Ann. Inst. H. Poincaré C
Anal. Non Linéaire 43 (2026), 883–925
DOI 10.4171/AIHPC/174

Stochastic homogenization of HJ equations: A differential game approach

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Abstract. We prove stochastic homogenization for a class of nonconvex and noncoercive first-order Hamilton–Jacobi equations in a finite-range dependence environment for Hamiltonians that can be expressed by a max-min formula. Exploiting the representation of solutions as value functions of differential games, we develop a game-theoretic approach to homogenization. We furthermore extend this result to a class of Lipschitz Hamiltonians that need not admit a global max-min representation. Our methods allow us to get a quantitative convergence rate for solutions with linear initial data toward the corresponding ones of the effective limit problem.

1. Introduction

In this paper we study the asymptotic behavior, as $\varepsilon \to 0^+$, of solutions to a stochastic Hamilton–Jacobi (HJ) equation of the form

$$\partial_t u^\varepsilon + H\Big(\frac{x}{\varepsilon}, D_x u^\varepsilon, \omega\Big) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \tag{HJ$_\varepsilon$}$$

for each fixed $T > 0$, where $H\colon \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a Lipschitz Hamiltonian admitting a max-min representation. The dependence on the random environment $(\Omega, \mathcal{F}, \mathbb{P})$ enters through the Hamiltonian $H(x, p, \omega)$, whose law is assumed to be *stationary*, i.e., invariant under spatial translations, and *ergodic*, i.e., any translation-invariant event has probability either 0 or 1. Under the additional assumptions that the random variables $(H(\cdot, p, \cdot))_{p\in\mathbb{R}^d}$ satisfy a finite-range dependence condition and that the underlying dynamics is oriented, we prove homogenization for (HJ$_\varepsilon$) (Theorem 2.2) and obtain a convergence rate, for solutions with linear initial data, toward the corresponding solutions of the effective limit problem (Theorem 2.4). This latter, stronger result is stable under local uniform convergence of suitable sequences of Hamiltonians of the above type. As a consequence, homogenization extends to the limiting Hamiltonians (Corollary 2.5), which in general cannot be expressed in max-min form.

A second extension in this direction is provided by Theorem 2.6, where we prove analogous results for a class of Lipschitz Hamiltonians that need not admit a global max-min

Mathematics Subject Classification 2020: 35B27 (primary); 35F21, 60G10, 60K37 (secondary).
Keywords: Hamilton–Jacobi equation, stochastic homogenization, stationary ergodic random environment, differential games, viscosity solution.

representation. Using an argument from [29, Section 5], these Hamiltonians can, however, be written in max-min form *locally in* p; this suffices for our proof strategy, which is tailored to this extension. The full set of assumptions and the precise statements of our homogenization results are presented in Section 2. We emphasize that the Hamiltonians considered here are noncoercive and nonconvex in p.

The coercivity of H in the momentum is a condition often assumed in the homogenization theory of first-order HJ equations. Its role is to provide uniform L^∞ bounds on the derivatives of solutions to (HJ_ε) and to an associated "cell" problem. The first homogenization results for equations of the form (HJ_ε) with coercive Hamiltonians were established in the periodic setting in the pioneering work [36] and later extended to the almost periodic case in [34]. The generalization of these results to the stationary ergodic setting was obtained in [39, 41] under the additional assumption that the Hamiltonian is convex in p. By exploiting the metric character of first-order HJ equations, homogenization was extended to quasiconvex Hamiltonians in [9, 26]. The question of whether homogenization holds in the stationary ergodic setting for coercive Hamiltonians that are nonconvex in the momentum remained open for about 15 years, until the third author provided in [43] the first counterexample to homogenization in dimensions $d > 1$. Feldman and Souganidis generalized this example and showed in [30] that homogenization can fail for Hamiltonians of the form $H(x, p, \omega) := G(p) + V(x, \omega)$ whenever G has a strict saddle point. This has shut the door to the possibility of having a general qualitative homogenization theory in the stationary ergodic setting in dimensions $d \geqslant 2$ – at least without imposing further mixing conditions on the stochastic environment – and stands in sharp contrast to the periodic case, where qualitative homogenization is known to hold for Hamiltonians solely coercive in the momentum, regardless of convexity [36].

On the positive side, homogenization of (HJ_ε) for coercive and nonconvex Hamiltonians of fairly general type has been established in one dimension in [11, 31], and in any space dimension for Hamiltonians of the form $H(x, p, \omega) = (|p|^2 - 1)^2 + V(x, \omega)$ in [10]. This result was generalized in [38], where the authors studied Hamiltonians of the form $H(x, p, \omega) = \Psi(|p|) + V(x, \omega)$ under suitable monotonicity assumptions on H. Further positive results in random environments satisfying a finite-range dependence condition were obtained in [7] for Hamiltonians that are positively homogeneous of degree $\alpha \geqslant 1$. Subsequently, the techniques from that work were adapted in [30] to address Hamiltonians with strictly star-shaped sublevel sets. Despite this significant progress, the general question of which equations of the form (HJ_ε) homogenize in the nonconvex case is still not fully understood.

When the coercivity condition of H in p is dropped, one loses control of the derivatives of solutions to (HJ_ε) and of the associated "cell" problem, which are no longer Lipschitz continuous in general. As a consequence, homogenization of (HJ_ε) is known to fail even in the periodic case, regardless of whether the Hamiltonian is convex in p; see, for instance, the introductions in [19, 20] and some examples in [15]. In this level of generality, additional conditions must be imposed to compensate for the lack of coercivity of the Hamiltonian. In the periodic and other compact settings, homogenization results of this type have been obtained in [3–6] and, more recently, in certain convex situations in [13], for a class of nonconvex Hamiltonians in dimension $d = 2$ in [19], and in other nonconvex cases in [15, 17]. When $H(x, p, \omega) := |p| + \langle V(x, \omega), p\rangle$, equation (HJ$_\varepsilon$) is known in the literature (up to a sign change) as the G*-equation*. Homogenization has been established both in the periodic setting [20, 40, 42] and in the stationary ergodic case [21, 37]; see also [22, 23] for quantitative results, under a smallness condition on the divergence of V, but without imposing $|V| < 1$, meaning that H is not assumed to be coercive in p.

This paper furnishes a new and fairly general class of nonconvex and noncoercive Hamiltonians for which (HJ$_\varepsilon$) homogenizes. Our first results, Theorems 2.2 and 2.4, establish homogenization and a quantitative convergence rate for solutions to (HJ$_\varepsilon$) with linear initial data for a class of nonconvex Lipschitz Hamiltonians arising from differential game theory. Specifically, we consider Hamiltonians of the form

$$H(x,p,\omega) := \max_{b\in B}\min_{a\in A}\big\{-\ell(x,a,b,\omega) - \langle f(a,b), p\rangle\big\} \quad \text{for all } (x,p,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega, \tag{H}$$

where the main assumptions are that the law of ℓ has finite-range dependence, in the spirit of [7, 30], and that there exist $\delta > 0$ and a direction $e \in \mathbb{S}^{d-1}$ such that

$$\langle f(a,b), e\rangle \geqslant \delta \quad \text{for all } a \in A,\, b \in B. \tag{f}$$

Notably, assumption (f) precludes a Hamiltonian of the form (H) from being coercive; see Remark 2.3. This is a significant point of originality that distinguishes our work from most contributions on stochastic homogenization.

Another important novelty lies in the proof technique. Indeed, thanks to the form (H) of the Hamiltonian, we can represent the solution of (HJ$_\varepsilon$) as the value function of a differential game, as explained in [29], and adopt a game-theoretic approach. Such an approach has rarely been used in the homogenization of nonconvex HJ equations (see, e.g., [15] in the periodic setting) and, to our knowledge, this is the first time it has been employed to obtain a positive result in the stochastic case. By analyzing optimal strategies, generated paths, and the dynamic programming principle, we show that solutions of (HJ$_\varepsilon$) exhibit asymptotic concentration and that their mean satisfies an approximate subadditive inequality. The homogenization result then follows from the local Lipschitz regularity of solutions to (HJ$_\varepsilon$).

The probabilistic arguments we employ are related to those used in [7, 8] and to their variation in [30], where the authors prove homogenization for several classes of first- and second-order nonconvex Hamilton–Jacobi equations. They consider an auxiliary stationary Hamilton–Jacobi equation, the *metric problem* [8] (resp., [7, 30]), whose solutions can be interpreted as the minimal cost of going from one point in space to another (resp., to a planar surface). By analogy with techniques from first-passage percolation [2, 35], they combine Azuma's inequality with a subadditive argument to prove homogenization of the metric problem and to obtain convergence rates and concentration estimates. They then use a PDE argument to relate the metric problem to the original Hamilton–Jacobi equation. In comparison, our proof presents several key differences. First, the concentration and subadditive techniques are applied to the value of a two-player zero-sum differential game, rather than to the cost of an optimal control considered in the metric problem. Indeed, the metric problem can be seen as a degenerate two-player game (where Player 2 has no actions), i.e., an optimal-control formulation. While this usually produces convex Hamiltonians, Armstrong–Cardaliaguet [7] showed that, under positive homogeneity, the metric problem extends to certain nonconvex cases, with homogenization obtained via quantitative concentration rather than exact subadditivity. Secondly, our arguments rely primarily on a game-theoretic approach, exploiting the monotonicity (in the preferred direction e) of optimal trajectories, rather than on PDE methods. Thirdly, we treat noncoercive Hamiltonians, whereas [7, 8, 30] assume coercivity. This leads to several difficulties, including the fact that the spatial Lipschitz constants of solutions to (HJ_ε) are not uniformly bounded with respect to ε.

As an interesting output of the quantitative homogenization rate in Theorem 2.4, we show that the homogenization results described above extend to Hamiltonians that arise as local uniform limits of suitable sequences of Hamiltonians of the form (H), see Corollary 2.5, and that, in general, need not admit the same max-min representation. A further result in this direction is given by Theorem 2.6, where we extend homogenization to a class of nonconvex and noncoercive Lipschitz Hamiltonians that are not necessarily given by a max-min formula. This makes the game-theoretic approach even more notable, as it applies to Hamiltonians that do not a priori arise from a differential game. For this extension, we adapt the argument introduced in [29, Section 5] to put these Hamiltonians into the form (H) when p is constrained within a ball B_R, but using it to prove homogenization in the noncoercive setting is nontrivial and, as far as we know, new. The difficulty lies in the fact that, due to the lack of coercivity of the Hamiltonian, the Lipschitz constants in x of solutions to (HJ_ε) are not uniformly bounded in $\varepsilon > 0$, but instead blow up at rate $1/\varepsilon$. In view of this, we tailored the proof of Theorem 2.2 to this extension, ensuring that the constants appearing in the crucial estimates underpinning our arguments depend only on parameters that remain controlled when we perform the localization argument.

Our work is closely connected to the joint paper [32] of the third author. There, the authors introduced a new model of discrete-time games, called *percolation games*. They established a condition, called "oriented assumption", under which the value of the n-stage game converges as $n \to \infty$. Moreover, they sketched a heuristic link between the existence of such a limit and stochastic homogenization, explaining how assumptions on the discrete game can be translated into assumptions on Hamiltonians. The present paper provides the first formal implementation of this program: we identify precisely which Hamiltonians correspond to "oriented games" and turn the convergence result for oriented games into a rigorous result in stochastic homogenization. In this sense, our paper constitutes the first "proof of concept" that the methodology outlined in [32] can be fully validated. We refer the reader to [32, Section 4] for a detailed presentation of the methodology. While the proof of Proposition 5.5, which constitutes the central result of our paper, shares several ingredients with Theorem 2.3 in [32], notably the use of concentration inequalities and subadditivity, the differential game and Hamilton–Jacobi framework call for substantially different techniques. In particular, viscosity solutions and comparison principles play a central role, and their use is especially delicate here due to the noncoercive nature of the Hamiltonians under consideration.

The paper is organized as follows. In Section 2 we present the notation, the standing assumptions, and the statements of our homogenization results, namely Theorems 2.2, 2.4, and 2.6 and Corollary 2.5. In Section 3 we present the reduction strategy we will follow to prove these results. Some proofs are deferred to Appendix B. In Section 4 we prove the probabilistic concentration result. Section 5 is devoted to the proofs of Theorems 2.2, 2.4, and 2.6 and Corollary 2.5. Appendix A contains the deterministic PDE results, along with their proofs, that we use in the paper.

2. Assumptions and main results

Throughout the paper, we will denote by $d \in \mathbb{N}$ the dimension of the ambient space. We will denote either by $B_r(x_0)$ or $B(x_0, r)$ (resp., $\bar{B}_r(x_0)$ or $\bar{B}(x_0, r)$) the open (resp., closed) ball in $\mathbb{R}^d$ of radius $r > 0$ centered at $x_0 \in \mathbb{R}^d$. When $x_0 = 0$, we will more simply write B_r (resp., $\bar{B}_r$). The symbol $|\cdot|$ will denote the norm in $\mathbb{R}^k$, for any $k \geqslant 1$. We will write $\varphi_n \rightrightarrows \varphi$ in $E \subseteq \mathbb{R}^k$ to mean that the sequence of functions $(\varphi_n)_n$ uniformly converges to φ on compact subsets of E. We will denote by $\mathrm{C}(X)$, $\mathrm{UC}(X)$, $\mathrm{BUC}(X)$, and $\mathrm{Lip}(X)$ the space of continuous, uniformly continuous, bounded uniformly continuous, and Lipschitz continuous functions on a metric space X, respectively.

We will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, where $\mathbb{P}$ is a probability measure and $\mathcal{F}$ is the σ-algebra of $\mathbb{P}$-measurable subsets of Ω. We will assume that $\mathbb{P}$ is *complete* in the usual measure-theoretic sense. We will denote by $\mathcal{B}(\mathbb{R}^k)$ the Borel σ-algebra on $\mathbb{R}^k$ and equip the product spaces $\mathbb{R}^d \times \Omega$ and $\mathbb{R}^d \times A \times B \times \Omega$ with the product σ-algebras $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$ and $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$, respectively.

We will assume that $\mathbb{P}$ is invariant under the action of a one-parameter group $(\tau_x)_{x \in \mathbb{R}^d}$ of transformations $\tau_x \colon \Omega \to \Omega$. More precisely, we assume that the mapping $(x, \omega) \mapsto \tau_x \omega$ from $\mathbb{R}^d \times \Omega$ to Ω is measurable; $\tau_0 = \mathrm{id}$; $\tau_{x+y} = \tau_x \circ \tau_y$ for every $x, y \in \mathbb{R}^d$; and $\mathbb{P}(\tau_x(E)) = \mathbb{P}(E)$, for every $E \in \mathcal{F}$ and $x \in \mathbb{R}^d$. Lastly, we will assume that the action of $(\tau_x)_{x \in \mathbb{R}^d}$ is *ergodic*, i.e., any measurable function $\varphi \colon \Omega \to \mathbb{R}$ satisfying $\mathbb{P}(\varphi(\tau_x \omega) = \varphi(\omega)) = 1$ for every fixed $x \in \mathbb{R}^d$ is almost surely equal to a constant.

A random process $f \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be *stationary* with respect to $(\tau_x)_{x \in \mathbb{R}^d}$ if $f(x, \omega) = f(0, \tau_x \omega)$ for all $(x, \omega) \in \mathbb{R}^d \times \Omega$. Moreover, whenever the action of $(\tau_x)_{x \in \mathbb{R}^d}$ is ergodic, we refer to f as a *stationary ergodic* process.

Let $(X_i)_{i \in \mathcal{I}}$ be a (possibly uncountable) family of jointly measurable functions from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}$. We will say that the random variables $(X_i)_{i \in \mathcal{I}}$ exhibit *long-range independence* (or, equivalently, have *finite range of dependence*) if there exists $\rho > 0$ such that, for all pair of sets $S, \widehat{S} \subseteq \mathbb{R}^d$ such that their Hausdorff distance $d_H(S, \widehat{S}) \geqslant \rho$, the generated

σ-algebras $\sigma(\{X_i(x,\cdot) : i \in \mathcal{I},\ x \in S\})$ and $\sigma(\{X_i(x,\cdot) : i \in \mathcal{I},\ x \in \widehat{S}\})$ are independent, in symbols,

$$\sigma(\{X_i(x,\cdot) : i\in\mathcal{I}, x\in S\}) \perp\!\!\!\perp \sigma(\{X_i(x,\cdot) : i\in\mathcal{I}, x\in\widehat{S}\}) \quad \text{whenever } d_H(S,\widehat{S}) \geqslant \rho. \qquad \text{(FRD)}$$

In this paper, we will be concerned with the Hamilton–Jacobi equation of the form

$$\partial_t u + H(x, D_x u, \omega) = 0, \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{2.1}$$

where the Hamiltonian $H: \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is assumed to be stationary with respect to shifts in the x-variable, i.e., $H(x+y, p, \omega) = H(x, p, \tau_y \omega)$ for every $x, y \in \mathbb{R}^d$, $p \in \mathbb{R}^d$, and $\omega \in \Omega$, and to belong to the class $\mathcal{H}$ defined as follows.

Definition 2.1. A function $H: \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be in the class $\mathcal{H}$ if it is jointly measurable and it satisfies the following conditions, for some constant $\beta > 0$:

(H1) $|H(x, p, \omega)| \leqslant \beta(1 + |p|)$ for all $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$;

(H2) $|H(x, p, \omega) - H(x, q, \omega)| \leqslant \beta|p - q|$ for all $x, p, q \in \mathbb{R}^d$;

(H3) $|H(x, p, \omega) - H(y, p, \omega)| \leqslant \beta|x - y|$ for all $x, y, p \in \mathbb{R}^d$.

Assumptions (H1)–(H3) guarantee well-posedness in $\mathrm{C}([0,T) \times \mathbb{R}^d)$, for every fixed $T > 0$, of the Cauchy problem associated with equation (2.1) when the initial datum is in $\mathrm{UC}(\mathbb{R}^d)$. Furthermore, the solutions are actually in $\mathrm{UC}([0,T) \times \mathbb{R}^d)$. Solutions, subsolutions, and supersolutions of (2.1) will be always understood in the viscosity sense, see [14, 16, 18, 24], and implicitly assumed continuous, if not otherwise specified.

The purpose of this paper is to prove a homogenization result for equation (2.1) for a subclass of stationary Hamiltonians belonging to $\mathcal{H}$ that arise from differential game theory and that can be expressed in the following max-min form:

$$H(x, p, \omega) := \max_{b\in B} \min_{a\in A} \big\{ -\ell(x, a, b, \omega) - \langle f(a, b), p \rangle \big\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}\times\mathbb{R}\times\Omega. \qquad \text{(H)}$$

Here A, B are compact subsets of $\mathbb{R}^m$, for some integer m, and the product space $\mathbb{R}^d \times A \times B \times \Omega$ is equipped with the product σ-algebra $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$. The mapping $f: A \times B \to \mathbb{R}^d$ is a continuous vector-valued function satisfying the following assumption:

(f) (oriented dynamics) the dynamics given by $f: A \times B \to \mathbb{R}^d$ is *oriented*, i.e., there exist $\delta > 0$ and a direction $e \in \mathbb{S}^{d-1}$ such that

$$\langle f(a, b), e \rangle \geqslant \delta \quad \text{for all } (a, b) \in A \times B.$$

For the *running cost* $\ell: \mathbb{R}^d \times A \times B \times \Omega \to \mathbb{R}$, we will assume it is jointly measurable and satisfies

(ℓ_1) $\ell(\cdot,\cdot,\cdot,\omega) \in \mathrm{BUC}(\mathbb{R}^d \times A \times B)$ for every $\omega \in \Omega$;

(ℓ_2) there exists a constant $\mathrm{Lip}(\ell) > 0$ such that

$$|\ell(x,a,b,\omega) - \ell(y,a,b,\omega)| \leqslant \mathrm{Lip}(\ell)|x-y|$$

for all $x, y \in \mathbb{R}^d$, $a \in A$, $b \in B$, $\omega \in \Omega$;

(ℓ_3) ℓ is stationary with respect to x, i.e.,

$$\ell(x,a,b,\omega) = \ell(0,a,b,\tau_x\omega)$$

for all $x \in \mathbb{R}^d$, $a \in A$, $b \in B$, and $\omega \in \Omega$.

Throughout the paper, we will denote by $\mathcal{H}_{\mathrm{dG}}$ the subclass of Hamiltonians in $\mathcal{H}$ that can be put in the form (H) with f and ℓ satisfying assumptions (f) and (ℓ_1)–(ℓ_2), respectively. A Hamiltonian H belonging to $\mathcal{H}_{\mathrm{dG}}$ will be furthermore termed stationary to mean that assumption (ℓ_3) is in force. In the sequel, we will denote by $\|\ell\|_\infty$ the L^∞-norm of ℓ on $\mathbb{R}^d \times A \times B \times \Omega$, which is finite due to (ℓ_1), (ℓ_3), and the ergodicity assumption on Ω.

The proof of our homogenization result relies crucially on the oriented-dynamics assumption (f) and on the following long-range independence hypothesis:

(ℓ_4) (long-range independence) the random variables $(\ell(\cdot,a,b,\cdot))_{(a,b)\in A\times B}$ from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}$ exhibit long-range independence, i.e., there exists $\rho > 0$ such that (FRD) holds with $\mathcal{I} := A \times B$ and $X_i := \ell(\cdot,a,b,\cdot)$ where $i = (a,b)$.

The specific form (H) of the Hamiltonian allows us to represent solutions to equation (2.1) via suitable formulae issuing from differential games; see [29]. Indeed, let us denote

$$\mathbb{A}(T) := \{a\colon [0,T] \to A : a \text{ measurable}\}, \quad \mathbb{B}(T) := \{b\colon [0,T] \to B : b \text{ measurable}\}.$$

The sets A and B are to be regarded as action sets for Players 1 and 2, respectively. A *nonanticipating strategy* for Player 1 is a function $\alpha\colon \mathbb{B}(T) \to \mathbb{A}(T)$ such that, for all $b_1, b_2 \in \mathbb{B}(T)$ and $\tau \in [0,T]$,

$$b_1(\cdot) = b_2(\cdot) \text{ in } [0,\tau] \implies \alpha[b_1](\cdot) = \alpha[b_2](\cdot) \text{ in } [0,\tau].$$

We will denote by $\Gamma(T)$ the family of such nonanticipating strategies for Player 1. For every fixed $\omega \in \Omega$ and every $(t,x) \in (0,+\infty) \times \mathbb{R}^d$, let us set

$$v(t,x,\omega) := \sup_{\alpha\in\Gamma(t)} \inf_{b\in\mathbb{B}(t)} \left\{ \int_0^t \ell(y_x(s),\alpha[b](s),b(s),\omega)\,ds + g(y_x(t)) \right\}, \tag{2.2}$$

where $y_x\colon [0,t] \to \mathbb{R}^d$ is the solution of the ODE

$$\begin{cases} \dot{y}_x(s) = f(\alpha[b](s),b(s)) & \text{in } [0,t], \\ y_x(0) = x. \end{cases} \tag{ODE}$$

The function v defined by (2.2) is usually called the *value function*. It is the unique continuous viscosity solution of the unscaled HJ equation (HJ$_\varepsilon$) (i.e., with $\varepsilon = 1$) satisfying the initial condition $v(0,\cdot,\omega) = g$ on $\mathbb{R}^d$ for every $\omega \in \Omega$. We refer the reader to Appendix A.3 for more details and relevant results.

Our main result reads as follows.

Theorem 2.2. *Let H be a stationary Hamiltonian belonging to $\mathcal{H}_{\mathrm{dG}}$ and satisfying hypotheses (ℓ_4) and (f). Then the HJ equation $(\mathrm{HJ}_\varepsilon)$ homogenizes, i.e., there exist a continuous function $\bar{H}:\mathbb{R}^d \to \mathbb{R}$, called* the effective Hamiltonian, *and a set $\widehat{\Omega}$ of probability 1 such that, for every fixed $\omega \in \widehat{\Omega}$ and every $g \in \mathrm{UC}(\mathbb{R}^d)$, the solutions $u^\varepsilon(\cdot,\cdot,\omega)$ of $(\mathrm{HJ}_\varepsilon)$ satisfying $u^\varepsilon(0,\cdot,\omega) = g$ converge, locally uniformly on $[0,T) \times \mathbb{R}^d$ as $\varepsilon \to 0^+$, to the unique solution $\bar{u}$ of*

$$\begin{cases} \partial_t \bar{u} + \bar{H}(D_x \bar{u}) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ \bar{u}(0,\cdot) = g & \text{in } \mathbb{R}. \end{cases}$$

Furthermore, $\bar{H}$ satisfies (H1) *and* (H2).

Remark 2.3. We stress that a Hamiltonian of the form (H) with f satisfying condition (f) is never coercive. Indeed,

$$\lim_{\lambda\to-\infty} H(x,\lambda e,\omega) = +\infty, \qquad \lim_{\lambda\to+\infty} H(x,\lambda e,\omega) = -\infty \quad \text{for every } (x,\omega) \in \mathbb{R}^d \times \Omega.$$

Theorem 2.2 is actually derived from the following stronger quantitative result.

Theorem 2.4. *Let H be a stationary Hamiltonian belonging to $\mathcal{H}_{\mathrm{dG}}$ and satisfying hypotheses (ℓ_4) and (f). Let us denote by $\tilde{u}^\varepsilon_\theta$ the solution of $(\mathrm{HJ}_\varepsilon)$ satisfying $\tilde{u}^\varepsilon(0,x,\omega) = \langle \theta, x\rangle$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. Then there exists a deterministic function $\bar{H}:\mathbb{R}^d \to \mathbb{R}$ such that, for every fixed $\theta \in \mathbb{R}^d$, $T > 0$, and $R > 0$, we have*

$$\mathbb{P}\Big(\sup_{[0,T]\times B_R} |\tilde{u}^\varepsilon_\theta(t,x,\omega) - \langle\theta,x\rangle + t\bar{H}(\theta)| \geqslant K(-\varepsilon \ln \varepsilon)^{1/2} \Big) \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2, \tag{2.3}$$

for some constant K depending on R, T, $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$.

This quantitative estimate yields the following interesting consequence.

Corollary 2.5. *Let $(H_n)_n$ be a sequence of stationary Hamiltonians belonging to $\mathcal{H}_{\mathrm{dG}}$ and satisfying hypotheses (ℓ_4) and (f). Let us assume that the associated quantities β_n, ρ_n, δ_n, $\mathrm{Lip}(\ell_n)$, and $\|f_n\|_\infty$ satisfy the following bounds:*

$$C := \sup_n(\beta_n + \rho_n + \mathrm{Lip}(\ell_n) + \|f_n\|_\infty) < +\infty, \qquad \delta := \inf_n \delta_n > 0.$$

Let us denote by $\bar{H}_n$ the effective Hamiltonian associated with H_n, for each $n \in \mathbb{N}$. If $H_n(\cdot,\cdot,\omega) \rightrightarrows H(\cdot,\cdot,\omega)$ in $\mathbb{R}^d \times \mathbb{R}^d$ for every $\omega \in \Omega$, the following hold:

(i) *there exists $\bar{H}:\mathbb{R}^d \to \mathbb{R}$ satisfying* (H1), (H2) *with $\beta := \sup_n \beta_n$ such $\bar{H}_n \rightrightarrows \bar{H}$ in $\mathbb{R}^d$;*

(ii) *for every $\theta \in \mathbb{R}^d$, the solution $\tilde{u}^\varepsilon_\theta$ of $(\mathrm{HJ}_\varepsilon)$ subject to $\tilde{u}^\varepsilon(0,x,\omega) = \langle\theta,x\rangle$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$ satisfies (2.3), for every fixed $T > 0$ and $R > 0$, for some constant K depending on R, T, $|\theta|$, d, C, δ, $\mathrm{Lip}(\ell)$.*

In particular, the HJ equation $(\mathrm{HJ}_\varepsilon)$ homogenizes with effective Hamiltonian $\bar{H}$.

We emphasize that the limiting H above is a stationary Hamiltonian belonging to $\mathcal{H}$, but it need not lie in $\mathcal{H}_{\mathrm{dG}}$; namely, it cannot, in general, be written in the max-min form (H).

By combining suitable Lipschitz bounds for solutions to (HJ_ε) with Lipschitz initial data with a localization argument inspired by [29, Section 5], we further establish the homogenization results above for a different subclass of Hamiltonians in $\mathcal{H}$ that intersects, but is not contained in, $\mathcal{H}_{\mathrm{dG}}$. This subclass is described in the next theorem.

Theorem 2.6. *Let G be a stationary Hamiltonian belonging to $\mathcal{H}$ and satisfying the following assumption:*

- (G1) *the random variables $(G(\cdot, p, \cdot))_{p\in\mathbb{R}^d}$ from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}$ exhibit long-range independence, i.e., there exists $\rho > 0$ such that* (FRD) *holds with $\mathcal{I} := \mathbb{R}^d$ and $X_i := G(\cdot, p, \cdot)$, where $i = p$.*

Then the quantitative estimate stated in Theorem 2.4 and the homogenization result stated in Theorem 2.2 hold for any Hamiltonian H of the form $H(x, p, \omega) := G(x, \pi(p), \omega) + \langle p, v\rangle$, where v is a nonzero vector in $\mathbb{R}^d$ and $\pi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear map such that $\pi(v) = 0$.

Examples of Hamiltonians G lying in $\mathcal{H}$ and satisfying (G1) are those of the form $G(x, p,\omega) := G_0(p) + V(x,\omega)$, where G_0 belongs to $\mathcal{H}$ and $V\colon\mathbb{R}^d \times \Omega \to \mathbb{R}$ is a stationary function, globally bounded and Lipschitz on $\mathbb{R}^d$, which satisfies (FRD) with $\mathcal{I} := \{0\}$ and $X_0 := V$.

3. Reduction arguments for homogenization

In this section we describe the reduction strategy that we will follow to prove Theorem 2.2. The first step consists in noticing that, in order to prove homogenization for equation (HJ_ε), it is enough to restrict to linear initial data instead of any $g \in \mathrm{UC}(\mathbb{R}^d)$. The precise statement is the following.

Theorem 3.1. *Let H satisfy hypotheses* (H1)–(H3) *and denote by $\tilde u^\varepsilon_\theta$ the unique continuous solution of equation* (HJ_ε) *satisfying $\tilde u^\varepsilon_\theta(0, x, \omega) = \langle\theta, x\rangle$ for all $(x, \omega) \in \mathbb{R}^d \times \Omega$ and for every fixed $\theta \in \mathbb{R}^d$ and $\varepsilon > 0$. Assume there exists a function $\bar H\colon\mathbb{R}^d \to \mathbb{R}$ such that, for every $\theta \in \mathbb{R}^d$, the following convergence takes place for every ω in a set Ω_θ of probability* 1*:*

$$\tilde u^\varepsilon_\theta(t, x, \omega) \rightrightarrows \langle\theta, x\rangle - t\bar H(\theta) \quad \text{in } [0, T) \times \mathbb{R}^d \text{ as } \varepsilon \to 0^+. \tag{3.1}$$

Then $\bar H$ satisfies conditions (H1)-(H2). *Furthermore, there exists a set $\widehat\Omega$ of probability* 1 *such that, for every fixed $\omega \in \widehat\Omega$ and every $g \in \mathrm{UC}(\mathbb{R}^d)$, the unique function $u^\varepsilon(\cdot, \cdot, \omega) \in$* $\mathrm{C}([0, T) \times \mathbb{R}^d)$ *which solves* (HJ_ε) *with initial condition $u^\varepsilon(0, \cdot, \omega) = g$ in $\mathbb{R}^d$ converges, locally uniformly in $[0, T) \times \mathbb{R}^d$ as $\varepsilon \to 0^+$, to the unique solution $\bar u \in \mathrm{C}([0, T) \times \mathbb{R}^d)$ of*

$$\partial_t\bar u + \bar H(D_x\bar u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \tag{3.2}$$

with the initial condition $\bar u(0, \cdot) = g$.

This reduction argument, which is essentially of deterministic nature,was previously contained in the pioneering work [36] on periodic homogenization, at least as far as first-order HJ equations are concerned. This holds, in fact, even in the case when equation (HJ_ε) presents an additional (vanishing) diffusive and possibly degenerate term. A proof of this can be found in [25] and is given for Hamiltonians that are coercive in the p-variable. Such a class does not cover the kind of Hamiltonians we consider here, as pointed out in Remark 2.3. Yet the extension follows by arguing as in [25], with the only difference that one has to use a different comparison principle, namely Theorem A.3, in place of [25, Proposition 2.4]. We refer the reader to Appendix B for the detailed argument.

Theorem 3.1 yields, in particular, that the effective Hamiltonian $\bar{H}$ is identified by the following almost sure limit:

$$\bar{H}(\theta) := -\lim_{\varepsilon\to 0} \tilde{u}^\varepsilon_\theta(1,0,\omega) \quad \text{for every fixed } \theta \in \mathbb{R}^d.$$

The second step in the reduction consists in observing that, in order to prove the local uniform convergence required to apply Theorem 3.1, it is enough to prove it for a fixed value of the time variable, that we chose equal to 1.

Lemma 3.2. *Let $\omega \in \Omega$ and $\theta \in \mathbb{R}^d$ be fixed, and assume that*

$$\limsup_{\varepsilon\to 0^+} \sup_{y\in B_R} |\tilde{u}^\varepsilon_\theta(1,y,\omega) - \langle\theta, y\rangle + \bar{H}(\theta)| = 0 \quad \textit{for every } R > 0. \tag{3.3}$$

Then, for every $T > 0$,

$$\tilde{u}^\varepsilon_\theta(t,x,\omega) \rightrightarrows \langle\theta, x\rangle - t\bar{H}(\theta) \quad \textit{in } [0,T)\times\mathbb{R}^d. \tag{3.4}$$

Proof. Since ω will remain fixed throughout the proof, we will omit it from our notation. Let us fix $\theta \in \mathbb{R}^d$. We first take note of the following scaling relations:

$$\begin{aligned}\tilde{u}^\varepsilon_\theta(t,x) &= \varepsilon\tilde{u}^1_\theta(t/\varepsilon, x/\varepsilon) = t(\varepsilon/t)\tilde{u}^1_\theta(t/\varepsilon, x/\varepsilon)\\ &= t\tilde{u}^{\varepsilon/t}_\theta(1, x/t) \quad \text{for all } t>0 \text{ and } x\in\mathbb{R}^d.\end{aligned}$$

Fix $T > 0$. Then, for every fixed $r \in (0,T)$, we obtain

$$\begin{aligned}&\sup_{r\leqslant t\leqslant T}\sup_{y\in B_R} |\tilde{u}^\varepsilon_\theta(t,y) - \langle\theta,y\rangle + t\bar{H}(\theta)|\\ &\quad= \sup_{r\leqslant t\leqslant T}\sup_{y\in B_R} |t(\tilde{u}^{\varepsilon/t}_\theta(1,y/t) - \langle\theta, y/t\rangle + \bar{H}(\theta))|\\ &\quad\leqslant T \sup_{\varepsilon/T\leqslant\eta\leqslant\varepsilon/r}\ \sup_{z\in B_{R/r}} |\tilde{u}^\eta_\theta(1,z) - \langle\theta,z\rangle + \bar{H}(\theta)|.\end{aligned} \tag{3.5}$$

By (3.3), the right-hand side goes to 0 as $\varepsilon \to 0^+$. On the other hand, in view of Proposition A.6 (i) and of the fact that $\tilde{u}^\varepsilon_\theta(0,x) = \langle\theta, x\rangle$ for all $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant r}\sup_{y\in\mathbb{R}^d} |\tilde{u}^\varepsilon_\theta(t,y) - \langle\theta,y\rangle + t\bar{H}(\theta)| &\leqslant r|\bar{H}(\theta)| + \sup_{0\leqslant t\leqslant r}\sup_{y\in\mathbb{R}^d} |\tilde{u}^\varepsilon_\theta(t,y) - \langle\theta,y\rangle|\\ &\leqslant r(|\bar{H}(\theta)| + \|\ell\|_\infty + \|f\|_\infty).\end{aligned}$$

Assertion (3.4) follows from this and (3.5) by the arbitrariness of the choice of $r \in (0, T)$. ∎

In order to simplify some arguments, we find it convenient to work with solutions with zero initial datum. We can always reduce to this case, without any loss of generality, by setting $u_\theta^\varepsilon(t,x,\omega) := \tilde{u}_\theta^\varepsilon(t,x,\omega) - \langle \theta, x\rangle$ for all $(t,x,\omega) \in [0,T) \times \mathbb{R}^d \times \Omega$. The function u_θ^ε is the unique continuous function which solves equation (HJ_ε) with $H_\theta := H(\cdot, \theta + \cdot, \omega)$ in place of H and which satisfies the initial condition $u_\theta^\varepsilon(0,x,\omega) = 0$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. Note that the Hamiltonian H_θ is still given by the max-min formula (H) where ℓ is replaced by $\ell_\theta(x,a,b,\omega) := \ell(x,a,b,\omega) + \langle f(a,b), \theta\rangle$. Furthermore, ℓ_θ satisfies the same conditions (ℓ_1)–(ℓ_4).

The last reduction remark consists in noticing that the following rescaling relation holds:

$$u_\theta^\varepsilon(1,x,\omega) = \varepsilon u_\theta(1/\varepsilon, x/\varepsilon, \omega) \quad \text{for all } x \in \mathbb{R}^d \text{ and } \varepsilon > 0,$$

where we have denoted the function u_θ^ε with $\varepsilon = 1$ by u_θ.

In the light of all this, the proof of Theorem 2.2 is thus reduced to show that, for every fixed $\theta \in \mathbb{R}^d$, there exists a set Ω_θ of probability 1 such that, for every $\omega \in \Omega_\theta$, we have

$$\begin{aligned}\limsup_{\varepsilon \to 0^+} \sup_{y \in B_R} |u_\theta^\varepsilon(1,y,\omega) + \bar{H}(\theta)| &= \limsup_{t \to +\infty} \sup_{y \in B_{tR}} \left| \frac{u_\theta(t,y,\omega)}{t} + \bar{H}(\theta) \right| \\ &= 0 \quad \text{for all } R > 0,\end{aligned}$$

for some deterministic function $\bar{H} \colon \mathbb{R}^d \to \mathbb{R}$.

4. Probabilistic concentration

In this section we will prove, by making use of Azuma's martingale inequality, that $u_\theta(t,0,\cdot)$ is concentrated. For this, we will take advantage of the fact that u_θ can be expressed via the differential game theoretic formula (2.2) with initial datum $g \equiv 0$ and running cost $\ell_\theta(x,a,b,\omega) := \ell(x,a,b,\omega) + \langle f(a,b), \theta\rangle$. Here is where assumptions (FRD) and (f) play a crucial role, by ensuring altogether that the solution of (HJ_ε) is robust with respect to local perturbations of the cost function ℓ.

Throughout this section, we will weaken the conditions on the running cost $\ell \colon \mathbb{R}^d \times A \times B \times \Omega \to \mathbb{R}$ and assume that ℓ is only jointly measurable and may therefore fail to satisfy conditions (ℓ_1)–(ℓ_2). In particular, no continuity and stationarity conditions with respect to x will be required.

We start by recalling a classic theorem on concentration of martingales, also known as Azuma's inequality.

Lemma 4.1 (Concentration of martingales [12, 33]). *Let $(X_n)_{n \in \mathbb{N}}$ be a martingale and $(c_n)_{n \in \mathbb{N}}$ a real sequence such that, for all $n \in \mathbb{N}$, $|X_n - X_{n+1}| \leqslant c_n$ almost surely. Then,* *for all* $n \in \mathbb{N}$ *and* $M > 0$,

$$\mathbb{P}(|X_n - X_0| \geqslant M) \leqslant 2\exp\left(\frac{-M^2}{2\sum_{m=0}^{n-1} c_m^2}\right).$$

The probabilistic concentration result mentioned before is stated as follows.

Proposition 4.2. *There exists a constant* $c = c(\rho, \delta, \mathrm{Lip}(\ell), \|f\|_\infty) > 0$*, only depending on* ρ*,* δ*,* $\mathrm{Lip}(\ell)$*, and* $\|f\|_\infty$*, such that, for all* $M > 0$ *and* $t \geqslant 1$*,*

$$\mathbb{P}(|u_\theta(t,0,\cdot) - U_\theta(t)| \geqslant M\sqrt{t}) \leqslant \exp(-cM^2),$$

where $U_\theta(t) := \mathbb{E}[u_\theta(t,0,\cdot)]$ *denotes the expectation of the random variable* $u_\theta(t,0,\cdot)$*.*

Remark 4.3. We have tailored the proof of Proposition 4.2 in such a way that the constant c appearing in the statement does not depend on $\|\ell\|_\infty$. This is crucial in view of the homogenization result provided in Theorem 2.6.

To prove Proposition 4.2, we need a technical lemma first. The result is deterministic, hence the dependence on ω will be omitted. It expresses that, thanks to the condition (f) on the dynamics, we can control the variation of the value function as we change the running cost on a strip that is orthogonal to the direction e.

Lemma 4.4. *For any given pair* $R < \widehat{R}$ *in* $\mathbb{R}$*, let us define the strip between* R *and* $\widehat{R}$ *as follows:*

$$S_{R,\widehat{R}} := \{x : \langle x, e\rangle \in [R, \widehat{R}]\}.$$

Let $\ell, \hat{\ell} \colon \mathbb{R}^d \times A \times B \to \mathbb{R}$ *be Borel-measurable bounded running costs and let* $f \colon A \times B \to \mathbb{R}^d$ *be a continuous vector-valued function satisfying condition (*f*). For every fixed* $\theta \in \mathbb{R}^d$*, let us denote by* $u_\theta(t,x)$*,* $\hat{u}_\theta(t,x)$ *the value functions defined via (2.2) with initial datum* $g \equiv 0$ *and running cost* $\ell_\theta(x,a,b,\omega) := \ell(x,a,b,\omega) + \langle f(a,b), \theta\rangle$ *and* $\hat{\ell}_\theta(x,a,b,\omega) := \hat{\ell}(x,a,b,\omega) + \langle f(a,b), \theta\rangle$*, respectively. If* $\ell = \hat{\ell}$ *on* $(\mathbb{R}^d \setminus S_{R,\widehat{R}}) \times A \times B$*, we have*

$$|u_\theta(t,x) - \hat{u}_\theta(t,x)| \leqslant \frac{\widehat{R} - R}{\delta}\|\ell - \hat{\ell}\|_\infty \quad \textit{for all } (t,x) \in (0,+\infty) \times \mathbb{R}^d.$$

We point out the generality of the previous statement: no continuity conditions on the running costs ℓ, $\hat{\ell}$ are assumed in the above statement, and the functions u_θ, $\hat{u}_\theta$ are still well defined.

Proof of Lemma 4.4. Fix $t > 0$ and $x, \theta \in \mathbb{R}^d$. Let $R < \widehat{R}$ be arbitrary in $\mathbb{R}$ and consider the strip $S_{R,\widehat{R}}$. Fix controls $\alpha \in \Gamma(t)$ and $b \in \mathbb{B}(t)$ and consider the solution $y_x \colon [0,t] \to \mathbb{R}^d$ of the ODE

$$\begin{cases} \dot{y}_x(s) = f(\alpha[b](s), b(s)) & \text{in } [0,t], \\ y_x(0) = x. \end{cases}$$

From the orientation of the game, the map $s \mapsto \langle y_x(s), e\rangle$ is strictly increasing in $[0,t]$. More precisely,

$$\frac{d}{ds}\langle y_x(s), e\rangle = \langle \dot{y}_x(s), e\rangle = \langle f(\alpha[b](s), b(s)), e\rangle \geqslant \delta \quad \text{for all } s \in [0,t]. \tag{4.1}$$

If $\langle y_x(0), e\rangle = \langle x, e\rangle \geqslant \widehat{R}$, we derive that the curve y_x always lies in $\mathbb{R}^d \setminus S_{R,\widehat{R}}$ and the assertion trivially follows since $\ell = \hat{\ell}$ on $(\mathbb{R}^d \setminus S_{R,\widehat{R}}) \times A \times B$. Let us then assume that $\langle y_x(0), e\rangle < \widehat{R}$ and define two exit times t_1 and t_2 as follows:

$$t_1 := \inf\{s \in [0,t] : \langle y_x(s), e\rangle > R\}, \quad t_2 := \sup\{s \in [0,t] : R < \langle y_x(s), e\rangle < \widehat{R}\},$$

where we agree that $t_1 = t_2 = t$ when the sets above are empty. Notice that $t_2 - t_1 \leqslant (\widehat{R} - R)/\delta$. Indeed, if $t_2 - t_1 > 0$, then $t_1 < t$ and, by continuity of y_x and (4.1), we have $\langle y_x(t_1), e\rangle = R$. From (4.1), we infer

$$\widehat{R} - R \geqslant \langle y_x(t_2) - y_x(t_1), e\rangle = \int_{t_1}^{t_2} \langle f(\alpha[b](s), b(s)), e\rangle \, ds \geqslant (t_2 - t_1)\delta,$$

as was claimed. Also, if $t_2 < t$, then, from (4.1), we have that $\langle y_x(s), e\rangle > \widehat{R}$ for every $s \in (t_2, t)$. Consider deterministic running costs $\ell, \hat{\ell} \colon \mathbb{R}^d \times A \times B \to \mathbb{R}$ such that $\ell = \hat{\ell}$ on $(\mathbb{R}^d \setminus S_{R,\widehat{R}}) \times A \times B$. Then, in view of the previous remarks, we get

$$\begin{aligned}
&\left| \int_0^t \big(\ell_\theta(y_x(s), \alpha(s), \beta(s)) - \hat{\ell}_\theta(y_x(s), \alpha(s), \beta(s))\big)\, ds \right| \\
&\quad = \left| \int_{t_1}^{t_2} \big(\ell(y_x(s), \alpha(s), \beta(s)) - \hat{\ell}(y_x(s), \alpha(s), \beta(s))\big)\, ds \right| \\
&\quad \leqslant (t_2 - t_1)\|\ell - \hat{\ell}\|_\infty \leqslant \frac{\widehat{R} - R}{\delta}\|\ell - \hat{\ell}\|_\infty.
\end{aligned}$$

The assertion easily follows from this by arbitrariness of the choice of the control $b \in \mathbb{B}(t)$ and of the strategy $\alpha \in \Gamma(t)$. ∎

In the proof of Proposition 4.2, we look at the conditional expectation of $u_\theta(t, 0, \cdot)$ given the running cost in a half-space that contains the origin and whose boundary is a hyperplane orthogonal to e. Considering a suitable increasing sequence of such half-spaces, we define the corresponding martingale and, by Lemma 4.4, observe that it has bounded differences. Then, applying Azuma's inequality, see Lemma 4.1, we conclude.

Proof of Proposition 4.2. Fix $t > 0$. Recall the long-range independence parameter $\rho \geqslant 1$. Denote $n := \lceil t \rceil$ and $C := \lceil \|f\|_\infty/\rho \rceil$, where $\lceil \cdot \rceil$ stands for the upper integer part. Note that, for all $\alpha \in \Gamma(t)$ and $b \in \mathbb{B}(t)$, the solution $y_0 \colon [0,t] \to \mathbb{R}^d$ of the ODE

$$\begin{cases} \dot{y}_0(s) = f(\alpha[b](s), b(s)) & \text{in } [0,t], \\ y_0(0) = 0 \end{cases}$$

satisfies that $|y_0(s)| \leqslant \rho C n$ for all $s \in [0,t]$. For $r \in \{1,2,\ldots,Cn\}$, let $\mathcal{F}_r$ be the σ-algebra generated by the random variables $\{\ell(x,a,b,\cdot) : a \in A,\ b \in B,\ x \in \mathbb{R}^d,\ \langle x,e\rangle \leqslant \rho r\}$, and let $\mathcal{F}_0$ be the trivial σ-algebra. We claim that, for all $0 \leqslant r < Cn$, we have that

$$\big|\mathbb{E}[u_\theta(t,0,\cdot)|\mathcal{F}_{r+1}] - \mathbb{E}[u_\theta(t,0,\cdot)|\mathcal{F}_r]\big| \leqslant \frac{12\rho^2}{\delta}\operatorname{Lip}(\ell). \tag{4.2}$$

Indeed, for $R < \hat{R}$ in $\mathbb{R}$, define the strip

$$S_{R,\hat{R}} := \big\{x \in \mathbb{R}^d : \langle x,e\rangle \in [R,\hat{R}]\big\}.$$

Fix $0 \leqslant r < Cn$, and consider $\hat{\ell}$ defined by $\hat{\ell}(x,a,b,\omega) := \ell(x - 3\rho e, a, b, \omega)$ for $(x,a,b,\omega) \in S_{\rho(r-1),\rho(r+2)} \times A \times B \times \Omega$, and $\hat{\ell}(x,a,b,\omega) := \ell(x,a,b,\omega)$ otherwise.

On the one hand, by Lemma 4.4, almost surely we have that

$$|u_\theta(t,0,\omega) - \hat{u}_\theta(t,0,\omega)| \leqslant \frac{3\rho}{\delta}\|\ell - \hat{\ell}\|_\infty y \leqslant \frac{3\rho}{\delta}\operatorname{Lip}(\ell)2\rho = \frac{6\rho^2}{\delta}\operatorname{Lip}(\ell), \tag{4.3}$$

where u_θ (resp., $\hat{u}_\theta$) is the value function associated via (2.2) with the running cost $\ell_\theta := \ell + \langle f(a,b),\theta\rangle$ (resp., $\hat{\ell}_\theta := \hat{\ell} + \langle f(a,b),\theta\rangle$) and initial datum $g \equiv 0$. On the other hand, $\hat{u}_\theta$ is independent of $\{\ell(x,a,b,\cdot) : a \in A,\ b \in B,\ x \in \mathbb{R}^d,\ \langle x,e\rangle \in [\rho r, \rho(r+1)]\}$. Indeed, by definition, $\hat{u}_\theta$ is measurable with respect to the σ-field generated by $\{\hat{\ell}(x,a,b,\cdot) : a \in A, b \in B,\ x \in \mathbb{R}^d\}$. Moreover, by long-range independence, $\hat{\ell}(x,a,b,\cdot)$ is independent of $\{\ell(x,a,b,\cdot) : a \in A,\ b \in B,\ x \in \mathbb{R}^d,\ \langle x,e\rangle \in [\rho r,\rho(r+1)]\}$. Therefore,

$$\mathbb{E}[\hat{u}_\theta(t,0,\cdot)|\mathcal{F}_{r+1}] = \mathbb{E}[\hat{u}_\theta(t,0,\cdot)|\mathcal{F}_r]. \tag{4.4}$$

Finally, using successively (4.3) and (4.4), we prove the claim in (4.2):

$$\begin{aligned}&\big|\mathbb{E}[u_\theta(t,0,\cdot)|\mathcal{F}_{r+1}] - \mathbb{E}[u_\theta(t,0,\cdot)|\mathcal{F}_r]\big| \\ &\quad\leqslant \big|\mathbb{E}[\hat{u}_\theta(t,0,\cdot)|\mathcal{F}_{r+1}] - \mathbb{E}[\hat{u}_\theta(t,0,\cdot)|\mathcal{F}_r]\big| + \frac{12\rho^2}{\delta}\operatorname{Lip}(\ell) = \frac{12\rho^2}{\delta}\operatorname{Lip}(\ell).\end{aligned}$$

For $r \leqslant Cn$, denote $W_r := \mathbb{E}[u_\theta(t,0,\cdot)|\mathcal{F}_r]$. The process $(W_r)_{r\leqslant Cn}$ is a martingale, and $W_0 = \mathbb{E}[u_\theta(t,0,\cdot)] = U_\theta(t)$. Moreover, since $u_\theta(t,0,\cdot)$ is $\mathcal{F}_{Cn}$-measurable, we have that $W_{Cn}(\cdot) = u_\theta(t,0,\cdot)$. By Azuma's inequality, see Lemma 4.1, applied to the martingale $(W_r)_{r\leqslant Cn}$ and (4.2), for all $M \geqslant 0$,

$$\mathbb{P}(|u_\theta(t,0,\cdot) - U_\theta(t)| \geqslant M) = \mathbb{P}(|W_{Cn} - W_0| \geqslant M) \leqslant 2\exp\bigg(-\frac{M^2}{2Cn(\frac{12\rho^2}{\delta}\operatorname{Lip}(\ell))^2}\bigg).$$

Therefore, there exists a constant $c > 0$, only depending on ρ, δ, $\operatorname{Lip}(\ell)$, and $\|f\|_\infty$, but not on θ, such that, for all $M > 0$ and $t \geqslant 1$, we have that

$$\mathbb{P}(|u_\theta(t,0,\cdot) - U_\theta(t)| \geqslant M\sqrt{t}) \leqslant \exp(-cM^2),$$

as was asserted. ∎

5. Proof of the homogenization results

This section is devoted to the proofs of our homogenization results, namely Theorems 2.2, 2.4, and 2.6 and Corollary 2.5. We will denote by H a stationary Hamiltonian belonging to the class $\mathcal{H}$, and by $u_\theta(\cdot,\cdot,\omega)$ the solution of the equation (HJ$_\varepsilon$) with $\varepsilon = 1$, $H_\theta := H(\cdot,\theta+\cdot,\omega)$ in place of H, and initial condition $u_\theta(0,x,\omega) = 0$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$.

Note that, since the initial condition is zero, the function u_θ is stationary, i.e., for all $(t,x,\omega) \in [0,+\infty) \times \mathbb{R}^d \times \Omega$ and $y \in \mathbb{R}^d$ we have $u_\theta(t,x+y,\omega) = u_\theta(t,x,\tau_y\omega)$, $\mathbb{P}$-almost surely in Ω.

We also recall that $U_\theta(t)$ denotes the expectation of the random variable $u_\theta(t,0,\cdot)$.

5.1. Proof of Theorem 2.2

In this subsection we will furthermore assume that H belongs to the class $\mathcal{H}_{\mathrm{dG}}$, so that $u_\theta(t,x,\omega)$ can be represented via (2.2) with initial datum $g \equiv 0$ and running cost

$$\ell_\theta(x,a,b,\omega) := \ell(x,a,b,\omega) + \langle f(a,b),\theta\rangle.$$

This allows us to make use of the results obtained in Section 4.

According to Section 3, the proof of Theorem 2.2 boils down to establishing the following result.

Proposition 5.1. *There exists a deterministic function $\bar{H}\colon \mathbb{R}^d \to \mathbb{R}$ such that, for every fixed $\theta \in \mathbb{R}^d$, we have*

$$\limsup_{\varepsilon\to 0} \sup_{y\in B_R} |u_\theta^\varepsilon(1,y,\omega) + \bar{H}(\theta)| = \limsup_{t\to+\infty} \sup_{y\in B_{tR}} \left|\frac{u_\theta(t,y,\omega)}{t} + \bar{H}(\theta)\right| = 0 \quad \textit{for all } R>0, \tag{5.1}$$

for every ω in a set Ω_θ of probability 1.

Proposition 5.1 is actually a consequence of the following stronger quantitative result.

Proposition 5.2. *There exists a deterministic function $\bar{H}\colon \mathbb{R}^d \to \mathbb{R}$ such that, for every fixed $\theta \in \mathbb{R}^d$ and $R > 0$, we have*

$$\mathbb{P}\left(\sup_{y\in B_{tR}} \left|\frac{u_\theta(t,y,\omega)}{t} + \bar{H}(\theta)\right| \geqslant K\left(\frac{\ln t}{t}\right)^{1/2}\right) \leqslant t^{-2} \quad \textit{for all } t \geqslant 2, \tag{5.2}$$

for some constant K depending on R, $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$.

We first show how Proposition 5.1 follows from Proposition 5.2. For later use, we isolate the argument in the following lemma. Note that u_θ satisfies the required hypotheses by Theorem A.5 and the condition $u_\theta(0,\cdot,\omega) = 0$ on $\mathbb{R}^d$.

Lemma 5.3. *Let $u\colon [0,+\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a function such that*

$$|u(t,x,\omega) - u(s,x,\omega)| \leqslant \kappa|t-s|, \quad |u(t,x,\omega)| \leqslant \kappa(1+t)$$

for all $t, s \in [0, +\infty)$, $x \in \mathbb{R}^d$, *and* $\omega \in \Omega$, *and for some constant* $\kappa > 0$. *Let us assume that* (5.2) *holds for some* $R > 0$ *and* $\bar{H}(\theta) \in \mathbb{R}$. *Then* (5.1) *holds* $\mathbb{P}$*-almost surely for the same* R *and* $\bar{H}(\theta)$.

Proof. Let us set

$$X_n(\omega) := \sup_{y \in B_{nR}} \left| \frac{u(n, y, \omega)}{n} + \bar{H}(\theta) \right| \quad \text{and} \quad \alpha_n := K\Big(\frac{\ln n}{n}\Big)^{1/2} \quad \text{for all } n \in \mathbb{N}.$$

We claim that $\limsup_n X_n(\omega) = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Indeed, note that

$$\Big\{\limsup_n X_n > 0\Big\} \subseteq \bigcap_{k=2}^{+\infty} \underbrace{\bigcup_{n=k}^{+\infty} \{X_n > \alpha_n\}}_{E_k}.$$

By hypothesis we have

$$\mathbb{P}\Big(\limsup_n X_n > 0\Big) \leqslant \lim_k \mathbb{P}(E_k) \leqslant \lim_k \sum_{n=k}^{+\infty} \frac{1}{n^2} = 0,$$

as was claimed. The assertion follows from this by applying the subsequent lemma with $v := u(\cdot, \cdot, \omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. ∎

Lemma 5.4. *Let* $v \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ *be a function such that*

$$|v(t, x) - v(s, x)| \leqslant \kappa |t - s|, \quad |v(t, x)| \leqslant \kappa(1 + t) \quad \textit{for all } t, s \in [0, +\infty) \textit{ and } x \in \mathbb{R}^d$$

for some constant $\kappa > 0$. *For every* $a \in \mathbb{R}$ *and* $R > 0$ *we have*

$$\limsup_{t \to +\infty} \sup_{x \in B_{tR}} \left| \frac{v(t, x)}{t} + a \right| = \limsup_{n \to +\infty} \sup_{x \in B_{nR}} \left| \frac{v(n, x)}{n} + a \right|. \tag{5.3}$$

Proof. It suffices to prove (5.3) with $\leqslant$ in place of $=$, the reverse inequality being obvious. For every $t \geqslant 1$ we have

$$\begin{aligned} \sup_{x \in B_{tR}} \left| \frac{v(t, x)}{t} + a \right| &\leqslant \sup_{x \in B_{\lceil t \rceil R}} \left| \frac{v(t, x)}{t} + a \right| \\ &\leqslant \sup_{x \in B_{\lceil t \rceil R}} \left| \frac{v(\lceil t \rceil, x)}{\lceil t \rceil} + a \right| + \sup_{x \in B_{\lceil t \rceil R}} \left| \frac{v(\lceil t \rceil, x)}{\lceil t \rceil} - \frac{v(t, x)}{t} \right|. \end{aligned}$$

Now

$$\begin{aligned} \sup_{x \in B_{\lceil t \rceil R}} \left| \frac{v(\lceil t \rceil, x)}{\lceil t \rceil} - \frac{v(t, x)}{t} \right| &\leqslant \sup_{x \in B_{\lceil t \rceil R}} \left| \frac{v(\lceil t \rceil, x)}{\lceil t \rceil} - \frac{v(t, x)}{\lceil t \rceil} \right| + \kappa(1 + t) \left| \frac{1}{\lceil t \rceil} - \frac{1}{t} \right| \\ &\leqslant \frac{\kappa}{\lceil t \rceil} + \frac{\kappa(1 + t)}{t \lceil t \rceil}, \end{aligned}$$

so the assertion follows by sending $t \to +\infty$. ∎

Let us now proceed to prove Proposition 5.2. To this aim, we start by stating the following fact.

Proposition 5.5. *The following statements hold:*

(i) *For all $\theta \in \mathbb{R}^d$, we have that $U_\theta(t)/t$ converges as t goes to infinity.*

(ii) *Let $\bar{H}: \mathbb{R}^d \to \mathbb{R}$ be defined by*

$$\bar{H}(\theta) := -\lim_{t\to+\infty} \frac{U_\theta(t)}{t} \quad \text{for all } \theta \in \mathbb{R}^d.$$

For every fixed $\theta \in \mathbb{R}^d$, there exists a constant K, depending on $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$, such that

$$\left|\frac{U_\theta(t)}{t} + \bar{H}(\theta)\right| \leqslant K\left(\frac{\ln t}{t}\right)^{1/2} \quad \text{for all } t \geqslant 2.$$

The proof of Proposition 5.5 consists in showing that $U_\theta(t)$ satisfies an approximated subadditive inequality. We postpone that proof and first explain how to use this fact to establish Proposition 5.2. In doing so, we also rely crucially on Proposition 4.2.

Proof of Proposition 5.2. According to Proposition 5.5, we define $\bar{H}: \mathbb{R}^d \to \mathbb{R}$ by setting

$$\bar{H}(\theta) := -\lim_{t\to+\infty} \frac{U_\theta(t)}{t} = -\lim_{t\to+\infty} \frac{\mathbb{E}[u_\theta(t,0,\cdot)]}{t} \quad \text{for all } \theta \in \mathbb{R}^d.$$

Consider $n \in \mathbb{N}$ and a discretization $Z_n := \frac{1}{n^2}\mathbb{Z}^d \cap B_{nR}$ of B_{nR} consisting of at most $(2n^3R)^d$ points and such that any point of B_{nR} is $(\sqrt{d}/n^2)$-close to Z_n. Consider $M > 0$ arbitrary. Then, for $n \in \mathbb{N}$, we focus on the event

$$\sup_{z\in Z_n} \left|\frac{u_\theta(n,z,\omega)}{n} + \bar{H}(\theta)\right| \geqslant M.$$

Using first the union bound, then Proposition 5.5 (ii), and last the stationarity of u_θ, we have that, for n large enough,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{z\in Z_n} \left|\frac{u_\theta(n,z,\cdot)}{n} + \bar{H}(\theta)\right| \geqslant M + K\Big(\frac{\ln n}{n}\Big)^{1/2}\Big) \\
&\leqslant \sum_{z\in Z_n} \mathbb{P}\Big(\left|\frac{u_\theta(n,z,\cdot)}{n} + \bar{H}(\theta)\right| \geqslant M + K\Big(\frac{\ln n}{n}\Big)^{1/2}\Big) \\
&\leqslant \sum_{z\in Z_n} \mathbb{P}\Big(\left|\frac{u_\theta(n,z,\cdot)}{n} - \frac{U_\theta(n)}{n}\right| \geqslant M\Big) \\
&= |Z_n|\mathbb{P}\Big(\left|\frac{u_\theta(n,0,\cdot)}{n} - \frac{U_\theta(n)}{n}\right| \geqslant M\Big).
\end{aligned}$$

Note that, by the choice of Z_n and Proposition 4.2,

$$|Z_n|\mathbb{P}\Big(\left|\frac{u_\theta(n,0,\cdot)}{n} - \frac{U_\theta(n)}{n}\right| \geqslant M\Big) \leqslant (2n^3R)^d \exp(-cM^2 n),$$

and combining with the previous inequality, we get

$$\mathbb{P}\Big(\sup_{z\in Z_n}\Big|\frac{u_\theta(n,z,\cdot)}{n}+\bar{H}(\theta)\Big|\geqslant M+K\Big(\frac{\ln n}{n}\Big)^{1/2}\Big)\leqslant (2n^3R)^d\exp(-cM^2n).$$

We deduce that there exists a constant $K'>K$ large enough, depending on R, $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$, such that

$$\mathbb{P}\Big(\sup_{z\in Z_n}\Big|\frac{u_\theta(n,z,\cdot)}{n}+\bar{H}(\theta)\Big|\geqslant M+K'\Big(\frac{\ln(n)}{n}\Big)^{1/2}\Big)$$
$$\leqslant \exp(-cM^2n)\quad \text{for all } n\geqslant 2. \tag{5.4}$$

By applying Theorem A.5 with H_θ in place of H we derive that $u_\theta(t,\cdot,\omega)$ is βt-Lipschitz in $\mathbb{R}^d$ and $u_\theta(\cdot,x,\omega)$ is $\beta(1+|\theta|)$-Lipschitz in $[0,+\infty)$.[1] We infer

$$|u_\theta(t,x,\omega)|=|u_\theta(t,x,\omega)-u_\theta(0,x,\omega)|\leqslant \beta t(1+|\theta|) \tag{5.5}$$

for all $(t,x,\omega)\in[0,+\infty)\times\mathbb{R}^d\times\Omega$. The latter implies

$$\begin{aligned}\sup_{y\in B_{tR}}\Big|\frac{u_\theta(t,y,\omega)}{t}+\bar{H}(\theta)\Big| &\leqslant \sup_{y\in B_{\lceil t\rceil R}}\Big|\frac{u_\theta(t,y,\omega)}{t}+\bar{H}(\theta)\Big|\\ &\leqslant \sup_{y\in B_{\lceil t\rceil R}}\Big|\frac{u_\theta(\lceil t\rceil,y,\omega)}{t}+\bar{H}(\theta)\Big|+\frac{\beta(1+|\theta|)}{t}\\ &\leqslant \sup_{y\in B_{\lceil t\rceil R}}\Big|\frac{u_\theta(\lceil t\rceil,y,\omega)}{\lceil t\rceil}+\bar{H}(\theta)\Big|+2\frac{\beta(1+|\theta|)}{t}\\ &\leqslant \sup_{z\in Z_{\lceil t\rceil}}\Big|\frac{u_\theta(\lceil t\rceil,z,\omega)}{\lceil t\rceil}+\bar{H}(\theta)\Big|+\beta\frac{\sqrt{d}}{\lceil t\rceil^2}+2\frac{\beta(1+|\theta|)}{t}.\end{aligned}$$

In view of (5.4) and of the fact that $1/t=o(\log(t)^{1/2}t^{-1/2})$, we deduce that there exists a constant K'', depending on R, $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$, such that, for all $M>0$,

$$\mathbb{P}\Big(\sup_{y\in B_{tR}}\Big|\frac{u_\theta(t,y,\omega)}{t}+\bar{H}(\theta)\Big|\geqslant M+K''\Big(\frac{\ln t}{t}\Big)^{1/2}\Big)\leqslant \exp(-cM^2t)\quad \text{for all } t\geqslant 2.$$

Taking $M:=(\ln t/t)^{1/2}\sqrt{2}c^{-1/2}$, we get the result. ∎

Let us turn back to the proof of Proposition 5.5.

Proof of Proposition 5.5. Fix $\theta\in\mathbb{R}^d$ and $t>0$. Denote $\mathbf{B}(t):=\bar{B}(0,t\|f\|_\infty)$. Note that, for all $\alpha\in\Gamma(t)$ and $b\in\mathbb{B}(t)$, the solution of the ODE

$$\begin{cases}\dot{y}_0(s)=f(\alpha[b](s),b(s)) & \text{in } [0,t],\\ y_0(0)=0,\end{cases}$$

[1] Since $g=0$ and the Hamiltonian H_θ satisfies (H1*)–(H3*) with $\beta_1:=\beta(1+|\theta|)$ and $\beta_2=\beta_3:=\beta$.

is such that, for all $s \in [0,t]$, we have that $y_0(s) \in \mathbf{B}(t)$. From Theorem A.5 with H_θ in place of H we derive that $u_\theta(t,\cdot,\omega)$ is βt-Lipschitz in $\mathbb{R}^d$ and $u_\theta(\cdot,x,\omega)$ is $\beta(1+|\theta|)$-Lipschitz in $[0,+\infty)$. We will discretize $\mathbf{B}(t)$ accordingly. Consider a finite set Z of size $\lceil 2\|f\|_\infty t(\beta t)\rceil^d$ such that any point of $\mathbf{B}(t)$ is $(\beta t)^{-1}$-close to a point in Z. Using the Lipschitz property of $u_\theta(t,\cdot,\omega)$, the union bound, the fact that variables $u_\theta(t,x,\omega)$ and $u_\theta(t,0,\omega)$ have the same distribution by stationarity and the concentration proven in Proposition 4.2, we get that, for all $M>0$,

$$\begin{aligned}\mathbb{P}(\exists x \in \mathbf{B}(t),\ |u_\theta(t,x,\cdot)-U_\theta(t)| \geqslant M) &\leqslant \mathbb{P}(\exists z \in Z,\ |u_\theta(t,z,\cdot)-U_\theta(t)| \geqslant M-1)\\ &\leqslant \sum_{z\in Z}\mathbb{P}(|u_\theta(t,z,\cdot)-U_\theta(t)| \geqslant M-1)\\ &= \sum_{z\in Z}\mathbb{P}(|u_\theta(t,0,\cdot)-U_\theta(t)| \geqslant M-1)\\ &\leqslant \lceil 2\|f\|_\infty \beta t^2\rceil^d \exp(-c(M-1)^2/t),\end{aligned}$$

where c is a constant depending on ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$, but not on θ.

Taking $M_t := (\ln(\lceil 2\|f\|_\infty \beta t^2\rceil^d t))^{1/2} c^{-1/2} t^{1/2} + 1$ in place of M in the above inequality, we get

$$\mathbb{P}(\exists x \in \mathbf{B}(t),\ |u_\theta(t,x,\cdot)-U_\theta(t)| \geqslant M_t) \leqslant t^{-2}.$$

In particular,

$$\begin{aligned}&\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) \leqslant U_\theta(t)-M_t\Big) \leqslant t^{-2},\\ &\mathbb{P}\Big(\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) \geqslant M_t+U_\theta(t)\Big) \leqslant t^{-2}.\end{aligned} \tag{5.6}$$

We claim that there exists a constant $\widehat{K}$ depending on $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$ such that

$$\begin{aligned}&\mathbb{E}\Big[\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\Big] \geqslant U_\theta(t)-\widehat{K}(t\ln t)^{1/2},\\ &\mathbb{E}\Big[\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\Big] \leqslant U_\theta(t)+\widehat{K}(t\ln t)^{1/2}\end{aligned} \tag{5.7}$$

for all $t \geqslant 1$. Let us prove the first inequality. We recall that $|u_\theta(t,x,\omega)| \leqslant \beta t(1+|\theta|)$, see (5.5), in particular $|U_\theta(t)| \leqslant \beta t(1+|\theta|)$. Then

$$\begin{aligned}\mathbb{E}\Big[\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\Big] \geqslant\ &\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) \leqslant U_\theta(t)-M_t\Big)(-\beta t(1+|\theta|))\\ &+\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) > U_\theta(t)-M_t\Big)(U_\theta(t)-M_t).\end{aligned}$$

Now we use the identity

$$\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) > U_\theta(t)-M_t\Big) = 1-\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) \leqslant U_\theta(t)-M_t\Big)$$

and the fact, observed above, that $-U_\theta \geqslant -\beta t(1+|\theta|)$. We get

$$\begin{aligned}\mathbb{E}\Big[\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\Big] \\ &\geqslant \mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) \leqslant U_\theta(t)-M_t\Big)(-\beta t(1+|\theta|)-U_\theta(t)+M_t)+U_\theta(t)-M_t \\ &\geqslant U_\theta(t)-M_t-\mathbb{P}\Big(\inf_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\leqslant U_\theta(t)-M_t\Big)2\beta t(1+|\theta|).\end{aligned}$$

In view of the first inequality in (5.6), we get the first inequality in (5.7).

To prove the second inequality in (5.7), we argue analogously. We have

$$\begin{aligned}\mathbb{E}\Big[\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\Big] &\leqslant \mathbb{P}\Big(\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot) < U_\theta(t)+M_t\Big)(U_\theta(t)+M_t) \\ &\quad + \mathbb{P}\Big(\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\geqslant U_\theta(t)+M_t\Big)\beta t(1+|\theta|) \\ &\leqslant U_\theta(t)+M_t+\mathbb{P}\Big(\sup_{x\in\mathbf{B}(t)} u_\theta(t,x,\cdot)\geqslant U_\theta(t)+M_t\Big)2\beta t(1+|\theta|),\end{aligned}$$

and the second inequality in (5.7) follows in view of the second inequality in (5.6).

We will now show that the sequence $(-U_\theta(n))_{n\in\mathbb{N}}$ is almost subadditive and therefore it has a limit. Let $n\geqslant 1$ and $m,n\in\mathbb{N}$. According to Theorems A.8 and A.7, we have the following formulae:

$$u_\theta(m,0,\omega)=\sup_{\alpha\in\Gamma(m)}\inf_{b\in\mathbb{B}(m)}\Big\{\int_0^m \ell_\theta(y_0(s),\alpha[b](s),b(s),\omega)\,ds\Big\} \tag{5.8}$$

and

$$\begin{aligned}&u_\theta(m+n,0,\omega)\\ &\quad=\sup_{\alpha\in\Gamma(m)}\inf_{b\in\mathbb{B}(m)}\Big\{\int_0^m \ell_\theta(y_0(s),\alpha[b](s),b(s),\omega)\,ds+u_\theta(n,y_0(m),\omega)\Big\}.\end{aligned} \tag{5.9}$$

By (5.8) and (5.9), we have

$$u_\theta(m+n,0,\omega)\geqslant u_\theta(m,0,\omega)+\inf_{x\in\mathbf{B}(m)} u_\theta(n,x,\omega).$$

By taking expectation and by using the first inequality in (5.7), we get

$$U_\theta(m+n)\geqslant U_\theta(m)+U_\theta(n)-\widehat{K}(n\ln(n))^{1/2}\quad\text{for all } n\in\mathbb{N}. \tag{5.10}$$

Set $a_n := -U_\theta(n)$ for all $n\in\mathbb{N}$, and $z(h):=\widehat{K}(h\ln(h))^{1/2}$ for all $h\geqslant 1$. By the previous inequality, the sequence (a_n) is *subadditive with an error term* z, i.e.,

$$a_{m+n}\leqslant a_m+a_n+z(m+n)\quad\text{for all } m,n\in\mathbb{N}. \tag{5.11}$$

Note that z is nonnegative, nondecreasing and $\int_1^{+\infty} z(h)/h^2\,dh < +\infty$. Furthermore, by Theorem A.5, the function U_θ is $\beta(1+|\theta|)$-Lipschitz in $[0,+\infty)$; in particular a_n/n is bounded since $U_\theta(0)=0$. By [27, Theorem 23, p. 162], the sequence $(a_n/n)_{n\in\mathbb{N}}$ converges to a limit, which we will call $\bar{H}(\theta)$. We want to estimate the rate of convergence. To this aim, we remark that inequality (5.11) gives by induction that, for all $p,n\in\mathbb{N}$,

$$a_{2^p n} \leqslant 2^p a_n + 2^p \hat{K} \sum_{k=1}^{p} 2^{-k/2}(n\ln(2^k n))^{1/2},$$

from which we derive

$$a_{2^p n} \leqslant 2^p a_n + 2^p A(n\ln(n))^{1/2} \quad \text{for all } n \geqslant 2,$$

with $A := \hat{K}\sum_{k=1}^{+\infty} 2^{-k/2}\sqrt{k+1}$. By dividing the above inequality by $2^p n$ and sending $p\to+\infty$, we end up with

$$\bar{H}(\theta) \leqslant a_n/n + A\ln(n)^{1/2} n^{-1/2} \quad \text{for all } n \geqslant 2. \tag{5.12}$$

Let us now estimate from below the term $\bar{H}(\theta) - a_n/n$. In view of (5.9), for every fixed $\delta > 0$ we can choose a strategy $\alpha\in\Gamma(m)$ such that

$$u_\theta(m+n,0,\omega) - \delta \leqslant \inf_{b\in\mathbb{B}(m)} \left\{ \int_0^m \ell_\theta(y_0(s),\alpha[b](s),b(s),\omega)\,ds + u_\theta(n,y_0(m),\omega) \right\}.$$

In view of (5.8) we get

$$u_\theta(m+n,0,\omega) - \delta \leqslant u_\theta(m,0,\omega) + \sup_{x\in\mathbf{B}(m)} u_\theta(n,x,\omega).$$

By taking expectation, by the arbitrariness of $\delta > 0$ and by using the second inequality in (5.7), we get

$$U_\theta(m+n) \leqslant U_\theta(m) + U_\theta(n) + \hat{K}(n\ln(n))^{1/2} \quad \text{for all } n\in\mathbb{N}.$$

By setting, as above, $a_n := -U_\theta(n)$ for all $n\in\mathbb{N}$, we obtain

$$a_m + a_n \leqslant a_{m+n} + \hat{K}(n\ln(n))^{1/2} \quad \text{for all } m,n\in\mathbb{N}.$$

By induction, we derive, for all $p,n\in\mathbb{N}$,

$$2^p a_n \leqslant a_{2^p n} + 2^p \hat{K} \sum_{k=1}^{p} 2^{-k/2}(n\ln(2^k n))^{1/2},$$

from which we derive

$$2^p a_n \leqslant a_{2^p n} + 2^p A(n\ln(n))^{1/2} \quad \text{for all } n \geqslant 2,$$

with $A := \widehat{K} \sum_{k=1}^{+\infty} 2^{-k/2}\sqrt{k+1}$. By dividing the above inequality by $2^p n$ and sending $p \to +\infty$, we end up with

$$a_n/n \leqslant \overline{H}(\theta) + A\ln(n)^{1/2}n^{-1/2} \quad \text{for all } n \geqslant 2. \tag{5.13}$$

By putting together inequalities (5.12) and (5.13) we finally get

$$|U_\theta(n)/n + \overline{H}(\theta)| \leqslant A\ln(n)^{1/2}n^{-1/2} \quad \text{for all } n \geqslant 2. \tag{5.14}$$

The assertion follows by the Lipschitz character of the function U_θ. ∎

Remark 5.6. We remark for further use what we have actually shown with (5.10) and (5.14): there exist constants $\widehat{K}$ and A, only depending on $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$, such that

$$U_\theta(m+n) \geqslant U_\theta(m) + U_\theta(n) - \widehat{K}(n\ln(n))^{1/2} \quad \text{for all } m,n \in \mathbb{N}$$

and

$$\left|U_\theta(n) - \lim_n \frac{U_\theta(n)}{n}\right| \leqslant A\ln(n)^{1/2}n^{-1/2} \quad \text{for all } n \geqslant 2.$$

5.2. Proof of Theorem 2.4

Let us fix $\theta \in \mathbb{R}^d$. We first remark that $u_\theta^\varepsilon(t,x,\omega) = \tilde{u}_\theta^\varepsilon(t,x,\omega) - \langle\theta, x\rangle$ and, by rescaling,

$$u_\theta^\varepsilon(t,x,\omega) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon, \omega) \quad \text{for all } (t,x,\omega) \in (0,+\infty)\times\mathbb{R}^d\times\Omega. \tag{5.15}$$

Let us fix $T > 0$ and $R > 0$. Proposition 5.2 with $t := 1/\varepsilon$ yields, in view of (5.15),

$$\mathbb{P}\Big(\sup_{x\in B_R} |u_\theta^\varepsilon(1,x,\omega) + \overline{H}(\theta)| \geqslant \widehat{K}(-\varepsilon\ln\varepsilon)^{1/2}\Big) \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2, \tag{5.16}$$

for some constant $\widehat{K}$ depending on R, $|\theta|$, d, β, ρ, δ, $\mathrm{Lip}(\ell)$, and $\|f\|_\infty$. From Theorem A.5 with $H(x/\varepsilon, \theta+p, \omega)$ in place of H, we derive that $u_\theta^\varepsilon(\cdot, x, \omega)$ is $\beta(1+|\theta|)$-Lipschitz in $[0,+\infty)$. In particular,

$$|u_\theta^\varepsilon(t,x,\omega) + t\overline{H}(\theta)| \leqslant |u_\theta^\varepsilon(1,x,\omega) + \overline{H}(\theta)| + 2\beta(1+|\theta|)T, \tag{5.17}$$

where we also used the fact that $\overline{H}$ enjoys (H1). Let us choose $K > 0$ large enough so that

$$K(-\varepsilon\ln\varepsilon)^{1/2} \geqslant \widehat{K}(-\varepsilon\ln\varepsilon)^{1/2} + 2\beta(1+|\theta|)T \quad \text{for all } \varepsilon \leqslant 1/2.$$

The choice of such a constant K clearly depends on T, $|\theta|$, β, and on R, d, ρ, δ, $\mathrm{Lip}(\ell)$, $\|f\|_\infty$ through $\widehat{K}$. In view of (5.17) and (5.16) we derive

$$\mathbb{P}\Big(\sup_{[0,T]\times B_R} |u_\theta^\varepsilon(t,x,\omega) + t\overline{H}(\theta)| \geqslant K(-\varepsilon\ln\varepsilon)^{1/2}\Big) \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2,$$

as was to be shown. ∎

5.3. Proof of Corollary 2.5

For every fixed $\theta \in \mathbb{R}^d$, $\varepsilon > 0$, and $n \in \mathbb{N}$, let us denote by $\tilde{u}^\varepsilon_{n_\theta}$ the solution of equation (HJ_ε) satisfying $\tilde{u}^\varepsilon_{n_\theta}(0,x,\omega) = \langle \theta, x\rangle$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. According to Theorem 2.4, for every fixed $T > 0$ and $R > 0$ there exists a constant K depending on R, T, $|\theta|$, d, C, δ, $\mathrm{Lip}(\ell)$ such that, for each $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{[0,T]\times B_R} |\tilde{u}^\varepsilon_{n_\theta}(t,x,\omega) - \langle\theta,x\rangle + t\bar{H}_n(\theta)| \geqslant K(-\varepsilon\ln\varepsilon)^{1/2}\Big) \\ \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2. \qquad (5.18)\end{aligned}$$

According to Proposition B.1, each $\bar{H}_n$ satisfies conditions (H1)–(H2) with $\beta := \sup_n \beta_n$. We infer that $(\bar{H}_n)_n$ is a sequence of equi-Lipschitz and locally equi-bounded functions on $\mathbb{R}^d$, hence pre-compact in $\mathrm{C}(\mathbb{R}^d)$ by the Ascoli–Arzelà theorem. Let G_1, G_2 be a pair of accumulation points in $\mathrm{C}(\mathbb{R}^d)$ for the sequence $(\bar{H}_n)_n$, i.e., there exist two diverging sequences $(n^1_k)_k$, $(n^2_k)_k$ such that $\lim_k \bar{H}_{n^i_k} = G_i$ for $i \in \{1,2\}$. Since $H_n(\cdot,\cdot,\omega) \rightrightarrows H(\cdot,\cdot,\omega)$ in $\mathbb{R}^d \times \mathbb{R}^d$ for every $\omega \in \Omega$, by the stability property of viscosity solutions we infer that $\tilde{u}^\varepsilon_{n_\theta}(\cdot,\cdot,\omega) \rightrightarrows \tilde{u}^\varepsilon_\theta(\cdot,\cdot,\omega)$ in $[0,+\infty) \times \mathbb{R}^d$ for every $\omega \in \Omega$, where $\tilde{u}^\varepsilon_\theta$ denotes the solution of equation (HJ_ε) satisfying $\tilde{u}^\varepsilon(0,x,\omega) = \langle\theta,x\rangle$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. By passing to the limit in (5.18) along the subsequence $(n^i_k)_k$ we infer

$$\begin{aligned}\mathbb{P}\Big(\sup_{[0,T]\times B_R} |\tilde{u}^\varepsilon_\theta(t,x,\omega) - \langle\theta,x\rangle + t\bar{G}_i(\theta)| \geqslant K(-\varepsilon\ln\varepsilon)^{1/2}\Big) \\ \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2. \qquad (5.19)\end{aligned}$$

By the triangular inequality, this implies in particular

$$\mathbb{P}(T|G_1(\theta) - G_2(\theta)| \geqslant K(-\varepsilon\ln\varepsilon)^{1/2}) \leqslant \varepsilon^2 \quad \text{for all } \varepsilon \leqslant 1/2,$$

which implies that $G_1(\theta) = G_2(\theta)$. By the arbitrariness of $\theta \in \mathbb{R}^d$, we conclude that the sequence $(\bar{H}_n)_n$ converges in $\mathrm{C}(\mathbb{R}^d)$ to a function $\bar{H}$, which satisfies (H1)–(H2) with $\beta := \sup_n \beta_n$. This proves items (i) and (ii).

Arguing as we did at the beginning of Section 5.2, we see that (5.19) implies inequality (5.2) in the statement of Proposition 5.2. The fact that the HJ equation (HJ_ε) homogenizes with effective Hamiltonian $\bar{H}$ follows from this in view of Lemma 5.3, of the arbitrariness of the choices of $\theta \in \mathbb{R}^d$ and $R > 0$, and of the reduction arguments presented in Section 3. ■

5.4. Proof of Theorem 2.6

Let G be a stationary Hamiltonian satisfying conditions (G1) and (H1)–(H3), for some $\beta > 0$. According to [29, Lemma 5.1], for every fixed $R > 0$ the Hamiltonian G can be represented as

$$G(x,p,\omega) = \max_{b\in B(R)} \min_{a\in A}\{-\ell(x,a,b,\omega) - \langle \hat{f}(a), p\rangle\} \quad \text{for all } (x,p,\omega) \in \mathbb{R}^d \times \bar{B}_R \times \Omega,$$

where $A := \bar{B}_1$, $B(R) := \bar{B}_R$, $\ell(x,a,b) := -G(x,b,\omega) + \beta\langle a,b\rangle$, and $\hat{f}(a) := -\beta a$. We derive that

$$H(x,p,\omega) = \max_{b\in B(R)}\min_{a\in A}\big\{-\ell(x,a,b,\omega) - \langle f(a),p\rangle\big\} \quad \text{for all } (x,p,\omega) \in \mathbb{R}^d \times \bar{B}_R \times \Omega,$$

where $f(a) := \pi^\top(\hat{f}(a)) + v$ and $\pi^\top$ denotes the transpose of the linear map π. It is clear that f satisfies condition (f) with $e := v/|v|$ and $\delta := |v|$, and ℓ satisfies conditions (ℓ_1)–(ℓ_4). Furthermore, $\|f\|_\infty \leqslant \beta + \delta$ and $\operatorname{Lip}(\ell) \leqslant \beta$, independently of the choice of $R > 0$ in the definition of the set $B(R) := \bar{B}_R$.

Now let us fix $\theta \in \mathbb{R}^d$ and denote by u_θ the solution of equation (HJ_ε) with $\varepsilon = 1$ and with $H(\cdot,\theta+\cdot,\cdot)$ in place of H, subject to the initial condition $u_\theta(0,x,\omega) = 0$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. For every fixed $s \geqslant 1$, define

$$H^s(x,p,\omega) := \max_{b\in B(\beta s)}\min_{a\in A}\big\{-\ell(x,a,b,\omega) - \langle f(a),p\rangle\big\}, \quad (x,p,\omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega.$$

Let us denote by $u^s_\theta(t,x,\omega)$ the solution of equation (HJ_ε) with $\varepsilon = 1$ and with $H^s(\cdot,\theta+\cdot,\cdot)$ in place of H, subject to the initial condition $u^s_\theta(0,x,\omega) = 0$ for all $(x,\omega) \in \mathbb{R}^d \times \Omega$. From Theorem A.5 (with $g = 0$, $\beta_1 := \beta(1+|\theta|)$, and $\beta_2 = \beta_3 := \beta$) we know that both $\|D_x u\|_{L^\infty([0,s]\times\mathbb{R}^d)}$ and $\|D_x u^s\|_{L^\infty([0,s]\times\mathbb{R}^d)}$ are bounded from above by βs. Since $H(x,p,\omega) = H^s(x,p,\omega)$ on $\mathbb{R}^d \times \bar{B}_{\beta s} \times \Omega$, we infer that $u^s_\theta(t,\cdot,\cdot) \equiv u_\theta(t,\cdot,\cdot)$ for all $0 \leqslant t \leqslant s$. By applying Proposition 4.2 to u^t_θ, we derive that there exists a constant $c > 0$, only depending on $|\theta|$, d, β, ρ, $\delta = |v|$ (notice that $\operatorname{Lip}(\ell)$ and $\|f\|_\infty$ are bounded above by $\beta + \delta$), such that, for all $M > 0$ and $t \geqslant 1$,

$$\mathbb{P}\big(|u_\theta(t,0,\cdot) - U_\theta(t)| \geqslant M\sqrt{t}\big) = \mathbb{P}\big(|u^t_\theta(t,0,\cdot) - U^t_\theta(t)| \geqslant M\sqrt{t}\big) \leqslant \exp(-cM^2).$$

Moreover, as underlined in Remark 5.6, there exists a constant $\widehat{K}$, only depending on $|\theta|$, d, β, ρ, $\delta = |v|$ (notice that $\operatorname{Lip}(\ell)$ and $\|f\|_\infty$ are bounded above by $\beta + \delta$), such that

$$U^s_\theta(m+n) \geqslant U^s_\theta(m) + U^s_\theta(n) - \widehat{K}(n\ln(n))^{1/2} \quad \text{for all } n,m \in \mathbb{N}.$$

Applying this to $s = m + n$ we get

$$U_\theta(m+n) \geqslant U_\theta(m) + U_\theta(n) - \widehat{K}(n\ln(n))^{1/2} \quad \text{for all } n,m \in \mathbb{N}.$$

By arguing as in the proof of Proposition 5.5 (see the paragraph after inequality (5.10)), we conclude that $-U_\theta(n)/n$ converges, as $n \to +\infty$, to a limit that we will call $\bar{H}(\theta)$. Furthermore, as stressed in Remark 5.6, there exists a constant A only depending on $|\theta|$, d, β, ρ, δ (since $\operatorname{Lip}(\ell)$ and $\|f\|_\infty$ are bounded from above by $\beta + \delta$) such that

$$|U_\theta(n)/n + \bar{H}(\theta)| \leqslant A\ln(n)^{1/2}n^{-1/2} \quad \text{for all } n \geqslant 2.$$

By Theorem A.5 again we have that, for all $(x,\omega) \in \mathbb{R}^d \times \Omega$, $u_\theta(\cdot,x,\omega)$, and hence U_θ, is $\beta(1+|\theta|)$-Lipschitz in $[0,+\infty)$. This proves that U_θ satisfies the statement Proposition 5.5.

The statement of Proposition 5.2 remains valid as well. Its proof uses only Propositions 4.2 and 5.5, the Lipschitz bounds from Theorem A.5, and the fact that H is a stationary Hamiltonian in $\mathcal{H}$. Consequently, the quantitative estimate in Theorem 2.4 follows by the same argument as in Section 5.2, and Proposition 5.1 follows via Lemma 5.3. By the reduction arguments in Section 3, the HJ equation (HJ_ε) homogenizes for such H with effective Hamiltonian $\bar{H}$. This completes the proof. ∎

A. PDE results

In this appendix we collect the PDE results used in the paper, which are of deterministic nature and which follow from those herein stated and proved by regarding ω as a fixed parameter. We will denote by H a continuous Hamiltonian defined on $\mathbb{R}^d \times \mathbb{R}^d$ and satisfying further assumptions that will be specified case by case. Throughout this section, we will denote by $\mathrm{LSC}(X)$ and $\mathrm{USC}(X)$ the space of lower semicontinuous and upper semicontinuous real functions on a metric space X, respectively.

Let $T > 0$ be fixed and consider the following evolutive HJ equation:

$$\partial_t u + H(x, Du) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d. \tag{HJ}$$

We will say that a function $v \in \mathrm{USC}((0,T) \times \mathbb{R}^d)$ is an (upper semicontinuous) *viscosity subsolution* of (HJ) if, for every $\phi \in \mathrm{C}^1((0,T) \times \mathbb{R}^d)$ such that $v - \phi$ attains a local maximum at $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$, we have

$$\partial_t \phi(t_0, x_0) + H(x_0, D_x \phi(t_0, x_0)) \leqslant 0.$$

Any such test function ϕ will be called *supertangent* to v at (t_0, x_0).

We will say that $w \in \mathrm{LSC}((0,T) \times \mathbb{R}^d)$ is a (lower semicontinuous) *viscosity supersolution* of (HJ) if, for every $\phi \in \mathrm{C}^1((0,T) \times \mathbb{R}^d)$ such that $w - \phi$ attains a local minimum at $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$, we have

$$\partial_t \phi(t_0, x_0) + H(x_0, D_x \phi(t_0, x_0)) \geqslant 0.$$

Any such test function ϕ will be called *subtangent* to w at (t_0, x_0). A continuous function on $(0,T) \times \mathbb{R}^d$ is a *viscosity solution* of (HJ) if it is both a viscosity sub- and supersolution. Solutions, subsolutions, and supersolutions will always be intended in the viscosity sense, hence the term *viscosity* will be omitted in the sequel.

A.1. Comparison principles

We start by stating and proving a comparison principle, which applies in particular to the case of bounded sub- and supersolutions. The proof is somewhat standard; we provide it below for the reader's convenience.

Theorem A.1. *Let H be a Hamiltonian satisfying the following assumptions, for some continuity modulus ω:*

(H2′) $|H(x,p) - H(x,q)| \leqslant \omega(|p-q|)$ *for all* $x, p, q \in \mathbb{R}^d$*;*

(H4′) $|H(x,p) - H(y,p)| \leqslant \omega(|x-y|(1+|p|))$ *for all* $x, y, p \in \mathbb{R}^d$*.*

Let $v \in \mathrm{USC}([0,T] \times \mathbb{R}^d)$ and $w \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$ be, respectively, a bounded-from-above subsolution and a bounded-from-below supersolution of (HJ)*. Then, for all $(t,x) \in (0,T) \times \mathbb{R}^d$,*

$$v(t,x) - w(t,x) \leqslant \sup_{\mathbb{R}^d}(v(0,\cdot) - w(0,\cdot)).$$

Proof. Since v is bounded from above, up to adding to v a suitable constant, we assume without any loss of generality that $\sup_{\mathbb{R}^d}(v(0,\cdot) - w(0,\cdot)) = 0$. The assertion is thus reduced to proving that $v \leqslant w$ in $(0,T) \times \mathbb{R}^d$. We proceed by contradiction. Assume that $v > w$ at some point of $(0,T) \times \mathbb{R}^d$. Up to translations, we can assume without loss of generality that this point has the form $(\bar{t}, 0)$ for some $\bar{t} \in (0,T)$. We will construct a point (x, y, p, q) where the continuity assumptions (H2′)–(H4) do not hold, leading to a contradiction.

For every $\eta, \varepsilon, b > 0$, consider the auxiliary function $\Phi\colon ([0,T) \times \mathbb{R}^d)^2 \to \mathbb{R}$ defined by

$$\Phi(t,x,s,y) := v(t,x) - w(s,y) - \frac{|x-y|^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon} - \eta(\phi(x) + \phi(y)) - \frac{b}{T-t} - \frac{b}{T-s},$$

where $\phi(x) := \sqrt{1+|x|^2}$.

Define $\theta := v(\bar{t},0) - w(\bar{t},0) > 0$. Then, since $\bar{t} \in (0,T)$, there exists $\delta > 0$ small enough such that, for all $\eta, b \in (0,\delta)$,

$$\Phi(\bar{t},0,\bar{t},0) = v(\bar{t},0) - w(\bar{t},0) - 2\eta\phi(0) - \frac{2b}{T-\bar{t}} > \frac{\theta}{2}.$$

Notice that

$$\Phi(t,x,s,y) \leqslant M - \eta(\phi(x)+\phi(y)) - \frac{b}{T-t} - \frac{b}{T-s} \quad \text{in } ([0,T) \times \mathbb{R}^d)^2 \tag{A.1}$$

with $M := (\|v^+\|_{L^\infty([0,T)\times\mathbb{R}^d)} + \|w^-\|_{L^\infty([0,T)\times\mathbb{R}^d)})$, where we have denoted by $v^+(x) := \max\{v(x), 0\}$ the positive part of v and by $w^-(x) = \max\{-w(x), 0\}$ the negative part of w. We derive that, for every $\varepsilon > 0$ and $\eta \in (0,\delta)$, there exists $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in ([0,T) \times \mathbb{R}^d)^2$, which also depend on η, such that

$$\Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \sup_{([0,T)\times\mathbb{R}^d)^2} \Phi \geqslant \Phi(\bar{t},0,\bar{t},0) > \frac{\theta}{2}.$$

In view of (A.1) we infer

$$\begin{aligned} \eta(\phi(x_\varepsilon)+\phi(y_\varepsilon)) + \frac{b}{T-t_\varepsilon} + \frac{b}{T-s_\varepsilon} &\leqslant \widetilde{M}, \\ \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} + \frac{|t_\varepsilon - s_\varepsilon|}{\varepsilon} &\leqslant \sqrt{\frac{2\widetilde{M}}{\varepsilon}} \end{aligned} \tag{A.2}$$

with $\widetilde{M} := M - \theta/2$. From the first inequality in (A.2) we derive that there exist constants $T_b \in (0,T)$, depending on $b \in (0,\delta)$, and $\rho_\eta > 0$, depending on η, such that $t_\varepsilon, s_\varepsilon \in [0, T_b]$ and $x_\varepsilon, y_\varepsilon \in B_{\rho_\eta}$. For every fixed $\eta \in (0,\delta)$, from [24, Lemma 3.1] we derive that, up to subsequences,

$$\lim_{\varepsilon\to 0}(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = (t_0, x_0, t_0, x_0) \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0 \tag{A.3}$$

for some $(t_0, x_0) \in [0,T)\times\mathbb{R}^d$ satisfying

$$v(t_0,x_0) - w(t_0,x_0) - 2\eta\phi(x_0) - \frac{2b}{T-t_0} = \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} \Phi(t,x,t,x) > \frac{\theta}{2}. \tag{A.4}$$

In particular, such a point (t_0,x_0) actually lies in $(0,T)\times\mathbb{R}^d$, i.e., $t_0 > 0$, since (A.4) implies $v(t_0,x_0) - w(t_0,x_0) > 0$. For every fixed $\eta \in (0,\delta)$, choose $\varepsilon_\eta > 0$ small enough so that $(t_\varepsilon, x_\varepsilon)$ and $(s_\varepsilon, y_\varepsilon)$ both belong to $(0,T)\times\mathbb{R}^d$ when $\varepsilon \in (0,\varepsilon_\eta)$. The function

$$\varphi(t,x) := w(s_\varepsilon,y_\varepsilon) + \frac{|x-y_\varepsilon|^2}{2\varepsilon} + \eta(\phi(x)+\phi(y_\varepsilon)) + \frac{|t-s_\varepsilon|^2}{2\varepsilon} + \frac{b}{T-t}$$

is a supertangent to $v(t,x)$ at the point $(t_\varepsilon,x_\varepsilon)$ and v is a subsolution of (HJ), while the function

$$\psi(s,y) := v(t_\varepsilon,x_\varepsilon) - \frac{|x_\varepsilon-y|^2}{2\varepsilon} - \eta(\phi(x_\varepsilon)+\phi(y)) - \frac{|t_\varepsilon-s|^2}{2\varepsilon} - \frac{b}{T-s}$$

is a subtangent to $w(s,y)$ at the point $(s_\varepsilon,y_\varepsilon)$ and w is a subsolution of (HJ). We infer

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(x_\varepsilon, p_\varepsilon + \eta D\phi(x_\varepsilon)) \leqslant -c, \tag{A.5}$$

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(y_\varepsilon, p_\varepsilon - \eta D\phi(y_\varepsilon)) \geqslant c, \tag{A.6}$$

where we have set

$$c := \frac{b}{T^2} \quad \text{and} \quad p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}.$$

By subtracting (A.5) from (A.6), we get, according to assumptions (H2′)–(H4),

$$\begin{aligned} 0 < 2c &\leqslant H(y_\varepsilon, p_\varepsilon - \eta D\phi(y_\varepsilon)) - H(x_\varepsilon, p_\varepsilon + \eta D\phi(x_\varepsilon)) \\ &\leqslant \omega(|x_\varepsilon - y_\varepsilon|(1+|p_\varepsilon|)) + 2\omega(\eta). \end{aligned}$$

Sending $\varepsilon \to 0^+$, we have that $\omega(|x_\varepsilon - y_\varepsilon|(1+|p_\varepsilon|))$ vanishes in view of (A.3). Sending $\eta \to 0^+$, we have that $\omega(\eta)$ vanishes. But this is a contradiction since $0 < c$. ∎

Next we provide a result which is usually used to prove the finite speed of propagation of first-order HJ equations; see for instance [18, Lemma 5.3].

Proposition A.2. *Let H be a Hamiltonian satisfying conditions* (H4) *and*

(H2) $|H(x,p) - H(x,q)| \leqslant \beta|p-q|$ *for all* $x, p, q \in \mathbb{R}^d$,

for some constant $\beta > 0$. *Let* $v \in \mathrm{USC}([0,T] \times \mathbb{R}^d)$ *and* $w \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$ *be, respectively, a sub- and a supersolution of* (HJ). *Then the function* $u := v - w$ *is an upper semicontinuous subsolution to*

$$\partial_t u - \beta|Du| = 0 \quad \text{in } (0,T) \times \mathbb{R}^d. \tag{A.7}$$

Proof. Let $\varphi \in \mathrm{C}^1((0,T) \times \mathbb{R}^d)$ be supertangent to u in a point $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ and let us assume that $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ is a strict local maximum point of $u - \varphi$. Let us choose $r > 0$ such that the open ball $B := B_r((t_0, x_0))$ of radius $r > 0$ centered at (t_0, x_0) is compactly contained in $(0,T) \times \mathbb{R}^d$ and (t_0, x_0) is the only maximum point of $u - \varphi$ in $\bar{B}$. Let us introduce the function

$$\Phi(t,x,s,y) := v(t,x) - w(s,y) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon} - \varphi(t,x) \quad \text{for } (t,x,s,y) \in \bar{B} \times \bar{B}.$$

By upper semicontinuity of Φ and compactness of the domain, the maximum of Φ on $\bar{B} \times \bar{B}$ is attained at (at least) a point $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in \bar{B} \times \bar{B}$. In view of [24, Lemma 3.1], we infer that

$$\lim_{\varepsilon \to 0} (t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = (t_0, x_0, t_0, x_0) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{A.8}$$

Choose $\varepsilon_0 > 0$ small enough so that $(x_\varepsilon, t_\varepsilon)$, $(y_\varepsilon, s_\varepsilon)$ belong to B for every $\varepsilon \in (0, \varepsilon_0)$. The function

$$\psi_1(t,x) := w(s_\varepsilon, y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} + \frac{|t - s_\varepsilon|^2}{2\varepsilon} + \varphi(t,x)$$

is a supertangent to $v(t,x)$ at $(t_\varepsilon, x_\varepsilon)$ and v is a subsolution to (HJ), hence

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(x_\varepsilon, p_\varepsilon + D_x \varphi(t_\varepsilon, x_\varepsilon)) \leqslant 0, \tag{A.9}$$

where we have set $p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$. Analogously,

$$\psi_2(s,y) := v(t_\varepsilon, x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} - \frac{|t_\varepsilon - s|^2}{2\varepsilon} - \varphi(t_\varepsilon, x_\varepsilon)$$

is a subtangent to $w(s,y)$ at the point $(s_\varepsilon, y_\varepsilon)$ and w is a supersolution to (HJ), hence

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(y_\varepsilon, p_\varepsilon) \geqslant 0. \tag{A.10}$$

By subtracting (A.10) from (A.9) and by taking into account conditions (H2) and (H4), we get

$$\begin{aligned} 0 &\geqslant \partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(x_\varepsilon, p_\varepsilon + D_x\varphi(t_\varepsilon, x_\varepsilon)) - H(y_\varepsilon, p_\varepsilon) \\ &\geqslant \partial_t \varphi(t_\varepsilon, x_\varepsilon) - \beta |D_x\varphi(t_\varepsilon, x_\varepsilon)| + H(x_\varepsilon, p_\varepsilon) - H(y_\varepsilon, p_\varepsilon) \\ &\geqslant \partial_t \varphi(t_\varepsilon, x_\varepsilon) - \beta |D_x\varphi(t_\varepsilon, x_\varepsilon)| - \omega(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|)). \end{aligned}$$

Now we send $\varepsilon \to 0^+$ to get, in view of (A.8),

$$0 \geqslant \partial_t \varphi(t_0, x_0) - \beta |D_x\varphi(t_0, x_0)|,$$

as was to be shown. ■

With the aid of the previous proposition, we can prove the following version of the comparison principle for unbounded sub- and supersolutions.

Theorem A.3. *Let H be a Hamiltonian satisfying assumptions* (H2) *and* (H4). *Let $v \in$ USC$([0,T] \times \mathbb{R}^d)$ and $w \in$ LSC$([0,T] \times \mathbb{R}^d)$ be, respectively, a sub- and a supersolution of* (HJ). *Then*

$$v(t,x) - w(t,x) \leqslant \sup_{\mathbb{R}^d}(v(0,\cdot) - w(0,\cdot)) \quad \textit{for every } (t,x) \in (0,T) \times \mathbb{R}^d.$$

Proof. We can assume $\sup_{\mathbb{R}^d}(v(0,\cdot) - w(0,\cdot)) < +\infty$, otherwise the assertion is trivially satisfied. Up to adding an appropriate constant to v, we can furthermore assume, without loss of generality, that $\sup_{\mathbb{R}^d}(v(0,\cdot) - w(0,\cdot)) = 0$. In view of Proposition A.2, the function $u := v - w$ is an upper semicontinuous subsolution of (A.7) satisfying $u(0,\cdot) \leqslant 0$ in $\mathbb{R}^d$. Let $\Psi \colon \mathbb{R} \to \mathbb{R}$ be a C^1, strictly increasing, and bounded function satisfying $\Psi(0) = 0$. It is easily seen, due to the positive 1-homogeneity of equation (A.7), that the function $v := (\Psi \circ u)(t,x)$ is a bounded, upper semicontinuous subsolution to (A.7) satisfying $v(0,\cdot) \leqslant 0$ on $\mathbb{R}^d$. By applying Theorem A.1 with $w(t,x) \equiv 0$, we derive that $v = (\Psi \circ u) \leqslant w = 0 = \Psi(0)$ in $[0,T) \times \mathbb{R}^d$. The assertion follows by the strict monotonicity of Ψ. ■

A.2. Existence results and Lipschitz estimates for solutions

Throughout this subsection, we will assume that the (deterministic) Hamiltonian H satisfies the following assumptions for some constants $\beta_1, \beta_2, \beta_3 > 0$:

(H1*) $|H(x,p)| \leqslant \beta_1(1 + |p|)$ for all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$;

(H2*) $|H(x,p) - H(x,q)| \leqslant \beta_2 |p - q|$ for all $x, p, q \in \mathbb{R}^d$;

(H3*) $|H(x,p) - H(y,p)| \leqslant \beta_3 |x - y|$ for all $x, y, p \in \mathbb{R}^d$.

We begin with the following existence and uniqueness result, where the uniqueness part is guaranteed by Theorem A.3.

Theorem A.4. *Let $g \in \mathrm{UC}(\mathbb{R}^d)$. For every $T > 0$, the problem*

$$\begin{cases} \partial_t u + H(x, Du) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(0,\cdot) = g(\cdot) & \text{on } \mathbb{R}^d, \end{cases} \tag{HJP}$$

has a unique viscosity solution in $\mathrm{C}([0,T) \times \mathbb{R}^d)$. Furthermore, this solution belongs to $\mathrm{UC}([0,T) \times \mathbb{R}^d)$.

Proof. The case $g \in \mathrm{BUC}(\mathbb{R}^d)$ is proved in [18, Theorem 7.1]. Let us then assume $g \in \mathrm{UC}(\mathbb{R}^d)$. Pick $\psi \in \mathrm{C}^{1,1}(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$ such that $\|\psi - g\|_\infty \leqslant 1$. In view of the previous step, the Cauchy problem (HJP) with $\widetilde{H}(x,p) := H(x, D\psi(x) + p)$ and $g - \psi$ in place of H and g, respectively, admits a unique solution $\tilde{u}(t,x)$ in $\mathrm{C}([0,T) \times \mathbb{R}^d)$. Furthermore, $\tilde{u}(t,x)$ in $\mathrm{UC}([0,T) \times \mathbb{R}^d)$. We derive that $u(t,x) := \tilde{u}(t,x) + \psi(x)$ belongs to $\mathrm{UC}([0,T) \times \mathbb{R}^d)$ and is a solution of the original Cauchy problem (HJP). ∎

We proceed to show suitable Lipschitz bounds for the solution of (HJP) when the initial datum is Lipschitz.

Theorem A.5. *Let $g \in \mathrm{Lip}(\mathbb{R}^d)$ and let u be the unique continuous function in $[0,+\infty) \times \mathbb{R}^d$ which solves the Cauchy problem (HJP) for every $T > 0$. Then u is Lipschitz in $[0,T) \times \mathbb{R}^d$ for every $T > 0$. More precisely,*

$$\|D_x u\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leqslant (\beta_3 T + \mathrm{Lip}(g)), \quad \|\partial_t u\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leqslant \beta_1(1 + \|Dg\|_\infty).$$

Proof. (i) Let us fix $h \in \mathbb{R}^d$ and set

$$v_h(t,x) := u(t, x+h) - \beta_3|h|t, \quad w_h(t,x) := u(t,x+h) + \beta_3|h|t$$

for every $(t,x) \in [0,+\infty) \times \mathbb{R}^d$. By exploiting assumption (H2*), it is easily seen that v_h and w_h are, respectively, a viscosity sub- and supersolution to (HJ). Indeed, the following inequalities hold in the viscosity sense:

$$\begin{aligned} \partial_t v_h + H(x, Dv_h) &= \partial_t u(t, x+h) - \beta_3|h| + H(x, Du(t,x+h)) \\ &\leqslant \partial_t u(t,x+h) + H(x+h, Du(t,x+h)) \leqslant 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \end{aligned}$$

thus showing that v_h is a subsolution of (HJ). The assertion for w_h can be proved analogously. By the comparison principle, see Theorem A.3, we infer that, for all $(t,x) \in [0,T) \times \mathbb{R}^d$,

$$\begin{aligned} u(t,x) - v_h(t,x) &\leqslant u(0,x) - v_h(0,x) = g(x) - g(x+h) \leqslant \mathrm{Lip}(g)|h|, \\ w_h(t,x) - u(t,x) &\leqslant w_h(0,x) - u(0,x) = g(x+h) - g(x) \leqslant \mathrm{Lip}(g)|h|, \end{aligned}$$

namely

$$|u(t,x+h) - u(t,x)| \leqslant (\beta_3 t + \mathrm{Lip}(g))|h| \quad \text{for all } (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

thus showing the first assertion.

(ii) Let us first assume that $g \in \mathrm{Lip}(\mathbb{R}^d) \cap \mathrm{C}^1(\mathbb{R}^d)$. By assumption (H1*) we have

$$|H(x, Dg(x))| \leqslant \beta_1(1 + |Dg(x)|) \leqslant \beta_1(1 + \|Dg\|_\infty) \quad \text{for all } x \in \mathbb{R}^d.$$

For notational simplicity, let us denote by M the rightmost term in the above inequality. It is easily seen that the functions

$$\underline{u}(t,x) := g(x) - Mt, \quad \bar{u}(t,x) := g(x) + Mt, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

are, respectively, a sub- and a supersolution of (HJP) for every $T > 0$. By the comparison principle, see Theorem A.3, we infer that

$$\underline{u}(t,x) \leqslant u(t,x) \leqslant \bar{u}(t,x) \quad \text{for all } (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

i.e.,

$$\|u(t,\cdot) - g\|_\infty \leqslant Mt \quad \text{for all } t \geqslant 0.$$

By applying the comparison principle again we get

$$\|u(t+h,\cdot) - u(t,\cdot)\|_\infty \leqslant \|u(h,\cdot) - u(0,\cdot)\|_\infty \leqslant Mh = \beta_1(1 + \|Dg\|_\infty)h \tag{A.11}$$

for all $t, h \geqslant 0$, meaning that u is $\beta_1(1 + \|Dg\|_\infty)$-Lipschitz in t.

The case $g \in \mathrm{Lip}(\mathbb{R}^d)$ can be obtained by approximation. Let us denote by g_n the mollification of g via a standard convolution kernel and by u_n the solution to the Cauchy problem (HJP) with g_n in place of g. Since $\|Dg_n\|_\infty \leqslant \|Dg\|_\infty$ for every $n \in \mathbb{N}$, we deduce from what is proved above that the functions u_n are $\beta_1(1 + \|Dg\|_\infty)$-Lipschitz in t and $(\beta_3 t + \mathrm{Lip}(g))$-Lipschitz in x, for every $n \in \mathbb{N}$. By the Ascoli–Arzelà theorem, the stability of the notion of viscosity solution, and the uniqueness of the continuous solution to the Cauchy problem associated with (HJP), we infer that the functions $(u_n)_n$ are converging, locally uniformly in $[0,+\infty) \times \mathbb{R}^d$, to the solution u of (HJP) with initial datum g. We derive that u satisfies (A.11) as well, as was to be shown. ∎

A.3. Differential games

Throughout this subsection we will work with a deterministic Hamiltonian H of the form

$$H(x,p) := \max_{b \in B} \min_{a \in A} \big\{ -\ell(x,a,b) - \langle f(a,b), p \rangle \big\} \quad \text{for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where A, B are compact subsets of $\mathbb{R}^m$, for some integer m, $f \colon A \times B \to \mathbb{R}^d$ is a continuous vector-valued function, and the running cost $\ell \colon \mathbb{R}^d \times A \times B \to \mathbb{R}$ is a bounded and continuous function satisfying assumption (ℓ_2) appearing in Section 2. We will denote by $\|\ell\|_\infty$ the L^∞-norm of ℓ on $\mathbb{R}^d \times A \times B$. The Hamiltonian H clearly satisfies the properties (H1)–(H3) listed in Section 2.

For such a Hamiltonian, the solutions to the Cauchy problem (HJP) can be represented through a max-min formula provided by differential game theory; see [29]. To this aim, we first observe that $u(t,x)$ is a viscosity solution of (HJP) if and only if $\check{u}(t,x) := -u(T-t,x)$ is a viscosity solution of equation (HJ) with $\check{H}(x,p) := H(x,-p)$ in place of H in the sense adopted in [29] (see items (a)–(b) at the end of page 774 in [29]), and satisfying the terminal condition $\check{u}(T,x) = -g(x)$; cf. [29, problem (HJ)]. Let us denote by

$$\mathbb{A}(T) := \{a\colon [0,T] \to A : aa \text{ measurable}\}, \quad \mathbb{B}(T) := \{b\colon [0,T] \to B : b \text{ measurable}\}.$$

The sets A and B are to be regarded as action sets for Players 1 and 2, respectively. A *nonanticipating strategy* for Player 1 is a function $\alpha\colon \mathbb{B}(T) \to \mathbb{A}(T)$ such that, for all $b_1, b_2 \in \mathbb{B}(T)$ and $\tau \in [0,T]$,

$$b_1(\cdot) = b_2(\cdot) \quad \text{in } [0,\tau] \implies \alpha[b_1](\cdot) = \alpha[b_2](\cdot) \quad \text{in } [0,\tau].$$

We will denote by $\Gamma(T)$ the family of such nonanticipating strategies for Player 1. For every $(t,x) \in (0,+\infty) \times \mathbb{R}^d$, the value function associated with this differential game is defined as

$$v(t,x) := \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} \left\{ \int_0^t \ell(y_x(s), \alpha[b](s), b(s))\, ds + g(y_x(t)) \right\}, \tag{A.12}$$

where $y_x\colon [0,t] \to \mathbb{R}^d$ is the solution of the ODE

$$\begin{cases} \dot{y}_x(s) = f(\alpha[b](s), b(s)) & \text{in } [0,t], \\ y_x(0) = x. \end{cases} \tag{ODE}$$

Proposition A.6. *The following hold:*

(i) *Let $g \in W^{1,\infty}(\mathbb{R}^d)$. Then $v \in W^{1,\infty}([0,T) \times \mathbb{R}^d)$ for every fixed $T > 0$. More precisely, for every $x, \hat{x} \in \mathbb{R}^d$ and $t, \hat{t} \in (0,T)$, we have*

$$\begin{aligned} |v(t,x)| &\leqslant T\|\ell\|_\infty + \|g\|_\infty, \\ |v(t,x) - v(\hat{t},\hat{x})| &\leqslant (T\,\mathrm{Lip}(\ell) + \mathrm{Lip}(g))|x - \hat{x}| \\ &\quad + \big(\|\ell\|_\infty + \|f\|_\infty (T\,\mathrm{Lip}(\ell) + \mathrm{Lip}(g))\big)|t - \hat{t}|. \end{aligned}$$

(ii) *Let $g \in \mathrm{BUC}(\mathbb{R}^d)$. Then $v \in \mathrm{BUC}([0,T) \times \mathbb{R}^d)$ for every fixed $T > 0$.*

Proof. The first part of item (i) follows directly from [29, Theorem 3.2] after observing that $v(t,x) = -V(T-t,x)$, where V is the function given by [29, formula (2.6)] with $Z := A$, $Y := B$, and $-\ell$ and $-g$ in place of ℓ and g, respectively. The first inequality in (i) can be easily deduced from (A.12) in view of the uniform bounds on ℓ and g. To derive the second inequality, we have to prove the same kind of Lipschitz bounds for the function V. This can be achieved by arguing as in [29, Proof of Theorem 3.2]. We sketch the proof here and refer to [29] for the details.

Let us consider the lower value

$$V(t,x) := \inf_{\alpha} \sup_{b} \left\{ \int_t^T -\ell(y_x(s), \alpha[b](s), b(s))\, ds - g(y_x(T)) \right\},$$

for the dynamics

$$\dot{y}_x(s) = f(\alpha[b](s), b(s)) \quad \text{in } [t,T], \qquad y_x(t) = x,$$

with b and $\alpha[b]$ taken from the corresponding admissible control classes of Players 2 and 1.

Lipschitz continuity in x (time t fixed). Fix $t \in (0,T)$ and $x_1, x_2 \in \mathbb{R}^d$. Run the same controls (α, β) after time t for both initial conditions. Since f does not depend on x, we have

$$|y_1(s) - y_2(s)| = |x_1 - x_2| \quad \text{for all } s \in [t,T].$$

This replaces the Grönwall inequality used between (3.16) and (3.17) in [29]. Proceeding as in [29, inequalities (3.17)–(3.20)], the running and terminal parts satisfy

$$\left| \int_t^T \ell(\alpha(s), \beta(s), y_1(s)) - \ell(\alpha(s), \beta(s), y_2(s))\, ds \right| \leqslant \mathrm{Lip}(\ell)(T-t)|x_1 - x_2|,$$
$$|g(y_1(T)) - g(y_2(T))| \leqslant \mathrm{Lip}(g)|x_1 - x_2|.$$

Taking the $\inf_\alpha \sup_\beta$ over admissible controls yields

$$|V(t,x_1) - V(t,x_2)| \leqslant (\mathrm{Lip}(\ell)(T-t) + \mathrm{Lip}(g))|x_1 - x_2|,$$

and hence $\sup_{t\in[0,T]} \mathrm{Lip}_x V(t,\cdot) \leqslant T\,\mathrm{Lip}(\ell) + \mathrm{Lip}(g)$.

Lipschitz continuity in t (state x fixed). Fix $t_1 < t_2$ in $(0,T)$. As in [29, inequality (3.17)], split the payoff difference into

- *Short interval* $[t_1, t_2]$: $\left| \int_{t_1}^{t_2} \ell(\alpha(s), \beta(s), y_{t_1}(s))\, ds \right| \leqslant \|\ell\|_\infty |t_2 - t_1|$.
- *Overlap* $[t_2, T]$: run the same controls after t_2. At time t_2,

$$y_{t_1}(t_2) - x = \int_{t_1}^{t_2} f(\alpha(s), \beta(s))\, ds, \qquad |y_{t_1}(t_2) - x| \leqslant \|f\|_\infty |t_2 - t_1|.$$

  Since f is x-independent, this offset is preserved for all $s \geqslant t_2$, i.e.,

$$|y_{t_1}(s) - y_{t_2}(s)| = |y_{t_1}(t_2) - x| \leqslant \|f\|_\infty |t_2 - t_1|, \qquad s \in [t_2, T].$$

  This replaces the Grönwall growth used between inequalities (3.18) and (3.21) in [29].

Consequently, exactly as in [29, inequalities (3.18)–(3.21)],

$$\left| \int_{t_2}^{T} \ell(s, y_{t_1}(s)) - \ell(s, y_{t_2}(s))\, ds \right| \leqslant \mathrm{Lip}(\ell)(T - t_2)\|f\|_\infty |t_2 - t_1|,$$
$$|g(y_{t_1}(T)) - g(\dot{y}t_2(T))| \leqslant \mathrm{Lip}(g)\|f\|_\infty |t_2 - t_1|.$$

Collecting the pieces and passing to the $\inf_\alpha \sup_\beta$ over admissible controls gives

$$|V(t_1, x) - V(t_2, x)| \leqslant \big(\|\ell\|_\infty + \|f\|_\infty(\mathrm{Lip}(\ell)(T - t_2) + \mathrm{Lip}(g))\big)|t_2 - t_1|.$$

In particular,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{Lip}_t\, V \leqslant \|\ell\|_\infty + \|f\|_\infty(T\,\mathrm{Lip}(\ell) + \mathrm{Lip}(g)).$$

This yields the Lipschitz bounds for v in (t, x) stated above.

Let us prove (ii). Let $(g_n)_n$ be a sequence of functions in $W^{1,\infty}(\mathbb{R}^d)$ such that $\|g_n - g\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $n \to +\infty$. For every fixed $(t, x) \in (0, T) \times \mathbb{R}^d$, any fixed strategy $\alpha \in \Gamma(t)$, and every control $b \in \mathbb{B}(t)$, let us set

$$J[\alpha, b, g](t, x) := \int_0^t \ell(y_x(s), \alpha[b](s), b(s))\, ds + g(y_x(t)),$$

where $y_x \colon [0, t] \to \mathbb{R}^d$ is the solution of (ODE). We have

$$J[\alpha, b, g_n](t, x) - \|g_n - g\|_\infty \leqslant J[\alpha, b, g](t, x) \leqslant J[\alpha, b, g_n](t, x) + \|g_n - g\|_\infty.$$

By taking the infimum with respect to $b \in \mathbb{B}(t)$ and then the supremum with respect to $\alpha \in \Gamma(t)$, we infer

$$\begin{aligned}\sup_{\alpha\in\Gamma(t)} \inf_{b\in\mathbb{B}(t)} J[\alpha, \hat{b}, g_n](t, x) - \|g_n - g\|_\infty &\leqslant \sup_{\alpha\in\Gamma(t)} \inf_{b\in\mathbb{B}(t)} J[\alpha, \hat{b}, g](t, x) \\ &\leqslant \sup_{\alpha\in\Gamma(t)} \inf_{b\in\mathbb{B}(t)} J[\alpha, b, g_n](t, x) + \|g_n - g\|_\infty.\end{aligned}$$

This means that $|v_n(t, x) - v(t, x)| \leqslant \|g_n - g\|_\infty$ for all $(t, x) \in [0, T) \times \mathbb{R}^d$, where v_n and v denote the value function associated to g_n and g, respectively. As a uniform limit of a sequence of equi-bounded Lipschitz functions, we conclude that v belongs to $\mathrm{BUC}([0, T) \times \mathbb{R}^d)$. ■

Via the same argument used in the proof of Proposition A.6-(i), we derive from [29, Theorem 3.1] the following fact, known as the dynamic programming principle.

Theorem A.7 (Dynamic programming principle). *Let $g \in \mathrm{BUC}(\mathbb{R}^d)$. For every fixed $x \in \mathbb{R}^d$ and $0 < \tau < T$, we have*

$$v(T, x) = \sup_{\alpha\in\Gamma(T-\tau)} \inf_{b\in\mathbb{B}(T-\tau)} \left\{ \int_0^{T-\tau} \ell(y_x(s), \alpha[b](s), b(s))\, ds + v(\tau, y_x(T - \tau)) \right\}. \quad \text{(A.13)}$$

The following holds.

Theorem A.8. *Let* $g \in \mathrm{BUC}(\mathbb{R}^d)$*. For every fixed* $T > 0$*, the unique continuous solution of* (HJP) *is given by* (A.12).

Proof. When g is additionally assumed Lipschitz continuous, the assertion follows directly from [29, Theorem 4.1] via the same change of variables used in the proof of Proposition A.6 (i) and in view of what we remarked at the beginning of this subsection. Let us assume $g \in \mathrm{BUC}(\mathbb{R}^d)$ and pick a sequence of functions $g_n \in W^{1,\infty}(\mathbb{R}^d)$ such that $\|g_n - g\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $n \to +\infty$. Let us denote by v and v_n the value functions associated via (A.12) to g and g_n, respectively. Arguing as in the proof of Proposition A.6 (ii), we derive $|v_n(t,x) - v(t,x)| \leqslant \|g_n - g\|_\infty$ for all $(t,x) \in [0,T) \times \mathbb{R}^d$. From the previous step we know that v_n solves (HJP) with initial datum g_n. By the stability of the notion of viscosity solution, we conclude that v solves (HJP). ∎

We now extend the previous result to the case of initial data that are not necessarily bounded. The result is the following.

Theorem A.9. *Let* $g \in \mathrm{UC}(\mathbb{R}^d)$*. For every fixed* $T > 0$*, the unique viscosity solution* $u \in \mathrm{C}([0,T) \times \mathbb{R}^d)$ *of the Cauchy problem* (HJP) *is given by the representation formula* (A.12)*. Furthermore,* u *satisfies the dynamic programming principle* (A.13).

Proof. Let us pick $\psi \in \mathrm{C}^{1,1}(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$ such that $\|\psi - g\|_\infty \leqslant 1$. In view of Theorem A.8, the unique solution $\tilde{u}(t,x)$ in $\mathrm{C}([0,T) \times \mathbb{R}^d)$ of the Cauchy problem (HJP) with $\widetilde{H}(x,p) := H(x, D\psi(x) + p)$ and $g - \psi$ in place of H and g, respectively, is given by the formula (A.12) with $\ell(y_x(s), \alpha[b](s), b(s)) + \langle f(\alpha[b](s), b(s)), D\psi(y_x(s))\rangle$ and $(g - \psi)(y_x(t))$ in place of $\ell(y_x(s), \alpha[b](s), b(s))$ and $g(y_x(t))$, respectively. For every fixed strategy $\alpha \in \Gamma(t)$ and every control $b \in \mathbb{B}(t)$, we have

$$
\begin{aligned}
&\int_0^t \big(\ell(y_x(s), \alpha[b](s), b(s)) + \langle f(\alpha[b](s), b(s)), D\psi(y_x(s))\rangle\big)\, ds \\
&\qquad = \int_0^t \Big(\ell(y_x(s), \alpha[b](s), b(s)) + \frac{d}{ds}\psi(y_x(s))\Big)\, ds \\
&\qquad = \int_0^t \ell(y_x(s), \alpha[b](s), b(s))\, ds + \psi(y_x(t)) - \psi(x).
\end{aligned}
$$

We infer

$$
\begin{aligned}
\tilde{u}(t,x) = \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} \Big\{ \int_0^t &\big(\ell(y_x(s), \alpha[b](s), b(s)) \\
&+ \langle f(\alpha[b](s), b(s)), D\psi(y_x(s))\rangle\big)\, ds + (g - \psi)(y_x(t)) \Big\} \\
= \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} &\Big\{ \int_0^t \ell(y_x(s), \alpha[b](s), b(s))\, ds + g(y_x(t)) \Big\} - \psi(x).
\end{aligned}
$$

The first assertion readily follows after observing that the function $u(t,x) := \tilde{u}(t,x) + \psi(x)$ is the continuous solution of the original Cauchy problem (HJP). The second assertion can be derived via the same argument from the fact that $\tilde{u}$ satisfies the dynamic programming principle (A.13). ∎

B. Proof of Theorem 3.1

In this section we prove Theorem 3.1. The result follows from a couple of preliminary propositions of deterministic nature, with ω treated as a fixed parameter. We will therefore omit it from our notation. We start with the following result.

Proposition B.1. *Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a continuous Hamiltonian satisfying conditions* (H1)–(H3) *for some $\beta > 0$. For every fixed $\theta \in \mathbb{R}^d$ and $\varepsilon > 0$, let us denote by $\tilde{u}^\varepsilon_\theta$ the unique continuous solution of equation* (HJ_ε) *satisfying $\tilde{u}^\varepsilon_\theta(0,x) = \langle \theta, x\rangle$ for all $x \in \mathbb{R}^d$. Assume there exist a dense subset D of $\mathbb{R}^d$ and a function $\overline{H} : D \to \mathbb{R}$ such that, for every $\theta \in D$, the following convergence takes place:*

$$\tilde{u}^\varepsilon_\theta(t,x) \rightrightarrows \langle \theta, x\rangle - t\overline{H}(\theta) \quad \text{in } [0,T) \times \mathbb{R}^d \text{ as } \varepsilon \to 0^+. \tag{B.1}$$

The following hold:

(i) *$\overline{H}$ satisfies conditions* (H1)–(H2) *on D with the same $\beta > 0$. In particular, it can be uniquely extended by continuity to the whole $\mathbb{R}^d$.*

(ii) *The convergence stated in* (B.1) *holds for any $\theta \in \mathbb{R}^d$.*

Proof. (i) Let us prove that $\overline{H}$ satisfies condition (H1) on D. To this aim, we fix $\theta \in D$ and remark that the functions u^+, u^- defined as

$$u^\pm(t,x) := \langle \theta, x\rangle \pm \beta(1 + |\theta|)t, \quad (t,x) \in [0,T) \times \mathbb{R}^d,$$

are, respectively, a continuous super- and subsolution of (HJ_ε) satisfying $u^\pm(0,x) = \langle \theta, x\rangle$, for every $\varepsilon > 0$ in view of assumption (H1). By Theorem A.3 we infer $u^- \leqslant \tilde{u}^\varepsilon_\theta \leqslant u^+$ in $[0,T) \times \mathbb{R}^d$ for every $\varepsilon > 0$. Hence

$$|\overline{H}(\theta)| = \lim_{\varepsilon \to 0^+} |\tilde{u}^\varepsilon_\theta(1,0)| \leqslant \beta(1 + |\theta|),$$

as was to be shown. Let us now show that $\overline{H}$ satisfies condition (H2) on D. Fix $\theta_1, \theta_2 \in D$ and set

$$u^{\varepsilon,\pm}_{\theta_2}(t,x) := \tilde{u}^\varepsilon_{\theta_2}(t,x) + \langle \theta_1 - \theta_2, x\rangle \pm \beta|\theta_2 - \theta_1|t, \quad (t,x) \in [0,T) \times \mathbb{R}^d.$$

The functions $u^{\varepsilon,+}_{\theta_2}$ and $u^{\varepsilon,-}_{\theta_2}$ are, respectively, a super- and a subsolution of (HJ_ε), in view of assumption (H2), which satisfy $u^{\varepsilon,\pm}_{\theta_2}(0,x) = \langle \theta_1, x\rangle$. By Theorem A.3 we derive that $u^{\varepsilon,-}_{\theta_2} \leqslant \tilde{u}^\varepsilon_{\theta_1} \leqslant u^{\varepsilon,+}_{\theta_2}$ in $[0,T) \times \mathbb{R}^d$, hence

$$|\overline{H}(\theta_1) - \overline{H}(\theta_2)| = \lim_{\varepsilon \to 0^+} |\tilde{u}^\varepsilon_{\theta_1}(1,0) - \tilde{u}^\varepsilon_{\theta_2}(1,0)| \leqslant \beta|\theta_2 - \theta_1|.$$

It is clear that such an $\overline{H}$ can be uniquely extended by continuity to the whole $\mathbb{R}^d$.

(ii) For every $\theta \in \mathbb{R}^d$, let us set $u^\varepsilon_\theta := \tilde{u}^\varepsilon_\theta - \langle \theta, x\rangle$. Then u^ε_θ is the solution of (HJ$_\varepsilon$) with $H(\cdot, \theta + \cdot)$ in place of H and initial datum $u^\varepsilon(0, x) = 0$ for all $x \in \mathbb{R}^d$. Let us fix $\theta \in \mathbb{R}^d \setminus D$ and choose a sequence $(\theta_n)_n$ in D converging to θ. Let us set

$$u^\varepsilon_{n,\pm}(t, x) := u^\varepsilon_{\theta_n}(t, x) \pm t\beta(|\theta_n - \theta|), \quad (t, x) \in [0, T) \times \mathbb{R}^d.$$

In view of assumption (H2), it is easy to check that $u^\varepsilon_{n,-}$ and $u^\varepsilon_{n,+}$ are, respectively, a sub- and a supersolution of (HJ$_\varepsilon$) with $H(\cdot, \theta + \cdot)$ in place of H and zero initial datum. By comparison, see Theorem A.3, we infer that $|u^\varepsilon_\theta(t, x) - u^\varepsilon_{\theta_n}(t, x)| \leqslant t\beta(|\theta_n - \theta|)$ for all $(t, x) \in [0, T) \times \mathbb{R}^d$, hence

$$\begin{aligned}
|\tilde{u}^\varepsilon_\theta(t, x) - \langle \theta, x\rangle + t\bar{H}(\theta)| &= |u^\varepsilon_\theta(t, x) + t\bar{H}(\theta)| \\
&\leqslant |u^\varepsilon_{\theta_n} + t\bar{H}(\theta_n)| + t|\bar{H}(\theta_n) - \bar{H}(\theta)| + t\beta(|\theta_n - \theta|) \\
&\leqslant |u^\varepsilon_{\theta_n}(t, x) + t\bar{H}(\theta_n)| + 2T\beta(|\theta_n - \theta|)
\end{aligned}$$

for all $(t, x) \in [0, T) \times \mathbb{R}^d$. The assertion follows by sending first $\varepsilon \to 0^+$ and then $n \to +\infty$. ∎

We will also need the following fact.

Proposition B.2. *Let us assume that all the hypotheses of Theorem 3.1 are in force. Let $g \in \mathrm{UC}(\mathbb{R}^d)$ and, for every $\varepsilon > 0$, let us denote by u^ε the unique function in $\mathrm{C}([0, T) \times \mathbb{R}^d)$ that solves (HJ$_\varepsilon$) subject to the initial condition $u^\varepsilon(0, \cdot) = g$. Set*

$$u^*(t, x) := \lim_{r\to 0} \sup\{u^\varepsilon(s, y) : (s, y) \in (t - r, t + r) \times B_r(x),\ 0 < \varepsilon < r\},$$
$$u_*(t, x) := \lim_{r\to 0} \inf\{u^\varepsilon(s, y) : (s, y) \in (t - r, t + r) \times B_r(x),\ 0 < \varepsilon < r\}.$$

Let us assume that u^* and u_* are finite valued. Then

(i) *$u^* \in \mathrm{USC}([0, T) \times \mathbb{R}^d)$ and it is a viscosity subsolution of (3.2);*

(ii) *$u_* \in \mathrm{LSC}([0, T) \times \mathbb{R}^d)$ and it is a viscosity supersolution of (3.2).*

Proof. The fact that u^* and u_* are upper and lower semicontinuous on $[0, T) \times \mathbb{R}^d$ is an immediate consequence of their definitions. Let us prove (i), i.e., that u^* is a subsolution of (3.2). The proof of (ii) is analogous.

We make use of Evans's perturbed test function method; see [28]. Let us assume, by contradiction, that u^* is not a subsolution of (3.2). Then there exists a function $\phi \in \mathrm{C}^1([0, T) \times \mathbb{R}^d)$ that is a strict supertangent of u^* at some point $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ and for which the subsolution test fails, i.e.,

$$\partial_t\phi(t_0, x_0) + \bar{H}(D_x\phi(t_0, x_0)) > 2\delta \tag{B.2}$$

for some $\delta > 0$. For $r > 0$ define $V_r := (t_0 - r, t_0 + r) \times B_r(x_0)$. Choose $r_0 > 0$ to be small enough so that V_{r_0} is compactly contained in $[0, T) \times \mathbb{R}^d$ and $u^* - \phi$ attains a strict local maximum at (t_0, x_0) in V_{r_0}. In particular, we have for every $r \in (0, r_0)$,

$$\max_{\partial V_r}(u^* - \phi) < \max_{\overline{V}_r}(u^* - \phi) = (u^* - \phi)(t_0, x_0). \tag{B.3}$$

Let us set $\theta := D_x\phi(t_0, x_0)$ and for every $\varepsilon > 0$ denote by $\tilde{u}^\varepsilon_\theta$ the unique continuous function in $[0, T) \times \mathbb{R}^d$ that solves (HJ_ε) subject to the initial condition $\tilde{u}^\varepsilon_\theta(0, x) = \langle\theta, x\rangle$. We claim that there is an $r \in (0, r_0)$ such that the function

$$\phi^\varepsilon(t, x) := \phi(t, x) + \tilde{u}^\varepsilon_\theta(t, x) - (\langle\theta, x\rangle - t\overline{H}(\theta))$$

is a supersolution of (HJ_ε) in V_r for every $\varepsilon > 0$ small enough. Indeed, by a direct computation we first get

$$\partial_t\phi^\varepsilon + H\Big(\frac{x}{\varepsilon}, D_x\phi^\varepsilon\Big) = \partial_t\phi + \overline{H}(\theta) + \partial_t\tilde{u}^\varepsilon_\theta + H\Big(\frac{x}{\varepsilon}, D_x\tilde{u}^\varepsilon_\theta + D_x\phi - \theta\Big) \tag{B.4}$$

in the viscosity sense in V_r. Using (B.2), the continuity of $\overline{H}$ and the fact that ϕ is of class C^1, we get that there is an $r \in (0, r_0)$ such that for all sufficiently small $\varepsilon > 0$ and all $(t, x) \in V_r$,

$$\partial_t\phi(t, x) + \overline{H}(\theta) > 2\delta.$$

Moreover, by taking into account (H2), we can further reduce r if necessary to get

$$H\Big(\frac{x}{\varepsilon}, D\tilde{u}^\varepsilon_\theta + D_x\phi - \theta\Big) > H\Big(\frac{x}{\varepsilon}, D\tilde{u}^\varepsilon_\theta\Big) - \delta \quad \text{in } V_r$$

in the viscosity sense. Plugging these relations into (B.4) and using the fact that $\tilde{u}^\varepsilon_\theta$ is a solution of (HJ_ε), we finally get

$$\partial_t\phi^\varepsilon + H\Big(\frac{x}{\varepsilon}, D_x\phi^\varepsilon\Big) > \delta + \partial_t\tilde{u}^\varepsilon_\theta + H\Big(\frac{x}{\varepsilon}, D\tilde{u}^\varepsilon_\theta\Big) = \delta > 0$$

in the viscosity sense in V_r, thus showing that ϕ^ε is a supersolution of (HJ_ε) in V_r. Now we need a comparison principle for equation (HJ_ε) in V_r applied to ϕ^ε and u^ε to infer that

$$\sup_{V_r}(u^\varepsilon - \phi^\varepsilon) \leqslant \max_{\partial V_r}(u^\varepsilon - \phi^\varepsilon).$$

Since condition (H2) is in force, the validity of this comparison principle is guaranteed by [24, Theorem 3.3 and Section 5.C]. Now notice that, by the assumption (3.1), $\phi^\varepsilon \rightrightarrows \phi$ in $\overline{V}_r$. Taking the limsup of the above inequality as $\varepsilon \to 0^+$ we obtain

$$\sup_{V_r}(u^* - \phi) \leqslant \limsup_{\varepsilon\to 0^+}\sup_{V_r}(u^\varepsilon - \phi^\varepsilon) \leqslant \limsup_{\varepsilon\to 0^+}\max_{\partial V_r}(u^\varepsilon - \phi^\varepsilon) \leqslant \max_{\partial V_r}(u^* - \phi),$$

in contradiction with (B.3). This proves that u^* is a subsolution of (3.2). ■

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. The fact that $\overline{H}$ satisfies (H1)–(H2) directly follows from Proposition B.1 by taking $D = \mathbb{R}^d$. We now proceed to prove the second part of the assertion. Let us take a dense and countable subset $D := (\theta_n)_n$ of $\mathbb{R}^d$ and set $\widehat{\Omega} := \bigcap_n \Omega_{\theta_n}$. Let us fix $\omega \in \widehat{\Omega}$. According to Proposition B.1, the convergence in (B.1) holds for every $\theta \in \mathbb{R}^d$. We are going to show that, for any such fixed $\omega \in \widehat{\Omega}$, the solutions $u^\varepsilon(\cdot, \cdot, \omega)$ to (HJ_ε) with initial datum $u^\varepsilon(0, \cdot, \omega) = g$ in $\mathbb{R}^d$ converge to the solution $\bar{u}$ of (3.2) with the same initial datum, for any $g \in \mathrm{UC}(\mathbb{R}^d)$. Since ω will remain fixed throughout the proof, we will omit it from our notation.

Let us first assume $g \in \mathrm{C}^1(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$. Take a constant M large enough so that

$$M > \|H(x, Dg(x))\|_\infty.$$

Then the functions $u_-(t,x) := g(x) - Mt$ and $u_+(t,x) := g(x) + Mt$ are, respectively, a Lipschitz continuous sub- and supersolution of (HJ_ε) for every $\varepsilon > 0$. By the comparison principle stated in Theorem A.3, we get $u_- \leqslant u^\varepsilon \leqslant u_+$ in $[0,T) \times \mathbb{R}^d$ for every $\varepsilon > 0$. By the definition of relaxed semilimits we infer

$$u_-(t,x) \leqslant u_*(t,x) \leqslant u^*(t,x) \leqslant u_+(t,x) \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}^d,$$

in particular, u_*, u^* satisfy $u_*(0,\cdot) = u^*(0,\cdot) = g$ on $\mathbb{R}^d$. By Proposition B.2, we know that u^* and u_* are, respectively, an upper semicontinuous subsolution and a lower semicontinuous supersolution of the effective equation (3.2). We can therefore apply Theorem A.3 again to obtain $u^* \leqslant u_*$ on $[0,T) \times \mathbb{R}^d$. Since the opposite inequality holds by the definition of upper and lower relaxed semilimits, we conclude that the function

$$\bar{u}(t,x) := u_*(t,x) = u^*(t,x) \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}^d$$

is the unique continuous viscosity solution of (3.2) such that $\bar{u}(0,\cdot) = g$ on $\mathbb{R}^d$. The fact that the relaxed semilimits coincide implies that the u^ε converge locally uniformly in $[0,T) \times \mathbb{R}^d$ to $\bar{u}$; see for instance [1, Lemma 6.2, p. 80].

When the initial datum g is just uniformly continuous on $\mathbb{R}^d$, the result easily follows from the above by approximating g with a sequence $(g_n)_n$ of initial data in $\mathrm{C}^1(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$ in such a way that $\|g_n - g\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $n \to +\infty$. Indeed, let us denote by u_n^ε and u^ε the solutions of (HJ_ε) with initial data g_n and g, and by $\bar{u}$ and $\bar{u}_n$ the viscosity solutions of (3.2) with initial data g and g_n, respectively. Fix a compact set K in $[0,T) \times \mathbb{R}^d$. In view of Theorem A.3, we get

$$\begin{aligned}\|u^\varepsilon - \bar{u}\|_{L^\infty(K)} &\leqslant \|u^\varepsilon - u_n^\varepsilon\|_{L^\infty(K)} + \|u_n^\varepsilon - \bar{u}_n\|_{L^\infty(K)} + \|u_n - \bar{u}\|_{L^\infty(K)} \\ &\leqslant 2\|g_n - g\|_{L^\infty(\mathbb{R}^d)} + \|u_n^\varepsilon - \bar{u}_n\|_{L^\infty(K)}.\end{aligned}$$

The assertion follows from this in view of the fact that $\|u_n^\varepsilon - \bar{u}_n\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0^+$, for any $n \in \mathbb{N}$, and by arbitrariness of the choice of K. ■

Acknowledgments. We thank the anonymous referees for their careful reading and many valuable comments. In particular, we are grateful to one referee for recognizing that our proof strategy could be leveraged to obtain, essentially for free, a quantitative rate of homogenization, and for suggesting the stability result leading to Corollary 2.5. AD is a member of the INdAM Research Group GNAMPA. He thanks Guy Barles for a long and valuable email exchange in the fall of 2023 concerning possible generalizations of the comparison principle stated in Theorem A.1. AD is particularly grateful to Guy Barles for carefully reading his attempts to generalize Theorem A.1 and for sharing his expertise on the matter, including the details needed to turn the argument sketched in [18, Remark 5.3] into a full proof. BZ is very grateful to Scott Armstrong and Pierre Cardaliaguet for all the enlightening discussions on homogenization theory.

Funding. This work was supported by Sapienza Università di Roma – Research Funds 2018 and 2019, by the French Agence Nationale de la Recherche (ANR) under grant ANR-21-CE40-0020 (CONVERGENCE project) and under grant ANR-17-EURE-0010 (Investissements d'Avenir program), and by the ERC CoG 863818 (ForM-SMArt) grant. It was partly done during a one-year visit by BZ to the Center for Mathematical Modeling (CMM) at the University of Chile in 2023, under the IRL program of CNRS.

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Received 7 August 2024; revised 22 October 2025; accepted 13 January 2026.

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