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On an inhomogeneous coagulation model with a differential sedimentation kernel

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Abstract. We study an inhomogeneous coagulation equation that contains a transport term in the spatial variable modeling the sedimentation of clusters. We prove local existence of mass-conserving solutions for a class of coagulation kernels for which in the space homogeneous case instantaneous gelation (i.e., instantaneous loss of mass) occurs. Our result holds true in particular for sum-type kernels of homogeneity greater than one, for which solutions do not exist at all in the spatially homogeneous case. Moreover, our result covers kernels that in addition vanish on the diagonal, which have been used to model the onset of rain and the behavior of air bubbles in water.

1. Introduction

1.1. Background

In [25] the authors suggest a space-dependent coagulation equation to model the onset of rain. Here, spherical particles of volume v move in space vertically, for example due to gravitation, and merge when their trajectories cross. This leads to the following inhomogeneous coagulation equation for the density f of particles of size v at the point x:

$$\partial_t f(x,v,t) + v^{\frac{2}{3}} \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(x,v-v',t) f(x,v',t)\,\mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t)\,\mathrm{d}v' \tag{1.1}$$

with a so-called differential sedimentation kernel of the form

$$K(v,v') = |v^{\frac{2}{3}} - v'^{\frac{2}{3}}|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2. \tag{1.2}$$

This choice of kernel is motivated by the following consideration (see [25]): the cross-section of interaction between two particles of radii r and r', and volume v and v', respectively, that merged upon touching is given by $\pi(r+r')^2 \approx (v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$.

Additionally, the velocity is approximately $v^{\frac{2}{3}}$, which represents the Stokes velocity of a rigid sphere with no slip boundary condition, and the collision rate between two particles is taken to be proportional to their relative velocities $|v^{\frac{2}{3}} - v'^{\frac{2}{3}}|$.

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The model is used to describe the behavior of air bubbles in water which move due to buoyancy and it is also valid for water droplets. We refer to [12, 20, 32] for more details. In [25], slip-flow corrections for water droplets are discussed and it is mentioned that this requires changing the power of the volume for the velocity. More precisely, the left-hand side of (1.1) becomes $\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t)$, for some $\alpha \in (0,1)$, and the kernel in (1.2) has the form $K(v,v') = |v^\alpha - v'^\alpha|(v^{\frac{1}{3}} + v'^{\frac{1}{3}})^2$.

The model in (1.1) with kernel (1.2) is referred to as the *free merging regime* in [25], since it is assumed in its derivation that the particles merge when their trajectories cross. When studying equations like (1.1), it is customary to look for stationary solutions of non-zero flux (cf. [34]) of the form $f \approx v^d$, for some $d \in \mathbb{R}$. One of the possible approaches is to compute them using the so-called Zakharov transform (see [39]) and using it we find the solution $f(x,v,t) \approx v^{-\frac{13}{6}}$. However, this approach can be made rigorous only if the integral containing the coagulation kernel in (1.1) is finite, which is not the case for the kernel (1.2). In order to be able to rigorously find a solution for kernels with the same homogeneity, the so-called *forced locality regime* (in which only particles of similar sizes can merge) is studied in [25]. More precisely, for the *forced locality regime* a cut-off in the coagulation kernel is introduced that makes the kernel vanish outside the region where $\frac{1}{q} < \frac{v'}{v} < q$, for some $q > 1$. With this cut-off, the integral containing the coagulation kernel converges and thus the stationary solution $f \approx v^{-\frac{13}{6}}$ is a valid solution.

Our main goal in this paper is to show that mass-conserving solutions exist, at least for a short time interval, for a class of inhomogeneous coagulation equations that includes example (1.1) with (1.2). At first glance this might look surprising since the homogeneity of the kernel in (1.2) is greater than 1. Indeed, it is well known that gelation (mass loss) occurs for the standard one-dimensional coagulation equation,

$$\partial_t f(v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(v-v',t) f(v',t)\,\mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(v,t) f(v',t)\,\mathrm{d}v',$$

when the coagulation kernel behaves like a power law of homogeneity $\gamma > 1$ (see for example [18, 19, 26]). In particular, for sum kernels of the form

$$K(v,v') = v^\gamma + v'^\gamma, \tag{1.3}$$

with $\gamma > 1$, gelation happens instantaneously. Actually, making use of this property, one can prove that solutions which belong to L^1 for the standard coagulation equation do not exist at all for kernels as in (1.3) (see [2, 3, 8, 38]). In addition, it has been proven in [14] that the instantaneous gelation phenomenon holds for Radon measure solutions of the standard coagulation equation with sum kernels of homogeneity greater than 1 which vanish on the diagonal, i.e., $K(v,v') = 0$, such as the kernel in (1.2).

Our main goal is then to prove that, in contrast to the homogeneous case, there exist, at least for short times, mass-conserving solutions to the inhomogeneous model

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(x,v-v',t) f(x,v',t)\,\mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t)\,\mathrm{d}v', \tag{1.4}$$

where $\alpha \in (0,1)$. Our proof holds for a rather general class of coagulation kernels (see Assumption 1.1), in particular kernels of the forms (1.2) and (1.3). Thus, the model (1.4) provides a coagulation model in which existence for kernels of the form (1.3) with $\gamma > 1$ holds, at least for short times.

1.2. Main result

Short-time existence of mass-conserving solutions for the inhomogeneous model. Our goal is to prove short-time existence of mass-conserving solutions for the inhomogeneous model

$$\partial_t f(x,v,t) + v^\alpha \partial_x f(x,v,t) = \frac{1}{2}\int_{(0,v)} K(v-v',v') f(x,v-v',t) f(x,v',t)\,\mathrm{d}v' - \int_{(0,\infty)} K(v,v') f(x,v,t) f(x,v',t)\,\mathrm{d}v', \tag{1.5}$$

where

$$\alpha \in (0,1).$$

Assumption 1.1 (Assumptions on the coagulation kernel). We assume that $K\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a symmetric and continuous function that satisfies

$$0 \le K(v,v') \le K_1(v^\gamma + v'^\gamma), \quad \text{with } \gamma \in [0, 1+\alpha) \tag{1.6}$$

for some constant $K_1 > 0$ and

$$K(v-v',v') \le K(v,v'), \quad \text{when } v' \in \left[0, \frac{v}{2}\right]. \tag{1.7}$$

Condition (1.7) is a rather standard assumption in the study of coagulation equations, see for example [27], and most of the kernels used in applications satisfy this condition, in particular, kernels of the form $K(v,v') = v^\gamma + v'^\gamma$ or (1.2). The condition $\gamma < 1 + \alpha$ in (1.6) is such that the transport term will control the contribution coming from the coagulation term.

Definition 1.2 (Mild solutions). Let $\alpha \in (0,1)$, $\gamma \in [0, 1+\alpha)$, and $m > \frac{\gamma+1}{\alpha}$. Let $T > 0$ and K satisfy Assumption 1.1. We say that a non-negative function $f \in \mathrm{C}([0,T] \times \mathbb{R} \times (0,\infty))$ such that

$$\sup_{t\in[0,T]} \int_{(0,\infty)} (1+v^\gamma) f(x,v,t)\,\mathrm{d}v \le \frac{C_T}{\max\{1, |x|^{m-\frac{\gamma+1}{\alpha}}\}}, \quad \text{for } x \in \mathbb{R},$$

is a mild solution of equation (1.5) if

$$\begin{aligned} f(x,v,t) - f(x-v^{\alpha}t,v,0)S[f](x,v,0,t) \\ = \frac{1}{2}\int_0^t \int_{(0,v)} S[f](x,v,s,t)K(v-v',v')f(x-(t-s)v^{\alpha},v-v',s) \\ \times f(x-(t-s)v^{\alpha},v',s)\,\mathrm{d}s, \end{aligned} \tag{1.8}$$

for all $t \in [0,T]$, $v \in (0,\infty)$, and $x \in \mathbb{R}$, where

$$S[f](x,v,s,t) := \mathrm{e}^{-\int_s^t a[f](x-v^{\alpha}(t-\tau),v,\tau)\,\mathrm{d}\tau}, \tag{1.9}$$

with

$$a[f](x,v,t) := \int_{(0,\infty)} K(v,v')f(x,v',t)\,\mathrm{d}v'.$$

Definition 1.3. We call $f \in C([0,T]\times\mathbb{R}\times(0,\infty))$ a mass-conserving solution of equation (1.5) if f is as in Definition 1.2 and satisfies in addition

$$\int_{\mathbb{R}}\int_{(0,\infty)} vf(x,v,t)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}}\int_{(0,\infty)} vf(x,v,0)\,\mathrm{d}v\,\mathrm{d}x$$

for all $t \in [0,T]$.

Theorem 1.4 (Local existence of solutions). *Let $\alpha \in (0,1)$, $\gamma \in [0,1+\alpha)$, $m \in \mathbb{N}$ even, and $p = \alpha m$ with $m > \max\{\frac{2\gamma+1}{\alpha}, \frac{2+\gamma}{\alpha}+3\}$. Let K satisfy Assumption 1.1 and $T > 0$ be sufficiently small. Let $f_{\mathrm{in}} \in C^1(\mathbb{R}\times(0,\infty))$ such that*

$$f_{\mathrm{in}}(x,v) \le \frac{C_0}{1+|x|^m+v^p}, \tag{1.10}$$

for some $C_0 > 0$ and all $x \in \mathbb{R}$, $v \in (0,\infty)$. Then there exists a mass-conserving solution f of (1.8) as in Definition 1.3 that satisfies

$$f(x,v,t) \le \frac{C}{1+|x|^m+v^p}, \tag{1.11}$$

for all $t \in [0,T]$, for some $C > 0$.

Remark 1.5. Theorem 1.4 is valid for coagulation kernels K as in (1.2), as well as coagulation kernels of the form $K(v,v') = v^{\gamma} + v'^{\gamma}$.

Remark 1.6. It is worthwhile mentioning that mass conservation will follow due to the fact that our solution will have sufficiently fast decay for large values of $|x|$ and v; see (1.11). For more details, see the proof of Theorem 1.4. In other words, if we denote by

$$T_{\min} := \sup_{t\ge 0}\left\{\int_{\mathbb{R}}\int_{(0,\infty)} vf(x,v,t)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}}\int_{(0,\infty)} vf(x,v,0)\,\mathrm{d}v\,\mathrm{d}x\right\},$$

then $T_{\min} > 0$ and the choice of $T_{\min}$ depends on the total mass of particles at initial time through the choice of C_0 in (1.10). We refer to Remark 2.13 and Proposition 2.19, (2.94), and Step 5, for more details on the choice of $T_{\min}$.

1.3. Existence theory for coagulation equations in the mathematical literature

When $\gamma < 1$, our result could be expected according to the general theory of existence for one-dimensional coagulation equations. This states that solutions exist for kernels that behave like power laws of homogeneity $\gamma < 1$; see for example [33] for existence of solutions and [22] for existence of self-similar profiles.

Some multi-dimensional coagulation models have been studied in the mathematical literature; see [15,21]. Moreover, several classes of coagulation models for the distribution of particles with space dependence have also been considered. In particular, models in which in addition to coagulation there is space diffusion of the aggregating particles can be found in [1, 6, 9, 27]. Models that contain coagulation of particles as well as transport terms (that might include sedimentation terms) were studied in [5,10,11,16,17,23]. In all the models mentioned above, the homogeneity of the coagulation kernel is either $\gamma < 1$, in which case the solutions are globally defined and preserve the total mass, or product-type kernels are discussed, for which solutions preserve the mass up to a certain point in time.

To our knowledge, the only exception that considers the case $\gamma > 1$ for space-dependent models is [24]. Indeed, existence of solutions for the discrete version of the model in (1.1) has been established for coagulation kernels of the form $K(i, j) = \sigma_{ij}|\nu_i - \nu_j|$, $i, j \in \mathbb{N}$. Here, ν_i is a non-negative function of volume which represents the sedimentation velocity of the particles and σ_{ij} can be estimated by a function of the form $C(i^\gamma + j^\gamma)$ with $\gamma < 1$. More precisely, the following discrete model was considered in [24]:

$$\partial_t f_i(x,t) + \nu_i \partial_x f_i(x,t) = \frac{1}{2}\sum_{j=1}^{i-1} K(i-j, j) f_{i-j}(x,t) f_j(x,t) - f_i(x,t)\sum_{j=1}^{\infty} K(i,j) f_j(x,t). \tag{1.12}$$

Assuming that ν_i scales like a power law i^β, we would obtain a coagulation kernel $K(i, j)$ that behaves like a homogeneous function with homogeneity $\beta + \gamma$ that might be larger than 1. In fact, the results in [24] hold for any choice of $\nu_i \geq 0$. In particular, the solutions constructed in [24] have total mass of particles that may change continuously in time. The main idea behind the proof in [24] is that one can make use of the transport term to control the mass of the clusters for fixed x. We denote by M the total mass at initial time, i.e., $M := \int_{\mathbb{R}} \sum_{i\in\mathbb{N}} i f_i(x,0)\,\mathrm{d}x$, multiply by i in (1.12), sum over i, and integrate over $(-\infty, x]$ and in t in order to obtain

$$\int_0^t \sum_j j\nu_j f_j(x,\tau)\,\mathrm{d}\tau \leq M. \tag{1.13}$$

If $\nu_i = i^\beta$, (1.13) yields a bound for a power larger than 1 for fixed x. This allows one to obtain an estimate of the loss term which implies that f does not vanish instantaneously. In addition, due to the discreteness of the equations, a diagonal argument can be used to obtain compactness for a sequence of solutions of an approximating scheme. However, this does not imply conservation of mass. By contrast, in the present paper we develop a theory of existence for mild solutions which conserve mass in finite time, i.e., $\int_{\mathbb{R}} \int_{(0,\infty)} v f(x,v,t)\,\mathrm{d}v\,\mathrm{d}x < \infty$.

The class of kernels considered in [24] has a non-empty intersection with the class of kernels considered in this paper and this intersection includes the coagulation kernel (1.2), which is relevant in the physical literature as explained above. However, neither of the two classes of kernels is included in the other. On the other hand, the sedimentation speed v_i can be an arbitrary power law in [24]. This opens the question of whether we can extend our result to powers of α which are larger than 1 for the model in (1.5). However, a different approach to obtaining information from the characteristics created by the transport term in (1.5) will be needed in this case.

Moreover, kernels of the form $K(v,v') = v^{\gamma} + v'^{\gamma}$, with $\gamma \in (1, 1+\alpha)$, also satisfy the conditions (1.6), (1.7). Solutions of the standard one-dimensional coagulation equation do not exist for these types of kernels; see [2,3,8,38]. This is to our knowledge the first result of existence of mass-conserving solutions involving sum kernels of homogeneity $\gamma > 1$, regardless of whether one considers one-dimensional or multi-dimensional coagulation models.

It is worth mentioning that our results also include the cases $\gamma = 0$ and $\gamma = 1$, which are normally studied separately in the literature due to their rich features, such as being able to predict the long-time behavior of solutions, see [4,7,29,30,37], or prove uniqueness of self-similar profiles, see [29,35,36], where the constant kernel and perturbations of the constant kernel are discussed.

The strategy for proving existence of solutions is to consider an iterative scheme based on a linear version of (1.5). For this equation, we are able to find a suitable supersolution, which in turn will provide sufficiently good moment estimates. This will give us compactness of the iterated sequence and enable us to pass to the limit in the equation. The idea of finding appropriate supersolutions was also used in [31], in this case for finding self-similar solutions with fat tails. The idea of considering a linear version of the model in order to better study properties of its solutions is common in the study of coagulation equations; see for example [35] where this idea is used to study uniqueness of solutions.

We present the proof of our main Theorem 1.4 in the following Section 2 with some technical computations moved to the appendix.

2. Proof of the main theorem

2.1. Formal approximation and discussion on the assumptions

Our approach to prove existence of a solution to (1.5) is based on constructing a suitable supersolution by approximating the coagulation term for large particles by a transport term. To motivate this we present in this subsection this formal approximation of (1.5). Similar computations can be found in [25, Section 4 and Appendix 1].

Suppose now that f is a solution of (1.5). Since we are interested in the behavior for large values of v, we can assume, due to the fast decay of $f(x, v, t)$, that the term $\int_{[\frac{v}{2},\infty)} K(v, v') f(x, v, t) f(x, v', t)\, \mathrm{d}v'$ gives a small contribution. This is consistent with the known results in coagulation equations. We can then approximate (1.5) via

$$\begin{aligned}
&\partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t)\\
&\quad = \int_{(0,\frac{v}{2})} K(v - v', v') f(x, v - v', t) f(x, v', t)\, \mathrm{d}v'\\
&\qquad - \int_{(0,\infty)} K(v, v') f(x, v, t) f(x, v', t)\, \mathrm{d}v'\\
&\quad = \int_{(0,\frac{v}{2})} [K(v - v', v') f(x, v - v', t) - K(v, v') f(x, v, t)] f(x, v', t)\, \mathrm{d}v'\\
&\qquad - \int_{[\frac{v}{2},\infty)} K(v, v') f(x, v, t) f(x, v', t)\, \mathrm{d}v'\\
&\quad \approx \int_{(0,\frac{v}{2})} [K(v - v', v') f(x, v - v', t) - K(v, v') f(x, v, t)] f(x, v', t)\, \mathrm{d}v'. \qquad (2.1)
\end{aligned}$$

Since our strategy relies on finding a suitable supersolution, it suffices to find a lower bound for (2.1). This is where assumption (1.7) is needed. We thus use that $K(v - v', v') \leq K(v, v')$ when $v' \in [0, \frac{v}{2}]$ and obtain

$$\begin{aligned}
&\partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t)\\
&\qquad - \int_{(0,\frac{v}{2})} [K(v - v', v') f(x, v - v', t) - K(v, v') f(x, v, t)] f(x, v', t)\, \mathrm{d}v'\\
&\quad \geq \partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t)\\
&\qquad - \int_{(0,\frac{v}{2})} K(v, v') [f(x, v - v', t) - f(x, v, t)] f(x, v', t)\, \mathrm{d}v'. \qquad (2.2)
\end{aligned}$$

Assuming now that the coagulation kernel K behaves like $v'^\gamma + v^\gamma$ and since $v' \in [0, \frac{v}{2}]$, we further deduce that

$$\begin{aligned}
&\partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t)\\
&\quad \approx \int_{(0,\frac{v}{2})} v^\gamma [f(x, v - v', t) - f(x, v, t)] f(x, v', t)\, \mathrm{d}v'\\
&\quad \approx -v^\gamma \int_{(0,\frac{v}{2})} \int_{v-v'}^{v} \partial_w f(x, w, t)\, \mathrm{d}w\ f(x, v', t)\, \mathrm{d}v'.
\end{aligned}$$

Assume that $\partial_w f(x, w, t)$ behaves similarly for $w \in [\frac{v}{2}, v]$ and thus

$$\partial_t f(x, v, t) + v^\alpha \partial_x f(x, v, t) \approx -v^\gamma \partial_v f(x, v, t) \int_{(0,\frac{v}{2})} v' f(x, v', t)\, \mathrm{d}v'. \qquad (2.3)$$

We denote by $M_1(x,t) := \int_{(0,\infty)} v' f(x,v',t)\,\mathrm{d}v'$ the first moment in v of f. We consider only large values of v so that we can safely assume that $\int_{(0,\frac{v}{2})} v' f(x,v',t)\,\mathrm{d}v'$ contains most of the mass. In this manner, we can further approximate (2.3) by

$$\partial_t f(x,v,t) + v^{\alpha}\partial_x f(x,v,t) \approx -v^{\gamma}\partial_v f(x,v,t) M_1(x,t). \tag{2.4}$$

Notice that in order for our approximation to hold, assumption (1.7) was needed in (2.2). Otherwise, an analogous approximation of the model could be obtained by replacing $v^{\gamma}\partial_v f(x,v,t)$ in (2.3) by $\partial_v(v^{\gamma} f(x,v,t))$. The approximation containing the term $\partial_v(v^{\gamma} f(x,v,t))$ is the one actually used in [25]. However, due to (1.7), the approximation used in (2.4) suffices in order to prove our desired result. Suppose now that $M_1(x,t)$ decays sufficiently fast for large values of x, that is, assume that

$$M_1(x,t) \leq \frac{L}{1+|x|^{\overline{m}}}, \tag{2.5}$$

for some sufficiently large $\overline{m}$ and some $L > 0$. Combining (2.5) with (2.4), we obtain that f should behave formally like the solution of the equation

$$\partial_t f(x,v,t) + v^{\alpha}\partial_x f(x,v,t) + \frac{L v^{\gamma}\partial_v f(x,v,t)}{1+|x|^{\overline{m}}} = 0. \tag{2.6}$$

This motivates the analysis of equation (2.14) below when trying to find a supersolution for a linear version of (1.5). In order to obtain behavior of the form (2.5) for $M_1(x,t)$, we have to work with functions f such that (1.11) holds.

2.2. Upper and lower bounds for the solution of the approximated model

To prove short-time existence of a solution to (1.5) we will set up an iterative scheme and derive, using (2.6), a uniform supersolution for the solutions of this scheme. More precisely, for $n \in \mathbb{N}$, we define inductively a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned}\partial_t f_{n+1}(x,v,t) &+ v^{\alpha}\partial_x f_{n+1}(x,v,t)\\ &= \frac{1}{2}\int_{(0,v)} K(v-v',v') f_{n+1}(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v'\\ &\quad - \int_{(0,\infty)} K(v,v') f_{n+1}(x,v,t) f_n(x,v',t)\,\mathrm{d}v',\end{aligned} \tag{2.7}$$

with $\alpha \in (0,1)$, and

$$f_{n+1}(x,v,0) = f_n(x,v,0) = f_{\mathrm{in}}(x,v) \leq \frac{C_0}{1+|x|^m+v^p}, \quad \text{for all } n \in \mathbb{N}. \tag{2.8}$$

We take f_0 to be a function such that

$$\partial_t f_0(x,v,t) + v^{\alpha}\partial_x f_0(x,v,t) = 0 \tag{2.9}$$

with

$$f_0(x, v, 0) = f_{\text{in}}(x, v) \leq \frac{C_0}{1 + |x|^m + v^p}. \tag{2.10}$$

Notice that, if we have equality in (2.10), the solution of (2.9) is

$$f_0(x, v, t) = \frac{C_0}{1 + |x - v^\alpha t|^m + v^p}.$$

Remark 2.1. In principle we have to prove that the sequence in (2.7) is well defined. A rigorous proof would work as follows. First we approximate K with a kernel K_N which is such that

$$K_N(v, v') = K(v, v')\chi_N(v + v'),$$

for $N > 1$ and where $\chi_N\colon [0, \infty) \to [0, 1]$ is a continuous function such that $\chi_N(x) = 1$, when $x \leq \frac{N}{2}$, and $\chi_N(x) = 0$, when $x \geq N$. Then we can establish with a standard fixed-point argument the existence and uniqueness of $f_{n,N}$. The key result is then that we obtain a uniform fast decaying bound for the sequence of solutions which is in particular independent of N. Then one can pass to the limit $N \to \infty$. Since this procedure is standard once one has the bounds on the solution, we omit the details here and work directly with K.

Notation and assumptions. Let $R > 0$. In the following we will denote

$$\begin{gathered} \xi_R \in C([0, \infty)), \quad \xi_R\colon [0, \infty) \to [0, 1] \\ \text{such that } \xi_R(v) = 1 \text{ if } v \geq 2R \text{ and } \xi_R(v) = 0 \text{ if } v \leq R. \end{gathered} \tag{2.11}$$

Furthermore, we assume

$$\begin{gathered} \alpha \in (0, 1), \quad \gamma \in [0, 1 + \alpha), \quad m \in \mathbb{N}, \quad m \text{ even}, \\ p = \alpha m, \quad m > \max\left\{\frac{\gamma + 1}{\alpha}, \frac{2}{\alpha} + 3\right\}, \end{gathered} \tag{2.12}$$

and define d via

$$\begin{cases} d = \left[\dfrac{2}{\alpha}\right] + 1 & \text{if } \left[\dfrac{2}{\alpha}\right] \text{ odd}, \\ d = \left[\dfrac{2}{\alpha}\right] + 2 & \text{if } \left[\dfrac{2}{\alpha}\right] \text{ even}, \end{cases} \tag{2.13}$$

where $[\cdot]$ denotes the floor function. Note that (2.12) and (2.13) in particular imply that $m > d + 1$.

The main goal in this section will be to derive estimates for the solution of a transport equation that approximates the coagulation equation. For $L > 0$ let G_L be the solution of

$$\begin{gathered} \partial_t G_L(x, v, t) + v^\alpha \partial_x G_L(x, v, t) + \frac{L v^\gamma}{1 + |x|^{m-d}} \xi_R(v) \partial_v G_L(x, v, t) = 0, \\ G_L(x, v, 0) = \frac{C_0}{1 + |x|^m + v^p}. \end{gathered} \tag{2.14}$$

We first study the properties of the backward characteristics for equation (2.14). To this end, we define X and V via

$$\begin{cases} \partial_t X(x,v,t) = -V^\alpha, & X(x,v,0) = x, \\ \partial_t V(x,v,t) = -\dfrac{LV^\gamma \xi_R(V)}{1+|X|^{m-d}}, & V(x,v,0) = v, \end{cases} \tag{2.15}$$

where d was defined in (2.13) and L is as in (2.14).

Proposition 2.2 (Properties of the characteristics). *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \geq 0$ the following estimates hold:*

$$(1-\delta)v \leq V(x,v,t) \leq (1+\delta)v, \tag{2.16}$$

$$(1-\delta)v^\alpha t \leq x - X(x,v,t) \leq (1+\delta)v^\alpha t, \tag{2.17}$$

$$\frac{\alpha}{18} v^{\alpha-1} t \leq -\partial_v X(x,v,t) \leq 18\alpha v^{\alpha-1} t. \tag{2.18}$$

Moreover, if $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$, then

$$\frac{1}{4} \leq \partial_v V(x,v,t) \leq \frac{9}{4}. \tag{2.19}$$

Otherwise, if $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$, then

$$|\partial_v V(x,v,t)| \leq 36L \max\{1, v^{\gamma-1} t\}, \tag{2.20}$$

$$\partial_v V(x,v,t) \leq 2. \tag{2.21}$$

Corollary 2.3. *As an immediate consequence of (2.17) we obtain the following estimates for all $t \geq 0$.*

If $x \leq 0$ then

$$\begin{aligned} |x| + (1-\delta)v^\alpha t = |x - (1-\delta)v^\alpha t| &\leq |X(x,v,t)| \\ &\leq |x| + (1+\delta)v^\alpha t = |x-(1+\delta)v^\alpha t|. \end{aligned} \tag{2.22}$$

If $x > 0$ then

$$|x-(1-\delta)v^\alpha t| \leq |X(x,v,t)| \leq |x-(1+\delta)v^\alpha t| \quad \text{if } t \geq \frac{x}{(1-\delta)v^\alpha}, \tag{2.23}$$

$$|x-(1+\delta)v^\alpha t| \leq |X(x,v,t)| \leq |x-(1-\delta)v^\alpha t| \quad \text{if } t \leq \frac{x}{(1+\delta)v^\alpha}, \tag{2.24}$$

and

$$|X(x,v,t)| \leq 3v^\alpha t \quad \text{if } t \in \left(\frac{x}{(1+\delta)v^\alpha}, \frac{x}{(1-\delta)v^\alpha}\right). \tag{2.25}$$

Proof of Proposition 2.2. Proof of (2.16). First, we see from (2.15) that $X(x,v,t) \leq x$ and $V(x,v,t) \leq v$ for all $t \geq 0$. Next, we define

$$\psi(x) := \int_{-\infty}^x \frac{L\,\mathrm{d}\xi}{1+|\xi|^{m-d}} \tag{2.26}$$

and $\Phi\colon (0,\infty)^2 \to \mathbb{R}$, $(z,v) \to \Phi(z,v)$ via

$$\Phi\big(\psi(X(x,v,t)) - \psi(x), v\big) := V(x,v,t). \tag{2.27}$$

Then it follows from (2.15) that

$$\begin{cases} \partial_z \Phi(\cdot, v) = \Phi^{\gamma-\alpha}\xi_R(\Phi), & z > 0, \\ \Phi(0,v) = v. \end{cases} \tag{2.28}$$

By integrating the ODE in (2.28) we deduce that

$$\Phi(z,v) = v \qquad \text{for all } z \geq 0,\ v \in [0,R], \tag{2.29}$$

$$\Phi(z,v) = \left(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z\right)^{\frac{1}{1-(\gamma-\alpha)}} \qquad \text{for all } z \geq 0,\ v \geq 2R, \tag{2.30}$$

$$\Phi(z,v) \leq \left(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z\right)^{\frac{1}{1-(\gamma-\alpha)}} \qquad \text{for all } z \geq 0,\ v \geq R, \tag{2.31}$$

$$\Phi(z,v) \geq v \qquad \text{for all } z \geq 0,\ v \geq R. \tag{2.32}$$

Notice also that $0 \leq \psi(x) \leq CL$, for some constant $C > 0$ which is independent of x. It thus suffices to consider values of z in the interval $[0, CL]$. Taking in particular R sufficiently large, it follows that $(1-\delta)v \leq \Phi(z,v) \leq (1+\delta)v$ for all $z \in [0, CL]$ and $v \geq 0$. Due to the definition of Φ in (2.27), estimate (2.16) follows.

Proof of (2.17). Estimate (2.17) follows then from (2.16) and the relation

$$X(x,v,t) - x = -\int_0^t V^\alpha(x,v,\tau)\,\mathrm{d}\tau,$$

together with the facts that $(1+\delta)^\alpha \leq 1+\delta$ and $(1-\delta)^\alpha \geq 1-\delta$.

Proof of (2.18)*:* In order to estimate the derivatives with respect to v of the characteristics we first prove that

$$\frac{1}{2} \leq \partial_v \Phi(z,v) \leq 2. \tag{2.33}$$

Estimate (2.33) is immediate if $v \leq R$. If $v \geq R$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}v}\right) = [(\gamma-\alpha)\Phi^{\gamma-\alpha-1}\xi_R(\Phi) + \Phi^{\gamma-\alpha}\xi_R'(\Phi)]\frac{\mathrm{d}\Phi}{\mathrm{d}v}, \quad \frac{\mathrm{d}\Phi}{\mathrm{d}v}(0) = 1$$

such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}v} = \exp\left(\int_0^z [(\gamma-\alpha)\Phi^{\gamma-\alpha-1}\xi_R(\Phi) + \Phi^{\gamma-\alpha}\xi_R'(\Phi)]\,\mathrm{d}s\right) \leq \exp(CR^{\gamma-\alpha-1})$$

and the upper bound in (2.33) follows since $\gamma < 1+\alpha$ and if R is sufficiently large. Analogously, we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}v} \geq \exp(-CR^{\gamma-\alpha-1})$$

and (2.33) follows. In order to proceed, we notice that

$$\int_x^X \frac{1}{\Phi(\psi(\xi)-\psi(x),v)^\alpha}\,\mathrm{d}\xi = -t. \tag{2.34}$$

Differentiating in v we obtain

$$\frac{\partial_v X(x,v,t)}{\Phi\big(\psi(X(x,v,t))-\psi(x),v\big)^\alpha} = \alpha \int_x^X \frac{\partial_v \Phi(\psi(\xi)-\psi(x),v)}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}\,\mathrm{d}\xi.$$

Since $\delta \in (0,\frac{1}{2})$, (2.16) and (2.17) imply that $\frac{v}{2} \le V(x,v,t) \le \frac{3v}{2}$ and that $\frac{v^\alpha t}{2} \le x - X(x,v,t) \le \frac{3v^\alpha t}{2}$. From this we deduce, using (2.17), the estimates for Φ, and the fact that $\alpha < 1$, that

$$\begin{aligned}-\partial_v X(x,v,t) &= \alpha \int_X^x \frac{\partial_v \Phi(\psi(\xi)-\psi(x),v)}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}\,\mathrm{d}\xi \times [\Phi(\psi(X(x,v,t))-\psi(x),v)]^\alpha \\ &\le 4\alpha 3^\alpha \int_X^x \frac{1}{v^{\alpha+1}}\,\mathrm{d}\xi v^\alpha \le 12\alpha v^{-1}[x-X(x,v,t)] \le 18\alpha v^{\alpha-1}t.\end{aligned}$$

Analogously we obtain

$$\begin{aligned}-\partial_v X &\ge \frac{\alpha}{2}\int_X^x \frac{1}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}\,\mathrm{d}\xi \times [\Phi(\psi(X(x,v,t))-\psi(x),v)]^\alpha \\ &\ge \frac{\alpha}{3^{\alpha+1}}\int_X^x \frac{1}{v^{\alpha+1}}\,\mathrm{d}\xi v^\alpha \ge \frac{\alpha}{9}v^{-1}[x-X(x,v,t)] \ge \frac{\alpha}{18}v^{\alpha-1}t,\end{aligned}$$

which concludes the proof of (2.18).

Proof of (2.19). We now prove that (2.19) holds if $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. From (2.27) we deduce that

$$\partial_v V(x,v,t) = \partial_v \Phi(\psi(X)-\psi(x),v) + \partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X. \tag{2.35}$$

Due to (2.33) it suffices to show that

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \le \frac{1}{4} \tag{2.36}$$

in order to conclude our proof. From (2.27), (2.28), and (2.16), we have

$$0 \le \partial_z \Phi(z,v) \le 2v^{\gamma-\alpha}.$$

Indeed, by (2.26) and (2.16) it holds that

$$\partial_z \Phi(z,v) \le \Phi(z,v)^{\gamma-\alpha} \le \max\Big\{2^{\alpha-\gamma}, \Big(\frac{3}{2}\Big)^{\gamma-\alpha}\Big\} v^{\gamma-\alpha} \le 2v^{\gamma-\alpha}.$$

By (2.18), it thus follows that

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \le \frac{36\alpha L v^{\gamma-\alpha}}{1+|X(t)|^{m-d}} v^{\alpha-1} t = \frac{36\alpha L v^{\gamma-1} t}{1+|X(t)|^{m-d}}. \tag{2.37}$$

We only analyze here the case when $v \ge R$, since by (2.28) we have that $\partial_z \Phi(z,v) = 0$ when $v \le R$ and thus there is nothing to prove in this case.

(1) Assume $x \leq (1-2\delta)v^\alpha t$. Then by (2.23) we have $|X(x,v,t)| \geq |x-(1-\delta)v^\alpha t| = (1-\delta)v^\alpha t - x \geq \delta v^\alpha t$ and we obtain

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq CL\frac{v^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}}.$$

Now, if $v^{\gamma-1}t \geq \frac{1}{4LC}$, then $v^\alpha t \geq \frac{1}{4LC}$ since $\alpha > \gamma - 1$ and we obtain

$$\begin{aligned}|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| &\leq CL\frac{v^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}} \leq \frac{CL}{\delta}v^{\gamma-\alpha-1}(\delta v^\alpha t)^{1+d-m}\\ &\leq C(L,\delta)v^{\gamma-\alpha-1} \leq \frac{1}{4}\end{aligned}$$

if $R > 0$ is sufficiently large. If $v^{\gamma-1}t \leq \frac{1}{4LC}$, (2.37) becomes

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq \frac{CLv^{\gamma-1}t}{1+(\delta v^\alpha t)^{m-d}} \leq \frac{1}{4}.$$

(2) Assume $x \geq (1+2\delta)v^\alpha t$. Then $|X(x,v,t)| \geq |x-(1+\delta)v^\alpha t| = x-(1+\delta)v^\alpha t \geq \delta v^\alpha t$ and we can conclude as before.

Proof of (2.20) *and* (2.21). If $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$, then from (2.37) it follows that

$$|\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X| \leq \frac{36Lv^{\gamma-1}t}{1+|X(t)|^{m-d}} \leq 36Lv^{\gamma-1}t$$

and (2.20) follows.

In order to prove that (2.21) holds, we notice that

$$\partial_z \Phi(\psi(X)-\psi(x),v)\psi'(X)\partial_v X \leq 0.$$

Combining this with (2.35) and (2.33), the conclusion follows. ∎

With the help of the characteristics, the solution G_L of (2.14) can be written as

$$G_L(x,v,t) = \frac{C_0}{1+|X(x,v,t)|^m + V(x,v,t)^p}. \tag{2.38}$$

Moreover, it holds that

$$\partial_v G_L(x,v,t) = -C_0\frac{[m|X|^{m-2}X\partial_v X + pV^{p-1}\partial_v V]}{(1+|X|^m+V^p)^2}(x,v,t). \tag{2.39}$$

This function G_L will be the main building block for constructing a uniform supersolution to the sequence $\{f_n\}_{n\in\mathbb{N}}$ in Section 2.3. We would like this supersolution to be decreasing in v for fixed x. Unfortunately, for $x > 0$ the function G_L is not decreasing in v.

Proposition 2.4. *In the following we characterize a local maximum of* G_L.

(a) *Given* $L > 0$ *and* $\delta \in (0, \frac{1}{2})$ *there exists a sufficiently large* $R > 0$ *such that for all* $t \in [0, 1]$ *the following holds. For every* $x \in \mathbb{R}$ *and* $t \in [0, 1]$ *there exists at least one point* $v_{\max}(x, t)$ *with the properties that*

$$v_{\max}(x,t)^{\alpha} \notin \left[\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}\right] \tag{2.40}$$

and

$$\partial_v G_L(x, v_{\max}(x,t), t) = 0. \tag{2.41}$$

Moreover, there exists a constant $K_{\max} > 0$*, which is independent of* x, t, δ, R, *and* L, *such that the following holds:*

$$\frac{1}{K_{\max}} x t^{\frac{1}{m-1}} \leq v_{\max}(x,t)^{\alpha} \leq K_{\max} x t^{\frac{1}{m-1}}. \tag{2.42}$$

(b) *Given* $L > 0$ *and* $\delta \in (0, \frac{1}{2})$ *there exists a sufficiently large* $R > 0$ *such that for all* $t \in [0, T]$*, with* T *sufficiently small, that is independent of* L, δ, *and* R, *there exists a unique point* $v_{\max}$ *with the properties* (2.40) *and* (2.41).

Proof. Let $v^{\alpha} \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ such that (2.41) holds. Notice that from (2.39), it follows that $m|X|^{m-2}X\partial_v X + pV^{p-1}\partial_v V = 0$. If R is sufficiently large, then the estimates from Proposition 2.2 hold. We have the following cases.

Case 1*:* $x - (1+2\delta)v^{\alpha}t \geq 0$. Then, by (2.17), it holds that $0 \leq x - (1+\delta)v^{\alpha}t \leq X(x,v,t) \leq x - (1-\delta)v^{\alpha}t$ and by (2.18), it follows that $\partial_v X \leq 0$. Thus, if (2.41) holds, we have that $m|X|^{m-1}|\partial_v X| = pV^{p-1}\partial_v V$. By Proposition 2.2 and since $\delta \in (0, \frac{1}{2})$, we have that there exists a constant $C > 0$, that is independent of δ, L, and R, such that

$$\frac{1}{C}|x - (1+\delta)v^{\alpha}t|^{m-1}v^{\alpha-1}t \leq m|X|^{m-1}|\partial_v X| = pV^{p-1}\partial_v V \leq Cv^{p-1}.$$

This implies that $\frac{1}{C}|x - (1+\delta)v^{\alpha}t|^{m-1}t \leq v^{\alpha(m-1)}$. Since $m-1$ is odd, $\delta < 1$, and $x - (1+2\delta)v^{\alpha}t \geq 0$, we further have that $xt^{\frac{1}{m-1}} \leq Cv^{\alpha} + 2v^{\alpha}t^{\frac{m}{m-1}}$. Since $t \leq 1$, we then obtain that $xt^{\frac{1}{m-1}} \leq Cv^{\alpha}$ and the lower bound in (2.42) follows. In order to obtain the upper bound in (2.42), we use similar computations together with the fact that $t \geq 0$ and that

$$C|x - (1-\delta)v^{\alpha}t|^{m-1}v^{\alpha-1}t \geq m|X|^{m-1}|\partial_v X| = pV^{p-1}\partial_v V \geq \frac{1}{C}v^{p-1}.$$

Case 2*:* $x - (1-2\delta)v^{\alpha}t \leq 0$. Then, by (2.17), it holds that $x - (1+\delta)v^{\alpha}t \leq X(x,v,t) \leq x - (1-\delta)v^{\alpha}t \leq 0$ and thus $m|X|^{m-2}X\partial_v X + pV^{p-1}\partial_v V = m|X|^{m-1}|\partial_v X| + pV^{p-1}\partial_v V > 0$ in this region. Using this and (2.39), it follows that $\partial_v G_L(x,v,t) < 0$.

In Appendix A, Proposition A.1, we will prove that for a sufficiently small $T > 0$, which is independent of L, δ, and R, we have that

$$\partial_v^2 G_L(x,v,t) < 0, \tag{2.43}$$

for all v that satisfy the estimate in (2.42) and all $t \in [0, T]$ which implies the uniqueness of such a point. This concludes our proof. ∎

The following lemma will also be needed in the construction of a supersolution in Section 2.3.

Proposition 2.5. *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0, T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that*

$$\partial_x G_L(x, v_{\max}(x,t), t) \leq 0,$$

where G_L is the solution of (2.14) *and $v_{\max}(x,t)$ was defined in* (2.41).

Proof. If R is sufficiently large, then the estimates from Proposition 2.2 hold. We first notice that due to (2.17) and (2.42) we have

$$X(x, v_{\max}(x,t), t) \geq x - 2v_{\max}(x,t)^{\alpha} t \geq x - 2K_{\max} x t^{1+\frac{1}{m-1}} > 0$$

if $t \in [0, T]$ and T is sufficiently small. Then we compute

$$\partial_x G_L(x, v, t) = -C_0 \frac{[m|X|^{m-2} X \partial_x X + pV^{p-1} \partial_x V]}{(1 + |X|^m + V^p)^2}(x, v, t),$$

and thus it suffices to prove that

$$(mX^{m-1} \partial_x X + pV^{p-1} \partial_x V)(x, v_{\max}(x,t), t) \geq 0. \tag{2.44}$$

Since $V \geq 0$ we have to prove that $\partial_x X(x, v_{\max}(x,t), t) \geq 0$ and $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. We start by analyzing $\partial_x X$.

We consider the case when $v \geq 2R$, since the other cases work similarly. Differentiating (2.34) with respect to x, keeping in mind that $x > 0$, we obtain

$$\frac{\partial_x X(x, v, t)}{\Phi\big(\psi(X(x,v,t)) - \psi(x), v\big)^{\alpha}} + \alpha \int_x^X \frac{\partial_z \Phi \partial_x \psi(x)\, d\xi}{\Phi(\psi(\xi) - \psi(x), v)^{\alpha+1}} - \frac{1}{\Phi(0, v)^{\alpha}} = 0.$$

We have that $\partial_z \Phi \geq 0$ due to (2.28). Since $X(x, v, t) \leq x$ and by (2.27) and (2.16) we obtain

$$\partial_x X(x, v, t) \geq \frac{\Phi(\psi(X(x,v,t)) - \psi(x), v)^{\alpha}}{v^{\alpha}} \geq \frac{1}{C} > 0.$$

Thus, in order for (2.44) to hold, what is left to prove is that $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. Integrating (2.15) over time and differentiating with respect to x, we obtain

$$V(x, v, t)^{-\gamma} \partial_x V(x, v, t) = \int_0^t \frac{L(m-d)|X|^{m-d-2} X \partial_x X\, d\xi}{(1 + X(\xi)^{m-d})^2}.$$

Since we have $X(x, v_{\max}(x,t), t) \geq 0$ and $\partial_x X(x, v, t) \geq 0$ we obtain $\partial_x V(x, v_{\max}(x,t), t) \geq 0$. ∎

2.3. Construction of a supersolution for a linear reformulation of the model

In this subsection we will prove that the solution G_L of (2.14), after suitable modifications, is a supersolution for problem (2.7).

Definition 2.6. Let $\delta \in (0, \frac{1}{2})$, $L > 0$, and R, which depends on L and δ, be as in Proposition 2.2. Let G_L be the solution of (2.14) with given δ, L, and R. We define the function H_L via

$$H_L(x, v, t) = \begin{cases} G_L(x, v, t) & \text{if } \partial_v G_L(x, v, t) \le 0 \\ & \text{or } x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t], \\ G_L(x, v_{\max}(x, t), t) & \text{otherwise,} \end{cases}$$

where $v_{\max}(x, t)$ was defined in (2.41).

By the choice of $v_{\max}$ in (2.41), if $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $v \le v_{\max}(x, t)$, then we have that $H_L(x, v, t) = G_L(x, v_{\max}(x, t), t)$. Moreover, H_L is decreasing in v for fixed x outside possibly a critical region where $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$.

Intuition behind the choice of our supersolution. We expect that for small times the coagulation operator gives a small contribution and solutions to (1.5) behave like solutions of

$$\partial_t h(x, v, t) + v^\alpha \partial_x h(x, v, t) = 0,$$
$$h(x, v, 0) = \frac{1}{1 + |x|^m + v^p},$$

that is, $h(x, v, t) = \frac{1}{1+|x-v^\alpha t|^m + v^p}$. However, this cannot be a good approximation for large v, since for large volumes the coagulation operator cannot be neglected. In order to find a supersolution for equation (2.7), motivated by the formal arguments presented in Section 2.1, a possible candidate is the function G_L that solves the transport equation (2.14). Unfortunately though, G_L is not monotone in the regions where $v \le v_{\max}$ and where $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$, but monotonicity is useful to control the coagulation operator. In the formal argument this was used in (2.4) in order to obtain (2.6). Thus, in Definition 2.6 we replace G_L by a constant function for $v \le v_{\max}$. Such a simple construction is however not possible in the region where $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. In the following we always have to deal with this region separately. Roughly speaking, in that region we can estimate the contribution due to the coagulation operator by terms proportional to $t^{\frac{1-\gamma}{\alpha}} G_L$. These terms can be controlled by an exponential factor in the time derivative and explain the choice of B_t in equation (2.66) below in the final definition of a supersolution.

We now collect some properties of the function H_L, which are independent of δ, L, and R.

Lemma 2.7. *Let $\delta \in (0, \frac{1}{2})$, $L > 0$, and R be as in Proposition 2.2. Let $T > 0$ be sufficiently small, independent of δ, L, and R, and H_L be as in Definition 2.6. Then there exists a* *constant* $K_2 > 0$*, which is independent of* δ*,* L*, and* R*, such that the following holds for* $t \in [0, T]$.

If $x > 0$ *then*

$$H_L(x, v - v', t) \leq K_2 H_L(x, v, t) \quad \textit{for all } v' \in \left(0, \frac{v}{2}\right). \tag{2.45}$$

If $x > 0$ *and for all* v *such that* $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ *then there exists a sufficiently large constant* $C_l > 0$ *such that*

$$-\partial_v H_L(x, v', t) \leq -K_2 \partial_v H_L(x, v, t) \quad \textit{for all } v \geq \max\{R, C_l v_{\max}(x, t)\} \quad \textit{and } v' \in \left(\frac{v}{2}, v\right). \tag{2.46}$$

If $x \leq 0$ *then*

$$-\partial_v H_L(x, v', t) \leq -K_2 \partial_v H_L(x, v, t) \quad \textit{for all } v \geq R, v' \in \left(\frac{v}{2}, v\right). \tag{2.47}$$

Proof of (2.45). We will prove that (2.45) holds by proving separately that there exists $K_2 > 0$, which is independent of δ, L, and R, such that the following hold.

If $x > 0$ and for all v such that $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ then

$$H_L(x, v - v', t) \leq K_2 H_L(x, v, t) \quad \text{for all } v' \in \left(0, \frac{v}{2}\right) \quad \text{and } v, v - v' \geq v_{\max}(x, t), \tag{2.48}$$

$$H_L(x, v_{\max}(x, t), t) \leq K_2 H_L(x, v, t) \quad \text{for all } v \in (v_{\max}(x, t), 2v_{\max}(x, t)), \tag{2.49}$$

If $x > 0$ and for all v such that $v^\alpha \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ then

$$H_L(x, v - v', t) \leq K_2 H_L(x, v, t) \quad \text{for all } v' \in \left(0, \frac{v}{2}\right). \tag{2.50}$$

Before beginning our proof, we make the following observation. Because the proof of each region when $v^\alpha \notin [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$ differs, we have to distinguish between different cases.

If $v^\alpha \geq \frac{x}{(1-2\delta)t}$ and $w \in [\frac{v}{2}, v]$, we have the following subcases:

(1.a) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \geq \frac{x}{(1-2\delta)t}$. Notice that in this region, $x - (1 - \delta)w^\alpha t \leq 0$.

(1.b) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$.

(1.c) $x > 0$, $v^\alpha \geq \frac{x}{(1-2\delta)t}$, and $w^\alpha \leq \frac{x}{(1+2\delta)t}$. Notice that in this region, $x - (1 + \delta)w^\alpha t \geq 0$.

The remaining case is

(2) $x > 0$, $v^\alpha \leq \frac{x}{(1+2\delta)t}$ and then $w^\alpha \leq \frac{x}{(1+2\delta)t}$.

Proof of (2.48). We will prove that there exists a constant $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C(1 + |x|^m + v^p)} \leq H_L(x, w, t) \leq \frac{C}{1 + |x|^m + v^p},$$

for all $w \in [\frac{v}{2}, v]$.

Case (1.a). Notice that because of (2.17), we are in the region where $X(x, w, t) \leq 0$. Because of (2.23), it follows that $|x - (1 - \delta)w^\alpha t| \leq |X(x, w, t)| \leq |x - (1 + \delta)w^\alpha t|$. Thus, due to (2.38), it suffices to show in this case that

$$|x - (1 + \delta)w^\alpha t|^m + w^p \leq C(|x|^m + v^p) \leq C(|x - (1 - \delta)w^\alpha t|^m + w^p). \tag{2.51}$$

To prove (2.51) we notice that it suffices to show that

$$|x - (1 + \delta)w^\alpha t|^m + w^p \leq C(|x|^m + w^p) \leq C(|x - (1 - \delta)w^\alpha t|^m + w^p)$$

since $w \in [\frac{v}{2}, v]$. Since $x - (1 - \delta)w^\alpha t \leq 0$ in this case, it holds that $|x - (1 - \delta)w^\alpha t| \leq w^\alpha$. In this case, we have that $|x| - (1 - \delta)w^\alpha t \leq |x - (1 - \delta)w^\alpha t| \leq w^\alpha$. This implies that $|x| \leq 2w^\alpha$. Notice that since $x \geq 0$ and $x - (1 - \delta)w^\alpha t \leq 0$, it immediately follows that $0 \leq x \leq w^\alpha$. However, we do a more general proof as a similar estimate will be needed in order to prove (2.47) or when $x - (1 + \delta)w^\alpha \geq 0$ later on.

Since $(|x| + (1 + \delta)w^\alpha t)^m \leq C_m(|x|^m + t^m w^p)$ and t is sufficiently small, we obtain

$$\begin{aligned} |x - (1 + \delta)w^\alpha t|^m + w^p &\leq (|x| + (1 + \delta)w^\alpha t)^m + w^p \leq C_m(|x|^m + t^m w^p) + w^p \\ &\leq C(|x|^m + w^p). \end{aligned}$$

Additionally, since $|x| \leq 2w^\alpha$, it holds that

$$|x - (1 - \delta)w^\alpha t|^m + w^p \geq w^p = \frac{w^p}{2} + \frac{w^p}{2} \geq \frac{1}{C}(|x|^m + w^p).$$

Case (1.b). From (2.17) and (2.25), it follows that $X(x, v, t) \leq 0$ and $|X(x, w, t)| \leq C w^\alpha t$. Since in this case $v^\alpha \geq \frac{x}{(1-2\delta)t}$, from (2.51) we know that there exists a constant $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C(1 + |x|^m + v^p)} \leq H_L(x, v, t).$$

Thus, in order for (2.48) to hold in this case, we need to prove that there exists a constant $C > 0$, which is independent of δ, L, and R, such that $H_L(x, w, t) \leq \frac{C}{1+|x|^m+v^p}$, for $w \in [\frac{v}{2}, v]$. More precisely, due to (2.38), it suffices to prove that

$$1 + |X(x, w, t)|^m + V(x, w, t)^p \geq \frac{1}{C}(1 + |x|^m + v^p). \tag{2.52}$$

Since $t \leq 1$, we have that $x \leq (1 + 2\delta)w^\alpha t \leq 2w^\alpha$ in this case. Due to (2.16), it holds that

$$|X(x, w, t)|^m + V(x, w, t)^p \geq V(x, w, t)^p \geq C w^p = \frac{C w^p}{2} + \frac{C w^p}{2} \geq C(|x|^m + w^p).$$

Case (1.c). From (2.17), it follows that $X(x,v,t) \leq 0$ and $X(x,w,t) \geq 0$. Since in this case $v^\alpha \geq \frac{x}{(1-2\delta)t}$, from (2.51) we know that there exists a constant $C > 0$, independent of δ, L, and R, such that $\frac{1}{C(1+|x|^m+v^p)} \leq H_L(x,v,t)$ and we need to prove that $H_L(x,w,t) \leq \frac{C}{1+|x|^m+v^p}$, for $w \in [\frac{v}{2}, v]$. More precisely, due to (2.24) and (2.38), it suffices to prove as before that $1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1 + |x|^m + v^p)$. Actually, we prove here a more general estimate that will be used in Case (2), namely

$$|x-(1-\delta)w^\alpha t|^m + w^p \leq C(|x|^m + v^p) \leq C(|x-(1+\delta)w^\alpha t|^m + w^p). \tag{2.53}$$

In order to prove (2.53), we distinguish between two cases:

(i) $|x-(1+\delta)w^\alpha t| \geq w^\alpha$. In this case, we have that $w^\alpha \leq |x-(1+\delta)w^\alpha t| \leq |x| + (1+\delta)w^\alpha t$, which implies that $\frac{1}{2}w^\alpha \leq |x|$. Thus $|x| - (1+\delta)w^\alpha t \geq |x| - \frac{w^\alpha}{4} \geq \frac{|x|}{2} > 0$ and $(|x|-(1+\delta)w^\alpha t)^m \geq \frac{|x|^m}{2^m}$. Since $|x|-(1+\delta)w^\alpha t > 0$ and remembering that m is even, we also have that $|x-(1+\delta)w^\alpha t|^m \geq (|x|-(1+\delta)w^\alpha t)^m$ and thus $|x-(1+\delta)w^\alpha t|^m \geq 2^{-m}|x|^m$. Additionally , since t is sufficiently small and $\frac{1}{2}w^\alpha \leq |x|$, it holds that $|x-(1-\delta)w^\alpha t| \leq |x| + w^\alpha t \leq 2|x|$.

(ii) $|x-(1+\delta)w^\alpha t| \leq w^\alpha$. This case can be treated as in the proof of (2.51).

Case (2). From (2.17), it follows that $X(x,v,t) \geq 0$ and $X(x,w,t) \geq 0$. Thus, from (2.24) and (2.53), the conclusion follows.

Proof of (2.49). Notice that due to (2.42) and since $v \in (v_{\max}(x,t), 2v_{\max}(x,t))$, we have that $x-(1+2\delta)v^\alpha t \geq 0$ if we choose t to be sufficiently small. Moreover, since $\delta < \frac{1}{2}$, t can be chosen independently of δ. Since $v > v_{\max}(x,t)$, it holds by (2.53) that $\frac{1}{C(1+|x|^m+v^p)} \leq H_L(x,v,t)$. Thus, since $v \in (v_{\max}(x,t), 2v_{\max}(x,t))$, we further have that $\frac{1}{C(1+|x|^m+v_{\max}(x,t)^p)} \leq H_L(x,v,t)$. From (2.38) and (2.24), it suffices to prove that there exists a constant $C > 1$, independent of δ, L, and R, such that

$$|x|^m + v_{\max}(x,t)^p \leq C(|x-(1+\delta)v_{\max}(x,t)^\alpha t|^m + v_{\max}(x,t)^p).$$

The inequality holds since due to (2.42) we can choose t sufficiently small such that $|x-(1+\delta)v_{\max}(x,t)^\alpha t| = x-(1+\delta)v_{\max}(x,t)^\alpha t \geq \frac{x}{2}$.

Proof of (2.50). Due to (2.38) and since $v - v' \in (\frac{v}{2}, v)$ it holds on one side that

$$H_L(x, v-v', t) \leq \frac{C}{1+(v-v')^p} \leq \frac{C}{1+v^p},$$

for some $C > 0$, independent of δ, L, and R. On the other side, similarly to (2.25) and from (2.38), we have

$$H_L(x,v,t) \geq \frac{C}{1+v^p t^m + v^p} \geq \frac{C}{1+v^p},$$

for some $C > 0$, independent of δ, L, and R. Combining the two inequalities, we can conclude that (2.50) holds.

Proof of (2.46).

Case (1.a). We will prove that in this case it holds that

$$\frac{1}{C}\frac{v^{p-1}}{(1+|x|^m+v^p)^2} \leq -\partial_v H_L(x,w,t) \leq C\frac{v^{p-1}}{(1+|x|^m+v^p)^2}, \tag{2.54}$$

for some $C>0$, independent of δ, L, and R, and for all $w \in [\frac{v}{2}, v]$. We remember we are in the case when $x-(1+\delta)w^\alpha t \leq x-(1-\delta)w^\alpha t \leq 0$ and thus due to (2.17) it holds that $|X|^{m-2}(x,w,t)X(x,w,t)\partial_v X(x,w,t)(x,w,t) = |X|^{m-1}|\partial_v X(x,w,t)|$. Due to (2.16)–(2.19) we thus have in this region that

$$\begin{aligned}
&-\frac{1}{C}(x-(1-\delta)w^\alpha t)^{m-1}w^{\alpha-1}t + \frac{1}{C}(1-\delta)w^{p-1}\\
&\quad\leq |X|^{m-2}X\partial_v X(x,w,t) + V^{p-1}\partial_v V(x,w,t)\\
&\quad\leq -C(x-(1+\delta)w^\alpha t)^{m-1}w^{\alpha-1}t + C(1+\delta)w^{p-1}.
\end{aligned}$$

Moreover, from (2.51), we have

$$|x-(1+\delta)w^\alpha t|^m + w^p \leq C(|x|^m+v^p) \leq C(|x-(1-\delta)w^\alpha t|^m + w^p).$$

In order to prove (2.54), due to (2.39), it then suffices to show in this case that

$$\begin{aligned}
&|x-(1+\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1}\\
&\quad\leq Cv^{p-1} \leq C(|x-(1-\delta)w^\alpha t|^{m-1}w^{\alpha-1}t + w^{p-1}),
\end{aligned} \tag{2.55}$$

for all $w\in[\frac{v}{2},v]$. We thus prove (2.55). We have

$$\begin{aligned}
&w^{\alpha-1}t|x-(1+\delta)w^\alpha t|^{m-1} + w^{p-1}\\
&\quad= w^{\alpha-1}(|x-(1+\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)}).
\end{aligned} \tag{2.56}$$

Since $w\in[\frac{v}{2},v]$, it suffices to prove

$$\tilde{J} := |x-(1+\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)} \leq Cw^{\alpha(m-1)}$$

and

$$|x-(1-\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)} \geq \frac{1}{C}w^{\alpha(m-1)}.$$

We are in the case $v \geq C_l v_{\max}(x,t)$, for some sufficiently large $C_l>0$ and $x>0$. We know that $v_{\max}(x,t)^\alpha \geq \frac{1}{K_{\max}}xt^{\frac{1}{m-1}}$, for $K_{\max}$ as in (2.42), and thus $w^\alpha \geq \frac{1}{C}xt^{\frac{1}{m-1}} \geq 0$ since $w\geq\frac{v}{2}$.

We have that $|x-(1-\delta)w^\alpha t|^{m-1}t + w^{\alpha(m-1)} \geq w^{\alpha(m-1)}$. On the other hand, since $0 \leq xt^{\frac{1}{m-1}} \leq Cw^\alpha$ and $t\leq 1$, it follows that

$$\tilde{J}(x,w,t) \leq C_m(|x|^{m-1}t + w^{\alpha(m-1)}t^m) + w^{\alpha(m-1)} \leq Cw^{\alpha(m-1)}.$$

Remark 2.8. Since $v^\alpha \geq \frac{x}{(1-2\delta)t}$, we know from (2.54) that

$$\frac{1}{C}\frac{v^{p-1}}{(1+|x|^m+v^p)^2} \leq -\partial_v H_L(x,v,t).$$

Thus, for Cases (1.b) and (1.c) we need to prove that $-\partial_v H_L(x,w,t) \leq \frac{Cv^{p-1}}{(1+|x|^m+v^p)^2}$, for some $C > 0$, which is independent of δ, L, and R.

Case (1.b). As mentioned above, we need to prove that $-\partial_v H_L(x,w,t) \leq \frac{Cv^{p-1}}{(1+|x|^m+v^p)^2}$ when $w^\alpha \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$, $w \in [\frac{v}{2}, v]$. More precisely, due to (2.39), it suffices to prove that

$$1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1+|x|^m+v^p)$$

and that

$$X(x,w,t)^{m-1}\partial_v X(x,w,t) + V(x,w,t)^{p-1}\partial_v V(x,w,t) \leq Cv^{p-1},$$

for all $w \in [\frac{v}{2}, v]$ such that $w^\alpha \in [\frac{x}{(1+2\delta)t}, \frac{x}{(1-2\delta)t}]$.

We know from (2.52) that the first inequality holds and thus we focus on proving the second inequality. Due to (2.16) and (2.21), it follows that

$$\begin{aligned} X(x,w,t)^{m-1}\partial_v X(x,w,t) &+ V(x,w,t)^{p-1}\partial_v V(x,w,t) \\ &\leq X(x,w,t)^{m-1}\partial_v X(x,w,t) + Cw^{p-1}, \end{aligned}$$

where $C > 0$ is independent of δ, L, and R. We now analyze the term $X(x,w,t)^{m-1}\partial_v X(x, w,t)$. From (2.25) and (2.18), it holds that

$$\begin{aligned} X(x,w,t)^{m-1}\partial_v X(x,w,t) &\leq |X(x,w,t)|^{m-1}|\partial_v X(x,w,t)| \\ &\leq Cw^{\alpha(m-1)}w^{\alpha-1}t^m \leq Cw^{p-1}, \end{aligned}$$

for some $C > 0$, which is independent of δ, L, and R, and we can conclude by using that $w \in [\frac{v}{2}, v]$.

Case (1.c). As before, by Remark 2.8, we only need to prove that $1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1+|x|^m+v^p)$ and that $X(x,w,t)^{m-1}\partial_v X(x,w,t) + V(x,w,t)^{p-1}\times \partial_v V(x,w,t) \leq Cv^{p-1}$ when $w^\alpha \leq \frac{x}{(1+2\delta)t}$, $w \in [\frac{v}{2}, v]$.

By (2.17) and (2.18) it holds that $X(x,w,t) \geq 0$ and $\partial_v X(x,w,t) \leq 0$ in this region. Moreover, due to (2.16) and (2.19), it follows that

$$\begin{aligned} X(x,w,t)^{m-1}\partial_v X(x,w,t) &+ V(x,w,t)^{p-1}\partial_v V(x,w,t) \\ &\leq V(x,w,t)^{p-1}\partial_v V(x,w,t) \leq Cw^{p-1}. \end{aligned}$$

The fact that $1 + |X(x,w,t)|^m + V(x,w,t)^p \geq C(1+|x|^m+v^p)$ in this region follows from (2.53).

Case (2). We will prove that (2.54) holds in this case too. Due to (2.17), we have that $X(x,v,t)$, $X(x,w,t) \geq 0$ in this region. Thus, from (2.24), it holds that $|x-(1+\delta)w^\alpha t| \leq |X(x,w,t)| \leq |x-(1-\delta)w^\alpha t|$. We know from (2.53) that $|x-(1-\delta)w^\alpha t|^m + w^p \leq C(|x|^m + v^p) \leq C(|x-(1+\delta)w^\alpha t|^m + w^p)$, for some constant $C > 0$, which is independent of δ, L, and R. Moreover, since $X(x,w,t) \geq 0$ and $\partial_w X(x,w,t) \leq 0$, we have that $X(x,w,t)\partial_w X(x,w,t) = -|X(x,w,t)||\partial_w X(x,w,t)|$. Thus, due to (2.39), it suffices to show in this case that there exists $C > 0$, independent of δ, L, and R such that

$$-|x-(1+\delta)w^\alpha t|^{m-1} w^{\alpha-1} t + w^{p-1} \leq C v^{p-1} \leq C(-|x-(1-\delta)w^\alpha t|^{m-1} w^{\alpha-1} t + w^{p-1}),$$

for all $w \in [\frac{v}{2}, v]$.

We remember we are in the case when $x-(1+\delta)w^\alpha t \geq 0$. It is clear that $-|x-(1+\delta)w^\alpha t|^{m-1} w^{\alpha-1} t + w^{p-1} \leq C v^{p-1}$. For the other inequality, due to (2.56) it suffices to prove the statement for $-|x-(1-\delta)w^\alpha t|^{m-1} t + w^{\alpha(m-1)}$. Since $x-(1-\delta)w^\alpha t \geq 0$ and using that $a^{m-1} + b^{m-1} \leq (a+b)^{m-1}$, for $a, b \geq 0$, we have that $x^{m-1} - ((1-\delta)w^\alpha t)^{m-1} \geq (x-(1-\delta)w^\alpha t)^{m-1}$. Thus, it holds that

$$\begin{aligned} &-|x-(1-\delta)w^\alpha t|^{m-1} t + w^{\alpha(m-1)} \\ &\quad \geq -|x|^{m-1} t + (1-\delta)^{m-1} w^{\alpha(m-1)} t^m + w^{\alpha(m-1)}. \end{aligned} \tag{2.57}$$

Since $w \geq \frac{v}{2} \geq \frac{C_l}{2} v_{\max}(x,t)$ in this case we have that $x t^{\frac{1}{m-1}} \leq \frac{2^\alpha K_{\max} w^\alpha}{C_l^\alpha}$, for a sufficiently large constant $C_l > 0$. Thus (2.57) becomes

$$-|x-(1-\delta)w^\alpha t|^{m-1} t + w^{\alpha(m-1)} \geq -\frac{2^{p-\alpha} K_{\max}^{m-1} w^{\alpha(m-1)}}{C_l^{p-\alpha}} + w^{\alpha(m-1)} \geq \frac{w^{\alpha(m-1)}}{2},$$

for C_l sufficiently large, thus concluding our proof.

Proof of (2.47). In order to prove (2.47), it is useful to notice that if $x \leq 0$, then (2.22) holds. We will prove that there exists $C > 0$, which is independent of δ, L, and R, such that

$$\frac{1}{C} \frac{v^{\alpha-1}|x|^{m-1} t + v^{p-1}}{(1+|x|^m + v^p)^2} \leq -\partial_v H_L(x,w,t) \leq C \frac{v^{\alpha-1}|x|^{m-1} t + v^{p-1}}{(1+|x|^m + v^p)^2}, \tag{2.58}$$

for all $w \in [\frac{v}{2}, v]$. Using similar computations to those for (2.53), we can prove that

$$(|x| + (1+\delta)w^\alpha t)^m + w^p \leq C(|x|^m + v^p) \leq C((|x| + (1-\delta)w^\alpha t)^m + w^p). \tag{2.59}$$

Since $X(x,w,t) \leq 0$ and $\partial_w X(x,w,t) \leq 0$ we have that $X(x,w,t)\partial_w X(x,w,t) = |X(x,w,t)||\partial_w X(x,w,t)|$. Due to (2.22) and (2.39), what is left to prove in order for (2.58) to hold is that

$$\begin{aligned} &(|x| + (1+\delta)w^\alpha t)^{m-1} w^{\alpha-1} t + w^{p-1} \\ &\quad \leq C(v^{\alpha-1}|x|^{m-1} t + v^{p-1}) \\ &\quad \leq C((|x| + (1-\delta)w^\alpha t)^{m-1} w^{\alpha-1} t + w^{p-1}), \end{aligned} \tag{2.60}$$

for some $C > 0$, independent of δ, L, and R, for all $w \in [\frac{v}{2}, v]$.

Thus, we only need to prove (2.60). For $x \leq 0$ and $w \in [\frac{v}{2}, v]$, we have that $(|x| + (1-\delta)w^\alpha t)^{m-1}t + w^{\alpha(m-1)} \geq |x|^{m-1}t + w^{\alpha(m-1)}$. Furthermore, since $t \leq 1$, it follows that

$$(|x| + (1+\delta)w^\alpha t)^{m-1}t + w^{\alpha(m-1)} \leq C_m|x|^{m-1}t + C_m w^{\alpha(m-1)}t^m + w^{\alpha(m-1)} \\ \leq C(|x|^{m-1}t + w^{\alpha(m-1)}). \qquad \blacksquare$$

We also prove some moment bounds for the function H_L, which are independent of δ, L, and R.

Lemma 2.9 (Moment estimates). *Let $T > 0$ be sufficiently small, independent of δ, L, and R. Then there exists $K_3 > 0$, which is independent of $\delta \in (0, \frac{1}{2})$, L, and R, such that for all $t \in [0, T]$ we have*

$$M_{1,L}(x,t) := \int_{(0,\infty)} vH_L(x,v,t)\,\mathrm{d}v \leq \frac{K_3 C_0}{1+|x|^{m-d}} \quad \textit{for } x \in \mathbb{R},\ t \geq 0. \tag{2.61}$$

In general, if $p > \max\{2\gamma+1, 2\}$, we have

$$M_{n,L}(x,t) := \int_{(0,\infty)} v^n H_L(x,v,t)\,\mathrm{d}v \leq \frac{K_3 C_0}{1+|x|^{m-\frac{n+1}{\alpha}}} \quad \textit{for } x \in \mathbb{R},\ t \geq 0, \tag{2.62}$$

for $n \in [0, \max\{2\gamma, 1\}]$.

Proof. We first recall that due to (2.16) we have $G_L(x,v,t) \leq \frac{C}{1+|X(x,v,t)|^m + v^p}$. We split the integral for $M_{n,L}$ as follows:

$$M_{n,L}(x,t) = \int_0^{v_{\max}(x,t)} v^n H_L(x,v,t)\,\mathrm{d}v + \int_{v_{\max}(x,t)}^\infty v^n H_L(x,v,t)\,\mathrm{d}v \\ =: M_{n,1}(x,t) + M_{n,2}(x,t),$$

where $v_{\max}(x,t) = 0$ if $x \leq 0$. From (2.16) and since $\delta \in (0, \frac{1}{2})$, we have that $\frac{v}{2} \leq V(x,v,t) \leq \frac{3v}{2}$. Due to (2.17) and since $\delta \in (0, \frac{1}{2})$, it holds that $X(x,v,t) \geq x - (1+\delta)v^\alpha t \geq x - 2v^\alpha t$. Then, by (2.42), we further deduce that

$$X(x, v_{\max}(x,t), t) \geq x - 2v_{\max}(x,t)^\alpha t \geq x - 2K_{\max} x t^{\frac{m}{m-1}} \geq \frac{x}{2} \geq 0,$$

for all $t \leq T$ if T is sufficiently small. Thus, by Definition 2.6 and (2.38), we obtain

$$M_{n,1}(x,t) = \int_0^{v_{\max}} v^n H_L(x,v,t)\,\mathrm{d}v = \int_0^{v_{\max}} v^n G_L(x, v_{\max}(x,t), t)\,\mathrm{d}v \\ \leq \frac{C v_{\max}^{n+1}}{1+|x|^m + v_{\max}^p} \leq \frac{C}{1+|x|^{m-\frac{n+1}{\alpha}}},$$

for some constant $C > 0$ which is independent of δ, L, and R.

To estimate $M_{n,2}$ we note that for all $x \in \mathbb{R}$ we have

$$M_{n,2}(x,t) \leq C \int_0^\infty \frac{v^n}{1+v^p}\,\mathrm{d}v \leq C$$

since $p-n>1$ by assumption.

To obtain a decay for large $|x|$ we consider first $x>1$ and use (2.23)–(2.24) to obtain

$$M_{n,2}(x,t) \leq C\left(\int_0^{(\frac{x}{(1+2\delta)t})^{\frac{1}{\alpha}}} \frac{v^n}{1+(x-(1+\delta)v^\alpha t)^m+v^p}\,\mathrm{d}v + \int_{(\frac{x}{(1+2\delta)t})^{\frac{1}{\alpha}}}^\infty v^{n-p}\,\mathrm{d}v\right).$$

Then, since $\delta < \frac{1}{2}$, we have

$$\int_{(\frac{x}{(1+2\delta)t})^{\frac{1}{\alpha}}}^\infty v^{n-p}\,\mathrm{d}v \leq \frac{1}{p-n-1}\left(\frac{(1+2\delta)t}{x}\right)^{m-\frac{n+1}{\alpha}} \leq \frac{2^{m-\frac{n+1}{\alpha}}}{p-n-1}\left(\frac{t}{x}\right)^{m-\frac{n+1}{\alpha}}.$$

For the other term, by using the change of variables $v=(\frac{x}{t})^{\frac{1}{\alpha}}\xi$, we find

$$\begin{aligned}
&\int_0^{(\frac{x}{(1+2\delta)t})^{\frac{1}{\alpha}}} \frac{v^n}{1+(x-(1+\delta)v^\alpha t)^m+v^p}\,\mathrm{d}v \\
&\quad\leq \left(\frac{x}{t}\right)^{\frac{1}{\alpha}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{1+x^m|1-(1+\delta)\xi^\alpha|^m+(\frac{x}{t})^{\frac{p}{\alpha}}\xi^p}\,\mathrm{d}\xi \\
&\quad\leq \frac{1}{x^{m-\frac{1}{\alpha}}}\frac{1}{t^{\frac{1}{\alpha}}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{|1-(1+\delta)\xi^\alpha|^m+(\frac{1}{t})^m\xi^p}\,\mathrm{d}\xi. \qquad (2.63)
\end{aligned}$$

Since $t\leq 1$ and $m>1$ is even we have that the following holds:

$$3^{m+1}\left[|1-(1+\delta)\xi^\alpha|^m+\frac{1}{t^m}\xi^p\right] \geq 1+\frac{1}{t^m}\xi^p, \qquad (2.64)$$

for $\xi \geq 0$. Indeed, if $|1-(1+\delta)\xi^\alpha| \geq \frac{1}{2}$, then (2.64) follows. Otherwise, if $|1-(1+\delta)\xi^\alpha| \leq \frac{1}{2}$, then $\frac{1}{2} \leq (1+\delta)\xi^\alpha \leq \frac{3}{2}$ and since $\delta<\frac{1}{2}$ it holds that $\xi^\alpha \geq \frac{1}{3}$. Thus, if we use in addition that $t \leq 1$, we have

$$|1-(1+\delta)\xi^\alpha|^m+\frac{1}{t^m}\xi^p \geq \frac{1}{2t^m}\xi^p+\frac{1}{2t^m}\xi^p \geq \frac{1}{3^{m+1}}+\frac{1}{2t^m}\xi^p$$

and thus (2.64) holds. It then follows that

$$\begin{aligned}
\frac{1}{x^{m-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{|1-\xi^\alpha|^m+(\frac{1}{t})^m\xi^p}\,\mathrm{d}\xi &\leq \frac{C}{x^{m-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}} \int_{(0,\infty)} \frac{(\frac{x}{t})^{\frac{n}{\alpha}}\xi^n}{1+(\frac{1}{t})^m\xi^p}\,\mathrm{d}\xi \\
&\leq \frac{C}{x^{m-\frac{n+1}{\alpha}}} \int_{(0,\infty)} \frac{\eta^n}{1+\eta^p}\,\mathrm{d}\eta \leq \frac{C}{x^{m-\frac{n+1}{\alpha}}},
\end{aligned}$$

for some constant $C>0$ which is independent of $\delta \in (0,\frac{1}{2})$, L, and R.

In the case $x \leq 0$ we can use (2.22) to obtain the estimate similarly to above without the need to split the integral. Estimate (2.61) follows from (2.62) by choosing $n=1$ and the fact that d in (2.13) satisfies $d>\frac{2}{\alpha}$. ∎

Remark 2.10. With similar computations to those used in Lemma 2.9, we can prove that

$$\int_{(0,\infty)} \frac{v^n}{1+|x|^m+v^p}\,\mathrm{d}v \le \frac{K_3}{1+|x|^{m-\frac{n+1}{\alpha}}}, \tag{2.65}$$

for all $x \in \mathbb{R}$ and for $n \in [0, \max\{2\gamma, 1\}]$. This is since we can take $t = 1$ in (2.63) in order to obtain

$$\begin{aligned}\int_0^{x^{\frac{1}{\alpha}}} \frac{v^n}{1+x^m+v^p}\,\mathrm{d}v &\le x^{\frac{1}{\alpha}} \int_{(0,\infty)} \frac{x^{\frac{n}{\alpha}}\xi^n}{1+x^m+x^{\frac{p}{\alpha}}\xi^p}\,\mathrm{d}\xi \\ &\le \frac{1}{x^{m-\frac{1+n}{\alpha}}} \int_{(0,\infty)} \frac{\xi^n}{1+\xi^p}\,\mathrm{d}\xi.\end{aligned}$$

We now define the function that we will prove is a supersolution for the problem (2.7).

Definition 2.11. Let $\delta \in (0, \frac{1}{4}]$ and $L = 4K_1K_2K_3C_0$, where C_0 is as in (2.38), K_1 is as in (1.6), K_2 is as in Lemma 2.7, and K_3 is as in Lemma 2.9. Denote by $H(x, v, t) := H_L(x, v, t)$, where H_L is as in Definition 2.6. Moreover, let B_t be

$$B_t := \mathrm{e}^{\lambda t+\lambda t^{\frac{\alpha+1-\gamma}{\alpha}}} \tag{2.66}$$

for some $\lambda > 0$. Then we define the function G via

$$G(x, v, t) = B_t H(x, v, t). \tag{2.67}$$

As mentioned before, with this construction G is decreasing in v for fixed x outside possibly a critical region where $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. In the following we will have to deal with this region separately.

Our key result is the following.

Proposition 2.12. *Let m, p, γ, α as in (2.12). Let $T > 0$ be sufficiently small and $\delta \in (0, \frac{1}{4}]$. There exists a sufficiently large $\lambda > 0$, which depends only on C_0 and the parameters m, γ, α, p, such that if $f_n \le G$, where the sequence $\{f_n\}_{n\in\mathbb{N}}$ was defined in (2.7) with initial condition as in (2.8), then $f_{n+1} \le G$, for all $n \in \mathbb{N}$ and all $t \in [0, T]$.*

Remark 2.13. Since the constants will play an important role in our proof, it is worthwhile mentioning for clarity which are the constants that the parameters in Proposition 2.12 depend on. Let C_0 be as in (2.38), K_1 as in (1.6), K_2 as in Lemma 2.7, and K_3 be as in Lemma 2.9. Notice that these constants do not depend on δ, L, or R from Proposition 2.2. In order to prove that G is a supersolution, we take $L = 4K_1K_2K_3C_0$. We then take λ to be sufficiently large depending on C_0, K_1, K_2, and K_3. We then take T to be such that $\max\{T, T^{\frac{\alpha+1-\gamma}{\alpha}}\} \le \frac{\ln(2)}{2\lambda}$, which implies that $B_t \le 2$, for all $t \in [0, T]$, where B_t was defined in (2.66).

Before we begin with the proof of Proposition 2.12, it is worthwhile noticing that we have some moment bounds for the function G in Definition 2.11 as a direct consequence of Lemma 2.9.

Lemma 2.14. *Let $T > 0$ be sufficiently small with K_3 as in Lemma 2.9. Then for all $t \in [0, T]$ we have*

$$M_1(x,t) := \int_{(0,\infty)} vG(x,v,t)\,\mathrm{d}v \le \frac{K_3 C_0 B_t}{1+|x|^{m-d}} \quad \text{for } x \in \mathbb{R},$$

where B_t was defined in (2.66). In general, if $p > \max\{2\gamma+1, 2\}$, we have

$$M_n(x,t) := \int_{(0,\infty)} v^n G(x,v,t)\,\mathrm{d}v \le \frac{K_3 C_0 B_t}{1+|x|^{m-\frac{n+1}{\alpha}}} \quad \text{for } x \in \mathbb{R},\ t \ge 0,$$

for $n \in [0, \max\{2\gamma, 1\}]$.

We now focus on proving Proposition 2.12.

Proof of Proposition 2.12. We now prove that G is a supersolution of the problem (2.7), that is, we show

$$\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) - \int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v' + \int_0^\infty K(v,v')G(x,v,t) f_n(x,v',t)\,\mathrm{d}v' \ge 0. \tag{2.68}$$

We prove (2.68) by showing first that

$$\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t) \ge 0, \tag{2.69}$$

for any $L > 0$ and where d was defined in (2.13), and then that

$$\begin{aligned} &\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) - \int_0^{\frac{v}{2}} K(v-v',v')G(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v' \\ &\quad + \int_0^\infty K(v,v')G(x,v,t) f_n(x,v',t)\,\mathrm{d}v' \\ &\quad \ge \partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t), \end{aligned} \tag{2.70}$$

for $L > 0$ as in Definition 2.11.

We now prove (2.69).

Assume $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $\partial_v G_L \le 0$ or $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. Then $G = B_t G_L$ and thus, using (2.14), it holds with $c_\alpha := \frac{\alpha+1-\gamma}{\alpha}$ that

$$\begin{aligned} &\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t) \\ &\quad = B_t \Big(\partial_t G_L(x,v,t) + v^\alpha \partial_x G_L(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G_L(x,v,t) \Big) \\ &\qquad + \lambda (1 + c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1}) B_t G_L \ge \lambda \mathrm{e}^{\lambda t} G_L(x,v,t) \ge 0. \end{aligned}$$

Notice that we did not use the contribution of the term $t^{\frac{\alpha+1-\gamma}{\alpha}}$ in the computations. This term will be needed later in the proof.

Assume now $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $\partial_v G_L > 0$ such that we have $G(x,v,t) = B_t G_L(x, v_{\max}(x,t), t)$. Then

$$\begin{aligned} &\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t) \\ &\quad = B_t\big(\partial_t G_L(x, v_{\max}(x,t), t) + v^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\big) \\ &\quad\quad + \lambda(1 + c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1}) B_t G_L(x, v_{\max}(x,t), t) \\ &\quad \geq B_t\big(\partial_t G_L(x, v_{\max}(x,t), t) + v^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\big). \end{aligned} \tag{2.71}$$

By the choice of $v_{\max}(x,t)$ we have that $v \leq v_{\max}(x,t)$ in the region where $\partial_v G_L(x,v,t) > 0$. Moreover, from Proposition 2.5, we know that $\partial_x G_L(x, v_{\max}(x,t), t) \leq 0$. Thus, from (2.71) and (2.14), we further obtain

$$\begin{aligned} &\partial_t G(x,v,t) + v^\alpha \partial_x G(x,v,t) + \frac{Lv^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t) \\ &\quad \geq B_t\big(\partial_t G_L(x, v_{\max}(x,t), t) + v_{\max}^\alpha \partial_x G_L(x, v_{\max}(x,t), t)\big) = 0 \end{aligned}$$

and (2.69) follows.

We are thus left to prove that (2.70) holds. We analyze the cases when $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ and $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ separately.

Proof of (2.70) *for* $x \notin [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. We have

$$\begin{aligned} &-\int_0^{\frac{v}{2}} K(v-v',v') G(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v' + \int_0^\infty K(v,v') G(x,v,t) f_n(x,v',t)\,\mathrm{d}v' \\ &\quad \geq -\int_0^{\frac{v}{2}} K(v,v')[G(x,v-v',t) - G(x,v,t)] f_n(x,v',t)\,\mathrm{d}v' \\ &\quad\quad + \int_0^{\frac{v}{2}} [K(v,v') - K(v-v',v')] G(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v'. \end{aligned} \tag{2.72}$$

Since $G, f_n \geq 0$ and $K(v-v',v') \leq K(v,v')$ when $v' \in [0, \frac{v}{2})$ by (1.7), it holds that

$$\int_0^{\frac{v}{2}} [K(v,v') - K(v-v',v')] G(x,v-v',t) f_n(x,v',t)\,\mathrm{d}v' \geq 0. \tag{2.73}$$

We know that $K(v,v') \leq K_1(v^\gamma + v'^\gamma)$, where K_1 is as in (1.6). Thus, when $v' \leq \frac{v}{2}$, it holds that $K(v,v') \leq 2K_1 v^\gamma$. Without loss of generality we assume in the following that $K(v,v') \leq v^\gamma$ when $v' \leq \frac{v}{2}$. This is in order to simplify the notation, but we allow L in Definition 2.11 to depend on K_1. Additionally, by (2.67), it holds that $\partial_v G \leq 0$ and thus $G(x,v-v',t) - G(x,v,t) \geq 0$, for $v \in (0, \frac{v}{2})$. Since $f_n \leq G$ and with (2.73) we deduce that

$$-\int_0^{\frac{v}{2}} v^\gamma[G(x,v-v',t)-G(x,v,t)]f_n(x,v',t)\,\mathrm{d}v' \geq -v^\gamma \int_0^{\frac{v}{2}} (G(x,v-v',t)-G(x,v,t))G(x,v',t)\,\mathrm{d}v'.$$

We use the following notation:

$$I_1 := v^\gamma \int_0^{\frac{v}{2}} (G(x,v-v',t)-G(x,v,t))G(x,v',t)\,\mathrm{d}v' = \xi_R(v)I_1 + (1-\xi_R(v))I_1$$

with ξ_R as in (2.11). Assume that H, as in Definition 2.11, satisfies

$$\xi_R(v)v^\gamma \int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v' \leq -\frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v H(x,v,t) + CB_t H(x,v,t) \tag{2.74}$$

and

$$(1-\xi_R(v))v^\gamma \int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v' \leq CB_t H(x,v,t), \tag{2.75}$$

for $L = 4K_1K_2K_3C_0$ as in Definition 2.11. Then

$$\begin{aligned}
&\partial_t G + v^\alpha \partial_x G - v^\gamma \int_0^{\frac{v}{2}} (G(x,v-v',t)-G(x,v,t))G(x,v',t)\,\mathrm{d}v' \\
&\quad \geq B_t\Big(\partial_t H + v^\alpha \partial_x H - v^\gamma \int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v'\Big) \\
&\quad + \lambda \mathrm{e}^{\lambda t + \lambda t^{\frac{\alpha+1-\gamma}{\alpha}}} H \\
&\quad \geq B_t\Big(\partial_t H + v^\alpha \partial_x H + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v H(x,v,t)\Big) \\
&\quad + B_t(-2CB_t+\lambda)H \geq \partial_t G + v^\alpha \partial_x G + \frac{Lv^\gamma}{1+|x|^{m-d}}\xi_R(v)\partial_v G
\end{aligned}$$

if $\lambda \geq 2CB_t$, which holds true if $\lambda = 4C$ and if $\max\{t, t^{\frac{\alpha+1-\gamma}{\alpha}}\} \leq \frac{\ln(2)}{2\lambda}$. This proves (2.70).

Proof of (2.75). We have the following cases:

Case (1.a)*: $x > 0$ and $v \leq v_{\max}(x,t)$.* In this case $H(x,v-v',t) = H(x,v,t)$, for $v' \in (0, \frac{v}{2}]$. Thus, (2.75) holds.

Case (1.b)*: $x > 0$ and $v_{\max}(x,t) \leq v \leq 2v_{\max}(x,t)$.* The proof of this case is the same as for Case (1.b) when proving that (2.74) holds. We thus postpone its proof.

Case (2): *Either* $\{x > 0$ and $2v_{\max}(x,t) \le v\}$ *or* $\{x \le 0\}$. Since $v \le 2R$ and $\gamma \ge 0$, it follows, using (2.48) and Lemma 2.14, that

$$\begin{aligned}(1-\xi_R(v))I_1 e^{-\lambda t} &\le (2R)^\gamma \int_0^{\frac{v}{2}} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\,\mathrm{d}v' \\ &\le (2R)^\gamma \int_0^{\frac{v}{2}} H(x,v-v',t)G(x,v',t)\,\mathrm{d}v' \\ &\le K_2(2R)^\gamma H(x,v,t) \int_0^{\frac{v}{2}} G(x,v',t)\,\mathrm{d}v' \\ &\le \frac{C_0 K_2 K_3 (2R)^\gamma B_t H(x,v,t)}{1+|x|^{m-\frac{1}{\alpha}}} \le C B_t H(x,v,t). \end{aligned} \tag{2.76}$$

Remark 2.15. Notice that the constant C in (2.76) depends on R from Proposition 2.2, on the constant K_2 from (2.48), and on the constant K_3 from Lemma 2.14. However, since L is fixed in Definition 2.11, R depends only on K_1, K_2, K_3, and C_0.

Proof of (2.74). We have the following cases:

Case (1.a): $x > 0$ *and* $v \le v_{\max}(x,t)$. Notice that in this case $H(x,v-v',t) = H(x,v,t)$, for $v' \in (0, \frac{v}{2}]$, and $\partial_v H(x,v,t) = 0$. Thus, (2.74) holds.

Case (1.b): $x > 0$ *and* $v_{\max}(x,t) \le v \le 2v_{\max}(x,t)$. We divide the integral on the left-hand side of (2.74) into the region $v' \in (0, v - v_{\max})$ and $v' \in (v - v_{\max}, \frac{v}{2})$, respectively. For $v' \in (v - v_{\max}, \frac{v}{2})$, using $\frac{v}{2} \le v_{\max}(x,t)$ and Lemma 2.14 with $n = 0$, we obtain

$$\begin{aligned}&\int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\,\mathrm{d}v' \\ &\quad\le H(x,v_{\max}(x,t),t) \int_0^{v_{\max}} G(x,v',t)\,\mathrm{d}v' \\ &\quad\le \frac{C_0 K_3 B_t H(x,v_{\max}(x,t),t)}{1+|x|^{m-\frac{1}{\alpha}}}.\end{aligned}$$

We have $0 \le v^\alpha \le 2v_{\max}^\alpha(x,t) \le 2K_{\max} x t^{\frac{1}{m-1}}$, with $K_{\max}$ as in (2.42).

Moreover, using (2.49) it follows that

$$\begin{aligned}&v^\gamma \int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x,v-v',t) - H(x,v,t))G(x,v',t)\,\mathrm{d}v' \\ &\quad\le \frac{2^\gamma C_0 K_2 K_3 B_t H(x,v,t) v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}}.\end{aligned}$$

Since $v_{\max}(x,t)^\alpha \le 2K_{\max} x t^{\frac{1}{m-1}}$, with $K_{\max}$ as in (2.42), we further obtain

$$\frac{H(x,v,t)v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}} \le \frac{C H(x,v,t) x^{\frac{\gamma}{\alpha}} t^{\frac{\gamma}{\alpha(m-1)}}}{1+x^{m-\frac{1}{\alpha}}} \le C t^{\frac{\gamma}{\alpha(m-1)}} H(x,v,t),$$

since $m > \frac{\gamma+1}{\alpha}$. Thus

$$v^\gamma \int_{v-v_{\max}(x,t)}^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v' \leq CB_t t^{\frac{\gamma}{\alpha(m-1)}} H(x,v,t). \quad (2.77)$$

We now estimate the integral

$$J := \int_0^{v-v_{\max}(x,t)} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v'.$$

As before, using (2.48) and Lemma 2.14, we find

$$\begin{aligned} J \leq \int_0^{v-v_{\max}(x,t)} H(x,v-v',t)G(x,v',t)\,\mathrm{d}v' &\leq \int_0^{v-v_{\max}(x,t)} K_2 H(x,v,t)G(x,v',t)\,\mathrm{d}v' \\ &\leq \frac{C_0 K_2 K_3 B_t H(x,v,t)}{1+x^{m-\frac{1}{\alpha}}}, \end{aligned}$$

and this implies, since $v \leq 2v_{\max}(x,t)$, that

$$\begin{aligned} &v^\gamma \int_0^{v-v_{\max}(x,t)} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v \\ &\quad \leq CB_t H(x,v,t)\frac{v_{\max}(x,t)^\gamma}{1+x^{m-\frac{1}{\alpha}}} \leq CB_t H(x,v,t)\frac{x^{\frac{\gamma}{\alpha}} t^{\frac{\gamma}{\alpha(m-1)}}}{1+x^{m-\frac{1}{\alpha}}}. \end{aligned} \quad (2.78)$$

Since $m > \frac{\gamma+1}{\alpha}$ it follows that (2.74) holds in this case.

Remark 2.16. Notice that the constant C in (2.77) and (2.78) depends only on the constant K_2 from (2.48) and on the constants C_0, K_3 from Lemma 2.14.

Case (1.c): $x > 0$ and $2v_{\max}(x,t) \leq v$ or $x < 0$. Notice that the computations used in Case (1.b) hold for any $v \leq C_l v_{\max}(x,t)$, for some fixed constant $C_l > 0$. We can thus assume without loss of generality in this case that $v > C_l v_{\max}(x,t)$ and that $v \geq R$ because of the presence of $\xi_R(v)$ in (2.74).

We have that $v - v' \in (\frac{v}{2}, v)$, for $v' \in (0, \frac{v}{2})$. It holds that

$$\begin{aligned} &\int_0^{\frac{v}{2}} (H(x,v-v',t)-H(x,v,t))G(x,v',t)\,\mathrm{d}v' \\ &\quad = -\int_0^{\frac{v}{2}} G(x,v',t)\int_{v-v'}^{v} \partial_v H(x,\tilde{v},t)\,\mathrm{d}\tilde{v}\,\mathrm{d}v'. \end{aligned}$$

We can then use (2.47), (2.46), and Lemma 2.14 with $n = 1$ in order to deduce that

$$\begin{aligned} &-v^\gamma \xi_R(v)\int_0^{\frac{v}{2}} G(x,v',t)\int_{v-v'}^{v} \partial_v H(x,\tilde{v},t)\,\mathrm{d}\tilde{v}\,\mathrm{d}v' \\ &\quad \leq -K_2 v^\gamma \xi_R(v)\partial_v H(x,v,t)\int_0^{\frac{v}{2}} v' G(x,v',t) \end{aligned}$$

$$\leq -\frac{C_0 K_2 K_3 B_t v^\gamma \xi_R(v)\partial_v H(x,v,t)}{1+|x|^{m-d}}$$
$$\leq -\frac{L v^\gamma \xi_R(v)\partial_v H(x,v,t)}{1+|x|^{m-d}},$$

where in the last inequality we used the definition of L in Definition 2.11 and that $B_t \leq 2$ by Remark 2.13. Thus (2.74) holds in this case.

Proof of (2.70) *for* $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$ *or, alternatively,* $v^\alpha t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$. We will assume that $v \geq 1$ and $\gamma > 1$, since the other cases are similar but easier to treat.

In order to prove (2.70) we will first show that

$$\int_0^{\frac{v}{2}} K(v,v') f_n(x,v',t) |H(x,v-v',t) - H(x,v,t)| \, \mathrm{d}v'$$
$$\leq CL(1 + t^{\frac{\alpha+1-\gamma}{\alpha}-1}) H(x,v,t) \tag{2.79}$$

and then that

$$\frac{v^\gamma}{1+|x|^{m-d}} |\partial_v H(x,v,t)| \leq CLt^{\frac{\alpha+1-\gamma}{\alpha}-1} H(x,v,t), \tag{2.80}$$

for H and L as in Definition 2.11. If (2.79) and (2.80) hold, then (2.70) holds. Indeed, arguing as in (2.72), (2.73) and using $f_n \leq G$, we find

$$(*) := \partial_t G + v^\alpha \partial_x G - \int_0^{\frac{v}{2}} K(v-v',v') G(x,v-v',t) f_n(x,v',t)\, dv'$$
$$+ \int_0^\infty K(v,v') G(x,v,t) f_n(x,v',t)\, dv'$$
$$\geq B_t \Big(\partial_t H + v^\alpha \partial_x H + \frac{L v^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v H(x,v,t) \Big)$$
$$- B_t \frac{L v^\gamma}{1+|x|^{m-d}} |\partial_v H(x,v,t)|$$
$$- B_t v^\gamma \int_0^{\frac{v}{2}} |H(x,v-v',t) - H(x,v,t)| f_n(x,v',t)\, \mathrm{d}v'$$
$$+ \lambda (c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1} + 1) B_t H.$$

Now using (2.79) and (2.80), we can conclude that

$$(*) \geq B_t \Big(\partial_t H + v^\alpha \partial_x H + \frac{L v^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v H(x,v,t) \Big)$$
$$+ B_t [-CL - 2LCt^{\frac{\alpha+1-\gamma}{\alpha}-1} + \lambda (c_\alpha t^{\frac{\alpha+1-\gamma}{\alpha}-1} + 1)] H$$
$$\geq \partial_t G + v^\alpha \partial_x G + \frac{L v^\gamma}{1+|x|^{m-d}} \xi_R(v) \partial_v G(x,v,t)$$

if λ is sufficiently large.

Remark 2.17. It is worthwhile mentioning that L depends only on K_1 from (1.6), K_2 from Lemma 2.7, and K_3 from Lemma 2.14; see Definition 2.11.

We now prove that (2.79) holds. Let $\eta \in (0,1)$ be fixed and sufficiently small. We want to bound the following terms:

$$\int_{\eta v}^{\frac{v}{2}} K(v,v') f_n(x,v',t)|H(x,v-v',t)-H(x,v,t)|\,\mathrm{d}v' \\ + \int_0^{\eta v} K(v,v') f_n(x,v',t)|H(x,v-v',t)-H(x,v,t)|\,\mathrm{d}v' =: J_1 + J_2.$$

We analyze each term separately. We notice that, since $\delta < 1$ and t is sufficiently small, we have that $v^\alpha \geq \frac{x}{(1+2\delta)t} \geq 2K_{\max} x t^{\frac{1}{m-1}} \geq 2v_{\max}^\alpha$ in this region, where $K_{\max}$ is as in (2.42). Thus we can assume in all the following that

$$v \geq 2v_{\max} \text{ and } v' \geq v_{\max}, \quad \text{for all } v' \in \Big[\frac{v}{2}, v\Big]. \tag{2.81}$$

Moreover, by (2.38), it holds that $G(x,v,t) = \frac{C_0 B_t}{1+|X|^m+V^p} \leq \frac{C_0 B_t}{1+V^p} \leq \frac{2^p C_0 B_t}{1+v^p}$. Using (2.50) and the fact that $B_t \leq 2$, for $t \leq T$, we have

$$\begin{aligned} J_1 &\leq C\int_{\eta v}^{\frac{v}{2}} v^\gamma [H(x,v-v',t)+H(x,v,t)]G(x,v',t)\,\mathrm{d}v' \\ &\leq CH(x,v,t)\int_{\eta v}^{\frac{v}{2}} \frac{v^\gamma}{1+v'^p}\,\mathrm{d}v' \\ &\leq C(\eta)H(x,v,t)\int_0^\infty \frac{v'^\gamma}{1+v'^p}\,\mathrm{d}v' \leq CH(x,v,t). \end{aligned}$$

For the second term, it holds that

$$J_2 \leq Cv^\gamma \int_0^{\eta v} |H(x,v-v',t)-H(x,v,t)|G(x,v',t)\,\mathrm{d}v'.$$

We remember we are in the region where $v^\alpha t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$ and thus $v'^\alpha \leq \eta^\alpha v^\alpha \leq \frac{\eta^\alpha x}{(1-2\delta)t}$. Due to (2.17), it follows that

$$X(x,v',t) \geq x-(1+\delta)v'^\alpha t \geq x - \frac{(1+\delta)\eta^\alpha x}{1-2\delta} = \frac{1-2\delta-(1+\delta)\eta^\alpha}{1-2\delta}x,$$

for all $v' \in [0,\eta v]$. Since $\delta \leq \frac{1}{4}$ it follows that we can choose η to be sufficiently small, but independent of δ, such that $1-2\delta-(1+\delta)\eta^\alpha \geq \frac{1}{4}$ and thus

$$X(x,v',t) \geq (1-2\delta-(1+\delta)\eta^\alpha)x \geq \frac{x}{4},$$

for all $v' \in [0,\eta v]$. Thus, since $v^\alpha t \in [\frac{x}{1+2\delta}, \frac{x}{1-2\delta}]$ and $\delta \leq \frac{1}{4}$, we have that $X(x,v',t) \geq \frac{v^\alpha t}{8}$. It follows that

$$J_2 \leq CB_t v^\gamma \int_0^{\eta v} \frac{\int_{v-v'}^{v} |\partial_{\tilde v} H(x,\tilde v,t)|\,\mathrm{d}\tilde v\,\mathrm{d}v'}{1+(v^\alpha t)^m + v'^p} \leq Cv^\gamma \int_0^{\eta v} \frac{\int_{v-v'}^{v} |\partial_{\tilde v} H(x,\tilde v,t)|\,\mathrm{d}\tilde v\,\mathrm{d}v'}{1+(v^\alpha t)^m + v'^p}$$

since $B_t \leq 2$. Now let $\tilde{v} \in [v - v', v]$, with $v' \in [0, \eta v]$. We have

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq C_0 \frac{|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V|}{(1 + |X|^m + V^p)^2}(x, \tilde{v}, t)$$
$$\leq C_0 \frac{|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V|}{(1 + V^p)^2}(x, \tilde{v}, t).$$

Assume $v^{\gamma-1} t \geq 1$ and remember we are in the case when $\gamma > 1$ and $v \geq 1$. Since $v' \leq \eta v$ it holds that $\frac{v}{2} \leq \tilde{v} \leq v$ for $\tilde{v} \in [v - v', v]$. In addition, by (2.81), we have that $\tilde{v} \geq v_{\max}$. If $\tilde{v}^\alpha \geq \frac{x}{1+2\delta}$, then from (2.20), we have

$$|m|X|^{m-2} X \partial_{\tilde{v}} X + V^{p-1} \partial_{\tilde{v}} V| \leq CL(|X(x, \tilde{v}, t)|^{m-1} \tilde{v}^{\alpha-1} t + \tilde{v}^{p+\gamma-2} t). \tag{2.82}$$

Otherwise, if $\tilde{v}^\alpha < \frac{x}{1+2\delta}$, then (2.82) still holds since $v^{\gamma-1} t \geq 1$.

Moreover, since $\frac{v}{2} \leq \tilde{v} \leq v$ for $\tilde{v} \in [v - v', v]$, it holds that $|X(x, \tilde{v}, t)| \leq C v^\alpha t \leq C v^\alpha$ and thus $|X(x, \tilde{v}, t)|^{m-1} \tilde{v}^{\alpha-1} t \leq C v^{p-1} t \leq C v^{p+\gamma-2} t$ since $\gamma > 1$. Thus

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq \frac{CLv^{p+\gamma-2} t}{(1 + v^p)^2}, \tag{2.83}$$

and thus it holds that

$$J_2 \leq CLv^{2(\gamma-1)} \frac{v^p t}{(1 + v^p)^2} \int_0^{\eta v} \frac{v' \, dv'}{1 + (v^\alpha t)^m + v'^p}.$$

By making the change of variables $v' = (1 + v^p t^m)^{\frac{1}{p}} \xi$, we further obtain

$$J_2 \leq CLv^{2(\gamma-1)} \frac{v^p}{(1 + v^p)^2} \frac{t}{(1 + v^p t^m)^{1-\frac{2}{p}}} \int_0^\infty \frac{\xi \, d\xi}{1 + \xi^p}$$
$$\leq CLv^{2(\gamma-1)} \frac{v^p}{(1 + v^p)^2} \frac{t}{(1 + v^p t^m)^{1-\frac{2}{p}}}.$$

Remembering the definition of H and since we have, due to (2.16), that

$$H(x, v, t) = \frac{C_0}{1 + |X(x, v, t)|^m + V(x, v, t)^p} \geq \frac{C}{1 + v^p}, \tag{2.84}$$

we deduce that

$$J_2 \leq \frac{CLv^{2(\gamma-1)} t}{(1 + v^p t^m)^{1-\frac{2}{p}}} H(x, v, t). \tag{2.85}$$

We now analyze the term $\frac{v^{2(\gamma-1)} t}{(1+v^p t^m)^{1-\frac{2}{p}}}$. It holds that

$$\frac{v^{2(\gamma-1)} t}{(1 + v^p t^m)^{1-\frac{2}{p}}} = \frac{(v^\alpha t)^{\frac{2(\gamma-1)}{\alpha}} t^{1-\frac{2(\gamma-1)}{\alpha}}}{(1 + (v^\alpha t)^m)^{1-\frac{2}{p}}} \leq t^{1-\frac{2(\gamma-1)}{\alpha}} \leq t^{-\frac{\gamma-1}{\alpha}}. \tag{2.86}$$

It holds that $-\frac{\gamma-1}{\alpha} > -1$ since $\gamma < \alpha + 1$ and thus we have that $e^{t^{1-\frac{\gamma-1}{\alpha}}} \leq C$. Then (2.79) follows from (2.85).

If $v^{\gamma-1}t \leq 1$, we use (2.20) and then (2.83) becomes

$$|\partial_{\tilde{v}} H(x, \tilde{v}, t)| \leq \frac{CLv^{p-1}}{(1+v^p)^2}. \tag{2.87}$$

We can conclude using the same computations as above and by noticing that $-\frac{(\gamma-1)}{\alpha} > -1$.

Finally, we need to prove that (2.80) holds in the case when $x \in [(1-2\delta)v^\alpha t, (1+2\delta)v^\alpha t]$. Assume first that $v^{\gamma-1}t \geq 1$. We have

$$\frac{v^\gamma}{1+|x|^{m-d}} |\partial_v H(x, v, t)| \leq \frac{Cv^\gamma |\partial_v H(x, v, t)|}{1+(v^\alpha t)^{m-d}}.$$

Making use of (2.83) and (2.84), we further obtain as before that

$$\frac{v^\gamma}{1+|x|^{m-d}} |\partial_v H(x, v, t)| \leq CLv^{2(\gamma-1)} \frac{v^p}{1+v^p} \frac{t}{1+(v^\alpha t)^{m-d}} H(x, v, t),$$

and we can then use similar arguments to (2.86). The case when $v^{\gamma-1}t \leq 1$ can be proved similarly using (2.87) instead of (2.83). This concludes our proof. ∎

2.4. Proof of Theorem 1.4

In this subsection, we finish the proof of Theorem 1.4 by establishing that there exists a limit for the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined in (2.7) and then passing to the limit in the equation. Our proof has analogies with the methods used to solve symmetric hyperbolic systems; see for example [28].

We first prove some bounds that are independent of t for the function G defined in Definition 2.11.

Lemma 2.18. *Let $T > 0$ be sufficiently small. Then it holds that*

$$G(x, v, t) \leq \frac{2^m C_0 B_t}{1+|x|^m + v^p} \leq \frac{2^{m+1} C_0}{1+|x|^m + v^p}, \tag{2.88}$$

$$G(x, v-v', t) \leq \frac{2^{m+1} K_2 C_0}{1+|x|^m + v^p} \quad \text{for all } v' \in \left(0, \frac{v}{2}\right), \tag{2.89}$$

for all $v > 0$, $x \in \mathbb{R}$, and all $t \in [0, T]$, where B_t was defined in (2.66), K_2 is as in Lemma 2.7, and C_0 is as in (2.8).

Proof. To prove (2.88) we notice first that it holds that $B_t \leq 2$ if we take $t \leq 1$ to be sufficiently small, where B_t was defined in (2.66).

We first consider the case $x > 0$ and $v \leq v_{\max}$, where $v_{\max}(x, t)$ was defined in (2.41).

From Definition 2.11 and the fact that if $v \leq v_{\max}(x,t)$ then we are in the region where $t \leq \frac{x}{v^{\alpha}(1+2\delta)}$, we can use the bound in (2.24) to deduce that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1+\delta)v_{\max}(x,t)^{\alpha}t|^m+(1-\delta)^p v_{\max}(x,t)^p},$$

when $v \leq v_{\max}(x,t)$, where C_0 is as in (2.38).

By (2.42), we have that $\frac{1}{K_{\max}} x t^{\frac{1}{m-1}} \leq v_{\max}(x,t) \leq K_{\max} x t^{\frac{1}{m-1}}$ and thus $x \geq x-(1+\delta)v_{\max}(x,t)^{\alpha}t \geq \frac{x}{2} > 0$ when $x>0$ if t is sufficiently small. Thus, using in addition that $v \leq v_{\max}$ and that $\delta < \frac{1}{2}$, it holds that $(1-\delta)^p v_{\max}^p \geq \frac{v^p}{2^p}$. Thus,

$$G(x,v,t) = G(x, v_{\max}(x,t), t) \leq \frac{2^m C_0}{1+|x|^m+v^p}.$$

We now consider the case when $t \in [\frac{x}{(1+2\delta)v^{\alpha}}, \frac{x}{(1-2\delta)v^{\alpha}}]$. From (2.38) and then using the fact that $x \leq 2v^{\alpha}t \leq v^{\alpha}$, it follows that

$$G(x,v,t) \leq \frac{2^p C_0}{1+v^p} \leq \frac{2^{p+1}C_0}{1+|x|^m+v^p}.$$

If $v^{\alpha}t \leq \frac{x}{1+2\delta}$, from (2.24) it holds that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1+\delta)v^{\alpha}t|^m+(1-\delta)^p v^p}$$

and (2.88) follows from (2.53).

Finally, if $v^{\alpha}t \geq \frac{x}{1-2\delta}$, from (2.23) it holds that

$$G(x,v,t) \leq \frac{C_0}{1+|x-(1-\delta)v^{\alpha}t|^m+(1-\delta)^p v^p}$$

and (2.88) follows from (2.51).

If $x \leq 0$, from (2.22) we have that $|X(x,v,t)| \geq |x-(1-\delta)v^{\alpha}t| = |x|+(1-\delta)v^{\alpha}t$ and the conclusion follows.

Formula (2.89) follows from (2.88), (2.45), and (2.59). ∎

Using these bounds, we can now prove that there exists a limit for the sequence $\{f_n\}_{n\in\mathbb{N}}$ that was defined in (2.7).

Proposition 2.19. *Let m, p, γ, α be as in (2.12). Assume in addition that $m \geq \frac{2\gamma+1}{\alpha}$. Let $T>0$ be sufficiently small. For every $\varepsilon>0$, there exists $n_{\varepsilon} \in \mathbb{N}$ such that, for every $n,m \geq n_{\varepsilon}$, it holds that $\|f_n - f_m\|_{\infty} := \sup_{t\in[0,T],x\in\mathbb{R},v\in(0,\infty)} |f_n(x,v,t)-f_m(x,v,t)| \leq \varepsilon$.*

Proof. Step 1 *(Setup)*. Let $t \geq 0$, $x \in \mathbb{R}$, and $v>0$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence defined in (2.7). For $n \in \mathbb{N}$, we denote

$$R_n(x,v,t) := f_{n+1}(x,v,t) - f_n(x,v,t).$$

Moreover, for two functions f, g, we denote

$$\mathbb{K}_1[f,g] := \int_0^{\frac{v}{2}} K(v-v',v')f(x,v',t)g(x,v-v',t)\,\mathrm{d}v',$$
$$\mathbb{K}_2[f,g] := \int_0^{\infty} K(v,v')f(x,v',t)g(x,v,t)\,\mathrm{d}v',$$

and

$$\mathbb{K}[f,g] := \mathbb{K}_1[f,g] - \mathbb{K}_2[f,g].$$

Using this notation, it holds that

$$\begin{aligned}
&\partial_t R_n(x,v,t) + v^\alpha \partial_x R_n(x,v,t) \\
&\quad = \mathbb{K}_1[f_n, f_{n+1}] - \mathbb{K}_1[f_{n-1}, f_n] - \mathbb{K}_2[f_n, f_{n+1}] + \mathbb{K}_2[f_{n-1}, f_n] \\
&\quad = \mathbb{K}_1[f_n, f_{n+1}] - \mathbb{K}_1[f_n, f_n] + \mathbb{K}_1[f_n, f_n] - \mathbb{K}_1[f_{n-1}, f_n] \\
&\qquad - \mathbb{K}_2[f_n, f_{n+1}] + \mathbb{K}_2[f_n, f_n] - \mathbb{K}_2[f_n, f_n] + \mathbb{K}_2[f_{n-1}, f_n] \\
&\quad = \mathbb{K}[f_n, R_n] + \mathbb{K}[R_{n-1}, f_n].
\end{aligned} \tag{2.90}$$

Remark 2.20. Notice that it suffices to analyze the term R_n in order to obtain the statement of Proposition 2.19. This is since we can repeat the computations in (2.90) to obtain

$$\partial_t[f_m - f_n] + v^\alpha \partial_x [f_m - f_n] = \mathbb{K}[f_{m-1}, f_m - f_n] + \mathbb{K}[f_{m-1} - f_{n-1}, f_n].$$

Since the estimates we will prove do not depend on $n, m \in \mathbb{N}$, we can reduce the problem to analyzing R_n in order to simplify the notation.

Notice the following. From (2.90) and (2.8), we have that R_n solves the following system:

$$\begin{cases}
\partial_t R_n(x,v,t) + v^\alpha \partial_x R_n(x,v,t) = \mathbb{K}_1[f_n, R_n] + \mathbb{K}_1[R_{n-1}, f_n] \\
\qquad\qquad\qquad\qquad\qquad\qquad - \mathbb{K}_2[f_n, R_n] - \mathbb{K}_2[R_{n-1}, f_n], \\
R_n(x,v,0) = 0.
\end{cases} \tag{2.91}$$

Since the system is linear in R_n, by Duhamel's principle, it suffices to derive estimates for

$$\begin{cases}
\partial_t R_n^s(x,v,t) + v^\alpha \partial_x R_n^s(x,v,t) = \mathbb{K}_1[f_n, R_n^s] - \mathbb{K}_2[f_n, R_n^s], \quad \text{for } t > s, \\
R_n^s(x,v,s) = \mathbb{K}_1[R_{n-1}, f_n](x,v,s) - \mathbb{K}_2[R_{n-1}, f_n](x,v,s).
\end{cases} \tag{2.92}$$

We now prove suitable estimates for the inhomogeneous part in (2.91), which in turn will give us suitable estimates for R_n.

Step 2 *(Induction basis).* It holds that

$$R_1(x,v,t) \le \frac{CT}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}, \tag{2.93}$$

for all $t \in [0,T]$.

We will prove (2.93) after (2.94) since the estimates are similar.

Step 3 *(Induction step).* It holds that

$$R_n(x, v, t) \leq \frac{(CT)^n}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}, \tag{2.94}$$

for all $t \in [0, T]$.

We assume by induction that there exists a constant $C > 0$ such that

$$R_{n-1}(x, v, t) \leq \frac{(CT)^{n-1}}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}. \tag{2.95}$$

We estimate the inhomogeneous terms $\mathbb{K}_1[R_{n-1}, f_n]$ and $\mathbb{K}_2[R_{n-1}, f_n]$. Assume that the following inequality holds:

$$|\mathbb{K}_1[R_{n-1}, f_n]| + |\mathbb{K}_2[R_{n-1}, f_n]| \leq \frac{C^n T^{n-1}}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}. \tag{2.96}$$

Notice that (2.92) is a linearized version of the spatially inhomogeneous coagulation equation as in (2.7) with initial condition satisfying the bound (2.96). Moreover, Proposition 2.12 holds for m replaced by $m - \frac{\gamma}{\alpha}$ and for $p = \alpha m$ replaced by $p = \alpha m - \gamma = p - \gamma$ since $m \geq \max\{\frac{2\gamma+1}{\alpha}, \frac{2+\gamma}{\alpha} + 3\}$ and thus $m - \frac{\gamma}{\alpha} \geq \max\{\frac{\gamma+1}{\alpha}, \frac{2}{\alpha} + 3\}$. By (2.96) and (2.92), we can thus apply Proposition 2.12 and (2.88) with m replaced by $m - \frac{\gamma}{\alpha}$. It follows that

$$R_n^s(x, v, t) \leq \frac{C^n T^{n-1}}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}, \tag{2.97}$$

for $t \in [0, T]$, if T is sufficiently small. Since from (2.91) and (2.92), we have that

$$R_n(x, v, t) = \int_0^t R_n^s(x, v, t) \, \mathrm{d}s,$$

the conclusion (2.94) follows from (2.97).

It remains to prove that (2.96) holds. We start with $\mathbb{K}_2[R_{n-1}, f_n]$. Using Proposition 2.12, (2.88) and the fact that $K(v, v') \leq K_1(v^\gamma + v'^\gamma)$ from (1.6), we have

$$\begin{aligned}
\mathbb{K}_2[R_{n-1}, f_n] &= \int_0^\infty K(v, v') R_{n-1}(x, v', t) f_n(x, v, t) \, \mathrm{d}v' \\
&\leq \frac{C v^\gamma}{1 + |x|^m + v^p} \int_0^\infty R_{n-1}(x, v', t) \, \mathrm{d}v' \\
&\quad + \frac{C}{1 + |x|^m + v^p} \int_0^\infty v'^\gamma R_{n-1}(x, v', t) \, \mathrm{d}v'.
\end{aligned} \tag{2.98}$$

Using assumption (2.95) and then (2.65), we further obtain

$$\begin{aligned}
\int_0^\infty (1 + v'^\gamma) R_{n-1}(x, v', t) \, \mathrm{d}v' &\leq (CT)^{n-1} \int_0^\infty \frac{v'^\gamma + 1}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v'^{p-\gamma}} \, \mathrm{d}v' \\
&\leq \frac{C^n T^{n-1}}{1 + |x|^{m-\frac{2\gamma+1}{\alpha}}} \leq C^n T^{n-1}.
\end{aligned} \tag{2.99}$$

Plugging (2.99) into (2.98), we deduce that

$$\mathbb{K}_2[R_{n-1}, f_n] \leq \frac{C^n T^{n-1}(v^\gamma + 1)}{1 + |x|^m + v^p}.$$

Analyzing separately the cases when $v^\alpha \leq |x|$ and $v^\alpha \geq |x|$ and since $p = \alpha m$, we have

$$\mathbb{K}_2[R_{n-1}, f_n] \leq \frac{C^n T^{n-1}(1 + v^\gamma)}{1 + |x|^m + v^p} \leq \frac{C^n T^{n-1}}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}.$$

Similarly, we can bound the term $\mathbb{K}_1[R_{n-1}, f_n]$. More precisely, since $v' \in (0, \frac{v}{2})$, we make use of Proposition 2.12 and (2.89) and, as before, it holds that

$$\begin{aligned}
\mathbb{K}_1[R_{n-1}, f_n] &= \int_0^{\frac{v}{2}} K(v - v', v') R_{n-1}(x, v', t) f_n(x, v - v', t) \, \mathrm{d}v' \\
&\leq \frac{C v^\gamma}{1 + |x|^m + v^p} \int_0^\infty R_{n-1}(x, v', t) \, \mathrm{d}v' \\
&\leq \frac{C^n T^{n-1}(v^\gamma + 1)}{1 + |x|^m + v^p}.
\end{aligned}$$

This concludes the proof of (2.94).

Step 4 *(Proof of* (2.93)*).* Using the definition of f_0 in (2.9), we obtain

$$\partial_t R_1(x, v, t) + v^\alpha \partial_x R_1(x, v, t) = \mathbb{K}[f_0, R_1] + \mathbb{K}[f_0, f_0].$$

Following the steps of the proof of (2.94), we obtain

$$\mathbb{K}[f_0, f_0] \leq \frac{C(1 + v^\gamma)}{1 + |x|^m + v^p} \leq \frac{C}{1 + |x|^{m-\frac{\gamma}{\alpha}} + v^{p-\gamma}}$$

and the conclusion follows by Duhamel's principle as before.

Step 5 *(Conclusion).* We combine Remark 2.20 with (2.93) and (2.94) and choose the time T in (2.94) to be sufficiently small, such that the right-hand side of (2.94) tends to zero as $n \to \infty$. ■

We are now able to conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. We first prove that if $t \leq T$ and T is sufficiently small, it holds that

$$f_0(x, v, t) \leq \frac{2C_0}{1 + |x|^m + v^p}. \tag{2.100}$$

From (2.9) and (2.10) it follows that $f_0(x, v, t) \leq \frac{C_0}{1+|x-v^\alpha t|^m + v^p}$. We first consider $x > 0$. If $v^\alpha t \leq x(1 - \frac{1}{\sqrt[m]{2}})$ then $x - v^\alpha t \geq \frac{x}{\sqrt[m]{2}}$ and thus

$$\frac{1}{1 + |x - v^\alpha t|^m + v^p} \leq \frac{1}{1 + \frac{x^m}{2} + v^p} \leq \frac{2}{1 + x^m + v^p}.$$

If $v^\alpha t \geq x(1 - \frac{1}{\sqrt[m]{2}})$ then $t \neq 0$ and since $t \leq 1$ is sufficiently small we have that $v^p \geq 2x^m$ and thus

$$\frac{1}{1 + |x - v^\alpha t|^m + v^p} \leq \frac{1}{1 + v^p} = \frac{1}{1 + \frac{v^p}{2} + \frac{v^p}{2}} \leq \frac{1}{1 + x^m + \frac{v^p}{2}} \leq \frac{2}{1 + x^m + v^p}.$$

If $x \leq 0$, then, since m is even, we have $(x - v^\alpha t)^m = (|x| + v^\alpha t)^m \geq x^m$ and the conclusion follows. Then, by (2.100) and (2.65), we obtain

$$\int_{(0,\infty)} v f_0(x, v, t)\, \mathrm{d}v \leq \frac{2K_3 C_0}{1 + |x|^{m - \frac{2}{\alpha}}}. \tag{2.101}$$

On the other hand, by Lemma 2.14, it follows that we can find a constant $K_3 > 0$ such that

$$\int_{(0,\infty)} v G(x, v, t)\, \mathrm{d}v \leq \frac{K_3 C_0 B_t}{1 + |x|^{m-d}}, \tag{2.102}$$

where d was defined in (2.13), B_t was defined in (2.66), and G was defined in Definition 2.11. Moreover, we can choose $t \leq 1$ to be sufficiently small (as in Remark 2.13) such that $B_t \leq 2$ and thus it holds that $\int_{(0,\infty)} v G(x, v, t)\, \mathrm{d}v \leq \frac{2K_3 C_0}{1+|x|^{m-d}}$.

We use induction in order to prove that $f_n \leq G$ for all $n \in \mathbb{N}$. By Proposition 2.12, the induction step holds true. For the induction basis, we need to prove that (2.68) holds true for $n = 0$. This is done with the same estimates as in Proposition 2.12 by using (2.101). Thus, if we take L in (2.14) to be as in Definition 2.11, namely $L = 4K_1 K_2 K_3 C_0$, where K_1 is as in (1.6), K_2 is as in Lemma 2.7, and K_3 is as in (2.102), we can conclude that

$$f_n(x, v, t) \leq G(x, v, t), \tag{2.103}$$

for all $n \in \mathbb{N}$.

From (2.103) and (2.88), it follows that there exists some $C > 0$ such that

$$f_n(x, v, t) \leq \frac{C}{1 + |x|^m + v^p}, \tag{2.104}$$

for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}$, and $v \in (0, \infty)$.

By Proposition 2.19 we have that there exists a limit of the sequence $\{f_n\}_{n \in \mathbb{N}}$. It remains to show that the limit of the sequence $\{f_n\}_{n \in \mathbb{N}}$ satisfies equation (1.8). We recall that since f_n satisfies (2.7), it also satisfies the mild formulation of equation (2.7), namely

$$\begin{aligned}
&f_n(x, v, t) - f_n(x - v^\alpha t, v, 0) S[f_{n-1}](x, v, 0, t) \\
&\quad = \frac{1}{2} \int_0^t \int_{(0,v)} S[f_{n-1}](x, v, s, t) K(v - v', v') f_n(x - (t - s)v^\alpha, v - v', s) \\
&\qquad \times f_{n-1}(x - (t - s)v^\alpha, v', s)\, \mathrm{d}s,
\end{aligned}$$

for all $t \in [0, T]$, $v \in (0, \infty)$, and $x \in \mathbb{R}$, where S is as in (1.9). With the bound on $\{f_n\}$ in (2.104) and the bound on the kernel (1.6) it is completely standard to pass to the limit in the equation. Mass conservation of f_n follows by testing with v in (2.7) and then integrating in v and x. Mass conservation of f then follows by passing to the limit as $n \to \infty$ in $\int_{\mathbb{R}} \int_{(0,\infty)} v f_n(x, v, t)\, \mathrm{d}v\, \mathrm{d}x$. We omit the details here.

Discussion on global-in-time existence of mass-conserving solutions. The strategy presented in this paper cannot easily be adapted to prove existence of mass-conserving solutions for larger times. Indeed, the estimates for the characteristics (Proposition 2.2) of equation (2.14) hold for all times. Our candidate G for a supersolution satisfies $\partial_v G = 0$ and changes its sign at

$$\bar{v}^{\alpha} = \frac{x t^{\frac{1}{m-1}}}{\alpha + t^{\frac{m}{m-1}}}.$$

For $t \gg 1$, this point can be approximated by $\bar{v}^{\alpha} \approx \frac{x}{t}$. In this region it is not clear whether the coagulation operator can be estimated by a transport operator, which was the main idea in the construction of a supersolution for small times. (See also the formal arguments in [13, Section 6.4].) In addition, for large times, $1 + |x - v^{\alpha} t|^m + v^p$ cannot be approximated by $1 + |x|^m + v^p$. This will imply that (2.61) will have a different upper bound and that the transport term in the v variable in (2.14) will change. ■

A. Estimates for the second-order derivative of G_L

We will prove in this appendix that (2.43) holds true.

Proposition A.1. *Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0, T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that*

$$\partial_v^2 G_L(x, v, t) < 0 \tag{A.1}$$

if

$$\frac{1}{K_{\max}} x t^{\frac{1}{m-1}} \leq v^{\alpha} \leq K_{\max} x t^{\frac{1}{m-1}}, \tag{A.2}$$

where $K_{\max}$ is as in (2.42).

Proof. Let $Q(x, v, t) = 1 + |X|^m + V^p$. Since $G_L = \frac{1}{Q}$, we have

$$\partial_v Q(x, v, t) = m|X|^{m-2} X \partial_v X + p V^{p-1} \partial_v V, \tag{A.3}$$

$$\begin{aligned} \partial_v^2 Q(x, v, t) &= m(m-1)|X|^{m-2} |\partial_v X|^2 + p(p-1) V^{p-2} (\partial_v V)^2 \\ &\quad + m|X|^{m-2} X \partial_v^2 X + p V^{p-1} \partial_v^2 V, \end{aligned} \tag{A.4}$$

and

$$\partial_v^2 G_L = 2\frac{|\partial_v Q|^2}{Q^3} - \frac{\partial_v^2 Q}{Q^2} = \frac{2|\partial_v Q|^2 - Q \partial_v^2 Q}{Q^3}. \tag{A.5}$$

It is worthwhile noticing that if v is as in (A.2), then $x \geq (1 + 2\delta) v^{\alpha} t$ if t is sufficiently small and thus (2.19) holds. We analyze first the term $2|\partial_v Q|^2$ in (A.5). Since $v^{\alpha} \leq K_{\max} x t^{\frac{1}{m-1}}$, we have that $0 \leq x - (1+\delta) v^{\alpha} t \leq X \leq x - (1-\delta) v^{\alpha} t \leq x$ and $x > 0$. Using Proposition 2.2 and (A.2) we deduce that

$$|\partial_v Q| \leq C_1 x^{m-1} v^{\alpha - 1} t + C_2 v^{p-1} \leq \frac{C x^m}{v} t^{\frac{m}{m-1}},$$

for some constants $C_1, C_2, C > 0$, and thus

$$2|\partial_v Q(x,v,t)|^2 \le \frac{Cx^{2m}}{v^2} t^{\frac{2m}{m-1}}. \tag{A.6}$$

We then analyze the terms in (A.4). We have $|X|^{m-2}|\partial_v X|^2 \ge 0$ and from Proposition 2.2 we know

$$V^{p-2}(\partial_v V)^2 \ge \frac{1}{C} v^{p-2}. \tag{A.7}$$

We are going to show that if t is sufficiently small and v as in (A.2) we have

$$\partial_v^2 X(x,v,t) \ge 0, \tag{A.8}$$

and that for any $\varepsilon > 0$ and v as in (A.2) we have

$$|V^{p-1}\partial_v^2 V| \le \varepsilon v^{p-2} \tag{A.9}$$

if R is sufficiently large and t is sufficiently small. Then, combining (A.7)–(A.9) with (A.2) and noticing that $Q \ge |X|^m \ge (x-(1+\delta)v^\alpha t)^m \ge \frac{x^m}{2^m}$ if t is sufficiently small, it follows that

$$Q\partial_v^2 Q \ge \frac{x^m}{C} v^{p-2} \ge \frac{x^{2m}}{Cv^2} t^{\frac{m}{m-1}}. \tag{A.10}$$

Now, (A.5), (A.6), and (A.10) imply that we can find a sufficiently small $T \in (0,1)$ such that (A.1) holds for all $t \le T$.

In all the following computations, we will assume for simplicity that $v \ge 2R$, with R as in Proposition 2.2. The remaining cases can be proved using similar computations.

Auxiliary result. Given $L > 0$ and $\delta \in (0, \frac{1}{2})$ there exists a sufficiently large $R > 0$ such that for all $t \in [0,T]$, with T sufficiently small, which is independent of L, δ, and R, it holds that

$$|\partial_v^2 X(x,v,t)| \le Cv^{\alpha-2} t \tag{A.11}$$

if v is as in (A.2).

In order to show (A.11) we first prove that there exists $C > 0$ such that

$$|\partial_v^2 \Phi(z,v)| \le Cv^{-1}, \quad \text{for } v \ge 2R, \tag{A.12}$$

where Φ is as (2.28). We differentiate two times with respect to v in (2.30) in order to deduce that

$$\begin{aligned}\partial_v^2 \Phi(z,v) &= (\gamma-\alpha)\big(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z\big)^{\frac{2(\gamma-\alpha)-1}{1-(\gamma-\alpha)}} v^{-2(\gamma-\alpha)} \\ &\quad - (\gamma-\alpha)\big(v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z\big)^{\frac{\gamma-\alpha}{1-(\gamma-\alpha)}} v^{-(\gamma-\alpha)-1}.\end{aligned}$$

Since $0 \le z \le C$ and we are in the case when $v \ge 2R$, we can choose $R > 0$ sufficiently large such that $v^{1-(\gamma-\alpha)} \le v^{1-(\gamma-\alpha)} + (1-(\gamma-\alpha))z \le 2v^{1-(\gamma-\alpha)}$ and thus $|\partial_v^2\Phi(z,v)| \le Cv^{-1}$.

In order to prove (A.11) we differentiate (2.34) twice with respect to v and obtain

$$\frac{\partial_v^2 X(x,v,t)}{\Phi\big(\psi(X(x,v,t))-\psi(x),v\big)^{\alpha}}-\alpha\frac{\partial_v X(x,v,t)[\partial_v\Phi+\partial_z\Phi\psi'(X)\partial_v X]}{\Phi\big(\psi(X(x,v,t))-\psi(x),v\big)^{\alpha+1}}$$
$$=\alpha\int_x^X\frac{\partial_v^2\Phi\,\mathrm{d}\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}+\alpha(\alpha+1)\int_X^x\frac{(\partial_v\Phi)^2\,\mathrm{d}\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+2}}$$
$$+\frac{\alpha\partial_v\Phi}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}}\partial_v X. \tag{A.13}$$

From (A.12), (2.17), and (2.16) we deduce that

$$\left|\int_x^X\frac{\partial_v^2\Phi\,\mathrm{d}\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+1}}\right|\le Cv^{-\alpha-2}[x-X]\le Cv^{-2}t. \tag{A.14}$$

Furthermore, Proposition 2.2 implies that

$$\frac{|\partial_v\Phi\partial_v X|}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}}\le Cv^{-2}t, \tag{A.15}$$

$$\int_X^x\frac{(\partial_v\Phi)^2\,\mathrm{d}\xi}{\Phi(\psi(\xi)-\psi(x),v)^{\alpha+2}}\le Cv^{-\alpha-2}[x-X]\le Cv^{-2}t \tag{A.16}$$

and that $\frac{1}{C}v^{\alpha}\le\Phi(\psi(X)-\psi(x),v)^{\alpha}\le Cv^{\alpha}$.

We claim that if v satisfies (A.2) it holds that

$$0\le\frac{\partial_z\Phi\psi'(X)|\partial_v X|^2}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}}\le Cv^{-2}t. \tag{A.17}$$

Indeed, Proposition 2.2 implies that $|\partial_v X|\le Cv^{\alpha-1}t$ and $\frac{1}{\Phi(\psi(X)-\psi(x),v)^{\alpha+1}}\le Cv^{-\alpha-1}$.

We use (A.2) again to deduce that we are in the region where $x-(1+2\delta)v^{\alpha}t\ge 0$ for sufficiently small t. From the estimate (2.36) we obtain

$$0\le\partial_z\Phi\psi'(X)|\partial_v X|\le C.$$

Thus, (A.17) follows.

Combining the estimates (A.14)–(A.17) and then making use of (A.13), we obtain (A.11).

Proof of (A.9). We only look at the case when $\gamma>1$. The case $\gamma\in[0,1]$ can be proved in a similar manner. As before, we assume in the following for simplicity that the constant L in (2.15) is $L=1$. If $\gamma\in(1,1+\alpha)$, then by integrating in (2.15) it follows that

$$V(x,v,t)^{1-\gamma}-v^{1-\gamma}=(\gamma-1)\int_0^t\frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}}$$

such that

$$V(x,v,t)=\frac{v}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t\frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{1}{\gamma-1}}}.$$

Differentiating in v, we obtain

$$\begin{aligned}\partial_v V(x,v,t) = &\frac{1}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{1}{\gamma-1}}}\\ &-\frac{(\gamma-1)vv^{\gamma-2}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{\gamma}{\gamma-1}}}\\ &+\frac{vv^{\gamma-1}\int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}}{\left(1+(\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}\right)^{\frac{\gamma}{\gamma-1}}} =: \frac{T_1}{T_2^{\frac{\gamma}{\gamma-1}}},\end{aligned}$$

where

$$T_1 := 1 + v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}$$

and

$$T_2 := 1 + (\gamma-1)v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}}.$$

Differentiating in v, we obtain

$$\partial_v^2 V(x,v,t) = \frac{\partial_v T_1}{T_2^{\frac{\gamma}{\gamma-1}}} - \frac{\gamma}{\gamma-1}\frac{T_1\partial_v T_2}{T_2^{\frac{2\gamma-1}{\gamma-1}}}.$$

It holds that

$$\begin{aligned}\partial_v T_1 = &\gamma v^{\gamma-1}\int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}\\ &-2v^\gamma \int_0^t \frac{\left|(m-d)|X|^{m-d-2}X\partial_v X\right|^2\mathrm{d}\xi}{(1+|X|^{m-d})^3}\\ &+v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v^2 X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}\\ &+v^\gamma \int_0^t \frac{(m-d)(m-d-1)|X|^{m-d-2}|\partial_v X|^2\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}.\end{aligned}$$

Moreover, we have

$$\partial_v T_2 = (\gamma-1)^2 v^{\gamma-2}\int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}} - (\gamma-1)v^{\gamma-1}\int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}.$$

We first prove the following estimates:

$$0 \le v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}} \le \frac{1}{2} \tag{A.18}$$

and

$$v^{\gamma}\left|\int_0^t \frac{(m-d)|X(x,v,\xi)|^{m-d-2}X(x,v,\xi)\partial_v X(x,v,\xi)\,\mathrm{d}\xi}{(1+|X(x,v,\xi)|^{m-d})^2}\right| \le \frac{1}{2}. \tag{A.19}$$

We first prove that (A.18) holds. For sufficiently small t, we are in the case when $x \ge (1+2\delta)v^{\alpha}t$ since v is as in (A.2). We use (2.17) and the fact that $x \ge (1+2\delta)v^{\alpha}t \ge (1+2\delta)v^{\alpha}s$ for all $s \in [0,t]$. Thus, $|X(s)| \ge |x-(1+\delta)v^{\alpha}s|$ when $x \ge (1+2\delta)v^{\alpha}t$, for all $s \in [0,t]$. It follows that

$$\begin{aligned} v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|X(x,v,\xi)|^{m-d}} &\le C v^{\gamma-1}\int_0^t \frac{\mathrm{d}\xi}{1+|x-(1+\delta)v^{\alpha}\xi|^{m-d}} \\ &\le C v^{\gamma-\alpha-1}\int_0^{v^{\alpha}t} \frac{\mathrm{d}z}{1+|x-(1+\delta)z|^{m-d}} \\ &\le C v^{\gamma-\alpha-1}, \end{aligned} \tag{A.20}$$

which, since $v \ge R$ and $\gamma < \alpha+1$, implies (A.18) if R is sufficiently large.

We now prove (A.19). As before, we consider the case when $x \ge (1+2\delta)v^{\alpha}t$ since we can choose t sufficiently small. It follows that

$$v^{\gamma}\left|\int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}\right| \le C v^{\gamma-1}\int_0^t \frac{|x-(1-\delta)v^{\alpha}\xi|^{m-d-1}v^{\alpha}\xi\,\mathrm{d}\xi}{(1+|x-(1+\delta)v^{\alpha}\xi|^{m-d})^2}.$$

If v is as in (A.2), then $x-(1+\delta)z \ge x-2z \ge \frac{x}{2}$ for sufficiently small t, for $z \in [0, v^{\alpha}t]$. By making the change of variables $z = v^{\alpha}\xi$ and using the fact that $x-(1-\delta)z \le x$ we further obtain

$$\begin{aligned} v^{\gamma-1}\int_0^t \frac{|x-(1-\delta)v^{\alpha}\xi|^{m-d-1}v^{\alpha}\xi\,\mathrm{d}\xi}{(1+|x-(1+\delta)v^{\alpha}\xi|^{m-d})^2} &\le C v^{\gamma-\alpha-1}\int_0^{v^{\alpha}t} \frac{|x-(1-\delta)z|^{m-d-1}z\,\mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2} \\ &\le C v^{\gamma-\alpha-1}\int_0^{\frac{x}{1+2\delta}} \frac{|x|^{m-d}\,\mathrm{d}z}{(1+|x|^{m-d})^2} \\ &\le \frac{C v^{\gamma-\alpha-1}|x|^{m-d+1}}{(1+|x|^{m-d})^2} \le C v^{\gamma-\alpha-1}, \end{aligned}$$

and thus (A.19) follows if we take R sufficiently large since $v \ge R$ and $\gamma < \alpha+1$.

From (A.18) and (A.19), it holds that $T_1 \in [\frac{1}{2}, 1]$ and $T_2 \in [\frac{1}{2}, \frac{3}{2}]$. From this we deduce that

$$|\partial_v^2 V(x,v,t)| \le C(|\partial_v T_1| + |\partial_v T_2|).$$

We analyze each term of $\partial_v T_1$ separately. Following the computations for (A.19), we deduce that

$$v^{\gamma-1}\left|\int_0^t \frac{(m-d)|X|^{m-d-2}X\partial_v X\,\mathrm{d}\xi}{(1+|X|^{m-d})^2}\right| \le C v^{\gamma-\alpha-1}v^{-1}. \tag{A.21}$$

Similarly, by making the change of variables $z = v^\alpha \xi$, it follows that

$$v^\gamma \int_0^t \frac{\left||X|^{m-d-2} X \partial_v X\right|^2 \mathrm{d}\xi}{(1+|X|^{m-d})^3} \le C v^\gamma \int_0^t \frac{(|x-(1-\delta)v^\alpha \xi|^{m-d-1} v^{\alpha-1}\xi)^2 \mathrm{d}\xi}{(1+|x-(1+\delta)v^\alpha\xi|^{m-d})^3}$$
$$\le C v^{\gamma-\alpha-2} \int_0^{v^\alpha t} \frac{(|x-(1-\delta)z|^{m-d-1} z)^2 \mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^3}$$
$$\le C v^{\gamma-\alpha-1} v^{-1}. \tag{A.22}$$

Moreover, (A.11) implies that

$$v^\gamma \left| \int_0^t \frac{(m-d)|X|^{m-d-2} X \partial_v^2 X \,\mathrm{d}\xi}{(1+|X|^{m-d})^2} \right| \le C v^{\alpha-2} v^\gamma \int_0^t \frac{(m-d)|X|^{m-d-1} \xi \,\mathrm{d}\xi}{(1+|X|^{m-d})^2}$$
$$\le C v^{-\alpha-2} v^\gamma \int_0^{v^\alpha t} \frac{|x-(1-\delta)z|^{m-d-1} z \,\mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2}$$
$$\le C v^{\gamma-\alpha-1} v^{-1}. \tag{A.23}$$

We now analyze the last term in $\partial_v T_1$. We have

$$v^\gamma \int_0^t \frac{|X|^{m-d-2} |\partial_v X|^2 \,\mathrm{d}\xi}{(1+|X|^{m-d})^2} \le C v^{\gamma-\alpha-2} \int_0^{v^\alpha t} \frac{|x-(1-\delta)z|^{m-d-2} z^2 \,\mathrm{d}z}{(1+|x-(1+\delta)z|^{m-d})^2}$$
$$\le C v^{\gamma-\alpha-1} v^{-1}. \tag{A.24}$$

We continue by finding upper bounds for each term in $\partial_v T_2$. From (A.20), it holds that

$$v^{\gamma-2} \int_0^t \frac{\mathrm{d}\xi}{1+|X|^{m-d}} \le C v^{\gamma-\alpha-1} v^{-1}, \tag{A.25}$$

which offers the right estimate in order to prove (A.9). Notice that the last term in $\partial_v T_2$, which is

$$v^{\gamma-1} \int_0^t \frac{|X|^{m-d-2} X \partial_v X \,\mathrm{d}\xi}{(1+|X|^{m-d})^2},$$

was analyzed in (A.21).

Combining (A.21)–(A.25) with the fact that $v \ge R$ and that $\gamma < \alpha + 1$, we obtain

$$|v^{p-1} \partial_v^2 V(x, v, t)| \le C v^{\gamma-\alpha-2} v^{p-1} \le \varepsilon v^{p-2},$$

if $v \ge R$ is sufficiently large. This concludes the proof of (A.9).

Proof of (A.8). From (2.15) it follows that

$$\partial_v^2 X(x, v, t) = \alpha(1-\alpha) \int_0^t V^{\alpha-2} (\partial_v V)^2 \,\mathrm{d}s - \alpha \int_0^t V^{\alpha-1} \partial_v^2 V \,\mathrm{d}s.$$

Now (A.9) implies that the second term on the right-hand side can be absorbed into the first one and thus (A.8) follows. ∎

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