to our advantage. As ϱ is bounded, we can use the growth condition stated in assumption **(P)**, the norm equivalence in inequality (2.2), and Hölder's inequality to obtain

$$\big\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})F'(\phi_h^{n-1})\}\big\|_{L^1(\mathcal{O})}^2\leq C+C\|\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2\int_{\mathcal{O}}\mathcal{I}_h\{F(\phi_h^{n-1})\}\,\mathrm{d}x. \tag{6.6}$$

Hence, the lower bound on F provides

$$\mathbb{E}\big[|R_{3,m}|^{\mathfrak{p}}\big]\leq C+C\sum_{n=1}^{m}\tau\mathbb{E}\big[\|\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\mathfrak{p}}\big].$$

Reusing inequality (6.6), we estimate $\mathbb{E}[|R_{4,m}|^{\mathfrak{p}}]$ via

$$\mathbb{E}\big[|R_{4,m}|^{\mathfrak{p}}\big]\leq C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\Big(\frac{|r_h^{n-1}|}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\big\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})F'(\phi_h^{n-1})\}\big\|_{L^1(\mathcal{O})}\Big)^{\mathfrak{p}}\Big]$$

$$\leq C + C\sum_{n=1}^{m}\tau\mathbb{E}\big[|r_h^{n-1}|^{2\mathfrak{p}}\big] + C\sum_{n=1}^{m}\tau\mathbb{E}\big[\|\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\mathfrak{p}}\big].$$

Similar to the lines of the proof of [44, Lemma 6.1], we obtain

$$\begin{aligned}
\mathbb{E}[|R_{5,m}|^{\mathfrak{p}}] &\leq C\tau^{\mathfrak{p}},\\
\mathbb{E}[|R_{6,m}|^{\mathfrak{p}}] &\leq C\sum_{n=1}^{m}\tau\mathbb{E}\big[|r_h^{n-1}|^{2\mathfrak{p}}\big] + C,\\
\mathbb{E}[|R_{7,m}|^{\mathfrak{p}}] &\leq C\tau^{\mathfrak{p}},\\
\mathbb{E}[|R_{8,m}|^{\mathfrak{p}}] &\leq C\sum_{n=1}^{m}\tau\mathbb{E}\big[|r_h^{n-1}|^{2\mathfrak{p}}\big] + C.
\end{aligned}$$

The remaining terms can be estimated analogously using estimate (2.10). In particular, we interpret $\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\cdot]|_\Gamma$ as a Hilbert–Schmidt operator onto $(L^2(\Gamma))^{d-1}$ and apply Lemma 2.4 to obtain

$$\begin{aligned}
\mathbb{E}\big[|R_9, m|^{\mathfrak{p}}\big] &\leq Cm^{\mathfrak{p}-1}\sum_{n=1}^{m}\mathbb{E}\Big[\big\|\nabla_\Gamma[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\big\|_{L^2(\Gamma)}^{2\mathfrak{p}}\Big]\\
&\leq C\sum_{n=1}^{m}\tau\mathbb{E}\Big[\big\|\mathcal{I}_h^\Gamma\{\varrho(\phi_h^{n-1})\}\big\|_{H^1(\Gamma)}^{2\mathfrak{p}}\Big(\sum_{k\in\mathbb{Z}}\lambda_k^2\|\mathfrak{g}_k\|_{W^{1,\infty}(\Gamma)}\Big)^{\mathfrak{p}}\Big]\\
&\leq C\sum_{n=1}^{m}\tau\mathbb{E}\big[(1+\|\nabla_\Gamma\phi_h^{n-1}\|_{L^2(\Gamma)}^{2\mathfrak{p}})\big].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\mathbb{E}\big[|R_{10,m}|^{\mathfrak{p}}\big] &\leq C + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau\|\nabla_\Gamma\phi_h^{n-1}\|_{L^2(\Gamma)}^{2\mathfrak{p}}\Big],\\
\mathbb{E}\big[|R_{11,m}|^{\mathfrak{p}}\big] &\leq C + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau\|\phi_h^{n-1}\|_{L^2(\Gamma)}^{2\mathfrak{p}}\Big],\\
\mathbb{E}\big[|R_{12,m}|^{\mathfrak{p}}\big] &\leq C + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau|s_h^{n-1}|^{2\mathfrak{p}}\Big] + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau\|\phi_h^{n-1}\|_{L^2(\Gamma)}^{2\mathfrak{p}}\Big],\\
\mathbb{E}\big[|R_{13,m}|^{\mathfrak{p}}\big] &\leq C\tau^{\mathfrak{p}},\\
\mathbb{E}\big[|R_{14,m}|^{\mathfrak{p}}\big] &\leq C + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau|s_h^{n-1}|^{2\mathfrak{p}}\Big],\\
\mathbb{E}\big[|R_{15,m}|^{\mathfrak{p}}\big] &\leq C\tau^{\mathfrak{p}},\\
\mathbb{E}\big[|R_{16,m}|^{\mathfrak{p}}\big] &\leq C + C\mathbb{E}\Big[\sum_{n=1}^{m}\tau|s_h^{n-1}|^{2\mathfrak{p}}\Big].
\end{aligned}$$

Hence, we obtain from estimate (6.4) after neglecting non-negative terms

$$\begin{aligned}
&\mathbb{E}\big[\|\phi_h^m\|_{H^1(\mathcal{O})}^{2p}\big] + \mathbb{E}\big[\|\phi_h^m\|_{H^1(\Gamma)}^{2\mathfrak{p}}\big] + \mathbb{E}\big[|r_h^m|^{2\mathfrak{p}}\big] + \mathbb{E}\big[|s_h^m|^{2\mathfrak{p}}\big] \\
&\quad \leq C\|\phi_h^0\|_{H^1(\mathcal{O})}^{2\mathfrak{p}} + C\|\phi_h^0\|_{H^1(\Gamma)}^{2\mathfrak{p}} + C|r_h^0|^{2\mathfrak{p}} + C|s_h^0|^{2\mathfrak{p}} + C\sum_{n=1}^{m}\tau\mathbb{E}\big[\|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{2\mathfrak{p}}\big] \\
&\qquad + C\sum_{n=1}^{m}\tau\mathbb{E}\big[\|\phi_h^{n-1}\|_{H^1(\Gamma)}^{2\mathfrak{p}}\big] + C + C\sum_{n=1}^{m}\tau\mathbb{E}\big[|r_h^{n-1}|^{2\mathfrak{p}}\big] \\
&\qquad + C\sum_{n=1}^{m}\tau\mathbb{E}\big[|s_h^{n-1}|^{2\mathfrak{p}}\big]
\end{aligned}$$

for all $m \in \{1,\dots,N\}$. As assumption (I) allows us to control the first four terms on the right-hand side independently of the discretization parameters, we can apply a discrete version of Gronwall's inequality and obtain estimate (6.2). In the second step, we shall now establish estimate (6.1) for $\mathfrak{p} \in 2\mathbb{N}$. Starting from estimate (6.4), we can take the maximum over $m \in \{1,\dots,N\}$ before taking the expected value. Reusing the above calculations, in particular estimate (6.2), provides the claim. ∎

In the next step, we shall analyze equation (3.4c) in more detail and verify that the additional terms $\Xi_{h,\mathcal{O}}^n$ and $\Xi_{h,\Gamma}^n$ introduced in the approximation of $F'(\phi)$ and $G'(\phi)$ (cf. equation (3.5)) will vanish for $\tau \searrow 0$.

Lemma 6.2. *Let the assumptions* (T), (S1), (S2), (P), (I), (C), *and* (D0)–(D3) *hold true. Then, for every* $1 \leq q < \infty$, *there exists a positive constant* C *independent of* h *and* τ *such that*

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N}\tau\|\Xi_{h,\mathcal{O}}^n\|_{h,\mathcal{O}}^2\Big)^q\Big] \leq C\tau^q, \tag{6.7a}$$

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N}\tau\|\Xi_{h,\Gamma}^n\|_{h,\mathcal{O}}^2\Big)^q\Big] \leq C\tau^q. \tag{6.7b}$$

Furthermore, the following estimate on the $L^2(\mathcal{O})$*-norm holds true:*

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N}\tau\|\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^q\Big] \leq C. \tag{6.8}$$

Proof. We start by establishing estimate (6.7a). Discussing both summands in the definition of $\Xi_{h,\mathcal{O}}^n$ separately, we obtain by using Hölder's inequality, the lower bound on $E_h^{\mathcal{O}}(\cdot)$, and the growth estimate for F'

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N}\tau\frac{|r_h^n|^2}{[E_h^{\mathcal{O}}(\phi_h^{n-1})]^3}\Big|\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\Big|^2\|\mathcal{I}_h\{F'(\phi_h^{n-1})\}\|_{h,\mathcal{O}}^2\Big)^q\Big]$$

$$\leq C\mathbb{E}\Big[\Big(\max_{1\leq n\leq N}|r_h^n|^2\big(1+\max_{1\leq n\leq N}\|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^6\big)\Big)^{2q}\Big]^{1/2}$$
$$\times\,\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\Big|^2\Big)^{2q}\Big]^{1/2}.$$

Here, the first factor on the right-hand side can immediately be controlled using Lemma 6.1. To control the second factor, we apply Lemma 2.4, the definition in (2.11), and assumptions **(C)** and **(D3)** to obtain

$$\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\Big|^2\Big)^{2q}\Big]$$
$$\leq C\mathbb{E}\Big[\sum_{n=1}^N\tau\Big|\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\Big|^{4q}\Big]$$
$$\leq C\tau^{2q}\sum_{n=1}^N\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\Big|\lambda_k\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\varrho(\phi_h^{n-1})\mathfrak{g}_k\}\,\mathrm{d}x\Big|^2\Big)^{2q}\Big]$$
$$\leq C\tau^{2q}\sum_{n=1}^N\tau\mathbb{E}\Big[\big(1+\|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{12q}\big)\Big(\sum_{k\in\mathbb{Z}_h}\lambda_k^2\|\mathfrak{g}_k\|_{L^\infty(\mathcal{O})}^2\Big)^{2q}\Big]\leq C\tau^{2q}.$$

The estimate for the second summand in equation (3.5a) can be deduced using similar considerations:

$$\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\frac{|r_h^n|^2}{E_h^{\mathcal{O}}(\phi_h^{n-1})}\big\|\mathcal{I}_h\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\big\|_{h,\mathcal{O}}^2\Big)^q\Big]$$
$$\leq C\mathbb{E}\Big[\max_{1\leq n\leq N}|r_h^n|^{4q}\Big]^{1/2}\mathbb{E}\Big[\sum_{n=1}^N\tau\big\|\mathcal{I}_h\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\big\|_{h,\mathcal{O}}^{4q}\Big]^{1/2}$$
$$\leq C\Big(\sum_{n=1}^N\tau\mathbb{E}\Big[\big(\tau\sum_{k\in\mathbb{Z}_h}\big\|\mathcal{I}_h\{F''(\phi_h^{n-1})\varrho(\phi_h^{n-1})\lambda_k\mathfrak{g}_k\}\big\|_{h,\mathcal{O}}^2\big)^{2q}\Big]\Big)^{1/2}\leq C\tau^q,$$

due to Hölder's inequality, Lemmas 2.4 and 6.1, and assumptions **(C)**, **(D3)**, and **(P)**. Estimate (6.7b) on $\Xi_{h,\Gamma}^n$ can be proven analogously by applying Lemma 2.4 to the operators $\mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\cdot]|_\Gamma\}$ and $\mathcal{I}_h^\Gamma\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\cdot]|_\Gamma\}$ and recalling inequality (2.10).

Choosing $\eta_h\equiv 1$ in equation (3.4c), we obtain

$$\Big|\int_{\mathcal{O}}\mu_h^n\,\mathrm{d}x\Big|\leq\Big|\int_\Gamma\theta_h^n\,\mathrm{d}\Gamma\Big|+\Big|\frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\}\,\mathrm{d}x\Big|$$
$$+\Big|\frac{r_h^n}{4[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\}\,\mathrm{d}x\Big|$$

$$
\begin{aligned}
&+\Big|\frac{r_h^n}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\{F''(\phi_h^{n-1})\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\,\mathrm{d}x\Big|\\
&+\Big|\frac{s_h^n}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big|\\
&+\Big|\frac{s_h^n}{4[E_h^{\Gamma}(\phi_h^{n-1})]^{3/2}}\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big|\\
&+\Big|\frac{s_h^n}{2\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G''(\phi_h^{n-1})[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\,\mathrm{d}\Gamma\Big|.
\end{aligned}
$$

Due to the above estimates, taking the second power on both sides, summing from $n=1$ to N, taking the q-th power, and taking the expected value provides

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\int_{\mathcal{O}}\mu_h^n\,\mathrm{d}x\Big|^2\Big)^q\Big]&\le C\,\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\|\theta_h^n\|_{L^2(\Gamma)}^2\Big)^q\Big]\\
&+C\,\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\}\,\mathrm{d}x\Big|^2\Big)^q\Big]\\
&+C\,\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\frac{s_h^n}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big|^2\Big)^q\Big]+C\tau^q.
\end{aligned}
$$

Using the growth conditions for F' and G' stated in assumption **(P)** and the regularity results established in Lemma 6.1, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\frac{r_h^n}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\{F'(\phi_h^{n-1})\}\,\mathrm{d}x\Big|^2\Big)^q\Big]\\
&\qquad\le C\,\mathbb{E}\Big[\max_{1\le n\le N}|r_h^n|^{2q}\big(1+\|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{6q}\big)\Big]\le C,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\Big|\frac{s_h^n}{\sqrt{E_h^{\Gamma}(\phi_h^{n-1})}}\int_{\Gamma}\mathcal{I}_h^{\Gamma}\{G'(\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big|^2\Big)^q\Big]\\
&\qquad\le C\,\mathbb{E}\Big[\max_{1\le n\le N}|s_h^n|^{2q}\big(1+\|\phi_h^{n-1}\|_{H^1(\Gamma)}^{6q}\big)\Big]\le C.
\end{aligned}
$$

Therefore, we can apply Poincaré's inequality to deduce estimate (6.8). ∎

Lemma 6.3. *Let the assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)**–**(D3)** *hold true. Then for all* $\alpha, p\ge 1$ *and* $1\le\beta\le 2$, *there exists a constant* $C>0$ *that is independent* *of h and τ such that*

$$\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \|\phi_h^{m+l} - \phi_h^m\|_{L^2(\mathcal{O})}^{2\alpha}\Big] \le C(l\tau)^{\alpha/2}, \tag{6.9a}$$

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N} \tau \|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2\beta}\Big)^p\Big] \le C\tau^{\beta p}, \tag{6.9b}$$

$$\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \|\phi_h^{m+l} - \phi_h^m\|_{L^2(\Gamma)}^{2\alpha}\Big] \le C(l\tau)^{\alpha} \tag{6.9c}$$

for all $l = 0, \dots, N$.

Proof. Summing equation (3.4a) from $n = m+1$ to $m + l \le N$, choosing $\psi_h = \phi_h^{m+l} - \phi_h^m$, taking the α-th power on both sides, multiplying by τ, summing the result from $m = 0$ to $N - l$, and computing the expected value, we obtain

$$\begin{aligned}\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \|\phi_h^{m+l} - \phi_h^m\|_{h,\mathcal{O}}^{2\alpha}\Big] &\le C\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \Big| \sum_{n=m+1}^{m+l} \tau \int_{\mathcal{O}} \nabla\mu_h^n \cdot \nabla(\phi_h^{m+l} - \phi_h^m)\,\mathrm{d}x\Big|^{\alpha}\Big] \\ &\quad + C\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \Big|\int_{\mathcal{O}} \sum_{n=m+1}^{m+l} \mathcal{I}_h\{\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau (\phi_h^{m+l} - \phi_h^m)\}\,\mathrm{d}x\Big|^{\alpha}\Big] \\ &=: A + B.\end{aligned} \tag{6.10}$$

Using Hölder's inequality multiple times and recalling the already established regularity results from Lemma 6.1, the first term can be controlled via

$$\begin{aligned}A &\le C\mathbb{E}\Big[\max_{0\le n\le N} \|\nabla\phi_h^n\|_{L^2(\mathcal{O})}^{\alpha} (l\tau)^{\alpha/2} \sum_{m=0}^{N-l} \tau \Big(\sum_{n=m+1}^{m+l} \tau \|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^{\alpha/2}\Big] \\ &\le C(l\tau)^{\alpha/2}.\end{aligned}$$

As shown in [44, Lemma 6.3], the second term is bounded by

$$\begin{aligned}B &\le \frac{1}{4}\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \|\phi_h^{m+l} - \phi_h^m\|_{h,\mathcal{O}}^{2\alpha}\Big] + C\sum_{m=0}^{N-l} \tau \mathbb{E}\Big[\Big\| \sum_{n=m+1}^{m+l} \mathcal{I}_h\{\Phi_h(\phi_h^{n-1})\blacktriangle^n \xi^\tau\}\Big\|_{h,\mathcal{O}}^{2\alpha}\Big] \\ &\le \frac{1}{4}\mathbb{E}\Big[\sum_{m=0}^{N-l} \tau \|\phi_h^{m+l} - \phi_h^m\|_{h,\mathcal{O}}^{2\alpha}\Big] \\ &\quad + C\sum_{m=0}^{N-l} \tau (l\tau)^{\alpha-1} \sum_{n=m+1}^{m+l} \tau \mathbb{E}\big[\|\mathcal{I}_h\{\varrho(\phi_h^{n-1})\}\|_{L^2(\mathcal{O})}^{2\alpha}\big]\end{aligned} \tag{6.11}$$

due to Young's inequality and Lemma 2.4. As $\varrho \in L^\infty(\mathbb{R})$, the second term can be controlled by $C(l\tau)^\alpha$. By applying norm equivalence (2.2), we obtain estimate (6.9a).
To prove estimate (6.9b), we set $\alpha = 2$ and $l = 1$ and obtain analogously to equation (6.10)

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau \|\phi_h^n - \phi_h^{n-1}\|_{h,\mathcal{O}}^4\Big)^p\Big] &\le C\,\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\Big|\tau\int_{\mathcal{O}} \nabla\mu_h^n \cdot \nabla(\phi_h^n - \phi_h^{n-1})\,\mathrm{d}x\Big|^2\Big)^p\Big] \\
&\quad + C\,\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\Big|\int_{\mathcal{O}} \mathcal{I}_h\{\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau(\phi_h^n - \phi_h^{n-1})\}\,\mathrm{d}x\Big|^2\Big)^p\Big] \\
&=: \widehat{A} + \widehat{B}.
\end{aligned}$$

Applying Hölder's inequality and the regularity results established in Lemma 6.1, we compute

$$\begin{aligned}
\widehat{A} &\le C\tau^{2p}\mathbb{E}\Big[\max_{1\le n\le N}\|\nabla\phi_h^n - \nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^{2p}\Big(\sum_{n=1}^N \tau\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^p\Big] \\
&\le C\tau^{2p}\mathbb{E}\Big[\Big(\sum_{n=1}^N \|\nabla\phi_h^n - \nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2\Big)^{2p}\Big]^{1/2}\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^{2p}\Big]^{1/2} \\
&\le C\tau^{2p}.
\end{aligned}$$

To obtain an estimate for $\widehat{B}$, we adapt estimate (6.11) and combine Young's inequality, Hölder's inequality, and Lemma 2.4 to obtain

$$\begin{aligned}
\widehat{B} &\le \frac{1}{4}\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\|\phi_h^n - \phi_h^{n-1}\|_{h,\mathcal{O}}^4\Big)^p\Big] + C\sum_{n=1}^N \tau\mathbb{E}\Big[\|\mathcal{I}_h\{\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau\}\|_{h,\mathcal{O}}^{4p}\Big] \\
&\le \frac{1}{4}\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\|\phi_h^n - \phi_h^{n-1}\|_{h,\mathcal{O}}^4\Big)^p\Big] + C\tau^{2p}.
\end{aligned}$$

Estimate (6.9b) for arbitrary $1 \le \beta \le 2$ then follows by equation (2.2) and Hölder's inequality.

The remaining estimate in (6.9c) can be shown in a similar manner using

$$\begin{aligned}
&C\,\mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\Big|\sum_{n=m+1}^{m+l}\tau\int_\Gamma \mathcal{I}_h^\Gamma\{\theta_h^n(\phi_h^{m+l} - \phi_h^m)\}\,\mathrm{d}\Gamma\Big|^\alpha\Big] \\
&\quad\le \frac{1}{4}\mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\|\phi_h^{m+l} - \phi_h^m\|_{h,\Gamma}^{2\alpha}\Big] + C(l\tau)^\alpha\mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\Big(\sum_{n=m+1}^{m+l}\tau\|\theta_h^n\|_{L^2(\Gamma)}^2\Big)^\alpha\Big] \\
&\quad\le \frac{1}{4}\mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\|\phi_h^{m+l} - \phi_h^m\|_{h,\Gamma}^{2\alpha}\Big] + C(l\tau)^\alpha
\end{aligned}$$

and

$$C\sum_{m=0}^{N-l}\tau\mathbb{E}\Big[\Big\|\sum_{n=m+1}^{m+l}\mathcal{I}_h^\Gamma\{[\Phi_h(\phi_h^{n-1})\blacktriangle^n\xi^\tau]|_\Gamma\}\Big\|_{h,\Gamma}^{2\alpha}\Big]$$
$$\le C\sum_{m=0}^{N-l}\tau(l\tau)^{\alpha-1}\sum_{n=m+1}^{m+l}\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}}\lambda_k^2\|\mathfrak{g}_k\|_{L^\infty(\Gamma)}^2\|\mathcal{I}_h^\Gamma\{\varrho(\phi_h^{n-1})\}\|_{L^2(\Gamma)}^2\Big)^\alpha\Big]$$
$$\le C(l\tau)^\alpha. \qquad \blacksquare$$

Next, we shall derive an estimate for the error introduced by the scalar auxiliary variables.

Lemma 6.4. *Let assumptions (T), (S1), (S2), (P), (I), (C), and (D0)–(D3) hold true. Then, for all $p\in[1,\infty)$, there exists a constant C that is independent of h and τ such that the estimates*

$$\mathbb{E}\Big[\max_{0\le m\le N}\Big|r_h^m-\sqrt{E_h^{\mathcal{O}}(\phi_h^m)}\Big|^p\Big]\le C\tau^{p\max\{\nu/2,\,1/8\}}, \tag{6.12a}$$

$$\mathbb{E}\Big[\max_{0\le m\le N}\Big|s_h^m-\sqrt{E_h^{\Gamma}(\phi_h^m)}\Big|^p\Big]\le C\tau^{\nu p/2}, \tag{6.12b}$$

hold true.

Proof. Adapting the ideas in [44, Lemma 6.5], we start with a Taylor expansion of $\sqrt{E_h^{\mathcal{O}}(\phi_h^n)}$ to quantify the approximation error for one time step, that is, the difference between the increments $r_h^n-r_h^{n-1}$ and $\sqrt{E_h^{\mathcal{O}}(\phi_h^n)}-\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}$:

$$\begin{aligned}
&\sqrt{E_h^{\mathcal{O}}(\phi_h^n)}-\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}\\
&\quad=\frac{1}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}(E_h^{\mathcal{O}}(\phi_h^n)-E_h^{\mathcal{O}}(\phi_h^{n-1}))-\frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}(E_h^{\mathcal{O}}(\phi_h^n)-E_h^{\mathcal{O}}(\phi_h^{n-1}))^2\\
&\qquad+\frac{1}{16[E_h^{\mathcal{O}}(\varphi_1)]^{5/2}}(E_h^{\mathcal{O}}(\phi_h^n)-E_h^{\mathcal{O}}(\phi_h^{n-1}))^3\\
&\quad=\frac{1}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\Big(\int_{\mathcal{O}}\mathcal{I}_h\Big\{F'(\phi_h^{n-1})(\phi_h^n-\phi_h^{n-1})+\frac{1}{2}F''(\phi_h^{n-1})(\phi_h^n-\phi^{n-1})^2\Big\}\,\mathrm{d}x\Big)\\
&\qquad+\frac{1}{2\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}}\int_{\mathcal{O}}\mathcal{I}_h\Big\{\frac{1}{2}(F''(\varphi_2)-F''(\phi_h^{n-1}))(\phi_h^n-\phi_h^{n-1})^2\Big\}\,\mathrm{d}x\\
&\qquad-\frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}}\Big(\int_{\mathcal{O}}\mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n-\phi_h^{n-1})\big\}\,\mathrm{d}x\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \Big) \\
&\qquad \times \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\varphi_3)(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x \Big) \\
&- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big(\frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\varphi_3)(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x \Big)^2 \\
&+ \frac{1}{16[E_h^{\mathcal{O}}(\varphi_1)]^{5/2}} \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\varphi_4)(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \Big)^3
\end{aligned} \tag{6.13}
$$

with $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \operatorname{conv}\{\phi_h^n, \phi_h^{n-1}\}$. Recalling the definition of the discrete Laplacian Δ_h in equation (2.5), we write equation (3.4a) as

$$
\phi_h^n - \phi_h^{n-1} = \tau \Delta_h \mu_h^n + \Phi_h(\phi_h^{n-1}) \blacktriangle^n \xi^\tau.
$$

Hence, we obtain from equation (6.13)

$$
\begin{aligned}
\sqrt{E_h^{\mathcal{O}}(\phi_h^n)} - \sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})} &= r_h^n - r_h \\
&+ \frac{1}{4\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\tau \Delta_h \mu_h^n\big\}\,\mathrm{d}x \\
&+ \frac{1}{4\sqrt{E_h^{\mathcal{O}}(\phi_h^{n-1})}} \int_{\mathcal{O}} \mathcal{I}_h\big\{(F''(\varphi_2) - F''(\phi_h^{n-1}))(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x \\
&- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \Big) \\
&\qquad \times \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})\tau \Delta_h \mu_h^n\big\}\,\mathrm{d}x \Big) \\
&- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\phi_h^{n-1})(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \Big) \\
&\qquad \times \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\varphi_3)(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x \Big) \\
&- \frac{1}{8[E_h^{\mathcal{O}}(\phi_h^{n-1})]^{3/2}} \Big(\frac{1}{2} \int_{\mathcal{O}} \mathcal{I}_h\big\{F''(\varphi_3)(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x \Big)^2 \\
&+ \frac{1}{16[E_h^{\mathcal{O}}(\varphi_1)]^{5/2}} \Big(\int_{\mathcal{O}} \mathcal{I}_h\big\{F'(\varphi_4)(\phi_h^n - \phi_h^{n-1})\big\}\,\mathrm{d}x \Big)^3 \\
&=: r_h^n - r_h^{n-1} + R_{1,n} + R_{2,n} + R_{3,n} + R_{4,n} + R_{5,n} + R_{6,n}.
\end{aligned} \tag{6.14}
$$

Summing equation (6.14) from $n = 1$ to $m \leq N$, noting that by definition $r_h^0 = \sqrt{E_h^{\mathcal{O}}(\phi_h^0)}$, taking the p-th power and the supremum over all $m \in \{0, \ldots, N\}$, and computing the expected value, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le m\le N}\Big|r_h^m-\sqrt{E_h^{\mathcal{O}}(\phi_h^m)}\Big|^p\Big]\\
&\quad\le C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{1,n}\Big|^p\Big]+C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{2,n}\Big|^p\Big]+C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{3,n}\Big|^p\Big]\\
&\quad+C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{4,n}\Big|^p\Big]+C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{5,n}\Big|^p\Big]+C\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{6,n}\Big|^p\Big].
\end{aligned}$$

Deriving estimates for this right-hand side is more intricate than for the one discussed in [44] as uniform bounds for $\Delta_h\mu_h^n$ are only available in the $(H^1(\mathcal{O}))'$-norm. It is, however, possible to derive a τ-dependent bound that provides sufficient regularity. Choosing $\psi_h\equiv\tau^{1/2}\Delta_h\mu_h^n$ in equation (3.4a), we obtain

$$\begin{aligned}
&\tau^{3/2}\|\Delta_h\mu_h^n\|_{h,\mathcal{O}}^2\\
&\quad=\tau^{1/2}\int_{\mathcal{O}}\mathcal{I}_h\big\{(\phi_h^n-\phi_h^{n-1})\Delta_h\mu_h^n\big\}\,\mathrm{d}x-\tau^{1/2}\int_{\mathcal{O}}\mathcal{I}_h\big\{\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau\Delta_h\mu_h^n\big\}\,\mathrm{d}x\\
&\quad\le\tau\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2+\frac12\|\nabla\phi_h^n-\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2\\
&\qquad+\frac12\|\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau)\|_{L^2(\mathcal{O})}^2.
\end{aligned}\tag{6.15}$$

Recalling the computation in estimate (6.5) and the regularity results from Lemma 6.1,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau^{3/2}\|\Delta_h\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^p\Big]\le C\mathbb{E}\Big[\Big(\sum_{n=1}^N\tau\|\nabla\mu_h^n\|_{L^2(\mathcal{O})}^2\Big)^p\Big]\\
&\quad+C\mathbb{E}\Big[\Big(\sum_{n=1}^N\|\nabla\phi_h^n-\nabla\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2\Big)^p\Big]+C\mathbb{E}\Big[\Big(\sum_{n=1}^N\|\nabla(\Phi_h(\phi_h^{n-1})\blacktriangle^n\boldsymbol{\xi}^\tau)\|_{L^2(\mathcal{O})}^2\Big)^p\Big]\\
&\le C
\end{aligned}\tag{6.16}$$

for any $p\ge1$. Using this estimate, we compute an upper bound for $R_{1,n}$ by applying Hölder's inequality, assumption **(P)**, Young's inequality, and Lemma 6.1:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{1,n}\Big|^p\Big]\\
&\quad\le C\mathbb{E}\Big[\Big|\sum_{n=1}^N\tau\|\Delta_h\mu_h^n\|_{L^2(\mathcal{O})}\|\mathcal{I}_h\{F''(\phi_h^{n-1})\}\|_{L^3(\mathcal{O})}\|\phi_h^n-\phi_h^{n-1}\|_{L^6(\mathcal{O})}\Big|^p\Big]\\
&\quad\le C\mathbb{E}\Big[\Big|\Big(1+\max_{0\le n\le N}\|\phi_h^n\|_{H^1(\mathcal{O})}^2\Big)\tau^{1/4}\\
&\qquad\times\Big(\sum_{n=1}^N\tau^{3/2}\|\Delta_h\mu_h^n\|_{L^2(\mathcal{O})}^2+\sum_{n=1}^N\|\phi_h^n-\phi_h^{n-1}\|_{H^1(\mathcal{O})}^2\Big|\Big)^p\Big]\le C\tau^{p/4}.
\end{aligned}$$

Concerning the second term $R_{2,n}$, we start by discussing the $C^{2,\nu}$-part F_1 of the potential F (cf. assumption **(P)**). Due to the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}\Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{(F_1''(\varphi_2) - F_1''(\phi_h^{n-1}))(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x\Big| &\leq C\int_{\mathcal{O}} \mathcal{I}_h\big\{|\phi_h^n - \phi_h^{n-1}|^{2+\nu}\big\}\,\mathrm{d}x \\ &\leq C\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^{3\nu/2}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^{(4-\nu)/2} =: (*).\end{aligned}$$

If $\nu \leq 0.8$, we may apply Young's inequality to deduce

$$(*) \leq C\tau^{\nu/2}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^2 + C\tau^{-3\nu^2/(8-6\nu)}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^{(8-2\nu)/(4-3\nu)},$$

where we will control the second term using estimate (6.9b) as $(8-2\nu)/(4-3\nu) \leq 4$. Unfortunately, the limited regularity of the phase-field parameter reduces gains for larger ν. Yet, we still can rely on the discrete $L^\infty(0,T;H^1(\mathcal{O}))$-bounds for $(\phi_h^n)_n$ and recover the scaling obtained for $\nu = 0.8$. In particular, we estimate

$$\begin{aligned}(*) \leq C\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^{(5\nu-4)/4}\big(&\tau^{4/10}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^2 \\ &+ \tau^{-6/10}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^4\big).\end{aligned}$$

To deal with the remaining part, we recall the growth condition for F_2''' and apply the Gagliardo–Nirenberg inequality to deduce

$$\begin{aligned}&\Big|\int_{\mathcal{O}} \mathcal{I}_h\big\{(F_2''(\varphi_2) - F_2''(\phi_h^{n-1}))(\phi_h^n - \phi_h^{n-1})^2\big\}\,\mathrm{d}x\Big| \\ &\quad\leq C(1 + \|\phi_h^n\|_{H^1(\mathcal{O})}^2 + \|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^2)\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^{5/2}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^{1/2} \\ &\quad\leq C(1 + \|\phi_h^n\|_{H^1(\mathcal{O})}^{11/4} + \|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{11/4})\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^{7/4}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^{1/2} \\ &\quad\leq C(1 + \|\phi_h^n\|_{H^1(\mathcal{O})}^{11/4} + \|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{11/4})\tau^{1/8}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\mathcal{O})}^2 \\ &\qquad + C(1 + \|\phi_h^n\|_{H^1(\mathcal{O})}^{11/4} + \|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^{11/4})\tau^{-7/8}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}^4.\end{aligned}$$

Combining the above results with the uniform lower bound on $E_h^{\mathcal{O}}(\cdot)$, Hölder's inequality, and Lemmas 6.1 and 6.3, we compute

$$\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{2,n}\Big|^p\Big] \leq C\tau^{p\min\{\nu/2,1/8\}}.$$

Combining Hölder's inequality, assumption **(P)**, Young's inequality, and Lemma 6.1, and estimate (6.16) provides

$$\begin{aligned}&\mathbb{E}\Big[\Big|\sum_{n=1}^N R_{3,n}\Big|^p\Big] \\ &\quad\leq C\mathbb{E}\Big[\Big(\sum_{n=1}^N \tau\|\mathcal{I}_h\{F'(\phi_h^{n-1})\}\|_{L^2(\mathcal{O})}^2\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\mathcal{O})}\|\Delta_h\mu_h^n\|_{L^2(\mathcal{O})}\Big)^p\Big]\end{aligned}$$

$$\leq C\mathbb{E}\Big[\Big(1+\max_{1\leq n\leq N}\|\phi_h^{n-1}\|_{H^1(\mathcal{O})}^6\Big)^p\tau^{p/4}$$
$$\times\Big(\sum_{n=1}^{N}(\tau^{3/2}\|\Delta_h\mu_h^n\|_{L^2(\mathcal{O})}^2+\|\phi_h^n-\phi_h^{n-1}\|_{L^2(\mathcal{O})}^2)\Big)^p\Big]\leq C\tau^{p/4}.$$

The remaining terms $R_{4,n}$, $R_{5,n}$, and $R_{6,n}$ can be treated similarly to $R_{2,n}$. In particular, applying the Gagliardo–Nirenberg inequality and assumption **(P)**,

$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N}R_{4,n}\Big|^p\Big]\leq C\mathbb{E}\Big[\Big(\Big(1+\max_{0\leq n\leq N}\|\phi_h^n\|_{H^1(\mathcal{O})}^5\Big)\sum_{n=1}^{N}\tau^{1/2}\|\phi_h^n-\phi_h^{n-1}\|_{H^1(\mathcal{O})}^2\Big)^p\Big]$$
$$+C\mathbb{E}\Big[\Big(\Big(1+\max_{0\leq n\leq N}\|\phi_h^n\|_{H^1(\mathcal{O})}^5\Big)\sum_{n=1}^{N}\tau^{-1/2}\|\phi_h^n-\phi_h^{n-1}\|_{L^2(\mathcal{O})}^4\Big)^p\Big]\leq C\tau^{p/2},$$
$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N}R_{5,n}\Big|^p\Big]\leq C\mathbb{E}\Big[\Big(\Big(1+\max_{0\leq n\leq N}\|\phi_h^n\|_{H^1(\mathcal{O})}^5\Big)\sum_{n=1}^{N}\tau^{1/2}\|\phi_h^n-\phi_h^{n-1}\|_{H^1(\mathcal{O})}^2\Big)^p\Big]$$
$$+C\mathbb{E}\Big[\Big(\Big(1+\max_{0\leq n\leq N}\|\phi_h^n\|_{H^1(\mathcal{O})}^5\Big)\sum_{n=1}^{N}\tau^{-1/2}\|\phi_h^n-\phi_h^{n-1}\|_{L^2(\mathcal{O})}^4\Big)^p\Big]\leq C\tau^{p/2},$$
$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N}R_{6,n}\Big|^p\Big]\leq C\mathbb{E}\Big[\Big(\max_{0\leq n\leq N}\big(1+\|\phi_h^n\|_{H^1(\mathcal{O})}^9\big)\sum_{n=1}^{N}\|\phi_h^n-\phi_h^{n-1}\|_{L^2(\mathcal{O})}^3\Big)^p\Big]\leq C\tau^{p/2}.$$

Combining the above estimates provides estimate (6.12a). To obtain estimate (6.12b), we follow a similar pathway and obtain

$$\sqrt{E_h^\Gamma(\phi_h^n)}-\sqrt{E_h^\Gamma(\phi_h^{n-1})}=s_h^n-s_h^{n-1}$$
$$-\frac{1}{4\sqrt{E_h^\Gamma(\phi_h^{n-1})}}\int_\Gamma\mathcal{I}_h^\Gamma\{G''(\phi_h^{n-1})(\phi_h^n-\phi_h^{n-1})\tau\theta_h^n\}\,\mathrm{d}\Gamma$$
$$+\frac{1}{4\sqrt{E_h^\Gamma(\phi_h^{n-1})}}\int_\Gamma\mathcal{I}_h^\Gamma\{(G''(\widehat{\varphi}_2)-G''(\phi_h^{n-1}))(\phi_h^n-\phi_h^{n-1})^2\}\,\mathrm{d}\Gamma$$
$$+\frac{1}{8[E_h^\Gamma(\phi_h^{n-1})]^{3/2}}\Big(\int_\Gamma\mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})(\phi_h^n-\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big)$$
$$\times\Big(\int_\Gamma\mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\tau\theta_h^n\}\,\mathrm{d}\Gamma\Big)$$
$$-\frac{1}{8[E_h^\Gamma(\phi_h^{n-1})]^{3/2}}\Big(\int_\Gamma\mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})(\phi_h^n-\phi_h^{n-1})\}\,\mathrm{d}\Gamma\Big)$$
$$\times\Big(\int_\Gamma\mathcal{I}_h^\Gamma\{G''(\widehat{\varphi}_3)(\phi_h^n-\phi_h^{n-1})^2\}\,\mathrm{d}\Gamma\Big)$$
$$-\frac{1}{8[E_h^\Gamma(\phi_h^{n-1})]^{3/2}}\Big(\frac{1}{2}\int_\Gamma\mathcal{I}_h^\Gamma\{G''(\widehat{\varphi}_3)(\phi_h^n-\phi_h^{n-1})^2\}\,\mathrm{d}\Gamma\Big)^2$$

$$+ \frac{1}{16[E_h^\Gamma(\widehat{\varphi}_1)]^{5/2}} \Big(\int_\Gamma \mathcal{I}_h^\Gamma \big\{ G'(\widehat{\varphi}_4)(\phi_h^n - \phi_h^{n-1}) \big\} \,\mathrm{d}\Gamma \Big)^3$$
$$=: s_h^n - s_h^{n-1} + \widehat{R}_{1,n} + \widehat{R}_{2,n} + \widehat{R}_{3,n} + \widehat{R}_{4,n} + \widehat{R}_{5,n} + \widehat{R}_{6,n}$$

with $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4 \in \operatorname{conv}\{\phi_h^n, \phi_h^{n-1}\}$. The error terms $\widehat{R}_{1,n}, \ldots, \widehat{R}_{6,n}$ can in principle be estimated similarly to $R_{1,n}, \ldots, R_{6,n}$. On the boundary, however, this requires only control over the potential θ_h^n instead of its discrete Laplacian. Furthermore, the Gagliardo–Nirenberg inequality in $d-1$ dimensions allows for better estimates. In particular, we obtain for the first term

$$\mathbb{E}\Big[\Big|\sum_{n=1}^N \widehat{R}_{1,n}\Big|^p\Big]$$
$$\leq C\,\mathbb{E}\Big[\Big|\big(1 + \max_{0\leq n\leq N} \|\phi_h^n\|_{H^1(\Gamma)}^2\big)\tau^{1/2} \sum_{n=1}^N \big(\tau\|\theta_h^n\|_{L^2(\Gamma)}^2 + \|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^2\big)\Big|^p\Big]$$
$$\leq C\tau^{p/2}.$$

To estimate the second term, we again use the decomposition of the potential G. Starting with the $C^{2,\nu}$-part, we compute by using the Gagliardo–Nirenberg inequality

$$\Big|\int_\Gamma \mathcal{I}_h^\Gamma \big\{ (G_1''(\widehat{\varphi}_2) - G_1''(\phi_h^{n-1}))(\phi_h^n - \phi_h^{n-1})^2 \big\} \,\mathrm{d}\Gamma\Big| \leq C \int_\Gamma \mathcal{I}_h^\Gamma \big\{ |\phi_h^n - \phi_h^{n-1}|^{2+\nu} \big\} \,\mathrm{d}\Gamma$$
$$\leq C\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^\nu \|\phi_h^n - \phi_h^{n-1}\|_{L^2(\Gamma)}^2$$
$$\leq C(\tau^{\nu/2}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^2 + \tau^{-\nu^2/(4-2\nu)}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\Gamma)}^{4/(2-\nu)}).$$

Concerning the C^3-part of G, we deduce

$$\Big|\int_\Gamma \mathcal{I}_h^\Gamma \big\{ (G_2''(\widehat{\varphi}_2) - G_2''(\phi_h^{n-1}))(\phi_h^n - \phi_h^{n-1})^2 \big\} \,\mathrm{d}\Gamma\Big|$$
$$\leq C(1 + \|\phi_h^n\|_{H^1(\Gamma)}^2 + \|\phi_h^{n-1}\|_{H^1(\Gamma)}^2)\|\phi_h^n - \phi_h^{n-1}\|_{L^{3+\delta}(\Gamma)}^3$$
$$\leq C(1 + \|\phi_h^n\|_{H^1(\Gamma)}^2 + \|\phi_h^{n-1}\|_{H^1(\Gamma)}^2)\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^{(3+3\delta)/(3+\delta)}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\Gamma)}^{6/(3+\delta)}$$
$$\leq C(1 + \|\phi_h^n\|_{H^1(\Gamma)}^2 + \|\phi_h^{n-1}\|_{H^1(\Gamma)}^2)\tau^{1/2}\|\phi_h^n - \phi_h^{n-1}\|_{H^1(\Gamma)}^2$$
$$+ C(1 + \|\phi_h^n\|_{H^1(\Gamma)}^2 + \|\phi_h^{n-1}\|_{H^1(\Gamma)}^2)\tau^{-\frac{3+3\delta}{6-2\delta}}\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\Gamma)}^{12/(3-\delta)}$$

for arbitrary $0 < \delta < 1$. Using estimate (6.9c), we obtain

$$\mathbb{E}\Big[\Big|\sum_{n=1}^N \widehat{R}_{2,n}\Big|^p\Big] \leq C\tau^{\nu p/2}$$

as $\nu \leq 1$. Furthermore, we have

$$\mathbb{E}\Big[\Big|\sum_{n=1}^N \widehat{R}_{3,n}\Big|^p\Big] \leq C\,\mathbb{E}\Big[\Big|\sum_{n=1}^N \tau\|\mathcal{I}_h^\Gamma\{G'(\phi_h^{n-1})\}\|_{L^2(\Gamma)}^2\|\phi_h^n - \phi_h^{n-1}\|_{L^2(\Gamma)}\|\theta_h^n\|_{L^2(\Gamma)}\Big|^p\Big]$$
$$\leq C\tau^{p/2},$$

$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N} \widehat{R}_{4,n}\Big|^p\Big] \le C\tau^{p/2},$$
$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N} \widehat{R}_{5,n}\Big|^p\Big] \le C\tau^{p/2},$$
$$\mathbb{E}\Big[\Big|\sum_{n=1}^{N} \widehat{R}_{6,n}\Big|^p\Big] \le C\tau^{p/2},$$

which concludes the proof. ∎

In the proof of Lemma 6.4, we relied on a bound on $\tau^{3/2}\|\Delta_h\mu_h^n\|_{h,\mathcal{O}}^2$ (cf. estimate (6.15)). When passing to the limit $\tau \searrow 0$, this bound loses its significance. Using a weaker spatial norm, however, allows us to obtain τ-independent bounds for the weak form of the Laplacian of μ_h^n. Introducing the linear form $l[\mu_h^n] : H^1(\mathcal{O}) \to \mathbb{R}$ given via

$$\psi \mapsto l[\mu_h^n](\psi) := \int_{\mathcal{O}} \nabla\mu_h^n \cdot \nabla\psi \,\mathrm{d}x \tag{6.17}$$

for all $\psi \in H^1(\mathcal{O})$, a straightforward computation provides the following result:

Corollary 6.5. *Let assumptions (T), (S1), (S2), (P), (I), (C), and (D0)–(D3) hold true. Then, for every $1 \le p < \infty$, there exists a constant $C \equiv C(p, T) > 0$ independent of h and τ such that the linear form $l[\mu_h^n] : H^1(\mathcal{O}) \to \mathbb{R}$ given in equation (6.17) satisfies*

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{N} \tau\|l[\mu_h^n]\|_{(H^1(\mathcal{O}))'}^2\Big)^p\Big] \le C.$$

7. Compactness properties of discrete solutions

In this section, we shall establish the tightness of the laws of the discrete solutions and deduce the existence of weakly and strongly converging subsequences.

Using the time-index free notation defined in equation (2.14), we restate the regularity results from the last section as follows:

$$\begin{aligned}
&\|\phi_h^{\tau,(\pm)}\|_{L^{2p}(\Omega;L^\infty(0,T;H^1(\mathcal{O})))} + \|\phi_h^{\tau,(\pm)}\|_{L^{2p}(\Omega;L^\infty(0,T;H^1(\Gamma)))} + \|r_h^{\tau,(\pm)}\|_{L^{2p}(\Omega;L^\infty(0,T))} \\
&\quad + \|s_h^{\tau,(\pm)}\|_{L^{2p}(\Omega;L^\infty(0,T))} + \|\mu_h^{\tau,+}\|_{L^{2p}(\Omega;L^2(0,T;H^1(\mathcal{O})))} + \|\theta_h^{\tau,+}\|_{L^{2p}(\Omega;L^2(0,T;L^2(\Gamma)))} \\
&\quad + \tau^{-1/2}\|\phi_h^{\tau,+} - \phi_h^{\tau,-}\|_{L^{2p}(\Omega;L^2(0,T;H^1(\Omega)))} \\
&\quad + \tau^{-1/2}\|\phi_h^{\tau,+} - \phi_h^{\tau,-}\|_{L^{2p}(\Omega;L^2(0,T;H^1(\Gamma)))} \\
&\quad + \tau^{-1/2}\|r_h^{\tau,+} - r_h^{\tau,-}\|_{L^{2p}(\Omega;L^2(0,T))} + \tau^{-1/2}\|s_h^{\tau,+} - s_h^{\tau,-}\|_{L^{2p}(\Omega;L^2(0,T))} \\
&\quad + \|l[\mu_h^{\tau,+}]\|_{L^{2p}(\Omega;L^2(0,T;(H^1(\mathcal{O}))'))} \le C,
\end{aligned} \tag{7.1a}$$

$$\|\Xi_{h,\mathcal{O}}^{\tau,+}\|_{L^{\mathfrak{p}}(\Omega;L^2(0,T;L^2(\mathcal{O})))} + \|\Xi_{h,\Gamma}^{\tau,+}\|_{L^{\mathfrak{p}}(\Omega;L^2(0,T;L^2(\Gamma)))} \leq C\tau^{1/2}, \quad (7.1b)$$

$$\|\phi_h^\tau\|_{L^{2\alpha}(\Omega;N^{1/4,2\alpha}(0,T;L^2(\mathcal{O})))} + \|\phi_h^\tau\|_{L^{2\alpha}(\Omega;N^{1/2,2\alpha}(0,T;L^2(\Gamma)))} \leq C, \quad (7.1c)$$

$$\|\phi_h^\tau\|_{L^{4\alpha}(\Omega;C^{0,(\alpha-1)/(4\alpha)}([0,T];L^2(\mathcal{O})))} + \|\phi_h^\tau\|_{L^{2\alpha}(\Omega;C^{0,(\alpha-1)/(2\alpha)}([0,T];L^2(\Gamma)))} \leq C, \quad (7.1d)$$

for $\mathfrak{p} \in [1,\infty)$ and $\alpha \in (1,\infty)$. While equation (7.1a) is a direct consequence of Lemmas 6.1 and 6.2 and Corollary 6.5, equation (7.1b) follows from Lemma 6.2 and equation (2.2), and equation (7.1c) follows from the results of Lemma 6.3 and [3, Lemma 3.2]. The last inequality (see equation (7.1d)) then follows from the embedding results in [60].

In the next step, we want to identify almost surely converging subsequences by applying Jakubowski's theorem (cf. [30]). In particular, we are interested in the convergence properties of $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma, r_h^\tau, s_h^\tau, \mu_h^{\tau,+}, l[\mu_h^{\tau,+}], \theta_h^{\tau,+})_{h,\tau}$ and the linear interpolation ξ_h^τ of $\{\xi_h^{m,\tau}\}_m$. Although the time-continuous processes ξ_h^τ are not martingales, we will later show that they converge toward martingales.

We start by establishing uniform estimates for ξ_h^τ.

Lemma 7.1. *Let assumptions (D0)–(D3) hold true. Then, the piecewise linear process ξ_h^τ satisfies*

$$\|\xi_h^\tau\|_{L^{2p}(\Omega;C^{0,(p-1)/(2p)}([0,T];H^2(\mathcal{O})))} \leq C$$

for arbitrary $p \in (1,\infty)$ with a constant $C > 0$ independent of h and τ.

Proof. Recalling the definition of $\xi_h^{m,\tau}$ (cf. equation (2.12)) and Lemma 2.4, we obtain

$$\begin{aligned}\mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\|\xi_h^{m+l,\tau}-\xi_h^{m,\tau}\|_{H^2(\mathcal{O})}^{2p}\Big] &= \mathbb{E}\Big[\sum_{m=0}^{N-l}\tau\Big\|\sum_{n=m+1}^{m+l}\sqrt{\tau}\sum_{k\in\mathbb{Z}_h}\lambda_k\mathfrak{g}_k\xi_h^{n,\tau}\Big\|_{H^2(\mathcal{O})}^{2p}\Big]\\ &\leq C\sum_{m=0}^{N-l}\tau(l\tau)^{p-1}\sum_{n=m+1}^{m+l}\tau\mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}_h}\|\lambda_k\mathfrak{g}_k\|_{H^2(\mathcal{O})}^2\Big)^p\Big]\\ &\leq C(l\tau)^p,\end{aligned}$$

which is sufficient to establish the result (cf. [3, Lemma 3.2]). ■

Lemma 7.1 in particular provides the tightness of the laws of $(\xi_h^\tau)_{h,\tau}$ in $C([0,T];H^1(\mathcal{O}))$. With the next lemma, we establish the tightness of the laws of $(\phi_h^\tau)_{h,\tau}$ and $([\phi_h^\tau]|_\Gamma)_{h,\tau}$ in $C([0,T];L^s(\mathcal{O}))$ and $C([0,T];L^r(\Gamma))$ with $s \in [1,\frac{2d}{d-2})$ and $r \in [1,\infty)$.

Lemma 7.2. *Let $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma)_{h,\tau}$ be a family of continuous, piecewise linear processes that satisfy the bounds stated in equation (7.1). Then the family of laws $(\nu_{\phi_h^\tau})_{h,\tau}$ generated by $(\phi_h^\tau)_{h,\tau}$ is tight on $C([0,T];L^s(\mathcal{O}))$ ($s \in [1,\frac{2d}{d-2})$) and the family of laws $(\nu_{[\phi_h^\tau]|_\Gamma})_{h,\tau}$ generated by $([\phi_h^\tau]|_\Gamma)_{h,\tau}$ is tight on $C([0,T];L^r(\Gamma))$ with $r \in [1,\infty)$.*

Proof. Recalling the compactness theorem by Simon (cf. [59]) we note that the closed ball $\overline{B}_R^{\mathcal{O}}$ in $L^\infty(0,T;H^1(\mathcal{O})) \cap C^{0,(\alpha-1)/(4\alpha)}([0,T];L^2(\mathcal{O}))$ is a compact subset of

$C([0,T];L^s(\mathcal{O}))$. Furthermore, the family of laws $(\nu_{\phi_h^\tau})_{h,\tau}$ satisfies for any $R>0$

$$\begin{aligned}&\nu_{\phi_h^\tau}(C([0,T];L^s(\mathcal{O}))\setminus \overline{B}_R^{\mathcal{O}})\\ &\quad = \mathbb{P}[\|\phi_h^\tau\|^{4\alpha}_{L^\infty(0,T;H^1(\mathcal{O}))}+\|\phi_h^\tau\|^{4\alpha}_{C^{0,(\alpha-1)/(4\alpha)}([0,T];L^2(\mathcal{O}))}>R^{4\alpha}]\\ &\quad \le R^{-4\alpha}\mathbb{E}[\|\phi_h^\tau\|^{4\alpha}_{L^\infty(0,T;H^1(\mathcal{O}))}+\|\phi_h^\tau\|^{4\alpha}_{C^{0,(\alpha-1)/(4\alpha)}([0,T];L^2(\mathcal{O}))}].\end{aligned}$$

The tightness of the family of laws $(\nu_{[\phi_h^\tau]|_\Gamma})_{h,\tau}$ on $C([0,T];L^r(\Gamma))$ follows by similar arguments. ∎

As the closed balls in $L^2(0,T)$, $L^2(0,T;H^1(\mathcal{O}))$, $L^2(0,T;(H^1(\mathcal{O}))')$, and $L^2(0,T;L^2(\Gamma))$ are compact in the weak topology, laws $(\nu_{r_h^\tau})_{h,\tau}$, $(\nu_{s_h^\tau})_{h,\tau}$, $(\nu_{\mu_h^{\tau,+}})_{h,\tau}$, $(\nu_{l[\mu_h^{\tau,+}]})_{h,\tau}$, and $(\nu_{\theta_h^{\tau,+}})_{h,\tau}$ generated by $(r_h^\tau)_{h,\tau}$, $(s_h^\tau)_{h,\tau}$, $(\mu_h^{\tau,+})_{h,\tau}$, $(l[\mu_h^{\tau,+}])_{h,\tau}$, and $(\theta_h^{\tau,+})_{h,\tau}$ are tight in the spaces $L^2(0,T)_{\mathrm{weak}}$, $L^2(0,T)_{\mathrm{weak}}$, $L^2(0,T;H^1(\mathcal{O}))_{\mathrm{weak}}$, $L^2(0,T;(H^1(\mathcal{O}))')_{\mathrm{weak}}$, and in $L^2(0,T;L^2(\Gamma))_{\mathrm{weak}}$ due to Markov's inequality and the bounds collected in equation (7.1). Hence, the joint laws $(\nu_h^\tau)_{h,\tau}$ of $(\phi_h^\tau)_{h,\tau}$, $([\phi_h^\tau]|_\Gamma)_{h,\tau}$, $(r_h^\tau)_{h,\tau}$, $(s_h^\tau)_{h,\tau}$, $(\mu_h^{\tau,+})_{h,\tau}$, $(l[\mu_h^{\tau,+}])_{h,\tau}$, $(\theta_h^{\tau,+})_{h,\tau}$, and $(\boldsymbol{\xi}_h^\tau)_{h,\tau}$ are tight on the path space

$$\begin{aligned}\mathcal{X} := {}& C([0,T];L^s(\mathcal{O}))\times C([0,T];L^r(\Gamma))\times L^2(0,T)_{\mathrm{weak}}\times L^2(0,T)_{\mathrm{weak}}\\ &\times L^2(0,T;H^1(\mathcal{O}))_{\mathrm{weak}}\times L^2(0,T;(H^1(\mathcal{O}))')_{\mathrm{weak}}\\ &\times L^2(0,T;L^2(\Gamma))_{\mathrm{weak}}\times C([0,T];H^1(\mathcal{O})),\end{aligned}$$

for $s\in[1,\frac{2d}{d-2})$ and $r\in[1,\infty)$. Therefore, we can apply Jakubowski's generalization of the Skorokhod theorem (cf. [30]) to obtain the following convergence result:

Theorem 7.3. *Let $(\phi_h^\tau,[\phi_h^\tau]|_\Gamma,r_h^\tau,s_h^\tau,\mu_h^{\tau,+},l[\mu_h^{\tau,+}],\theta_h^{\tau,+})_{h,\tau}$ satisfy the estimates given in equation (7.1) and let the family of discrete approximations $(\boldsymbol{\xi}_h^\tau)_{h,\tau}$ of the $\mathcal{Q}$-Wiener process satisfy the bounds in Lemma 7.1. Then there exists a subsequence*

$$\begin{aligned}&(\phi_j,[\phi_j]|_\Gamma,r_j,s_j,\mu_j^+,l[\mu_j^+],\theta_j^+,\boldsymbol{\xi}_j)_j\\ &\qquad := (\phi_{h_j}^{\tau_j},[\phi_{h_j}^{\tau_j}]|_\Gamma,r_{h_j}^{\tau_j},s_{h_j}^{\tau_j},\mu_{h_j}^{\tau_j,+},l[\mu_{h_j}^{\tau_j,+}],\theta_{h_j}^{\tau_j,+},\boldsymbol{\xi}_{h_j}^{\tau_j})_j,\end{aligned}$$

a stochastic basis $(\widetilde{\Omega},\widetilde{\mathcal{A}},\widetilde{\mathbb{P}})$, a sequence of random variables

$$(\widetilde{\phi}_j,[\widetilde{\phi}_j]|_\Gamma,\widetilde{r}_j,\widetilde{s}_j,\widetilde{\mu}_j^+,l[\widetilde{\mu}_j^+],\widetilde{\theta}_j^+,\widetilde{\boldsymbol{\xi}}_j)\;:\;\widetilde{\Omega}\to\mathcal{X},$$

and random variables

$$(\widetilde{\phi},\widetilde{\phi}_\Gamma,\widetilde{r},\widetilde{s},\widetilde{\mu},\widetilde{L},\widetilde{\theta},\widetilde{W})\;:\;\widetilde{\Omega}\to\mathcal{X}$$

such that the following holds:

- *The law of $(\widetilde{\phi}_j,[\widetilde{\phi}_j]|_\Gamma,\widetilde{r}_j,\widetilde{s}_j,\widetilde{\mu}_j^+,l[\widetilde{\mu}_j^+],\widetilde{\theta}_j^+,\widetilde{\boldsymbol{\xi}}_j)$ on $\mathcal{X}$ under $\widetilde{\mathbb{P}}$ coincides with for any $j\in\mathbb{N}$ with the law of $(\phi_j,[\phi_j]|_\Gamma,r_j,s_j,\mu_j^+,l[\mu_j^+],\theta_j^+,\boldsymbol{\xi}_j)$ under $\mathbb{P}$.*

- *The sequence* $(\widetilde{\phi}_j, [\widetilde{\phi}_j]|_\Gamma, \widetilde{r}_j, \widetilde{s}_j, \widetilde{\mu}_j^+, l[\widetilde{\mu}_j^+], \widetilde{\theta}_j^+, \widetilde{\boldsymbol{\xi}}_j)$ *converges* $\widetilde{\mathbb{P}}$*-almost surely toward* $(\widetilde{\phi}, \widetilde{\phi_\Gamma}, \widetilde{r}, \widetilde{s}, \widetilde{\mu}, \widetilde{L}, \widetilde{\theta}, \widetilde{W})$ *in the topology of* $\mathcal{X}$.

Proof. As the combined laws on $\mathcal{X}$ are tight, Jakuboswki's theorem provides the existence of a stochastic basis and a sequence of random variables $(\widetilde{\phi}_j, \widetilde{\phi_{\Gamma,j}}, \widetilde{r}_j, \widetilde{s}_j, \widetilde{\mu}_j^+, \widetilde{L}_j^+, \widetilde{\theta}_j^+, \widetilde{\xi})$ on $\widetilde{\Omega}$ with the stated convergence properties. Hence, it only remains to identify $\widetilde{\phi_{\Gamma,j}}$ and $\widetilde{L}_j^+$ with $[\widetilde{\phi}_j]|_\Gamma$ and $l[\widetilde{\mu}_j^+]$. As the laws of $\widetilde{\phi}_j$ and ϕ_j coincide, $\widetilde{\phi}_j$ is $\widetilde{\mathbb{P}}$-almost surely a piecewise linear (time) interpolation of a $U_h^{\mathcal{O}}$-valued random variable. Hence, the trace of $\widetilde{\phi}_j$ is well defined as a continuous mapping from $C(\overline{\Omega})$ to $C(\Gamma)$. Therefore, by identity of laws, we may identify $\widetilde{\phi_{\Gamma,j}}$ as $[\widetilde{\phi}_j]|_\Gamma$. Similarly, we deduce that $\widetilde{\mu}_j^+$ and $\widetilde{L}_j^+$ are $\widetilde{\mathbb{P}}$-almost surely piecewise constant in time. Using again the identity of laws, we can identify $\widetilde{L}_j(\psi)$ with $\int_{\mathcal{O}} \nabla\widetilde{\mu}_j^+ \cdot \nabla\psi \,\mathrm{d}x = l[\widetilde{\mu}_j^+](\psi)$ for all $\psi \in H^1(\mathcal{O})$ $\widetilde{\mathbb{P}}$-almost surely. ■

Remark 7.4. The restriction to subsequences in Theorem 7.3 is necessary, as Jakubowski's theorem uses a generalization of Prokhorov's theorem to deduce convergence in distribution for a subsequence from uniform bounds. For $(\xi_h^\tau)_{h,\tau}$, convergence in distribution can already be deduced from Donsker's invariance theorem.

As the random variables introduced in Theorem 7.3 are $\widetilde{\mathbb{P}}$-almost surely continuous and piecewise linear (or left continuous and piecewise constant, respectively), we can evaluate them at the nodes of the time grid and recover the remaining time interpolants. Analogously to equation (3.5), we introduce the $\widetilde{\mathbb{P}}$-almost surely left continuous and piecewise constant in time finite-element-valued random variables $\widetilde{\Xi}_{j,\mathcal{O}}^+$ and $\widetilde{\Xi}_{j,\Gamma}^+$. Due to the identity of laws, the constructed sequences satisfy the bounds collected in equation (7.1) with respect to the new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$. Hence, we obtain the following additional convergence properties:

Lemma 7.5. *Let* $(\widetilde{\phi}_j^{(\pm)})_{j\in\mathbb{N}}$, $([\widetilde{\phi}_j^{(\pm)}]|_\Gamma)_{j\in\mathbb{N}}$, $(\widetilde{r}_j^{(\pm)})_{j\in\mathbb{N}}$, $(\widetilde{s}_j^{(\pm)})_{j\in\mathbb{N}}$, $(\widetilde{\mu}_j^+)_{j\in\mathbb{N}}$, $(l[\widetilde{\mu}_j^+])_{j\in\mathbb{N}}$, *and* $(\widetilde{\theta}_j^+)_{j\in\mathbb{N}}$ *be the sequences defined by interpolation of the sequences from Theorem 7.3. Furthermore, let assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)**–**(D3)** *hold true. Then there exist functions*

$$\widetilde{\phi} \in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T;H^1(\mathcal{O}))) \cap L^{4\mathfrak{p}}(\widetilde{\Omega}; C^{0,(\mathfrak{p}-1)/(4\mathfrak{p})}([0,T];L^2(\mathcal{O}))), \tag{7.2a}$$

$$\widetilde{\phi_\Gamma} \in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T;H^1(\Gamma))) \cap L^{2\mathfrak{p}}(\widetilde{\Omega}; C^{0,(\mathfrak{p}-1)/(2\mathfrak{p})}([0,T];L^2(\Gamma))), \tag{7.2b}$$

$$\widetilde{r} \in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T)), \tag{7.2c}$$

$$\widetilde{s} \in L^{2\mathfrak{p}}_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T)), \tag{7.2d}$$

$$\widetilde{\mu} \in L^{2\mathfrak{p}}(\widetilde{\Omega}; L^2(0,T;H^1(\mathcal{O}))), \tag{7.2e}$$

$$\widetilde{L} \in L^{2\mathfrak{p}}(\widetilde{\Omega}; L^2(0,T;(H^1(\mathcal{O}))')), \tag{7.2f}$$

$$\widetilde{\theta} \in L^{2\mathfrak{p}}(\widetilde{\Omega}; L^2(0,T;L^2(\Gamma))) \tag{7.2g}$$

for any $\mathfrak{p} \in (1,\infty)$ *such that* $[\widetilde{\phi}]|_\Gamma = \widetilde{\phi_\Gamma}$ $\widetilde{\mathbb{P}}$*-almost surely almost everywhere on* $(0,T)\times\Gamma$, $\widetilde{r} = \sqrt{\int_{\mathcal{O}} F(\widetilde{\phi})\,\mathrm{d}x}$ *and* $\widetilde{s} = \sqrt{\int_\Gamma G(\widetilde{\phi_\Gamma})\,\mathrm{d}\Gamma}$ $\widetilde{\mathbb{P}}$*-almost surely almost everywhere on* $(0,T)$, *and* $\widetilde{L} = l[\widetilde{\mu}]$ $\widetilde{\mathbb{P}}$*-almost surely almost everywhere on* $(0,T)\times\mathcal{O}$*. Furthermore, for* $j\to\infty$*, the following convergence statements hold true along a subsequence:*

$$\widetilde{\phi}_j \to \widetilde{\phi} \qquad \text{in } L^p(\widetilde{\Omega}; C([0,T]; L^s(\mathcal{O}))), \tag{7.3a}$$

$$\widetilde{\phi}_j^{(\pm)} \to \widetilde{\phi} \qquad \text{in } L^p(\widetilde{\Omega}; L^p(0,T; L^s(\mathcal{O}))), \tag{7.3b}$$

$$\widetilde{\phi}_j^{(\pm)} \overset{*}{\rightharpoonup} \widetilde{\phi} \qquad \text{in } L^p_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T; H^1(\mathcal{O}))), \tag{7.3c}$$

$$[\widetilde{\phi}_j]|_\Gamma \to \widetilde{\phi}_\Gamma \qquad \text{in } L^p(\widetilde{\Omega}; C([0,T]; L^r(\Gamma))), \tag{7.3d}$$

$$[\widetilde{\phi}_j^{(\pm)}]|_\Gamma \to \widetilde{\phi}_\Gamma \qquad \text{in } L^p(\widetilde{\Omega}; L^p(0,T; L^r(\Gamma))) \tag{7.3e}$$

$$[\widetilde{\phi}_j^{(\pm)}]|_\Gamma \overset{*}{\rightharpoonup} \widetilde{\phi}_\Gamma \qquad \text{in } L^p_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T; H^1(\Gamma))), \tag{7.3f}$$

$$\widetilde{\mu}_j^+ \rightharpoonup \widetilde{\mu} \qquad \text{in } L^p(\widetilde{\Omega}; L^2(0,T; H^1(\mathcal{O}))), \tag{7.3g}$$

$$\widetilde{\theta}_j^+ \rightharpoonup \widetilde{\theta} \qquad \text{in } L^p(\widetilde{\Omega}; L^2(0,T; L^2(\Gamma))), \tag{7.3h}$$

$$\int_{\mathcal{O}} \mathcal{I}_{h_j}\{F(\widetilde{\phi}_j^{(\pm)})\}\,\mathrm{d}x \to \int_{\mathcal{O}} F(\widetilde{\phi})\,\mathrm{d}x \qquad \text{in } L^p(\widetilde{\Omega}; L^p(0,T)), \tag{7.3i}$$

$$\widetilde{r}_j^{(\pm)} \to \widetilde{r} \qquad \text{in } L^p(\widetilde{\Omega}; L^p(0,T)), \tag{7.3j}$$

$$\widetilde{r}_j^{(\pm)} \overset{*}{\rightharpoonup} \widetilde{r} \qquad \text{in } L^p_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T)), \tag{7.3k}$$

$$\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{G([\widetilde{\phi}_j^{(\pm)}]|_\Gamma)\}\,\mathrm{d}\Gamma \to \int_\Gamma G(\widetilde{\phi}_\Gamma)\,\mathrm{d}\Gamma \qquad \text{in } L^p(\widetilde{\Omega}, L^p(0,T)), \tag{7.3l}$$

$$\widetilde{s}_j^{(\pm)} \to \widetilde{s} \qquad \text{in } L^p(\widetilde{\Omega}; L^p(0,T)), \tag{7.3m}$$

$$\widetilde{s}_j^{(\pm)} \overset{*}{\rightharpoonup} \widetilde{s} \qquad \text{in } L^p_{\text{weak-}(*)}(\widetilde{\Omega}; L^\infty(0,T)), \tag{7.3n}$$

$$\widetilde{\Xi}_{j,\mathcal{O}}^+ \to 0 \qquad \text{in } L^p(\widetilde{\Omega}; L^2(0,T; L^2(\mathcal{O}))), \tag{7.3o}$$

$$\widetilde{\Xi}_{j,\Gamma}^+ \to 0 \qquad \text{in } L^p(\widetilde{\Omega}; L^2(0,T; L^2(\Gamma))) \tag{7.3p}$$

for $p\in[1,\infty)$*,* $s\in[1,\frac{2d}{d-2})$*, and* $r\in[1,\infty)$*.*

Proof. Due to Theorem 7.3, we have $\widetilde{\phi}_j\to\widetilde{\phi}$ in $C([0,T];L^s(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-almost surely. By Vitali's convergence theorem and the bounds in equation (7.1), we obtain the strong convergence expressed in equation (7.3a). This strong convergence provides the existence of a subsequence $\widetilde{\phi}_j\to\widetilde{\phi}$ pointwise almost everywhere in $\widetilde{\Omega}\times[0,T]\times\mathcal{O}$. As equation (7.1a) entails

$$\|\widetilde{\phi}_j^{(\pm)} - \widetilde{\phi}_j\|_{L^{2p}(\widetilde{\Omega};L^2(0,T;H^1(\mathcal{O})))} \to 0,$$

there exists a subsequence $(\widetilde{\phi}_j^{(\pm)})_{j\in\mathbb{N}}$ converging pointwise almost everywhere toward $\widetilde{\phi}$. Hence, we can again combine the uniform bounds from equation (7.1a) and Vitali's convergence theorem to establish equation (7.3b). Similar arguments provide equations (7.3d) and (7.3e). Finally, choosing $\boldsymbol{\psi} \in L^2(\widetilde{\Omega}; L^2(0,T; (C^\infty(\overline{\mathcal{O}}))^d))$, we obtain

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big[\int_0^T \int_{\mathcal{O}} \widetilde{\phi} \operatorname{div} \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t\Big] &\leftarrow \widetilde{\mathbb{E}}\Big[\int_0^T \int_{\mathcal{O}} \widetilde{\phi}_j^{(\pm)} \operatorname{div} \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t\Big] \\
&= \widetilde{\mathbb{E}}\Big[-\int_0^T \int_{\mathcal{O}} \nabla \widetilde{\phi}_j^{(\pm)} \cdot \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Gamma} [\widetilde{\phi}_j^{(\pm)}]|_\Gamma \boldsymbol{\psi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \, \mathrm{d}t\Big] \\
&\to \widetilde{\mathbb{E}}\Big[-\int_0^T \int_{\mathcal{O}} \nabla \widetilde{\phi} \cdot \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Gamma} \widetilde{\phi}_\Gamma \boldsymbol{\psi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \, \mathrm{d}t\Big] \\
&= \widetilde{\mathbb{E}}\Big[\int_0^T \int_{\mathcal{O}} \widetilde{\phi} \operatorname{div} \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t\Big] + \widetilde{\mathbb{E}}\Big[\int_0^T \int_{\Gamma} (\widetilde{\phi}_\Gamma - [\widetilde{\phi}]|_\Gamma) \boldsymbol{\theta} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \, \mathrm{d}t\Big],
\end{aligned}$$

allowing us to identify $\widetilde{\phi}_\Gamma$ with $[\widetilde{\phi}]|_\Gamma$.

The weak and weak* convergence results stated in equations (7.3c), (7.3f), (7.3g), (7.3h), (7.3k), and (7.3n) are a direct consequence of the uniform bounds in equation (7.1a).

The convergence results in equations (7.3i)–(7.3j) can be obtained as follows (see also [44]): By [43, estimate (4.8)], we have

$$\operatorname*{ess\,sup}_{t \in (0,T)} \int_{\mathcal{O}} |(1 - \mathcal{I}_{h_j})\{F(\widetilde{\phi}_j)\}| \, \mathrm{d}x \leq C h_j |\mathcal{O}| + C h_j \operatorname*{ess\,sup}_{t \in (0,T)} \|\widetilde{\phi}_j^{(\pm)}\|_{H^1(\mathcal{O})}^6,$$

which indicates that the interpolation operator $\mathcal{I}_{h_j}$ is negligible when passing to the limit. Starting from a pointwise almost-everywhere convergent subsequence, we may use the continuity of F, the growth condition in assumption **(P)**, and Vitali's convergence theorem to establish equation (7.3i). The strong convergence follows from equation (7.3i) and Lemma 6.4. This also allows for the identification of $\widetilde{r}$ with $\sqrt{\int_{\mathcal{O}} F(\widetilde{\phi}) \, \mathrm{d}x}$. Similar arguments provide equations (7.3l) and (7.3m) and the identification of $\widetilde{s}$. The strong convergence of $\Xi_{j,\mathcal{O}}^+$ and $\Xi_{j,\Gamma}^+$ stated in equations (7.3o) and (7.3p) follows from equation (7.1b). ■

8. Passage to the limit

The goal of this section is the identification of the limits of the (discrete) stochastic integrals. We shall use a more precise notation for our time grid and write t_j^n $(n = 0, \ldots, N_j)$ for the nodes of the equidistant grid obtained using the step size τ_j. Denoting the increments $\widetilde{\boldsymbol{\xi}}_j(t_j^n) - \widetilde{\boldsymbol{\xi}}_j(t_j^{n-1})$ by $\blacktriangle^n \widetilde{\boldsymbol{\xi}}^j$ and using the stochastic processes defined in the last section, we can rewrite equations (3.4a) and (3.4b) as

$$\begin{aligned}
\int_{\mathcal{O}} \mathcal{I}_{h_j}\{(\widetilde{\phi}_j(t) - \widetilde{\phi}_j^-)\psi_{h_j}\} \, \mathrm{d}x + (t - t_j^{n-1}) \int_{\mathcal{O}} \nabla \widetilde{\mu}_j^+ \cdot \nabla \psi_{h_j} \, \mathrm{d}x \\
= \frac{t - t_j^{n-1}}{\tau_j} \int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j^-) \blacktriangle^n \widetilde{\boldsymbol{\xi}}^j \psi_{h_j}\} \, \mathrm{d}x, \qquad (8.1a)
\end{aligned}$$

$$\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\widetilde{\phi}_j(t)-\widetilde{\phi}_j^-]|_\Gamma \widehat{\psi}_{h_j}\}\,\mathrm{d}\Gamma + (t-t_j^{n-1})\int_\Gamma \mathcal{I}_h^\Gamma\{\widetilde{\theta}_j^+\widehat{\psi}_{h_j}\}\,\mathrm{d}\Gamma$$
$$= \frac{t-t_j^{n-1}}{\tau_j}\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\Phi_{h_j}(\widetilde{\phi}_j^-)\blacktriangle^n\widetilde{\boldsymbol{\xi}}^j]|_\Gamma\widehat{\psi}_{h_j}\}\,\mathrm{d}\Gamma, \tag{8.1b}$$

for all $\psi_{h_j}\in U_{h_j}^{\mathcal{O}}$ and $\widehat{\psi}_{h_j}\in U_{h_j}^\Gamma$. The relation between $\widetilde{\phi}_j$, $\widetilde{r}_j^+$, $\widetilde{s}_j^+$, $\widetilde{\mu}_j^+$, and $\widetilde{\theta}_j^+$ in the interval (t_j^{n-1},t_j^n) is given by

$$\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\widetilde{\mu}_j^+\eta_{h_j}\}\,\mathrm{d}x + \int_\Gamma \mathcal{I}_{h_j}^\Gamma\{\widetilde{\theta}_j^+[\eta_{h_j}]|_\Gamma\}\,\mathrm{d}\Gamma$$
$$= \int_{\mathcal{O}}\nabla\widetilde{\phi}_j^+\cdot\nabla\eta_{h_j}\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma[\widetilde{\phi}_j^+]|_\Gamma\cdot\nabla_\Gamma[\eta_{h_j}]|_\Gamma\,\mathrm{d}\Gamma$$
$$+\frac{\widetilde{r}_j^+}{\sqrt{E_{h_j}^{\mathcal{O}}(\widetilde{\phi}_j^-)}}\int_{\mathcal{O}}\mathcal{I}_{h_j}\{F'(\widetilde{\phi}_j^-)\eta_{h_j}\}\,\mathrm{d}x + \int_{\mathcal{O}}\mathcal{I}_{h_j}\{\widetilde{\Xi}_{j,\mathcal{O}}^+\eta_{h_j}\}\,\mathrm{d}x$$
$$+\frac{\widetilde{s}_j^+}{\sqrt{E_{h_j}^\Gamma([\widetilde{\phi}_j^-]|_\Gamma)}}\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{G'([\widetilde{\phi}_j^-]|_\Gamma)[\eta_{h_j}]|_\Gamma\}\,\mathrm{d}\Gamma$$
$$+\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{\widetilde{\Xi}_{j,\Gamma}^+[\eta_{h_j}]|_\Gamma\}\,\mathrm{d}\Gamma \tag{8.1c}$$

for all $\eta_{h_j}\in U_{h_j}^{\mathcal{O}}$.

As the convergence properties collected in Lemma 7.5 allow to pass to the limit $j\to\infty$ in equation (8.1c) and the left-hand sides of equations (8.1a) and (8.1b), it remains to identify the limit of the right-hand sides of equations (8.1a) and (8.1b) as suitable Itô integrals.

We start by introducing the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ as the augmentation of the filtration generated by $\widetilde{\phi}$, $\widetilde{L}$, $\widetilde{\mu}$, $\widetilde{\phi_\Gamma}$, $\widetilde{\theta}$, and $\widetilde{W}$ and show that $\widetilde{W}$ is a $\mathcal{Q}$-Wiener process.

Lemma 8.1. *The process $\widetilde{W}$ obtained in Theorem 7.3 is a $\mathcal{Q}$-Wiener process adapted to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ and can be written as*

$$\widetilde{W}=\sum_{k\in\mathbb{Z}}\lambda_k\mathfrak{g}_k\widetilde{\beta}_k.$$

Here $(\widetilde{\beta}_k)_{k\in\mathbb{Z}}$ is a family of independently and identically distributed Brownian motions with respect to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$.

Proof. As the laws of $\boldsymbol{\xi}_{h_j}^{\tau_j}$ and $\widetilde{\boldsymbol{\xi}}_j$ coincide, we have $\widetilde{\mathbb{P}}$-almost surely that $\widetilde{\boldsymbol{\xi}}_j$ is piecewise linear in time and satisfies

$$\widetilde{\boldsymbol{\xi}}_j(t_j^m)=\sum_{n=1}^m\sqrt{\tau_j}\sum_{k\in\mathbb{Z}_{h_j}}\mathfrak{g}_k\widetilde{\xi}_k^{n,\tau_j},$$

where $\widetilde{\xi}_k^{n,\tau_j}$ are mutually independent random variables on $\widetilde{\Omega}$ satisfying assumption **(D2)**. Although $\widetilde{\boldsymbol{\xi}}_j$ is not a martingale, we can use the $\widetilde{\mathbb{P}}$-almost sure convergence of $\widetilde{\phi}_j$, $[\widetilde{\phi}_j]|_\Gamma$, $l[\widetilde{\mu}_j^+]$, $\widetilde{\mu}_j^+$, $\widetilde{\theta}_j^+$, and $\widetilde{\boldsymbol{\xi}}_j$ established in Theorem 7.3 to show that the limit process $\widetilde{W}$ is a martingale. According to [49, Lemma 5.8] it suffices to show that

$$\widetilde{\mathbb{E}}\Big[(\widetilde{W}(t_2) - \widetilde{W}(t_1)) \prod_{i=1}^{q} \prod_{\iota=1}^{o} f_{i\iota}(\Psi_{i\iota}(\widetilde{\phi}(s_\iota), \widetilde{\phi_\Gamma}(s_\iota), \widetilde{L}(s_\iota), \widetilde{\mu}(s_\iota), \widetilde{\theta}(s_\iota), \widetilde{W}(s_\iota)))\Big] \\ = 0$$

for all times $0 \leq s_1 < \ldots < s_q \leq t_1 < t_2 \leq T$, all $f_{i\iota} \in C_b(\mathbb{R})$, and for all $\Psi_{1\iota}, \ldots, \Psi_{r\iota} \in \mathcal{A}_{s_\iota}$, where $\mathcal{A}_{s_\iota}$ is a subset of real-valued functions on

$$\widehat{\mathcal{X}}_{s_\iota} := C([0,s_\iota]; L^s(\mathcal{O})) \times C([0,s_\iota]; L^r(\Gamma)) \times L^2(0,s_\iota; (H^1(\mathcal{O}))')_{\text{weak}} \\ \times L^2(0,s_\iota; H^1(\mathcal{O}))_{\text{weak}} \times L^2(0,s_\iota; L^2(\Gamma))_{\text{weak}} \times C([0,s_\iota]; H^1(\mathcal{O}))$$

for which the σ-algebra of $\mathcal{A}_{s_\iota}$ equals the Borel σ-algebra on $\widehat{\mathcal{X}}_{s_\iota}$. As explained in [49, Example 5.9], these functions can always be chosen in a manner such that they are continuous with respect to weakly converging sequences. Hence, using the abbreviations

$$\mathfrak{f}^{qo} := \prod_{i=1}^{q} \prod_{\iota=1}^{o} f_{i\iota}(\Psi_{i\iota}(\widetilde{\phi}(s_\iota), \widetilde{\phi_\Gamma}(s_\iota), \widetilde{L}(s_\iota), \widetilde{\mu}(s_\iota), \widetilde{\theta}(s_\iota), \widetilde{W}(s_\iota))), \tag{8.2a}$$

$$\mathfrak{f}_j^{qo} := \prod_{i=1}^{q} \prod_{\iota=1}^{o} f_{i\iota}(\Psi_{i\iota}(\widetilde{\phi}_j(s_\iota), [\widetilde{\phi}_j]|_\Gamma(s_\iota), l[\widetilde{\mu}_j^+](s_\iota), \widetilde{\mu}_j^+(s_\iota), \widetilde{\theta}_j^+(s_\iota), \widetilde{\boldsymbol{\xi}}_j(s_\iota))) \tag{8.2b}$$

for times $0 \leq s_1 < \ldots < s_q \leq t_1 < t_2$, the uniform integrability of $\widetilde{\boldsymbol{\xi}}_j$ established in Lemma 7.1, and a Vitali argument, we obtain

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big[(\widetilde{W}(t_2) - \widetilde{W}(t_1)) \prod_{i=1}^{q} \prod_{\iota=1}^{o} f_{i\iota}(\Psi_{i\iota}(\widetilde{\phi}(s_\iota), \widetilde{\phi_\Gamma}(s_\iota), \widetilde{L}(s_\iota), \widetilde{\mu}(s_\iota), \widetilde{\theta}(s_\iota), \widetilde{W}(s_\iota)))\Big] \\
&\quad = \lim_{j\to\infty} \widetilde{\mathbb{E}}[(\widetilde{\boldsymbol{\xi}}_j(t_2) - \widetilde{\boldsymbol{\xi}}_j(t_1))\mathfrak{f}_j^{qo}] \\
&\quad = \lim_{j\to\infty} \widetilde{\mathbb{E}}[(\widetilde{\boldsymbol{\xi}}_j(t_j^n) - \widetilde{\boldsymbol{\xi}}_j(t_j^m))\mathfrak{f}_j^{qo}] + \lim_{j\to\infty} \widetilde{\mathbb{E}}[(\widetilde{\boldsymbol{\xi}}_j(t_2) - \widetilde{\boldsymbol{\xi}}_j(t_j^n))\mathfrak{f}_j^{qo}] \\
&\qquad - \lim_{j\to\infty} \widetilde{\mathbb{E}}[(\widetilde{\boldsymbol{\xi}}_j(t_1) - \widetilde{\boldsymbol{\xi}}_j(t_j^m))\mathfrak{f}_j^{qo}] \\
&\quad = \lim_{j\to\infty} \widetilde{\mathbb{E}}[\sum_{a=m+1}^{n} \sqrt{\tau_j} \sum_{k\in\mathbb{Z}_{h_j}} \mathfrak{g}_k \widetilde{\xi}_k^{a,\tau_j}] = 0.
\end{aligned}$$

Here, we compared t_1 and t_2 to grid points t_j^n and t_j^m satisfying $0 \leq t_j^n - t_2 \leq \tau_j$ and $0 \leq t_j^m - t_1 \leq \tau_j$ and used the continuity of $\widetilde{\boldsymbol{\xi}}_j$. We now define $\widetilde{\beta}_k(t) := \lambda_k^{-1} \int_{\mathcal{O}} \widetilde{W}(t,x)\mathfrak{g}_k(x)\,\mathrm{d}x$ and use similar arguments to show that

$$\widetilde{\beta}_k(t)\widetilde{\beta}_l(t) - \delta_{kl}t$$

with δ_{kl} denoting the usual Kronecker delta is a martingale. Hence, by Levy's characterization of Brownian motions, we obtain the result. ∎

In the next step, we shall show that the limit processes of the left-hand sides of equations (8.1a) and (8.1b) are martingales with respect to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ and identify the corresponding quadratic variation processes. Hence, we consider for arbitrary but fixed $v \in C^\infty(\overline{\mathcal{O}})$ and $w \in C^\infty(\Gamma)$ the families of processes

$$\widetilde{M}_j^v(t) := \int_{\mathcal{O}} \mathcal{I}_{h_j}\{(\widetilde{\phi}_j(t) - \widetilde{\phi}_j(0))v\}\,\mathrm{d}x + \int_0^t \int_{\mathcal{O}} \nabla\widetilde{\mu}_j^+ \cdot \nabla\mathcal{I}_{h_j}\{v\}\,\mathrm{d}x\,\mathrm{d}s, \tag{8.3a}$$

$$\widetilde{N}_j^w(t) := \int_{\Gamma} \mathcal{I}_{h_j}^\Gamma\{(\widetilde{\phi}_j(t) - \widetilde{\phi}_j(0))w\}\,\mathrm{d}\Gamma + \int_0^t \int_{\Gamma} \mathcal{I}_h^\Gamma\{\widetilde{\theta}_j^+ w\}\,\mathrm{d}\Gamma\,\mathrm{d}s. \tag{8.3b}$$

We shall now show that these processes converge toward the processes

$$\widetilde{M}^v(t) := \int_{\mathcal{O}} (\widetilde{\phi}(t) - \widetilde{\phi}(0))v\,\mathrm{d}x + \int_0^t \int_{\mathcal{O}} \nabla\widetilde{\mu} \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}s, \tag{8.4a}$$

$$\widetilde{N}^w(t) := \int_{\Gamma} (\widetilde{\phi}_\Gamma(t) - \widetilde{\phi}_\Gamma(0))w\,\mathrm{d}\Gamma + \int_0^t \int_{\Gamma} \widetilde{\theta} w\,\mathrm{d}\Gamma\,\mathrm{d}s, \tag{8.4b}$$

which are martingales with respect to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$.

Lemma 8.2. *Let the assumptions of Lemma 7.5 hold true. Then, the processes $\widetilde{M}^v$ and $\widetilde{N}^w$ defined in equation (8.4) with $v \in C(\overline{\mathcal{O}})$ and $w \in C(\Gamma)$ are martingales with respect to the filtration $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$.*

Proof. We start by showing that the piecewise linear process $\widetilde{M}_j^v$ converges toward $\widetilde{M}^v$ in $L^p(\widetilde{\Omega})$ for $t \in [0, T]$:

$$\begin{aligned}
&\widetilde{\mathbb{E}}[|\widetilde{M}^v(t) - \widetilde{M}_j^v(t)|^p] \\
&\quad\le C\widetilde{\mathbb{E}}\Big[\Big|\int_{\mathcal{O}} (\widetilde{\phi}(t) - \widetilde{\phi}_j(t) - \widetilde{\phi}(0) + \widetilde{\phi}_j(0))v\,\mathrm{d}x\Big|^p\Big] \\
&\qquad + C\widetilde{\mathbb{E}}\Big[\Big|\int_{\mathcal{O}} (\widetilde{\phi}_j(t) - \widetilde{\phi}_j(0))(1 - \mathcal{I}_{h_j})\{v\}\,\mathrm{d}x\Big|^p\Big] \\
&\qquad + C\widetilde{\mathbb{E}}\Big[\Big|\int_{\mathcal{O}} (1 - \mathcal{I}_{h_j})\{(\widetilde{\phi}_j(t) - \widetilde{\phi}_j(0))\mathcal{I}_{h_j}\{v\}\}\,\mathrm{d}x\Big|^p\Big] \\
&\qquad + C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t \int_{\mathcal{O}} \nabla(\widetilde{\mu} - \widetilde{\mu}_j^+) \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] \\
&\qquad + C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t \int_{\mathcal{O}} \nabla\widetilde{\mu}_j^+ \cdot \nabla(1 - \mathcal{I}_{h_j})\{v\}\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] \to 0,
\end{aligned}$$

due to the convergence results from Theorem 7.3, Lemma 2.1, the regularity results collected in equation (7.1), and the standard error estimates for the nodal interpolation operator (see, e.g., [10]). Hence, we can repeat the arguments from the proof of Lemma 8.1 and show that

$$\widetilde{\mathbb{E}}[(\widetilde{M}^v(t_2) - \widetilde{M}^v(t_1))\mathfrak{f}^{qo}] = \lim_{j\to\infty} \widetilde{\mathbb{E}}[(\widetilde{M}_j^v(t_j^n) - \widetilde{M}_j^v(t_j^m))\mathfrak{f}_j^{qo}]$$
$$= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\sum_{a=m+1}^{n} \int_{\mathcal{O}} \Phi_{h_j}(\widetilde{\phi}_j(t_j^{a-1}))\blacktriangle^a \widetilde{\xi}^j \,\mathrm{d}x\Big] = 0,$$

due to assumption **(D2)**. Here, we again used grid points t_j^n and t_j^m satisfying $0 \leq t_j^n - t_2 \leq \tau_j$ and $0 \leq t_j^m - t_1 \leq \tau_j$ and the abbreviations in equation (8.2) for times $0 \leq s_1 < \ldots < s_q \leq t_1 < t_2$. This establishes the martingale property of $\widetilde{M}^v$. The properties of $\widetilde{N}^w$ follow by similar arguments. ■

In the next step, we shall identify the quadratic variation processes $\langle\widetilde{M}^v\rangle$ and $\langle\widetilde{N}^w\rangle$ of $\widetilde{M}^v$ and $\widetilde{N}^w$, as well as their cross-variation processes $\langle\widetilde{M}^v, \widetilde{\beta}_k\rangle$ and $\langle\widetilde{N}^w, \widetilde{\beta}_k\rangle$. Natural candidates for the quadratic and cross variation processes are

$$\langle\widetilde{M}^v\rangle(t) := \int_0^t \sum_{k\in\mathbb{Z}} \Big(\int_{\mathcal{O}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x\Big)^2 \mathrm{d}s, \tag{8.5a}$$

$$\langle\widetilde{M}^v, \widetilde{\beta}_k\rangle(t) := \int_0^t \int_{\mathcal{O}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x\,\mathrm{d}s, \tag{8.5b}$$

$$\langle\widetilde{N}^w\rangle(t) := \int_0^t \sum_{k\in\mathbb{Z}} \Big(\int_{\Gamma} \varrho(\widetilde{\phi}_\Gamma)\lambda_k [\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma\Big)^2 \mathrm{d}s, \tag{8.5c}$$

$$\langle\widetilde{N}^w, \widetilde{\beta}_k\rangle(t) := \int_0^t \int_{\Gamma} \varrho(\widetilde{\phi}_\Gamma)\lambda_k [\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma\,\mathrm{d}s. \tag{8.5d}$$

Lemma 8.3. *The processes defined in equation* (8.5) *are the quadratic and cross-variation processes of the martingales* $\widetilde{M}^v$ *and* $\widetilde{N}^w$ *defined in equation* (8.4).

Proof. We start by showing that $(\widetilde{M}^v)^2 - \langle\widetilde{M}^v\rangle$ is a martingale, that is, using the abbreviations introduced in equation (8.2) we shall show that

$$\widetilde{\mathbb{E}}[((\widetilde{M}^v(t_2))^2 - (\widetilde{M}^v(t_1))^2 - \langle\widetilde{M}^v\rangle(t_2) + \langle\widetilde{M}^v\rangle(t_1))\mathfrak{f}^{qo}] = 0.$$

For this reason, we introduce a family of approximations $(\langle\widetilde{M}_j^v\rangle)_{j\in\mathbb{N}}$ of $\langle\widetilde{M}^v\rangle$ via

$$\langle\widetilde{M}_j^v\rangle(t_j^m) := \int_0^{t_j^m} \sum_{k\in\mathbb{Z}_{h_j}} \Big(\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k v\} \,\mathrm{d}x\Big)^2 \mathrm{d}s$$

and prove that $\langle\widetilde{M}_j^v\rangle(t_j^m)$ converges in $L^p(\widetilde{\Omega})$ toward $\langle\widetilde{M}^v\rangle(t)$ for $t_j^m \searrow t$. We decompose the discretization error as follows:

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\big[|\langle\widetilde{M}^v\rangle(t)-\langle\widetilde{M}_j^v\rangle(t_j^m)|^p\big]\\
&\quad\le C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t\sum_{k\in\mathbb{Z}}\Big(\int_{\mathcal{O}}\varrho(\widetilde{\phi})\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\Big)^2\mathrm{d}s-\int_0^t\sum_{k\in\mathbb{Z}_{h_j}}\Big(\int_{\mathcal{O}}\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\Big)^2\Big|^p\Big]\\
&\qquad+C\widetilde{\mathbb{E}}\Big[\Big|\int_t^{t_j^m}\sum_{k\in\mathbb{Z}_{h_j}}\Big(\int_{\mathcal{O}}\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\Big)^2\mathrm{d}s\Big|^p\Big]\\
&\qquad+C\widetilde{\mathbb{E}}\Big[\Big|\int_0^{t_j^m}\sum_{k\in\mathbb{Z}_{h_j}}\Big[\Big(\int_{\mathcal{O}}\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\Big)^2-\Big(\int_{\mathcal{O}}\mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\}\,\mathrm{d}x\Big)^2\Big]\mathrm{d}s\Big|^p\Big]\\
&\quad=:Q_1^{\mathcal{O}}+Q_2^{\mathcal{O}}+Q_3^{\mathcal{O}}.
\end{aligned}
$$

From assumptions **(D3)** and **(C)** and Lemma 7.5, we obtain that $Q_1^{\mathcal{O}}$ vanishes. The bounds stated in assumption **(C)** yield $Q_2^{\mathcal{O}}\le C(v)|t_j^m-t|^p\to 0$. To show that also $Q_3^{\mathcal{O}}$ vanishes, we use assumption **(C)**, the standard error estimates for the interpolation operator (see, e.g., [10]), and Lemma 2.1 to compute

$$
\begin{aligned}
&\Big|\int_{\mathcal{O}}\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\,\mathrm{d}x-\int_{\mathcal{O}}\mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\}\,\mathrm{d}x\Big|\\
&\quad\le\Big|\int_{\mathcal{O}}(1-\mathcal{I}_{h_j})\{\varrho(\widetilde{\phi}_j^-)\}\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\Big|+\Big|\int_{\mathcal{O}}\mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\}\lambda_k(1-\mathcal{I}_{h_j})\{\mathfrak{g}_k v\}\,\mathrm{d}x\Big|\\
&\qquad+\Big|\int_{\mathcal{O}}(1-\mathcal{I}_{h_j})\{\mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\}\mathcal{I}_{h_j}\{\lambda_k\mathfrak{g}_k v\}\}\,\mathrm{d}x\Big|\\
&\quad\le Ch_j\|\nabla\widetilde{\phi}_j^-\|_{L^2(\mathcal{O})}|\lambda_k|\|\mathfrak{g}_k\|_{L^\infty(\mathcal{O})}\|v\|_{L^2(\mathcal{O})}+Ch_j|\lambda_k|\|\mathfrak{g}_k\|_{W^{1,\infty}(\mathcal{O})}\|v\|_{W^{1,\infty}(\mathcal{O})}\\
&\qquad+Ch_j|\lambda_k|\|\mathfrak{g}_k\|_{W^{1,\infty}(\mathcal{O})}\|v\|_{W^{1,\infty}(\mathcal{O})}.
\end{aligned}\tag{8.6}
$$

Hence,

$$
\begin{aligned}
Q_3^{\mathcal{O}}&\le C\widetilde{\mathbb{E}}\Big[\Big|\int_0^{t_j^m}\sum_{k\in\mathbb{Z}_{h_j}}\Big|\int_{\mathcal{O}}(1+\mathcal{I}_{h_j})\{\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\}\,\mathrm{d}x\Big|h_j|\lambda_k|\|\mathfrak{g}_k\|_{W^{2,\infty}(\mathcal{O})}\\
&\qquad\times\|v\|_{W^{1,\infty}(\mathcal{O})}(1+\|\nabla\widetilde{\phi}_j^-\|_{L^2(\mathcal{O})})\,\mathrm{d}s\Big|^p\Big]\\
&\le C(v)\widetilde{\mathbb{E}}\big[\big|h_j\sum_{k\in\mathbb{Z}}|\lambda_k|^2\|\mathfrak{g}_k\|^2_{W^{2,\infty}(\mathcal{O})}\int_0^{t_j^m}(1+\|\nabla\widetilde{\phi}_j^-\|_{L^2(\mathcal{O})})\,\mathrm{d}s\big|^p\big]\to 0.
\end{aligned}
$$

Having established the convergence in $L^p(\widetilde{\Omega})$, we have $\widetilde{\mathbb{P}}$-almost sure convergence for a subsequence. Using similar arguments, we also obtain the uniform bounds for higher moments of $\langle\widetilde{M}_j^v\rangle$. Hence, we can reuse the ideas from the proof of Lemma 8.2 and deduce for $0\le t_j^n-t_2\le\tau_j$ and $0\le t_j^m-t_1\le\tau_j$

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[((\widetilde{M}^v(t_2))^2-(\widetilde{M}^v(t_1))^2-\langle\widetilde{M}^v\rangle(t_2)+\langle\widetilde{M}^v\rangle(t_1))\mathfrak{f}^{qo}]\\
&\quad=\lim_{j\to\infty}\widetilde{\mathbb{E}}[((\widetilde{M}_j^v(t_j^n))^2-(\widetilde{M}_j^v(t_j^m))^2-\langle\widetilde{M}_j^v\rangle(t_j^n)+\langle\widetilde{M}_j^v\rangle(t_j^m))\mathfrak{f}_j^{qo}]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\Big(\sum_{a=m+1}^{n}\Big(\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{a-1}))\blacktriangle^a \widetilde{\boldsymbol{\xi}}^j v\}\,\mathrm{d}x\Big)^2 \\
&\qquad - \sum_{a=m+1}^{n} \tau_j \sum_{k\in\mathbb{Z}_{h_j}} \Big(\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j(t_j^{a-1}))\lambda_k \mathfrak{g}_k v\}\,\mathrm{d}x\Big)^2\Big)\mathfrak{f}_j^{qo}\Big] = 0.
\end{aligned}
$$

Here we used the mutual independence of the stochastic increments and $\widetilde{\mathbb{E}}[|\widetilde{\xi}_k^{n,\tau_j}|^2] = 1$ (cf. assumption (D2)).

We shall now apply similar arguments to show that $\langle\widetilde{M}^v, \widetilde{\beta}_k\rangle$ defined in equation (8.5b) is indeed the cross-variation process. We start by defining suitable time-discrete approximations for time points t_j^m via

$$
\langle\widetilde{M}_j^v, \sum_{a=1}^{m}\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j}\rangle := \begin{cases} \int_0^{t_j^m}\int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s & \text{if } k\in\mathbb{Z}_{h_j}, \\ 0 & \text{else,}\end{cases}
$$

and show that this approximation converges toward $\langle\widetilde{M}^v, \widetilde{\beta}_k\rangle(t)$ for $t_j^m \searrow t$. As $\bigcup_{h>0}\mathbb{Z}_h = \mathbb{Z}$, we can assume without loss of generality that $k\in\mathbb{Z}_{h_j}$. Hence,

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathcal{O}} \varrho(\widetilde{\phi})\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\,\mathrm{d}s - \int_0^{t_j^m}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] \\
&\quad \le C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathcal{O}}(\varrho(\widetilde{\phi})-\varrho(\widetilde{\phi}_j^-))\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] + C\widetilde{\mathbb{E}}\Big[\Big|\int_t^{t_j^m}\int_{\mathcal{O}}\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] \\
&\qquad + C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathcal{O}}(1-\mathcal{I}_{h_j})\{\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\Big|^p\Big] \\
&\quad =: R_1^{\mathcal{O}} + R_2^{\mathcal{O}} + R_3^{\mathcal{O}}.
\end{aligned}
$$

As before, $R_1^{\mathcal{O}}$ vanishes due to assumption (C) and Lemma 7.5, while $R_2^{\mathcal{O}}$ vanishes as $R_2^{\mathcal{O}} \le C(v)|t_j^m - t|^p$. To treat $R_3^{\mathcal{O}}$, we use the estimates in equation (8.6). Hence, we can again argue for $0 \le t_j^n - t_2 \le \tau_j$ and $0 \le t_j^m - t_1 \le \tau_j$ as follows:

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[(\widetilde{M}^v(t_2)\widetilde{\beta}_k(t_2) - \widetilde{M}^v(t_1)\widetilde{\beta}_k(t_1) - \langle\widetilde{M}^v,\widetilde{\beta}_k\rangle(t_2) + \langle\widetilde{M}^v,\widetilde{\beta}_k\rangle(t_1))\mathfrak{f}^{qo}] \\
&\quad = \lim_{j\to\infty}\widetilde{\mathbb{E}}\Big[\Big(\widetilde{M}_j^v(t_j^n)\sum_{a=1}^{n}\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j} - \widetilde{M}_j^v(t_j^m)\sum_{a=1}^{m}\sqrt{\tau_j}\widetilde{\xi}_j^{a,\tau_j} \\
&\qquad\quad - \int_{t_j^m}^{t_j^n}\int_{\mathcal{O}}\lambda_k\mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\Big)\mathfrak{f}_j^{qo}\Big] \\
&\quad = \lim_{j\to\infty}\widetilde{\mathbb{E}}\Big[\sum_{b=1}^{n}\int_{\mathcal{O}}\mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\blacktriangle^b\widetilde{\boldsymbol{\xi}}^j v\}\,\mathrm{d}x\sum_{a=1}^{n}\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j}\mathfrak{f}_j^{qo}\Big] \\
&\qquad - \lim_{j\to\infty}\widetilde{\mathbb{E}}\Big[\sum_{b=1}^{m}\int_{\mathcal{O}}\mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\blacktriangle^b\widetilde{\boldsymbol{\xi}}^j v\}\,\mathrm{d}x\sum_{a=1}^{m}\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j}\mathfrak{f}_j^{qo}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\int_{t_j^m}^{t_j^n} \int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\,\mathfrak{f}_j^{qo}\Big]\\
&= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\sum_{b=m+1}^{n}\sum_{a=1}^{n}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\blacktriangle^b\widetilde{\boldsymbol{\xi}}^j\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j} v\}\,\mathrm{d}x\,\mathfrak{f}_j^{qo}\Big]\\
&\quad+ \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\sum_{b=1}^{m}\sum_{a=m+1}^{n}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\blacktriangle^b\widetilde{\boldsymbol{\xi}}^j\sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j} v\}\,\mathrm{d}x\,\mathfrak{f}_j^{qo}\Big]\\
&\quad- \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\int_{t_j^m}^{t_j^n} \int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\,\mathfrak{f}_j^{qo}\Big]\\
&= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\sum_{b=m+1}^{n}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\blacktriangle^b\widetilde{\boldsymbol{\xi}}^j\sqrt{\tau_j}\widetilde{\xi}_k^{b,\tau_j} v\}\,\mathrm{d}x\,\mathfrak{f}_j^{qo}\Big]\\
&\quad- \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\int_{t_j^m}^{t_j^n} \int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\,\mathfrak{f}_j^{qo}\Big]\\
&= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\sum_{b=m+1}^{n}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\lambda_k\mathfrak{g}_k\widetilde{\xi}_k^{b,\tau_j}\tau_j\widetilde{\xi}_k^{b,\tau_j} v\}\,\mathrm{d}x\,\mathfrak{f}_j^{qo}\Big]\\
&\quad- \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\int_{t_j^m}^{t_j^n} \int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\,\mathfrak{f}_j^{qo}\Big]\\
&= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\Big(\sum_{b=m+1}^{n}\int_{\mathcal{O}} \mathcal{I}_{h_j}\{\Phi_{h_j}(\widetilde{\phi}_j(t_j^{b-1}))\lambda_k\mathfrak{g}_k\tau_j v\}\,\mathrm{d}x\\
&\quad- \int_{t_j^m}^{t_j^n} \int_{\mathcal{O}} \lambda_k \mathcal{I}_{h_j}\{\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k v\}\,\mathrm{d}x\,\mathrm{d}s\Big)\mathfrak{f}_j^{qo}\Big] = 0.
\end{aligned}
$$

Here, we used equation (8.3a) and assumption **(D2)**.

We repeat the above arguments to show that $\langle\widetilde{N}^w\rangle$ and $\langle\widetilde{N}^w, \widetilde{\beta}_k\rangle$ defined in equations (8.5c) and (8.5d) respectively are indeed the quadratic and cross-variation processes of the martingale $\widetilde{N}^w$. We again introduce a family of approximations $(\langle\widetilde{N}_j^w\rangle)_{j\in\mathbb{N}}$ of $\langle\!\langle\widetilde{N}^w\rangle\!\rangle$ via

$$
\langle\widetilde{N}_j^w\rangle(t_j^m) := \int_0^{t_j^m} \sum_{k\in\mathbb{Z}_{h_j}} \Big(\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\varrho(\widetilde{\phi}_j^-)\lambda_k\mathfrak{g}_k]|_\Gamma w\}\,\mathrm{d}\Gamma\Big)^2\,\mathrm{d}s.
$$

To establish convergence, we again decompose the error as follows:

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[|\langle\widetilde{N}^w\rangle(t) - \langle\widetilde{N}_j^w\rangle(t_j^m)|^p]\\
&\quad\le C\widetilde{\mathbb{E}}\Big[\Big|\int_0^t \sum_{k\in\mathbb{Z}}\Big(\int_\Gamma \varrho(\widetilde{\phi}_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w\,\mathrm{d}\Gamma\Big)^2\,\mathrm{d}s\\
&\qquad\quad- \int_0^t \sum_{k\in\mathbb{Z}_{h_j}}\Big(\int_\Gamma \varrho([\widetilde{\phi}_j^-]|_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w\,\mathrm{d}\Gamma\Big)^2\,\mathrm{d}s\Big|^p\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ C\widetilde{\mathbb{E}}\Big[\Big|\int_t^{t_j^m} \sum_{k\in\mathbb{Z}_{h_j}} \big([\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma\big)^2 \,\mathrm{d}s\Big|^p\Big] \\
&+ C\widetilde{\mathbb{E}}\Big[\Big|\int_0^{t_j^m} \sum_{k\in\mathbb{Z}_{h_j}} \Big[\Big(\int_\Gamma \varrho([\widetilde{\phi}_j^-]|_\Gamma)\lambda_k [\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma\Big)^2 \\
&- \Big(\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma\Big)^2\Big] \,\mathrm{d}s\Big|^p\Big] \\
=:{}& Q_1^\Gamma + Q_2^\Gamma + Q_3^\Gamma.
\end{aligned}
$$

As before, Q_1^Γ vanishes due to Lemma 7.5 and assumptions **(D3)** and **(C)**. The term Q_2^Γ can be treated analogously to Q_2. To show that Q_3^Γ vanishes, we note that $\widetilde{\phi}_j$ are finite element functions and $\mathfrak{g}_k$ are bounded in $W^{2,\infty}(\mathcal{O})$, that is, they are continuous on $\overline{\mathcal{O}}$. Together with equation (2.1), assumption **(C)**, and the arguments from equation (8.6), this provides

$$
\begin{aligned}
&\Big|\int_\Gamma \varrho([\widetilde{\phi}_j^-]|_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma - \int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma\Big| \\
&\quad \le \Big|\int_\Gamma \varrho([\widetilde{\phi}_j^-]|_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma - \int_\Gamma \mathcal{I}_{h_j}^\Gamma\{\varrho([\widetilde{\phi}_j^-]|_\Gamma)\}\lambda_k \mathcal{I}_{h_j}^\Gamma\{[\mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma\Big| \\
&\qquad + \Big|\int_\Gamma (1-\mathcal{I}_{h_j}^\Gamma)\{\mathcal{I}_{h_j}^\Gamma\{\varrho([\widetilde{\phi}_j^-]|_\Gamma)\}\lambda_k \mathcal{I}_{h_j}^\Gamma\{[\mathfrak{g}_k]|_\Gamma w\}\} \,\mathrm{d}\Gamma\Big| \\
&\quad \le C h_j |\lambda_k| \|[\mathfrak{g}_k]|_\Gamma\|_{W^{1,\infty}(\Gamma)} \|w\|_{W^{1,\infty}(\Gamma)} (1 + \|\nabla_\Gamma \widetilde{\phi}_j^-\|_{L^2(\Gamma)}).
\end{aligned}
$$

As $\|[\mathfrak{g}_k]|_\Gamma\|_{W^{1,\infty}(\Gamma)} \le C\|\mathfrak{g}_k\|_{W^{2,\infty}(\mathcal{O})}$, we have

$$
Q_3^\Gamma \le C(w)\widetilde{\mathbb{E}}\Big[\Big|h_j \sum_{k\in\mathbb{Z}} |\lambda_k|^2 \|\mathfrak{g}_k\|_{W^{2,\infty}(\mathcal{O})}^2 \int_0^t (1 + \|\nabla_\Gamma \widetilde{\phi}_j^-\|_{L^2(\Gamma)}) \,\mathrm{d}s\Big|^p\Big] \to 0.
$$

As before, we use this convergence property to show that $(\widetilde{N}^w)^2 - \langle\!\langle \widetilde{N}^w \rangle\!\rangle$ is a martingale by showing this property for its approximations and using the convergence. For $0 \le t_j^n - t_2 \le \tau_j$ and $0 \le t_j^m - t_1 \le \tau_j$ we compute

$$
\begin{aligned}
&\widetilde{\mathbb{E}}[((\widetilde{N}^w(t_2))^2 - (\widetilde{N}^w(t_1))^2 - \langle\!\langle \widetilde{N}^w \rangle\!\rangle(t_2) + \langle\!\langle \widetilde{N}^w \rangle\!\rangle(t_1))\mathfrak{f}^{qo}] \\
&\quad = \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\Big(\sum_{a=m+1}^n \Big(\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\Phi_{h_j}(\widetilde{\phi}_j(t_j^{a-1}))\blacktriangle^a \widetilde{\xi}^j]|_\Gamma w\} \,\mathrm{d}\Gamma\Big)^2 \\
&\qquad - \sum_{a=m+1}^n \tau_j \sum_{k\in\mathbb{Z}_{h_j}} \Big(\int_\Gamma \mathcal{I}_{h_j}^\Gamma\{[\varrho(\widetilde{\phi}_j(t_j^{a-1}))\lambda_k \mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma\Big)^2\Big)\mathfrak{f}_j^{qo}\Big] = 0
\end{aligned}
$$

due to assumption **(D2)**. To identify the cross variations of $\widetilde{N}^w$, we define suitable approximations for time points t_j^m via

$$\langle \widetilde{N}_j^w, \sum_{a=1}^m \sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j} \rangle := \begin{cases} \int_0^{t_j^m} \int_\Gamma \mathcal{I}_{h_j}^\Gamma \{[\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma \,\mathrm{d}s & \text{if } k \in \mathbb{Z}_{h_j}, \\ 0 & \text{else.} \end{cases}$$

Assuming without loss of generality that $k \in \mathbb{Z}_{h_j}$ and arguing like before shows

$$\widetilde{\mathbb{E}}\Big[\Big|\int_0^t \int_\Gamma \varrho(\widetilde{\phi}_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma \,\mathrm{d}s - \int_0^{t_j^m} \int_\Gamma \mathcal{I}_{h_j}^\Gamma \{[\varrho(\widetilde{\phi}_j^-)\lambda_k \mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma \,\mathrm{d}s\Big|^p\Big] \to 0,$$

and hence,

$$\begin{aligned} &\widetilde{\mathbb{E}}[(\widetilde{N}^w(t_2)\widetilde{\beta}_k(t_2) - \widetilde{N}^w(t_1)\widetilde{\beta}_k(t_1) - \langle \widetilde{N}^w, \widetilde{\beta}_k\rangle(t_2) + \langle \widetilde{N}^w, \widetilde{\beta}_k\rangle(t_1))\mathfrak{f}^{qo}] \\ &= \lim_{j\to\infty} \widetilde{\mathbb{E}}\Big[\Big(\widetilde{N}_j^w(t_j^n)\sum_{a=1}^n \sqrt{\tau_j}\widetilde{\xi}_k^{a,\tau_j} - \widetilde{N}_j^w(t_j^m)\sum_{a=1}^m \sqrt{\tau_j}\widetilde{\xi}_j^{a,\tau_j} \\ &\quad - \int_{t_j^m}^{t_j^n} \int_\Gamma \lambda_k \mathcal{I}_{h_j}^\Gamma \{[\varrho(\widetilde{\phi}_j^-)\mathfrak{g}_k]|_\Gamma w\} \,\mathrm{d}\Gamma \,\mathrm{d}s\Big)\mathfrak{f}_j^{qo}\Big] = 0. \end{aligned}$$

■

Having obtained explicit expressions for the quadratic and cross variations of the martingales $\widetilde{M}^v$ and $\widetilde{N}^w$, we can follow the approach introduced in [11, 27, 48] (see also [9, 26]) to show that $\widetilde{M}^v$ and $\widetilde{N}^w$ can be written as Itô integrals with the Wiener process $\widetilde{W}$.

Lemma 8.4. *Let the assumptions of Theorem 7.3 and Lemma 7.5 hold true. Then we have, for $\widetilde{M}^v$ and $\widetilde{N}^w$,*

$$\widetilde{M}^v(t) = \int_0^t \int_\mathcal{O} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k, \tag{8.7a}$$

$$\widetilde{N}^w(t) = \int_0^t \int_\Gamma \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi}_\Gamma)\lambda_k[\mathfrak{g}_k]|_\Gamma w \,\mathrm{d}\Gamma \,\mathrm{d}\widetilde{\beta}_k. \tag{8.7b}$$

Proof. As a martingale with vanishing quadratic variation is almost surely constant, it suffices to show that the quadratic variation of

$$\widetilde{M}^v(t) - \int_0^t \int_\mathcal{O} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k$$

vanishes to establish equation (8.7a). We get

$$\begin{aligned} &\Big\langle \widetilde{M}^v - \int_0^{(\cdot)} \int_\mathcal{O} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k\Big\rangle(t) \\ &= \langle \widetilde{M}^v\rangle(t) + \Big\langle \int_0^{(\cdot)} \int_\mathcal{O} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k\Big\rangle(t) \end{aligned}$$

$$- 2\Big\langle \widetilde{M}^v, \int_0^{(\cdot)} \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k \Big\rangle(t). \tag{8.8}$$

To find a suitable expression for the last term on the right-hand side of equation (8.8), we apply the cross-variation formula (see, e.g., [31]) to obtain

$$\Big\langle \widetilde{M}^v, \int_0^{(\cdot)} \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k \Big\rangle(t) = \int_0^t \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\langle\!\langle \widetilde{M}^v, \widetilde{\beta}_k \rangle\!\rangle(s).$$

By equation (8.5b) and assumption **(C)**, we deduce that the process $[0,T] \ni s \mapsto \langle\!\langle \widetilde{M}^v, \widetilde{\beta}_k \rangle\!\rangle(s)$ is absolutely continuous. Hence, we have

$$\mathrm{d}\langle\!\langle \widetilde{M}^v, \widetilde{\beta}_k \rangle\!\rangle(s) = \lambda_k \int_{\mathcal{O}} \varrho(\widetilde{\phi}) \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}s$$

and consequently,

$$\Big\langle \widetilde{M}^v, \int_0^{(\cdot)} \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k \Big\rangle(t) = \int_0^t \sum_{k\in\mathbb{Z}} |\lambda_k|^2 \Big(\int_{\mathcal{O}} \varrho(\widetilde{\phi}) \mathfrak{g}_k v \,\mathrm{d}x \Big)^2 \,\mathrm{d}s.$$

Together with equation (8.5a) and

$$\Big\langle \int_0^{(\cdot)} \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \varrho(\widetilde{\phi})\lambda_k \mathfrak{g}_k v \,\mathrm{d}x \,\mathrm{d}\widetilde{\beta}_k \Big\rangle(t) = \int_0^t \sum_{k\in\mathbb{Z}} |\lambda_k|^2 \Big(\int_{\mathcal{O}} \varrho(\widetilde{\phi}) \mathfrak{g}_k v \,\mathrm{d}x \Big)^2 \,\mathrm{d}s,$$

this provides equation (8.7a). Repeating the above arguments also provides equation (8.7b). ∎

We conclude the proof of Theorem 4.2 by passing to the limit in equation (8.1c). For any $v \in C^\infty\{\overline{\mathcal{O}}\}$, we choose $\mathcal{I}_{h_j}\{v\}$ multiplied by a sufficiently regular function from $\widetilde{\Omega} \times [0,T]$ to $\mathbb{R}$ as a test function in equation (8.1c). An application of the convergence results collected in Lemma 7.5 yields the following result:

Lemma 8.5. *Let the assumptions of Theorem 7.3 and Lemma 7.5 hold true. Then the limit processes $\widetilde{\phi}$, $\widetilde{\phi}_\Gamma$, $\widetilde{\mu}$, and $\widetilde{\theta}$ satisfy*

$$\begin{aligned} \int_{\mathcal{O}} \widetilde{\mu}\eta \,\mathrm{d}x + \int_\Gamma \widetilde{\theta}[\eta]|_\Gamma \,\mathrm{d}\Gamma = \int_{\mathcal{O}} \nabla\widetilde{\phi}\cdot\nabla\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\widetilde{\phi}_\Gamma \cdot \nabla_\Gamma[\eta]|_\Gamma \,\mathrm{d}\Gamma \,\mathrm{d}\Gamma \\ + \int_{\mathcal{O}} F'(\widetilde{\phi})\eta \,\mathrm{d}x + \int_\Gamma G'(\widetilde{\phi}_\Gamma)[\eta]|_\Gamma \end{aligned}$$

$\widetilde{\mathbb{P}}$-almost surely for almost all $t \in [0,T]$ and all $\eta \in C^\infty(\overline{\mathcal{O}})$.

A density argument shows that the martingale solutions satisfy the formulation stated in Theorem 4.2. Hence, it remains to establish the pathwise uniqueness.

9. Pathwise uniqueness of martingale solutions

In this section we shall prove that the martingale solutions established in Section 8 are pathwise unique and hence complete the proof of Theorem 4.2. As the Allen–Cahn-type equation on Γ can be interpreted as an L^2-gradient flow and the Cahn–Hilliard equation on $\mathcal{O}$ is a gradient flow with respect to the H^{-1}-norm, we shall estimate the difference between two possible solutions using a combination of these norms. To define the dual norm on $\mathcal{O}$, we follow the ideas used in [15] to prove the uniqueness of the deterministic problem and define

$$\begin{aligned}&\operatorname{dom}\mathfrak{D} := \{\varphi^* \in (H^1(\mathcal{O}))' : (\varphi^*)_{\mathcal{O}} = 0\},\\&\mathfrak{D} : \operatorname{dom}\mathfrak{D} \to \{\varphi \in H^1(\mathcal{O}) : (\varphi)_{\mathcal{O}} = 0\}\end{aligned}$$

by setting, for $\varphi^* \in \operatorname{dom}\mathfrak{D}$,

$$\begin{aligned}&\mathfrak{D}\varphi^* \in H^1(\mathcal{O}), \quad (\mathfrak{D}\varphi^*)_{\mathcal{O}} = 0,\\&\int_{\mathcal{O}} \nabla\mathfrak{D}\varphi^* \cdot \nabla z \,\mathrm{d}x = \langle \varphi^*, z\rangle, \quad \forall z \in H^1(\mathcal{O}).\end{aligned} \tag{9.1}$$

Hence, $\mathfrak{D}\varphi^*$ is the solution to the generalized Neumann problem for $-\Delta$ with datum φ^* satisfying $(\varphi^*)_{\mathcal{O}} = 0$. From equation (9.1), we immediately obtain the identity

$$\langle \psi^*, \mathfrak{D}\varphi^*\rangle = \langle \varphi^*, \mathfrak{D}\psi^*\rangle = \int_{\mathcal{O}} \nabla\mathfrak{D}\psi^* \cdot \nabla\mathfrak{D}\varphi^* \,\mathrm{d}x \quad \text{for } \psi^*, \varphi^* \in \operatorname{dom}\mathfrak{D}.$$

Using $\mathfrak{D}$, we define a norm $\|\cdot\|_*$ on $(H^1(\mathcal{O}))'$ that is equivalent to the usual dual norm via

$$\|\varphi^*\|_*^2 := \|\nabla\mathfrak{D}(\varphi^* - (\varphi^*)_{\mathcal{O}})\|_{L^2(\mathcal{O})}^2 + |(\varphi^*)_{\mathcal{O}}|^2 \tag{9.2}$$

and denote the corresponding seminorm by

$$|\varphi^*|_*^2 := \|\nabla\mathfrak{D}(\varphi^* - (\varphi^*)_{\mathcal{O}})\|_{L^2(\mathcal{O})}^2.$$

Lemma 9.1. *Let assumptions* **(T)**, **(S1)**, **(S2)**, **(P)**, **(I)**, **(C)**, *and* **(D0)**–**(D3)** *hold true. Then, martingale solutions to system* (4.1) *are pathwise unique.*

Proof. Let $(\widetilde{\phi}_1, \widetilde{\phi}_{\Gamma 1}, \widetilde{\mu}_1, \widetilde{\theta}_1)$ and $(\widetilde{\phi}_2, \widetilde{\phi}_{\Gamma 2}, \widetilde{\mu}_2, \widetilde{\theta}_2)$ be two martingale solutions to system (4.1) with the same initial data ϕ_0, the same probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, and the same $\mathcal{Q}$-Wiener process $\widetilde{W}$. To show that these solutions are identical, we will make use of a local monotonicity argument (cf. [37, 53] for a generalization) and apply a Gronwall-type argument. Hence, we introduce

$$\begin{aligned}\mathfrak{N}(\varphi^*, \varphi_\Gamma^*) &:= \|\varphi^*\|_*^2 + \|\varphi_\Gamma^*\|_{L^2(\Gamma)}^2 + 2|\Gamma|(\varphi^*)_{\mathcal{O}}^2 + 2(\varphi^*)_{\mathcal{O}}(\varphi_\Gamma^*)_\Gamma\\&\geq \|\varphi^*\|_*^2 + \frac{1}{2}\|\varphi_\Gamma^*\|_{L^2(\Gamma)}^2\end{aligned}$$

and apply Itô's formula to the expression

$$\begin{aligned}&\mathfrak{R}(t,(\widetilde{\phi}_1,\widetilde{\phi}_{\Gamma 1})-(\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 2}))\\&\quad:=\frac{1}{2}e^{-\int_0^t(\widehat{C}+\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}s}\mathfrak{R}(\widetilde{\phi}_1-\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2})\end{aligned}$$

with a suitable positive constant $\widehat{C}$ and a measurable mapping ϑ, which we will define later. As both solutions share the same initial conditions, we obtain for $\widehat{T}\in[0,T]$,

$$\begin{aligned}&\widetilde{\mathbb{E}}[\mathfrak{R}(\widehat{T},(\widetilde{\phi}_1(\widehat{T}),\widetilde{\phi}_{\Gamma 1}(\widehat{T}))-(\widetilde{\phi}_2(\widehat{T}),\widetilde{\phi}_{\Gamma 2}(\widehat{T})))]\\
&\quad=\widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}}[-(\widehat{C}+\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))]\mathfrak{R}(t,(\widetilde{\phi}_1,\widetilde{\phi}_{\Gamma 1})-(\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}t\Big]\\
&\qquad+\widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}}e^{-\int_0^t(\widehat{C}+\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}s}\Big(-\int_{\mathcal{O}}(\widetilde{\phi}_1-\widetilde{\phi}_2-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})(\widetilde{\mu}_1-\widetilde{\mu}_2)\,\mathrm{d}x\Big)\mathrm{d}t\Big]\\
&\qquad+\widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}}e^{-\int_0^t(\widehat{C}+\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}s}\Big(-\int_{\Gamma}(\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})(\widetilde{\theta}_1-\widetilde{\theta}_2)\,\mathrm{d}\Gamma\Big)\mathrm{d}t\Big]\\
&\qquad+\frac{1}{2}\widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}}e^{-\int_0^t(\widehat{C}+\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}s}\Big(\sum_{k\in\mathbb{Z}}\|(\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k\|_*^2\\
&\qquad\quad+\sum_{k\in\mathbb{Z}}\|[(\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k]|_\Gamma\|_{L^2(\Gamma)}^2+2|\Gamma|\sum_{k\in\mathbb{Z}}((\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k)_{\mathcal{O}}^2\\
&\qquad\quad+2\sum_{k\in\mathbb{Z}}((\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k)_{\mathcal{O}}([(\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k]|_\Gamma)_\Gamma\Big)\,\mathrm{d}t\Big]\\
&\quad=:I_1+I_2+I_3+I_4.\end{aligned}$$

We start by estimating I_2+I_3: as $\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}}$ is $\widetilde{\mathbb{P}}$-almost surely the trace of $\widetilde{\phi}_1-\widetilde{\phi}_2-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}}$, we obtain

$$\begin{aligned}&-\int_{\mathcal{O}}(\widetilde{\phi}_1-\widetilde{\phi}_2-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})(\widetilde{\mu}_1-\widetilde{\mu}_2)\,\mathrm{d}x\\
&\qquad-\int_{\Gamma}(\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})(\widetilde{\theta}_1-\widetilde{\theta}_2)\,\mathrm{d}\Gamma\\
&\quad=-\|\nabla\widetilde{\phi}_1-\nabla\widetilde{\phi}_2\|_{L^2(\mathcal{O})}^2-\|\nabla_\Gamma\widetilde{\phi}_{\Gamma 1}-\nabla_\Gamma\widetilde{\phi}_{\Gamma 2}\|_{L^2(\Gamma)}^2\\
&\qquad-\int_{\mathcal{O}}(F'(\widetilde{\phi}_1)-F'(\widetilde{\phi}_2))(\widetilde{\phi}_1-\widetilde{\phi}_2-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})\,\mathrm{d}x\\
&\qquad-\int_{\Gamma}(G'(\widetilde{\phi}_{\Gamma 1})-G'(\widetilde{\phi}_{\Gamma 2}))(\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}-(\widetilde{\phi}_1-\widetilde{\phi}_2)_{\mathcal{O}})\,\mathrm{d}\Gamma\\
&\quad=:R_1+R_2+R_3+R_4.\end{aligned}$$

Similarly to [44], we use Hölder's inequality, assumption **(P)**, the standard Sobolev embeddings, and Young's inequality to deduce, for $0 < \alpha \ll 1$,

$$\begin{aligned}|R_4| &\le C\|1 + |\widetilde{\phi}_{\Gamma 1}|^2 + |\widetilde{\phi}_{\Gamma 2}|^2\|_{L^3(\Gamma)}\|\widetilde{\phi}_{\Gamma 1} - \widetilde{\phi}_{\Gamma 2}\|_{L^2(\Gamma)}(\|\widetilde{\phi}_{\Gamma 1} - \widetilde{\phi}_{\Gamma 2}\|_{H^1(\Gamma)} \\ &\quad + |(\widetilde{\phi}_1 - \widetilde{\phi}_2)_{\mathcal{O}}|) \\ &\le C_\alpha(1 + \|\widetilde{\phi}_{\Gamma 1}\|^4_{H^1(\Gamma)} + \|\widetilde{\phi}_{\Gamma 2}\|^4_{H^1(\Gamma)})(\|\widetilde{\phi}_{\Gamma 1} - \widetilde{\phi}_{\Gamma 2}\|^2_{L^2(\Gamma)} + \|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_*) \\ &\quad + \alpha\|\nabla_\Gamma\widetilde{\phi}_{\Gamma 1} - \nabla_\Gamma\widetilde{\phi}_{\Gamma 2}\|^2_{L^2(\Gamma)}.\end{aligned}$$

From assumption **(P)**, Hölder's inequality, Poincaré's inequality, and Young's inequality, we obtain

$$\begin{aligned}|R_3| &\le C\|1 + |\widetilde{\phi}_1|^2 + |\widetilde{\phi}_2|^2\|_{L^3(\mathcal{O})}\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|_{L^2(\mathcal{O})}\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|_{L^2(\mathcal{O})} \\ &\le C_\alpha(1 + \|\widetilde{\phi}_1\|^4_{H^1(\mathcal{O})} + \|\widetilde{\phi}_2\|^4_{H^1(\mathcal{O})})\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_{L^2(\mathcal{O})} \\ &\quad + \alpha\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

Recalling the definition of $\|\cdot\|_*$ in equation (9.2), we compute

$$\begin{aligned}\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_{L^2(\mathcal{O})} &= \|\widetilde{\phi}_1 - \widetilde{\phi}_2 - (\widetilde{\phi}_1 - \widetilde{\phi}_2)_{\mathcal{O}}\|^2_{L^2(\mathcal{O})} + (\widetilde{\phi}_1 - \widetilde{\phi}_2)^2_{\mathcal{O}} \\ &\le |\widetilde{\phi}_1 - \widetilde{\phi}_2|_*\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|_{L^2(\mathcal{O})} + (\widetilde{\phi}_1 - \widetilde{\phi}_2)^2_{\mathcal{O}}.\end{aligned} \tag{9.3}$$

Hence, by Young's inequality, we obtain

$$|R_3| \le C_\alpha(1 + \|\widetilde{\phi}_1\|^8_{H^1(\mathcal{O})} + \|\widetilde{\phi}_2\|^8_{H^1(\mathcal{O})})\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_* + 2\alpha\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|^2_{L^2(\mathcal{O})}.$$

We continue by deducing estimates for I_4. Using the Lipschitz continuity of ϱ (cf. assumption **(C)**), we obtain using assumption **(D3)**, estimate (9.3), and Young's inequality

$$\begin{aligned}\frac{1}{2}\sum_{k\in\mathbb{Z}}\|(\varrho(\widetilde{\phi}_1) - \varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k\|^2_* &\le C\sum_{k\in\mathbb{Z}}\|(\varrho(\widetilde{\phi}_1) - \varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k\|^2_{L^2(\mathcal{O})} \\ &\le C\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_{L^2(\mathcal{O})} \\ &\le C_\alpha\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_* + \alpha\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

Similar computations provide $\widetilde{\mathbb{P}}$-almost surely

$$|\Gamma|\sum_{k\in\mathbb{Z}}((\varrho(\widetilde{\phi}_1) - \varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k)^2_{\mathcal{O}} \le C_\alpha\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_* + \alpha\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|^2_{L^2(\mathcal{O})},$$

$$\frac{1}{2}\sum_{k\in\mathbb{Z}}\|[(\varrho(\widetilde{\phi}_1)-\varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k]|_\Gamma\|^2_{L^2(\Gamma)} \le C\|\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}\|^2_{L^2(\Gamma)},$$

and

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}((\varrho(\widetilde{\phi}_1) - \varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k)_{\mathcal{O}}([(\varrho(\widetilde{\phi}_1) - \varrho(\widetilde{\phi}_2))\lambda_k\mathfrak{g}_k]|_\Gamma)_\Gamma \\ &\quad\le C_\alpha\|\widetilde{\phi}_1 - \widetilde{\phi}_2\|^2_* + C\|\widetilde{\phi}_{\Gamma 1} - \widetilde{\phi}_{\Gamma 2}\|^2_{L^2(\Gamma)} + \alpha\|\nabla\widetilde{\phi}_1 - \nabla\widetilde{\phi}_2\|^2_{L^2(\Gamma)}.\end{aligned}$$

Combining the above results, we obtain for α sufficiently small

$$
\begin{aligned}
&I_2 + I_3 + I_4 \\
&\le \widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}} e^{-\int_0^t (\widehat{C}-\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}s}(\|\widetilde{\phi}_1-\widetilde{\phi}_2\|_*^2+\|\widetilde{\phi}_{\Gamma 1}-\widetilde{\phi}_{\Gamma 2}\|_{L^2(\Gamma)}^2) \\
&\quad \times (C_1 + C_2(1+\|\widetilde{\phi}_1\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_2\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_{\Gamma 1}\|_{H^1(\Gamma)}^4+\|\widetilde{\phi}_{\Gamma 2}\|_{H^1(\Gamma)}^4))\,\mathrm{d}t\Big] \\
&\le \widetilde{\mathbb{E}}\Big[\int_0^{\widehat{T}} (C_1 + C_2(1+\|\widetilde{\phi}_1\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_2\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_{\Gamma 1}\|_{H^1(\Gamma)}^4+\|\widetilde{\phi}_{\Gamma 2}\|_{H^1(\Gamma)}^4)) \\
&\quad \times \mathfrak{R}(t,(\widetilde{\phi}_1,\widetilde{\phi}_{\Gamma 1})-(\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 2}))\,\mathrm{d}t\Big].
\end{aligned}
$$

Choosing

$$\vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2}) = C_2(\|\widetilde{\phi}_1\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_2\|_{H^1(\mathcal{O})}^8+\|\widetilde{\phi}_{\Gamma 1}\|_{H^1(\Gamma)}^4+\|\widetilde{\phi}_{\Gamma 2}\|_{H^1(\Gamma)}^4)$$

and $\widehat{C}$ sufficiently large, we obtain

$$\widetilde{\mathbb{E}}[\mathfrak{R}(\widehat{T},(\widetilde{\phi}_1(\widehat{T}),\widetilde{\phi}_{\Gamma 1}(\widehat{T}))-(\widetilde{\phi}_2(\widehat{T}),\widetilde{\phi}_{\Gamma 2}(\widehat{T})))] \le 0.$$

Due to the already established regularity of the martingale solutions to equation (4.1), we have

$$\int_0^T \vartheta(\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 1},\widetilde{\phi}_{\Gamma 2})\,\mathrm{d}t < \infty$$

$\widetilde{\mathbb{P}}$-almost surely. Together with the continuity $(\widetilde{\phi}_1,\widetilde{\phi}_{\Gamma 1})$ and $(\widetilde{\phi}_2,\widetilde{\phi}_{\Gamma 2})$ in $L^2(\mathcal{O})\times L^2(\Gamma)$ this implies the pathwise uniqueness of martingale solutions. ■

10. Convergence toward strong solutions

In this section, we shall prove Theorem 4.3. Hence, we assume that for a given a filtered probability space $(\Omega,\mathcal{A},\mathcal{F},\mathbb{P})$ with a $\mathcal{Q}$-Wiener process W satisfying assumptions **(W1)** and **(W2)**, we have a finite-dimensional approximation

$$\xi_h^{m,\tau} = \sum_{k\in\mathbb{Z}_h} (W(t^m),\mathfrak{g}_k)_{L^2(\mathcal{O})}\mathfrak{g}_k, \tag{10.1}$$

satisfying assumptions **(D0)**, **(D1)**, **(D2)**, and **(D3)**. As this is a specialization of the previously discussed setting, the prior results remain valid. In particular, we still have the pathwise uniqueness of martingale solutions. Hence, we can apply a generalization of the Gyöngy–Krylov characterization of convergence in probability (cf. [25]) to the setting of quasi-Polish spaces that was established in [9].

As shown before, our numerical scheme (see scheme (3.4)) has a pathwise-unique solution for any h and τ. Hence, there exists a sequence of stochastic processes defined on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ satisfying

$$\int_{\mathcal{O}} \mathcal{I}_h\{(\phi_h^\tau(t) - \phi_h^{\tau,-})\psi_h\}\,\mathrm{d}x + (t - t^{n-1}) \int_{\mathcal{O}} \nabla\mu_h^{\tau,+} \cdot \nabla\psi_h\,\mathrm{d}x$$
$$= \frac{t - t^{n-1}}{\tau} \int_{\mathcal{O}} \mathcal{I}_h\{\Phi_h(\phi_h^{\tau,-})(W(t) - W(t^{n-1}))\psi_h\}\,\mathrm{d}x, \tag{10.2a}$$
$$\int_\Gamma \mathcal{I}_h^\Gamma\{[\phi_h^\tau(t) - \phi_h^{\tau,-}]|_\Gamma \widehat{\psi}_h\}\,\mathrm{d}\Gamma + (t - t^{n-1}) \int_\Gamma \mathcal{I}_h^\Gamma\{\theta_h^{\tau,+}\widehat{\psi}_h\}\,\mathrm{d}\Gamma$$
$$= \frac{t - t^{n-1}}{\tau} \int_\Gamma \mathcal{I}_h^\Gamma\{[\Phi_h(\phi_h^{\tau,-})(W(t) - W(t^{n-1}))]|_\Gamma \widehat{\psi}_h\}\,\mathrm{d}\Gamma, \tag{10.2b}$$

and

$$\int_{\mathcal{O}} \mathcal{I}_h\{\mu_h^{\tau,+}\eta_h\}\,\mathrm{d}x + \int_\Gamma \mathcal{I}_h^\Gamma\{\theta_h^{\tau,+}[\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma$$
$$= \int_{\mathcal{O}} \nabla\phi_h^{\tau,+} \cdot \nabla\eta_h\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma[\phi_h^{\tau,+}]|_\Gamma \cdot \nabla_\Gamma[\eta_h]|_\Gamma\,\mathrm{d}\Gamma$$
$$+ \frac{r_h^{\tau,+}}{\sqrt{E_h^{\mathcal{O}}(\phi_h^{\tau,-})}} \int_{\mathcal{O}} \mathcal{I}_h\{F'(\phi_h^{\tau,-})\eta_h\}\,\mathrm{d}x + \int_{\mathcal{O}} \mathcal{I}_h\{\Xi_{h,\mathcal{O}}^{\tau,+}\eta_h\}\,\mathrm{d}x$$
$$+ \frac{s_h^{\tau,+}}{\sqrt{E_h^\Gamma([\phi_h^{\tau,-}]|_\Gamma)}} \int_\Gamma \mathcal{I}_h^\Gamma\{G'([\phi_h^{\tau,-}]|_\Gamma)[\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma$$
$$+ \int_\Gamma \mathcal{I}_h^\Gamma\{\Xi_{h,\Gamma}^{\tau,+}[\eta_h]|_\Gamma\}\,\mathrm{d}\Gamma \tag{10.2c}$$

for all $\psi_h, \eta_h \in U_h^{\mathcal{O}}$ and $\widehat{\psi}_h \in U_h^\Gamma$. Here, $\Xi_{h,\mathcal{O}}^{\tau,+}$ and $\Xi_{h,\Gamma}^{\tau,+}$ are the piecewise constant-in-time interpolations of $\Xi_{h,\mathcal{O}}^n$ and $\Xi_{h,\Gamma}^n$, which were defined in equation (3.5). As shown in Lemma 6.2, these terms will vanish for $\tau \searrow 0$.

From this sequence of stochastic processes, we extract an arbitrary pair of subsequences, which we shall denote by

$$a_\alpha := (\phi_{h_\alpha}^{\tau_\alpha}, [\phi_{h_\alpha}^{\tau_\alpha}]|_\Gamma, r_{h_\alpha}^{\tau_\alpha}, s_{h_\alpha}^{\tau_\alpha}, \mu_{h_\alpha}^{\tau_\alpha,+}, l[\mu_{h_\alpha}^{\tau_\alpha,+}], \theta_{h,\alpha}^{\tau_\alpha,+})_{\alpha\in\mathbb{N}}$$

and

$$a_\beta := (\phi_{h_\beta}^{\tau_\beta}, [\phi_{h_\beta}^{\tau_\beta}]|_\Gamma, r_{h_\beta}^{\tau_\beta}, s_{h_\beta}^{\tau_\beta}, \mu_{h_\beta}^{\tau_\beta,+}, l[\mu_{h_\beta}^{\tau_\beta,+}], \theta_{h,\beta}^{\tau_\beta,+})_{\beta\in\mathbb{N}}.$$

For these subsequences, we consider their joint laws $(\nu_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}}$ on $\mathcal{Y} \times \mathcal{Y}$ with

$$\mathcal{Y} := C([0,T]; L^s(\mathcal{O})) \times C([0,T]; L^r(\Gamma)) \times L^2(0,T)_{\mathrm{weak}} \times L^2(0,T)_{\mathrm{weak}}$$
$$\times L^2(0,T; H^1(\mathcal{O}))_{\mathrm{weak}} \times L^2(0,T; (H^1(\mathcal{O}))')_{\mathrm{weak}} \times L^2(0,T; L^2(\Gamma))$$

and $s \in [1, \frac{2d}{d-2})$ and $r \in [1, \frac{2(d-1)}{d-3})$. In the following we will show that for the sequence of joint laws, there exists a further subsequence that converges weakly to a probability measure ν such that

$$\nu((a,b) \in \mathcal{Y} \times \mathcal{Y} \,:\, a = b) = 1. \tag{10.3}$$

Then [9, Proposition A.4] provides $\mathbb{P}$-almost-sure convergence in the topology of $\mathcal{Y}$ for a subsequence $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma, r_h^\tau, s_h^\tau, \mu_h^{\tau,+}, l[\mu_h^{\tau,+}], \theta_h^{\tau,+})_{h,\tau}$. Together with the pathwise uniqueness, this guarantees convergence for the complete sequence.

To establish equation (10.3), we define the extended path space

$$\widehat{\mathcal{X}} := \mathcal{Y} \times \mathcal{Y} \times C([0,T]; H^1(\mathcal{O})),$$

and consider the sequence

$$(z_{\alpha\beta})_{\alpha,\beta\in\mathbb{N}} := (a_\alpha, a_\beta, W)_{\alpha,\beta\in\mathbb{N}}.$$

Recalling the arguments from Section 7, we note that the joint laws of this sequence are tight on $\widehat{\mathcal{X}}$. Hence, repeating the arguments of Theorem 7.3, we obtain for a subsequence $(z_{\alpha_i\beta_i})_{i\in\mathbb{N}}$ the existence of a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$ and a sequence of random variables

$$(\widetilde{z}_i)_{i\in\mathbb{N}} = (\widetilde{a}_{\alpha_i}, \widetilde{a}_{\beta_i}, \widetilde{W}_i)_{i\in\mathbb{N}}$$

defined on $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$ that has the same law as $(z_{\alpha_i\beta_i})_{i\in\mathbb{N}}$ and converges toward random variables

$$(\widetilde{\phi}, \widetilde{\phi_\Gamma}, \widetilde{r}, \widetilde{s}, \widetilde{\mu}, \widetilde{L}, \widetilde{\theta}, \widehat{\phi}, \widehat{\phi_\Gamma}, \widehat{R}, \widehat{s}, \widehat{\mu}, \widehat{L}, \widehat{\theta}, \widetilde{W}).$$

Arguing as in Lemma 7.5, we obtain

$$[\widetilde{\phi}]|_\Gamma = \widetilde{\phi_\Gamma}, \widetilde{r} = \sqrt{\int_{\mathcal{O}} F(\widetilde{\phi})\,\mathrm{d}x}, \widetilde{s} = \sqrt{\int_\Gamma G(\widetilde{\phi_\Gamma})\,\mathrm{d}\Gamma}, \widetilde{L} = l[\widetilde{\mu}],$$
$$[\widehat{\phi}]|_\Gamma = \widehat{\phi_\Gamma}, \widehat{r} = \sqrt{\int_{\mathcal{O}} F(\widehat{\phi})\,\mathrm{d}x}, \widehat{s} = \sqrt{\int_\Gamma G(\widehat{\phi_\Gamma})\,\mathrm{d}\Gamma}, \quad \text{and} \quad \widehat{L} = l[\widehat{\mu}]$$

$\widetilde{\mathbb{P}}$-almost surely almost everywhere. Following the arguments of Section 8, we can show that $(\widetilde{\phi}, \widetilde{\phi_\Gamma}, \widetilde{\mu}, \widetilde{\theta}, \widetilde{W})$ and $(\widehat{\phi}, \widehat{\phi_\Gamma}, \widehat{\mu}, \widehat{\theta}, \widetilde{W})$ are both martingale solutions satisfying equation (4.1) with the same initial conditions and the same Wiener process. Due to the pathwise uniqueness established in Section 9, we have $\widetilde{\phi} = \widehat{\phi}$, $\widetilde{\phi_\Gamma} = \widehat{\phi_\Gamma}$, $\widetilde{\mu} = \widehat{\mu}$, $\widetilde{\theta} = \widehat{\theta}$ $\widetilde{\mathbb{P}}$-almost surely. As a consequence, we also obtain $\widetilde{r} = \widehat{r}$, $\widetilde{s} = \widehat{s}$, and $\widetilde{L} = \widehat{L}$. Hence, by equality of laws, we obtain equation (10.3).

As this guarantees $\mathbb{P}$-almost-sure convergence in the topology of $\mathcal{Y}$ for a subsequence of $(\phi_h^\tau, [\phi_h^\tau]|_\Gamma, r_h^\tau, s_h^\tau, \mu_h^{\tau,+}, l[\mu_h^{\tau,+}], \theta_h^{\tau,+})_{h,\tau}$, we have a convergence result similar to the one stated in Theorem 7.3 without introducing a new probability space. Consequently, we can repeat the arguments of the previous sections to conclude the proof of Theorem 4.3.

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