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Renormalized Bogoliubov theory for the Nelson model

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Abstract. We consider the time evolution of the renormalized Nelson model, which describes N bosons linearly coupled to a quantized scalar field, in the mean-field limit of many particles $N \gg 1$ with coupling constant proportional to $N^{-1/2}$. First, we show that initial states exhibiting Bose–Einstein condensation for the particles and approximating a coherent state for the quantum field retain their structure under the many-body time evolution. Concretely, the dynamics of the reduced densities are approximated by solutions of two coupled PDEs, the Schrödinger–Klein–Gordon equations. Second, we construct a renormalized Bogoliubov evolution that describes the quantum fluctuations around the Schrödinger–Klein–Gordon equations. This evolution is used to extend the approximation of the evolved many-body state to the full norm topology. In summary, we provide a comprehensive analysis of the Nelson model that reveals the role of renormalization in the mean-field Bogoliubov theory.

1. Introduction and main results

We study the effective behavior of a large number of bosonic particles in weak interaction with a quantized scalar field. Microscopically, such a system is described by the Nelson Hamiltonian. This was first introduced in 1964 in the mathematical physics literature by Nelson [82], and provides an example of rigorous renormalization in quantum field theory: the formal Hamiltonian needs to be corrected by the divergent self-energy of the particles to obtain a self-adjoint operator and associated unitary dynamics. Renormalization plays a crucial role not only in (mathematical) physics, but it also led, perhaps unexpectedly, to groundbreaking advances in pure and applied mathematics, from stochastic and non-linear partial differential equations to dynamical systems and geometry (see [13, 17, 55, 71] for some celebrated examples). A deeper understanding of renormalization is thus of great relevance for both mathematics and physics. In this paper, we clarify the role played by renormalization in the mean-field Bogoliubov theory for the Nelson model.

What came to be known as Bogoliubov theory was introduced in the 1940s as a heuristic approach to the analysis of excitations in the condensed Bose gas [10]. After the successful creation of Bose–Einstein condensates in laboratory experiments during the

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1990s, these ideas have regained significant interest from the mathematical physics community. This led to the development of a larger research endeavor aimed at providing a rigorous justification of Bogoliubov's approach, starting from the many-body Schrödinger theory. For the low-energy excitation spectrum of large bosonic systems, Bogoliubov theory was first justified by Seiringer [90] and Grech and Seiringer [46]. Regarding the time evolution of excitations in many-particle systems, pioneering results were obtained by Ginibre and Velo [41, 42] and Grillakis, Machedon, and Margetis [51, 52]. Over the past decade, there has been substantial progress in developing refined methods for the derivation of Bogoliubov's theory and in extending the analysis to cover more singular interactions, e.g., [7, 14, 68, 69], and we refer to Section 1.3 for a detailed overview. In the present paper, we continue this effort by establishing Bogoliubov's approximation for the time evolution in the Nelson model. What sets this problem qualitatively apart from previous works is the need for renormalization of the underlying many-body Hamiltonian at finite particle number and the corresponding Bogoliubov evolution. Our results shed new light on the behavior of such systems and we hope that they pave the way for further investigation of the interplay between many-body effects and singular particle-field interactions. It is worth mentioning that such systems are of continued relevance in physics. To cite a recent example from condensed matter theory, the interplay of many-body effects and singular particle-field interactions is crucial in quantum fluids of light [21, 36, 39]. There, a renormalized Bogoliubov theory of the condensate-phonon interaction is necessary to explain the properties of concrete polariton-exciton condensates [39].

Now in more detail, we investigate the dynamical evolution of the Nelson model in a mean-field limit in which the number of particles, denoted by N, becomes large while the coupling to the scalar field is proportional to $1/\sqrt{N}$. Our first result concerns the dynamics of the one-particle and one-field-mode reduced density matrices. We assume that, at the initial time, the particles exhibit Bose–Einstein condensation and the quantum field is approximately in a coherent state. As a starting point, we prove that the time evolution of the reduced density matrices of such initial states can be described by a condensate wave function and a classical scalar field that solve a system of two coupled PDEs, the Schrödinger–Klein–Gordon equations, with errors that tend to zero as $N \to \infty$. The renormalization on the microscopic level, interestingly, does not appear in the mean-field equations, even though it affects the initial microscopic states for which one has convergence. This was observed earlier by Ammari and one of the authors in [2], who proved a similar statement using semiclassical techniques and without quantitative bounds. In this article we employ different techniques that yield an explicit rate of convergence for initial states satisfying an energy condition.

Our second result constitutes the main novelty of the paper, and concerns the Bogoliubov dynamics for the renormalized Nelson model, modeling the quantum fluctuations around the Schrödinger–Klein–Gordon mean-field dynamics. Remarkably, such a Bogoliubov theory requires a renormalization similar to the one of the microscopic model itself. However, the underlying dependence on the mean-field dynamics makes its construction more challenging from both a conceptual and a technical point of view. The renormalized

Bogoliubov dynamics is then used to construct an approximation in norm of the time-evolved many-body state.

1.1. The Nelson Hamiltonian

We consider the massive Nelson model in the mean-field regime. It describes a system of N non-relativistic bosonic particles that are linearly coupled to a scalar quantum field, whose states are elements of the Hilbert space

$$\mathcal{H}_N = \bigotimes_{\mathrm{sym}}^{N} L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3)$ is the bosonic Fock space over $L^2(\mathbb{R}^3)$ with vacuum state Ω. The state of the system evolves according to the Schrödinger equation

$$i\partial_t \Psi_N(t) = H_N \Psi_N(t). \tag{1.1}$$

Formally, the Nelson Hamiltonian H_N is given by the expression

$$\sum_{j=1}^{N}\Big[-\Delta_j + N^{-1/2}\int_{\mathbb{R}^3} \mathrm{d}k\, \omega^{-1/2}(k)(e^{-ikx_j}a_k^* + e^{ikx_j}a_k)\Big] + \mathrm{d}\Gamma_a(\omega), \tag{1.2}$$

where $x_1, \dots, x_N$ denote the variables of the particles, $\omega(k) = \sqrt{k^2+1}$, and $\mathrm{d}\Gamma_a(\omega) = \int_{\mathbb{R}^3} \mathrm{d}k\, \omega(k) a_k^* a_k$ is the second quantization of the multiplication operator ω, describing the energy of the quantum field. The annihilation and creation operators are defined as the operator-valued distributions

$$\begin{aligned}(a_k\Psi_N)^{(n)}(X_N, K_n) &:= \sqrt{n+1}\,\Psi_N^{(n+1)}(X_N, k, K_n),\\ (a_k^*\Psi_N)^{(n)}(X_N, K_n) &:= n^{-\frac{1}{2}} \sum_{j=1}^{n} \delta(k-k_j)\Psi_N^{(n-1)}(X_N, K_n \setminus k_j),\end{aligned}$$

with $\Psi_N^{(n)} \in \bigotimes_{\mathrm{sym}}^{N} L^2(\mathbb{R}^3) \otimes \bigotimes_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3)$ and $X_N = (x_1, \dots, x_N)$, $K_n = (k_1, \dots, k_n)$. They satisfy the canonical commutation relations

$$[a_k, a_l^*] = \delta(k-l), \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

This definition of H_N is only formal, since no domain has been specified. The quadratic form associated to the expression is ill defined on the form domain of the non-interacting Hamiltonian, and while it may be defined on more regular states, this makes it unbounded from below and not closable. However, this problem can be remedied by renormalization [82]: Denote by H_N^{Λ} the version of (1.2) with $\omega^{-1/2}$ replaced by $\omega^{-1/2}\mathbb{1}_{|k|\le\Lambda}$ in the interaction and

$$E^{\Lambda} := \int_{|k|\le\Lambda} \frac{\mathrm{d}k}{\omega(k)(k^2+\omega(k))}. \tag{1.3}$$

Then there exists a self-adjoint operator H_N, $D(H_N)$ so that

$$e^{-itH_N} := \operatorname*{s-lim}_{\Lambda\to\infty} e^{-itH_N^\Lambda} e^{-itE^\Lambda}. \tag{1.4}$$

We take this as the definition of the Nelson Hamiltonian H_N. While the correct renormalization constant E^Λ could, in principle, depend on N, the coupling constant $N^{-1/2}$ in H_N^Λ ensures it can be chosen independently of N. The operator H_N can be characterized further by applying a dressing transformation, see [48, 82] and Lemma 3.1, or by an alternative approach related to boundary conditions [60]. It is important to note the effect of the renormalization on the domain of H_N. While the operators H_N^Λ all share the domain of the free Hamiltonian, it holds that $D(H_N^{1/2}) \cap H^1(\mathbb{R}^{3N}) \otimes \mathcal{F} = \{0\}$, i.e., even the form domain of H_N is completely different from that of the free Hamiltonian [48, 60].

An important role in our analysis is played by a unitary transformation, usually called the dressing transformation, that relates the renormalized Nelson Hamiltonian to an operator whose quadratic form is more explicit and, importantly, comparable to that of the free Hamiltonian. Although our main results below can be stated without any reference to this transformation, it is crucial in their proofs.

1.2. Main results

In this section we state our results on the approximation for the Nelson time evolution, first on the level of reduced densities by the mean-field equations, and then in norm by a renormalized Bogoliubov evolution.

Mean-field approximation. We are interested in the evolution of many-body states in which the particles form a Bose–Einstein condensate and the field is approximately in a coherent state. To be more precise, let us define the unitary Weyl operator

$$W(f) := \exp\biggl(\int_{\mathbb{R}^3} \mathrm{d}k \, \bigl(f(k)a_k^* - \overline{f(k)}a_k\bigr)\biggr). \tag{1.5}$$

The initial states we have in mind are of the form

$$\Psi_N \approx u^{\otimes N} \otimes W(\sqrt{N}\alpha)\Omega \tag{1.6}$$

with Ω being the vacuum in $\mathcal{F}$ and $u, \alpha \in L^2(\mathbb{R}^3)$. We will show that this product-like structure is preserved during the time evolution and that

$$e^{-itH_N}\Psi_N \approx u_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega, \tag{1.7}$$

where $(u_t, \alpha_t) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ solve the Schrödinger–Klein–Gordon (SKG) equations

$$\begin{cases} i\partial_t u_t(x) = \Bigl(-\Delta + \phi_{\alpha_t}(x) - \frac{1}{2}\langle u_t, \phi_{\alpha_t} u_t\rangle_{L^2}\Bigr)u_t(x), \\ i\partial_t \alpha_t(k) = \omega(k)\alpha_t(k) + \langle u_t, G_{(\cdot)}(k)u_t\rangle_{L^2}, \end{cases} \tag{1.8}$$

where

$$G_x(k) := \frac{1}{\sqrt{\omega(k)}} e^{-ikx} \quad \text{and} \quad \phi_\alpha(x) := 2\,\mathrm{Re}\langle G_x, \alpha\rangle_{L^2(\mathbb{R}^3)}. \tag{1.9}$$

These equations are the Hamiltonian equations of the energy (with the usual symplectic form (5.6))

$$\mathcal{E}(u,\alpha) := \langle u, (-\Delta + \phi_\alpha)u\rangle_{L^2(\mathbb{R}^3)} + \langle \alpha, \omega\alpha\rangle_{L^2(\mathbb{R}^3)}. \tag{1.10}$$

We denote the flow of solutions to the SKG equations by $\mathfrak{s}[t](u,\alpha) = (u_t, \alpha_t)$, that is, (u_t, α_t) solves (1.8) with initial conditions $(u_t, \alpha_t)|_{t=0} = (u, \alpha)$. In our main results, we will use that $\mathfrak{s}$ is well defined on $H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$, where H^3 denotes the L^2-Sobolev space and $\mathfrak{h}_{5/2}$ is the weighted space with norm $\|\alpha\|_{\mathfrak{h}_{5/2}} = \|\omega^{5/2}\alpha\|_{L^2(\mathbb{R}^3)}$. Moreover, the SKG flow conserves the energy $\mathcal{E}$ and the L^2-norm of u_t. These statements follow from the more general well-posedness result summarized in Proposition 2.3.

Our first result states that the one-particle reduced density matrices of a state $e^{-itH_N}\Psi_N$ are close to those of the product state (1.7) if this holds at the initial time and if the energy expectation per particle of Ψ_N is close to the initial mean-field energy. A convenient measure for the convergence of reduced densities is given by the functional

$$\beta[\Psi_N, (u,\alpha)] := \langle \Psi_N, (q_u)_1 \Psi_N\rangle_{\mathcal{H}_N} + N^{-1} \| \mathcal{N}_a^{1/2} W^*(\sqrt{N}\alpha)\Psi_N\|^2_{\mathcal{H}_N}, \tag{1.11}$$

where $(q_u)_1$ denotes the orthogonal projection $q_u = \mathbb{1} - |u\rangle\langle u|$ on $L^2(\mathbb{R}^3)$ acting on the first particle's variable x_1, and $\mathcal{N}_a = \int_{\mathbb{R}^3} \mathrm{d}k\, a_k^* a_k$ is the number operator on $\mathcal{F}$. The functional β counts the number of particles in states orthogonal to u and the number of field modes outside of the coherent state $W(\sqrt{N}\alpha)\Omega$, both relative to the total number N of particles. In particular, $\beta[\Psi_N, (u,\alpha)] = 0$ for states of the product form (1.6).

Theorem 1.1. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2(\mathbb{R}^3)} = 1$ and let $(u_t, \alpha_t) := \mathfrak{s}[t](u,\alpha)$ denote the solution of (1.8) with initial data (u,α). Then there exists a constant $C > 0$ such that for all $N \geq 1$, $t \in \mathbb{R}$ and $\Psi_N \in D(H_N^{1/2})$ with $\|\Psi_N\|_{\mathcal{H}_N} = 1$,*

$$\begin{aligned} &\beta\big[e^{-itH_N}\Psi_N, \mathfrak{s}[t](u,\alpha)\big] \\ &\quad \leq e^{CR(t)}\big(|N^{-1}\langle \Psi_N, H_N\Psi_N\rangle_{\mathcal{H}_N} - \mathcal{E}(u,\alpha)| + \max_{j=1,2}(\beta[\Psi_N,(u,\alpha)] + N^{-1})^{1/j}\big) \end{aligned}$$

with $R(t) := 1 + \int_0^{|t|} (\|u_s\|^{10}_{H^3(\mathbb{R}^3)} + \|\alpha_s\|^{10}_{\mathfrak{h}_{5/2}})\,\mathrm{d}s$, and $\mathcal{E}$ defined by (1.10).

The proof of the theorem is given in Section 3.4. We note that $R(t)$ does not grow faster than polynomially in time by Proposition 2.3. Initial states of interest are of course those for which the right-hand side is small. It is important to mention that this is not the case if Ψ_N is exactly of the product form (1.6). Indeed, due to the singular nature of the Nelson Hamiltonian, such states are not in the form domain of H_N; see [48,60]. Next, we provide an example of initial states in the form domain of the Nelson Hamiltonian that are close to product states. To this end, we modify the large momenta by means of a dressing transformation.

Proposition 1.2. *For* $(u,\alpha)\in L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, $K\geq 0$, *let* $B_{K,x}=(k^2+\omega)^{-1}G_{K,x}\mathbb{1}_{|k|\geq K}$ *and define*

$$\Psi_{N,K} := W^*\Big(N^{-1/2}\sum_{j=1}^N B_{K,x_j}\Big)(u^{\otimes N}\otimes W(\sqrt{N}\alpha)\Omega).$$

There exists a constant $C>0$ *such that for all* $(u,\alpha)\in H^1(\mathbb{R}^3)\oplus\mathfrak{h}_1$ *with* $\|u\|_{L^2(\mathbb{R}^3)}=1$, $K>0$, *and* $N\geq 1$

$$\beta[\Psi_{N,K},(u,\alpha)]\leq CK^{-1}(1+\|\alpha\|_{L^2(\mathbb{R}^3)}),$$
$$|N^{-1}\langle\Psi_{N,K},H_N\Psi_{N,K}\rangle-\mathcal{E}(u,\alpha)|\leq C\Big(K^{-1}+\frac{1+\ln K}{N}\Big)(\|u\|^2_{H^1(\mathbb{R}^3)}+\|\alpha\|^2_{\mathfrak{h}_1}).$$

The proof is given in Appendix A. Note that the transformation $W^*(N^{-\frac12}\sum_{j=1}^N B_{K,x_j})$ converges strongly to the identity as $K\to\infty$, and thus $\Psi_{N,K}$ for large K is close to a product state in the norm topology also. For initial states $\Psi_{K,N}$ with $K=N$, Theorem 1.1 simplifies to the following form.

Corollary 1.3. *Let* $(u,\alpha)\in H^3(\mathbb{R}^3)\oplus\mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2(\mathbb{R}^3)}=1$ *and let* $\Psi_{N,N}$ *be the state defined in Proposition 1.2 for* $K=N$. *Then there exist constants* $C>0$, $\delta>0$ *so that*

$$\beta\big[e^{-itH_N}\Psi_{N,N},\mathfrak{s}[t](u,\alpha)\big]\leq e^{C(1+|t|^\delta)}N^{-1/2}.$$

Proof. The corollary is a direct consequence of Theorem 1.1, Proposition 1.2, and the bound on the solutions to the SKG equations of Proposition 2.3. ∎

Remark 1.1 (Convergence of reduced densities). We briefly explain how Theorem 1.1 relates to the approximation of reduced densities. To this end, recall the definition of the reduced one-particle density matrix for the bosons,

$$\gamma^{(1,0)}_{\Psi_N}=N\,\mathrm{Tr}_{2,\dots,N}\otimes\mathrm{Tr}_{\mathcal{F}}(|\Psi_N\rangle\langle\Psi_N|),$$

where $\mathrm{Tr}_{2,\dots,N}$ is the partial trace with respect to $(x_2,\dots,x_N)$, and the definition of the reduced density of the field, given in terms of its integral kernel

$$\gamma^{(0,1)}_{\Psi_N}(k,l)=\langle\Psi_N,a_l^*a_k\Psi_N\rangle_{\mathcal{H}_N}.$$

Their distance, measured in trace norm, to the density operators obtained from the solutions of the SKG equations can be controlled by β from (1.11) via the inequalities [65, Lem. VII.2]

$$\mathrm{Tr}\big|\gamma^{(1,0)}_{\Psi_N}-N|u\rangle\langle u|\big|\leq N\sqrt{8\langle\Psi_N,(q_u)_1\Psi_N\rangle_{\mathcal{H}_N}},$$
$$\mathrm{Tr}\big|\gamma^{(0,1)}_{\Psi_N}-N|\alpha\rangle\langle\alpha|\big|\leq 3\|\mathcal{N}_a^{1/2}W^*(\sqrt{N}\alpha)\Psi_N\|^2_{\mathcal{H}_N}$$
$$\qquad+6\|\alpha\|_{L^2(\mathbb{R}^3)}\|\mathcal{N}_a^{1/2}W^*(\sqrt{N}\alpha)\Psi_N\|_{\mathcal{H}_N}.$$

For $\Psi_N(t) = e^{-itH_N}\Psi_N$ and (u_t, α_t), we consequently get a bound on the difference of reduced densities in terms of the right-hand side of the bound in Theorem 1.1. That is, for suitable initial states (such as those of Corollary 1.3), the average boson behaves like u_t and there are on average N field modes behaving like α_t.

Bogoliubov approximation. We now define the renormalized Nelson–Bogoliubov evolution and explain its role in approximating the fluctuations in the Nelson dynamics. We gather the quantum fluctuations around the condensate with wave function $u \in L^2(\mathbb{R}^3)$ and the coherent state associated with the field $\sqrt{N}\alpha \in L^2(\mathbb{R}^3)$ in an element $\chi \in \mathscr{F} \otimes \mathscr{F}$ that is orthogonal to u in every variable $x_1, \dots, x_N$. That is,

$$\chi \in \bigoplus_{k=0}^{\infty} \bigotimes_{\text{sym}}^{k} \{u\}^{\perp} \otimes \mathscr{F} =: \mathscr{F}_{\perp u} \otimes \mathscr{F}. \tag{1.13}$$

For any $\Psi_N \in \mathscr{H}_N$ one obtains such a $\chi = X_{u,\alpha}\Psi_N$ using a variant of the excitation map introduced in [69] (see Section 2.1 for the precise definition). To describe the inverse of this map, let $\chi^{(k)}$ denote the component of χ in the kth summand of (1.13). If $\chi^{(k)} = 0$ for $k > N$, we can reconstruct Ψ_N as

$$\Psi_N = W(\sqrt{N}\alpha) \sum_{k=0}^{N} u^{\otimes N-k} \otimes_{\mathrm{s}} \chi^{(k)} =: X_{u,\alpha}^{*}\chi, \tag{1.14}$$

where the symmetric tensor product has to be understood as the tensor product of the orthogonal subspaces of $L^2(\mathbb{R}^3)$, $\operatorname{span}(u)$, and $\{u\}^{\perp}$, so that each summand yields an element of $\bigotimes_{\text{sym}}^{N} L^2(\mathbb{R}^3) \otimes \mathscr{F}$, on which the Weyl operator $W(\sqrt{N}\alpha)$ acts in the second tensor factor.[1] Note that the product state (1.6) would correspond to $\chi = \Omega \otimes \Omega$.

Now let $\Psi_N(t)$ and (u_t, α_t) be solutions of (1.1) and (1.8) with suitable initial conditions Ψ_N and (u, α), and consider the fluctuation vector $\chi(t)$ satisfying $\Psi_N(t) = X_{u_t,\alpha_t}^{*}\chi(t)$. This vector is an element of $\mathscr{F}_{\perp u_t} \otimes \mathscr{F}$, which we can naturally identify with a subspace of the double Fock space

$$\mathscr{F} \otimes \mathscr{F} \cong \bigoplus_{n=0}^{\infty} \bigotimes_{\text{sym}}^{n} (L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)).$$

We will denote the creation and annihilation operators on the first factor in $\mathscr{F} \otimes \mathscr{F}$, associated with excitations of the bosons, by b^* and b.

The dynamics of the fluctuations $\chi(t)$ will be approximated by a time-dependent Bogoliubov transformation. Roughly speaking, Bogoliubov transformations on $\mathscr{F} \otimes \mathscr{F}$ are unitary maps that are determined (up to a phase) by a map on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. This

[1] More precisely, we set $u^{\otimes N-k} \otimes_{\mathrm{s}} \chi^{(k)} = P_{\text{sym}}^{N}(u^{\otimes N-k} \otimes \chi^{(k)})$, where the tensor product is taken with respect to the spaces $\operatorname{span}(u)^{\otimes N-k}$ and $(\{u\}^{\perp})^{\otimes k} \otimes \mathscr{F}$ and where P_{sym}^{N} is the orthogonal projection onto the symmetric subspace of $\bigotimes^{N} L^2(\mathbb{R}^3)$, while it acts as the identity on $\mathscr{F}$.

property makes Bogoliubov transformations much simpler in terms of complexity. To be more precise, define the joint creation operator of the excitations and the field by

$$c^*(f \oplus g) = b^*(f) + a^*(g),$$

and the annihilation operator as its adjoint. A Bogoliubov transformation on $\mathcal{F} \otimes \mathcal{F}$ is a unitary map $\mathbb{U}$ with the property that

$$\mathbb{U}^* c^*(f \oplus g)\mathbb{U} = c^*(\mathfrak{u}(f \oplus g)) + c(\mathfrak{v}(\overline{f \oplus g})) \tag{1.15}$$

for some bounded linear maps $\mathfrak{u}, \mathfrak{v}$: $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. In other words, conjugation of a^*, b^* by $\mathbb{U}$ maps these to linear combinations of a, a^*, b, b^* with modified arguments. For a more detailed introduction of Bogoliubov transformations and related concepts, we refer to [91] and [12, Sect. 4].

The generators of Bogoliubov transformations are (formally) Hamiltonians quadratic in the creation and annihilation operators, a, a^*, b, b^*. For the Nelson model with ultraviolet cutoff, such a quadratic generator can be obtained from the full Hamiltonian [35] following the approximation ideas of Bogoliubov [10]. The Nelson–Bogoliubov Hamiltonian with cutoff $\Lambda \in (0, \infty)$ is defined by

$$\begin{aligned}\mathbb{H}^\Lambda_{u,\alpha}(t) := &\int_{\mathbb{R}^3} \mathrm{d}x\, b^*_x h_{\alpha_t} b_x + \int_{\mathbb{R}^3} \mathrm{d}k\, \omega(k) a^*_k a_k \\ &+ \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}k \left((q_{u_t} G^\Lambda_{(\cdot)}(k) u_t)(x) a^*_k b^*_x + (q_{u_t} \overline{G^\Lambda_{(\cdot)}(k)} u_t)(x) a_k b^*_x \right) + \text{h.c.},\end{aligned} \tag{1.16}$$

where $G^\Lambda_x(k) = G_x(k)\mathbb{1}_{|k|\le\Lambda}$ and $h_{\alpha_t} = -\Delta + \phi_{\alpha_t} - \frac{1}{2}\langle u_t, \phi_{\alpha_t} u_t\rangle$ with $G_x(k)$ and ϕ_{α_t} given by (1.9). Here, $(u_t, \alpha_t) := \mathfrak{s}[t](u, \alpha)$, and the notation h.c. denotes the Hermitian conjugate of the preceding term. Let us emphasize that $\mathbb{H}^\Lambda_{u,\alpha}(t)$ is always defined with respect to the mean-field flow (1.8) with the subscript (u, α) referring to the initial conditions. Further, let $\mathbb{U}^\Lambda_{u,\alpha}(t)$ be the unique unitary propagator (with initial time $t = 0$) on the double Fock space $\mathcal{F} \otimes \mathcal{F}$ associated with $\mathbb{H}^\Lambda_{u,\alpha}(t)$. For a discussion of its existence, we refer to [35, Thm. 4.1] or Proposition 5.2 below with $\theta = 0$. Our next result states the existence of a renormalized Nelson–Bogoliubov time evolution in the limit $\Lambda \to \infty$.

Theorem 1.4. *Let $(u, \alpha) \in H^3 \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2(\mathbb{R}^3)} = 1$ and let $(u_t, \alpha_t) := \mathfrak{s}[t](u, \alpha)$ denote the solution of (1.8). Let E^Λ be given by (1.3) and $V \in L^2(\mathbb{R}^3, \mathbb{R})$ by (3.2). Then*

$$\mathbb{U}_{u,\alpha}(t) := \operatorname*{s-lim}_{\Lambda\to\infty} \mathbb{U}^\Lambda_{u,\alpha}(t) e^{-iE^\Lambda t - \frac{i}{2}\int_0^t \mathrm{d}s\, \langle u_s, V * |u_s|^2 u_s\rangle}$$

exists for all $t \in \mathbb{R}$. Moreover, $\mathbb{U}_{u,\alpha}(t)$ has the following properties:

(i) *$\mathbb{U}_{u,\alpha}(t)$ is unitary and strongly continuous in t,*

(ii) *$\mathbb{U}_{u,\alpha}(t)(\mathcal{F}_{\perp u} \otimes \mathcal{F}) \subseteq \mathcal{F}_{\perp u_t} \otimes \mathcal{F}$,*

(iii) *$\mathbb{U}_{u,\alpha}(t)$ is a Bogoliubov transformation on $\mathcal{F} \otimes \mathcal{F}$.*

Our second main result is a norm approximation of the dynamics generated by the Nelson Hamiltonian. It states that the fluctuations around the condensate wave function $u_t \in L^2(\mathbb{R}^3)$ and the coherent state associated with the field mode $\sqrt{N}\alpha_t \in L^2(\mathbb{R}^3)$ are effectively described by the renormalized Nelson–Bogoliubov evolution introduced in Theorem 1.4. Together with the fact that $\mathbb{U}_{u,\alpha}(t)$ is a Bogoliubov transformation, this implies an approximation of the Nelson evolution in terms of a transformation of the form (1.15).

Theorem 1.5. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2(\mathbb{R}^3)} = 1$ and let $(u_t,\alpha_t) := \mathfrak{s}[t](u,\alpha)$ denote the solution of* (1.8)*. There exists a quadratic form $\delta \geq 1$ whose domain is dense in $\mathcal{F}_{\perp u} \otimes \mathcal{F}$ and a constant $C > 0$ so that for all $t \in \mathbb{R}$, $N \geq 1$, and $\chi \in \mathcal{F}_{\perp u} \otimes \mathcal{F}$ in the domain of δ, with $\|\chi\|_{\mathcal{F}\otimes\mathcal{F}} = 1$, we have for $\Psi_N = X^*_{u,\alpha}\chi$ given by* (1.14),

$$\left\| e^{-itH_N}\Psi_N - W(\sqrt{N}\alpha_t)\sum_{k=0}^{N} u_t^{\otimes N-k} \otimes_{\mathrm{s}} (\mathbb{U}_{u,\alpha}(t)\chi)^{(k)} \right\|_{\mathcal{H}_N} \leq e^{CR(t)}\delta(\chi)^{1/2}\frac{\sqrt{\ln N}}{N^{1/4}},$$

where $R(t) := 1 + \int_0^{|t|}(\|u_s\|^{10}_{H^3(\mathbb{R}^3)} + \|\alpha_s\|^{10}_{\mathfrak{h}_{5/2}})\,\mathrm{d}s$.

This theorem is proved in Section 4.3.

Remark 1.2. The quadratic form δ is constructed explicitly using a unitary Bogoliubov transformation $\mathbb{W}$ that implements a dressing on the level of the fluctuation vector χ. With this transformation, which is introduced in Proposition 4.3 as $\mathbb{W}^{\infty}_{u,\alpha}(1)$, δ is given by

$$\delta(\chi) := \|\chi\|^2_{\mathcal{F}\otimes\mathcal{F}} + \langle \mathbb{W}\chi, (\mathcal{N}^3 + \mathrm{d}\Gamma_b(-\Delta) + \mathrm{d}\Gamma_a(\omega))\mathbb{W}\chi\rangle_{\mathcal{F}\otimes\mathcal{F}},$$

i.e., it measures the expectation of the third moment of the number of excitations and the energy of the excitations and the field after dressing with $\mathbb{W}$. In Proposition 4.3, we also show that $\mathbb{W}$ preserves the domain of $\mathcal{N}^{3/2}$. This is relevant, as it implies that the norm of the initial state Ψ_N in Theorem 1.5 approaches 1 as $N \to \infty$. More precisely, it implies that $\|X^*_{u,\alpha}\chi\|_{\mathcal{H}_N} \geq \|\chi\|_{\mathcal{F}\otimes\mathcal{F}} - CN^{-3}\delta(\chi)$ for some $C > 0$. On the other hand, let us note that we do not expect that $\mathbb{W}$ preserves the domain of $\mathrm{d}\Gamma_b(-\Delta)$.

Remark 1.3. By the density of the domain of δ, Theorem 1.5 extends naturally to all initial states with a finite number of excitations. Specifically, for any Ψ_N such that $X_{u,\alpha}\Psi_N$ has a well-defined limit as $N \to \infty$, we have

$$\lim_{N\to\infty} \|e^{-itH_N}\Psi_N - X^*_{u_t,\alpha_t}\mathbb{U}_{u,\alpha}(t)\chi\|_{\mathcal{H}_N} = 0.$$

1.3. Comparison with the literature

The broader subject of this work, the justification of the time-dependent mean-field, and Bogoliubov approximations, has been addressed extensively in the literature, mainly in the context of the Bose gas with two-body pair interaction. The situation of particles coupled to a quantum field has been explored to a much lesser extent. Below we give a brief overview of the literature on this topic and other works related to this study.

The first works on the mean-field approximation of reduced densities for the many-body Bose gas with two-body interaction date back to the 1970s and 1980s by Hepp, Ginibre, Velo, and Spohn [41, 42, 56, 92]. The question was revived in the early 2000s [4,33] and within the next years, new techniques were developed to obtain explicit rates of convergence [22,86,89] and to cover more singular two-body potentials, in particular those converging to a Dirac-delta potential [5,31,32,58,87]. Since then, this topic has continued to be actively studied and we recommend [6, 45, 81] for comprehensive surveys of recent works. Fluctuations around the time-dependent mean-field equations were considered first in [41,42,49,51,52]. Since [68,69], this subject has gained increased interest which has led to further extensions and refinements in the derivation of the Bogoliubov approximation; see e.g., [9,14,18,23,29,50,59,73–78,85]. Higher-order corrections to Bogoliubov theory have been obtained in [11, 12, 43, 44]. Let us note that Bogoliubov theory also plays a crucial role in the description of the excitation spectrum of large bosonic systems. While this has been extensively studied for bosons with two-body potentials [7, 8, 16, 30, 46, 54, 69, 79, 80, 88, 90], we are not aware of any results concerning the spectral properties of many bosons coupled to a quantum field.

The derivation of the SKG equations starting from the renormalized Nelson model, in the same limit as considered in this work, has been addressed previously by [2]. Using techniques from semiclassical theory, the authors demonstrate that the Wigner measure associated with the many-body dynamics evolves in the limit $N \to \infty$ in accordance with the push-forward of a Wigner measure under the SKG flow. Since convergence of the Wigner measure implies weak-$*$ convergence for the reduced densities, this statement is comparable to Theorem 1.1 of the present work. Unlike the approach taken in [2], which provides a limit result without explicit error estimates, our method allows us to determine an explicit rate of convergence for the reduced densities. On the other hand, the results in [2] apply to a wider class of initial states. Regarding our second result, the construction of the renormalized Nelson–Bogoliubov Hamiltonian and the norm approximation, we are not aware of any prior work that has addressed this problem.

More results have been obtained for models with regular particle-field interactions (i.e., without need for renormalization): For the regularized Nelson model with ultraviolet cutoff, derivations of the corresponding mean-field dynamics were obtained in [1, 34, 65] and the validity of the Bogoliubov approximation, as well as higher-order corrections, was established in [35]. In addition, the regularized Nelson model was also studied in a many-fermion limit that is closely linked to a semiclassical limit [64]. Other particle-field systems, such as the Fröhlich model and the Pauli–Fierz Hamiltonian, have been studied in the scaling regime of the present article too; see [63, 66] for the mean-field approximation and [61] for an approximation of the Fröhlich dynamics in norm. The dressed Nelson Hamiltonian, which will play a crucial role in our analysis (see Section 1.4), has similar regularity properties to the Fröhlich Hamiltonian, as both are given in terms of perturbations of the non-interacting quadratic form. However, the dressed Nelson Hamiltonian has a more complicated structure than the Fröhlich Hamiltonian since it is not linear in creation and annihilation operators. This makes the analysis of the time evolution more involved even on the level of the dressed Hamiltonian.

Within the broader scope of deriving effective equations from particle-field models, it is worth noting the following works. The subject of [20, 25, 26, 40] is a partially classical limit of a class of models (covering the regularized Nelson, and the Fröhlich, and the Pauli–Fierz models), where a fixed number of particles is weakly coupled to a quantum scalar field with high occupation number. For the Fröhlich Hamiltonian specifically, the time evolution has been actively studied in the strong coupling regime also [37, 38, 47, 62, 67, 72]. While the resulting effective equations are of a similar form to the SKG equations, the strong coupling limit is accompanied by a separation of timescales between the particle and the field, a feature that is absent in the mean-field limit. The papers [19, 27, 57, 93] focus on the derivation of effective pair particle potentials arising from the particle-field interaction, in suitable weak-coupling and adiabatic limits.

Finally, for an overview of results on the renormalized Nelson model not directly linked to the derivation of effective equations, we refer to the discussion of [70].

1.4. Outline of the proofs

General idea. The Hamiltonian expressed formally in (1.2) can be represented in terms of an operator H_N^{D} with a more regular and explicitly given quadratic form, conjugated with a unitary dressing transformation W^{D}; see Lemma 3.1. By unitarity of W^{D}, this allows us to relate the Nelson dynamics to the dynamics generated by the dressed Hamiltonian via

$$e^{-itH_N} = (W^{\mathrm{D}})^* e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}. \tag{1.17}$$

The general strategy of the proof is to analyze the mean-field and norm approximations of the dressed time evolution $e^{-itH_N^{\mathrm{D}}}$, and then connect the corresponding dressed mean-field and Bogoliubov evolutions to the original (undressed) ones. To accomplish this, we will introduce approximations of the dressing transformation that relate the dressed and undressed effective evolutions, in analogy to the relation shown in (1.17). Denoting the approximations of the dressing transformation by $\mathfrak{D}$ and $\mathbb{W}$, and the dressed mean-field flow and the dressed Bogoliubov evolution as $\mathfrak{s}^{\mathrm{D}}[t]$ and $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$, then the connection between the effective evolutions can be expressed as

$$\mathfrak{s}[t] = \mathfrak{D}^{-1} \circ \mathfrak{s}^{\mathrm{D}}[t] \circ \mathfrak{D} \quad \text{and} \quad \mathbb{U}_{u,\alpha}(t) = \mathbb{W}^* \mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t) \mathbb{W}. \tag{1.18}$$

To determine $\mathfrak{D}$ and $\mathbb{W}$, we view $W^{\mathrm{D}} = W^{\mathrm{D}}(1)$ as the special case of a quantum evolution operator $W^{\mathrm{D}}(\theta)$ with "time" θ and examine its mean-field and Bogoliubov approximations. The motivation for this stems from the observation that $W^{\mathrm{D}}(\theta)$ is a unitary group that is generated by a field operator resembling the interaction term in (1.2), but with a square-integrable form factor (this was exploited in [2] for the mean-field flow).

In the proof of Theorem 1.4, we establish an identity that is similar to (1.18) but for $\mathbb{U}^{\Lambda}_{u,\alpha}(t)e^{-itE_\Lambda}$ and with Λ-dependent versions of $\mathbb{W}$ and $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$, and then show that the cutoff can be removed for the conjugated dressed evolution.

Mean-field approximation. In order to derive Theorem 1.1, we estimate the difference of the dressed dynamics $e^{-itH_N^{\mathrm{D}}}$ applied to the dressed initial state $W^{\mathrm{D}}\Psi_N$ to the corresponding mean-field equations introduced in (5.2), whose flow is denoted by $\mathfrak{s}^{\mathrm{D}}[t]$, in Theorem 3.2. The proof relies on the use of the excitation map and estimates on the generator of the fluctuation dynamics. To pass from the approximation of $e^{-itH_N^{\mathrm{D}}}$ to that of e^{-itH_N}, we then expand on the idea that the dressing W^{D} itself can be approximated by the mean-field dressing transformation $\mathfrak{D}$. This is the subject of Lemmas 3.5 and 3.6, with the latter providing the reason for the required energy condition in Theorem 1.1. Since $\mathfrak{D}$ intertwines between the dressed and the undressed mean-field evolutions, i.e., $\mathfrak{s}[t] = \mathfrak{D}^{-1} \circ \mathfrak{s}^{\mathrm{D}}[t] \circ \mathfrak{D}$, this allows us to translate the approximation result of the dressed dynamics to the desired result on the undressed ones.

Bogoliubov approximation. To prove the norm approximation, we start again from the analysis of the dressed dynamics, where we now examine the fluctuation vector $\chi^{\mathrm{D}}(t) = X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)} e^{-itH_N^{\mathrm{D}}} X_{u,\alpha}^* \chi$ associated with the dressed mean-field flow and compare it with the effective evolution $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)\chi$. Here, $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$ is obtained from the Bogoliubov approximation of the dressed dynamics. The statement analogous to Theorem 1.5 for the dressed dynamics is given in Theorem 4.2, whose proof is based on estimates of the difference of the generator of the dressed fluctuations and the quadratic generator of $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$. To establish a connection of this result to the one for the undressed dynamics, we use (1.17) together with a norm approximation for the dressing transformation W^{D}. To this end, we elevate the mean-field approximation $\mathfrak{D}$ to the level of the fluctuations by implementing a Bogoliubov transformation $\mathbb{W}$. For the definition of $\mathbb{W}$, we need to introduce a non-autonomous flow of Bogoliubov transformations on $\mathcal{F} \otimes \mathcal{F}$, which is the content of Proposition 4.3. Lemma 4.5 then demonstrates that $\mathbb{W}$ indeed offers a norm approximation of W^{D}, in the sense that $X_{\mathfrak{D}(u,\alpha)} W^{\mathrm{D}} X_{u,\alpha}^* \chi \approx \mathbb{W}\chi$ as $N \to \infty$ for suitable $\chi \in \mathcal{F} \otimes \mathcal{F}$. As a final step, we argue that $\mathbb{W}$ intertwines between the dressed and the undressed Bogoliubov evolutions, i.e., that $\mathbb{U}_{u,\alpha}(t) = \mathbb{W}^* \mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)\mathbb{W}$, with the precise version of this identity given in (5.11). A schematic representation of the Bogoliubov approximation is illustrated in Figure 1.

Renormalization. For the purpose of the norm approximation, we could take (1.18) as the definition of $\mathbb{U}_{u,\alpha}(t)$ in Theorem 1.5. However, we also want to elucidate the relation of this evolution to the one that can be formally derived by applying the quadratic Bogoliubov approximation to (1.2). This relation is given by Theorem 1.4. To this end, one needs to introduce an ultraviolet cutoff Λ, since it is not clear that the Nelson–Bogoliubov Hamiltonian (1.16) defines a self-adjoint operator for $\Lambda = \infty$. With a cutoff Λ, one might expect that the Bogoliubov approximation of the Nelson dynamics is given exactly by the Bogoliubov approximation of the dressed Hamiltonian, conjugated with the approximation of the dressing. However, the Bogoliubov evolution is fixed only up to a phase, so the identity may only hold for an appropriate choice of such a phase. This is the content of Proposition 5.3, where we show that the correct choice of phase $e^{-i\int_0^t \mathrm{d}s\, E^{\Lambda}(s)}$ is such

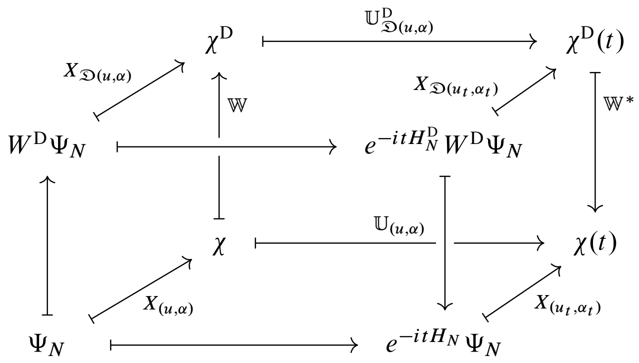

Figure 1. Schematic representation of the approximations. The diagram commutes up to small errors in N. The exact dynamics are represented in the front plane, with the Bogoliubov approximations in the second plane. Note the dependence of the excitation maps on the solutions to the mean-field equations $(u_t, \alpha_t) := \mathfrak{s}[t](u, \alpha)$.

that $E^\Lambda(s) \to \infty$ as $\Lambda \to \infty$. This is analogous to the renormalization of H_N, as stated in (1.4). Due to the dependence of the unitary evolutions in (1.18) on the corresponding mean-field flows, the renormalization of the Bogoliubov evolution is conceptually and technically more challenging compared to the renormalization of H_N.

2. Preliminaries

2.1. Fock spaces and excitation map

We recall the Fock space $\mathcal{F} = \bigoplus_{k=0}^{\infty} \bigotimes_{\mathrm{sym}}^{k} L^2(\mathbb{R}^3)$ and define the (truncated) Fock spaces for the excitations of the particles,

$$\mathcal{F}_{\perp u}^{(k)} = \bigotimes_{\mathrm{sym}}^{k} \{u\}^\perp, \quad \mathcal{F}_{\perp u}^{\leq N} = \bigoplus_{k=0}^{N} \mathcal{F}_{\perp u}^{(k)}, \quad \mathcal{F}_{\perp u} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\perp u}^{(k)},$$

for $\{u\}^\perp = \{\varphi \in L^2(\mathbb{R}^3) : \langle \varphi, u \rangle = 0\}$. The relevant double Fock spaces for the Nelson model are

$$\mathcal{F}_{\perp u}^{\leq N} \otimes \mathcal{F}, \quad \mathcal{F}_{\perp u} \otimes \mathcal{F}, \quad \text{and} \quad \mathcal{F} \otimes \mathcal{F}, \tag{2.1}$$

where the first factor always refers to the excitations of the N bosonic particles, while the second factor describes the excitations of the quantum field. If the context is not unambiguous, we will write $\mathcal{F} = \mathcal{F}_b$ for the particles and $\mathcal{F} = \mathcal{F}_a$ for the quantum field. In the order of (2.1), we refer to the Fock spaces as the truncated excitation Fock space, excitation Fock space, and double Fock space. Moreover, we denote by b_x, b_x^*, $\mathcal{N}_b$ and a_k, a_k^*, $\mathcal{N}_a$ the annihilation, creation, and number operators on $\mathcal{F}_b$ and $\mathcal{F}_a$, respectively.

For $f \in L^2(\mathbb{R}^3)$ let

$$a(f) := \int_{\mathbb{R}^3} \mathrm{d}k\, \overline{f(k)} a_k, \quad a^*(f) := \int_{\mathbb{R}^3} \mathrm{d}k\, f(k) a_k^*,$$
$$b(f) := \int_{\mathbb{R}^3} \mathrm{d}x\, \overline{f(x)} b_x, \quad b^*(f) := \int_{\mathbb{R}^3} \mathrm{d}x\, f(x) b_x^*$$

denote the bosonic annihilation and creation operators. For a self-adjoint operator T: $D(T) \to L^2(\mathbb{R}^3)$, we denote by $\mathrm{d}\Gamma(T)$ the self-adjoint second quantization of T on $\mathcal{F}$. Depending on which factor of $\mathcal{F} \otimes \mathcal{F}$ this acts on, we write

$$\mathrm{d}\Gamma_a(T) := \int_{\mathbb{R}^6} \mathrm{d}k\, \mathrm{d}l\, T(k,l) a_k^* a_l \quad \text{and} \quad \mathrm{d}\Gamma_b(T) := \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}y\, T(x,y) b_x^* b_y,$$

where $T(\cdot,\cdot)$ is the Schwartz kernel of T. With this, we introduce the notation

$$\mathcal{N} := \mathcal{N}_b + \mathcal{N}_a, \quad \mathbb{T} := \mathrm{d}\Gamma_a(-\Delta) + \mathrm{d}\Gamma_a(\omega). \tag{2.2}$$

We define the field operator by

$$\widehat{\Phi}(f) := a(f) + a^*(f). \tag{2.3}$$

Using this definition, the Weyl operator introduced in (1.5) can be expressed as

$$W(f) = e^{-i\widehat{\Phi}(if)}.$$

It satisfies

$$W^{-1}(f) = W(-f), \quad W(f)W(g) = W(f+g)e^{-i\,\mathrm{Im}\langle f,g\rangle}, \tag{2.4}$$

as well as the shift property

$$W^*(f) a_k W(f) = a_k + f(k). \tag{2.5}$$

As an important tool in our analysis, we introduce a variant of the *excitation map* introduced in [68, 69]. Specifically, we define $X_{u,\alpha} := X_u \otimes W(\sqrt{N}\alpha)^*$, where X_u denotes the usual excitation map for N bosons [68, 69] and $W(\sqrt{N}\alpha)$ is a Weyl operator. Let us explain this in more detail: In the Nelson model, the excitation map factors out a condensate with wave function u and a coherent state with field mode $\sqrt{N}\alpha$. For $u, \alpha \in L^2(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$, it is defined as the map

$$X_{u,\alpha}\colon \mathcal{H}_N \to \mathcal{F}_{\perp u} \otimes \mathcal{F}$$

with $\Psi_N \mapsto (\chi^{(k)})_{k=0}^N$ given by

$$\chi^{(k)} = \binom{N}{k}^{1/2} \prod_{i=1}^{k} (q_u)_i \langle u^{\otimes(N-k)}, W^*(\sqrt{N}\alpha)\Psi_N \rangle_{L^2(\mathbb{R}^{3(N-k)})} \quad \in \mathcal{F}_{\perp u}^{(k)} \otimes \mathcal{F}, \tag{2.6}$$

where $(q_u)_i$ is the orthogonal projection $q_u = \mathbb{1} - |u\rangle\langle u|$ acting on the ith particle coordinate x_i. Here, the partial inner product is taken with respect to the particle coordinates $x_{k+1}, \dots, x_N$. The adjoint of $X_{u,\alpha}$ is given by (1.14), and it holds that

$$X_{u,\alpha}^* X_{u,\alpha} = \mathbb{1}_{\mathcal{H}_N}, \quad X_{u,\alpha} X_{u,\alpha}^* = \mathbb{1}_{\mathcal{F}_{\perp u}^{\leq N} \otimes \mathcal{F}}, \tag{2.7}$$

in particular $X_{u,\alpha} \colon \mathcal{H}_N \to \mathcal{F}_{\perp u}^{\leq N} \otimes \mathcal{F}$ is unitary. Written in terms of creation and annihilation operators, the excitation map acts as

$$X_{u,\alpha} \Psi_N = \left(\bigoplus_{k=0}^{N} q_u^{\otimes k} \frac{b(u)^{N-k}}{\sqrt{(N-k)!}} \right) \otimes W^*(\sqrt{N}\alpha)\Psi_N.$$

This leads to the following useful relations [68, 69].

Lemma 2.1. *As identities on $\mathcal{F}_{\perp u} \otimes \mathcal{F}$, we have for all $f, g \in \{u\}^\perp$,*

$$X_{u,\alpha} b^*(u) b(u) X_{u,\alpha}^* = [N - \mathcal{N}_b]_+, \qquad X_{u,\alpha} b^*(u) b(f) X_{u,\alpha}^* = [N - \mathcal{N}_b]_+^{1/2} b(f),$$
$$X_{u,\alpha} b^*(f) b(g) X_{u,\alpha}^* = b^*(f) b(g), \qquad X_{u,\alpha} b^*(f) b(u) X_{u,\alpha}^* = b^*(f) [N - \mathcal{N}_b]_+^{1/2},$$

where $[a]_+ = \max\{a, 0\}$. Moreover, for all $h \in L^2(\mathbb{R}^3)$,

$$X_{u,\alpha} a(h) X_{u,\alpha}^* = a(h) + \sqrt{N}\langle h, \alpha \rangle.$$

For reference, we also state the following identity for the derivative of the excitation map, leaving the proof to the reader (see [68, eq. (40)] for the N-particle Bose case).

Lemma 2.2. *Let $\mathbb{R} \ni s \mapsto (u_s, \alpha_s)$ be differentiable with $(\partial_s u_s, \partial_s \alpha_s) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, and consider the family of unitaries $(X_{u_s,\alpha_s})_{s \in \mathbb{R}}$. Then, as an identity of operators $\mathcal{F}_{\perp u_s}^{\leq N} \otimes \mathcal{F} \to \mathcal{F}^{\leq N} \otimes \mathcal{F}$, we have*

$$\begin{aligned} i(\partial_s X_{u_s,\alpha_s}) X_{u_s,\alpha_s}^* &= \big(-\sqrt{N}\widehat{\Phi}(i\partial_s \alpha_s) - N \operatorname{Re}\langle \alpha_s, i\partial_s \alpha_s \rangle \big) W(\sqrt{N}\alpha_t)^* \\ &\quad + b^*(u_s) b(q_{u_s} i\partial_s u_s) - \langle i\partial_s u_s, u_s \rangle (N - \mathcal{N}_b) \\ &\quad - \big(\sqrt{N - \mathcal{N}_b}\, b(q_{u_s} i\partial_s u_s) + \text{h.c.} \big). \end{aligned}$$

To measure the distance between reduced densities, we introduced the functional β in (1.11). In Section 3, we need a second functional given by

$$\gamma[\Psi_N, (u, \alpha)] := \|\nabla_1 (q_u)_1 \Psi_N\|^2 + N^{-1} \|\mathrm{d}\Gamma_a(\omega)^{1/2} W^*(\sqrt{N}\alpha)\Psi_N\|^2. \tag{2.8}$$

Essentially, γ is the mean kinetic energy of particles outside the condensate state u and field modes outside the coherent state $W(\sqrt{N}\alpha)\Omega$. Using Lemma 2.1, they can be expressed in terms of the excitation map and the operators from (2.2) as

$$\beta[\Psi_N, (u, \alpha)] = \frac{1}{N} \langle X_{u,\alpha}\Psi_N, \mathcal{N} X_{u,\alpha}\Psi_N \rangle_{\mathcal{H}_N}, \tag{2.9a}$$

$$\gamma[\Psi_N, (u, \alpha)] = \frac{1}{N} \langle X_{u,\alpha}\Psi_N, \mathbb{T} X_{u,\alpha}\Psi_N \rangle_{\mathcal{H}_N}. \tag{2.9b}$$

2.2. Notation

We recall the definitions $\omega(k) = \sqrt{k^2+1}$ and

$$G_x(k) = \frac{1}{\sqrt{\omega(k)}} e^{-ikx} \quad \text{and} \quad B_x(k) = \frac{G_x(k)}{k^2+\omega(k)} = \frac{e^{-ikx}}{\sqrt{\omega(k)}(k^2+\omega(k))}. \tag{2.10}$$

Moreover, we adopt the following notational conventions. The space $H^s(\mathbb{R}^3)$, for $s \in \mathbb{R}$, denotes the non-homogeneous L^2-Sobolev space, while $\mathfrak{h}_s$, for $s \in \mathbb{R}$, is the weighted L^2-space with norm $\|\alpha\|_{\mathfrak{h}_s} = \|\omega^s \alpha\|_{L^2}$. The symbol $\|\cdot\|$ represents the norm of either $\mathcal{H}_N$ or $\mathcal{F} \otimes \mathcal{F}$, depending on context. For a quadratic form with domain $Q(A)$ associated with an operator A: $Q(A) \to Q(A)'$, we write $A + \text{h.c.}$ to denote the quadratic form

$$\langle \psi, (A + \text{h.c.})\psi \rangle = 2\,\mathrm{Re}\langle \psi, A\psi \rangle, \quad \psi \in Q(A).$$

Lastly, the letter C denotes a generic constant whose value may change within a sequence of inequalities; for instance, in $X \le CY \le CZ$, the two occurrences of C may denote different numbers.

2.3. The SKG equations

The next statement recaps the well-posedness theory of the non-linear SKG equations,

$$\begin{cases} i\partial_t u_t(x) = \Big(-\Delta + \phi_{\alpha_t}(x) - \frac{1}{2}\langle u_t, \phi_{\alpha_t} u_t\rangle\Big) u_t(x), \\ i\partial_t \alpha_t(k) = \omega(k)\alpha_t(k) + \langle u_t, G_{(\cdot)}(k) u_t \rangle, \\ (u_t, \alpha_t)|_{t=0} = (u, \alpha). \end{cases} \tag{2.11}$$

Proposition 2.3. *For any $s \ge 0$ the Cauchy problem (2.11) is globally well posed in the space $H^s(\mathbb{R}^3) \oplus \mathfrak{h}_{s-1/2}$. The solutions satisfy $\|u_t\|_{L^2} = \|u\|_{L^2}$ and, for $s \ge 1$, $\mathcal{E}(u_t, \alpha_t) = \mathcal{E}(u, \alpha)$ with $\mathcal{E}$ defined by (1.10).*

In addition, for any integer $s \ge 1$, there exists δ such that for any $M > 0$ there exists C so that for all $t \in \mathbb{R}$ and $\|(u,\alpha)\|_{H^s \oplus \mathfrak{h}_{s-1/2}} \le M$, the solution (u_t, α_t) satisfies

$$\|u_t\|_{H^s} + \|\alpha_t\|_{\mathfrak{h}_{s-1/2}} \le \begin{cases} C & \text{if } s = 1, \\ C(1+|t|)^\delta & \text{otherwise.} \end{cases}$$

Proof. The well-posedness, together with the conservation properties is a special case of [84, Thm. 1.4]; see also [3, 24]. The estimate on the norms for $s = 1$ follows from the conservation properties and an application of the bound (6.2e). The polynomial-in-time bounds for $s > 1$ can be proved by an iterative argument, adapting the approach of [37] for the Landau–Pekar equations; see also [53] for a different approach. ∎

3. The mean-field approximation

In this section we study the approximation of e^{-itH_N} on the level of reduced densities and prove Theorem 1.1. We first consider the evolution generated by the Nelson Hamiltonian after dressing it with a suitable unitary transformation. Even though the dressed Hamiltonian H_N^{D} has a more complicated structure than the original Hamiltonian, it contains interaction terms that are less singular and is thus better suited for our analysis. In fact, H_N^{D} is a perturbation of the non-interacting Hamiltonian in the sense of quadratic forms.

In order to approximate the evolution $e^{-itH_N^{\mathrm{D}}}$, it is necessary to replace the mean-field equations by their dressed variant, as already observed in [2]. The approximation result for the reduced densities of $e^{-itH_N^{\mathrm{D}}}W^{\mathrm{D}}\Psi_N$ using the dressed mean-field equations, with a statement analogous to that of Theorem 1.1, is given in Theorem 3.2. We then study the dressing transformations, and state in Lemma 3.5 that they can be approximated by a mean-field dressing flow in a similar way. The proof of Theorem 1.1 is given by combining these results in Section 3.4.

3.1. Dressed dynamics on the microscopic level

With $B_x(k)$ given by (2.10) and the field operator $\widehat{\Phi}$ from (2.3), we consider the family of unitary dressing transformations

$$W^{\mathrm{D}}(\theta) := \exp\Big(\frac{-i\theta}{\sqrt{N}}\sum_{j=1}^{N}\widehat{\Phi}(iB_{x_j})\Big) \tag{3.1}$$

and set $W^{\mathrm{D}} = W^{\mathrm{D}}(1)$. This transformation goes back to Gross and Nelson [82], and the following lemma recalls a well-known relation between the renormalized Nelson Hamiltonian and the dressed Nelson Hamiltonian.

Lemma 3.1. *Consider the symmetric quadratic form defined on the form domain of the free operator* $\mathrm{d}\Gamma_a(\omega) + \sum_{i=1}^{N}(-\Delta_i)$,

$$\begin{aligned} H_N^{\mathrm{D}} :=\ & \mathrm{d}\Gamma_a(\omega) + \sum_{i=1}^{N}(-\Delta_i) + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\hat{A}_{x_i} + \frac{1}{N}\sum_{1\le i<j\le N}V(x_i-x_j) \\ & + \frac{1}{N}\sum_{i=1}^{N}\big(a(kB_{x_i})^2 + 2a^*(kB_{x_i})a(kB_{x_i}) + a^*(kB_{x_i})^2\big), \end{aligned}$$

where

$$\begin{aligned} \hat{A}_x &:= -2(i\nabla_x\cdot a(kB_x) + a^*(kB_x)\cdot i\nabla_x), \\ V(x) &:= -4\,\mathrm{Re}\langle G_x, B_0\rangle + 2\,\mathrm{Re}\langle \omega B_x, B_0\rangle. \end{aligned} \tag{3.2}$$

There exists a unique self-adjoint operator H_N^{D}, $D(H_N^{\mathrm{D}})$ *whose quadratic form coincides with the above, and we have*

$$H_N = (W^{\mathrm{D}})^* H_N^{\mathrm{D}} W^{\mathrm{D}}.$$

Proof. Formally, this follows from the definition of the Weyl operators and a direct computation. The precise statement is a corollary to the original construction of the renormalized Nelson Hamiltonian [82], refined in [48]. There, one considers the operator $H^{\mathrm{D}}_{N,K}$ related to a dressing transformation with an infrared cutoff K (as in Proposition 1.2). This is used to bound the interaction terms relative to the form of the non-interacting operator with bound less than 1, for K sufficiently large (see [48, Thm. 3.3]). Transforming $H^{\mathrm{D}}_{N,K}$ with the dressing transformation on momenta below K gives the formula above. This does not change the form domain by [48, Thm. 4.1, Lem. C.4]. ∎

3.2. Mean-field approximation of the dressed dynamics

Given the dressed Nelson Hamiltonian H^{D}_N, one can derive an associated mean-field energy by projecting onto states of the product-like form (1.6) with given one-particle functions u, α. The dressed mean-field equations, the Hamiltonian equations associated with this energy, take the form

$$\begin{cases} i\partial_t u_t(x) = h_{u_t,\alpha_t} u_t(x), \\ i\partial_t \alpha_t(k) = \omega(k)\alpha_t + 2\langle u_t, kB_{(\cdot)}(k)(-i\nabla + F_{\alpha_t})u_t\rangle, \\ (u_0,\alpha_0) = (u,\alpha), \end{cases} \tag{3.3}$$

where

$$h_{u,\alpha} := -\Delta + A_\alpha + (F_\alpha)^2 + V * |u|^2 - \mu_{u,\alpha}, \tag{3.4a}$$

$$A_{\alpha,x} := 2(-i\nabla_x)\langle kB_x,\alpha\rangle + 2\overline{\langle kB_x,\alpha\rangle}(-i\nabla_x), \tag{3.4b}$$

$$F_\alpha(x) := 2\,\mathrm{Re}\langle kB_x,\alpha\rangle, \tag{3.4c}$$

$$\mu_{u,\alpha} := \frac{1}{2}\langle u, V * |u|^2 u\rangle + \mathrm{Re}\langle \alpha, f_u\rangle + \mathrm{Re}\langle \alpha, g_{u,\alpha}\rangle, \tag{3.4d}$$

$$f_u(k) := 2\langle u, kB_{(\cdot)}(k)(-i\nabla)u\rangle, \tag{3.4e}$$

$$g_{u,\alpha}(k) := 2\langle u, kB_{(\cdot)}(k)F_\alpha u\rangle. \tag{3.4f}$$

We denote the associated flow by $\mathfrak{s}^{\mathrm{D}}[t](u,\alpha) = (u_t,\alpha_t)$, that is, (u_t,α_t) solves (3.3) with initial conditions $(u_t,\alpha_t)|_{t=0} = (u,\alpha)$ (existence of this flow is the special case $\theta = 1$ of Lemma 5.1).

Remark 3.1. The notation (u_t,α_t) is used to denote different flows throughout the paper. For example, in Theorems 1.1, 1.4, and 1.5, we defined $(u_t,\alpha_t) := \mathfrak{s}[t](u,\alpha)$, while in Theorems 3.2 and 4.2, we set $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$. In each case, (u_t,α_t) serves as a local placeholder for a specific mean-field flow.

In the next statement we compare the evolution generated by H^{D}_N with the dressed mean-field flow $\mathfrak{s}^{\mathrm{D}}[t]$. To this end, we recall (1.11) for the definition of the functional β and (2.8) for the definition of the functional γ. Also note that by Lemma 3.1 we have the identity $e^{-itH_N}\Psi_N = (W^{\mathrm{D}})^* e^{-itH^{\mathrm{D}}_N} W^{\mathrm{D}}\Psi_N$, which explains why we now consider initial states of the form $W^{\mathrm{D}}\Psi_N$.

The following statement is the main result of this section.

Theorem 3.2. *Let* $(u,\alpha) \in H^3 \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$ *and let* $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ *denote the solution to* (3.3) *for initial conditions* (u,α). *There exists a constant* $C > 0$ *such that for all* $\Psi_N \in D(H_N^{1/2})$ *with* $\|\Psi_N\| = 1$, $N \geq 1$, *and* $t \in \mathbb{R}$, *we have*

$$\begin{aligned}\beta\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] + \gamma\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] \\ \leq e^{CR^{\mathrm{D}}(t)}\big(\beta[W^{\mathrm{D}}\Psi_N,(u,\alpha)] + \gamma[W^{\mathrm{D}}\Psi_N,(u,\alpha)] + N^{-1}\big),\end{aligned}$$

where $R^{\mathrm{D}}(t) := 1 + \int_0^{|t|} \|u_s\|_{H^3}^2 (1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}})^2 \,\mathrm{d}s$.

To prepare the proof of the theorem, we introduce the fluctuation generator associated with $e^{-itH_N^{\mathrm{D}}}$ and $\mathfrak{s}^{\mathrm{D}}[t]$. Recalling the definition of the excitation map (2.6), and fixing $\Psi_N \in \mathcal{H}_N$, $(u,\alpha) \in H^3 \oplus \mathfrak{h}_{5/2}$, we consider the fluctuation vector

$$\chi^{\mathrm{D}}(t) := X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)} e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}} \Psi_N. \tag{3.5}$$

A simple computation shows that $\chi^{\mathrm{D}}(t)$ satisfies the equation $i\partial_t \chi^{\mathrm{D}}(t) = H_{u,\alpha}^{\mathrm{D},\leq N}(t)\chi^{\mathrm{D}}(t)$ with

$$H_{u,\alpha}^{\mathrm{D},\leq N}(t) = i\dot{X}_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)}(X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)})^* + X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)} H_N^{\mathrm{D}} (X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)})^*. \tag{3.6}$$

Note that $H_{u,\alpha}^{\mathrm{D},\leq N}(t)$ maps $\mathcal{F}_{\perp u_t}^{\leq N} \otimes \mathcal{F}$ into $\mathcal{F}^{\leq N} \otimes \mathcal{F}$, but for convenience, we will write it as the restriction of a symmetric operator $H_{u,\alpha}^{\mathrm{D}}(t)\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$, that is,

$$H_{u,\alpha}^{\mathrm{D},\leq N}(t) = H_{u,\alpha}^{\mathrm{D}}(t) \restriction \mathcal{F}_{\perp u_t}^{\leq N} \otimes \mathcal{F}. \tag{3.7}$$

The explicit expression for $H_{u,\alpha}^{\mathrm{D}}(t)$ is given in Section 6.4. The fluctuation vector $\chi^{\mathrm{D}}(t)$ then satisfies the Schrödinger type equation

$$\begin{cases} i\partial_t \chi^{\mathrm{D}}(t) = H_{u,\alpha}^{\mathrm{D}}(t)\chi^{\mathrm{D}}(t), \\ \chi^{\mathrm{D}}(0) = X_{u,\alpha} W^{\mathrm{D}} \Psi_N. \end{cases} \tag{3.8}$$

In Lemma 6.11 we will prove the following bounds:

$$\pm(H_{u,\alpha}^{\mathrm{D}}(t) - \mathbb{T}) \leq \frac{1}{2}\mathbb{T} + C(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{3.9a}$$

$$\pm i[\mathcal{N}, H_{u,\alpha}^{\mathrm{D}}(t)] \leq \frac{1}{2}\mathbb{T} + C(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{3.9b}$$

$$\pm\frac{\mathrm{d}}{\mathrm{d}t} H_{u,\alpha}^{\mathrm{D}}(t) \leq \frac{1}{2}\mathbb{T} + C\rho(t)(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{3.9c}$$

where $\mathcal{N}$ and $\mathbb{T}$ are defined as in (2.2), $\rho(t) = \|u_t\|_{H^3}^2 (1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$, $(u_t,\alpha_t) = \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$. Equipped with these estimates we can now come to the proof of Theorem 3.2, whose strategy is inspired by [15, 61, 78].

Proof of Theorem 3.2. Consider $(u_t, \alpha_t) = \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ and the fluctuation vector $\chi^{\mathrm{D}}(t)$ given by (3.5) for initial states (u,α) and Ψ_N as stated in the hypothesis. Note that by definition, $\chi^{\mathrm{D}}(t) \in \mathcal{F}_{\perp u_t}^{\leq N} \otimes \mathcal{F}$. Relations (2.9a) and (2.9b) imply that

$$\beta\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] = \frac{1}{N}\langle \chi^{\mathrm{D}}(t), \mathcal{N}\chi^{\mathrm{D}}(t)\rangle, \tag{3.10a}$$

$$\gamma\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] = \frac{1}{N}\langle \chi^{\mathrm{D}}(t), \mathbb{T}\chi^{\mathrm{D}}(t)\rangle. \tag{3.10b}$$

From (3.9a) and $\mathbb{1}_{\mathcal{N}_b \leq N}\chi^{\mathrm{D}}(t) = \chi^{\mathrm{D}}(t)$ it follows that

$$\begin{aligned}
&N\beta\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] + N\gamma\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] \\
&\quad = \langle \chi^{\mathrm{D}}(t), (\mathbb{T} + \mathcal{N})\chi^{\mathrm{D}}(t)\rangle \\
&\quad \leq 2\langle \chi^{\mathrm{D}}(t), H^{\mathrm{D}}_{u,\alpha}(t)\chi^{\mathrm{D}}(t)\rangle + C\langle \chi^{\mathrm{D}}(t), (\mathcal{N}+1)\chi^{\mathrm{D}}(t)\rangle =: f(t).
\end{aligned}$$

We proceed by estimating the time derivative of $f(t)$ in order to conclude via Grönwall's inequality. Using (3.8) and with the aid of (3.9a)–(3.9c), one computes

$$\begin{aligned}
|\dot f(t)| &= \Big|2\Big\langle \chi^{\mathrm{D}}(t), \Big(\frac{\mathrm{d}}{\mathrm{d}t} H^{\mathrm{D}}_{u,\alpha}(t)\Big)\chi^{\mathrm{D}}(t)\Big\rangle + C\langle \chi^{\mathrm{D}}(t), i[H^{\mathrm{D}}_{u,\alpha}(t), \mathcal{N}]\chi^{\mathrm{D}}(t)\rangle\Big| \\
&\leq C\langle \chi^{\mathrm{D}}(t), \mathbb{T}\chi^{\mathrm{D}}(t)\rangle + C\rho(t)\langle \chi^{\mathrm{D}}(t), (\mathcal{N}+1)\chi^{\mathrm{D}}(t)\rangle \\
&\leq C\langle \chi^{\mathrm{D}}(t), H^{\mathrm{D}}_{u,\alpha}(t)\chi^{\mathrm{D}}(t)\rangle + C\rho(t)\langle \chi^{\mathrm{D}}(t), (\mathcal{N}+1)\chi^{\mathrm{D}}(t)\rangle \leq C\rho(t) f(t),
\end{aligned}$$

where we used that $\rho(t) \geq 1$. Grönwall's inequality thus implies $f(t) \leq e^{C\int_0^{|t|}\rho(s)\,\mathrm{d}s} f(0)$ and using (3.9a) again, together with (3.10a), (3.10b), we arrive at

$$\begin{aligned}
\beta\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)\big] &\leq Ce^{C\int_0^{|t|}\rho(s)\,\mathrm{d}s}\frac{1}{N}\langle \chi^{\mathrm{D}}(0), (\mathbb{T} + \mathcal{N} + 1)\chi^{\mathrm{D}}(0)\rangle \\
&= e^{CR^{\mathrm{D}}(t)}\big(\beta[W^{\mathrm{D}}\Psi_N, (u,\alpha)] + \gamma[W^{\mathrm{D}}\Psi_N, (u,\alpha)] + N^{-1}\big).
\end{aligned}$$

This completes the proof of Theorem 3.2. ∎

3.3. Mean-field approximation of the dressing transformation

The dressing transformation W^{D} is generated by an operator that looks like the interaction term of the Nelson Hamiltonian, but has the regular form factor iB_x. There are thus mean-field equations associated to the dynamics $\theta \mapsto W^{\mathrm{D}}(\theta)$, (3.1), as in the case of the (dressed) Nelson Hamiltonian. The mean-field equations corresponding to the dressing transformation are given by

$$\begin{cases}
i\partial_\theta u^\theta(x) = \tau_{u^\theta,\alpha^\theta}(x)u^\theta(x), \\
\partial_\theta \alpha^\theta(k) = B_0(k)\widehat{|u^\theta|^2}(k), \\
(u^\theta, \alpha^\theta)|_{\theta=0} = (u,\alpha),
\end{cases} \tag{3.11}$$

where we introduced

$$\tau_{u,\alpha}(x) := \tilde{\phi}_\alpha(x) - \frac{1}{2}\langle u, \tilde{\phi}_\alpha u\rangle, \quad \tilde{\phi}_\alpha(x) := 2\,\mathrm{Re}\langle iB_x, \alpha\rangle. \tag{3.12}$$

We denote by $\mathfrak{D}[\theta]$ the flow corresponding to this equation, i.e.,

$$\mathfrak{D}[\theta](u,\alpha) = (u^\theta, \alpha^\theta), \tag{3.13}$$

where $(u^\theta, \alpha^\theta)$ is the solution to (3.11) with initial condition (u,α). Being the flow of an autonomous system of equations, we have $\mathfrak{D}[\theta] \circ \mathfrak{D}[-\theta] = 1$. For $\theta = 1$ we use the shorthand $\mathfrak{D} = \mathfrak{D}[1]$.

In fact, $\mathfrak{D}[\theta]$ can be determined explicitly following [2, Lem. III.11]. Since $\tau_{u^\theta,\alpha^\theta}$ is real, the solution satisfies $|u^\theta|^2 = |u|^2$, and then the equation for α can be solved for

$$\alpha^\theta(k) = \alpha(k) + \theta B_0(k)\widehat{|u|^2}(k).$$

Since B_0 is an even function,

$$\mathrm{Re}\langle iB_x, B_0\widehat{|u|^2}\rangle = \mathrm{Im}\int \mathrm{d}y\, \widehat{B_0^2}(y-x)|u|^2(y) = 0.$$

Hence, we have $\tilde{\phi}_{\alpha^\theta} = \tilde{\phi}_\alpha$, and one can simplify the equations using $\tau_{u^\theta,\alpha^\theta} = \tau_{u,\alpha}$. The system of ordinary differential equations (3.11) for each (x,k) is then solved explicitly by

$$(u^\theta, \alpha^\theta) = \mathfrak{D}[\theta](u,\alpha) = (e^{-i\theta\tau_{u,\alpha}}u, \alpha + \theta B_0(\cdot)\widehat{|u|^2}). \tag{3.14}$$

The flow $\mathfrak{D}[\theta]$ preserves the relevant spaces of Cauchy data for the SKG equations (cf. Proposition 2.3).

Lemma 3.3. *Let $n \geq 1$ be an integer and $n-1 < s < n+1$. There exists C so that for all $(u,\alpha) \in H^n(\mathbb{R}^3) \oplus \mathfrak{h}_s$ and $|\theta| \leq 1$ the functions $(u^\theta, \alpha^\theta) = \mathfrak{D}[\theta](u,\alpha)$ satisfy*

$$\|u^\theta\|_{H^n} \leq C\|u\|_{H^n}\|\alpha\|_{\mathfrak{h}_s}^n \quad \textit{and} \quad \|\alpha^\theta\|_{\mathfrak{h}_s} \leq C(\|u\|_{H^n}^2 + \|\alpha\|_{\mathfrak{h}_s}).$$

Proof. We use the explicit form (3.13) of $\mathfrak{D}[\theta]$ and some straightforward estimates (some of which will be proved in Section 6.2). We have $e^{i\theta\tau_{u,\alpha}}\nabla^m u^\theta = (\nabla - i\theta\nabla\tau_{u,\alpha})^m u$ for $m \leq n$. With $\nabla\tau_{u,\alpha}(x) = -2\,\mathrm{Re}\langle kB_x, \alpha\rangle$ we thus have

$$\|\nabla^m u^\theta\|_{L^2} \leq C\sum_{\ell=0}^{m}\sum_{j_1+\cdots+j_\ell\leq m}\left\|\left(\prod_{i=0}^{\ell}\langle|k|^{j_i}B_0, |\alpha|\rangle\right)\nabla^{m-\sum j_i}u\right\|_{L^2} \leq C\|u\|_{H^m}\|\alpha\|_{\mathfrak{h}_s}^m,$$

by Lemma 6.3 (where we used that $s > n-1$). For α^θ we have

$$\|\alpha^\theta\|_{\mathfrak{h}_s} \leq \|\alpha\|_{\mathfrak{h}_s} + |\theta|\,\|B_0(k)\widehat{|u|^2}\|_{\mathfrak{h}_s} \leq \|\alpha\|_{\mathfrak{h}_s} + |\theta|\,\||u|^2\|_{H^{s-5/2}}.$$

For $n=1$ we simply bound the last term by $\|u\|_{L^4}^2 \leq C\|u\|_{H^1}^2$, and for $n \geq 2$ we use that

$$\|u^2\|_{H^{s-5/2}} \leq \|u^2\|_{H^n} \leq C\|u\|_{H^n}^2.$$

This proves the claim. ∎

To connect the statements of Theorems 3.2 and 1.1, we make use of the fact that the dressing flow $\mathfrak{D}$ interpolates between the SKG flow (2.11) and the dressed mean-field flow (3.3). This is a direct consequence of Lemma 5.1 for $\theta = 1$.

Lemma 3.4. *For all $t \in \mathbb{R}$, we have*

$$\mathfrak{s}^{\mathrm{D}}[t] \circ \mathfrak{D} = \mathfrak{D} \circ \mathfrak{s}[t].$$

The next lemma is an analogue to Theorem 3.2 for the dynamics $W^{\mathrm{D}}(\theta)$ in "time" θ.

Lemma 3.5. *There exists a constant $C > 0$ such that for all $(u, \alpha) \in H^1(\mathbb{R}^3) \oplus \mathfrak{h}_0$ with $\|u\|_{L^2} = 1$, $\Psi_N \in \mathcal{H}_N$ with $\|\Psi_N\| = 1$, $N \geq 1$, and $|\theta| \leq 1$, we have*

$$\beta\big[W^{\mathrm{D}}(\theta)\Psi_N, \mathfrak{D}[\theta](u,\alpha)\big] \leq e^{C(\|u\|_{H^1}^2 + \|\alpha\|_{\mathfrak{h}_0})}\big(\beta[\Psi_N, (u,\alpha)] + N^{-1}\big).$$

To prove the lemma, we introduce, in close analogy to the discussion after Theorem 3.2, the fluctuation generator associated with $W^{\mathrm{D}}(\theta)$ and $\mathfrak{D}[\theta]$. For $\Psi_N \in \mathcal{H}_N$ and $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus \mathfrak{h}_0$, we consider the fluctuation vector

$$\zeta(\theta) := X_{\mathfrak{D}[\theta](u,\alpha)} W^{\mathrm{D}}(\theta)\Psi_N. \tag{3.15}$$

A short computation shows that $i\partial_\theta \zeta(\theta) = D_{u,\alpha}^{\leq N}(\theta)\zeta(\theta)$ with

$$D_{u,\alpha}^{\leq N}(\theta) = X_{\mathfrak{D}[\theta](u,\alpha)}\Big(\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\widehat{\Phi}(iB_{x_j})\Big)(X_{\mathfrak{D}[\theta](u,\alpha)})^* + i(\partial_\theta X_{\mathfrak{D}[\theta](u,\alpha)})(X_{\mathfrak{D}[\theta](u,\alpha)})^*,$$

acting as an operator $\mathcal{F}_{\perp u}^{\leq N} \otimes \mathcal{F} \to \mathcal{F}^{\leq N} \otimes \mathcal{F}$. As before, it is convenient to write $D_{u,\alpha}^{\leq N}(\theta)$ as the restriction of a suitable symmetric operator $D_{u,\alpha}(\theta)\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$. To this end, we define

$$\begin{aligned} D_{u,\alpha}(\theta) := \mathrm{d}\Gamma_b(\tau_{u,\alpha}) &+ \Big(\int \mathrm{d}x\,\mathrm{d}k\,\kappa_{u^\theta}(k,x) a_k^* b_x^* \Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+^{1/2} + \mathrm{h.c.}\Big) \\ &- \Big(\int \mathrm{d}x\,\mathrm{d}k\,\kappa_{u^\theta}(-k,x) a_k b_x^* \Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+^{1/2} + \mathrm{h.c.}\Big) \\ &+ N^{-1/2}\int \mathrm{d}x\, b_x^*\big(q_{u^\theta}\widehat{\Phi}(iB_x)q_{u^\theta} - \langle u^\theta, \widehat{\Phi}(iB_{(\cdot)})u^\theta\rangle\big)b_x, \end{aligned} \tag{3.16}$$

where $\tau_{u,\alpha}$ is defined by (3.12) and

$$\kappa_u(k,x) := (q_u iB_{(\cdot)}(k)u)(x).$$

Using (2.5) and Lemmas 2.1 and 2.2 to compute $D_{u,\alpha}^{\leq N}(\theta)$, one verifies the claimed identity, namely that $D_{u,\alpha}^{\leq N}(\theta) = D_{u,\alpha}(\theta) \restriction \mathcal{F}_{\perp u}^{\leq N} \otimes \mathcal{F}$.

Proof of Lemma 3.5. Consider the fluctuation vector $\zeta(\theta)$ given by (3.15). Using (2.9a), we can express the relevant β functional as

$$\beta\big[W^{\mathrm{D}}(\theta)\Psi_N, \mathfrak{D}[\theta](u,\alpha)\big] = \frac{1}{N}\langle \zeta(\theta), \mathcal{N}\zeta(\theta)\rangle. \tag{3.17}$$

We use $i\partial_\theta \zeta(\theta) = D_{u,\alpha}(\theta)\zeta(\theta)$, with $D_{u,\alpha}(\theta)$ given by (3.16), in combination with the commutator bound (which is stated precisely and proved in Lemma 6.13)

$$\pm i[\mathcal{N}, D_{u,\alpha}(\theta)] \le C(\|u\|_{H^1}^2 + \|\alpha\|_{\mathfrak{h}_0})(\mathcal{N}+1)\Big(1+\Big(\frac{1}{N}\mathcal{N}_b\Big)^{1/2}\Big).$$

Together with $\zeta(\theta) = \mathbb{1}_{\mathcal{N}_b\le N}\zeta(\theta)$, this implies

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\theta}\langle\zeta(\theta), \mathcal{N}\zeta(\theta)\rangle\Big| \le C(\|u\|_{H^1}^2 + \|\alpha\|_{\mathfrak{h}_0})\langle\zeta(\theta), (\mathcal{N}+1)\zeta(\theta)\rangle,$$

and by applying Grönwall's inequality, we obtain for $|\theta| \le 1$,

$$\langle\zeta(\theta), (\mathcal{N}+1)\zeta(\theta)\rangle \le e^{C|\theta|(\|u\|_{H^1}^2+\|\alpha\|_{\mathfrak{h}_0})}\langle\zeta(0), (\mathcal{N}+1)\zeta(0)\rangle.$$

In combination with (3.17) and the fact that $\|\zeta(0)\| = 1$, we can derive the desired bound. ∎

As a final preparation for the proof of Theorem 1.1, the following lemma gives an upper bound for the functional γ, defined in (2.8), when evaluated for the dressed states $W^{\mathrm{D}}\Psi_N$ and $\mathfrak{D}(u,\alpha)$ in terms of the energy difference of the microscopic and mean-field models evaluated in the states Ψ_N and (u,α), without dressing.

Lemma 3.6. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$ and $\mathcal{E}$ be given by (1.10). There exists a constant $C > 0$ such that for all $\Psi_N \in D(H_N^{1/2})$ with $\|\Psi_N\| = 1$, $N \ge 1$, we have*

$$\begin{aligned}&\gamma[W^{\mathrm{D}}\Psi_N, \mathfrak{D}(u,\alpha)]\\ &\quad\le C\big(|N^{-1}\langle\Psi_N, H_N\Psi_N\rangle - \mathcal{E}(u,\alpha)| + \max_{j=1,2}(\beta[\Psi_N,(u,\alpha)] + N^{-1})^{j/2}\big).\end{aligned}$$

For the proof of Lemma 3.6, which is given in Section 6.5, it is important to note that the strategy used to prove Theorem 3.2 does not work. The reason for this is that the operator $\mathbb{T}$ is not dominated by the generator $D_{u,\alpha}(\theta)$, and hence we do not have analogous estimates to (3.9a)–(3.9c) at our disposal. Therefore we rely on a different type of energy estimate, motivated by ideas from [58].

3.4. Proof of Theorem 1.1

Combining the results of the previous two sections, we can prove our first main theorem.

Proof of Theorem 1.1. For $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$, denote by

$$(u_t,\alpha_t) := \mathfrak{s}[t](u,\alpha), \quad (u^{\mathrm{D}},\alpha^{\mathrm{D}}) := \mathfrak{D}(u,\alpha), \quad (u_t^{\mathrm{D}},\alpha_t^{\mathrm{D}}) := \mathfrak{s}^{\mathrm{D}}[t]\circ\mathfrak{D}(u,\alpha)$$

the solutions of the SKG equations (2.11), and the dressed SKG equations (3.3) with initial data transformed by the mean-field dressing transformation (3.13). Also recall that by Lemma 3.4,

$$\mathfrak{D}(u_t,\alpha_t) = \mathfrak{s}^{\mathrm{D}}[t](u^{\mathrm{D}},\alpha^{\mathrm{D}}). \tag{3.18}$$

Since $W^{\mathrm{D}}(-1)W^{\mathrm{D}}(1) = \mathbb{1}$ and $\mathfrak{D}[-1] \circ \mathfrak{D} = \mathbb{1}$, we can use Lemma 3.5 for $\theta = -1$ and with $\Psi_N \to W^{\mathrm{D}} e^{-itH_N}\Psi_N$, and $(u,\alpha) \to \mathfrak{D}(u,\alpha)$, we get

$$\beta[e^{-itH_N}\Psi_N, (u_t,\alpha_t)] \leq e^{C(\|u_t^{\mathrm{D}}\|_{H^1}^2 + \|\alpha_t^{\mathrm{D}}\|_{\mathfrak{h}_0})}(\beta[W^{\mathrm{D}} e^{-itH_N}\Psi_N, \mathfrak{D}(u_t,\alpha_t)] + N^{-1}).$$

The exponential factor is uniformly bounded in t, since by Lemma 3.3 and Proposition 2.3,

$$\begin{aligned} \|u_t^{\mathrm{D}}\|_{H^1}^2 &\leq C\|u_t\|_{H^1}^2\|\alpha_t\|_{\mathfrak{h}_{1/2}}^2 \leq C, \\ \|\alpha_t^{\mathrm{D}}\|_{\mathfrak{h}_0} &\leq C(\|u_t\|_{H^1}^2 + \|\alpha_t\|_{\mathfrak{h}_{1/2}}) \leq C. \end{aligned} \tag{3.19}$$

In view of $H_N = (W^{\mathrm{D}})^* H_N^{\mathrm{D}} W^{\mathrm{D}}$ (see Lemma 3.1) and (3.18), we can proceed using Theorem 3.2 to estimate

$$\begin{aligned} &\beta\big[e^{-itH_N^{\mathrm{D}}} W^{\mathrm{D}}\Psi_N, \mathfrak{s}^{\mathrm{D}}[t](u^{\mathrm{D}},\alpha^{\mathrm{D}})\big] \\ &\quad\leq e^{CR^{\mathrm{D}}(t)}\big(\beta[W^{\mathrm{D}}\Psi_N, (u^{\mathrm{D}},\alpha^{\mathrm{D}})] + \gamma[W^{\mathrm{D}}\Psi_N, (u^{\mathrm{D}},\alpha^{\mathrm{D}})] + N^{-1}\big), \end{aligned}$$

where $R^{\mathrm{D}}(t) = 1 + \int_0^{|t|} \|u_s^{\mathrm{D}}\|_{H^3}^2 (1 + \|\alpha_s^{\mathrm{D}}\|_{\mathfrak{h}_{3/2}})^2 \,\mathrm{d}s$. The β functional on the right-hand side is estimated with the aid of Lemma 3.5,

$$\beta[W^{\mathrm{D}}\Psi_N, \mathfrak{D}(u,\alpha)] \leq C(\beta[\Psi_N,(u,\alpha)] + N^{-1}).$$

The γ functional is bounded by Lemma 3.6, which yields altogether

$$\begin{aligned} \beta[e^{-itH_N}\Psi_N, (u_t,\alpha_t)] \leq e^{CR^{\mathrm{D}}(t)}\Big(&|N^{-1}\langle\Psi_N, H_N\Psi_N\rangle - \mathcal{E}(u,\alpha)| \\ &+ \max_{j=1,2}(\beta[\Psi_N,(u,\alpha)] + N^{-1})^{j/2}\Big). \end{aligned}$$

It remains to relate the time-dependent pre-factor to the solution of the SKG equation. Using Lemma 3.3 and the fact that $(u_s^{\mathrm{D}}, \alpha_s^{\mathrm{D}}) = \mathfrak{D}(u_s,\alpha_s)$ we have

$$\|u_s^{\mathrm{D}}\|_{H^3}(1 + \|\alpha_s^{\mathrm{D}}\|_{\mathfrak{h}_{3/2}}) \leq C\|u_s\|_{H^3}\|\alpha_s\|_{\mathfrak{h}_{5/2}}^3(1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}} + \|u_s\|_{H^1}^2).$$

By Proposition 2.3 we have $\|u_s\|_{H^1}^2 \leq C$. Young's inequality then yields

$$R^{\mathrm{D}}(t) \leq 1 + C\int_0^{|t|}(\|u_s\|_{H^3}^{10} + \|\alpha_s\|_{\mathfrak{h}_{5/2}}^{10})\,\mathrm{d}s, \tag{3.20}$$

and this proves the claim. ■

4. Bogoliubov theory and the norm approximation

In this section we study the Bogoliubov approximation of e^{-itH_N} and prove Theorem 1.5. Our strategy is similar to the case of the mean-field approximation discussed in Section 3.

We start by introducing the dressed Bogoliubov Hamiltonian $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ and the associated Fock space evolution $\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)$, which describe the fluctuations around the mean-field solution for the dressed dynamics. In Theorem 4.2 we provide a statement analogous to Theorem 1.5 for the dressed case. In Section 4.2 we then study the norm approximation of the dressing transformation, which is given again in terms of a suitable Bogoliubov-type evolution. This evolution describes the fluctuations with respect to the mean-field dressing $\mathfrak{D}[\theta]$. We use the norm approximation of the dressing to relate the statements of Theorem 1.5 and 4.2. For that purpose, it is crucial to observe that the Bogoliubov approximation of $W^{\mathrm{D}}(\theta)$ in fact interpolates between the dressed and undressed Bogoliubov evolutions. This is stated in Proposition 4.6, whose proof is given in Section 5.

4.1. Norm approximation of the dressed dynamics

We introduce the Bogoliubov evolution describing the fluctuations associated with the dressed Nelson Hamiltonian H^{D}_N and the dressed mean-field equations (5.2). To this end, we consider the quadratic approximation of the fluctuation generator $H^{\mathrm{D}}_{u,\alpha}(t)$ introduced in (3.6) (see Lemma 6.10 for the explicit form of $H^{\mathrm{D}}_{u,\alpha}(t)$). For $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ and $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$, we introduce the quadratic operator acting on $\mathcal{F} \otimes \mathcal{F}$ given by

$$\begin{aligned}\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) := {} & \mathrm{d}\Gamma_b(h_t) + \mathbb{K}^{(1)}_{u_t} + (\mathbb{K}^{(2)}_{u_t} + \mathrm{h.c.}) + \mathrm{d}\Gamma_a(\omega) \\ & + \int \mathrm{d}x\,\mathrm{d}k\,\big((q_{u_t} L_{\alpha_t}(k) u_t)(x) a^*_k b^*_x + (q_{u_t} L_{\alpha_t}(k)^* u_t)(x) a_k b^*_x\big) + \mathrm{h.c.} \\ & + \int \mathrm{d}k\,\mathrm{d}l\,\big(-2M_{u_t}(k,-l) a^*_k a_l + M_{u_t}(k,l) a^*_k a^*_l + M_{u_t}(-k,-l) a_k a_l\big),\end{aligned} \tag{4.1}$$

with $h_t := h_{u_t,\alpha_t}$ as defined in (3.4a),

$$(L_\alpha(k)u)(x) := 2kB_x(k)\big((-i\nabla + F_\alpha(x))u\big)(x), \tag{4.2a}$$

$$M_u(k,l) := \langle u, kB_{(\cdot)}(k) \cdot lB_{(\cdot)}(l) u\rangle, \tag{4.2b}$$

with F_α given by (3.4c), and

$$\mathbb{K}^{(1)}_u := \int \mathrm{d}x\,\mathrm{d}y\, K^{(1)}_u(x,y) b^*_x b_y, \quad \mathbb{K}^{(2)}_u := \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, K^{(2)}_u(x,y) b^*_x b^*_y, \tag{4.2c}$$

where

$$\begin{aligned} K^{(1)}_u &:= q_u \widetilde{K}^{(1)}_u q_u, & \widetilde{K}^{(1)}_u(x,y) &:= u(x)V(x-y)\overline{u(y)}, \\ K^{(2)}_u &:= (q_u \otimes q_u)\widetilde{K}^{(2)}_u, & \widetilde{K}^{(2)}_u(x,y) &:= u(x)V(x-y)u(y), \end{aligned}$$

with $q_u = 1 - |u\rangle\langle u|$ and $V(x)$ defined in (3.2).

The next proposition on the evolution generated by the operator $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ is the special case $\theta = 1$, $\Lambda = \infty$ of Proposition 5.2 below.

Proposition 4.1. *Let* $(u, \alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$ *and let* $(u_t, \alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u, \alpha)$ *be given by* (3.3). *For every* $\Psi_0 \in D((\mathcal{N} + \mathbb{T})^{1/2})$ *there exists a unique solution to the Cauchy problem*

$$\begin{cases} i\partial_t \Psi(t) = \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)\Psi(t), \\ \Psi(0) = \Psi_0, \end{cases}$$

such that $\Psi \in C(\mathbb{R}, \mathcal{F} \otimes \mathcal{F}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}, D((\mathcal{N} + \mathbb{T})^{1/2}))$. *The solution map* $\Psi_0 \mapsto \Psi(t)$ *extends to a unitary* $\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)$ *on* $\mathcal{F} \otimes \mathcal{F}$ *satisfying* $\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)(\mathcal{F}_{\perp u} \otimes \mathcal{F}) \subseteq \mathcal{F}_{\perp u_t} \otimes \mathcal{F}$. *Moreover, for every* $\ell \in \mathbb{N}$ *there is a constant* $C(\ell)$ *such that for all* $t \in \mathbb{R}$,

$$\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)^*(\mathbb{T} + \mathcal{N} + 1)\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t) \le e^{C(1)R^{\mathrm{D}}(t)}(\mathbb{T} + \mathcal{N} + 1),$$
$$\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)^*(\mathcal{N} + 1)^\ell\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t) \le e^{C(\ell)R^{\mathrm{D}}(t)}(\mathcal{N} + 1)^\ell,$$

in the sense of quadratic forms on $\mathcal{F} \otimes \mathcal{F}$, *with*

$$R^{\mathrm{D}}(t) := 1 + \int_0^{|t|} \|u_s\|^2_{H^3}(1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}})^2\, \mathrm{d}s.$$

The next theorem is our main statement of this section, making precise that the evolution $\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)$ describes the fluctuations of the dressed Nelson dynamics around the dressed mean-field solutions.

Theorem 4.2. *Let* $(u, \alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$, *and* $\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)$ *the unitary defined in Proposition* 4.1. *There exists a constant* $C > 0$ *such that for all* $\chi \in \mathcal{F}_{\perp u} \otimes \mathcal{F}$ *with* $\|\chi\| = 1$, $N \ge 1$, *and* $t \in \mathbb{R}$,

$$\|e^{-itH^{\mathrm{D}}_N} X^*_{u,\alpha}\chi - X^*_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)}\mathbb{U}^{\mathrm{D}}_{u,\alpha}(t)\chi\| \le e^{CR^{\mathrm{D}}(t)}\delta^{\mathrm{D}}(\chi)^{1/2}\frac{\sqrt{\log N}}{N^{1/4}},$$

where $R^{\mathrm{D}}(t) := 1 + \int_0^{|t|} \|u_s\|^2_{H^3}(1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}})^2\, \mathrm{d}s$ *with* $(u_s, \alpha_s) := \mathfrak{s}^{\mathrm{D}}[s](u, \alpha)$ *and*

$$\delta^{\mathrm{D}}(\chi) := \|(1 + \mathcal{N}^3 + \mathrm{d}\Gamma_b(-\Delta) + \mathrm{d}\Gamma_a(\omega))^{1/2}\chi\|^2.$$

The proof of the theorem relies on a technical bound on the difference of $H^{\mathrm{D}}_{u,\alpha}(t)$ and its quadratic approximation $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ in terms of the operators $\mathcal{N}$ and $\mathbb{T}$. In detail, there exists a constant $C > 0$, such that for all $\chi \in \mathcal{F}^{\le N} \otimes \mathcal{F}$, $\phi \in \mathcal{F} \otimes \mathcal{F}$,

$$\begin{aligned} &|\langle \chi, (H^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t))\phi\rangle| \\ &\quad \le C\rho(t)\frac{\ln N}{N^{1/2}}\|(\mathcal{N} + \mathbb{T} + 1)^{1/2}\chi\|\|(\mathcal{N}^3 + \mathbb{T} + 1)^{1/2}\phi\|, \end{aligned} \tag{4.3}$$

with $\rho(t) = \|u_t\|^2_{H^3}(1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$. The precise statement and its proof are given in Lemma 6.12. Note that we choose to distribute the higher moments of $\mathcal{N}$ unequally in (4.3) because below we will rely on the estimates provided by Theorem 3.2 and Proposition 4.1 that control the higher moments of $\mathcal{N}$ during the Bogoliubov dynamics, but only its first moment during the many-body evolution.

Proof of Theorem 4.2. Using that the excitation map is isometric, we can write the norm difference as

$$\|e^{-itH_N^{\mathrm{D}}} X_{u,\alpha}^* \chi - X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)}^* \mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\| = \|\chi^{\mathrm{D}}(t) - \mathbb{1}_{\mathcal{N}_b \le N} \mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\|,$$

where $\chi^{\mathrm{D}}(t) = X_{\mathfrak{s}^{\mathrm{D}}[t](u,\alpha)} e^{-itH_N^{\mathrm{D}}} X_{u,\alpha}^* \chi$, and by (2.7), $\chi^{\mathrm{D}}(0) = \mathbb{1}_{\mathcal{N}_b \le N}\chi$. Then, since $\mathbb{1}_{\mathcal{N}_b \le N}\chi^{\mathrm{D}}(t) = \chi^{\mathrm{D}}(t)$, we can omit the projection on the right-hand side by increasing the value of the norm. Recalling (3.8), we now use $i\partial_t \chi^{\mathrm{D}}(t) = H_{u,\alpha}^{\mathrm{D}}(t)\chi^{\mathrm{D}}(t)$ and Proposition 4.1 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\chi^{\mathrm{D}}(t) - \mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\|^2 = 2\,\mathrm{Im}\langle \chi^{\mathrm{D}}(t), (H_{u,\alpha}^{\mathrm{D}}(t) - \mathbb{H}_{u,\alpha}^{\mathrm{D}}(t))\mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\rangle.$$

Using $\mathbb{1}_{\mathcal{N}_b \le N}\chi^{\mathrm{D}}(t) = \chi^{\mathrm{D}}(t)$ again, we can apply (4.3) to bound the right-hand side, so that

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\chi^{\mathrm{D}}(t) - \mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\|^2\right| \\
&\quad \le C\rho(t)\|(\mathcal{N} + \mathbb{T} + 1)^{1/2}\chi^{\mathrm{D}}(t)\|\,\|(\mathcal{N}^3 + \mathbb{T} + 1)^{1/2}\mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\|\frac{\ln N}{\sqrt{N}}.
\end{aligned}$$

By Proposition 4.1, we have

$$\|(\mathcal{N}^3 + \mathbb{T} + 1)^{1/2}\mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\| \le e^{CR^{\mathrm{D}}(t)}\|(\mathcal{N}^3 + \mathbb{T} + 1)^{1/2}\chi\|$$

and using (3.10a), (3.10b), and Theorem 3.2, we obtain

$$\langle \chi^{\mathrm{D}}(t), (\mathcal{N} + \mathbb{T} + 1)\chi^{\mathrm{D}}(t)\rangle \le e^{CR^{\mathrm{D}}(t)}\langle \chi^{\mathrm{D}}(0), (\mathcal{N} + \mathbb{T} + 1)\chi^{\mathrm{D}}(0)\rangle.$$

Since $\|\chi^{\mathrm{D}}(0) - \chi\|^2 \le N^{-1}\|\mathcal{N}^{1/2}\chi\|^2$, the fundamental theorem of calculus and the above estimates, together with $\frac{\mathrm{d}}{\mathrm{d}t}R^{\mathrm{D}}(t) = \rho(t)$, imply the desired bound, namely

$$\begin{aligned}
\|\chi^{\mathrm{D}}(t) - \mathbb{U}_{u,\alpha}^{\mathrm{D}}(t)\chi\|^2 &\le \|\chi^{\mathrm{D}}(0) - \chi\|^2 + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|\chi^{\mathrm{D}}(s) - \mathbb{U}_{u,\alpha}^{\mathrm{D}}(s)\chi\|^2\right| \mathrm{d}s \\
&\le e^{CR^{\mathrm{D}}(t)}\|(\mathcal{N}^3 + \mathbb{T} + 1)^{1/2}\chi\|^2 N^{-1/2}\ln N,
\end{aligned}$$

which concludes the proof of Theorem 4.2. ∎

4.2. Norm approximation of the dressing transformation

We now consider the dressing transformation on the level of the fluctuations. The effective dressing transformation is used for two purposes. First, it allows for a norm approximation of $W^{\mathrm{D}}(\theta)$ and, second, it intertwines the undressed and dressed Bogoliubov evolutions. These statements are made precise in Lemma 4.4 and Proposition 4.6, respectively. Since the undressed and the dressed Bogoliubov Hamiltonians are both quadratic operators on $\mathcal{F} \otimes \mathcal{F}$, see (1.16) and (4.1), it is natural to choose the dressing transformation that interpolates between the two itself as an evolution generated by a quadratic operator. The right candidate for this is the Bogoliubov-type approximation of the microscopic dressing $W^{\mathrm{D}}(\theta)$ associated with the mean-field flow $\mathfrak{D}[\theta]$ (more precisely, the evolution generated by the quadratic approximation of the fluctuation generator (3.16)).

For $(u,\alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, $(u^\theta, \alpha^\theta) := \mathfrak{D}[\theta](u,\alpha)$, $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$, consider the quadratic operator

$$\mathbb{D}^{\Lambda}_{u,\alpha}(\theta) := \mathrm{d}\Gamma_b(\tau_{u,\alpha}) + \left(\int \mathrm{d}x\, \mathrm{d}k \left(\kappa^{\Lambda}_{u^\theta}(k,x) a^*_k b^*_x - \kappa^{\Lambda}_{u^\theta}(-k,x) a_k b^*_x \right) + \mathrm{h.c.} \right) \tag{4.4}$$

with $\tau_{u,\alpha}$ defined in (3.12) and

$$\kappa^{\Lambda}_u(k,x) := (q_u i B^{\Lambda}_{(\cdot)}(k) u)(x), \quad B^{\Lambda}_x(k) := \mathbb{1}_{|k|\le \Lambda} B_x(k).$$

Since the unitary generated by $\mathbb{D}^{\Lambda}_{u,\alpha}$ will play an important role in the renormalization of the Nelson–Bogoliubov Hamiltonian in Section 5, we introduce the kernel κ^{Λ}_u with a cutoff $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$. The next proposition states the existence and some important properties of this unitary evolution.

Proposition 4.3. *Let $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$ and denote $(u^\theta, \alpha^\theta) := \mathfrak{D}[\theta](u,\alpha)$ the solution to (3.11) with initial datum (u,α). For every $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$ and $\Psi_0 \in D(\mathcal{N}^{1/2})$ there exists a unique solution to the Cauchy problem*

$$\begin{cases} i\partial_\theta \Psi(\theta) = \mathbb{D}^{\Lambda}_{u,\alpha}(\theta)\Psi(\theta), \\ \Psi(0) = \Psi_0, \end{cases}$$

such that $\Psi \in C(\mathbb{R}, \mathcal{F} \otimes \mathcal{F}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}, D(\mathcal{N}^{1/2}))$. The solution map $\Psi_0 \mapsto \Psi(\theta)$ defines a unitary $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ on $\mathcal{F} \otimes \mathcal{F}$ with the following properties:

- (i) $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)(\mathcal{F}_{\perp u} \otimes \mathcal{F}) = \mathcal{F}_{\perp u^\theta} \otimes \mathcal{F}$.
- (ii) $(u,\alpha,\Lambda) \mapsto \mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ *is strongly continuous.*
- (iii) $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ *is a Bogoliubov transformation.*
- (iv) *For every $\ell \in \mathbb{N}$ there exists a constant $C(\ell) > 0$ such that for all $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$, $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$, and $|\theta| \le 1$,*

$$\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^* (\mathcal{N}+1)^\ell \mathbb{W}^{\Lambda}_{u,\alpha}(\theta) \le e^{C(\ell)(\|u\|^2_{H^1} + \|\alpha\|_{L^2})} (\mathcal{N}+1)^\ell, \tag{4.5a}$$

$$\mathbb{W}^{\Lambda}_{u,\alpha}(\theta) (\mathcal{N}+1)^\ell \mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^* \le e^{C(\ell)(\|u\|^2_{H^1} + \|\alpha\|_{L^2})} (\mathcal{N}+1)^\ell, \tag{4.5b}$$

in the sense of quadratic forms on $\mathcal{F} \otimes \mathcal{F}$.

Proof. Existence and uniqueness of the solution follow from [68, Thm. 8] in combination with the bounds from Lemma 6.13. As a consequence, there exists a two-parameter flow $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta,\theta')$ that for $\theta,\theta',\theta'' \in \mathbb{R}$ satisfies $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta,\theta'')\mathbb{W}^{\Lambda}_{u,\alpha}(\theta'',\theta') = \mathbb{W}^{\Lambda}_{u,\alpha}(\theta,\theta')$. Indeed, define $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta'+\vartheta,\theta')\Psi(\theta')$ to be the solution of $i\partial_\vartheta \chi(\vartheta) = \mathbb{D}^{\Lambda}_{u,\alpha}(\theta'+\vartheta)\chi(\vartheta)$ with $\chi(0) = \Psi(\theta')$. Then the flow property follows from uniqueness of the solution, since

$\mathbb{W}^{\Lambda}_{u,\alpha}(\theta,\theta')\Psi(\theta')$ and $\Psi(\theta)$ are both solutions that agree at $\theta=\theta'$. Since $\mathbb{D}^{\Lambda}_{u,\alpha}(\theta)$ defines a symmetric quadratic form on $D(\mathcal{N}^{1/2})$, the flow maps $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta,\theta')$ are unitary.

To show the mapping property, consider the orthogonal projection $\Gamma(q_{u^\theta})$ defined by $\Gamma(q_{u^\theta})\restriction \mathcal{F}^{(n)}\otimes\mathcal{F} = (q_{u^\theta})^{\otimes n}\otimes 1$ and satisfying $i\frac{\mathrm{d}}{\mathrm{d}\theta}\Gamma(q_{u^\theta}) = -[\mathrm{d}\Gamma(\tau_{u,\alpha}),\Gamma(q_{u^\theta})]$. Since $\Gamma(q_{u^\theta})(\mathcal{F}\otimes\mathcal{F}) = \mathcal{F}_{\perp u^\theta}\otimes\mathcal{F}$, proving that for all $\Psi\in D(\mathcal{N}^{1/2})$,

$$i\frac{\mathrm{d}}{\mathrm{d}\theta}\|\Gamma(q_{u^\theta})\Psi(\theta)\|^2 = -\langle\Psi(\theta),[\mathbb{D}^{\Lambda}_{u,\alpha}(\theta)-\mathrm{d}\Gamma(\tau_{u,\alpha}),\Gamma(q_{u^\theta})]\Psi(\theta)\rangle = 0 \tag{4.6}$$

will imply $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)(\mathcal{F}_{\perp u}\otimes\mathcal{F})\subset\mathcal{F}_{\perp u^\theta}\otimes\mathcal{F}$. This holds, since $\kappa^{\Lambda}_{u^\theta}(k,\cdot)\in\operatorname{Ran}(q_{u^\theta})$ and thus

$$\begin{aligned}\left[\int \mathrm{d}k\,\mathrm{d}x\,\kappa^{\Lambda}_{u^\theta}(k,x)a^*_k b^*_x,\Gamma(q_{u^\theta})\right] &= \int \mathrm{d}k\, a^*_k[b^*(\kappa^{\Lambda}_{u^\theta}(k,\cdot)),\Gamma(q_{u^\theta})]\\ &= \int \mathrm{d}k\, a^*_k b^*((1-q_{u^\theta})\kappa^{\Lambda}_{u^\theta}(k,\cdot))\Gamma(q_{u^\theta}) = 0,\end{aligned}$$

with similar calculations for the other terms in the commutator. Applying the same argument to $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^* = \mathbb{W}^{\Lambda}_{u,\alpha}(\theta',\theta)|_{\theta'=0}$ shows that $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^*(\mathcal{F}_{\perp u^\theta}\otimes\mathcal{F})\subset\mathcal{F}_{\perp u}\otimes\mathcal{F}$, which gives the equality.

We now prove continuity of $(u,\alpha,\Lambda)\mapsto\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)\Psi$ for every $\Psi\in\mathcal{F}\otimes\mathcal{F}$, $\theta\in\mathbb{R}$. Let $u',\alpha'\in L^2(\mathbb{R}^3)$, $\Lambda'\in\mathbb{R}_+\cup\{\infty\}$. By uniqueness of the solution, Duhamel's formula

$$\begin{aligned}&\langle\Phi,(1-\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^*\mathbb{W}^{\Lambda'}_{u',\alpha'}(\theta))\Psi\rangle\\ &\quad= -i\int_0^\theta \mathrm{d}\eta\,\langle\Phi,\mathbb{W}^{\Lambda}_{u,\alpha}(\eta)^*(\mathbb{D}^{\Lambda}_{u,\alpha}(\eta)-\mathbb{D}^{\Lambda'}_{u',\alpha'}(\eta))\mathbb{W}^{\Lambda'}_{u',\alpha'}(\eta)\Psi\rangle\end{aligned} \tag{4.7}$$

holds for all $\Psi,\Phi\in D(\mathcal{N}^{1/2})$. As $(u',\alpha',\Lambda')\to(u,\alpha,\Lambda)$, $\tau_{u,\alpha}$ tends to $\tau_{u',\alpha'}$ in $L^\infty(\mathbb{R}^3)$ and $\kappa^{\Lambda'}_{u'}$ tends to κ^{Λ}_{u} in $L^2(\mathbb{R}^3\times\mathbb{R}^3)$, as one easily verifies. Thus, $\mathbb{D}^{\Lambda}_{u,\alpha}(\eta)-\mathbb{D}^{\Lambda'}_{u',\alpha'}(\eta)$ tends to zero as a quadratic form on $D(\mathcal{N}^{1/2})$ and since $\mathbb{W}^{\Lambda}_{u,\alpha}(\eta)$ preserves $D(\mathcal{N}^{1/2})$ the difference in (4.7) tends to zero. Since $1-\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^*\mathbb{W}^{\Lambda'}_{u',\alpha'}(\theta)$ is uniformly bounded, we can extend the convergence to arbitrary $\Phi,\Psi\in\mathcal{F}\otimes\mathcal{F}$, so $\mathbb{W}^{\Lambda'}_{u',\alpha'}(\theta)\to\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ in the weak operator topology. Since this is a family of unitary operators, this implies convergence in the strong operator topology.

The fact that $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ is a Bogoliubov transformation can be deduced from the property that $\tau_{u,\alpha}\in L^2(\mathbb{R}^3)$ and $\kappa^{\Lambda}_{u}\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, which holds for all $\Lambda\in\mathbb{R}_+\cup\{\infty\}$. This is a well-known result; see e.g., [12, Lem. 4.8].

To demonstrate the final statement, we rely on the fact that $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ is a Bogoliubov transformation. This implies the existence of bounded linear maps $\mathfrak{u},\mathfrak{v}\colon L^2\oplus L^2\to L^2\oplus L^2$, $\mathfrak{v}\in\mathfrak{S}^2(L^2\oplus L^2)$, such that

$$\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^*c^*(f\oplus g)\mathbb{W}^{\Lambda}_{u,\alpha}(\theta) = c^*(\mathfrak{u}(f\oplus g)) + c(\mathfrak{v}\overline{(f\oplus g)})$$

for all $f,g\in L^2$, where $c^*(f\oplus g) = b^*(f)+a^*(g)$. Using this relation, one can deduce (see [12, Lem. 4.4]) that for all $\ell\in\mathbb{N}$,

$$\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)^*(\mathcal{N}+1)^\ell\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)\le\ell^\ell(1+2\|\mathfrak{v}\|_{\mathfrak{S}_2}+\|\mathfrak{u}\|)^\ell(\mathcal{N}+1)^\ell$$

as quadratic forms on $\mathcal{F} \otimes \mathcal{F}$. To bound the norms on the right-hand side, we rely on the fact that

$$\|\mathfrak{v}\|_{\mathfrak{S}_2}^2 = \|\mathcal{N}^{1/2}\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)\Omega\|^2 \le e^{C(\|u\|_{H^1}^2+\|\alpha\|_{L^2})},$$

where $\|\mathcal{N}^{1/2}\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)\Omega\|^2$ is bounded by estimating the derivative with respect to θ, using the bounds from Lemma 6.13, and then applying Grönwall's inequality. Together, this proves (4.5a). Now using the fact that the inverse of a Bogoliubov transformation is again a Bogoliubov transformation, the proof of (4.5b) can be carried out in a similar fashion. In fact, it is easy to show that

$$\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)c^*(f \oplus g)\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)^* = c^*(\mathfrak{u}^*(f \oplus g)) - c(\overline{\mathfrak{v}^*(\bar{f} \oplus \bar{g})})$$

and thus we can follow the same steps as in the proof of (4.5a). ∎

Lemma 4.4. *Let* $(u, \alpha) \in H^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ *with* $\|u\|_{L^2} = 1$*, and* $\Lambda \in (0, \infty)$*. Then for all* $\theta \in \mathbb{R}$ *and* $r \in [0, 1]$ *we have*

$$\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)D(\mathbb{T}^r) \subset D(\mathbb{T}^r),$$

where $\mathbb{T} = \mathrm{d}\Gamma_b(-\Delta) + \mathrm{d}\Gamma_a(\omega)$.

Proof. For $u \in H^2(\mathbb{R}^3)$ and finite Λ, the coefficients of the generator, $\tau_{u,\alpha}(x)$ and $\kappa_{u^\theta}^{\Lambda}(k, x)$, are elements of $H^2(\mathbb{R}^3)$ in x and compactly supported in k. The commutator $[\mathbb{T}, \mathbb{D}_{u,\alpha}^{\Lambda}(\theta)]$ is thus $\mathbb{T}$-bounded. From this, the claim follows from Grönwall's inequality and interpolation. ∎

The next lemma shows that the dressing transformation $W^{\mathrm{D}}(\theta)$ introduced in (3.1) is effectively described by the evolution $\mathbb{W}_{u,\alpha}^{\infty}(\theta)$ introduced in Proposition 4.3. The proof of the lemma follows a similar strategy to the proof of Theorem 4.2.

Lemma 4.5. *There exists a constant* $C > 0$ *such that for all* $(u, \alpha) \in H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ *with* $\|u\|_{L^2} = 1$*,* $\chi \in \mathcal{F}_{\perp u} \otimes \mathcal{F}$ *with* $\|\chi\| = 1$*,* $N \ge 1$*, and* $|\theta| \le 1$*,*

$$\|W^{\mathrm{D}}(\theta)X_{u,\alpha}^*\chi - X_{\mathfrak{D}[\theta](u,\alpha)}^*\mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi\| \le e^{C(\|u\|_{H^1}^2+\|\alpha\|_{L^2})}\|(1 + \mathcal{N})^{\frac{3}{2}}\chi\|N^{-\frac{1}{2}}.$$

Proof. Let $\zeta(\theta) = X_{\mathfrak{D}[\theta](u,\alpha)}W^{\mathrm{D}}(\theta)X_{u,\alpha}^*\chi$ with $\zeta(0) = \mathbb{1}_{\mathcal{N}_b \le N}\chi$ and write the norm difference as

$$\|W^{\mathrm{D}}(\theta)X_{u,\alpha}^*\chi - X_{\mathfrak{D}[\theta](u,\alpha)}^*\mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi\| = \|\zeta(\theta) - \mathbb{1}_{\mathcal{N}_b \le N}\mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi\|.$$

Observe that dropping the projection $\mathbb{1}_{\mathcal{N}_b \le N}$ from the norm increases its value, since $\mathbb{1}_{\mathcal{N}_b \le N}\zeta(\theta) = \zeta(\theta)$. Using $i\partial_\theta\zeta(\theta) = D_{u,\alpha}(\theta)\zeta(\theta)$, with $D_{u,\alpha}(\theta)$ given by (3.16), and Proposition 4.3, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|\zeta(\theta) - \mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi\|^2 = 2\,\mathrm{Im}\langle\zeta(\theta) - \mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi, (D_{u,\alpha}(\theta) - \mathbb{D}_{u,\alpha}^{\infty}(\theta))\mathbb{W}_{u,\alpha}^{\infty}(\theta)\chi\rangle.$$

To bound the right-hand side, we employ Lemma 6.13, which states the existence of a constant C, such that for all $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$, $\phi, \chi \in \mathcal{F} \otimes \mathcal{F}$,

$$|\langle \phi, (D_{u,\alpha}(\theta) - \mathbb{D}^{\infty}_{u,\alpha}(\theta))\chi\rangle| \le C\|\phi\|\|(\mathcal{N}+1)^{3/2}\chi\|N^{-1/2}.$$

With this at hand, the proof is readily finished, as

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\|\zeta(\theta) - \mathbb{W}^{\infty}_{u,\alpha}(\theta)\chi\|^2\right| \le C\|\zeta(\theta) - \mathbb{W}^{\infty}_{u,\alpha}(\theta)\chi\|\|(\mathcal{N}+1)^{3/2}\mathbb{W}^{\infty}_{u,\alpha}(\theta)\chi\|N^{-1/2},$$

so using that $\|\zeta(0) - \chi\| = \|\mathbb{1}_{\mathcal{N}_b > N}\chi\| \le N^{-3}\|\mathcal{N}^{3/2}\chi\|^2$ and integrating leads to

$$\|\zeta(\theta) - \mathbb{W}^{\infty}_{u,\alpha}(\theta)\chi\| \le C\|(\mathcal{N}+1)^{3/2}\mathbb{W}^{\infty}_{u,\alpha}(\theta)\chi\|N^{-1/2}.$$

The desired result now follows from (4.5a). ∎

After having established that $\mathbb{W}^{\infty}_{u,\alpha}(\theta)$ provides a norm approximation for $W^{\mathrm{D}}(\theta)$, the next proposition clarifies its second key characteristic: namely, how the dressing $\mathbb{W}^{\infty}_{u,\alpha}(1)$ intertwines between the dressed and the undressed Bogoliubov evolutions. Since $\mathbb{W}^{\infty}_{u,\alpha}(1)$ is the unique N-independent Bogoliubov-type approximation for the microscopic dressing W^{D}, it naturally serves as the correct dressing at the level of the Bogoliubov evolutions, as Bogoliubov transformations are closed under composition. This is the content of the next statement, which is the analogous statement to Lemma 3.4 for the mean-field flows at the level of the fluctuations. The proof of the proposition, along with the proof of the existence of $\mathbb{U}_{u,\alpha}(t)$ (Theorem 1.4), is the subject of Section 5 (see Proposition 5.3).

Proposition 4.6. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$, and $\mathbb{U}(t)$, $\mathbb{U}^{\mathrm{D}}(t)$, $\mathbb{W}^{\infty}(\theta)$ the evolutions introduced in Theorem 1.4 and Propositions 4.1 and 4.3. For every $t \in \mathbb{R}$, we have*

$$\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)\mathbb{W}^{\infty}_{u,\alpha}(1) = \mathbb{W}^{\infty}_{\mathfrak{s}[t](u,\alpha)}(1)\mathbb{U}_{u,\alpha}(t).$$

Observe that the actions of the mean-field flows, with respect to which we are considering fluctuations, on both sides of the identity agree. Indeed, on the left, (u,α) is first evolved to $\mathfrak{D}(u,\alpha)$, by the definition of the generator $\mathbb{D}^{\infty}_{u,\alpha}(\theta)$, which is taken as the initial reference state in the dressed Bogoliubov transformation $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$, whose generator includes the evolution $\mathfrak{s}^{\mathrm{D}}$. On the right-hand side, the reference states evolve according to $(u,\alpha) \mapsto \mathfrak{s}[t](u,\alpha) \mapsto \mathfrak{D}[1] \circ \mathfrak{s}[t](u,\alpha)$, which equals $\mathfrak{s}^{\mathrm{D}}[t] \circ \mathfrak{D}(u,\alpha)$ by Lemma 3.4.

The next statement is the last step in the preparation for the proof of Theorem 1.5. It is an immediate consequence of Propositions 4.6, 4.1, and 4.3.

Corollary 4.7. *Let $\mathbb{U}_{u,\alpha}(t)$ be the unitary of Theorem 1.4 and $(u_t,\alpha_t) := \mathfrak{s}[t](u,\alpha)$ with (u,α) as stated in the hypothesis. For every $\ell \in \mathbb{N}$ there exists a constant $C(\ell)$ so that for all $|t| \ge 0$,*

$$\mathbb{U}_{u,\alpha}(t)^*(\mathcal{N}+1)^{\ell}\mathbb{U}_{u,\alpha}(t) \le e^{C(\ell)R(t)}(\mathcal{N}+1)^{\ell}$$

as quadratic forms on $\mathcal{F} \otimes \mathcal{F}$, where $R(t) := 1 + \int_0^{|t|}(\|u_s\|^{10}_{H^3} + \|\alpha_s\|^{10}_{\mathfrak{h}_{5/2}})\,\mathrm{d}s$.

Proof. Starting from the identity of Proposition 4.6 and acting with $\mathbb{W}^{\infty}_{\mathfrak{s}[t](u,\alpha)}(1)^*$ on both sides, we can express $\mathbb{U}_{u,\alpha}(t)$ in terms of three Bogoliubov evolutions, for each of which the corresponding bounds on number operators have been established in (4.5a), (4.5b), and Proposition 4.1. Collecting the right constant, this gives

$$\mathbb{U}_{u,\alpha}(t)^*(\mathcal{N}+1)^{\ell}\mathbb{U}_{u,\alpha}(t) \le e^{C(\ell)(R^{\mathrm{D}}(t)+\|u_t^{\mathrm{D}}\|^2_{H^1}+\|\alpha_t^{\mathrm{D}}\|_{L^2}+\|u\|^2_{H^1}+\|\alpha\|_{L^2})}(\mathcal{N}+1)^{\ell},$$

with $R^{\mathrm{D}}(t) := 1+\int_0^{|t|}\|u_s^{\mathrm{D}}\|^2_{H^3}(1+\|\alpha_s^{\mathrm{D}}\|_{\mathfrak{h}_{3/2}})^2\,\mathrm{d}s$, where $(u_t^{\mathrm{D}},\alpha_t^{\mathrm{D}}) = \mathfrak{s}^{\mathrm{D}}[t]\circ\mathfrak{D}(u,\alpha)$. That the exponential factor is bounded by $e^{C(\ell)R(t)}$ follows from (3.19) and (3.20). ■

4.3. Proof of Theorem 1.5

We are now ready to prove our second main result.

Proof of Theorem 1.5. Let $(u,\alpha)\in H^3(\mathbb{R}^3)\oplus\mathfrak{h}_{5/2}$ with $\|u\|_{L^2}=1$, $\chi\in\mathcal{F}_{\perp u}\otimes\mathcal{F}$, as given by the hypothesis. Let $\mathbb{W}^{\infty}_{u,\alpha}(1)$ be the Bogoliubov approximation of the dressing transformation of Proposition 4.3 and $\mathbb{U}_{u,\alpha}(t)$ be the Nelson–Bogoliubov dynamics given by Theorem 1.4, and $\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)$ be the dressed Bogoliubov dynamics of Proposition 4.1.

Using (1.17), we can write the norm difference that we want to estimate in terms of the dressed dynamics,

$$\mathcal{D} := \|e^{-itH_N^{\mathrm{D}}}W^{\mathrm{D}}X^*_{u,\alpha}\chi - W^{\mathrm{D}}X^*_{\mathfrak{s}[t](u,\alpha)}\mathbb{U}_{u,\alpha}(t)\chi\|.$$

We split this into the difference of the dressed dynamics and its Bogoliubov approximation, and the corresponding approximation of the dressing transformation, i.e., we estimate $\mathcal{D}\le\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3$ with

$$\begin{aligned}
\mathcal{D}_1 &:= \|e^{-itH_N^{\mathrm{D}}}W^{\mathrm{D}}X^*_{u,\alpha}\chi - e^{-itH_N^{\mathrm{D}}}X^*_{\mathfrak{D}(u,\alpha)}\mathbb{W}^{\infty}_{u,\alpha}(1)\chi\|\\
\mathcal{D}_2 &:= \|e^{-itH_N^{\mathrm{D}}}X^*_{\mathfrak{D}(u,\alpha)}\mathbb{W}^{\infty}_{u,\alpha}(1)\chi - X^*_{\mathfrak{s}^{\mathrm{D}}[t]\circ\mathfrak{D}(u,\alpha)}\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)\mathbb{W}^{\infty}_{u,\alpha}(1)\chi\|\\
\mathcal{D}_3 &:= \|X^*_{\mathfrak{s}^{\mathrm{D}}[t]\circ\mathfrak{D}(u,\alpha)}\mathbb{U}^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(t)\mathbb{W}^{\infty}_{u,\alpha}(1)\chi - W^{\mathrm{D}}X^*_{\mathfrak{s}[t](u,\alpha)}\mathbb{U}_{u,\alpha}(t)\chi\|.
\end{aligned}$$

Recalling $\mathfrak{D}=\mathfrak{D}[1]$ and applying Lemma 4.5 with $\theta=1$ immediately gives

$$\mathcal{D}_1 = \|W^{\mathrm{D}}X^*_{u,\alpha}\chi - X^*_{\mathfrak{D}(u,\alpha)}\mathbb{W}^{\infty}_{u,\alpha}(1)\chi\| \le C\|(\mathcal{N}+1)^{3/2}\chi\|N^{-1/2}.$$

By Proposition 4.6 and Lemma 3.4, we have

$$\mathcal{D}_3 = \|X^*_{\mathfrak{D}\circ\mathfrak{s}[t](u,\alpha)}\mathbb{W}^{\infty}_{\mathfrak{s}[t](u,\alpha)}(1)\mathbb{U}_{u,\alpha}(t)\chi - W^{\mathrm{D}}X^*_{\mathfrak{s}[t](u,\alpha)}\mathbb{U}_{u,\alpha}(t)\chi\|,$$

and applying Lemma 4.5, Proposition 2.3 then yields (here, $(u_t,\alpha_t)=\mathfrak{s}[t](u,\alpha)$)

$$\mathcal{D}_3 \le e^{C(\|u_t\|^2_{H^1}+\|\alpha_t\|_{L^2})}\|(\mathcal{N}+1)^{3/2}\mathbb{U}_{u,\alpha}(t)\chi\|N^{-1/2} \le C\|(\mathcal{N}+1)^{3/2}\mathbb{U}_{u,\alpha}(t)\chi\|N^{-1/2}.$$

Using Corollary 4.7 we arrive at the desired bound $\mathcal{D}_3\le e^{CR(t)}\|(\mathcal{N}+1)^{3/2}\chi\|N^{-1/2}$. To estimate $\mathcal{D}_2$, we apply Theorem 4.2 (note that $\mathfrak{D}(u,\alpha)\in H^3\oplus\mathfrak{h}_{5/2}$ by Lemma 3.3 and $(u,\alpha)\in H^3\oplus\mathfrak{h}_{5/2}$) so that

$$\mathcal{D}_2 \le e^{CR^{\mathrm{D}}(t)}\delta^{\mathrm{D}}(\mathbb{W}^{\infty}_{u,\alpha}(1)\chi)^{1/2}\sqrt{\ln N}\,N^{-1/4}$$

with $R^{\mathrm{D}}(t) = 1 + \int_0^{|t|} \|u_s^{\mathrm{D}}\|_{H^3}^2 (1 + \|\alpha_s^{\mathrm{D}}\|_{\mathfrak{h}_{3/2}})^2 \,\mathrm{d}s$ and $(u_t^{\mathrm{D}}, \alpha_t^{\mathrm{D}}) = \mathfrak{s}^{\mathrm{D}}[t]\mathfrak{D}(u,\alpha)$, which equals $\mathfrak{D}(\mathfrak{s}[t](u,\alpha))$. In view of the bound on $R^{\mathrm{D}}(t)$ stated in (3.20), this yields the claimed bound with

$$\delta(\chi) = \delta^{\mathrm{D}}(\mathbb{W}_{u,\alpha}^{\infty}(1)\chi).$$

The domain of δ is dense in $\mathcal{F} \otimes \mathcal{F}$ since it is the image under the continuous map $\mathbb{W}_{u,\alpha}^{\infty}(1)^*$ of the dense set $D((\mathcal{N}^3 + \mathbb{T})^{1/2})$. The set $D((\mathcal{N}^3 + \mathbb{T})^{1/2}) \cap \mathcal{F}_{\perp u^{\mathrm{D}}} \otimes \mathcal{F}$ is also dense in $\mathcal{F}_{\perp u^{\mathrm{D}}} \otimes \mathcal{F}$, as $\Gamma(q_{u^{\mathrm{D}}})$ is continuous and leaves $D((\mathcal{N}^3 + \mathbb{T})^{1/2})$ invariant for $u^{\mathrm{D}} \in H^1$. By the mapping property from Proposition 4.3, we have $\mathbb{W}_{u,\alpha}^{\infty}(1)^* \mathcal{F}_{\perp u^{\mathrm{D}}} \otimes \mathcal{F} = \mathcal{F}_{\perp u} \otimes \mathcal{F}$, so the image of $D((\mathcal{N}^3 + \mathbb{T})^{1/2})$ is also dense in $\mathcal{F}_{\perp u} \otimes \mathcal{F}$. This completes the proof. ∎

5. Renormalization of the Nelson–Bogoliubov evolution

This section is dedicated to proving the existence of the renormalized Nelson–Bogoliubov evolution, which is stated in Theorem 1.4. To accomplish this, we consider a family of Bogoliubov Hamiltonians $\mathbb{H}_{u,\alpha,\theta}^{\Lambda}$ that interpolate between $\mathbb{H}_{u,\alpha}^{\Lambda}$ and $\mathbb{H}_{u,\alpha}^{\mathrm{D}}$ with "dressing parameter" $\theta \in [0,1]$ and UV cutoff $\Lambda < \infty$. The key point is that for $\theta = 1$ we can remove the cutoff, i.e., $\mathbb{H}_{u,\alpha,1}^{\infty} = \mathbb{H}_{u,\alpha}^{\mathrm{D}}$, and that the Bogoliubov transformations $\mathbb{W}_{u,\alpha}^{\Lambda}(\theta)$ associated with the dressing interpolate between the different members of $(\mathbb{H}_{u,\alpha,\theta}^{\Lambda})_{\theta\in[0,1]}$. We exploit this property to prove Theorem 1.4 in Section 5.2 by defining $\mathbb{U}_{u,\alpha}(t)$ as the Bogoliubov transformation generated by $\mathbb{H}_{u,\alpha}^{\mathrm{D}}$ and transformed by the Bogoliubov approximation of the dressing. Along with the existence of $\mathbb{U}_{u,\alpha}(t)$, this also proves Proposition 4.6.

Even though the general strategy in this section is motivated by Nelson's original approach for renormalizing the Nelson Hamiltonian, the argument is more involved on the level of the fluctuations. This is because the dressing transformation and the different Nelson–Bogoliubov evolutions are all generated by non-autonomous equations.

5.1. Bogoliubov Hamiltonians with $\theta \in [0,1]$

In order to define the Bogoliubov Hamiltonians $\mathbb{H}_{u,\alpha}^{\Lambda}(t)$, we first introduce the partially dressed mean-field flow

$$\mathfrak{s}_\theta[t] = \mathfrak{D}[\theta] \circ \mathfrak{s}[t] \circ \mathfrak{D}[-\theta], \tag{5.1}$$

where $\mathfrak{s}$ is the flow of the Schrödinger–Klein–Gordon system (2.11). For the next statement recall the definitions below (3.3) and $\phi_\alpha(x) = 2\,\mathrm{Re}\langle G_x, \alpha\rangle$.

Lemma 5.1. *For $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2}$, the function $(u_t, \alpha_t) := \mathfrak{s}_\theta[t](u,\alpha)$ is the unique solution to the Hamiltonian equations*

$$\begin{cases} i\partial_t u_t(x) = h_{u_t,\alpha_t,\theta} u_t(x), \\ i\partial_t \alpha_t(k) = \omega(k)\alpha_t + (1-\theta)\langle u_t, G_{(\cdot)}(k) u_t\rangle \\ \qquad\qquad + 2\theta\langle u_t, kB_{(\cdot)}(k)(-i\nabla + \theta F_{\alpha_t}) u_t\rangle, \end{cases} \tag{5.2}$$

with conserved energy

$$\mathcal{E}_\theta(u,\alpha) := \Big\langle u, \Big(-\Delta + (1-\theta)\phi_\alpha + \theta A_\alpha + \theta^2 (F_\alpha)^2 + \frac{1}{2} V_\theta * |u|^2\Big) u\Big\rangle + \langle \alpha, \omega\alpha\rangle, \quad (5.3)$$

where

$$h_{u,\alpha,\theta} := -\Delta + (1-\theta)\phi_\alpha + \theta A_\alpha + \theta^2 (F_\alpha)^2 + V_\theta * |u|^2 - \mu_{u,\alpha,\theta}, \quad (5.4a)$$

$$\mu_{u,\alpha,\theta} := \frac{1-\theta}{2}\langle u, \phi_\alpha u\rangle + \frac{1}{2}\langle u, V_\theta * |u|^2 u\rangle + \theta\,\mathrm{Re}\langle \alpha, f_u + \theta g_{u,\alpha}\rangle, \quad (5.4b)$$

$$V_\theta(x) := -4\theta\,\mathrm{Re}\langle G_x, B_0\rangle + 2\theta^2\,\mathrm{Re}\langle B_x, \omega B_0\rangle. \quad (5.4c)$$

Remark 5.1. Since, for $\theta = 1$, equations (5.2) coincide with the dressed mean-field equations (3.3), we have $s_1[t] = \mathfrak{s}^{\mathrm{D}}[t]$. Lemma 5.1 thus provides a proof of Lemma 3.4.

Proof of Lemma 5.1. One checks by direct calculation [2, Prop. III.12] that

$$\mathcal{E}_\theta = \mathcal{E}_0 \circ \mathfrak{D}[-\theta]. \quad (5.5)$$

Since $\mathfrak{s}$ is a Hamiltonian flow of $\mathcal{E}_0$ with respect to the symplectic form

$$\sigma((u,\alpha),(u',\alpha')) = 2\,\mathrm{Im}(\langle u,u'\rangle_{L^2} + \langle \alpha,\alpha'\rangle_{L^2}), \quad (5.6)$$

and the equations (5.2) are the Hamiltonian equations for $\mathcal{E}_\theta$, the claim should follow by showing that $\mathfrak{D}$ acts by symplectic transformations. However, since the involved spaces are infinite-dimensional, we need to take care of some domain issues. These are addressed by [28, Lem. 6.9], so we will check the hypothesis of this lemma (the reader might find it helpful to consult the examples given in [28, Sect. 6.5]).

(1) $(u,\alpha) \mapsto \mathfrak{D}[\theta](u,\alpha)$ must be differentiable with derivative continuous on $E = H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2}$, and symplectic. The derivative of the flow is

$$\partial_{(v,\eta)}\mathfrak{D}[\theta](u,\alpha) = \big(e^{-i\theta\tau_{u,\alpha}} v - i\theta(\tilde{\phi}_\eta - 2\,\mathrm{Re}\langle u, \tilde{\phi}_\alpha v\rangle) u^\theta, \gamma + \theta B_0 \widehat{2\mathrm{Re}\,\bar{u}v}\big). \quad (5.7)$$

As a linear function of (v,η), this is continuous on $H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2}$, for all $(u,\alpha) \in H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2}$, as follows from the bounds of Lemma 6.3. Using formula (5.7), one also checks that

$$\sigma(\partial_{(v,\eta)}\mathfrak{D}, \partial_{(v',\eta')}\mathfrak{D}) = \sigma((v,\eta),(v',\eta')),$$

by using that B_0 is an even function and observing that the mixed terms in v, η' cancel each other (see also [2, Prop. IV.1]), i.e., $\mathfrak{D}$ is symplectic in the sense of [28, Def. 6.5].

(2) The domain $\mathcal{D}$ on which the Hamiltonian vector field (given by equation (5.2)) is defined must be invariant under the flow $\mathfrak{D}$. We take $\mathcal{D} = H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$, so this follows from Lemma 3.3.

(3) The derivative of $\mathcal{E}_\theta$ on $\mathcal{D}$ should be compatible with the symplectic structure as in [28, Def. 6.5]. Since in our case the symplectic map is simply $\mathcal{J} = 2i$ with range $H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2} \subset E'$, we must check that for $(u, \alpha) \in \mathcal{D}$, the right-hand side of equation (5.2) is an element of $H^1(\mathbb{R}^3) \oplus \mathfrak{h}_{1/2}$. This follows from the bounds of Lemmas 6.3 and 6.4.

Thus the hypothesis of [28, Lem. 6.9] is satisfied and this implies the claim. ∎

Next we introduce a family of θ- and t-dependent quadratic operators $\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(\theta)$ on $\mathcal{F} \otimes \mathcal{F}$ that is associated with the mean-field flow $\mathfrak{s}_\theta[t](u, \alpha)$ and defined such that

$$\mathbb{H}^{\Lambda}_{u,\alpha,0}(t) = \mathbb{H}^{\Lambda}_{u,\alpha}(t) \quad \text{and} \quad \mathbb{H}^{\infty}_{u,\alpha,1}(t) = \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$$

for the Nelson–Bogoliubov Hamiltonian (1.16) and the dressed Bogoliubov Hamiltonian (4.1). These are essentially the Bogoliubov Hamiltonians associated with the partially dressed Hamiltonians $W^{\mathrm{D}}(\theta) H_N W^{\mathrm{D}}(\theta)^*$. It is important to note that for $\theta \in [0, 1)$ the operator needs to be defined with a UV cutoff $\Lambda < \infty$, while for $\theta = 1$ the definition also makes sense for $\Lambda = \infty$.

Now, more concretely, for (u, α) and $\theta \in [0, 1]$, let $(u_t, \alpha_t) := \mathfrak{s}_\theta[t](u, \alpha)$ denote the solution to (5.2), and define

$$\begin{aligned}\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t) &:= \mathrm{d}\Gamma_b(h_{u_t,\alpha_t,\theta}) + \mathbb{K}^{(1),\Lambda}_{\theta,u_t} + (\mathbb{K}^{(2),\Lambda}_{\theta,u_t} + \text{h.c.}) + \mathrm{d}\Gamma_a(\omega) \\ &\quad + \int \mathrm{d}x\, \mathrm{d}k\, \big((q_{u_t} L^{\Lambda}_{\theta,\alpha_t}(k) u_t)(x) a^*_k b^*_x + (q_{u_t} L^{\Lambda}_{\theta,\alpha_t}(k)^* u_t)(x) a_k b^*_x\big) + \text{h.c.} \\ &\quad + \theta^2 \int \mathrm{d}k\, \mathrm{d}l\, \big(-2M^{\Lambda}_{u_t}(k,-l) a^*_k a_l + M^{\Lambda}_{u_t}(k,l) a^*_k a^*_l + M^{\Lambda}_{u_t}(-k,-l) a_k a_l\big),\end{aligned} \tag{5.8}$$

with $h_{u_t,\alpha_t,\theta}$ defined in (5.4a),

$$(L^{\Lambda}_{\theta,\alpha}(k)u)(x) := (1-\theta) G^{\Lambda}_x(k) + 2B^{\Lambda}_x(k) k \big((-i\theta\nabla + \theta^2 F_\alpha(x))u\big)(x), \tag{5.9a}$$

$$M^{\Lambda}_u(k,l) := \langle u, k B^{\Lambda}_{(\cdot)}(k) \cdot l B^{\Lambda}_{(\cdot)}(l) u\rangle, \tag{5.9b}$$

and

$$\mathbb{K}^{(1),\Lambda}_{\theta,u} := \int \mathrm{d}x\, \mathrm{d}y\, K^{(1),\Lambda}_{\theta,u}(x,y) b^*_x b_y, \quad \mathbb{K}^{(2),\Lambda}_{\theta,u} := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, K^{(2),\Lambda}_{\theta,u}(x,y) b^*_x b^*_y,$$

where

$$K^{(1),\Lambda}_{\theta,u} := q_u \widetilde{K}^{(1),\Lambda}_{\theta,u} q_u, \qquad \widetilde{K}^{(1),\Lambda}_{\theta,u}(x,y) := u(x) V^{\Lambda}_\theta(x-y) \overline{u(y)}, \tag{5.9c}$$

$$K^{(2),\Lambda}_{\theta,u} := (q_u \otimes q_u) \widetilde{K}^{(2),\Lambda}_{\theta,u}, \quad \widetilde{K}^{(2),\Lambda}_{\theta,u}(x,y) := u(x) V^{\Lambda}_\theta(x-y) u(y), \tag{5.9d}$$

with

$$V^{\Lambda}_\theta(x) := -4\theta\, \mathrm{Re}\langle G^{\Lambda}_x, B_0\rangle + 2\theta^2\, \mathrm{Re}\langle B^{\Lambda}_x, \omega B_0\rangle.$$

For $\theta = 1$ and $\Lambda = \infty$, these definitions coincide with those from Section 4.1. For $\theta \neq 1$, the cutoff $\Lambda < \infty$ is necessary, since the term involving $G^{\Lambda}_x(k)$ in (5.9a) does not yield a quadratic form on $D(\mathbb{T}^{1/2})$ for $\Lambda = \infty$. All other terms are in fact unproblematic for $\Lambda = \infty$ even if $\theta \neq 1$. For the purpose of the proof of Proposition 5.3, where cancellations between different terms are important, we work with the definition given above.

The next proposition states the existence and suitable properties of the unitary time evolution generated by $\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t)$.

Proposition 5.2. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$ and let $(u_t, \alpha_t) := \mathfrak{s}_\theta[t](u,\alpha)$ be given by (5.1). Moreover, let $\theta = 1$ and $\Lambda = \infty$ or $\theta \neq 1$ and $\Lambda \in (0,\infty)$. For every $\Psi \in D((\mathcal{N} + \mathbb{T})^{1/2})$ there exists a unique solution to the Cauchy problem*

$$\begin{cases} i\partial_t \Psi(t) = \mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t)\Psi(t), \\ \Psi(0) = \Psi_0, \end{cases}$$

such that $\Psi \in C(\mathbb{R}, \mathcal{F} \otimes \mathcal{F}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}, D((\mathcal{N} + \mathbb{T})^{1/2}))$. The solution map $\Psi_0 \mapsto \Psi(t)$ extends to a unitary $\mathbb{U}^{\Lambda}_{u,\alpha,\theta}(t)$ on $\mathcal{F} \otimes \mathcal{F}$ satisfying $\mathbb{U}^{\Lambda}_{u,\alpha,\theta}(t)(\mathcal{F}_{\perp u} \otimes \mathcal{F}) \subseteq \mathcal{F}_{\perp u_t} \otimes \mathcal{F}$. Moreover, for $\theta = 1$ we have the following properties:

(i) *There is a constant $C > 0$ such that for all $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$ and $t \in \mathbb{R}$,*

$$\mathbb{U}^{\Lambda}_{u,\alpha,1}(t)^*(\mathbb{T} + \mathcal{N})\mathbb{U}^{\Lambda}_{u,\alpha,1}(t) \leq e^{CR^{\mathrm{D}}(t)}(\mathbb{T} + \mathcal{N} + 1)$$

in the sense of quadratic forms on $\mathcal{F} \otimes \mathcal{F}$, with $R^{\mathrm{D}}(t) := 1 + \int_0^{|t|} \|u_s\|^2_{H^3}(1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}})^2 \,\mathrm{d}s$.

(ii) *For every $t \in \mathbb{R}$,*

$$\mathbb{U}^{\infty}_{u,\alpha,1}(t) = \operatorname*{s-lim}_{\Lambda\to\infty} \mathbb{U}^{\Lambda}_{u,\alpha,1}(t).$$

(iii) *$\mathbb{U}^{\Lambda}_{u,\alpha,1}(t)$ is a Bogoliubov transformation for all $t \in \mathbb{R}$, $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$.*

(iv) *For every $\ell \in \mathbb{N}$ there is a constant $C(\ell)$ such that for all $t \in \mathbb{R}$ and $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$,*

$$\mathbb{U}^{\Lambda}_{u,\alpha,1}(t)^*(\mathcal{N} + 1)^\ell \mathbb{U}^{\Lambda}_{u,\alpha,1}(t) \leq e^{C(\ell)R^{\mathrm{D}}(t)}(\mathcal{N} + 1)^\ell$$

in the sense of quadratic forms on $\mathcal{F} \otimes \mathcal{F}$, with $R^{\mathrm{D}}(t)$ as in (i).

Proof. Existence and uniqueness of the dynamics and property (i) follow from [68, Thm. 8] and the bounds of Lemma 6.8. Note that the existence of the unitary $\mathbb{U}^{\Lambda}_{u,\alpha,\theta}(t)$ follows by the same reasoning as in the proof of Proposition 4.3 and that the mapping property is obtained by a similar argument to (4.6).

To prove (ii) use Duhamel's formula for $\Psi, \Xi \in D((\mathcal{N} + \mathbb{T})^{1/2})$ together with Lemma 6.9 to obtain

$$\begin{aligned} &|\langle \Xi, (1 - \mathbb{U}^{\infty}_{u,\alpha,1}(t)^*\mathbb{U}^{\Lambda}_{u,\alpha,1}(t))\Psi\rangle| \\ &\quad \leq \int_0^{|t|} \mathrm{d}s\, |\langle \Xi, \mathbb{U}^{\infty}_{u,\alpha,1}(s)^*(\mathbb{H}^{\infty}_{u,\alpha,1}(s) - \mathbb{H}^{\Lambda}_{u,\alpha,1}(s))\mathbb{U}^{\Lambda}_{u,\alpha,1}(s)\Psi\rangle| \\ &\quad \leq \varepsilon_\Lambda \int_0^{|t|} \mathrm{d}s\, e^{CR^{\mathrm{D}}(s)} \|(\mathbb{T} + \mathcal{N} + 1)^{\frac{1}{2}}\mathbb{U}^{\infty}_{u,\alpha,1}(s)\Xi\| \|(\mathbb{T} + \mathcal{N} + 1)^{\frac{1}{2}}\mathbb{U}^{\Lambda}_{u,\alpha,1}(s)\Psi\|, \end{aligned}$$

where $R^{\mathrm{D}}(s) = \|u_s\|_{H^3}^2(1 + \|\alpha_s\|_{\mathfrak{h}_{3/2}})^2$ and $\varepsilon_\Lambda \to 0$ as $\Lambda \to 0$. The right-hand side thus converges to zero by property (i). This implies strong convergence of $\mathbb{U}^\Lambda_{u,\alpha,1}$ to $\mathbb{U}^\infty_{u,\alpha,1}$ by unitarity of $\mathbb{U}^\Lambda_{u,\alpha,1}$, $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$, as argued in the proof of Proposition 4.3.

The statement that $\mathbb{U}^\Lambda_{u,\alpha,1}(t)$ for finite Λ is a Bogoliubov transformation is essentially a consequence of the square-integrability of the kernels in (5.8) appearing in the terms with b^*b^*, b^*a^*, and a^*a^*; see [12, Lem. 4.8] for a detailed argument. For $\Lambda = \infty$, however, the kernel corresponding to a^*b^* fails to meet the Hilbert–Schmidt criterion. In this case, the statement follows from a suitable approximation argument, see Lemma B.1, together with the fact that $\mathbb{U}^\Lambda_{u,\alpha,1}(t) \to \mathbb{U}^\infty_{u,\alpha,1}(t)$ converges strongly.

The final statement can be derived using the same logic as in the proof of Proposition 4.3, in combination with Proposition 5.2 (i). ∎

5.2. Dressing identity and proof of Theorem 1.4

We can now prepare for the proof of Theorem 1.4. We will start by making precise how the unitaries $\mathbb{W}^\Lambda_{u,\alpha}(\theta)$ interpolate between the Bogoliubov dynamics $\mathbb{U}^\Lambda_{u,\alpha,\theta}(t)$ for different θ. For $\theta = 1$ we can then take the limit $\Lambda \to \infty$ of $\mathbb{U}^\Lambda_{u,\alpha,1}(t)$ to obtain information on the behavior of $\mathbb{U}^\Lambda_{u,\alpha,0}(t)$ as $\Lambda \to \infty$.

Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$, and consider the evolutions

$$\mathbb{U}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t) \quad \text{and} \quad \mathbb{W}^\Lambda_{\mathfrak{s}_0[t](u,\alpha)}(\theta), \tag{5.10}$$

with generators $\mathbb{H}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t)$ and $\mathbb{D}^\Lambda_{\mathfrak{s}_0[t](u,\alpha)}(\theta)$, respectively (see Propositions 5.2 and 4.3). It is important to keep in mind that the subscripts refer to the initial condition of the mean-field flow, which is used to define the generator of each evolution.

The next result shows that the two flows in (5.10) commute, up to a global phase. The additional phase is due to the fact that we wrote $\mathbb{H}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t)$ in normal order, which is not preserved by the transformations. As discussed below, this identity is the key ingredient for the renormalization of the Nelson–Bogoliubov Hamiltonian.

Proposition 5.3. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$ and $(u_t,\alpha_t) := \mathfrak{s}[t](u,\alpha)$ be the solution of (1.8) with initial datum (u,α). If we denote*

$$E^\Lambda_\theta(t) := (2\theta - \theta^2)\langle G^\Lambda_0, B^\Lambda_0\rangle + \frac{1}{2}\langle u_t, V^\Lambda_\theta * |u_t|^2 u_t\rangle,$$

then for all $t, \theta \in \mathbb{R}$ and $\Lambda \in (0,\infty)$ we have the identity

$$\mathbb{U}^\Lambda_{u,\alpha,0}(t) e^{-i\int_0^t \mathrm{d}s\, E^\Lambda_\theta(s)} = \mathbb{W}^\Lambda_{\mathfrak{s}_0[t](u,\alpha)}(\theta)^* \mathbb{U}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t) \mathbb{W}^\Lambda_{u,\alpha}(\theta).$$

This proposition follows from the uniqueness of both sides by comparing their derivatives. The lengthy calculation is given in Section 5.3. Assuming this for now, we can prove Theorem 1.4 and Proposition 4.6.

Proofs of Theorem 1.4 *and Proposition* 4.6. By strong continuity of $\mathbb{W}^{\Lambda}_{u,\alpha}(\theta)$ in Λ (see Lemma 4.3) and of $\mathbb{U}^{\Lambda}_{u,\alpha,1}(t)$ (Proposition 5.2), we have

$$\operatorname*{s-lim}_{\Lambda\to\infty} \mathbb{W}^{\Lambda}_{\mathfrak{s}_0[t](u,\alpha)}(1)^* \mathbb{U}^{\Lambda}_{\mathfrak{D}(u,\alpha),1}(t) \mathbb{W}^{\Lambda}_{u,\alpha}(0) = \mathbb{W}^{\infty}_{\mathfrak{s}_0[t](u,\alpha)}(1)^* \mathbb{U}^{\infty}_{\mathfrak{D}(u,\alpha),1}(t) \mathbb{W}^{\infty}_{u,\alpha}(0).$$

By Proposition 5.3 with $\theta = 1$ and $\mathbb{U}^{\Lambda}_{u,\alpha}(t) = \mathbb{U}^{\Lambda}_{u,\alpha,0}(t)$ this shows that

$$\mathbb{U}_{u,\alpha}(t) = \operatorname*{s-lim}_{\Lambda\to\infty} \mathbb{U}^{\Lambda}_{u,\alpha,0}(t) e^{-i\int_0^t \mathrm{d}s\, E_1^{\Lambda}(s)} = \mathbb{W}^{\infty}_{\mathfrak{s}_0[t](u,\alpha)}(1)^* \mathbb{U}^{\infty}_{\mathfrak{D}(u,\alpha),1}(t) \mathbb{W}^{\infty}_{u,\alpha}(0). \quad (5.11)$$

Strong continuity of $t \mapsto \mathbb{U}_{u,\alpha}(t)$ follows since the right-hand side is strongly continuous, as for $\Psi \in \mathcal{F} \otimes \mathcal{F}$ the maps

$$t \mapsto \mathbb{W}^{\infty}_{\mathfrak{s}_0[t](u,\alpha)}(1)^*\Psi, \quad t \mapsto \mathbb{U}^{\infty}_{\mathfrak{D}(u,\alpha),1}(t)\Psi, \quad t \mapsto \mathbb{W}^{\infty}_{u,\alpha}(0)\Psi$$

are continuous. In order to relate the phase to the one given in the statement of the theorem, we recall from (1.3) that $\langle G_0^{\Lambda}, B_0^{\Lambda}\rangle = E^{\Lambda}$. The convergence of $\langle u_s, V^{\Lambda} * |u_s|^2 u_s\rangle$ for $\Lambda \to \infty$ follows from Lemma 6.4 and Proposition 2.3.

To show the mapping property for $\mathbb{U}_{u,\alpha}(t)$, consider $\Gamma(q_{u_t})$ as in the proof of Proposition 4.3 for $(u_t, \alpha_t) = \mathfrak{s}[t](u,\alpha)$. By the same argument as in (4.6) one shows that $\|\Gamma(q_{u_t})\mathbb{U}^{\Lambda}_{u,\alpha,0}(t)\Psi\| = 0$ for every $\Psi \in \mathcal{F} \otimes \mathcal{F}$ and all $\Lambda \in \mathbb{R}_+$, $t \in \mathbb{R}$. The desired result then follows from $\|\Gamma(q_{u_t})\mathbb{U}^{\Lambda}_{u,\alpha,0}(t)\Psi\| \to \|\Gamma(q_{u_t})\mathbb{U}_{u,\alpha}(t)\Psi\|^2$ as $\Lambda \to \infty$.

The property that $\mathbb{U}_{u,\alpha}(t)$ is a Bogoliubov transformation is a direct consequence of the result that it is a composition of three Bogoliubov transformations. ∎

Remark 5.2 (On the renormalized Nelson–Bogoliubov Hamiltonian). If the evolution operator $\mathbb{U}_{u,\alpha}(t)$ were a semigroup, we could deduce from Theorem 1.4 the existence of a generator $\mathbb{H}_{u,\alpha}$ that would represent the renormalization of the Nelson–Bogoliubov Hamiltonian (1.16). But $\mathbb{U}_{u,\alpha}(t)$ is associated with a non-autonomous evolution equation and $\mathbb{H}_{u,\alpha}$ should depend on the time t. In this setting, only the following, weaker theory is available (see [83] for details). Consider the extension of $\mathbb{U}_{u,\alpha}(t)$ to a two-parameter family $\mathbb{U}_{u,\alpha}(t,s)$ with $\mathbb{U}_{u,\alpha}(t,0) = \mathbb{U}_{u,\alpha}(t)$. On the Banach space $C_\infty(\mathbb{R}, \mathcal{F} \otimes \mathcal{F})$ of continuous $\mathcal{F} \otimes \mathcal{F}$-valued functions tending to zero at infinity, we can define the corresponding evolution semigroup of isometries by

$$(T(t)\chi)(s) = \mathbb{U}_{u,\alpha}(s, s-t)\chi(s-t).$$

Then T has a generator A, $D(A) \subset C_\infty(\mathbb{R}, \mathcal{F} \otimes \mathcal{F})$, which corresponds formally to

$$(A\chi)(s) = \frac{\mathrm{d}}{\mathrm{d}t} T(t)\Big|_{t=0} \chi(s) = \Big(-i\mathbb{H}_{u,\alpha}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\Big)\chi(s).$$

However, we do not have any information concerning the domain or self-adjointness of $\mathbb{H}_{u,\alpha}(t)$ (one could consider using [83, Thm. 2.9], but this does not apply since the solutions provided by Proposition 5.2 need not be differentiable in the norm of $\mathcal{F} \otimes \mathcal{F}$).

5.3. Proof of the dressing identity

In this section we derive the dressing identity for the Nelson–Bogoliubov evolution, that is, we prove Proposition 5.3. To enhance the clarity of the presentation, we will use the shorthand notation

$$\begin{aligned} \mathbb{U}_\theta^\Lambda(t) &:= \mathbb{U}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t), \quad \mathbb{W}_t^\Lambda(\theta) := \mathbb{W}^\Lambda_{\mathfrak{s}_0[t](u,\alpha)}(\theta), \\ \mathbb{H}_\theta^\Lambda(t) &:= \mathbb{H}^\Lambda_{\mathfrak{D}[\theta](u,\alpha),\theta}(t), \quad \mathbb{D}_t^\Lambda(\theta) := \mathbb{D}^\Lambda_{\mathfrak{s}_0[t](u,\alpha)}(\theta). \end{aligned} \tag{5.12}$$

Proof of Proposition 5.3. Adopting (5.12), the identity we aim to prove becomes

$$\mathbb{U}_0^\Lambda(t) e^{-i\int_0^t \mathrm{d}s\, E_1^\Lambda(s)} = \mathbb{W}_t^\Lambda(\theta)^* \mathbb{U}_\theta^\Lambda(t) \mathbb{W}_0^\Lambda(\theta).$$

The idea is to prove a differential version of this identity after taking derivatives in t and θ. To make this precise, first note that the coefficients of $\mathbb{D}_t^\Lambda(\theta)$ given in (4.4) depend on u, α in a differentiable way. Since for $(u,\alpha) \in H^2(\mathbb{R}^3) \oplus \mathfrak{h}_1$ the flow $\mathfrak{s}_\theta[t](u,\alpha)$ is differentiable in t in the L^2-sense, this implies that for any $\Psi \in D(\mathcal{N})$ the mapping $t \mapsto \mathbb{D}_t^\Lambda(\theta)\Psi$ is differentiable in $\mathcal{F} \otimes \mathcal{F}$. Since $\mathbb{W}_t^\Lambda(\theta)$ preserves $D(\mathcal{N})$, we deduce from Duhamel's formula (4.7) that $\mathbb{W}_t^\Lambda(\theta)\Psi$ is differentiable in t for $\Psi \in D(\mathcal{N})$ and

$$(i\partial_t \mathbb{W}_t^\Lambda(\theta)^*)\mathbb{W}_t^\Lambda(\theta)\Psi = -\int_0^\theta \mathrm{d}\eta\, \mathbb{W}_t^\Lambda(\eta)^* (\partial_t \mathbb{D}_t^\Lambda)(\eta) \mathbb{W}_t^\Lambda(\eta)\Psi.$$

Denote by $\mathbb{V}_\theta^\Lambda(t) = \mathbb{W}_t^\Lambda(\theta)^* \mathbb{U}_\theta^\Lambda(t) \mathbb{W}_0^\Lambda(\theta)$ the right-hand side of the identity we want to prove. It follows from our previous considerations and Lemma 4.4 that for $\Psi \in D((\mathcal{N} + \mathbb{T})^{1/2})$ we have

$$\begin{aligned} i\partial_t \mathbb{V}_\theta^\Lambda(t)\Psi = &\ (i\partial_t \mathbb{W}_t^\Lambda(\theta)^*)\mathbb{U}_\theta^\Lambda(t)\mathbb{W}_0^\Lambda(\theta)\Psi + \mathbb{W}_t^\Lambda(\theta)^*\mathbb{H}_\theta^\Lambda(t)\mathbb{U}_\theta^\Lambda(t)\mathbb{W}_0^\Lambda(\theta)\Psi \\ =: &\ \mathbb{B}_\theta(t)\mathbb{V}_\theta^\Lambda(t)\Psi \end{aligned}$$

in $D((\mathcal{N} + \mathbb{T})^{-1/2})$. By uniqueness of the solutions proved in Proposition 5.2 our claim will follow if we can show that the generators of $\mathbb{U}_0^\Lambda(t)e^{-i\int_0^t \mathrm{d}s\, E_\theta^\Lambda(s)}$ and $\mathbb{V}_\theta^\Lambda(t)$ are equal, that is, for every $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{H}_0^\Lambda(t) + E_\theta^\Lambda(t) &= \mathbb{B}_\theta(t) \\ &= -\int_0^\theta \mathrm{d}\eta\, \mathbb{W}_t^\Lambda(\eta)^*(\partial_t \mathbb{D}_t^\Lambda)(\eta)\mathbb{W}_t^\Lambda(\eta) + \mathbb{W}_t^\Lambda(\theta)^*\mathbb{H}_\theta^\Lambda(t)\mathbb{W}_t^\Lambda(\theta). \end{aligned}$$

Equality holds for $\theta = 0$ since $E_0^\Lambda = 0$ and $\mathbb{W}_t^\Lambda(0) = 1$, so it is sufficient to prove that for all $\Psi, \Xi \in D((\mathcal{N} + \mathbb{T})^{1/2})$,

$$\begin{aligned} 0 &= i\partial_\theta \langle \Xi, (\mathbb{H}_0^\Lambda(t) + E_\theta^\Lambda(t) - \mathbb{B}_\theta(t))\Psi\rangle \\ &= \langle \mathbb{W}_t^\Lambda(\theta)\Xi, (i\partial_\theta E_\theta^\Lambda(t) + i\partial_t \mathbb{D}_t^\Lambda(\theta) - i\partial_\theta \mathbb{H}_\theta^\Lambda(t) - [\mathbb{H}_\theta^\Lambda(t), \mathbb{D}_t^\Lambda(\theta)])\mathbb{W}_t^\Lambda(\theta)\Psi\rangle, \end{aligned} \tag{5.13}$$

where we anticipated differentiability of $\mathbb{H}_\theta^\Lambda$ which follows easily from the explicit calculation of its derivative below.

The remainder of the proof is an explicit calculation of this quadratic form. For ease of presentation, we set, using that the flows commute,

$$(u_t^\theta, \alpha_t^\theta) := \mathfrak{s}_\theta[t] \circ \mathfrak{D}[\theta](u, \alpha) = \mathfrak{D}[\theta] \circ \mathfrak{s}_0[t](u, \alpha). \tag{5.14}$$

Moreover, we do not make the dependence of the different objects on t, θ, Λ explicit everywhere and adopt the following shorthand notation:

$$\begin{aligned} h_\theta &:= h_{u_t^\theta, \alpha_t^\theta, \theta}, & L_\theta^\Lambda(k) &:= (1-\theta) G_{(\cdot)}^\Lambda + 2 B_{(\cdot)}^\Lambda(k) k(-i\theta\nabla + \theta^2 F_{\alpha_t^\theta}(\cdot)), \\ \tau_t &:= \tau_{u_t, \alpha_t}, & M^\Lambda(k,l) &:= \langle u_t^\theta, k B_{(\cdot)}^\Lambda(k) l B_{(\cdot)}^\Lambda(l) u_t^\theta \rangle, \\ q &:= q_{u_t^\theta} = 1 - |u_t^\theta\rangle\langle u_t^\theta|, & K_\theta^{(1),\Lambda} &:= K_{\theta, u_t^\theta}^{(1),\Lambda}, \\ \kappa_t^\Lambda(k,x) &:= (q i B_{(\cdot)}^\Lambda(k) u_t^\theta)(x), & K_\theta^{(2),\Lambda} &:= K_{\theta, u_t^\theta}^{(2),\Lambda}. \end{aligned}$$

Note that $M^\Lambda(k,l)$ is independent of θ as the flow $\mathfrak{D}$ preserves the modulus of u.

After commuting $\mathbb{H}_\theta$ and $\mathbb{D}_t$, the expression from (5.13) takes the form

$$\begin{aligned} &-i\partial_t \mathbb{D}_t^\Lambda(\theta) + i\partial_\theta \mathbb{H}_\theta^\Lambda(t) + [\mathbb{H}_\theta^\Lambda(t), \mathbb{D}_t^\Lambda(\theta)] \\ &\quad = \int \mathrm{d}x\, b_x^* (-i\partial_t \tau_t + i\partial_\theta h_\theta + [h_\theta, \tau_t]) b_x && (5.15a) \\ &\quad\quad + \int \mathrm{d}x\, \mathrm{d}y\, D(x,y) b_x^* b_y + i E_b + \Big(\int \mathrm{d}x\, \mathrm{d}y\, \widetilde{D}(x,y) b_x^* b_y^* - \text{h.c.} \Big) && (5.15b) \\ &\quad\quad + \int \mathrm{d}k\, \mathrm{d}x\, (X(k,x) a_k^* b_x^* + \widetilde{X}(k,x) a_k b_x^*) - \text{h.c.} && (5.15c) \\ &\quad\quad + \int \mathrm{d}k\, \mathrm{d}l\, A(k,l) a_k^* a_l + i E_a + \Big(\int \mathrm{d}k\, \mathrm{d}l\, \tilde{A}(k,l) a_k^* a_l^* - \text{h.c.} \Big), && (5.15d) \end{aligned}$$

in the sense of sesquilinear forms on $D((\mathcal{N} + \mathbb{T})^{1/2})$. We will now show that all the coefficients of creation and annihilation operators vanish, and that $E_a + E_b = \partial_\theta E_\theta^\Lambda(t)$.

Mean-field part (5.15a). By (3.11), Lemma 5.1, and (5.14) we have $\partial_\theta u_t^\theta = -i\tau_t u_t^\theta$ and $\partial_t u_t^\theta = -i h_\theta u_t^\theta$, and since the derivatives commute

$$0 = (\partial_t \partial_\theta - \partial_\theta \partial_t) u_t^\theta = (-(i\partial_t \tau_t) + i(\partial_\theta h_\theta) + [h_\theta, \tau_t]) u_t^\theta.$$

Since (5.14) is a bijection, this implies that (5.15a) vanishes.

Terms quadratic in b, b^*, and E_b (5.15b). In the commutator $[\mathbb{H}_\theta^\Lambda(t), \mathbb{D}_t^\Lambda(\theta)]$ a quadratic term in b, b^* can arise either by commuting two terms with $b^\# a^\#$, or terms with $b^\# b^\#$. In the first case, the coefficients are combinations of $L_\theta^\Lambda(k)$ and $\kappa_t^\Lambda(k,\cdot)$ integrated over k. In the latter case, some terms have already been taken into account in (5.15a) and only the commutators of $K_\theta^{(1),\Lambda}$, $K_\theta^{(2),\Lambda}$ with τ_t remain. Combining these with the derivatives of $K_\theta^{(1),\Lambda}$, $K_\theta^{(2),\Lambda}$ and putting them in normal order yields

$$\begin{aligned}
D(x,y) &= i\partial_\theta K_\theta^{\Lambda,(1)}(x,y) - (\tau_t(x)-\tau_t(y))K_\theta^{\Lambda,(1)}(x,y)\\
&\quad - \int \mathrm{d}k\,\big(q(L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*)u_t^\theta\big)(x)\overline{\kappa_t^\Lambda(k,y)}\\
&\quad + \int \mathrm{d}k\,\kappa_t^\Lambda(k,x)\overline{\big(q(L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*)u_t^\theta\big)(y)},\\
E_b &= i\int \mathrm{d}x\,\mathrm{d}k\,\overline{\kappa_t^\Lambda(k,x)}\big(q(L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*)\big)u_t^\theta(x),\\
\widetilde{D}(x,y) &= \frac{i}{2}\partial_\theta K_\theta^{(2),\Lambda}(x,y) - \frac{1}{2}(\tau_t(x)+\tau_t(y))K_\theta^{(2),\Lambda}(x,y)\\
&\quad + \int \mathrm{d}k\,\big(q(L_\theta^\Lambda(k)^*+L_\theta^\Lambda(-k))u_t^\theta\big)(x)\kappa_t^\Lambda(k,y).
\end{aligned}$$

We now show that $D(x,y)=0$. First, we may observe that $i\partial_\theta u_t^\theta = \tau_t u_t^\theta$ and $q = 1 - |u_t^\theta\rangle\langle u_t^\theta|$ imply for any operator T_θ the identity

$$i\partial_\theta(qT_\theta u_t^\theta) = \tau_t qT_\theta u_t^\theta + q[T_\theta,\tau_t]u_t^\theta + q(i\partial_\theta T_\theta)u_t^\theta. \tag{5.16}$$

Since $[V_\theta^\Lambda,\tau_t]=0$, this gives us

$$\begin{aligned}
&i\partial_\theta K_\theta^{(1),\Lambda}(x,y) - (\tau_t(x)-\tau_t(y))K_\theta^{(1),\Lambda}(x,y)\\
&\quad = \int \mathrm{d}z\,\mathrm{d}z'\,q(x,z)u_t^\theta(z)(i\partial_\theta V_\theta^\Lambda(z-z'))\bar{u}_t^\theta(z')q(z',y).
\end{aligned}\tag{5.17}$$

To evaluate the terms involving $L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*$, we first calculate using $\overline{B_x^\Lambda}(-k) = B_x^\Lambda(k)$,

$$(L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*)u_t^\theta(x) = ((1-\theta)2G_x^\Lambda(k) + 2\theta k^2 B_x^\Lambda(k))u_t^\theta(x).$$

This gives

$$\begin{aligned}
&-\int \mathrm{d}k\,\big(q(L_\theta^\Lambda(k)+L_\theta^\Lambda(-k)^*)u\big)(x)\overline{\kappa_t^\Lambda(k,y)}\\
&\quad = \int \mathrm{d}z\,\mathrm{d}z'\,q(x,z)u_t^\theta(z)\big(2i(1-\theta)\langle B_{z'}^\Lambda,G_z^\Lambda\rangle + 2i\theta\langle k^2B_{z'}^\Lambda,B_z^\Lambda\rangle\big)\bar{u}_t^\theta(z')q(z',y).
\end{aligned}$$

Adding this and minus its complex conjugate with x, y exchanged (which leads to an exchange of z,z') gives with $(k^2+\omega(k))B_x(k) = G_x(k)$,

$$\int \mathrm{d}z\,\mathrm{d}z'\,q(x,z)u_t^\theta(z)\underbrace{\big(4i(1-\theta)\,\mathrm{Re}\langle B_{z'}^\Lambda,G_z^\Lambda\rangle + 4i\theta\,\mathrm{Re}\langle k^2B_{z'}^\Lambda,B_z^\Lambda\rangle\big)}_{=4i\,\mathrm{Re}\langle B_{z'}^\Lambda,G_z^\Lambda\rangle - 4\theta i\,\mathrm{Re}\langle\omega B_{z'}^\Lambda,B_z^\Lambda\rangle = -i\partial_\theta V_\theta^\Lambda}\bar{u}_t^\theta(z')q(z',y).$$

Combined with (5.17), this shows that $D\equiv 0$.

For $\widetilde{D}$ this follows from the same calculation, using that $\langle B_x^\Lambda, G_y^\Lambda\rangle = \widehat{B_0^\Lambda G_0^\Lambda}(x-y)$ is real valued since G, B are even functions. By the same reasoning, the value of the constant is

$$\begin{aligned} E_b &= -i \int \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}z'\, q(x,z) u_t^\theta(z)(2i(1-\theta)\langle B_{z'}^\Lambda, G_z^\Lambda\rangle + 2i\theta\langle k^2 B_{z'}^\Lambda, B_z^\Lambda\rangle) \\ &\qquad \times \bar{u}_t^\theta(z') q(z',x) \\ &= -\frac{i}{2}\int \mathrm{d}z\, \mathrm{d}z'\, q(z',z)(-i\partial_\theta V_\theta^\Lambda(z-z') u_t^\theta(z)\bar{u}_t^\theta(z')) \\ &= 2\operatorname{Re}\langle G_0^\Lambda, B_0^\Lambda\rangle - 2\theta \operatorname{Re}\langle \omega B_0^\Lambda, B_0^\Lambda\rangle + \partial_\theta \frac{1}{2}\langle u_t^0, V_\theta^\Lambda * |u_t^0|^2 u_t^0\rangle, \end{aligned} \tag{5.18}$$

where we used that $|u_t^\theta|^2$ is independent of θ and integrates to 1.

Mixed terms in a, a^*, b, b^* (5.15c). Mixed terms with, say, $a_k^* b_x^*$ arise from the derivatives of the respective terms in $\mathbb{H}_\theta^\Lambda$, $\mathbb{D}_t^\Lambda$, and from the commutator $[\mathbb{H}_\theta^\Lambda, \mathbb{D}_t^\Lambda]$ if one commutes a term with one $a^\#$ and one $b^\#$ with terms with two $a^\#$'s or two $b^\#$'s. The commutator

$$\Big[\int \mathrm{d}k\, \mathrm{d}l\, (2M^\Lambda(k,-l) a_k^* a_l^* + M^\Lambda(k,l) a_k^* a_l^* + \text{h.c.}), \\ \int \mathrm{d}x\, \mathrm{d}m\, (\kappa_t^\Lambda(m,x) a_m^* b_x^* - \kappa_t^\Lambda(-m,x) a_m b_x^*) + \text{h.c.}\Big]$$

vanishes identically, as one easily checks.

We group the remaining terms into two parts, $X = X^{(1)} + X^{(2)}$, which vanish separately. Spelling things out, we set

$$\begin{aligned} X^{(1)}(k,x) &:= (i\partial_\theta - \tau_t(x))(q L_\theta^\Lambda(k) u_t^\theta)(x) + (-i\partial_t + h_\theta + \omega(k))\kappa_t^\Lambda(k,x), \\ X^{(2)}(k,x) &:= \int \mathrm{d}y\, (K_\theta^{(1),\Lambda}(x,y)\kappa_t^\Lambda(k,y) + K_\theta^{(2),\Lambda}(x,y)\overline{\kappa_t^\Lambda(-k,y)}), \\ \widetilde{X}^{(1)}(k,x) &:= (i\partial_\theta - \tau_t(x))(q L_\theta^\Lambda(k)^* u_t^\theta)(x) + (i\partial_t - h_\theta + \omega(k))\kappa_t^\Lambda(-k,x), \\ \widetilde{X}^{(2)}(k,x) &:= -X^{(2)}(-k,x). \end{aligned}$$

To see that $X^{(1)}(k,x) = 0$, we use (5.16) with $T_\theta = L_\theta^\Lambda(k)$ and the identity

$$\nabla\tau_t(x) = \nabla 2\operatorname{Re}\langle i B_x, \alpha_t^\theta\rangle = 2\operatorname{Re}\langle k B_x, \alpha_t^\theta\rangle = F_{\alpha_t^\theta}(x)$$

to obtain

$$\begin{aligned} (i\partial_\theta - \tau_t(x))(q L_\theta^\Lambda(k) u_t^\theta) &= q[L_\theta^\Lambda(k), \tau_t] u_t^\theta + q(i\partial_\theta L_\theta^\Lambda(k)) u_t^\theta \\ &= q(2\theta B_{(\cdot)}^\Lambda(k) k[-i\nabla, \tau_t] - i G_{(\cdot)}^\Lambda(k) \\ &\qquad + 2i B_{(\cdot)}^\Lambda(k) k(-i\nabla + 2\theta F_{\alpha_t^\theta})) u_t^\theta \\ &= q(-i G_{(\cdot)}^\Lambda(k) + 2B_{(\cdot)}^\Lambda(k) k\nabla + 2i\theta B_{(\cdot)}^\Lambda(k) k F_{\alpha_t^\theta}) u_t^\theta. \end{aligned} \tag{5.19}$$

Using

$$[-\Delta, B_x^\Lambda(k)] = k^2 B_x^\Lambda(k) + 2kB_x^\Lambda(k)i\nabla,$$
$$[A_\alpha, B_x^\Lambda(k)] = [2(-i\nabla)\langle kB_x, \alpha\rangle + \text{h.c.}, B_x^\Lambda(k)] = -2kB_x^\Lambda(k)\underbrace{2\,\mathrm{Re}\langle kB_x, \alpha\rangle}_{=F_\alpha(x)},$$

we find in the same way that

$$\begin{aligned}(-i\partial_t + h_\theta + \omega(k))\kappa_t^\Lambda(k,x) &= q(i\omega(k)B_{(\cdot)}^\Lambda + [-\Delta + \theta A_{\alpha_t^\theta}, iB_{(\cdot)}^\Lambda(k)])u_t^\theta \\ &= q(iG_{(\cdot)}^\Lambda - 2B_{(\cdot)}^\Lambda(k)k\nabla - 2i\theta B_{(\cdot)}^\Lambda(k)kF_{\alpha_t^\theta})u_t^\theta.\end{aligned}$$

This equals the negative of (5.19), so $X^{(1)} \equiv 0$.

The equality $X^{(2)} \equiv 0$ follows simply by expanding the expressions:

$$\begin{aligned}&\int \mathrm{d}y\, K_\theta^{(2),\Lambda}(x,y)\overline{\kappa_t^\Lambda(-k,y)} \\ &\quad = \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}z''\, q(x,z)q(y,z')u_t^\theta(z)u_t^\theta(z')V_\theta^\Lambda(z-z')\overline{iB_{z''}^\Lambda(-k)u_t^\theta(z'')q(y,z'')} \\ &\quad = \int \mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}z''\, q(x,z)q(z'',z')u_t^\theta(z)u_t^\theta(z')V_\theta^\Lambda(z-z')\bar{u}_t^\theta(z'')(-iB_{z''}^\Lambda(k)) \\ &\quad = -\int \mathrm{d}z\,\mathrm{d}z'\, q(x,z)u_t^\theta(z)|u_t^\theta(z')|^2 V_\theta^\Lambda(z-z')iB_{z'}^\Lambda(k) \\ &\qquad + i\int \mathrm{d}z\, q(x,z)(V_\theta^\Lambda * |u_t^\theta|^2)(z)\langle u_t^\theta, B_\cdot^\Lambda(k)u_t^\theta\rangle \\ &\quad = -\int \mathrm{d}y\, K_\theta^{(1),\Lambda}(x,y)\kappa_t^\Lambda(k,y),\end{aligned}$$

where the last equality is obtained by performing the same calculation for $K^{(1)}\kappa$, which just changes the location of some complex conjugates. This implies vanishing of $\widetilde{X}^{(2)}$ and the argument for $\widetilde{X}^{(1)}$ is completely analogous to that for $X^{(1)}$.

Terms quadratic in a, a^* (5.15d). The only way to obtain a term with two $a^\#$'s from the commutator $[\mathbb{H}_\theta^\Lambda, \mathbb{D}_t^\Lambda]$ is to commute two terms with one $a^\#$ and one $b^\#$ each. Since the coefficient $M^\Lambda(k,l)$ (5.9b) of the terms with two $a^\#$'s in $\mathbb{H}_\theta^\Lambda(t)$ is independent of θ, we obtain, for the coefficients in (5.15d),

$$\begin{aligned}A(k,l) &= -4i\theta M^\Lambda(k,-l) \\ &\qquad - \int \mathrm{d}x\,\Big((qL_\theta^\Lambda(k)u_t^\theta)(x)\overline{\kappa_t^\Lambda(l,x)} + \overline{(qL_\theta^\Lambda(k)^*u_t^\theta)(x)}\kappa_t^\Lambda(-l,x)\Big) \\ &\qquad + \int \mathrm{d}x\,\Big(\overline{(qL_\theta^\Lambda(l)u_t^\theta)(x)}\kappa_t^\Lambda(k,x) + (qL_\theta^\Lambda(l)^*u_t^\theta)(x)\overline{\kappa_t^\Lambda(-k,x)}\Big),\end{aligned} \tag{5.20}$$

$$E_a = -i\int \mathrm{d}k\,\mathrm{d}x\,\Big(\overline{(qL_\theta^\Lambda(k)u_t^\theta)(x)}\kappa_t^\Lambda(k,x) + (qL_\theta^\Lambda(k)^*u_t^\theta)(x)\overline{\kappa_t^\Lambda(-k,x)}\Big)$$

$$\tilde{A}(k,l) = 2i\theta M^\Lambda(k,l) + \int \mathrm{d}x\,\Big(\overline{(qL_\theta^\Lambda(k)^*u_t^\theta)(x)}\kappa_t^\Lambda(l,x) + (qL_\theta^\Lambda(k)u_t^\theta)(x)\overline{\kappa_t^\Lambda(-l,x)}\Big).$$

To see that $A(k,l) = 0$, we first calculate using that $q^2 = q = 1 - |u_t^\theta\rangle\langle u_t^\theta|$ and

$$\begin{aligned}
\int \mathrm{d}x\, \big((qL_\theta^\Lambda(k)u_t^\theta)(x)\overline{\kappa_t^\Lambda(l,x)} + \overline{qL_\theta^\Lambda(k)^* u_t^\theta(x)}\kappa_t^\Lambda(-l,x)\big) \\
&= \int \mathrm{d}z\, \mathrm{d}z'\, q(z',z)\big((L_\theta^\Lambda(k)u_t^\theta)(z)(-iB_{z'}^\Lambda(-l))\bar{u}_t^\theta(z') \\
&\qquad + \overline{(L_\theta^\Lambda(k)^* u_t^\theta)(z')}iB_z^\Lambda(-l)u_t^\theta(z)\big) \\
&= \langle u_t^\theta, i[L_\theta^\Lambda(k), B_{(\cdot)}^\Lambda(-l)]u_t^\theta\rangle \\
&= 2i\theta\langle u_t^\theta, B_{(\cdot)}^\Lambda(-l)lkB_{(\cdot)}^\Lambda(k)u_t^\theta\rangle = -2i\theta M^\Lambda(k,-l).
\end{aligned}$$

The second line in (5.20) is the complex conjugate of this with k, l exchanged, so it equals $2i\theta M^\Lambda(k,-l)$. This implies that $A \equiv 0$. The argument for $\tilde{A} \equiv 0$ is essentially the same.

It remains to evaluate E_a. We have by the calculation of $A(k,l)$,

$$E_a = 2\theta \int \mathrm{d}k\, M^\Lambda(k,-k) = -2\theta\langle k^2 B_0^\Lambda, B_0^\Lambda\rangle.$$

Consequently, with (5.18),

$$\begin{aligned}
E_a + E_b &= 2\,\mathrm{Re}\langle G_0^\Lambda, B_0^\Lambda\rangle - 2\theta\,\mathrm{Re}\langle (k^2+\omega)B_0^\Lambda, B_0^\Lambda\rangle + \partial_\theta \frac{1}{2}\langle u_t^0, V_\theta^\Lambda * |u_t^0|^2 u_t^0\rangle \\
&= (2-2\theta)\,\mathrm{Re}\langle G_0^\Lambda, B_0^\Lambda\rangle + \partial_\theta \frac{1}{2}\langle u_t^0, V_\theta^\Lambda * |u_t^0|^2 u_t^0\rangle = \partial_\theta E_\theta^\Lambda(t).
\end{aligned}$$

This completes the proof of the proposition. ∎

6. Estimates for the generators

In this section we establish the inequalities on the different generators of the dynamics considered in the previous sections. This includes the generators of the fluctuation dynamics for $e^{-itH_N^{\mathrm{D}}}$ and its Bogoliubov approximation used in the proofs of Theorems 3.2 and 4.2, given in Sections 6.4 and 6.3, respectively. Similar bounds for the generators associated with the dressing flow and its Bogoliubov approximation are given in Section 6.5.

6.1. Fock space operator bounds

We start by proving a general bound for operators on Fock space that will prove very useful.

Lemma 6.1. *Let $n_a, n_b, m_a, m_b \in \mathbb{N}_0$, $M = n_a + n_b + m_a + m_b$, $s, t \in \mathbb{R}$ and*

$$T\colon L^2(\mathbb{R}^3)^{\otimes m_b} \otimes \mathfrak{h}_t^{\otimes m_a} \to L^2(\mathbb{R}^3)^{\otimes n_b} \otimes \mathfrak{h}_{-s}^{\otimes n_a}$$

be a bounded operator of norm τ with an integral kernel $T((K,X),(L,Y)) \in \mathcal{S}'(\mathbb{R}^{3M})$. Set

$$A_n(K) = \prod_{i=1}^n a_{k_i} \quad \text{and} \quad B_n(X) = \prod_{i=1}^n b_{x_i}.$$

Then for all $0 \le r_b \le n_b + m_b$, $0 \le r_a \le n_a + m_a$,

$$\begin{aligned}&\left|\left\langle \chi, \int_{\mathbb{R}^{3M}} T(X,K,Y,L) B^*_{n_b}(X) A^*_{n_a}(K) A_{m_a}(L) B_{m_b}(Y)\, \mathrm{d}X\, \mathrm{d}K\, \mathrm{d}Y\, \mathrm{d}L\, \xi \right\rangle\right| \\ &\quad \le \tau \|(\mathcal{N}_b + M)^{\frac{n_b+m_b-r_b}{2}} (\mathcal{N}_a + M + 1)^{r_a}\, \mathrm{d}\Gamma_a\,(\omega^{2s})^{\frac{n_a}{2}} \chi\| \\ &\qquad \times \|(\mathcal{N}_b + M)^{\frac{r_b}{2}} (\mathcal{N}_a + 1)^{-r_a}\, \mathrm{d}\Gamma_a\,(\omega^{2t})^{\frac{m_a}{2}} \xi\|.\end{aligned}$$

Proof. To keep the notation manageable we give the proof in the case $n_b = m_b = 0$; the generalization is straightforward. Set $n = n_a$, $m = m_a$, and let $\chi \in D(\mathrm{d}\Gamma(\omega^{2s})^{n/2})$. Then

$$\begin{aligned}\int \left\| \prod_{i=1}^n \omega^s(k_i) a_{k_i} \chi \right\|^2 \mathrm{d}K &= \int \left(\prod_{i=1}^n \omega^{2s}(k_i) \right) \langle \chi, a^*_{k_1} \cdots a^*_{k_n} a_{k_n} \cdots a_{k_1} \chi \rangle\, \mathrm{d}K \\ &= \|\mathrm{d}\Gamma_a(\omega^{2s})^{n/2} \chi\|^2,\end{aligned}$$

so

$$(k_1, \dots, k_n) \mapsto \left(\prod_{i=1}^n \omega^s(k_i) a_{k_i} \right) \chi \in L^2(\mathbb{R}^{3n}, \mathcal{F} \otimes \mathcal{F}),$$

and analogously for ξ. Hence

$$\begin{aligned}&\left|\left\langle \chi, \int T(K,L) A^*_n(K) A_m(L)\, \mathrm{d}K\, \mathrm{d}L\, \xi \right\rangle\right| \\ &\quad = \left| \int \mathrm{d}K\, \mathrm{d}L \left\langle (\mathcal{N}_a + 1)^{r_a} \left(\prod_{i=1}^n \omega^s(k_i) a_{k_i} \right) \chi, \right.\right. \\ &\qquad\qquad \left.\left. T(K,L) (\mathcal{N}_a + 1)^{-r_a} \left(\prod_{i=1}^n \omega^{-s}(k_i) \right) A_m(L) \xi \right\rangle \right| \\ &\quad \le \|\mathrm{d}\Gamma_a(\omega^{2s})^{\frac{n}{2}} (\mathcal{N}_a + n + 1)^{r_a} \chi\| \\ &\qquad \times \left\| \int \left(\prod_{i=1}^n \omega^{-s}(k_i) \right) T(K,L) A_m(L) (\mathcal{N}_a + m + 1)^{-r_a} \xi\, \mathrm{d}L \right\|_{L^2(\mathbb{R}^{3n}_K, \mathcal{F}\otimes\mathcal{F})} \\ &\quad \le \|T\|_{\mathfrak{h}_t^{\otimes m} \to \mathfrak{h}_{-s}^{\otimes n}} \|\mathrm{d}\Gamma_a(\omega^{2s})^{\frac{n}{2}} (\mathcal{N}_a + M + 1)^{r_a} \chi\|\, \|\mathrm{d}\Gamma_a(\omega^{2t})^{\frac{m}{2}} (\mathcal{N}_a + 1)^{-r_a} \xi\|.\end{aligned}$$

This proves the claim. ∎

Two special cases of the previous lemma that we use frequently are given separately below.

Lemma 6.2. *For any* $s \in \mathbb{R}$ *and* $\chi, \xi \in \mathcal{F} \otimes \mathcal{F}$, *we have*

$$|\langle \chi, \mathrm{d}\Gamma_a(\omega^{s/2}) \xi \rangle| \le \|\mathrm{d}\Gamma_a(\omega^s)^{1/2} \chi\| \|\mathcal{N}_a^{1/2} \xi\|, \tag{6.1a}$$

$$\|\mathrm{d}\Gamma_a(\omega^{s/2}) \chi\| \le \|\mathcal{N}_a^{1/2} \mathrm{d}\Gamma_a(\omega^s)^{1/2} \chi\|. \tag{6.1b}$$

Proof. The first inequality is a special case of Lemma 6.1 ($t = 0$, $T\alpha = \omega^s \alpha$, $m_a = n_a = 1$, $r_a = 0$). The second inequality follows from the first by taking the supremum over $\|\xi\| = 1$. ∎

6.2. Preliminary estimates

Here we provide some bounds on the terms appearing in the mean-field Hamiltonians as well as the kernels of the Bogoliubov Hamiltonians. These will later be combined with the operator bounds from the previous section to prove the estimates for the generators.

Lemma 6.3. *Let G, B, and V be defined as in (2.10) and (3.2). For every $s > 0$ there exists a constant $C > 0$ such that*

$$\|\widehat{V}\|_{L^{1+s}} \leq C, \tag{6.2a}$$

$$\|kB_0\|_{\mathfrak{h}_{-s}} \leq C, \tag{6.2b}$$

$$\forall n \in \mathbb{N}_0,\ \langle |k|^n B_0, |\alpha|\rangle \leq C\|\alpha\|_{\mathfrak{h}_{(n-1)+s}}, \tag{6.2c}$$

$$\forall n \in \{1,2,3\},\ \|\langle k^n B_{(\cdot)}, \alpha\rangle u\|_{L^2} \leq C\|\alpha\|_{\mathfrak{h}_{(n-2)/2+s}}\|u\|_{H^{n/2}} \tag{6.2d}$$

for all $u \in H^n(\mathbb{R}^3)$ and $\alpha \in \mathfrak{h}_{n-1+s}$. Moreover, for every $\varepsilon > 0$ there exists $C > 0$ so that for $u \in H^1(\mathbb{R}^3)$, $\alpha \in \mathfrak{h}_{1/2}$,

$$\|\langle G_{(\cdot)}, \alpha\rangle u\|_{L^2} \leq \varepsilon(\|u\|^2_{H^1} + \|\alpha\|^2_{\mathfrak{h}_{1/2}}) + C\|u\|^2_{L^2}. \tag{6.2e}$$

Proof. In view of the formula for V, we have the identity $(2\pi)^{3/2}\widehat{V}(k) = -4G_0(k)B_0(k) + 2\omega(k)B_0^2(k)$, and the first three inequalities then follow immediately from the integrability properties of $G_0(k) = \omega(k)^{-1/2}$ and $B_0(k) = (k^2+\omega(k))^{-1}\omega(k)^{-1/2}$.

For (6.2d), we use the Fourier representation in x together with Parseval to write

$$\begin{aligned}\|\langle k^n B_{(\cdot)}, \alpha\rangle u\|^2_{L^2} &= \frac{1}{(2\pi)^3}\left\|\int \mathrm{d}k\, k^n B_0(k)\alpha(k)\int \mathrm{d}p\, e^{ip(\cdot)}\hat{u}(p-k)\right\|^2_{L^2} \\ &= \left\|\int \mathrm{d}k\, k^n B_0(k)\alpha(k)\hat{u}(\cdot - k)\right\|^2_{L^2} \\ &= \int \mathrm{d}p\,\mathrm{d}k\,\mathrm{d}\ell\, \frac{k^n B_0(k)\overline{\alpha(k)}}{\omega(p-k)^{n/2}}\frac{\ell^n B_0(\ell)\alpha(\ell)}{\omega(p-\ell)^{n/2}} \\ &\qquad \times \omega(p-k)^{\frac{n}{2}}\hat{u}(p-k)\omega(p-\ell)^{\frac{n}{2}}\overline{\hat{u}(p-\ell)}.\end{aligned}$$

Since $k \mapsto |k|^n B_0(k)^2\omega(k)^{2-2s}$ and $k \mapsto \omega(k)^{-n}$ are radial and decreasing functions for $s > 0$ and $n \in \{1,2,3\}$, it follows by symmetric rearrangement that

$$\sup_{p\in\mathbb{R}^3}\int \mathrm{d}k\, \frac{|k|^n B_0(k)^2\omega(k)^{2-2s}}{\omega(p-k)^n} \leq \int \mathrm{d}k\, B_0(k)^2\omega(k)^{2-2s} \leq C. \tag{6.3}$$

With Cauchy–Schwarz we thus find

$$\begin{aligned}&\|\langle k^n B_{(\cdot)}, \alpha\rangle u\|^2_{L^2} \\ &\quad\leq \int \mathrm{d}p\,\mathrm{d}k\, d\ell\, \frac{|k|^n B_0(k)^2\omega(k)^{2-2s}}{\omega(p-k)^n}|\ell|^n\omega(\ell)^{2s-2}|\alpha(\ell)|^2\omega(p-\ell)^n|\hat{u}(p-\ell)|^2 \\ &\quad\leq C\int \mathrm{d}\ell\, \omega(\ell)^{n+2s-2}|\alpha(\ell)|^2\int \mathrm{d}p\,(|p|^2+1)^{\frac{n}{2}}|\hat{u}(p)|^2 \leq C\|\alpha\|^2_{\mathfrak{h}_{n/2+s-1}}\|u\|^2_{H^{n/2}}.\end{aligned}$$

The final inequality (6.2e) is proved in a similar way, but to obtain the small constant in front of the energy norm we start by introducing a cutoff $\Lambda < \infty$. We then bound

$$\|\langle G_{(\cdot)}, \alpha\rangle u\|_{L^2} \le \|\langle \mathbb{1}_{|\cdot|>\Lambda} G_{(\cdot)}, \alpha\rangle u\|_{L^2} + \frac{\varepsilon}{2}\|\alpha\|^2_{\mathfrak{h}_{1/2}} + \frac{1}{2\varepsilon}\|\mathbb{1}_{|\cdot|\le\Lambda} G_0\|^2_{\mathfrak{h}_{-1/2}}\|u\|_{L^2}.$$

Now the first term is treated exactly as for (6.2d), which yields

$$\|\langle \mathbb{1}_{|\cdot|>\Lambda} G_{(\cdot)}, \alpha\rangle u\|^2_{L^2} \le \|\alpha\|^2_{\mathfrak{h}_{1/2}}\|u\|^2_{H^1}\int_{|k|>\Lambda} \mathrm{d}k\, \frac{|G_0(k)|^2}{\omega(k)^3}. \tag{6.4}$$

Choosing Λ so that the final integral is less than or equal to ε^2 proves the claim. ∎

The next lemma collects bounds for the different potentials that appear in the mean-field equations introduced in Section 3.2.

Lemma 6.4. *Let V be defined by (3.2) and f_u by (3.4e). There exists a constant $C > 0$ such that for all $u \in H^1(\mathbb{R}^3)$ with $\|u\|_{L^2} = 1$,*

$$\|V * |u|^2\|_{L^\infty} + \|V^2 * |u|^2\|_{L^\infty} + \|\nabla(V * |u|^2)\|_{L^\infty} \le C\|u\|^{3/2}_{H^1}, \tag{6.5a}$$

$$\|f_u\|_{L^\infty} + \|f_u\|_{\mathfrak{h}_{1/2}} \le C\|u\|^2_{H^1}. \tag{6.5b}$$

Moreover, for every $s > 0$ there exists a constant $C > 0$ such that for all $u \in H^1(\mathbb{R}^3)$, $\|u\|_{L^2} = 1$, and $\alpha \in \mathfrak{h}_{1+s}$, the objects F_α, $g_{u,\alpha}$, $\mu_{u,\alpha}$ defined in (3.4c)–(3.4f) satisfy

$$\|F_\alpha\|_{L^\infty} \le C\|\alpha\|_{\mathfrak{h}_s}, \qquad \|g_{u,\alpha}\|_{L^\infty} + \|g_{u,\alpha}\|_{\mathfrak{h}_{1/2}} \le C\|u\|_{H^1}\|\alpha\|_{\mathfrak{h}_s},$$
$$\|\nabla F_\alpha\|_{L^\infty} \le C\|\alpha\|_{\mathfrak{h}_{1+s}}, \qquad |\mu_{u,\alpha}| \le C\|u\|^2_{H^1}\|\alpha\|^2_{\mathfrak{h}_s}.$$

Proof. For the first term in (6.5a), applying Young's inequality and Parseval,

$$\|V * |u|^2\|_{L^\infty} \le \|V\|_{L^2}\||u|^2\|_{L^2} = \|\widehat{V}\|_{L^2}\|u\|^2_{L^4} \overset{(6.2a)}{\le} C\|u\|^2_{L^4},$$

and then the Cauchy–Schwarz and Sobolev inequalities yield

$$\|u\|^2_{L^4} \le \|u\|^{1/2}_{L^2}\|u\|^{3/2}_{L^6} \le C\|u\|^{3/2}_{H^1}. \tag{6.7}$$

For the convolution involving V^2, we proceed similarly, and obtain

$$\|V^2 * |u|^2\|_{L^\infty} \le \|V^2\|_{L^2}\||u|^2\|_{L^2} = \|V\|^2_{L^4}\|u\|^2_{L^4} \le C\|V\|^2_{L^4}\|u\|^{3/2}_{H^1}.$$

By Hausdorff–Young, the Fourier transform is bounded from $L^{4/3}$ to L^4, so by (6.2a),

$$\|V\|_{L^4} \le C\|\widehat{V}\|_{L^{4/3}} \le C.$$

For the convolution involving the gradient, we use the same inequalities to estimate

$$\|\nabla(V * |u|^2)\|_{L^\infty} \le 2\|V\|_{L^4}\|\bar{u}\nabla u\|_{L^{4/3}} \le C\|\bar{u}\nabla u\|_{L^{4/3}}.$$

With Hölder and (6.7) we conclude that

$$\|\bar{u}\nabla u\|_{L^{4/3}} \le C\|u\|_{L^4}\|\nabla u\|_{L^2} \le C\|u\|_{H^1}^{3/2} \tag{6.8}$$

and thus obtain (6.5a).

In view of $f_u(k) = \langle u, kB_{(\cdot)}(k)(-i\nabla)u\rangle$, the bound on $\|f_u\|_{L^\infty}$ is obvious. To bound the $\mathfrak{h}_{1/2}$-norm of f_u, we use that

$$\|f_u\sqrt{\omega}\|_{L^2} \le 2\|kB_0\sqrt{\omega}\|_{L^4}\|\mathcal{F}[\bar{u}\nabla u]\|_{L^4}.$$

This implies the bound by the Hausdorff–Young inequality and (6.8).

The bounds on $F_\alpha(x) = 2\,\mathrm{Re}\langle kB_x, \alpha\rangle$ and ∇F_α follow directly from the fact that $|k|B(k) \in \mathfrak{h}_{-s}$. The bound on F_α implies the bound on the L^∞-norm of $g_{u,\alpha} = 2\langle u, kB_{(\cdot)}(k)F_\alpha u\rangle$. For the $\mathfrak{h}_{1/2}$-norm, writing $g_{u,\alpha} = 2kB_0 \cdot \mathcal{F}[F_\alpha|u|^2]$ gives

$$\|g_{u,\alpha}\sqrt{\omega}\|_{L^2} \le 2\|kB_0\sqrt{\omega}\|_{L^4}\|\mathcal{F}[F_\alpha|u|^2]\|_{L^4} \le C\|F_\alpha|u|^2\|_{L^{4/3}} \le C\|F_\alpha\|_{L^\infty}\||u|^2\|_{L^{4/3}}.$$

The claimed bound then follows from (6.7) by Hölder's inequality, since $\|u\|_{H^1} \ge 1$.

The estimate for $\mu_{u,\alpha} = \frac{1}{2}\langle u, V * |u|^2 u\rangle + \mathrm{Re}\langle\alpha, f_u\rangle + \mathrm{Re}\langle\alpha, g_{u,\alpha}\rangle$ follows from the previous bounds. ∎

We have similar bounds on the mean-field Hamiltonian.

Lemma 6.5. *Let $h_{u,\alpha}$ be defined by (3.4a). For every $s > 0$ there is a constant $C > 0$ such that for all $(u,\alpha) \in H^3 \oplus \mathfrak{h}_{1+s}$ with $\|u\|_{L^2} = 1$,*

$$\|h_{u,\alpha}u\|_{L^2} \le C(\|u\|_{H^2} + \|u\|_{H^1}^2)(1 + \|\alpha\|_{\mathfrak{h}_s}^2),$$
$$\|\nabla h_{u,\alpha}u\|_{L^2} \le C(\|u\|_{H^3} + \|u\|_{H^1}^{5/2})(1 + \|\alpha\|_{\mathfrak{h}_{1+s}}\|\alpha\|_{\mathfrak{h}_s}^2).$$

Proof. The proof follows from Lemmas 6.3 and 6.4 in combination with (we use $\|\cdot\|_{H^s} \ge 1$ and $\|\cdot\|_{\mathfrak{h}_s} \ge \|\cdot\|_{\mathfrak{h}_r}$ for $s \ge r$)

$$\begin{aligned}\|h_{u,\alpha}u\|_{L^2} &\le \|\Delta u\|_{L^2} + 2\|\langle k^2 B_{(\cdot)}, \alpha\rangle u\|_{L^2} + 4\langle |k|B_0, |\alpha|\rangle\|\nabla u\|_{L^2}\\ &\quad + \|F_\alpha\|_{L^\infty}^2 + \|V * |u|^2\|_{L^\infty} + |\mu_{u,\alpha}|\\ &\le C(\|u\|_{H^2} + \|u\|_{H^1}^2)(1 + \|\alpha\|_{\mathfrak{h}_s}^2)\end{aligned}$$

and

$$\begin{aligned}\|\nabla h_{u,\alpha}u\|_{L^2} &\le \|\nabla\Delta u\|_{L^2} + 2\|\langle k^3 B_{(\cdot)}, \alpha\rangle u\|_{L^2} + 6\|\langle k^2 B_{(\cdot)}, \alpha\rangle\nabla u\|_{L^2}\\ &\quad + 4\langle |k|B_0, |\alpha|\rangle\|\Delta u\|_{L^2} + 2\|F_\alpha\|_{L^\infty}\|\nabla F_\alpha\|_{L^\infty} + \|F_\alpha\|_{L^\infty}^2\|\nabla u\|_{L^2}\\ &\quad + \|\nabla(V * |u|^2)\|_{L^\infty} + \|V * |u|^2\|_{L^\infty}\|\nabla u\|_{L^2} + |\mu_{u,\alpha}|\|\nabla u\|_{L^2}\\ &\le C(\|u\|_{H^3} + \|u\|_{H^1}^{5/2})(1 + \|\alpha\|_{\mathfrak{h}_{1+s}}\|\alpha\|_{\mathfrak{h}_s}^2).\end{aligned}$$

∎

Next, we state suitable bounds for the time derivatives of u_t, α_t, and μ_{u_t,α_t}. Note that the constant C in the bound is uniform in t but not in (u,α), as it depends on the energy of the initial condition.

Lemma 6.6. *Let* $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$ *and* $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ *denote the solution to* (3.3). *Let* μ_{u_t,α_t} *be defined as in* (3.4d). *There exists a constant* $C > 0$ *such that for all* $|t| \geq 0$,

$$\|\dot{u}_t\|_{H^1} + \|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}} + |\dot{\mu}_{u_t,\alpha_t}| \leq C\|u_t\|_{H^3}(1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}}).$$

Proof. We use that $\|\dot{u}_t\|_{H^1}^2 = \|\dot{u}_t\|_{L^2}^2 + \|\nabla \dot{u}_t\|_{L^2}^2$ and $i\dot{u}_t = h_{u_t,\alpha_t}u_t$. Since, by Proposition 2.3 and Lemma 3.3, $\|u_t\|_{H^1} + \|\alpha_t\|_{\mathfrak{h}_{1/2}} \leq C$, we obtain from Lemma 6.5 for $s = \frac{1}{2}$

$$\|h_{u_t,\alpha_t}u_t\|_{L^2} \leq C\|u_t\|_{H^2} \quad \text{and} \quad \|\nabla h_{u_t,\alpha_t}u_t\|_{L^2} \leq C\|u_t\|_{H^3}(1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}}).$$

With the aid of Lemma 6.4, one easily verifies

$$\|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}} \leq \|\alpha_t\|_{\mathfrak{h}_{3/2}} + \|u_t\|_{H^1}^2\|\alpha_t\|_{\mathfrak{h}_{1/2}}.$$

Recall that $\mu_{u,\alpha} = \frac{1}{2}\langle u, V * |u|^2 u\rangle + \mathrm{Re}\langle \alpha, f_u\rangle + \mathrm{Re}\langle \alpha, g_{u,\alpha}\rangle$. Since V is an even function,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle u, V * |u|^2 u\rangle\right| = 4|\,\mathrm{Re}\langle \dot{u}_t, V * |u_t|^2 u_t\rangle| \leq 4\|V * |u_t|^2\|_{L^\infty}\|\dot{u}_t\|_{L^2} \leq C\|\dot{u}_t\|_{L^2}.$$

We further estimate

$$\begin{aligned}
\|\dot{f}_{u_t}\|_{\mathfrak{h}_{-1/2}} &\leq 2\|kB_0\|_{\mathfrak{h}_{-1/2}}(\|\dot{u}_t\|_{L^2}\|u_t\|_{H^1} + \|\dot{u}_t\|_{H^1}\|u_t\|_{L^2}) \leq C\|\dot{u}_t\|_{H^1},\\
\|\dot{g}_{u_t,\alpha_t}\|_{\mathfrak{h}_{-1/2}} &\leq 4\|\langle \dot{u}_t, kB_{(\cdot)}\cdot F_{\alpha_t}u_t\rangle\|_{\mathfrak{h}_{-1/2}} + 2\|\langle u_t, kB_{(\cdot)}\cdot F_{\dot{\alpha}_t}u_t\rangle\|_{\mathfrak{h}_{-1/2}}\\
&\leq (2\|\dot{u}_t\|_{L^2}\|F_{\alpha_t}\|_{L^\infty} + \|F_{\dot{\alpha}_t}\|_{L^\infty})\|kB_0\|_{\mathfrak{h}_{-1/2}}.
\end{aligned}$$

With (6.2c) we have $\|F_{\dot{\alpha}_t}\|_{L^\infty} \leq C\|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}}$, and hence

$$\begin{aligned}
|\dot{\mu}_{u_t,\alpha_t}| &\leq C\|\dot{u}_t\|_{L^2} + \|\dot{\alpha}_t\|_{L^2}\|f_{u_t} + g_{u_t,\alpha_t}\|_{L^2} + \|\alpha_t\|_{\mathfrak{h}_{1/2}}\|\dot{f}_{u_t} + \dot{g}_{u_t,\alpha_t}\|_{\mathfrak{h}_{-1/2}}\\
&\leq C(\|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}} + \|\dot{u}_t\|_{H^1}).
\end{aligned}$$

This completes the proof of the lemma. ∎

The next lemma summarizes estimates for the different kernels (and their time derivatives) that appear in the (dressed) Bogoliubov Hamiltonian (4.1) and the fluctuation Hamiltonian introduced in (3.7), given explicitly by (6.24).

We introduce the integral kernels

$$N_u(x,k,l) := (q_u kB_{(\cdot)}(k)\cdot lB_{(\cdot)}(l)u)(x), \tag{6.9a}$$

$$Q_u(x,y,k,l) := (q_u kB_{(\cdot)}(k)\cdot lB_{(\cdot)}(l)q_u)(x,y), \tag{6.9b}$$

where we note that $kB_x(k)\cdot lB_x(l)$ acts as a multiplication operator in x. Recalling the definition $(L_\alpha(k)f)(x) = 2kB_x(k)((-i\nabla + F_\alpha(x))f)(x)$ from (4.2a), we set

$$\ell_t^{(1)}(x,k) := (q_{u_t}L_{\alpha_t}(k)u_t)(x), \quad \ell_t^{(2)}(x,k) := (q_{u_t}L_{\alpha_t}(k)^* u_t)(x).$$

Moreover, for $u \in L^2(\mathbb{R}^3)$ and $\{u\}^\perp \subset L^2(\mathbb{R}^3)$, we consider the operators

$$\begin{aligned}
&K_u^{(3)}\colon \{u\}^\perp \to \{u\}^\perp \otimes \{u\}^\perp,\\
&\psi \mapsto (K_u^{(3)}\psi)(x_1,x_2) := (q_u)_1(q_u)_2 W_u(x_1,x_2)u(x_1)(q_u\psi)(x_2), && (6.10a)\\
&K_u^{(4)}\colon \{u\}^\perp \otimes \{u\}^\perp \to \{u\}^\perp \otimes \{u\}^\perp,\\
&\psi \mapsto (K_u^{(4)}\psi)(x_1,x_2) := (q_u)_1(q_u)_2 W_u(x_1,x_2)(q_u \otimes q_u\psi)(x_1,x_2). && (6.10b)
\end{aligned}$$

with, for V defined in (3.2),

$$W_u(x_1,x_2) := V(x_1-x_2) - V * |u|^2(x_1) - V * |u|^2(x_2) + \langle u, V * |u|^2 u\rangle. \quad (6.11)$$

Lemma 6.7. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$, denote by $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ the solution to (3.3), and $\rho(t) = \|u_t\|_{H^3}^2(1+\|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$. There exists a constant $C > 0$ such that for all $|t| \geq 0$,*

$$\begin{aligned}
&\|K_{u_t}^{(2)}\|_{L^2(\mathbb{R}^6)} \leq C, && \|\dot K_{u_t}^{(2)}\|_{L^2(\mathbb{R}^6)} \leq C\sqrt{\rho(t)}, && (6.12a)\\
&\|K_{u_t}^{(1)}\|_{L^2\to L^2} \leq C, && \|\dot K_{u_t}^{(1)}\|_{L^2\to L^2} \leq C\sqrt{\rho(t)}, && (6.12b)\\
&\|\ell_t^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/4}} \leq C, && \|\dot\ell_t^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/4}} \leq C\sqrt{\rho(t)}, && (6.12c)\\
&\|\ell_t^{(2)}\|_{\mathfrak{h}_{1/4}\to L^2}^2 \leq C, && \|\dot\ell_t^{(2)}\|_{\mathfrak{h}_{1/4}\to L^2} \leq C\sqrt{\rho(t)}, && (6.12d)\\
&\|M_{u_t}\|_{(L^2)^{\otimes 2}} \leq C, && \|\dot M_{u_t}\|_{(L^2)^{\otimes 2}} \leq C\sqrt{\rho(t)}, && (6.12e)\\
&\|K_{u_t}^{(3)}\|_{L^2\to L^2\otimes L^2} \leq C, && \|\dot K_{u_t}^{(3)}\|_{L^2\to L^2\otimes L^2} \leq C\sqrt{\rho(t)}, && (6.12f)\\
&\|N_{u_t}\|_{L^2\otimes\mathfrak{h}_{-1/8}^{\otimes 2}} \leq C, && \|\dot N_{u_t}\|_{L^2\otimes\mathfrak{h}_{-1/8}^{\otimes 2}} \leq C\sqrt{\rho(t)}, && (6.12g)\\
&\|Q_{u_t}\|_{L^2\otimes\mathfrak{h}_{1/8}^{\otimes 2}\to L^2} \leq C, && \|\dot Q_{u_t}\|_{L^2\otimes\mathfrak{h}_{1/8}^{\otimes 2}\to L^2} \leq C\sqrt{\rho(t)}, && (6.12h)
\end{aligned}$$

where L^2 stands for $L^2(\mathbb{R}^3)$.

Let us note the evident fact that the bounds (6.12a)–(6.12e) also hold uniformly for the Λ-dependent kernels introduced in (5.8). For instance, $\|K_{1,u_t}^{(2),\Lambda}\|_{L^2(\mathbb{R}^6)} \leq C$ for all $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$ with $K_{1,u_t}^{(2),\Lambda}$ defined by (5.9d). While we do not state them explicitly here, such uniform bounds will be used in the proofs of Lemmas 6.8 and 6.9.

Proof. Recall that $\|u_t\|_{L^2} = 1$ and $\|u_t\|_{H_1} + \|\alpha_t\|_{\mathfrak{h}_{1/2}} \leq C$ for all $|t| \geq 0$ by Proposition 2.3 and Lemma 3.3. We go through the claimed bounds line by line.

Line (6.12a). We use that $\|q_{u_t}\| = 1$, so $\|K_{u_t}^{(2)}\|_{L^2(\mathbb{R}^6)}^2 \leq \|\widetilde K_{u_t}^{(2)}\|_{L^2(\mathbb{R}^6)}$ and

$$\begin{aligned}
\|\widetilde K_{u_t}^{(2)}\|_{L^2(\mathbb{R}^6)}^2 &= \int \mathrm{d}x\,\mathrm{d}y\,|u_t(x)|^2 V^2(x-y)|u_t(y)|^2\\
&\leq \|V^2 * |u_t|^2\|_{L^\infty}\|u_t\|_{L^2}^2 \leq C && (6.13)
\end{aligned}$$

by Lemma 6.4. Invoking

$$\dot{K}^{(2)}_{u_t} = \dot{q}_{u_t} \otimes q_{u_t} \widetilde{K}^{(2)}_{u_t} + q_{u_t} \otimes \dot{q}_{u_t} \widetilde{K}^{(2)}_{u_t} + q_{u_t} \otimes q_{u_t} \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{K}^{(2)}_{u_t},$$

together with $\dot{q}_{u_t} = -\dot{p}_{u_t}$, $\|\dot{p}_{u_t}\|_{L^2\to L^2} \le 2\|\dot{u}_t\|_{L^2}$, and $\|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{K}^{(2)}_{u_t}\|_{L^2(\mathbb{R}^6)} \le C\|\dot{u}_t\|_{L^2}$ as in (6.13), we obtain

$$\|\dot{K}^{(2)}_{u_t}\|_{L^2(\mathbb{R}^6)} \le C\|\dot{u}_t\|_{L^2} \le C\sqrt{\rho(t)},$$

where we employed Lemma 6.6 in the last step.

Line (6.12b). Since $|K^{(1)}_{u_t}(x,y)| = |K^{(2)}_{u_t}(x,y)|$ we can use that the operator norm is bounded by the Hilbert–Schmidt norm, $\|K^{(1)}_{u_t}\|_{L^2\to L^2} \le \|K^{(2)}_{u_t}\|_{L^2(\mathbb{R}^6)}$, so that we can apply the previous bounds. The time derivative is bounded analogously.

Line (6.12c). We recall $\ell^{(1)}_t(x,k) = (q_{u_t} k B_{(\cdot)}(k)\cdot(-i\nabla + F_{\alpha_t})u_t)(x)$ and estimate

$$\|\ell^{(1)}_t\|_{L^2\otimes\mathfrak{h}_{-1/4}} \le 2\|kB_0\|_{\mathfrak{h}_{-1/4}}\|(-i\nabla + F_{\alpha_t})u_t\|_{L^2} \le C(\|u_t\|_{H^1} + \|F_{\alpha_t}\|_{L^\infty}) \le C,$$

and similarly for

$$\begin{aligned}\|\dot{\ell}^{(1)}_t\|_{L^2\otimes\mathfrak{h}_{-1/4}} &\le 2\|kB_0\|_{\mathfrak{h}_{-1/4}}\left\|\frac{\mathrm{d}}{\mathrm{d}t} q_{u_t} e^{-ik(\cdot)}(-i\nabla + F_{\alpha_t})u_t\right\|_{L^2}\\ &\le C(\|\dot{u}_t\|_{H^1} + \|F_{\dot{\alpha}_t}\|_{L^\infty}) \le C(\|\dot{u}_t\|_{H^1} + \|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}}) \le C\sqrt{\rho(t)},\end{aligned}$$

where we used $\|\dot{q}_{u_t}\|_{L^2\to L^2} \le 2\|\dot{u}_t\|_{L^2} \le 2\|\dot{u}_t\|_{H^1}$ and $\|F_{\dot{\alpha}_t}\|_{L^\infty} \le C\|\dot{\alpha}_t\|_{\mathfrak{h}_{1/2}}$.

Line (6.12d). We have $\ell^{(2)}_t(x,k) = \ell^{(1)}_t(x,k) + 2(q_{u_t} k^2 B_{(\cdot)}(k) u_t)(x)$, and thus

$$\begin{aligned}\|\ell^{(2)}_t\|_{\mathfrak{h}_{1/4}\to L^2} &\le \|\ell^{(1)}\|_{\mathfrak{h}_{1/4}\to L^2} + 2\sup_{\|\eta\|_{\mathfrak{h}_{1/4}}=1}\left\|\int \mathrm{d}k\, k^2 B_{(\cdot)}(k)\eta(k)u_t\right\|_{L^2}\\ &\le \|\ell^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/4}} + C\|u_t\|_{H^1}\end{aligned}$$

by (6.2d) (with $\alpha = \eta$, $u = u_t$, $n = 2$, and $s = 1/4$). Similarly, for the time derivative,

$$\|\dot{\ell}^{(2)}_t\|_{\mathfrak{h}_{1/4}\to L^2} \le \|\dot{\ell}^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/4}} + C\|\dot{u}_t\|_{H^1} \le C\sqrt{\rho(t)}.$$

Line (6.12e). The estimate

$$\begin{aligned}|M_{u_t}(k,l)| &= \left|\frac{kB_0(k)\cdot lB_0(l)(2\pi)^{3/2}(k+l)\cdot\mathcal{F}[i\nabla|u_t|^2](k+l)}{|k+l|^2}\right|\\ &\le C\|u_t\nabla u_t\|_{L^1}\frac{|k|B_0(k)|l|B_0(l)}{|k+l|}\end{aligned}$$

and the Hardy–Littlewood–Sobolev inequality imply

$$\|M_{u_t}\|_{L^2(\mathbb{R}^6)} \le C\|u_t\|^2_{H^1}\||\cdot|B_0\|^2_{L^3} \le C\|u_t\|^2_{H^1}.$$

For $\dot{M}_{u_t}$ we obtain by the same argument

$$\|\dot{M}_{u_t}\|_{L^2(\mathbb{R}^6)} \le C\|u_t\|_{H^1}\|\dot{u}_t\|_{H^1},$$

which implies the claim by Lemma 6.6.

Line (6.12f). Recall the definition of W_u in (6.11), and denote $q_i = (q_{u_t})_i$. Using $\|u_t\|_{H^1} \leq C$ we get

$$\|K_{u_t}^{(3)}\|_{L^2\to L^2\otimes L^2} = \sup_{\|\psi\|=1} \|q_1q_2W_{u_t}(x_1,x_2)u_t(x_1)(q_{u_t}\psi)(x_2)\|_{L^2\otimes L^2} \\ \leq C\|q_{u_t}\|^3_{L^2\to L^2}(\|V^2 * |u_t|^2\|^{1/2}_{L^\infty} + \|V * |u_t|^2\|_{L^\infty}) \leq C \quad (6.14)$$

by Lemma 6.4. For the norm of the time derivative, one computes

$$(\dot{K}_{u_t}^{(3)}\psi)(x_1,x_2) \\ = q_1q_2\dot{W}_{u_t}(x_1,x_2)u_t(x_1)(q\psi)(x_2) \\ + \dot{q}_1q_2W_{u_t}(x_1,x_2)u_t(x_1)(q\psi)(x_2) + q_1\dot{q}_2W_{u_t}(x_1,x_2)u_t(x_1)(q\psi)(x_2) \\ + q_1q_2W_{u_t}(x_1,x_2)\dot{u}_t(x_1)(q\psi)(x_2) + q_1q_2W_{u_t}(x_1,x_2)u_t(x_1)(\dot{q}\psi)(x_2)$$

where each term can be estimated similarly to (6.14). Using Lemma 6.4 in

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}V * |u_t|^2\right\|_{L^\infty} \leq 2\|\dot{u}_t\|_{L^2}\|V^2 * |u_t|^2\|^{1/2}_{L^\infty} \leq C\|\dot{u}_t\|_{L^2}, \quad (6.15)$$

one obtains $\|\dot{W}_{u_t}\|_{L^\infty(\mathbb{R}^6)} \leq C\|\dot{u}_t\|_{L^2}$. Together with $\|\dot{q}_{u_t}\|_{L^2\to L^2} \leq 2\|\dot{u}_t\|_{L^2}$, this leads to $\|\dot{K}_{u_t}^{(3)}\|_{L^2\to L^2\otimes L^2} \leq C\sqrt{\rho(t)}$.

Lines (6.12g) *and* (6.12h). Recalling the definitions of N_t, Q_t in (6.9a), (6.9b) it follows readily that

$$\|N_{u_t}\|_{L^2\otimes\mathfrak{h}^{\otimes 2}_{-1/8}} + \|Q_{u_t}\|_{L^2\otimes\mathfrak{h}^{\otimes 2}_{1/8}\to L^2} \leq C\|kB_0\|^2_{\mathfrak{h}_{-1/8}} \leq C,$$
$$\|\dot{N}_{u_t}\|_{L^2\otimes\mathfrak{h}^{\otimes 2}_{-1/8}} + \|\dot{Q}_{u_t}\|_{L^2\otimes\mathfrak{h}^{\otimes 2}_{1/8}\to L^2} \leq C\|kB_0\|^2_{\mathfrak{h}_{-1/8}}\|\dot{u}_t\|_{L^2} \leq C\sqrt{\rho(t)}.$$

This completes the proof of the lemma. ■

6.3. Estimates for the Bogoliubov Hamiltonians

The first lemma provides bounds on the dressed Bogoliubov Hamiltonian (4.1) and its time derivative, its difference from the operator $\mathbb{T} = \mathrm{d}\Gamma_b(-\Delta) + \mathrm{d}\Gamma_a(\omega)$, and its commutator with the total number operator $\mathcal{N} = \mathcal{N}_b + \mathcal{N}_a$. These imply existence and uniqueness of the associated dynamics, as explained in Proposition 5.2. Similar bounds also hold for the family of interpolating Bogoliubov Hamiltonians (5.8) (recall that $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) = \mathbb{H}^{\infty}_{u,\alpha,1}(t)$). Note that for $\theta = 1$ the bounds in part (b) are uniform in Λ.

Lemma 6.8. *For $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ defined by* (4.1) *and $\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t)$ defined by* (5.8)*, we have the following bounds:*

(a) *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$ and $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ denote the solution to* (3.3)*. There exists a constant $C > 0$ such that for all $t \in \mathbb{R}$,*

$$\pm(\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{T}) \leq \frac{1}{2}\mathbb{T} + C(\mathcal{N} + 1), \quad (6.16a)$$

$$\pm i[\mathcal{N}, \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)] \leq \frac{1}{2}\mathbb{T} + C(\mathcal{N}+1), \tag{6.16b}$$

$$\pm \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) \leq \frac{1}{2}\mathbb{T} + C\rho(t)(\mathcal{N}+1), \tag{6.16c}$$

as quadratic forms on $\mathcal{F} \otimes \mathcal{F}$*, where* $\rho(t) = \|u_t\|^2_{H^3}(1+\|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$.

(b) *Let* $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$ *and* $(u_t, \alpha_t) := \mathfrak{s}_\theta[t](u,\alpha)$ *as defined in* (5.1)*. There exists a constant* $C > 0$ *such that for all* $t \in \mathbb{R}$*,* $|\theta| \leq 1$ *and* $\Lambda \in \mathbb{R}_+$*,*

$$\pm(\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t) - \mathbb{T}) \leq \frac{1}{2}\mathbb{T} + C(1+|1-\theta|\Lambda)(\mathcal{N}+1),$$

$$\pm i[\mathcal{N}, \mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t)] \leq \frac{1}{2}\mathbb{T} + C(1+|1-\theta|\Lambda)(\mathcal{N}+1),$$

$$\pm \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t) \leq \frac{1}{2}\mathbb{T} + C(1+|1-\theta|\Lambda)\rho(t)(\mathcal{N}+1),$$

as quadratic forms on $\mathcal{F} \otimes \mathcal{F}$*, where* $\rho(t) = \|u_t\|^2_{H^3}(1+\|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$.

Proof. The proof follows essentially by combining the operator bounds of Lemma 6.1 with the kernel bounds of Lemma 6.7. We give the details below.

Proof of (6.16a). Recall that

$$\begin{aligned}\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{T} &= \mathrm{d}\Gamma_b(A_{\alpha_t} + F^2_{\alpha_t} - \mu_{u_t,\alpha_t}) + \mathbb{K}^{(1)}_{u_t} + (\mathbb{K}^{(2)}_{u_t} + \text{h.c.}) \\ &\quad + \left(\int \mathrm{d}k\, \mathrm{d}x\, (\ell^{(1)}_t(x,k) b^*_x a^*_k + \ell^{(2)}_t(x,k) b^*_x a_k) + \text{h.c.}\right) \\ &\quad + \int \mathrm{d}k\, \mathrm{d}l\, M_{u_t}(k,l)\mathcal{A}_{kl}.\end{aligned} \tag{6.17}$$

Since $A_{\alpha_t} = 2(-i\nabla_x)\cdot\langle kB_x, \alpha_t\rangle + \text{h.c.}$, the first term in the first line is bounded, using the Cauchy–Schwarz inequality, by

$$\begin{aligned}\pm\mathrm{d}\Gamma_b(A_{\alpha_t} + F^2_{\alpha_t} - \mu_{u_t,\alpha_t}) &\leq \varepsilon\, \mathrm{d}\Gamma_b(-\Delta) + \left(\frac{4}{\varepsilon}\langle |k|B_0, |\alpha_t|\rangle^2 + \|F_{\alpha_t}\|^2_{L^\infty} + |\mu_{u_t,\alpha_t}|\right)\mathcal{N}_b \\ &\leq \varepsilon\, \mathrm{d}\Gamma_b(-\Delta) + \frac{C}{\varepsilon}\mathcal{N}_b,\end{aligned}$$

where the last bound follows from Lemmas 6.3 and 6.4, and $\|u_t\|_{H^1} + \|\alpha_t\|_{\mathfrak{h}_{1/2}} \leq C$.

For the second and third terms in the first line, we apply Lemmas 6.1 and 6.7 to get

$$\pm\mathbb{K}^{(1)}_{u_t} \leq \|K^{(1)}_{u_t}\|_{L^2\to L^2}(\mathcal{N}_b+1) \leq C(\mathcal{N}_b+1), \tag{6.18a}$$

$$\pm(\mathbb{K}^{(2)}_{u_t} + \text{h.c.}) \leq 2\|K^{(2)}_{u_t}\|_{L^2(\mathbb{R}^6)}(\mathcal{N}_b+1) \leq C(\mathcal{N}_b+1). \tag{6.18b}$$

For the mixed quadratic terms, we use Lemma 6.1 choosing $m_a = m_b = 0$, $n_a = n_b = 1$, $s = 1/2$, and $r_a = 0$, $r_b = 1$ for the term involving $b^*_x a^*_k$, and $m_a = 1 = n_b$,

$n_a = m_b = 0$, $t = 1/2$, and $r_a = r_b = 0$ for the term involving $b_x^* a_k$. This gives

$$\begin{aligned}\pm\Big(\int \mathrm{d}k\,\mathrm{d}x\,(\ell_t^{(1)}(x,k)b_x^*a_k^* + \ell_t^{(2)}(x,k)b_x^*a_k) + \text{h.c.}\Big)\\ \le \varepsilon\,\mathrm{d}\Gamma_a(\omega) + C\varepsilon^{-1}(\|\ell_t^{(1)}\|^2_{L^2\otimes\mathfrak{h}_{-1/2}} + \|\ell_t^{(2)}\|^2_{\mathfrak{h}_{1/2}\to L^2})(\mathcal{N}_b+1)\\ \le \varepsilon\,\mathrm{d}\Gamma_a(\omega) + C\varepsilon^{-1}(\mathcal{N}_b+1),\end{aligned} \tag{6.19}$$

where again we used Lemma 6.7 and monotonicity of the $\mathfrak{h}_s$-norms.

In the last term in (6.17) we have $\mathcal{A}_{kl} = -2a_k^* a_{-l} + a_k^* a_l^* + a_{-l}a_{-k}$, so using that the operator norm is bounded by the Hilbert–Schmidt norm, we have with Lemma 6.1 and (6.12e),

$$\pm\int \mathrm{d}k\,\mathrm{d}l\; M_{u_t}(k,l)\mathcal{A}_{k,l} \le C(\mathcal{N}_a+1). \tag{6.20}$$

Proof of (6.16b). The commutator is easily found to be

$$\begin{aligned}[\mathcal{N},\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)] = 2\Big(\int \mathrm{d}k\,\mathrm{d}x\;\ell_t^{(1)}(x,k)b_x^*a_k^* - \text{h.c.}\Big)\\ + 2\int \mathrm{d}k\,\mathrm{d}l\; M_{u_t}(k,l)(a_k^*a_l^* - a_{-k}a_{-l}),\end{aligned}$$

which can be estimated exactly as in (6.19) and (6.20).

Proof of (6.16c). We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t) = \mathrm{d}\Gamma_b(\dot h_{u_t,\alpha_t}) + \dot{\mathbb{K}}^{(1)}_{u_t} + (\dot{\mathbb{K}}^{(2)}_{u_t} + \text{h.c.})\\ + \Big(\int \mathrm{d}k\,\mathrm{d}x\,(\dot\ell_t^{(1)}(x,k)b_x^*a_k^* + \dot\ell_t^{(2)}(x,k)b_x^*a_k) + \text{h.c.}\Big)\\ + \int \mathrm{d}k\,\mathrm{d}l\;\dot M_{u_t}(k,l)\mathcal{A}_{kl},\end{aligned} \tag{6.21}$$

with

$$\dot h_{u_t,\alpha_t} = A_{\dot\alpha_t} + 2F_{\alpha_t}\cdot F_{\dot\alpha_t} + \frac{\mathrm{d}}{\mathrm{d}t}V * |u_t|^2 - \dot\mu_{u_t,\alpha_t}.$$

Since $A_{\dot\alpha_t} = 2(-i\nabla_x)\cdot\langle kB_x,\dot\alpha_t\rangle + \text{h.c.}$, we can use Cauchy–Schwarz and Lemma 6.3 to obtain

$$\pm\mathrm{d}\Gamma_b(A_{\dot\alpha_t}) \le \varepsilon\,\mathrm{d}\Gamma_b(-\Delta) + \frac{C}{\varepsilon}\|\dot\alpha_t\|^2_{\mathfrak{h}_{1/2}}\mathcal{N}_b.$$

Recalling $F_{\dot\alpha_t}(x) = 2\,\mathrm{Re}\langle kB_x,\dot\alpha_t\rangle$ we estimate

$$\pm\mathrm{d}\Gamma_b(F_{\alpha_t}\cdot F_{\dot\alpha_t}) \le \|F_{\alpha_t}\|_{L^\infty}\|F_{\dot\alpha_t}\|_{L^\infty}\mathcal{N}_b \le C\|\dot\alpha_t\|_{\mathfrak{h}_{1/2}}\mathcal{N}_b.$$

The time derivative of the convolution is estimated in (6.15), and thus

$$\pm\mathrm{d}\Gamma_b\Big(\frac{\mathrm{d}}{\mathrm{d}t}V * |u_t|^2\Big) \le C\|\dot u_t\|_{L^2}\mathcal{N}_b.$$

Combining the above estimates, we arrive at

$$\pm \mathrm{d}\Gamma_b(\dot{h}_{u_t,\alpha_t}) \leq \varepsilon\, \mathrm{d}\Gamma_b(-\Delta) + \frac{C}{\varepsilon}(\|\dot{\alpha}_t\|^2_{\mathfrak{h}_{1/2}} + \|\dot{u}_t\|_{L^2})\mathcal{N}_b.$$

Similarly to (6.18a) and (6.18b) we bound the remaining terms in the first line in (6.21):

$$\pm \dot{\mathbb{K}}^{(1)}_{u_t} \leq \|\dot{K}^{(1)}_{u_t}\|_{L^2\to L^2}(\mathcal{N}_b + 1) \leq C\sqrt{\rho(t)}(\mathcal{N}_b + 1),$$
$$\pm(\dot{\mathbb{K}}^{(2)}_{u_t} + \text{h.c.}) \leq 2\|\dot{K}^{(2)}_{u_t}\|_{L^2(\mathbb{R}^6)}(\mathcal{N}_b + 1) \leq C\sqrt{\rho(t)}(\mathcal{N}_b + 1).$$

By the reasoning of (6.19) we obtain

$$\pm\Big(\int \mathrm{d}k\, \mathrm{d}x\, (\dot{\ell}^{(1)}_t(x,k) b^*_x a^*_k + \dot{\ell}^{(2)}_t(x,k) b^*_x a_k) + \text{h.c.}\Big) \leq \varepsilon\, \mathrm{d}\Gamma_a(\omega) + \frac{C\rho(t)}{\varepsilon}(\mathcal{N}_b + 1),$$

and analogously to (6.20) one also verifies that

$$\pm \int \mathrm{d}k\, \mathrm{d}l\, \dot{M}_{u_t}(k,l)\mathcal{A}_{kl} \leq C\rho(t)(\mathcal{N}_a + 1).$$

The above estimates prove (6.16c) and thus complete the proof of part (a) of the lemma.

Turning to part (b), first note that for $\theta = 1$ the proof is verbatim the same as that for part (a). To see this, recall that all kernels that appear in $\mathbb{H}^{\Lambda}_{u,\alpha,1}(t)$ satisfy the same bounds as in Lemma 6.7, as explained thereafter (in particular, they are uniformly bounded in Λ).

The crucial difference for $\theta \neq 1$, apart from the trivial θ-dependence of the kernels, is the appearance of the term

$$(1-\theta)\int \mathrm{d}k\, \mathrm{d}x\, \big((q_{u_t} G^{\Lambda}_x(k) u_t) a^*_k b^*_x + (q_{u_t} \overline{G^{\Lambda}_x(k)} u_t) a^*_k b_x\big) + \text{h.c.}$$

in $\mathbb{H}^{\Lambda}_{u,\alpha,\theta}$ which is not present when $\theta = 1$ (hence this term did not appear in the proof of part (a)). Since $\|q_{u_t} G^{\Lambda}_{(\cdot)} u_t\|_{L^2\otimes L^2} \leq \|G^{\Lambda}_0\|_{L^2} \leq C\Lambda$, we can bound the above term by $C\Lambda(\mathcal{N} + 1)$. All other contributions in $\mathbb{H}^{\Lambda}_{u,\alpha,\theta}(t)$ are estimated as for $\theta = 1$, i.e., they are uniformly bounded in Λ. This explains the Λ-dependent upper bound in (6.16a). The bounds for the commutator and the time derivative are obtained in the same way. ■

The next lemma was used in the proof of Proposition 5.2 to show that $\mathbb{U}^{\Lambda}_1(t) \to \mathbb{U}^{\infty}_1(t)$ strongly as $\Lambda \to \infty$.

Lemma 6.9. *Let* $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ *with* $\|u\|_{L^2} = 1$, $(u_t,\alpha_t) := \mathfrak{s}_1[t](u,\alpha)$ *as defined in* (5.1) *and* $\rho(t) = \|u_t\|^2_{H^3}(1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$. *There is a family* $\varepsilon_\Lambda > 0$ *with* $\varepsilon_\Lambda \xrightarrow{\Lambda\to\infty} 0$ *such that*

$$|\langle \chi, (\mathbb{H}^{\infty}_{u,\alpha,1}(t) - \mathbb{H}^{\Lambda}_{u,\alpha,1}(t))\phi\rangle| \leq \varepsilon_\Lambda e^{C\int_0^{|t|}\rho(s)\,\mathrm{d}s}\|(\mathbb{T} + \mathcal{N} + 1)^{1/2}\chi\|\|(\mathbb{T} + \mathcal{N} + 1)^{1/2}\phi\|$$

for all $t \in \mathbb{R}$ *and* $\chi, \phi \in D((\mathbb{T} + \mathcal{N})^{1/2})$.

Proof. The difference $\mathbb{H}^{\infty}_{u,\alpha,1}(t) - \mathbb{H}^{\Lambda}_{u,\alpha,1}(t)$ is up to the term $\mathrm{d}\Gamma_b(h_{u_t,\alpha_t}) + \mathrm{d}\Gamma_a(\omega)$ precisely of the same form as (5.8), with all kernels replaced by kernels of the form $K^{(1),\infty}_{u_t,1} - K^{(1),\Lambda}_{u_t,1}$ and $M^{\infty}_{u_t,1}(k,l) - M^{\Lambda}_{u_t,1}(k,l)$, and analogously for the other terms. The claimed bound is now obtained following the same steps as in the proof of (6.16a) and taking into account that, by Lemma 6.7 and continuity in Λ, the norms of the kernel differences all vanish as $\Lambda \to \infty$. ∎

6.4. Fluctuation generator for the dressed dynamics

We start by stating the precise form of the fluctuation generator $H^{\mathrm{D}}_{u,\alpha}(t)$ introduced in (3.7). For $[a]_+ = \max\{0, a\}$ and $(u_t, \alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$, we set

$$
\begin{aligned}
H_0(t) := {}& \mathrm{d}\Gamma_a(\omega) + \mathrm{d}\Gamma_b(h_{u_t,\alpha_t}) + \Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+ \mathbb{K}^{(1)}_{u_t} \\
& + \Big(\mathbb{K}^{(2)}_{u_t} \frac{\sqrt{[(N-\mathcal{N}_b)(N-\mathcal{N}_b-1)]_+}}{N} + \text{h.c.}\Big) \\
& + \Big(\int \mathrm{d}k\, \mathrm{d}x\, (q_{u_t} L_{\alpha_t}(k) u_t)(x) b^*_x a^*_k \sqrt{\Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+} + \text{h.c.}\Big) \\
& + \Big(\int \mathrm{d}k\, \mathrm{d}x\, (q_{u_t} L_{\alpha_t}(k)^* u_t)(x) b^*_x a_k \sqrt{\Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+} + \text{h.c.}\Big) \\
& + \int \mathrm{d}k\, \mathrm{d}l\, M_{u_t}(k,l) \Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+ (-2a^*_k a_{-l} + a^*_k a^*_l + a_{-k} a_{-l}),
\end{aligned}
\tag{6.22}
$$

with $L_{\alpha_t}(k)$, $M_{u_t}(k,l)$, and $\mathbb{K}^{(j)}_{u_t}$ defined in (4.2a), (3.4a), and (4.2c), respectively. Note that up to the N-dependent factors $H_0(t)$ coincides with $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$. We further introduce the operators

$$
\begin{aligned}
\mathbb{K}^{(3)}_{u_t} &:= \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, K^{(3)}_{u_t}(x_1,x_2,x_3) b^*_{x_1} b^*_{x_2} b_{x_3}, \\
\mathbb{K}^{(4)}_{u_t} &:= \frac{1}{2} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4\, K^{(4)}_{u_t}(x_1,x_2,x_3,x_4) b^*_{x_1} b^*_{x_2} b_{x_3} b_{x_4},
\end{aligned}
$$

where for $u \in L^2(\mathbb{R}^3)$ and $\{u\}^\perp \subset L^2(\mathbb{R}^3)$ we used the kernels of the operators $K^{(3)}_u$, $K^{(4)}_u$ introduced in (6.10a), (6.10b). Lastly, recall (6.9a) and (6.9b) and let

$$
J_{u,\alpha}(x,k,y) := 2(q_u k B_{(\cdot)}(k) \cdot (-i\nabla + F_\alpha) q_u)(x,y). \tag{6.23}
$$

The next lemma provides an explicit formula for the fluctuation generator $H^{\mathrm{D},\le N}_{u,\alpha}(t)$ defined in (3.6).

Lemma 6.10. *For $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$, let $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ be the solution to (3.3). The operator $H^{\mathrm{D},\le N}_{u,\alpha}(t)\colon \mathcal{F}^{\le N}_{\perp u_t} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ defined by (3.6) satisfies the identity $H^{\mathrm{D},\le N}_{u,\alpha}(t) = H^{\mathrm{D}}_{u,\alpha}(t) \restriction \mathcal{F}^{\le N}_{\perp u_t} \otimes \mathcal{F}$, where $H^{\mathrm{D}}_{u,\alpha}(t)\colon \mathcal{F} \otimes \mathcal{F} \to$*

$\mathcal{F} \otimes \mathcal{F}$ is given by

$$H^{\mathrm{D}}_{u,\alpha}(t) := \sum_{j=0}^{5} H_j(t) \tag{6.24}$$

with $H_0(t)$ defined by (6.22), and

$$\begin{aligned}
H_1(t) &:= -\frac{1}{N}\mathrm{d}\Gamma_b(V * |u_t|^2 - \mu_{u_t,\alpha_t}) + \Big(\mathbb{K}^{(3)}_{u_t}\frac{\sqrt{[N-\mathcal{N}_b]_+}}{N} + \text{h.c.}\Big) + \frac{1}{N}\mathbb{K}^{(4)}_{u_t},\\
H_2(t) &:= -\frac{1}{\sqrt{N}}\mathcal{N}_b\widehat{\Phi}(f_{u_t} + g_{u_t,\alpha_t}),\\
H_3(t) &:= \frac{1}{\sqrt{N}}\int \mathrm{d}k\,\mathrm{d}x\,\mathrm{d}y\; J_{u_t,\alpha_t}(x,k,y)b^*_x b_y a^*_k + \text{h.c.},\\
H_4(t) &:= \frac{1}{\sqrt{N}}\int \mathrm{d}k\,\mathrm{d}l\,\mathrm{d}x\,\Big(N_{u_t}(x,k,l)b^*_x\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}}\\
&\qquad\qquad + \overline{N_{u_t}(x,k,l)}\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}} b_x\Big)\mathcal{A}_{kl},\\
H_5(t) &:= \frac{1}{N}\int \mathrm{d}k\,\mathrm{d}l\,\mathrm{d}x\,\mathrm{d}y\; Q_{u_t}(x,y,k,l)b^*_x b_y \mathcal{A}_{kl},
\end{aligned}$$

where $\mathcal{A}_{kl} = -2a^*_k a_{-l} + a^*_k a^*_l + a_{-k}a_{-l}$.

Proof. The proof follows from a straightforward but lengthy computation using the Weyl operator shift property (2.5) and Lemmas 2.1 and 2.2. We omit the details; similar computations in the context of N-particle Bose systems can be found in [12, 68]. ∎

The next lemma provides estimates for the fluctuation Hamiltonian that are an important ingredient of the proof of Theorem 3.2.

Lemma 6.11. *Let $(u,\alpha) \in H^3(\mathbb{R}^3) \oplus \mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$, $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ denote the solution to (3.3). There exists a constant $C > 0$ such that for all $t \in \mathbb{R}$,*

$$\pm(H^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{T}) \le \frac{1}{2}\mathbb{T} + C(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{6.25a}$$

$$\pm i[\mathcal{N}, H^{\mathrm{D}}_{u,\alpha}(t)] \le \frac{1}{2}\mathbb{T} + C(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{6.25b}$$

$$\pm\frac{\mathrm{d}}{\mathrm{d}t}H^{\mathrm{D}}_{u,\alpha}(t) \le \frac{1}{2}\mathbb{T} + C\rho(t)(\mathcal{N}+1)\Big(1 + \frac{1}{N}\mathcal{N}_b\Big)^2, \tag{6.25c}$$

as quadratic forms on $\mathcal{F} \otimes \mathcal{F}$, where $\rho(t) := \|u_t\|^2_{H^3}(1 + \|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$.

Proof of (6.25a). Comparing (6.22) with (6.17) we see that $H_0(t)$ differs from $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ only by the factors

$$0 \le \Big[1 - \frac{\mathcal{N}_b}{N}\Big]_+ \le 1, \quad 0 \le \frac{\sqrt{[(N-\mathcal{N}_b)(N-\mathcal{N}_b-1)]_+}}{N} \le 1,$$

and is thus estimated in analogy to the proof of Lemma 6.8.

For later purposes we consider $\langle \chi, H_1(t)\phi\rangle$ for $\chi, \phi \in \mathcal{F} \otimes \mathcal{F}$. The first two terms in $H_1(t)$ are estimated by

$$\Big|\Big\langle \chi, \frac{1}{N}\mathrm{d}\Gamma_b(V * |u_t|^2 - \mu_{u_t,\alpha_t})\phi\Big\rangle\Big| \le \frac{C}{N}\|\mathcal{N}_b^{\frac{1}{2}}\chi\|\|\mathcal{N}_b^{\frac{1}{2}}\phi\|, \tag{6.26}$$

where we used Lemma 6.4, and

$$\begin{aligned}
&\Big|\Big\langle \chi, \Big(\mathbb{K}^{(3)}_{u_t}\frac{\sqrt{[N-\mathcal{N}_b]_+}}{N} + \text{h.c.}\Big)\phi\Big\rangle\Big| \\
&\quad \le \frac{1}{\sqrt{N}}\|K^{(3)}_{u_t}\|_{L^2\to L^2\otimes L^2}\|(\mathcal{N}_b+1)^{\frac{1}{2}}\chi\|\|(\mathcal{N}_b+1)\phi\| \\
&\quad \le \frac{C}{\sqrt{N}}\|(\mathcal{N}_b+1)^{\frac{1}{2}}\chi\|\|(\mathcal{N}_b+1)\phi\|, \qquad (6.27)
\end{aligned}$$

by Lemmas 6.1 and 6.7.

To estimate the term involving $\mathbb{K}^{(4)}_{u_t}$ we recall (3.2) and write $W_{u_t}(x,y) = W^{\mathrm{b}}_{u_t}(x,y) - 4\,\mathrm{Re}\langle G_x, B_y\rangle$, where $W^{\mathrm{b}}_{u_t}(x,y)$ is pointwise bounded. Using the symmetry of $\chi^{(n)}, \phi^{(n)} \in \mathcal{F}^{(n)} \otimes \mathcal{F}$ in the n particle coordinates, we find

$$\langle \chi, \mathbb{K}^{(4)}_{u_t}\phi\rangle = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}\langle q_1q_2\chi^{(n)}, (W^{\mathrm{b}}_{u_t}(x_1,x_2) - 4\,\mathrm{Re}\langle G_{x_1}, B_{x_2}\rangle)q_1q_2\phi^{(n)}\rangle,$$

with

$$\Big|\sum_{n=2}^{\infty}\frac{n(n-1)}{2}\langle q_1q_2\chi^{(n)}, W^{\mathrm{b}}_{u_t}(x_1,x_2)q_1q_2\phi^{(n)}\rangle\Big| \le C\|\mathcal{N}_b^{\frac{1}{2}}\chi\|\|\mathcal{N}_b^{\frac{3}{2}}\phi\|.$$

We use the Cauchy–Schwarz inequality and the fact that

$$e^{ikx} = (1+(-i\nabla_1 - k)^2)^{-\frac{1}{2}}e^{ikx}(1-\Delta_1)^{\frac{1}{2}} \tag{6.28}$$

to bound the remaining term by

$$\begin{aligned}
&\Big|\sum_{n=2}^{\infty}\frac{n(n-1)}{2}\langle q_1q_2\chi^{(n)}, \mathrm{Re}\langle G_{x_1}, B_{x_2}\rangle q_1q_2\phi^{(n)}\rangle\Big| \\
&\quad \le C\Big(\sum_{n=0}^{\infty}n^3\|\phi^{(n)}\|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}n\|\,\mathrm{Re}\langle G_{x_1}, B_{x_2}\rangle q_1q_2\chi^{(n)}\|^2\Big)^{\frac{1}{2}} \\
&\quad \le C\|\mathcal{N}_b^{3/2}\phi\|\Big(\sum_{n=0}^{\infty}n\Big\|\int \mathrm{d}k\,(1+(-i\nabla_1-k)^2)^{-\frac{1}{2}}\overline{G_{x_1}(k)} \\
&\qquad\qquad \times B_{x_2}(k)(1-\Delta_1)^{\frac{1}{2}}q_1q_2\chi^{(n)}\Big\|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

The last factor is bounded by

$$\sup_{p\in\mathbb{R}^3}\left\{\int \mathrm{d}k\, \frac{G_0(k)B_0(k)}{(1+(p-k)^2)^{\frac12}}\right\}\left(\sum_{n=0}^{\infty} n\|(1-\Delta_1)^{\frac12}q_1\phi^{(n)}\|^2\right)^{\frac12} \le C\|\mathrm{d}\Gamma_b(1-\Delta)^{\frac12}\chi\|,$$

where the supremum over $p\in\mathbb{R}^3$ is finite by the same argument as in (6.3), and where we further used

$$\|(1-\Delta_1)^{\frac12}q_1\chi^{(n)}(t)\| \le C\|u_t\|_{H^1}\|(1-\Delta_1)^{\frac12}\chi^{(n)}(t)\|. \tag{6.29}$$

Adding up the relevant terms, we arrive at the desired estimate

$$\begin{aligned}&|\langle\chi, H_1(t)\phi\rangle|\\ &\quad\le C\|(\mathcal{N}+\mathbb{T}+1)^{\frac12}\chi\|\left(\frac{1}{\sqrt{N}}\|(\mathcal{N}_b+1)\phi\|+\frac{1}{N}\|(\mathcal{N}_b+1)\mathcal{N}_b^{\frac12}\phi\|\right).\end{aligned} \tag{6.30}$$

The bound for $H_2(t)$ is straightforward:

$$|\langle\chi, H_2(t)\phi\rangle| \le C\frac{1}{\sqrt{N}}\|(\mathcal{N}+1)^{\frac12}\chi\|\|(\mathcal{N}+1)\phi\|, \tag{6.31}$$

where we used that $\|f_{u_t}+g_{u_t,\alpha_t}\|_{L^2}\le C$ by Lemma 6.7.

With the definition of the kernel of $H_3(t)$ in (6.23), we write

$$\langle\chi, H_3(t)\chi\rangle = \sum_{n=1}^{\infty}\frac{2n}{\sqrt{N}}\int \mathrm{d}k\, kB_0(k)\cdot 2\,\mathrm{Re}\langle q_1\chi^{(n)}, e^{-ikx_1}(-i\nabla_1+F_{\alpha_t}(x_1))q_1a_k^*\chi^{(n)}\rangle,$$

with $\chi^{(n)}\in\mathcal{F}^{(n)}\otimes\mathcal{F}$. With $\|F_{\alpha_t}\|_{L^\infty}\le C$ and $\|kB_0\|_{\mathfrak{h}_{-1/4}}\le C$, we obtain by Cauchy–Schwarz,

$$\begin{aligned}|\langle\chi, H_3(t)\chi\rangle| &\le \sum_{n=1}^{\infty}\frac{4n}{\sqrt{N}}\|(-i\nabla_1+F_{\alpha_t}(x_1))q_1\chi^{(n)}\|\int \mathrm{d}k\,|kB_0(k)|\|a_k\chi^{(n)}\|\\ &\overset{(6.29)}{\le} C\sum_{n=1}^{\infty}\frac{4n}{\sqrt{N}}\|(1-\Delta_1)^{\frac12}\chi^{(n)}\|\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac12}\chi^{(n)}\|\\ &\le \frac{C}{\sqrt{N}}\|\mathrm{d}\Gamma_b(1-\Delta)^{\frac12}\chi\|\|\mathcal{N}_b^{\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac12}\chi\|\\ &\overset{(6.1a)}{\le} \frac{C}{\sqrt{N}}\|\mathrm{d}\Gamma_b(1-\Delta)^{\frac12}\chi\|\|d\Gamma_a(\omega)^{\frac12}\chi\|^{\frac12}\|\mathcal{N}_a^{\frac12}\mathcal{N}_b\chi\|^{\frac12},\end{aligned} \tag{6.32}$$

which implies the desired bound.

To bound $H_4(t)$, we apply Lemma 6.1 (for instance, for the term involving $b_x^*a_k^*a_{-l}$ we choose $m_b=0$, $n_b=m_a=n_a=1$, $s=t=1/4$, and $r_a=r_b=0$) and use from Lemma 6.7 that

$$\|N_t\|_{\mathfrak{h}_{1/4}\to L^2\otimes\mathfrak{h}_{-1/4}}+\|N_t\|_{L^2\otimes\mathfrak{h}_{1/4}^{\otimes 2}\to\mathbb{C}} \le 2\|N_t\|_{L^2\to\mathfrak{h}_{-1/4}^{\otimes 2}} \le 2\|N_t\|_{L^2\otimes\mathfrak{h}_{-1/8}^{\otimes 2}} \le C.$$

This implies

$$|\langle \chi, H_4(t)\chi\rangle| \le C\big[\|(\mathcal{N}_b+1)^{\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac12}\chi\|\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac12}\chi\| + \|(1+\mathcal{N}_b)^{\frac12}(1+\mathcal{N}_a)^{\frac12}\chi\|\|(1+\mathcal{N}_a)^{-\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})\chi\| + \|(\mathcal{N}_b+1)^{\frac12}(\mathcal{N}_a+1)^{\frac12}\chi\|\|(\mathcal{N}_a+1)^{-\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})\chi\|\big].$$

The desired bound now follows easily from (6.1a). For $H_5(t)$, we use that

$$\|Q_t\|_{L^2\otimes\mathfrak{h}_{1/4}\to L^2\otimes\mathfrak{h}_{-1/4}} + \|Q_t\|_{L^2\otimes\mathfrak{h}_{1/4}^{\otimes 2}\to L^2} \le C\|Q_t\|_{L^2\otimes\mathfrak{h}_{1/8}^{\otimes 2}\to L^2} \le C,$$

by Lemma 6.7, and Lemma 6.1 imply

$$|\langle \chi, H_5(t)\chi\rangle| \le \frac{C}{N}\big[\|(\mathcal{N}_b+1)^{\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac12}\chi\|^2 + \|(\mathcal{N}_b+1)(\mathcal{N}_a+1)^{\frac12}\chi\|\|(\mathcal{N}_a+1)^{-\frac12}\mathrm{d}\Gamma_a(\sqrt{\omega})\chi\|\big].$$

The desired bound now follows from Lemma 6.2.

Proof of (6.25b). The commutation relations imply, e.g., $[\mathcal{N}, a_k] = -a_k$, so the non-zero terms in the commutator $[\mathcal{N}, H^{\mathrm{D}}_{u,\alpha}(t)]$ have the same kernels, up to signs, as those in $H^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{T}$. They can thus be estimated as in the proof of (6.25a), and we omit the details.

Proof of (6.25c). This inequality is obtained following similar steps to the proof of (6.25a) with some obvious modifications, like the use of the bounds for the time derivatives in Lemma 6.7 and the use of $\|\dot{q}_{u_t}\|_{H^1} \le C\sqrt{\rho(t)}$, cf. Lemma 6.6. ∎

The next lemma shows that the fluctuation generator can be approximated by the Bogoliubov Hamiltonian $\mathbb{H}^{\mathrm{D}}_{u,\alpha}(t)$ for large N, when tested on suitable states.

Lemma 6.12. *Let $(u,\alpha) \in H^3(\mathbb{R}^3)\oplus\mathfrak{h}_{5/2}$ with $\|u\|_{L^2} = 1$ and $(u_t,\alpha_t) := \mathfrak{s}^{\mathrm{D}}[t](u,\alpha)$ denote the solution to* (3.3)*. There exists a constant $C > 0$ such that*

$$|\langle \chi, (H^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t))\phi\rangle| \le C\rho(t)N^{-\frac12}\ln N\|(\mathcal{N}+\mathbb{T}+1)^{\frac12}\chi\|\|(\mathcal{N}^3+\mathbb{T}+1)^{\frac12}\phi\|$$

for all $\chi \in \mathcal{F}^{\le N}\otimes\mathcal{F}$ and $\phi\in\mathcal{F}\otimes\mathcal{F}$, where $\rho(t) := \|u_t\|^2_{H^3}(1+\|\alpha_t\|_{\mathfrak{h}_{3/2}})^2$.

Proof. Recalling the definitions of the fluctuation generator (6.24) and the Bogoliubov Hamiltonian (6.17), we write

$$\langle \chi, (H^{\mathrm{D}}_{u,\alpha}(t) - \mathbb{H}^{\mathrm{D}}_{u,\alpha}(t))\phi\rangle = \Big\langle \chi, \mathbb{K}^{(1)}_{u_t}\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+ - 1\Big)\phi\Big\rangle \tag{6.33a}$$

$$+ \big\langle \chi, \big(\mathbb{K}^{(2)}_{u_t}\big(N^{-1}\sqrt{[(N-\mathcal{N}_b)(N-\mathcal{N}_b-1)]_+} - 1\big) + \mathrm{h.c.}\big)\phi\big\rangle \tag{6.33b}$$

$$+ \Big\langle \chi, \Big(\int \mathrm{d}x\,\mathrm{d}k\,(q_{u_t}L_{\alpha_t}(k)u_t)(x)a^*_k b^*_x\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac12} - 1\Big) + \mathrm{h.c.}\Big)\phi\Big\rangle \tag{6.33c}$$

$$+\left\langle \chi, \left(\int \mathrm{d}x\, \mathrm{d}k\, (q_{u_t} L_{\alpha_t}(k)^* u_t)(x) a_k b_x^* \left(\left[1 - \frac{\mathcal{N}_b}{N}\right]_+^{\frac{1}{2}} - 1 \right) + \text{h.c.} \right) \phi \right\rangle \quad (6.33\text{d})$$

$$+\left\langle \chi, \int \mathrm{d}k\, \mathrm{d}l\, M_{u_t}(k,l) \mathcal{A}_{kl} \left(\left[1 - \frac{\mathcal{N}_b}{N}\right]_+ - 1 \right) \phi \right\rangle \quad (6.33\text{e})$$

$$+ \langle \chi, (H_1(t) + H_2(t) + H_3(t) + H_4(t) + H_5(t))\phi \rangle. \quad (6.33\text{f})$$

When bounding these terms, we have to take care to put any powers of the number operator exceeding one-half to the right, i.e., on ϕ. At the same time, the power of $\mathcal{N} + \mathbb{T}$ acting on ϕ cannot exceed one-half, either. For the terms in (6.33a)–(6.33e) the estimates are rather straightforward and given at the end of the proof. The most difficult estimate is that for the term coming from $H_3(t)$, which is also responsible for the presence of the factor $\ln N$ in the statement.

Term $H_3(t)$. Recall the expression for the kernel $J_{u,\alpha} = 2(q_u k B_{(\cdot)}(k) \cdot (-i\nabla + F_\alpha) q_u)$, which multiplies $b_x^* b_y a_k^*$ in $H_3(t)$. This term is problematic, for when the gradient acts on ϕ we cannot put further powers of $\mathcal{N}$ on ϕ while keeping control by $\|(\mathcal{N}^3 + \mathbb{T})^{1/2}\phi\|$. We deal with this problem by using the identity

$$k \cdot B_{x_1}(k)(-i\nabla_1 + F_{\alpha_t}(x_1)) a_k^* = (-i\nabla_1 + F_{\alpha_t}(x_1)) \cdot k B_{x_1}(k) + a_k^* k^2 B_{x_1}(k)$$

and splitting the momentum integration into $|k| \le \Lambda$ and $|k| > \Lambda$. Together with the adjoint expression, which is less of a problem, this gives

$$\frac{1}{2}\langle \chi, H_3(t)\phi \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} n \left\langle (-i\nabla_1 + F_{\alpha_t}(x_1)) q_1 \chi^{(n)}, \int \mathrm{d}k\, k \overline{B_{x_1}(k)} a_k q_1 \phi^{(n)} \right\rangle \quad (6.34\text{a})$$

$$+ \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} n \left\langle \chi^{(n)}, q_1 \int_{|k| \ge \Lambda} \mathrm{d}k\, k B_{x_1}(k) a_k^* (-i\nabla_1 + F_{\alpha_t}(x_1)) q_1 \phi^{(n)} \right\rangle \quad (6.34\text{b})$$

$$+ \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} n \left\langle (-i\nabla_1 + F_{\alpha_t}(x_1)) q_1 \chi^{(n)}, \int_{|k| \le \Lambda} \mathrm{d}k\, k B_{x_1}(k) a_k^* q_1 \phi^{(n)} \right\rangle \quad (6.34\text{c})$$

$$+ \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} n \left\langle \int_{|k| \le \Lambda} \mathrm{d}k\, k^2 \overline{B_{x_1}(k)} a_k q_1 \chi^{(n)}, q_1 \phi^{(n)} \right\rangle. \quad (6.34\text{d})$$

In the first line, the gradient acts on χ, so we can simply bound it as in (6.32):

$$|(6.34\text{a})| \le C N^{-\frac{1}{2}} \|\mathrm{d}\Gamma_b(1-\Delta)^{\frac{1}{2}} \chi\| \|\mathrm{d}\Gamma_a(\omega)^{\frac{1}{2}} \phi\|^{\frac{1}{2}} \|(\mathcal{N}+1)^{3/2}\phi\|^{\frac{1}{2}}.$$

In the second line, we can use that $\chi^{(n)} = 0$ for $n > N$ to remove a factor of $(n/N)^{1/2}$, since the lower cutoff Λ will give us a small pre-factor. With the Cauchy–Schwarz inequality and Lemma 6.1 this gives

$$|(6.34\text{b})| \le \sum_{n=0}^{\infty} n^{\frac{1}{2}} \left| \left\langle \chi^{(n)}, q_1 \int_{|k| \ge \Lambda} \mathrm{d}k\, k B_{x_1}(k) a_k^* (-i\nabla_1 + F_{\alpha_t}(x_1)) q_1 \phi^{(n)} \right\rangle \right|$$

$$\le C \left(\sum_{n=0}^{\infty} \left\| \int_{|k| \ge \Lambda} \mathrm{d}k\, k \overline{B_{x_1}(k)} a_k q_1 \chi^{(n)} \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} n \|(1-\Delta_1)^{\frac{1}{2}} \phi^{(n)}\|^2 \right)^{\frac{1}{2}}$$

$$\leq C\|\mathbb{1}_{|\cdot|\geq\Lambda}\omega^{-\frac{1}{2}}kB_0\|_{L^2}\|\mathrm{d}\Gamma_a(\omega)^{\frac{1}{2}}\chi\|\|\mathrm{d}\Gamma_b(1-\Delta)^{\frac{1}{2}}\phi\|$$
$$\leq C\Lambda^{-\frac{1}{2}}\|\mathbb{T}^{\frac{1}{2}}\chi\|\|(\mathcal{N}+\mathbb{T})^{\frac{1}{2}}\phi\|,$$

where we used $\|\mathbb{1}_{|\cdot|\geq\Lambda}\omega^{-1/2}kB_0\|_{L^2}\leq\sqrt{4\pi/\Lambda}$ in the last step. Lemma 6.1 together with $\|\mathbb{1}_{|\cdot|\leq\Lambda}kB_0\|_2\leq\sqrt{4\pi\ln\Lambda}$ yields for the third line,

$$|(6.34c)|\leq CN^{-\frac{1}{2}}CN^{-\frac{1}{2}}\sqrt{\ln\Lambda}\|\mathrm{d}\Gamma_b(1-\Delta)^{\frac{1}{2}}\chi\|\,\|(\mathcal{N}+1)\phi\|.$$

It remains to bound the last line, (6.34d). Here, we will need to use the regularity of χ in x to improve the integrability of $k^2B_x(k)$. Using the identity (6.28) to this end, we obtain

$$\begin{aligned}\left\|\int_{|k|\leq\Lambda}\mathrm{d}k\,k^2\overline{B_{x_1}(k)}a_k\chi\right\|^2&=\int_{|k|\leq\Lambda}\mathrm{d}k\int_{|l|\leq\Lambda}\mathrm{d}l\,k^2B_0(k)l^2B_0(l)\langle e^{ikx_1}a_k\chi,e^{ilx_1}a_l\chi\rangle\\&=\int_{|k|\leq\Lambda}\mathrm{d}k\int_{|l|\leq\Lambda}\mathrm{d}l\,k^2B_0(k)l^2B_0(l)\big\langle((-i\nabla_1-l)^2+1)^{-\frac{1}{2}}e^{ikx_1}(1-\Delta_1)^{\frac{1}{2}}a_k\chi,\\&\qquad((-i\nabla_1-k)^2+1)^{-\frac{1}{2}}e^{ilx_1}(1-\Delta_1)^{\frac{1}{2}}a_l\chi\big\rangle\\&\leq\int_{|k|\leq\Lambda}\mathrm{d}k\int_{|l|\leq\Lambda}\mathrm{d}l\,k^4B_0(k)^2\|((-i\nabla_1-k)^2+1)^{-\frac{1}{2}}e^{ilx_1}(1-\Delta_1)^{\frac{1}{2}}a_l\chi\|^2\\&\leq\sup_{p\in\mathbb{R}^3}\left\{\int_{|k|\leq\Lambda}\mathrm{d}k\,\frac{k^4B_0(k)^2}{1+(p-k)^2}\right\}\|\mathcal{N}_a^{\frac{1}{2}}(1-\Delta_1)^{\frac{1}{2}}\chi\|^2.\end{aligned}$$

By symmetric rearrangement (similarly to (6.3)) the supremum over $p\in\mathbb{R}^3$ is bounded by a constant times $\ln\Lambda$. With this inequality, we can estimate the remaining term by

$$\begin{aligned}|(6.34d)|&=\frac{1}{\sqrt{N}}\sum_{n=1}^{\infty}n\left|\left\langle\int_{|k|\leq\Lambda}\mathrm{d}k\,k^2\overline{B_{x_1}(k)}a_k\mathcal{N}_a^{-\frac{1}{2}}q_1\chi^{(n)},q_1(\mathcal{N}_a+1)^{\frac{1}{2}}\phi^{(n)}\right\rangle\right|\\&\leq C\frac{\sqrt{\ln\Lambda}}{\sqrt{N}}\left(\sum_{n=0}^{\infty}n\|(1-\Delta_1)^{\frac{1}{2}}q_1\chi^{(n)}\|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}n\|(\mathcal{N}_a+1)^{\frac{1}{2}}\phi^{(n)}\|^2\right)^{\frac{1}{2}}\\&\leq C\frac{\sqrt{\ln\Lambda}}{\sqrt{N}}\|(\mathcal{N}+\mathbb{T})^{\frac{1}{2}}\chi\|\|(\mathcal{N}+1)\phi\|.\end{aligned}$$

If we choose the cutoff parameter $\Lambda=N$ we thus arrive at

$$|\langle\chi,H_3(t)\phi\rangle|\leq CN^{-\frac{1}{2}}\|(\mathcal{N}+\mathbb{T}+1)^{\frac{1}{2}}\chi\|\big(\|(\mathcal{N}^3+\mathbb{T}+1)^{\frac{1}{2}}\chi\|+\sqrt{\ln N}\|(\mathcal{N}+1)\phi\|\big).$$

The terms $H_j(t)$ for $j\neq 3$ are somewhat easier to treat, since if there are any gradients (as in $H_2(t)$, via f_u) they act on u_t and not χ,ϕ. The powers of ω needed to render the kernels integrable are strictly less than 1, so the possibility of distributing factors of $\mathcal{N}$ given by Lemma 6.1 is sufficient to treat these terms, as we now show.

Term $H_1(t)+H_2(t)$. For this contribution we can use the already established bounds from the proof of Lemma 6.11, that is, (6.30) and (6.31), respectively.

Term $H_4(t)$. We use $\|N_t\|_{\mathfrak{h}_{1/8}\to L^2\otimes\mathfrak{h}_{-1/8}} + \|N_t\|_{\mathfrak{h}_{1/8}^{\otimes 2}\to L^2} \le \|N_t\|_{L^2\otimes\mathfrak{h}_{-1/8}^{\otimes 2}} \le C$ (see Lemma 6.7) and Lemma 6.1 to get

$$|\langle \chi, H_4(t)\phi\rangle| \le \frac{C}{\sqrt{N}}\big[\|(\mathcal{N}+1)^{\frac{1}{2}}\chi\|\|\mathrm{d}\Gamma_a(\omega^{\frac{1}{4}})\phi\| + \|(\mathcal{N}+1)^{-\frac{1}{2}}\mathrm{d}\Gamma_a(\sqrt{\omega})\chi\|\|(\mathcal{N}+1)\phi\| + \|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}\chi\|\,\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}(\mathcal{N}+1)^{\frac{1}{2}}\phi\|\big].$$

By means of (6.1b) we then obtain

$$|\langle \chi, H_4(t)\phi\rangle| \le CN^{-\frac{1}{2}}\|(\mathcal{N}+\mathbb{T}+1)^{\frac{1}{2}}\chi\|\|(\mathcal{N}^3+\mathbb{T}+1)^{\frac{1}{2}}\phi\|.$$

Term $H_5(t)$. Recalling that $|k|B_0(k)\in\mathfrak{h}_{-s}$ for $s>0$, Lemma 6.1 gives

$$|\langle \chi, H_5(t)\phi\rangle| \lesssim CN^{-1}\big[\|\mathcal{N}_b^{\frac{1}{2}}\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}\chi\|\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}\mathcal{N}_b^{\frac{1}{2}}\phi\| + \|\mathcal{N}_b^{\frac{1}{2}}(\mathcal{N}_a+1)^{-\frac{1}{2}}\mathrm{d}\Gamma_a(\sqrt{\omega})\chi\|\|(\mathcal{N}+1)\phi\| + \|\mathcal{N}_b\chi\|\|\mathrm{d}\Gamma_a(\omega^{1/4})\phi\|\big].$$

Since $\chi = \mathbb{1}_{\mathcal{N}_b\le N}\chi$, equation (6.1b) leads to

$$|\langle \chi, H_5(t)\phi\rangle| \le CN^{-\frac{1}{2}}\|(\mathcal{N}+\mathbb{T}+1)^{\frac{1}{2}}\chi\|\|(\mathcal{N}^3+\mathbb{T}+1)^{\frac{1}{2}}\phi\|.$$

We conclude by estimating the terms from (6.33a)–(6.33e) by using that

$$\pm\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+ - 1\Big) \le N^{-1}\mathcal{N}_b, \tag{6.35a}$$

$$\pm\big(N^{-1}\sqrt{[(N-\mathcal{N}_b)(N-\mathcal{N}_b-1)]_+} - 1\big) \le CN^{-1}\mathcal{N}_b. \tag{6.35b}$$

Terms (6.33a) and (6.33b). Using (6.26), (6.27), and $\mathbb{1}_{\mathcal{N}_b\le N}\chi = \chi$, we can estimate the first two lines by

$$|(6.33a)| + |(6.33b)| \le CN^{-\frac{1}{2}}\|(\mathcal{N}+1)^{\frac{1}{2}}\chi\|\|(\mathcal{N}+1)\phi\|.$$

Term (6.33c). Using Lemmas 6.7 and 6.1 we arrive at

$$\begin{aligned}
|(6.33c)| &= 2\Big|\mathrm{Re}\Big\langle \chi, \int \mathrm{d}x\,\mathrm{d}k\,\ell^{(1)}(x,k)a_k^* b_x^*\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}} - 1\Big)\phi\Big\rangle\Big| \\
&\le \|\ell^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/2}}\|\mathrm{d}\Gamma_a(\omega)^{\frac{1}{2}}\chi\|\Big\|(\mathcal{N}+1)^{\frac{1}{2}}\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}} - 1\Big)\phi\Big\| \\
&\quad + \|\ell^{(1)}\|_{L^2\otimes\mathfrak{h}_{-1/4}}\Big\|\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}} - 1\Big)\chi\Big\|\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}\mathcal{N}_b^{\frac{1}{2}}\phi\| \\
&\overset{(6.35a)}{\le} CN^{-1}\|\mathrm{d}\Gamma_a(\omega)^{\frac{1}{2}}\chi\|\|(\mathcal{N}+1)^{\frac{3}{2}}\phi\| + CN^{-1}\|\mathcal{N}_b\chi\|\|\mathrm{d}\Gamma_a(\sqrt{\omega})^{\frac{1}{2}}\mathcal{N}_b^{\frac{1}{2}}\phi\| \\
&\overset{(6.1b)}{\le} CN^{-\frac{1}{2}}(\|\mathrm{d}\Gamma_a(\omega)^{\frac{1}{2}}\chi\| + \|N^{-\frac{1}{2}}\mathcal{N}_b\chi\|)(\|(\mathcal{N}+1)^{3/2}\phi\| + \|\mathbb{T}^{\frac{1}{2}}\phi\|).
\end{aligned}$$

This implies the claimed bound since $\mathbb{1}_{\mathcal{N}_b\le N}\chi = \chi$.

Term (6.33d). This term is treated in close analogy to the previous one, leading to

$$|(6.33d)| \leq N^{-\frac{1}{2}} \|(\mathcal{N} + \mathbb{T} + 1)^{\frac{1}{2}} \chi\| \|(\mathcal{N}^3 + \mathbb{T} + 1)^{\frac{1}{2}} \phi\|.$$

Term (6.33e). By means of Lemmas 6.7 and 6.1, and (6.35a), we get

$$|(6.33e)| \leq C N^{-1} \|(\mathcal{N} + 1)^{\frac{1}{2}} \chi\| \|(\mathcal{N} + 1)^{3/2} \phi\|.$$

This completes the proof of the lemma. ∎

6.5. Estimates for the dressing transformation

In this section, we will derive estimates for the fluctuation generator associated with the dressing transformation $D_{u,\alpha}$ defined in (3.16), and its quadratic approximation $\mathbb{D}^{\Lambda}_{u,\alpha}$ from (4.4). We also give the proof of Lemma 3.6.

Lemma 6.13. *Let $\mathbb{D}^{\Lambda}_{u,\alpha}(\theta)$ and $D_{u,\alpha}(\theta)$ be defined by (4.4) and (3.16). There exists a constant $C > 0$, such that for all $(u, \alpha) \in H^1(\mathbb{R}^3) \oplus \mathfrak{h}_0$ with $\|u\|_{L^2} = 1$, $|\theta| \leq 1$, and $\Lambda \in \mathbb{R}_+ \cup \{\infty\}$,*

$$\begin{aligned}
\pm \mathbb{D}^{\Lambda}_{u,\alpha}(\theta) &\leq C(\|u\|^2_{H^1} + \|\alpha\|_{\mathfrak{h}_0})(\mathcal{N} + 1),\\
\pm i[\mathcal{N}, \mathbb{D}^{\Lambda}_{u,\alpha}(\theta)] &\leq C(\|u\|^2_{H^1} + \|\alpha\|_{\mathfrak{h}_0})(\mathcal{N} + 1),\\
\pm \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{D}^{\Lambda}_{u,\alpha}(\theta) &\leq C(\|u\|^{3/2}_{H^1} + \|\alpha\|_{\mathfrak{h}_0})(\mathcal{N} + 1),\\
\pm i[\mathcal{N}, D_{u,\alpha}(\theta)] &\leq C(\|u\|^2_{H^1} + \|\alpha\|_{\mathfrak{h}_0})(\mathcal{N} + 1)\Big(1 + \Big(\frac{1}{N}\mathcal{N}_b\Big)^{\frac{1}{2}}\Big),
\end{aligned}$$

in the sense of quadratic forms on $\mathcal{F} \otimes \mathcal{F}$ and

$$|\langle \phi, (D_{u,\alpha}(\theta) - \mathbb{D}^{\infty}_{u,\alpha}(\theta))\chi\rangle| \leq C \|\phi\| \|(\mathcal{N} + 1)^{3/2} \chi\| N^{-\frac{1}{2}}$$

for all $\phi, \chi \in \mathcal{F} \otimes \mathcal{F}$.

Proof. Recall that $\mathbb{D}^{\Lambda}_{u,\alpha}(\theta)$ and $D_{u,\alpha}(\theta)$ are defined with respect to the mean-field flow $(u^\theta, \alpha^\theta) := \mathfrak{D}[\theta](u, \alpha)$ and that $|u^\theta| = |u|$ by (3.14). One readily shows that $\|\tau_{u,\alpha}\|_{L^\infty} \leq 3\|B_0\|_{L^2} \|\alpha\|_{L^2}$ and $\|\kappa^{\Lambda}_{u^\theta}\|_{L^2(\mathbb{R}^6)} \leq \|B_0\|_{L^2}$. By means of (3.14) and (6.7) one further obtains

$$\|\partial_\theta \kappa^{\Lambda}_{u^\theta}\|_{L^2(\mathbb{R}^6)} = \|\tau_{u,\alpha} \kappa^{\Lambda}_{u^\theta}\|_{L^2(\mathbb{R}^6)} \leq 3\|B_0\|^2_{L^2} \|\alpha\|_{L^2}.$$

Using this, the estimates involving $\mathbb{D}^{\Lambda}_{u,\alpha}(\theta)$ follow standard bounds for creation and annihilation operators that are special cases of Lemma 6.1.

Since $0 \leq [1 - \frac{\mathcal{N}_b}{N}]_+ \leq 1$, the commutator of $\mathcal{N}$ with the first two terms in (3.16) can be bounded as before. Now let us denote the last term in (3.16), which is cubic in the creation/annihilation operators, by $D^{(3)}_{u,\alpha}(\theta)$. Using the canonical commutation relations and again standard estimates for creation and annihilation operators, one obtains

$$\pm i[\mathcal{N}, D^{(3)}_{u,\alpha}(\theta)] \leq 4\|B_0\|_{L^2} N^{-\frac{1}{2}} \mathcal{N}_b \mathcal{N}_a^{\frac{1}{2}}.$$

This proves the bound on the commutator $[\mathcal{N}, D^{(3)}_{u,\alpha}]$.

To show the last inequality, write

$$D_{u,\alpha}(\theta) - \mathbb{D}^{\infty}_{u,\alpha}(\theta) = \int \mathrm{d}x\, \mathrm{d}k\, (\kappa^{\infty}_{u^\theta}(k,x)a^*_k - \kappa^{\infty}_{u^\theta}(-k,x)a_k)b^*_x\Big(\Big[1-\frac{\mathcal{N}_b}{N}\Big]_+^{\frac{1}{2}} - 1\Big)$$
$$+ \int \mathrm{d}x\, \mathrm{d}k\, (\overline{\kappa^{\infty}_{u^\theta}(k,x)}a_k - \overline{\kappa^{\infty}_{u^\theta}(-k,x)}a^*_k)b_x\Big(\Big[1-\frac{\mathcal{N}_b - 1}{N}\Big]_+^{\frac{1}{2}} - 1\Big) + D^{(3)}_{u,\alpha}(\theta).$$

It is straightforward to show that

$$|\langle \phi, D^{(3)}_{u,\alpha}(\theta)\chi\rangle| \le 4\|B_0\|_{L^2}\|\phi\|\|\mathcal{N}_b\mathcal{N}_a^{\frac{1}{2}}\chi\|N^{-\frac{1}{2}}.$$

For the first two terms, the bounds $\|\kappa^{\infty}_{u^\theta}\|_{L^2(\mathbb{R}^6)} \le \|B_0\|_{L^2}$ and $([1-\frac{\mathcal{N}_b - j}{N}]_+^{1/2} - 1)^2 \le CN^{-1}(\mathcal{N}_b + 1)$ for $j \in \{0,1\}$ imply that

$$\Big|\Big\langle \phi, \int \mathrm{d}x\, \mathrm{d}k\, (\kappa^{\infty}_{u^\theta}(k,x)a^*_k - \kappa^{\infty}_{u^\theta}(-k,x)a_k)b^\bullet_x\Big(\Big[1-\frac{\mathcal{N}_b - j}{N}\Big]_+^{\frac{1}{2}} - 1\Big)\chi\Big\rangle\Big|$$
$$\le C\|\phi\|\|(\mathcal{N}_b+1)^{\frac{1}{2}}\mathcal{N}_b^{\frac{1}{2}}\mathcal{N}_a^{\frac{1}{2}}\chi\|N^{-\frac{1}{2}}$$

where $\bullet \in \{\varnothing, *\}$. This completes the proof of the lemma. ∎

We now turn to the proof of Lemma 3.6, which relates the energy of excitations in $W^{\mathrm{D}}\Psi_N$ to the difference of the energy of Ψ to its mean-field energy. As explained below the statement of the lemma, it is not possible to apply the strategy of the proof of Theorem 3.2 because $\mathbb{T}$ is not dominated by the generator $D_{u,\alpha}(\theta)$ in (3.16). Instead, our proof relies on comparing $X^*_{\mathfrak{D}}\mathbb{T}X_{\mathfrak{D}}$ directly with the difference between the many-body energy per particle and the dressed mean-field energy $\mathcal{E}_1$, evaluated at $\mathfrak{D}(u,\alpha)$. Energy estimates of this kind were previously used in a different context in [58].

Proof of Lemma 3.6. We recall Lemma 3.1 and the fact that $\mathcal{E} = \mathcal{E}_1 \circ \mathfrak{D}$ as shown by equation (5.5) for $\mathcal{E}_0 = \mathcal{E}$. With this at hand, we write the difference between the many-body energy per particle and the mean-field energy as

$$N^{-1}\langle \Psi_N, H_N\Psi_N\rangle - \mathcal{E}(u,\alpha) = N^{-1}\langle W^{\mathrm{D}}\Psi_N, H^{\mathrm{D}}_N W^{\mathrm{D}}\Psi_N\rangle - \mathcal{E}_1 \circ \mathfrak{D}(u,\alpha). \quad (6.36)$$

Moreover, for $\zeta = X_{\mathfrak{D}(u,\alpha)}W^{\mathrm{D}}\Psi$, we can use (2.9b) to write the relevant γ functional as

$$\gamma[W^{\mathrm{D}}\Psi_N, \mathfrak{D}(u,\alpha)] = N^{-1}\langle \zeta, \mathbb{T}\zeta\rangle. \quad (6.37)$$

To relate the expressions on the right-hand side of (6.36) and (6.37), we make use of the excitation map $X_{\mathfrak{D}(u,\alpha)}$. To do so, we rewrite H^{D}_N in terms of the fluctuation generator $H^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(0)$ from (6.24). This will allow us to employ previously established estimates.

To ease the notation, from now on we set $(u^{\mathrm{D}}, \alpha^{\mathrm{D}}) := \mathfrak{D}(u,\alpha)$ and the shorthand $q := q_{u^{\mathrm{D}}}$, $h := h_{\mathfrak{D}(u,\alpha)}$, $f := f_{u^{\mathrm{D}}}$, $g := g_{u^{\mathrm{D}},\alpha^{\mathrm{D}}}$ (see (3.4a)–(3.4f) for the definitions of these objects). We can use (2.5) and Lemma 2.1 to obtain

$$\begin{aligned} X_{\mathfrak{D}(u,\alpha)} H_N^{\mathrm{D}} X^*_{\mathfrak{D}(u,\alpha)} = {} & H^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(0) + \langle u^{\mathrm{D}}, h u^{\mathrm{D}}\rangle (N - \mathcal{N}_b) + N\langle \alpha^{\mathrm{D}}, \omega\alpha^{\mathrm{D}}\rangle \\ & + \sqrt{N}\Phi(\omega\alpha^{\mathrm{D}} + f + g) - b^*(u^{\mathrm{D}}) b(q h u^{\mathrm{D}}) \\ & + \sqrt{N - \mathcal{N}_b}\, b(q h u^{\mathrm{D}}) + b^*(q h u^{\mathrm{D}})\sqrt{N - \mathcal{N}_b}. \end{aligned}$$

From the first inequality of Lemma 6.11, the fact that $\mathbb{1}_{\mathcal{N}_b \le N}\zeta = \zeta$, and (6.35a), we get

$$\langle \zeta, \mathbb{T}\zeta\rangle \le 2\langle \zeta, H^{\mathrm{D}}_{\mathfrak{D}(u,\alpha)}(0)\zeta\rangle + C\langle \zeta, \mathcal{N}\zeta\rangle.$$

With formula (5.3) for the dressed mean-field energy $\mathcal{E}_1$, and using the inverse triangle inequality (with separate but similar calculations for both signs of the expression inside the absolute value), we arrive at

$$\begin{aligned} & N^{-1}\langle \zeta, \mathbb{T}\zeta\rangle - 2|N^{-1}\langle \Psi_N, H_N \Psi_N\rangle - \mathcal{E}(u,\alpha)| \\ & \quad \le C N^{-1}\langle \zeta, \mathcal{N}\zeta\rangle + C\Big| -N^{-1}\langle u^{\mathrm{D}}, h u^{\mathrm{D}}\rangle\langle \zeta, \mathcal{N}_b\zeta\rangle - N^{-1}\langle \zeta, b^*(u^{\mathrm{D}}) b(q h u^{\mathrm{D}})\zeta\rangle \\ & \qquad + N^{-\frac{1}{2}}\langle \zeta, \Phi(\omega\alpha^{\mathrm{D}} + f + g)\zeta\rangle \\ & \qquad + 2N^{-\frac{1}{2}}\operatorname{Re}\Big\langle \zeta, \sqrt{1 - \frac{\mathcal{N}_b}{N}}\, b(q h u^{\mathrm{D}})\zeta\Big\rangle\Big|. \end{aligned} \tag{6.38}$$

We bound the terms on the right-hand side of (6.38) by (using that $\|\zeta\| = \|\Psi_N\| = 1$)

$$\begin{aligned} |(6.38)| \le {} & C\langle \zeta, \mathcal{N}\zeta\rangle N^{-1}(1 + \|h u^{\mathrm{D}}\|_{L^2}) \\ & + C N^{-\frac{1}{2}}(\|\omega\alpha^{\mathrm{D}} + f + g\|_{L^2}\|\mathcal{N}_a^{\frac{1}{2}}\zeta\| + C\|\mathcal{N}_b^{\frac{1}{2}}\zeta\|). \end{aligned}$$

By Lemmas 6.4 and 6.5, the norms of $f, g, h u^{\mathrm{D}}$ are bounded in terms of the $H^2 \oplus \mathfrak{h}_{1/2}$-norm of $(u^{\mathrm{D}}, \alpha^{\mathrm{D}})$, which by Lemma 3.3 is controlled by the norm of $(u,\alpha) \in H^2 \oplus \mathfrak{h}_{3/2}$. Thus there exists a constant C, depending on this norm, so that

$$|(6.38)| \le C(N^{-1}\langle \zeta, \mathcal{N}\zeta\rangle + (N^{-1}\langle \zeta, \mathcal{N}\zeta\rangle)^{\frac{1}{2}}). \tag{6.39}$$

Moreover, by (2.9a) and Lemma 3.5 for $\theta = 1$, we have

$$N^{-1}\langle \zeta, \mathcal{N}\zeta\rangle = \beta[W^{\mathrm{D}}\Psi_N, \mathfrak{D}(u,\alpha)] \le C(\beta[\Psi_N, (u,\alpha)] + N^{-1}).$$

Combined with (6.39) this proves the statement of the lemma. ∎

A. Initial states

Proof of Proposition 1.2. Let

$$W^{\mathrm{D}}_{\ge K} := W\Big(N^{-1/2}\sum_{j=1}^{N} B_{K,x_j}\Big) \quad \text{and} \quad W^{\mathrm{D}}_{\ge K,x_j} := W(N^{-1/2} B_{K,x_j}).$$

The first inequality of the proposition can be obtained similarly to [63, Prop. II.2]. More explicitly, we use (2.4) and (2.5) to estimate

$$N^{-1}\|\mathcal{N}_a^{\frac{1}{2}} W^*(\sqrt{N}\alpha)\Psi_{N,K}\| = N^{-1}\int \mathrm{d}^3k\, \|a_k(W^{\mathrm{D}}_{\geq K})^*(u^{\otimes N}\otimes\Omega)\|^2$$
$$\leq \|B_{K,0}\|^2_{L^2} \leq CK^{-2}.$$

By means of

$$W^{\mathrm{D}}_{\geq K}(q_u)_1(W^{\mathrm{D}}_{\geq K})^* = (q_u)_1 + |u\rangle\langle u|_1 - W^{\mathrm{D}}_{\geq K,x_1}|u\rangle\langle u|_1(W^{\mathrm{D}}_{\geq K,x_1})^*$$

and

$$(1-(W^{\mathrm{D}}_{\geq K,x_1}))(1-(W^{\mathrm{D}}_{\geq K,x_1})^*) \leq N^{-1}\widehat{\Phi}(iB_{K,x_1})^2,$$

we get

$$\begin{aligned}|\langle\Psi_{N,K},(q_u)_1\Psi_{N,K}\rangle| &\leq 2\|(1-(W^{\mathrm{D}}_{\geq K,x_1})^*)u^{\otimes N}\otimes W(\sqrt{N}\alpha)\Omega\|\\ &\leq 2N^{-\frac{1}{2}}\|B_{K,0}\|_{L^2}\|(\mathcal{N}_a+1)^{\frac{1}{2}}u^{\otimes N}\otimes W(\sqrt{N}\alpha)\Omega\|\\ &\leq CK^{-1}(N^{-\frac{1}{2}}+\|\alpha\|_{L^2}),\end{aligned}$$

and thus $\beta[\Psi_{N,K},u,\alpha] \leq CK^{-1}(1+\|\alpha\|_{L^2})$. Since $W^{\mathrm{D}}_{\geq K} = \prod_{j=1}^N W^{\mathrm{D}}_{\geq K,x_j}$, the transformation relations of the dressing transformation from [48, Sect. II] lead to

$$\begin{aligned}W^{\mathrm{D}}_{\geq K}H_N(W^{\mathrm{D}}_{\geq K})^* = \sum_{j=1}^N\Big[&-\Delta_j + N^{-\frac{1}{2}}\widehat{\Phi}(\mathbb{1}(|\cdot|\leq K)G_{x_j})\\ &+ N^{-1}(a(kB_{K,x_j})^2 + \text{h.c.} + 2a^*(kB_{K,x_j})a(kB_{K,x_j}))\\ &- 2N^{-\frac{1}{2}}(i\nabla_{x_j}\cdot a(kB_{K,x_j}) + a^*(kB_{K,x_j})\cdot i\nabla_{x_j})\Big]\\ &+ N^{-1}\sum_{i<j}V_K(x_i-x_j) + \mathrm{d}\Gamma_a(\omega) + E_K,\end{aligned}$$

with $V_K(x_i-x_j) = 2\,\mathrm{Re}\langle B_{K,x_i},\omega B_{K,x_j}\rangle - 4\,\mathrm{Re}\langle G_{x_i},B_{K,x_j}\rangle$ and $E_K = \int\frac{\mathbb{1}_{|k|\leq K}\,\mathrm{d}k}{\omega(k)(k^2+\omega(k))}$. Using the shift property of the Weyl operator (2.5), we write the expectation value of the energy per particle as

$$\begin{aligned}N^{-1}\langle\Psi_{N,K},H_N\Psi_{N,K}\rangle &= N^{-1}E_K + \mathcal{E}(u,\alpha)\\ &\quad + \Big\langle u,\Big(2\,\mathrm{Re}\langle G_{(\cdot)},\alpha_{\geq K}\rangle + A_{\alpha_{\geq K},(\cdot)} + F^2_{\alpha_{\geq K}} + \frac{1}{2}V_K * |u|^2\Big)u\Big\rangle,\end{aligned}$$

with $\alpha_{\geq K} = \mathbb{1}_{|\cdot|\geq K}\alpha$ and A_α, F_α as defined in (3.4b), (3.4c). Note that $|E_K| \leq C(1+\ln K)$. By means of $\sup_{x\in\mathbb{R}^3}|\langle kB_x,\alpha_{\geq K}\rangle| \leq CK^{-1}\|\alpha\|_{\mathfrak{h}_1}$, $\|V_K\| \leq CK^{-3/2}$, and $\|u\|_{L^2} = 1$ we get

$$|\langle u, A_{\alpha_{\geq K},(\cdot)}u\rangle| \leq CK^{-1}\|\alpha\|_{\mathfrak{h}_1}\|u\|_{H^1}, \qquad \|F_{\alpha_{\geq K}}\|_{L^\infty} \leq CK^{-1}\|\alpha\|_{\mathfrak{h}_1}$$

and

$$\|V_K * |u|^2\|_{L^\infty} \leq \|V_K\|_{L^2}\||u|^2\|_{L^2} \leq CK^{-3/2}\|u\|_{H^1}^2.$$

Inequality (6.4) with $G_{(\cdot)} \mapsto \omega^{-1/2}G_{(\cdot)}$ and $\alpha \mapsto \omega^{1/2}\alpha$ leads to

$$\|\operatorname{Re}\langle G_{(\cdot)}, \alpha_{\geq K}\rangle u\|_{L^2} \leq CK^{-1}\|\alpha\|_{\mathfrak{h}_1}\|u\|_{H^1}.$$

In total, we obtain

$$|N^{-1}\langle \Psi_{N,K}, H_N\Psi_{N,K}\rangle - \mathcal{E}(u,\alpha)| \leq C(K^{-1} + N^{-1}(1+\ln K))(\|u\|_{H^1}^2 + \|\alpha\|_{\mathfrak{h}_1}^2). \quad ■$$

B. Bogoliubov transformations

For a linear map T on a complex Hilbert space, we denote by $\bar{T}f = \overline{T\bar{f}}$ its complex conjugate.

Lemma B.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathbb{U}_n$ with $n \in \mathbb{N}$ be a family of unitary Bogoliubov transformations on the Fock space over $\mathcal{H}$, that is, there exist $\mathfrak{u}_n$ linear, bounded and $\mathfrak{v}_n$ Hilbert–Schmidt, so that*

$$\mathbb{U}_n^* a^*(f)\mathbb{U}_n = a^*(\mathfrak{u}_n f) + a(\mathfrak{v}_n \bar{f}), \quad \mathbb{U}_n^* a(f)\mathbb{U}_n = a(\bar{\mathfrak{u}}_n f) + a^*(\overline{\mathfrak{v}_n f}).$$

Assume that $\mathbb{U}_\infty = \text{s-lim}_{n\to\infty}\mathbb{U}_n$ exists, and moreover there exists a self-adjoint A, $D(A)$ with $A \geq 1$ and $C > 0$ so that for all $n \in \mathbb{N}\cup\{\infty\}$ and $\Psi \in D(\mathrm{d}\Gamma(A)^{1/2})$ it holds that

$$\langle \mathbb{U}_n\Psi, \mathcal{N}\mathbb{U}_n\Psi\rangle \leq C\langle\Psi, (1+\mathrm{d}\Gamma(A))\Psi\rangle.$$

Then $\mathbb{U}_\infty$ is a Bogoliubov transformation and the corresponding maps $\mathfrak{u}$, $\mathfrak{v}$ satisfy the bounds $\|\mathfrak{u}\|_{\mathcal{H}\to\mathcal{H}} \leq C+1$, $\|\mathfrak{v}\|_{\mathfrak{S}_2(\mathcal{H})} \leq C$.

Proof. We start by showing that for $\Psi, \Phi \in D(\mathrm{d}\Gamma(A)^{1/2})$, $f \in \mathcal{H}$, and $\bullet \in \{\varnothing, *\}$,

$$\lim_{n\to\infty}\langle\Phi, \mathbb{U}_n^* a^\bullet(f)\mathbb{U}_n\Psi\rangle = \langle\Phi, \mathbb{U}_\infty^* a^\bullet(f)\mathbb{U}_\infty\Psi\rangle.$$

To see this, note that

$$\begin{aligned}
&|\langle\Phi, \mathbb{U}_n^* a^\bullet(f)\mathbb{U}_n\Psi\rangle - \langle\Phi, \mathbb{U}_\infty^* a^\bullet(f)\mathbb{U}_\infty\Psi\rangle| \\
&\quad\leq C\|f\|_{\mathcal{H}}\big(\|(\mathbb{U}_n - \mathbb{U}_\infty)\Phi\|_{\mathcal{F}}\|(1+\mathrm{d}\Gamma(A))^{\frac{1}{2}}\Psi\|_{\mathcal{F}} \\
&\qquad\qquad + \|(\mathbb{U}_n - \mathbb{U}_\infty)\Psi\|_{\mathcal{F}}\|(1+\mathrm{d}\Gamma(A))^{\frac{1}{2}}\Phi\|_{\mathcal{F}}\big),
\end{aligned}$$

which tends to zero since $\mathbb{U}_n$ converges strongly to $\mathbb{U}_\infty$.

Now let $f \in D(A)$, so $a^*(f)\Omega \in D(\mathrm{d}\Gamma(A)^{1/2})$. Then we have, using that $\mathbb{U}_n$ is a Bogoliubov transformation for $n \in \mathbb{N}$,

$$\begin{aligned}
\langle a^*(f)\Omega, \mathbb{U}_\infty a^*(g)\mathbb{U}_\infty^*\Omega\rangle &= \lim_{n\to\infty}\langle a^*(f)\Omega, (a^*(\mathfrak{u}_n g) + a(\mathfrak{v}_n\bar{g}))\Omega\rangle = \lim_{n\to\infty}\langle f, \mathfrak{u}_n g\rangle, \\
\langle a^*(f)\Omega, \mathbb{U}_\infty a(g)\mathbb{U}_\infty^*\Omega\rangle &= \lim_{n\to\infty}\langle a^*(f)\Omega, (a(\bar{\mathfrak{u}}_n g) + a^*(\overline{\mathfrak{v}_n g}))\Omega\rangle = \lim_{n\to\infty}\langle f, \overline{\mathfrak{v}_n g}\rangle.
\end{aligned}$$

Since, moreover,

$$|\langle a^*(f)\Omega, \mathbb{U}_\infty a^\bullet(g)\mathbb{U}_\infty^*\Omega\rangle| \le \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}} \overbrace{\|(1+\mathcal{N}^{\frac{1}{2}})\mathbb{U}_\infty\Omega\|_{\mathcal{F}}}^{\le C+1},$$

the operators $\mathfrak{u}_n$, $\mathfrak{v}_n$ converge weakly to operators $\mathfrak{u}$, $\mathfrak{v}$ with norm less than $C+1$. Weak convergence of $\mathfrak{u}_n$, $\mathfrak{v}_n$ implies that for $\Phi \in D(\mathcal{N}^{1/2})$, $\Psi \in \mathcal{F}$, and $f \in \mathcal{H}$,

$$\lim_{n\to\infty} \langle \Phi, a(\mathfrak{u}_n f)\Psi\rangle = \langle \Phi, a(\mathfrak{u} f)\Psi\rangle, \qquad \lim_{\Lambda\to\infty} \langle \Phi, a(\mathfrak{v}_n f)\Psi\rangle = \langle \Phi, a(\mathfrak{v} f)\Psi\rangle,$$

and thus for $\Phi, \Psi \in D(\mathcal{N}^{1/2})$,

$$\langle \Phi, \mathbb{U}_\infty^* a^*(f)\mathbb{U}_\infty\Psi\rangle = \langle a(\mathfrak{u} f)\Phi, \Psi\rangle + \langle \Phi, a(\mathfrak{v}\bar{f})\Psi\rangle = \langle \Phi, (a^*(\mathfrak{u} f) + a(\mathfrak{v}\bar{f}))\Psi\rangle,$$

and similarly for $a(f)$. Moreover, we have

$$\|\mathfrak{v}_n\|_{\mathfrak{S}_2}^2 = \|\mathcal{N}^{\frac{1}{2}}\mathbb{U}_n\Omega\|^2 \le C,$$

so the sequence $\mathfrak{v}_n$ is bounded in $\mathfrak{S}_2$, whence it has a subsequence that converges weakly in $\mathfrak{S}_2$. Since $\langle f, \mathfrak{v}_n g\rangle = \mathrm{Tr}(|f\rangle\langle g|\mathfrak{v}_n)$, the limit must be $\mathfrak{v}$, so $\mathfrak{v} \in \mathfrak{S}_2$ with norm less than C. This proves the claim. ∎

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