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Boundary regularity of uniformly rotating vortex patches and an unstable elliptic free boundary problem

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Abstract. In this paper, we consider the sign-changing free boundary problem related to the uniformly rotating vortex patch solutions of the two-dimensional incompressible Euler equations. We prove that the boundary of the vortex patch locally forms a 90° corner near singular boundary points, if the patch is Lipschitz.

1. Introduction

Consider the two-dimensional incompressible Euler equations on the plane,

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p, & x \in \mathbb{R}^2, \\ \nabla \cdot v = 0, & \\ v \to 0, & |x| \to \infty, \end{cases}$$

where $v = (v^1, v^2)^\intercal \in \mathbb{R}^2$ is the velocity of the fluid and p its pressure.Throughout this paper, we shall mainly work with the vorticity formulation of the two-dimensional incompressible Euler equations,

$$\begin{cases} \partial_t \omega + (v \cdot \nabla) \omega = 0, & x \in \mathbb{R}^2, \\ v = -\nabla^\perp (-\Delta)^{-1} \omega, & (a_1, a_2)^\perp := (-a_2, a_1)^\intercal, \end{cases} \tag{1.1}$$

with the vorticity field given by $\omega(t, x) := \partial_1 v^2 - \partial_2 v^1$, in which the velocity is uniquely determined by the vorticity due to the *stream-vorticity* formula. Equation (1.1) is clearly a closed PDE system concerning ω only. The global well-posedness of smooth solutions is well established, as well as the Yudovich-type weak solutions $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. For necessary background and a review on the classical results concerning the Euler equations, we refer to [21, 28] for interested readers.

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Particular interest focuses on the vortex patch solutions of (1.1) in the form of

$$\omega(t,x)=\sum_{j=1}^{N}\kappa_j\mathbf{I}_{D_j(t)},$$

where N is a positive integer, each $\kappa_1,\ldots,\kappa_N$ is a non-zero constant, $D_1,\ldots,D_N$ are disjoint bounded domains and $\mathbf{I}_D$ denotes the characteristic function of the domain D. The global existence and uniqueness of vortex patch solutions follow from [21] straightforwardly. The global regularity of vortex patches was first proved by Chemin [7] provided $\partial D\in C^{k,\alpha}$, $k\geq 1, 0<\alpha<1$; see other proofs given by Bertozzi and Constantin [3] and Serfati [32]. Recently, Kiselev and Luo [26] established the global regularity of the patch if the initial data are of Sobolev classes.

On the other hand, Chemin [8] studied the preservation of singularities on the boundary of an evolving patch; see also [10] where Elgindi and Jeong considered the well-/ill-posedness of specific singular structures. The global regularity of the patch solution of a two-dimensional generalized transport equation was proved by Verdera in [34].

Kiselev and Sverak [27] studied the small-scale formation of the two-dimensional incompressible Euler equations, namely the infinite time blow-up solutions which satisfy $\|\nabla\omega(t)\|_\infty\sim C_1e^{e^{C_2t}}$. Furthermore, Kiselev and Li [25] proved that an analogous phenomenon that the curvature of the boundary tends to infinity happens for vortex patches. There is a long-standing conjecture that evolving vortex patches would eventually weakly converge to some steady vortex patches. This conjecture seems beyond the current capability of PDEs, but also indicates that the singular uniformly rotating vortex patches may occur as the global attractors in the class of vortex patch solutions.

Our main interest will focus on steady vortex patches. Being aware of the rotation and translation invariance of the system, our consideration mainly address the uniformly rotating vortex patches around the origin,

$$\omega(t,x)=\omega_0(e^{-i\Omega t}x),$$

which are relative equilibria of the two-dimensional incompressible Euler equations, i.e., the vorticity configuration is invariant up to rigid motions, where $\Omega\in\mathbb{R}$ is the angular velocity and $\omega_0=\mathbf{I}_D$ is the initial patch. This setting holds due to the rotational invariance of $\mathbb{R}^2$ and conservation of the center of vorticity. By the incompressible condition, there exists a scalar function Ψ, referred to as the stream function, such that $v=-\nabla^{\perp}\Psi$. Then the uniformly rotating vortex patch is determined by a non-linear elliptic problem concerning the relative stream function $\psi(x)=\Psi(x)+\frac{\Omega}{2}|x|^2$.

We call $P\in\mathbb{R}^2$ a stagnation point of the fluid if $v(P)=0$, or equivalently $\nabla\psi(P)=0$ in the rotating coordinate with the angular velocity Ω. In general ψ is called the relative stream function, but we reserve the name stream function instead, throughout this paper, for simplicity.

Suppose D is a bounded domain enclosed by a rectifiable Jordan curve. It is not hard to verify that $\omega=\mathbf{I}_D$ is a uniformly rotating vortex patch if and only if ψ solves the elliptic free boundary problem

$$\begin{cases} -\Delta\psi = \mathbf{I}_D - 2\Omega, & x \in \mathbb{R}^2, \\ \nabla\Big(\psi - \dfrac{\Omega}{2}|x|^2\Big) \to 0, & |x| \to \infty, \\ \psi = 0, & x \in \partial D, \end{cases} \tag{1.2}$$

in which the vortical domain D is the main unknown.

Denote the unit disk by B_1. Clearly the circular patch $\omega = \mathbf{I}_{B_1}$ (the Rankine vortex) is a uniformly rotating vortex patch solution for any $\Omega \in \mathbb{R}$. Another explicit non-trivial uniformly rotating vortex patch is the Kirchhoff elliptic patch,

$$\omega = \mathbf{I}_{E_{a,b}}, \quad E_{a,b} := \Big\{(x_1, x_2) \in \mathbb{R}^2 \;\Big|\; \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1\Big\},$$

where the angular velocity satisfies $0 < \Omega = \frac{ab}{(a+b)^2} \le \frac{1}{4}$.

We note that the vortical domains may have a complex topology. When the domain D is not simply connected, it is sufficient to slightly modify equation (1.2) to

$$\begin{cases} -\Delta\psi = \mathbf{I}_D - 2\Omega, & x \in \mathbb{R}^2, \\ \nabla\Big(\psi - \dfrac{\Omega}{2}|x|^2\Big) \to 0, & |x| \to \infty, \\ \psi = c_j, & x \in \Gamma_j, \end{cases} \tag{1.3}$$

in which each closed curve Γ_j is a connected component of ∂D, and c_j is a constant, $1 \le j \le n$, where n is the number of connected components of ∂D. Clearly, the annular patches

$$\omega = \mathbf{I}_{C_{b,1}}, \quad C_{b,1} := \big\{x \in \mathbb{R}^2 \;\big|\; b \le |x| \le 1\big\}$$

are trivial solutions of (1.3), holding for any $\Omega \in \mathbb{R}$ and $b \in (0, 1)$. Beyond these explicit solutions, lots of uniformly rotating vortex patches are obtained via the local bifurcation approach [4, 6, 9, 18–20]. All of these solutions are sufficiently close to the explicit solutions, such as the circular/annular/elliptic vortex patches, in the sense of boundary perturbation.

Furthermore, Hassainia, Masmoudi and Wheeler [17] studied the global continuation of the m-fold local bifurcation curve $\mathscr{S}_m$ parameterized by $s \in \mathbb{R}$, emanating from the Rankine vortex for $m \ge 3$. They showed that the end of bifurcation curves has the following alternatives:

(1) Smooth vortex patch with overhanging profiles, i.e., the boundary cannot be parameterized as a graph in polar coordinates.

(2) Limiting V-states, i.e., singular vortex patch where the boundary is parameterized as a non-C^1 graph in polar coordinates.

Together with the numerical observations given in [30, 35], they made the following conjecture.

Conjecture. *The singular solution with* 90° *corners seen in numerics for* $m \geq 3$ *exists as the (weak) limit of patches along the* m*-fold branch* $\mathscr{S}_m$ *as* $s \to \infty$.

This conjecture is highly challenging since the nature of the problem essentially depends on the global geometric property of the special solutions emanating from the disk, which have not been obtained explicitly yet. Some recent progress on the quantitative properties of the m-fold rotating vortex patch has been made by Park [31] for sufficiently large m via a variational argument based on the optimal transport. However, it seems not easy to gain uniform estimates on the higher-order derivatives of the graph function.

On the other hand, inspired by the remarkable progress on the study of the unstable obstacle problem [2, 29], it is an interesting problem to employ the techniques developed by the community of the free boundary problem to study the boundary regularity of uniformly rotating vortex patches, which motivates the present paper. Since our approach follows a flavor of variation, the vortex patches we studied are not necessarily located on the bifurcation curves emanating from the unit disk.

Our consideration is based on the observation that the uniformly rotating vortex patch gives rise to the following elliptic unstable free boundary problem:

$$\begin{cases} -\Delta u = \lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c} & \text{in } \mathbb{R}^2, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.4}$$

in which $\lambda_1, \lambda_2 > 0$ are prescribed constants and D is an unknown bounded domain. By the maximum principle, clearly u is positive in the domain D, while the sign of the solution u may change in the complementary domain D^c. It is worth pointing out that equation (1.4) indeed describes the simply connected rotating vortex patch, or vortex patch solutions of (1.1) consist of simply connected components. However, since our analysis is indeed local, namely we focus on the singular points on a connected component $\Gamma \subset \partial D$, thus we can add a constant such that the stream function $u = 0$ on Γ; then the conclusion also holds for non-simply-vortical domains.

We shall focus on the regularity of the free boundary $\partial D \subset \{u = 0\}$ via tools developed in the study of obstacle-type problems since the pioneer work of Caffarelli [5], in particular, the Weiss-type monotonicity formula. See [13, 14] and references therein for more recent advances. To the best of the authors' knowledge, there are only a few results on the regularity of uniformly rotating vortex patches, except for those sufficiently close to explicit solutions. Hmidi, Mateu and Verdera [20] proved that the boundary of uniformly rotating vortex patches on the local branch emanating from the Rankine vortex is C^∞ smooth. The smoothness was very soon improved to real-analytic by Castro, Córdoba and Gómez-Serrano [6]. On the other hand, regarding the unstable obstacle problem, results obtained in [2, 29] could not be implemented here straightforwardly since the level set $\{u = 0\}$ does not completely coincide with the free boundary, and behaviors of the stream function are essentially different, and the minimal/maximal solutions hardly exist.

A uniformly rotating vortex patch is a special class of the sign-changing free boundary problem if and only if $\Omega \in (0, \frac{1}{2})$. Here we mainly focus on the singularity occupying the

Lipschitz boundary of the vortex patch with $\Omega \in (0, \frac{1}{2})$. A domain is called a Lipschitz domain if its boundary is a Lipschitz curve. With the Lipschitz smoothness assumption on the boundary, we can give a complete description of the boundary regularity of uniformly rotating vortex patches.

The main result of this paper is given as follows.

Theorem 1.1. *Suppose D is a Lipschitz domain and $\omega = \mathbf{I}_D$ is a uniformly rotating vortex patch with angular velocity $0 < \Omega < \frac{1}{2}$. Then the singular set $S \subset \partial D$ contains at most finitely many points, and $\partial D \setminus S$ is smooth. Near each singular point $x \in S$, for sufficiently small $r > 0$, $\partial D \cap B(x, r)$ consists of two C^1 arcs meeting at the right angle. In particular, if there is no stagnation point on the boundary, the boundary is C^∞ smooth.*

Remark 1.2. The conclusion also holds for symmetric rotating vortex pairs by the same argument. It holds regardless of the solutions located on the bifurcation curves. The result recovers numerical observations obtained in [35]. We expect the analysis to be useful in solving the conjecture on the limiting V-states, but a more delicate quantitative analysis on the geometry of the vortical domain seems necessary.

While Theorem 1.1 suggests that certain singularities are restricted, others might persist in Lipschitz domains when $\Omega \notin (0, \frac{1}{2})$. However, a remarkable rigidity result by Gómez-Serrano, Park, Shi, and Yao [16] rules out another type of singularity.

Theorem A. *Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Assume that $\omega = \mathbf{I}_D$ is a stationary/uniformly rotating vortex patch of (1.3) for some $\Omega \in \mathbb{R}$. Then D must be radially symmetric if $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, +\infty)$ and radially symmetric up to a translation if $\Omega = 0$.*

This result has recently been extended to domains with Jordan curve boundaries in [12].

We would like to close the introduction with some remarks concerning the existence of singular vortex patches. While plenty of results have been obtained, it is also worth pointing out that our consideration here is a priori, i.e., we assume the free boundary problem possesses a solution (u, D). Whether or not a singular steady vortex patch really exists remains an open problem to the best of our knowledge. Of course, one could apply a variational method to study the existence of rotating vortex patches as in [33], where the existence of uniformly rotating vortex patches could follow from maximizing the energy function penalized by the angular momentum,

$$E(\omega) = \frac{1}{2} \int_{\mathbb{R}^2} \omega(-\Delta)^{-1} \omega \, d\mu - \frac{\Omega}{2} \int_{\mathbb{R}^2} |x|^2 \omega \, d\mu,$$

among the rearrangement class of prescribed vorticity and angular impulse. But it seems highly non-trivial to determine the regularity of the boundary, except if the vortex patch is highly concentrated. Indeed, for steady vortex patches in a bounded domain, Turkington [33] obtained the existence of steady vortex patches and C^1 regularity of the boundary of vortical domains when these vortex patches are highly concentrated. He also proposed a conjecture that the singular set on the boundary of a steady vortex patch has Hausdorff dimension zero if these patches are not concentrated. So, if the conjecture in [17] holds, it verifies the existence of singular uniformly rotating patches, which may be of independent interest. We also note that García and Haziot [15] studied the global continuation of the local branch emanating from a pair of corotating point vortices in a very recent work. More limiting V-states occur and their geometry is much more complicated if D_1 and D_2 are allowed to touch each other in several ways.

This paper is organized as follows. In Section 2 we consider a sign-changing unstable elliptic free boundary problem and derive the Weiss-type monotonicity formula. The classification of singular points is obtained due to their blow-up limits. In Section 3 we prove uniform regularity near singular points where u has super-characteristic growth, and then finish the proof of Theorem 1.1.

2. A sign-changing two-phase unstable elliptic free boundary problem: Weiss-type monotonicity formula and classification of blow-up limits

The uniformly rotating vortex patch problem (1.2) gives rise to the two-phase unstable elliptic free boundary problem as follows:

$$\begin{cases} -\Delta u = \lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c} & \text{in } \mathbb{R}^n, n \geq 2, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{2.1}$$

where $\lambda_1, \lambda_2 > 0$ are prescribed constants and the bounded domain D is the main unknown. Here $D^c := \mathbb{R}^n \setminus D$ denotes the complementary set of the domain D.

Being aware that u may change its sign in $\bar{D}^c$, in general $\bar{D}^c$ does not coincide with the set $\{u < 0\}$. Thus (2.1) is rather different from the classical unstable obstacle problem which has been extensively investigated since the seminal works [2, 29], since equation (2.1) concerns not only the function u but also the domain D.

Assume a weak solution $(u, D) \in H^1(\mathbb{R}^n) \times \mathbb{O}(\mathbb{R}^n)$ exists for equation (2.1), where $\mathbb{O}(\mathbb{R}^n) := \{D \subset \mathbb{R}^n \text{ is a bounded domain}\}$. By standard elliptic regularity theory, we have $u \in C^{1,\alpha}(\mathbb{R}^2)$ for any $\alpha \in (0,1)$ and u is real-analytic in either D or $\bar{D}^c$.

Employing the implicit function theorem on a regular point $x \in \partial D$, i.e., $|\nabla u(x)| \neq 0$, there exists $\delta > 0$ such that $\partial D \cap B_r(x)$ is a $C^{1,\alpha}$ curve for some $0 < \alpha < 1$. Further employing the classical bootstrap argument, see for example [24], the regularity of this local curve can be improved to real-analytic.

On the other hand, when D is not C^1 smooth, the singular set of ∂D, defined by

$$\mathcal{S}^u := \big\{x \in \partial D \;\big|\; |\nabla u(x)| = 0\big\},$$

and the geometric properties of the boundary nearby are our main interest. We begin by establishing the Weiss-type monotonicity formula for (2.1) as follows.

Lemma 2.1. *Suppose that (u, D) is a weak solution of (2.1). For any $x_0 \in \mathbb{R}^n$, let*

$$\Phi_{x_0}(r) = r^{-n-2} \int_{B_r(x_0)} |\nabla u|^2 - 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c})\, d\mathcal{H}^n - 2r^{-n-3} \int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{n-1},$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. Then for any $0 < \rho < \delta$, the following Weiss-type monotonicity formula holds:

$$\Phi_{x_0}(\delta) - \Phi_{x_0}(\rho) = \int_\rho^\delta \int_{\partial B_r(x_0)} 2r^{-n-2}\Big(\nabla u \cdot \nu - \frac{2u}{r}\Big)^2\, d\mathcal{H}^{n-1}\, dr \geq 0. \tag{2.2}$$

We begin by proving the following lemma.

Lemma 2.2. *If u is a weak solution of (2.1), then*

$$\int_{\mathbb{R}^n} |\nabla u|^2 \operatorname{div} \vec{X} - 2(\nabla u)^\intercal \mathcal{D}\vec{X}\nabla u - 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c}) \operatorname{div} \vec{X} = 0 \tag{2.3}$$

holds for all $\vec{X} = (X_1, \dots, X_n) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, where $\mathcal{D}\vec{X} = (\partial_i X_j)_{1 \leq i,j \leq n}$.

Proof. Since $\vec{X}$ is compactly supported and $\nabla u \in L^2(\mathbb{R}^2)$, by a straightforward computation, we get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \operatorname{div}(|\nabla u|^2 \vec{X}) \\
&= \int_{\mathbb{R}^n} 2\sum_{i=1}^n \sum_{j=1}^n \partial_i u \partial_{ij} u X_j + |\nabla u|^2 \operatorname{div} \vec{X} \\
&= \int_{\mathbb{R}^n} -2\sum_{i=1}^n \sum_{j=1}^n (\partial_{ii} u \partial_j u X_j + \partial_i u \partial_i X_j \partial_j u) + |\nabla u|^2 \operatorname{div} \vec{X} \\
&= \int_{\mathbb{R}^n} -2\Delta u(\nabla u \cdot \vec{X}) - 2(\nabla u)^\intercal \mathcal{D}\vec{X}\nabla u + |\nabla u|^2 \operatorname{div} \vec{X} \\
&= \int_{\mathbb{R}^n} 2\lambda_1 \mathbf{I}_D \nabla u \cdot \vec{X} - 2\lambda_2 \mathbf{I}_{D^c} \nabla u \cdot \vec{X} - 2(\nabla u)^\intercal \mathcal{D}\vec{X}\nabla u + |\nabla u|^2 \operatorname{div} \vec{X} \\
&= \int_{\mathbb{R}^n} |\nabla u|^2 \operatorname{div} \vec{X} - 2(\nabla u)^\intercal \mathcal{D}\vec{X}\nabla u - 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c}) \operatorname{div} \vec{X}.
\end{aligned}$$

Then the proof is complete. ■

Now we are going to prove the Weiss-type monotonicity formula of the sign-changing elliptic free boundary problem.

Proof of Lemma 2.1. Suppose that r, k are positive constants and let

$$\eta_k(x) = \max\big\{0, \min\{1, (r - |x|)k\}\big\}, \quad \vec{X}_k(x) = \eta_k(x)x = (\eta_k(x)x_1, \dots, \eta_k(x)x_n).$$

A straightforward computation shows that

$$\operatorname{div} \vec{X}_k(x) = \sum_{i=1}^{n} \partial_i(\eta_k(x)x_i) = n\eta_k(x) + \nabla\eta_k(x)\cdot x,$$

$$\begin{aligned}(\nabla u)^{\mathsf{T}} \mathcal{D}\vec{X}_k \nabla u &= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i u \partial_j(\eta_k(x)x_i)\partial_j u \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i u \eta_k \delta_{ij}\partial_j u + \partial_i u \partial_j(\eta_k)x_i\partial_j u \\ &= |\nabla u|^2\eta_k + (\nabla u\cdot x)(\nabla u\cdot\nabla\eta_k).\end{aligned}$$

According to (2.3), one has

$$\begin{aligned}0 = &\int_{\mathbb{R}^n} n|\nabla u|^2\eta_k + |\nabla u|^2\nabla\eta_k\cdot x - 2|\nabla u|^2\eta_k - 2(\nabla u\cdot x)(\nabla u\cdot\nabla\eta_k) \\ &- \int_{\mathbb{R}^n} 2(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})(n\eta_k + \nabla\eta_k\cdot x)u.\end{aligned}$$

Due to the definition of the cutoff function $\eta_k(x)$, we get $\eta_k = 1$, $|x| \le r - \frac{1}{k}$ and $\eta_k = 0$, $|x| \ge r$. Now letting $k \to +\infty$, we obtain

$$\begin{aligned}0 = &\int_{B_r(0)} (n-2)|\nabla u|^2 - 2nu(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^n \\ &- r\int_{\partial B_r(0)} |\nabla u|^2 - 2(\nabla u\cdot\nu)^2 - 2u(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^{n-1},\end{aligned} \tag{2.4}$$

for a.e. $r > 0$, where ν is the unit outward normal vector of $B_r(0)$. Then one has

$$\begin{aligned}\frac{d}{dr}\Phi_{x_0}(r) = &-(n+2)r^{-n-3}\int_{B_r(x_0)} |\nabla u|^2 - 2u(\lambda_1\mathbf{I}_D - 2\lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^n \\ &+ r^{-n-2}\int_{\partial B_r(x_0)} |\nabla u|^2 - 2u(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^{n-1} \\ &+ 8r^{-5}\int_{\partial B_1(0)} u^2(x_0+rx)\, d\mathcal{H}^{n-1} \\ &- 4r^{-4}\int_{\partial B_1(0)} u(x_0+rx)(\nabla u(x_0+rx)\cdot x)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Inserting (2.4) into above equation, one has

$$\begin{aligned}\Phi'_{x_0}(r) = &-r^{-n-3}\int_{B_r(x_0)} (n-2)|\nabla u|^2 - 2nu(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^n \\ &- 4r^{-n-3}\int_{B_r(x_0)} |\nabla u|^2 - u(\lambda_1\mathbf{I}_D - \lambda_2\mathbf{I}_{D^c})\, d\mathcal{H}^n\end{aligned}$$

$$
\begin{aligned}
&+ r^{-n-2}\int_{\partial B_r(x_0)} |\nabla u|^2 - 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c})\, d\mathcal{H}^{n-1} \\
&+ 8r^{-n-4}\int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{n-1} - 4r^{-n-3}\int_{\partial B_r(x_0)} u(\nabla u\cdot\nu)\, d\mathcal{H}^{n-1} \\
&= 2r^{-n-2}\int_{\partial B_r(x_0)} (\nabla u\cdot\nu)^2\, d\mathcal{H}^{n-1} + 8r^{-n-4}\int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{n-1} \\
&- 8r^{-n-3}\int_{\partial B_r(x_0)} u(\nabla u\cdot\nu)\, d\mathcal{H}^{n-1} \\
&= 2r^{-n-2}\int_{\partial B_r(x_0)} \left(\nabla u\cdot\nu - \frac{2u}{r}\right)^2 d\mathcal{H}^{n-1},
\end{aligned}
\tag{2.5}
$$

in which we used the identity

$$
\begin{aligned}
\int_{\partial B_r(x_0)} u(\nabla u\cdot\nu)\, d\mathcal{H}^{n-1} &= \int_{B_r(x_0)} |\nabla u|^2 + u\Delta u\, d\mathcal{H}^{n} \\
&= \int_{B_r(x_0)} |\nabla u|^2 - u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c})\, d\mathcal{H}^{n}.
\end{aligned}
$$

Equation (2.2) is obtained by integrating equation (2.5). ■

Lemma 2.1 leads to a classification of the blow-up limits of u at singular points on the free boundary.

Proposition 2.3 (Classification of blow-up limits). *Suppose u is a solution of* (2.1) *and $x_0 \in \mathcal{S}^u$. The following alternative cases occur:*

(1) *In the case of $\Phi_{x_0}(0+) = -\infty$, one has $\lim_{r\to 0} r^{-3-n}\int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{n-1} = +\infty$, and each limit of*

$$
v_r(x) = \frac{u(x_0 + rx)}{S(x_0, r)}
$$

as $r \to 0$ is a homogeneous harmonic polynomial of degree two, where

$$
S(x_0, r) := \left(r^{1-n}\int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{n-1}\right)^{1/2}.
$$

(2) *In the case of $\Phi_{x_0}(0+) > -\infty$, for any given $R > 0$, there exists $\tau(R) > 0$ such that for $r < \tau(R)$,*

$$
u_r(x) := \frac{u(x_0 + rx)}{r^2}
$$

is bounded in $W^{1,2}(B_R(0))$. Let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence such that $\{u_{r_j}\}_{j\in\mathbb{N}}$ converges to u_0 weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Then u_0 is a degree-two homogeneous solution of

$$
\begin{cases}
-\Delta u_0 = \lambda_1 \mathbf{I}_{D_*^1} - \lambda_2 \mathbf{I}_{D_*^2} & \text{in } \mathbb{R}^n, \\
u_0 = 0 & \text{on } \partial D_*^1 \cup \partial D_*^2,
\end{cases}
$$

where D_*^1 and D_*^2 are open cones satisfying $D_*^1 \cap D_*^2 = \emptyset$, $\mathbb{R}^2 \setminus (D_*^1 \cup D_*^2) \subset \{u_0 = 0\}$ and $u_0 > 0$ in D_*^1. Here, $\lambda_1 \mathbf{I}_{D_*^1} - \lambda_2 \mathbf{I}_{D_*^2} = \lim_{j\to\infty}(\lambda_1 \mathbf{I}_{D_{r_j}} - \lambda_2 \mathbf{I}_{D_{r_j}^c})$ in the sense of distribution and $D_r = \{x \mid x_0 + rx \in D\}$.

Remark 2.4. When $n = 2$ and $D_*^1 \neq \emptyset$, we get $|\mathbb{R}^2 \setminus (D_*^1 \cup D_*^2)| = 0$. Indeed, direct calculation shows that for all $x \notin D_*^1 \cup D_*^2 \cup \{0\}$, $|\nabla u(x)| \neq 0$. Therefore, u_0 is a solution to the equation

$$-\Delta u_0 = \lambda_1 \mathbf{I}_{D_*^1} - \lambda_2 \mathbf{I}_{(D_*^1)^c} \quad \text{in } \mathbb{R}^2,$$

with $u_0 > 0$ in D_*^1 and $u_0 = 0$ on ∂D_*^1.

Proof of Proposition 2.3. (1) If $\Phi_{x_0}(0+) = -\infty$, we suppose towards a contradiction that $\limsup_{r\to 0} S(x_0, r) \leq M r^2$ for some constant M, and there exists an $r_0 > 0$ such that, for $r < r_0$,

$$\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \leq M^2 r^{n+3}.$$

Thus

$$\begin{aligned}
\int_{B_r(x_0)} |u| \, d\mathcal{H}^n &= \int_0^r \int_{\partial B_t(x_0)} |u| \, d\mathcal{H}^{n-1} \, dt \\
&\leq C_1 \int_0^r \left(\int_{\partial B_t(x_0)} u^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} t^{\frac{n-1}{2}} \, dt \\
&\leq C_2 \int_0^r t^{\frac{n+3}{2}} t^{\frac{n-1}{2}} \, dt \\
&\leq C_3 r^{n+2}.
\end{aligned}$$

Consequently,

$$\left| \frac{1}{r^{n+2}} \int_{B_r(x_0)} 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c}) \, d\mathcal{H}^n \right| \leq C_4.$$

We conclude that

$$\begin{aligned}
\Phi_{x_0}(r) &\geq -\left| r^{-n-2} \int_{B_r(x_0)} 2u(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c}) \, d\mathcal{H}^n \right| - 2r^{-n-3} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \\
&\geq -C_4 - 2M^2,
\end{aligned}$$

for all $r < r_0$, which contradicts the assumption that $\Phi_{x_0}(0+) = -\infty$. Therefore, $\limsup_{r\to 0} \frac{S(x_0,r)}{r^2} = +\infty$. We pick up a sequence $\{r_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} r_j = 0$ and $\lim_{j\to\infty} \frac{S(x_0,r_j)}{r_j^2} = +\infty$. Let

$$v_k(x) = \frac{u(x_0 + r_k x)}{S(x_0, r_k)}.$$

Then $\|v_k\|_{L^2(\partial B_1(0))} = 1$. We may assume that $\lim_{k\to\infty} v_k \to v_0$ weakly in $L^2(\partial B_1(0))$.

Let $K = \frac{\max\{\lambda_1,\lambda_2\}}{2n}$. Then $u_r^+ + K|x|^2$ and $u_r^- + K|x|^2$ are sub-harmonic functions which imply

$$\frac{1}{t^{n-1}}\int_{\partial B_t(0)} u_r^+ + K|x|^2\, d\mathcal{H}^{n-1},$$
$$\frac{1}{t^{n-1}}\int_{\partial B_t(0)} u_r^- + K|x|^2\, d\mathcal{H}^{n-1},$$

are monotone increasing in t. Therefore,

$$\begin{aligned}\int_{\partial B_t(0)} u_r^+\, d\mathcal{H}^{n-1} &\le t^{n-1}\int_{\partial B_1(0)} u_r^+ + K|x|^2\, d\mathcal{H}^{n-1}\\ &\le C_5 t^{n-1}(\|u_r\|_{L^2(\partial B_1(0))}+1),\end{aligned}$$

for some constant C_5. Thus we get

$$\int_{B_1(0)} u_r^+\, d\mathcal{H}^n = \int_0^1\int_{\partial B_t(0)} u_r^+\, d\mathcal{H}^{n-1}\, dt \le \frac{C_5}{n}(\|u_r\|_{L^2(\partial B_1(0))}+1).$$

Analogously, we have

$$\int_{B_1(0)} u_r^-\, d\mathcal{H}^n \le \frac{C_6}{n}(\|u_r\|_{L^2(\partial B_1(0))}+1).$$

By the monotonicity formula (2.2) we have

$$\int_{B_1(0)} (|\nabla u_{r_k}|^2 + 2\lambda_1 \mathbf{I}_{D_{r_k}} u_{r_k} - 2\lambda_2 \mathbf{I}_{D_{r_k}^c} u_{r_k})\, d\mathcal{H}^n - 2\int_{\partial B_1(0)} u_{r_k}^2\, d\mathcal{H}^{n-1} \le \Phi_{x_0}(r_1).$$

Let $T_k = \frac{S(x_0,r_k)}{r_k^2}$ and $v_k(x) = \frac{u(x_0+r_k x)}{S(x_0,r_k)}$. Then

$$\int_{B_1(0)} |\nabla v_k|^2\, d\mathcal{H}^n \le T_k^{-2}\Phi_{x_0}(r_1) + C T_k^{-2}\int_{B_1(0)} |u_{r_k}|\, d\mathcal{H}^n + 2\int_{\partial B_1(0)} v_k^2\, d\mathcal{H}^{n-1}.$$

Note that $\lim_{k\to\infty} T_k = +\infty$ and

$$T_k^{-1}\int_{B_1(0)} |u_{r_k}|\, d\mathcal{H}^{n-1} \le C T_k^{-1}(\|u_{r_k}\|_{L^2(\partial B_1(0))}+1) \le C(1+T_k^{-1}).$$

Thus

$$\sup_{k\in\mathbb{N}} \|\nabla v_k\|_{L^2(B_1(0))} \le C_6,$$

for some constant C_6. By the Poincaré–Steklov inequality (see, e.g., [11, Lemma 3.30]) we have

$$\|v_k\|_{L^2(B_1(0))} \le C_7(\|\nabla v_k\|_{L^2(B_1(0))} + \|v_k\|_{L^2(\partial B_1(0))}) \le C_8,$$

where C_8 does not depend on k. Thus we may assume that v_k converges to v_0 weakly in $W^{1,2}(B_1(0))$. Letting $k\to\infty$, we obtain that v_0 is harmonic and

$$\int_{B_1(0)} |\nabla v_0|^2\, d\mathcal{H}^n \le 2\int_{\partial B_1(0)} v_0^2\, d\mathcal{H}^{n-1}.$$

Since $|\Delta v_k| \le \max\{\lambda_1, \lambda_2\}$, by the elliptic regularity and the Sobolev embedding theorem, for any given $\alpha \in (0,1)$, there exists a constant C_9 such that

$$\sup_{k\in\mathbb{N}} \|v_k\|_{C^{1,\alpha}(B_1(0))} \le C_9.$$

We may assume that v_k converges to v_0 in $C^{1,\beta}(B_1(0))$ for some $\beta \in (0,\alpha)$. Notice that $v_k(0) = 0$ and $\nabla v_k(0) = 0$, and we have $v_0(0) = 0$ and $\nabla v_0(0) = 0$. It follows from the frequency formula (see, e.g., [29, Lemma 4.2]) that v_0 is a harmonic polynomial of degree two.

(2) If $\Phi_{x_0}(0+) > -\infty$, there exists a constant $M < \infty$ such that

$$\limsup_{r\to 0} r^{-3-n} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} < M.$$

Thus we can choose $r_0 > 0$ small enough such that $\|u_r\|_{L^2(\partial B_1(0))} \le M + 1$ for $r \in (0, r_0)$. Without loss of generality, we set $R = 1$. Similar to case (1), we have $\int_{B_1(0)} |u_r| \, d\mathcal{H}^n < C_{10}$ for some constant C_{10}. It follows from the monotonicity formula that there exists a constant C_{11} such that

$$\int_{B_1(0)} |\nabla u_r(x)|^2 \le \Phi_{x_0}(r) + 2\max\{\lambda_1,\lambda_2\} \int_{B_1(0)} |u_r| \, d\mathcal{H}^2 + 2\int_{\partial B_1(x)} u_r^2 \, d\mathcal{H}^{n-1} \le C_{11},$$

for all $r \in (0, r_0)$. Thus $\{u_r\}$ is uniformly bounded in $W^{1,2}(B_1(0))$. We may choose a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that, for some $u_0 \in W^{1,2}(B_1(0))$, $u_{r_k} \to u_0$ weakly in $W^{1,2}(B_1(0))$, $u_{r_k} \to u_0$ in $L^2(B_1(0))$ and $u_{r_k} \to u_0$ in $L^2(\partial B_1(0))$. Moreover, for any given $\sigma > \rho > 0$, we have

$$\int_\rho^\delta \int_{\partial B_r(x_0)} 2\Big(\nabla u_{r_k} \cdot \nu - \frac{2u_{r_k}}{r}\Big)^2 \, d\mathcal{H}^{n-1} \, dr = \Phi_{x_0}(r_k\delta) - \Phi_{x_0}(r_k\rho) \to 0,$$

as $k \to \infty$. Letting $k \to \infty$, we have

$$\nabla u_0(x) \cdot x - 2u_0(x) = 0$$

for all $x \in \mathbb{R}^n$, and thus u_0 is a homogeneous function of degree two.

On the other hand, u_{r_k} is a weak solution to equation

$$-\Delta u_{r_k} = \lambda_1 \mathbf{I}_{D_{r_k}} - \lambda_2 \mathbf{I}_{(D_{r_k})^c},$$

with $u_{r_k} = 0$ on ∂D_{r_k}, where $D_{r_k} = \{x \in \mathbb{R}^n \mid x_0 + r_k x \in D\}$. By the weak-* compactness of L^∞, we may assume that $\lambda_1 \mathbf{I}_{D_{r_k}} - \lambda_2 \mathbf{I}_{(D_{r_k})^c}$ weak-* converges to some functions $\nu(x) \in L^\infty(\mathbb{R}^n)$, and u_0 is a weak solution to equation

$$-\Delta u_0 = \nu.$$

By the elliptic regularity theory, for any given $\alpha \in (0,1)$, $\{u_{r_k}\}$ is bounded in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. Thus we may assume u_{r_k} converges to u_0 in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\beta \in (0,\alpha)$. We first show that in each connected component of $\{u_0 \neq 0\}$, either $-\Delta u_0 \equiv \lambda_1$ or $-\Delta u_0 \equiv -\lambda_2$. Indeed, if $u_0(x_1) < 0$, there exists an r_0 such that $u_0 < 0$ in $B_{2r_0}(x_1)$. By the $C^{1,\alpha}$ convergence of u_{r_k}, there exists a constant $N \in \mathbb{N}$ such that for $k \geq N$, $u_{r_k} < 0$ in $B_{r_0}(x_1)$. Thus $-\Delta u_{r_k} = -\lambda_2$ in $B_{r_0}(x_1)$ for $k \geq N$. Let $v_k = u_{r_k} - \frac{\lambda_2}{2n}|x|^2$. Then $\{v_k\}_{k>N}$ are harmonic functions and $v_k \to u_0 - \frac{\lambda_2}{2n}|x|^2$ in $C^{1,\alpha}(B_{r_0}(x_1))$. It follows that $u_0 - \frac{\lambda_2}{2n}|x|^2$ is harmonic in $B_{r_0}(x_1)$, which implies $-\Delta u_0 = -\lambda_2$ in $B_{r_0}(x_1)$. Therefore, $\nu \equiv -\lambda_2$ in $\{u_0 < 0\}$. If $u_0(x_2) > 0$, there exists an r_0 such that $u_0 > 0$ in $B_{2r_0}(x_2)$. By the $C^{1,\alpha}$ convergence of u_{r_k}, there exists a constant $N \in \mathbb{N}$ such that for $k \geq N$, $u_{r_k} > 0$ in $B_{r_0}(x_2)$. Note that for each $k \geq N$, $-\Delta u_{r_k} = \lambda_1$ or $-\Delta u_{r_k} = -\lambda_2$ in $B_{r_0}(x_2)$. Choosing a subsequence if necessary, we may assume that $-\Delta u_{r_k} = \lambda_1$ or $-\Delta u_{r_k} = -\lambda_2$ for all $k \geq N$. Thus $-\Delta u_0 = \lambda_1$ or $\Delta u_0 = -\lambda_2$ in $B_{r_0}(x_2)$. Therefore, in each connected component of $\{u_0 > 0\}$, either $\nu \equiv \lambda_1$ or $\nu \equiv -\lambda_2$.

Next, we show that $\nu(x) = 0$ for almost all $x \in \{u_0 = 0\}$. Since the set $\{u_0 > 0\}$ is $C^{1,\alpha}$ smooth, it is easy to see that the Hausdorff dimension of $\{u = 0\} \cap \{|\nabla u| \neq 0\}$ is $n-1$, so the Lebesgue measure of $\{u_0 = 0\} \cap \{|\nabla u_0| \neq 0\}$ is 0. In $\{u_0 = 0\} \cap \{|\nabla u_0| = 0\}$ we have $D^2 u_0(x) = 0$ a.e., which implies that $\Delta u_0(x) = 0$ for almost all $x \in \{u_0 = 0\} \cap \{|\nabla u_0| = 0\}$. Therefore, $\Delta u_0(x) = 0$ for almost all $x \in \{u_0 = 0\}$.

Since u_0 is homogeneous of degree two, $\nu(rx) = \nu(x)$ for all $x \neq 0$ and $r > 0$. Therefore, $\nu = \lambda_1 \mathbf{I}_{D^1_*} - \lambda_2 \mathbf{I}_{D^2_*}$, where D^1_* and D^2_* are open cones, $D^1_* \cap D^2_* = \emptyset$, $\mathbb{R}^n \setminus (D^1_* \cup D^2_*) \subset \{u_0 = 0\}$, $u_0 > 0$ in D^1_*, and $u_0 = 0$ on $\partial D^1_* \cup \partial D^2_*$. ∎

Proposition 2.3 implies the singular points on the free boundary would be classified into the following two classes:

(1) singular points where u has super-characteristic growth (the blow-up limit I),

(2) singular points where u has characteristic growth (the blow-up limit II).

3. Singularities of the two-dimensional unstable elliptic free boundary problem

The proof of our main results will be accomplished by the following lemmas. Firstly, we focus on the two-dimensional unstable elliptic free boundary problem

$$\begin{cases} -\Delta\psi = \lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c} & \text{in } \mathbb{R}^2, \\ \nabla(\psi - \frac{\Omega}{2}|x|^2) \to 0, & |x| \to \infty, \\ \psi = 0 & \text{on } \partial D, \end{cases} \tag{3.1}$$

where D is a bounded domain. Due to the maximum principle, we have $\psi > 0$ in D.

3.1. Blow-up analysis

Suppose $x_0 \in \mathcal{S}^{\psi}$. It follows from the monotonicity formula that

$$\Phi^{\psi}_{x_0}(r) = \frac{1}{r^4}\int_{B_r(x_0)} |\nabla\psi|^2 - 2\psi(\lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c})\, d\mathcal{H}^2 - \frac{2}{r^5}\int_{\partial B_r(x_0)} \psi^2\, d\mathcal{H}^1$$

is monotone increasing. Proposition 2.3 implies the following alternatives occur, hence we need to consider them case by case.

Blow-up limit I. If $\Phi^{\psi}_{x_0}(0+) = -\infty$. One has

$$\lim_{r\to 0} r^{-5}\int_{\partial B_r(x_0)} \psi^2\, d\mathcal{H}^1 = +\infty.$$

Proposition 2.3 implies that, as $r \to 0$, any limit function of

$$\frac{\psi(x_0 + rx)}{S(x_0, r)}$$

must be a homogeneous harmonic polynomial of the form

$$\frac{x_1x_2 + A(x_1^2 - x_2^2)}{\|x_1x_2 + A(x_1^2 - x_2^2)\|_{L^2(\partial B_1)}}, \quad A \in \mathbb{R},$$

where $S(x_0, r) = (\frac{1}{r}\int_{\partial B_r(x_0)} \psi^2\, d\mathcal{H}^1)^{\frac{1}{2}}$. Up to a rotation, we may assume that $A = 0$. Let D_0 be the limit of D_r. Then $D_0 = \{(x_1, x_2) \mid x_1x_2 > 0\}$, or $D_0 = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$, or $D_0 = \{(x_1, x_2) \mid x_1 < 0, x_2 < 0\}$.

Blow-up limit II. If $\Phi^{\psi}_{x_0}(0+) > -\infty$, by Proposition 2.3 one has that

$$\psi_r(x) := \frac{\psi(x_0 + rx)}{r^2}$$

is bounded in $W^{1,2}(B_R)$ for any given $R > 0$, and there exists a subsequence $\{\psi_{r_j}\}$ weakly converging to a degree-two homogeneous function ψ_0 satisfying

$$\begin{cases} -\Delta\psi_0 = \lambda_1 \mathbf{I}_{D^1_*} - \lambda_2 \mathbf{I}_{D^2_*}, & x \in \mathbb{R}^2, \\ \psi_0 = 0, & x \in \partial D_*, \end{cases} \tag{3.2}$$

where $D_r := \{x \mid x_0 + rx \in D\}$ and the blow-up limit $\lambda_1 \mathbf{I}_{D^1_*} - \lambda_2 \mathbf{I}_{D^2_*} := \lim_{j\to\infty} \lambda_1 \mathbf{I}_{D_{r_j}} - \lambda_2 \mathbf{I}_{(D_{r_j})^c}$ depends on the extraction of the subsequence $r_j \to 0$. Let $\psi_0 = r^2 f(\theta)$. Recall the polar coordinate formula of the Laplacian $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$. In any given connected component of D^1_*, we solve the equation $f''(\theta) + 4f(\theta) = -\lambda_1$ and obtain that $f(\theta) = C_1 \sin(2\theta + C_2) - \frac{\lambda_1}{4}$ for some constants C_1 and C_2, and similarly in a connected component of D^2_*, $f(\theta) = C_3 \sin(2\theta + C_4) + \frac{\lambda_2}{4}$. Since $\psi \in C^{1,\alpha}$, we have $f \in C^{1,\alpha}$.

If $D_*^1 \neq \emptyset$, then it easy to see that for all $x \in \partial D_*^1 \cup \partial D_*^2$ and $x \neq 0$, we have $|\nabla u(x)| \neq 0$. Thus $|\mathbb{R}^2 \setminus (D_*^1 \cup D_*^2)| = 0$. A direct computation shows that equation (3.2) is equivalent to the following ODE concerning $f(\theta)$:

$$\begin{cases} -f''(\theta) - 4f(\theta) = \lambda_1, & \theta_{2i} \le \theta \le \theta_{2i+1}, \\ -f''(\theta) - 4f(\theta) = -\lambda_2, & \theta_{2i+1} \le \theta \le \theta_{2i+2}, \\ f(\theta_{2i}) = f(\theta_{2i+1}) = f(\theta_{2i+2}) = 0, \end{cases} \tag{3.3}$$

for $i = 0, \dots, N$, where f is positive in $(\theta_{2i}, \theta_{2i+1})$.

Without loss of generality, we let $\theta_0 = 0$ and $\theta_{2N+2} = 2\pi$. By the standard ODE technique, we can write down the solution in each sector explicitly as follows:

$$f(\theta) = \begin{cases} -\frac{\lambda_1}{4} + \frac{\lambda_1}{4\sin 2(\theta_{2i+1}-\theta_{2i})}((\sin 2\theta_{2i+1} - \sin 2\theta_{2i})\cos 2\theta \\ \qquad + (\cos 2\theta_{2i} - \cos 2\theta_{2i+1})\sin 2\theta), \\ \qquad\qquad \text{for } \theta \in (\theta_{2i}, \theta_{2i+1}), \\ \frac{\lambda_2}{4} - \frac{\lambda_2}{4\sin 2(\theta_{2i+2}-\theta_{2i+1})}((\sin 2\theta_{2i+2} - \sin 2\theta_{2i+1})\cos 2\theta \\ \qquad + (\cos 2\theta_{2i+1} - \cos 2\theta_{2i+2})\sin 2\theta), \\ \qquad\qquad \text{for } \theta \in (\theta_{2i+1}, \theta_{2i+2}). \end{cases}$$

It is easy to see that $\theta_{2i+1} - \theta_{2i+2} \in (0, \frac{\pi}{2})$, otherwise f will change sign. On the other hand, f can change its sign in $(\theta_{2i+1}, \theta_{2i+2})$. Note that $f'(\theta_{2i+1}) < 0 < f'(\theta_{2i+2})$. If $f < 0$ in $(\theta_{2i+1}, \theta_{2i+2})$, then we have $\theta_{2i+2} - \theta_{2i+1} \in (0, \frac{\pi}{2})$; otherwise it follows that $\theta_{2i+2} - \theta_{2i+1} \in (\pi, \frac{3\pi}{2})$. Since $f \in C^{1,\alpha}$, there is

$$\frac{\lambda_1}{2\sin 2\alpha_i}(-1 + \cos 2\alpha_i) = f'(\theta_{2i+1}) = -\frac{\lambda_2}{2\sin 2\beta_i}(1 - \cos 2\beta_i),$$
$$\alpha_i := \theta_{2i+1} - \theta_{2i}, \ \beta_i := \theta_{2i+2} - \theta_{2i+1}.$$

It follows that

$$\frac{\tan \alpha_i}{\tan \beta_i} = \frac{\lambda_2}{\lambda_1} > 0, \quad i = 0, \dots, N, \tag{3.4}$$

which are independent of the index i. Similarly, the continuity of $f'(\theta)$ implies

$$\frac{\tan \alpha_{i+1}}{\tan \beta_i} = \frac{\tan \alpha_i}{\tan \beta_{i-1}} = \frac{\lambda_2}{\lambda_1} > 0, \quad i = 1, \dots, N-1. \tag{3.5}$$

From equalities (3.4) and (3.5) it follows that these angles α_i coincide with each other and $|\beta_i - \beta_j| \in \{0, \pi\}$ for all i, j. Therefore, we have the following alternatives:

(1) There exists a super-harmonic sector of ψ_0 with the opening angle denoted by $\beta > \pi$ without loss of generality. The complement consists of N sectors with

opening angle α and another $N-1$ sectors with opening angle $\beta-\pi$ which occur alternately. Here α and β satisfy

$$\alpha+\beta=\frac{N+1}{N}\pi \quad \text{and} \quad \frac{\tan\alpha}{\tan\beta}=\frac{\lambda_2}{\lambda_1}.$$

In particular, the pattern is Z_2-symmetric.

(2) The pattern is N-fold symmetric where super-harmonic and sub-harmonic domains of ψ_0, denoted by $\{(r,\theta)\in\mathbb{R}^2 \mid r>0,\ 0<\theta<\alpha\}$ and $\{(r,\theta)\in R^2 \mid r>0,\ \alpha<\theta<\alpha+\beta\}$, and their copies by rotating by $\frac{N}{2\pi}k$, $k=1,\dots,N-1$ respectively, occur alternately, where $N(\alpha+\beta)=2\pi$.

If $D_*^1=\emptyset$, we can easily show that one of the following holds:

(1) $D_*^2=\emptyset$, then $\phi_0\equiv 0$.

(2) $\overline{D_*^2}=\mathbb{R}^2$, then $\phi_0(x_1,x_2)=Ax_1^2-Bx_2^2$ after a rotation, where $A>0$ and $B\geq 0$ are constants such that $A-B=\frac{\lambda_2}{2}$.

(3) $D_*^2\neq\emptyset$, $\overline{D_*^2}\neq\mathbb{R}^2$, then $\psi_0(x_1,x_2)=\frac{\lambda_2}{2}(x_1^+)^2$ after a rotation, where x_1^+ denotes the positive part of x_1.

Due to the above discussion, we can show the following lemma.

Lemma 3.1. *Suppose $x^0\in S^\psi$ is a singular point. Regarding the blow-up limit* (2) *in Proposition* 2.3, *for the equation* (3.3) *we have the following conclusions:*

(1) *If $D_*^1\neq\emptyset$, then one of the following two options holds:*

(a) *Both D_*^1 and D_*^2 consist of N (≥ 3) open cones, which are distributed alternately, and $|\mathbb{R}^2\setminus(D_*^1\cup D_*^2)|=0$.*

(b) *Concerning the opening angles of these cones, one of the following two alternatives holds:*

(i) *The opening angles of all cones in D_*^1 are $\alpha\in(0,\frac{\pi}{2})$, and the opening angles of all cones in D_*^2 are $\beta\in(0,\frac{\pi}{2})$ (Figure 1a);*

(ii) *The opening angles of all cones in D_*^1 are $\alpha\in(0,\frac{\pi}{2})$, $N-1$ cones of D_*^2 have the same opening angle $\beta\in(0,\frac{\pi}{2})$ and the remaining one has an opening angle $\beta+\pi$, while ψ_0 changes its sign in the cone with an opening angle $\beta+\pi$ (Figure 1b).*

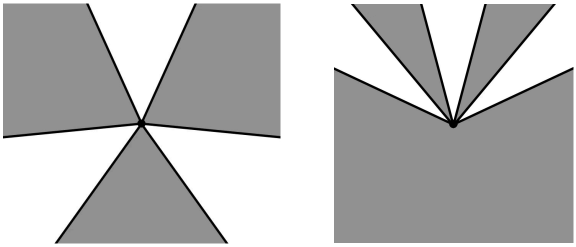
α and β depend on λ_1, λ_2 and N.

(2) *If $D_*^1=\emptyset$, then one of the following holds:*

(a) *$D_*^2=\emptyset$, then $\psi_0\equiv 0$.*

(b) *$\overline{D_*^2}=\mathbb{R}^2$, then $\psi_0(x_1,x_2)=Ax_1^2-Bx_2^2$ after a rotation, where $A>0$ and $B\geq 0$ are constants such that $A-B=\frac{\lambda_2}{2}$.*

(c) *$D_*^2\neq\emptyset$, $\overline{D_*^2}\neq\mathbb{R}^2$, then $\psi_0(x_1,x_2)=\frac{\lambda_2}{2}(x_1^+)^2$ after a rotation.*



(a) D^1_*: white **(b)** D^2_*: grey

Figure 1. Classification of the blow-up limits II

We summarize the above conclusions in the following theorem.

Theorem 3.2. *Suppose $D \subset \mathbb{R}^n$ is a bounded domain and (u, D) solves the two-dimensional unstable elliptic free boundary problem*

$$\begin{cases} -\Delta\psi = \lambda_1 \mathbf{I}_D - \lambda_2 \mathbf{I}_{D^c} & \text{in } \mathbb{R}^2, \\ \psi = 0 & \text{on } \partial D, \end{cases} \qquad \lambda_1, \lambda_2 > 0.$$

Then the regular part of the free boundary $\partial D \setminus \mathcal{S}$ is locally real-analytic, where

$$\mathcal{S} := \{x \in \partial D \mid |\nabla\psi(x)| = 0\}.$$

For any given $x_0 \in \mathcal{S}$, one of the following holds:

(1) $\lim_{r\to 0} r^{-5} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^1 \to +\infty$ *as* $r \to 0+$,

$$\phi_r(x) := \frac{\psi(x_0 + rx)}{S(x_0, r)}$$

is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$, and each limit of $\phi_r(x)$ as $r \to 0$ is a homogeneous harmonic polynomial of degree two, where

$$S(x_0, r) := \left(r^{-1} \int_{\partial B_r(x_0)} \psi^2 \, d\mathcal{H}^1\right)^{\frac{1}{2}}.$$

(2) $\limsup_{r\to 0} r^{-5} \int_{\partial B_r(x_0)} \psi^2 \, d\mathcal{H}^1 < \infty$,

$$\psi_r(x) := \frac{\psi(x_0 + rx)}{r^2}$$

is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, and each limit of $\psi_r(x)$ as $r \to 0$ is a degree-two homogeneous function satisfying

$$\begin{cases} -\Delta\psi_0 = \lambda_1 \mathbf{I}_{D^1_*} - \lambda_2 \mathbf{I}_{D^2_*} & \text{in } \mathbb{R}^n, \\ \psi_0 = 0 & \text{on } \partial D^1_* \cup \partial D^2_*, \end{cases}$$

where D_*^1 and D_*^2 are open cones satisfying $D_*^1 \cap D_*^2 = \emptyset$, $\mathbb{R}^2 \setminus (D_*^1 \cup D_*^2) \subset \{\psi_0 = 0\}$ and $\psi_0 > 0$ in D_*^1. The complete classification of the solutions ψ_0 is given in Lemma 3.1.

The proof concerned with the regular portion is standard, thus we just outline it here.

By the standard elliptic regularity theorem, we have $u \in C^{1,\alpha}$ for $0 < \alpha < 1$ due to (1.4). For the free boundary ∂D near a regular point $x_0 \in \partial D$, i.e., $|\nabla u(x_0)| \neq 0$ and $u(x_0) = 0$, by applying the implicit function theorem we have ∂D is locally a $C^{1,\alpha}$ curve near x_0, namely there exists $r > 0$ such that $\partial D \cap B_r(x_0)$ is parameterized by the graph of a $C^{1,\alpha}$ function. Following the idea in the proof of [23, Theorem 3.1], we apply the hodograph transform and the local real-analyticity of the curve follows.

Remark 3.3. If D is unbounded, under the assumption that $u > 0$ in D, the same conclusions as Theorem 3.2 also hold.

Moreover, suppose that the domain D is Lipschitz; then the second case in Theorem 3.2 can be ruled out.

Proposition 3.4. *Under the assumptions of Theorem 3.2, if D is Lipschitz and $x_0 \in \partial D$ is a singular point, then the blow-up limit II will not occur.*

Proof. Suppose towards a contradiction that we have a blow-up limit II at some singular point $x_0 \in \partial D$. Without loss of generality we may assume that $x_0 = 0$ and the boundary of D near x_0 can be written as $\{(x_1, x_2) \mid x_1 \in (-\delta, \delta), x_2 = f(x_1)\}$, where f is a Lipschitz function and L is the Lipschitz constant. Then we have $\{(x_1, x_2) \mid x_1 \in (-\delta, \delta),\ f(x_1) < x_2 < \delta\} \subset D$ and $\{(x_1, x_2) \mid x_1 \in (-\delta, \delta),\ -\delta < x_2 \le f(x_1)\} \subset D^c$. Suppose that $\{\psi_{r_j}\} = \frac{\psi(r_j x)}{r_j^2}$ is a blow-up sequence converging to a function ψ_0 that is homogeneous of degree two, with

$$-\Delta\psi_0 = \lambda_1 \mathbf{I}_{D_*^1} + \lambda_2 \mathbf{I}_{D_*^2},$$

where D_*^1 and D_*^2 are open cones satisfying $D_*^1 \cap D_*^2 = \emptyset$, $\mathbb{R}^2 \setminus (D_*^1 \cup D_*^2) \subset \{\phi_0 = 0\}$ and $\psi_0 = 0$ on $\partial D_*^1 \cup \partial D_*^2$. It is easy to see that ψ_{r_j} satisfies

$$-\Delta\psi_{r_j} = \lambda_1 \mathbf{I}_{D_j} + \lambda_2 \mathbf{I}_{D_j^2},$$

where $D_j := \{x \in \mathbb{R}^2 \mid r_j x \in D\}$ and the boundary of D_j can be represented by the graph of a Lipschitz function f_j, where $f_j(x_1) = \frac{f(r_j x)}{r_j}$. Moreover, we have

$$\left\{(x_1, x_2) \,\middle|\, x_1 \in (-\delta r_j^{-1},\ \delta r_j^{-1}),\ f_j(x_1) < x_2 < \delta r_j^{-1}\right\} \subset D_j,$$
$$\left\{(x_1, x_2) \,\middle|\, x_1 \in (-\delta r_j^{-1},\ \delta r_j^{-1}),\ -\delta r_j^{-1} < x_2 \le f_j(x_1)\right\} \subset D_j^c.$$

Since $\|f_j\|_{C^{0,1}} \le L$ for all $j \in \mathbb{N}$, passing if necessary to a subsequence, f_j converges uniformly to a Lipschitz function f_0 with Lipschitz constant L. Let $x = (x_1, x_2)$ be such that $x_2 > f_0(x_1)$. It follows from the uniform convergence of f_j that there exists a constant

$\tau > 0$ such that $B_\tau(x) \subset D_j$ for all large j. Thus $\psi_{r_j} > 0$ and $-\Delta\psi_{r_j} = \lambda_1$ in $B_\tau(x)$. We conclude that $\psi_0 \geq 0$ and $-\Delta\psi_0 = \lambda_1$ in $B_\tau(x)$. By the maximal principle, we have $\psi_0 > 0$ in $B_\tau(x)$. Since x is arbitrary, it follows that

$$\big\{(x_1, x_2) \bigm| x_2 > f_0(x_1)\big\} \subset D_*^1.$$

Similarly, we have $\{(x_1, x_2) \mid x_2 < f_0(x_1)\} \subset D_*^2$. Therefore

$$D_*^1 = \big\{(x_1, x_2) \bigm| x_2 > f_0(x_1)\big\}, \quad D_*^2 = \big\{(x_1, x_2) \bigm| x_2 < f_0(x_1)\big\},$$

which leads to a contradiction with Lemma 3.1. ∎

Corollary 3.5. *Suppose that D is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, and (D, ψ) is a solution to the uniformly rotating vortex patch problem with $\Omega \in (0, \frac{1}{2})$:*

$$\begin{cases} -\Delta\psi = \mathbf{I}_D - 2\Omega, & x \in \mathbb{R}^2, \\ \nabla\Big(\psi - \dfrac{\Omega}{2}|x|^2\Big) \to 0, & |x| \to \infty, \\ \psi = 0, & x \in \partial D. \end{cases}$$

Then for each $x_0 \in \partial D \cap \{|\nabla\psi| = 0\}$, we have

$$\lim_{r \to 0} r^{-5} \int_{\partial B_r(x_0)} \psi^2 \, d\mathcal{H}^1 \to +\infty$$

as $r \to 0$ and

$$\psi_r(x) := \frac{u(x_0 + rx)}{S(x_0, r)}$$

is uniformly bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$. Moreover, up to sequence of r, each limit ψ_0 of $\psi_r(x)$ as $r \to 0$ is a homogeneous harmonic polynomial of degree two, where

$$S(x_0, r) := \Big(r^{-1} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^1\Big)^{\frac{1}{2}}.$$

3.2. Uniform regularity

The Weiss-type monotonicity formula, as well as the classification of the blow-up limits of singular points, provides some geometric information concerning the 0-level set of the blow-up limit function. In general, the limits may not be unique, i.e., the limiting solution may depend on the choice of subsequence $\{r_j\}_{j \in \mathbb{N}}$.

For the singular points where ψ has super-characteristic growth (*the blow-up limit (I)*), in this subsection we further prove the uniqueness of the blow-up limit and the uniform regularity of the free boundary ∂D near the singularities. The situation is similar to Theorem A established in [1], where the proof essentially depends on the uniqueness of the generalized Newtonian potential of the function $-\mathbf{I}_{\{x_1 x_2 > 0\}}$.

It suffices for us to consider the specific equation

$$-\Delta\psi = \mathbf{I}_D - 2\Omega, \quad \psi = 0 \text{ on } \partial D,$$

where D is simply connected. Since the free boundary near the singular points of blow-up limit (1) is always cross-like, it consists of sectors with the opening angle $\frac{\pi}{2}$ which is independent of the values $\lambda_1, \lambda_2 > 0$.

Lemma 3.6. *Let ψ be a solution of (3.1) and*

$$S^{\psi}(x^0, r) \geq \frac{r^2}{\delta}$$

for some $0 < \delta \leq \delta_0$, $0 < r \leq r_0$, $\sup_{B_3(x_0)} |\psi| \leq M$ and $\psi(x^0) = |\nabla\psi(x^0)| = 0$. Then

(1) *there exists a second-order homogeneous harmonic polynomial $p^{x^0,\psi} = p$ such that for each $\alpha \in (0, \frac{1}{2})$ and each $\beta \in (0, 1)$,*

$$\left\| \frac{\psi(x^0 + sx)}{\sup_{B_s(x^0)} |\psi|} - p \right\|_{C^{1,\beta}(\overline{B}_1)} \leq C(M, \alpha, \beta) \Big(\frac{\delta}{1 + \delta \log(\frac{r}{s})} \Big)^{\alpha};$$

(2) *the set $\{\psi = 0\} \cap B_{r_0}(x^0)$ consists of two C^1 curves intersecting each other at right angles at x^0.*

It suffices for us to consider the generalized Newtonian potential

$$-\Delta z = \mathbf{I}_{\mathbf{C}} - 2\Omega, \quad \mathbf{C} := \big\{ (r, \theta) \subset \mathbb{R}^2 \,\big|\, r > 0,\ 0 < \theta < \tfrac{\pi}{2} \big\}. \tag{3.6}$$

Note that the right-hand side of (3.6) is of class L^{∞} and homogeneous of degree zero. By [22], z is uniquely representable in the form

$$z(x) = p(x) \log|x| + |x|^2 \phi\Big(\frac{x}{|x|}\Big) + q_1(x), \quad x \in \mathbb{R}^2,$$

where p is a homogeneous harmonic polynomial of degree two, $\phi \in C^1(S^1)$ and $q_1(x) = a \cdot x + b$. Suppose $z(0) = \nabla z(0) = 0$. It is sufficient to compute in the form of class $p(x) \log|x| + |x|^2 \phi(\frac{x}{|x|})$. Consider (3.6):

$$-\Big(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}\Big) z = \begin{cases} 1 - 2\Omega, & 0 < \theta < \dfrac{\pi}{2}, \\ -2\Omega, & \dfrac{\pi}{2} < \theta < 2\pi, \end{cases}$$

in polar coordinates, in which z is of the form

$$\begin{aligned} z(x_1, x_2) &= (A(x_1^2 - x_2^2) + 2Bx_1x_2) \log|x| + |x|^2 \phi\Big(\frac{x}{|x|}\Big) \\ &= r^2 \log r (A \cos 2\theta + B \sin 2\theta) + r^2 \phi(\theta), \end{aligned}$$

with constants $A, B \in \mathbb{R}$. A straightforward computation shows that

$$-4(A\cos 2\theta + B\sin 2\theta) - \phi''(\theta) - 4\phi(\theta) = \begin{cases} 1-2\Omega, & 0<\theta<\frac{\pi}{2}, \\ -2\Omega, & \frac{\pi}{2}<\theta<2\pi. \end{cases}$$

Consider the Fourier series

$$\phi(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta.$$

For $n \neq 2$, one has

$$A_0 = \frac{1-2\Omega}{2} - 3\Omega, \quad A_n = \frac{1}{n\pi}\sin n\frac{\pi}{2}, \quad B_n = \frac{1}{n\pi}\Big(1-\cos n\frac{\pi}{2}\Big), \quad n \geq 1,\ n \neq 2$$

and

$$A = 0, \quad B = \frac{1}{2\pi}.$$

Note that $r^2(A_2\cos 2\theta + B_2 \sin 2\theta)$ is a harmonic function. We let

$$\begin{aligned} z(r,\theta) = r^2\Bigg(\frac{1-8\Omega}{4} + \frac{1}{\pi}\sum_{n\geq 1, n\neq 2}\frac{1}{n}\Big(\sin\frac{n\pi}{2}\cos n\theta + \Big(1-\cos\frac{n\pi}{2}\Big)\sin n\theta\Big)\Bigg) \\ + \frac{r^2\log r}{2\pi}\sin 2\theta - \frac{1}{4\pi}r^2\sin 2\theta. \end{aligned} \tag{3.7}$$

Lemma 3.7. *The potential z given in (3.7) is the unique solution of equation (3.6) satisfying*

(1) $z(0) = |\nabla z(0)| = 0$,
(2) $\lim_{x\to\infty}\frac{z(x)}{|x|^3} = 0$,
(3) $\Pi(z) = 0$,
(4) $\Pi(z_{\frac{1}{2}}) = \frac{\log 2}{\pi}x_1x_2$,
(5) $\tau(z_{\frac{1}{2}}) = \frac{\log 2}{\pi}$,

where $\mathbb{P}$ is the space of second-order homogeneous harmonic polynomials and Π: $W^{2,2}(B_1) \to \mathbb{P}$ is a projection such that $\Pi(u)$ is the unique minimizer of

$$p \to \int_{B_1} |D^2v - D^2p|^2$$

on $\mathbb{P}$ and $\tau(u) \geq 0$ is defined by $\Pi(u) = \tau(u)p$, $p \in \mathbb{P}$, $\sup_{B_1}|p| = 1$.

Proof. Suppose $h := \Pi(z)$. Since h is harmonic, we let

$$D^2h = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Since h is the minimizer of the mapping $p \to \int_{B_1} |D^2 v - D^2 p|^2$, we have

$$0 = \partial_b \int_{B_1} |D^2 v - D^2 h|^2 = 4b - 4\int_{B_1} \partial_{12} z$$
$$= 4b - \int_0^1 \int_{S^1} \Big(\frac{\sin 2\theta}{2} \Big(\partial_{rr} z + \frac{1}{r^2}\partial_{\theta\theta} \Big) z(r,\theta) - \frac{1}{r}\partial_{r\theta} z(r,\theta) + \frac{1}{r^2}\partial_\theta z(r,\theta) + \frac{\sin 2\theta}{2r}\partial_r z(r,\theta) \Big) r\, dr\, d\theta.$$

Being aware that only $\frac{r^2 \log r}{2\pi}\sin 2\theta - \frac{1}{4\pi} r^2 \sin 2\theta$ provides non-zero integrals, we have $b = 0$ since

$$\int_{B_1} \partial_{12}\Big(\frac{\log(x_1^2 + x_2^2)}{2\pi} x_1 x_2 - \frac{1}{2\pi} x_1 x_2 \Big)\, dx_1\, dx_2$$
$$= -\frac{1}{2} + \frac{1}{2\pi}\int_{B_1} \Big(2 + \log(x_1^2 + x_2^2) - \frac{4x_1^2 x_2^2}{(x_1^2 + x_2^2)^2} \Big)\, dx_1\, dx_2 = 0,$$

while $a = 0$ is obtained similarly by considering

$$0 = \partial_a \int_{B_1} |D^2 v - D^2 h|^2 = 4a.$$

Therefore, one has $h := \Pi(z) = 0$ due to h being a second-order polynomial. Moreover, we have

$$\frac{z(sx_1, sx_2)}{s^2} = z(x_1, x_2) - \frac{x_1 x_2}{\pi}\log s.$$

Then it follows that

$$\Pi(z_{\frac{1}{2}}) = \Pi(z) - \Pi\Big(\frac{x_1 x_2 \log(\frac{1}{2})}{\pi} \Big) = \frac{\log 2}{\pi} x_1 x_2.$$

If z_1 and z_2 are solutions of (3.7) and $\eta := z_1 - z_2$ is a second-order harmonic polynomial, then $|\nabla \eta(0)| = \eta(0) = 0$ and $\Pi(\eta) = 0$ imply $z_1 = z_2$, namely z is the unique solution. ∎

Lemma 3.8. *Suppose ψ solves (3.1), $x_0 \in \mathcal{S}^u$ and $\sup_{B_3(x_0)} |u| \le M < +\infty$. Then*

$$\Big(\int_{B_1} \Big| D^2 \frac{u(x^0 + rx)}{r^2} - D^2 \Pi\Big(\frac{u(x^0 + rx)}{r^2} \Big) \Big|^p \Big)^{\frac{1}{p}} \le C(M, p)$$

and

$$\Big\| \frac{u(x^0 + rx)}{r^2} - \Pi\Big(\frac{u(x^0 + rx)}{r^2} \Big) \Big\|_{C^{1,\beta}(\bar{B}_1)} \le C(M, \beta).$$

Lemma 3.9. *For each $\varepsilon > 0$, $M < +\infty$, $\alpha \in [1, +\infty)$ and $\beta \in (0,1)$, there exist $r_0, \delta > 0$ with the following property: suppose that $0 < r \le r_0$ and that ψ is a solution of (3.1) satisfying $\sup_{B_2(x_0)} |u| \le M$, $u(x) = |\nabla u(x)| = 0$ and*

$$\mathcal{L}^2\big((\{u(x + r\cdot) > 0\} \Delta \{x_1 x_2 > 0\}) \cap B_1 \big) \le \delta.$$

Then

$$\left\|\frac{u(x+r\cdot)}{r^2}-\Pi\Big(\frac{u(x+r\cdot)}{r^2}\Big)-z\right\|_{C^{1,\beta}(\bar{B}_1)}\le\varepsilon.$$

Proof. Suppose that $r_j\to 0$ such that

$$\mathcal{L}^n(\{u(x+r\cdot)>0\}\Delta\{x_1x_2>0\})\to 0 \quad \text{as } j\to\infty$$

and

$$\frac{u_j(x^j+r_j\cdot)}{r_j^2}-\Pi\Big(\frac{u_j(x^j+r_j\cdot)}{r_j^2}\Big)\to\tilde{z} \text{ strongly in } C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and weakly in } W^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n).$$

Now consider $\widetilde{N}$ be the Newtonian potential of Δu_j, i.e.,

$$\widetilde{N}(y):=\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|y-\xi|\Delta u_j(\xi)\,d\xi.$$

Let $N(y)=\widetilde{N}(y)-\widetilde{N}(x^j)-\nabla\widetilde{N}(x^j)(y-x^j)$. Then $h(y)=u_j(y)-N(y)$ is a harmonic function. We have

$$|u_j(y)-N(y)-\mathcal{D}^2h(x^j)(y-x^j)(y-x^j)|\le C|x^j-y|^3 \quad \text{in } B_{\frac{1}{2}}(x^j).$$

For the scaled function $v_j(y)=\frac{\psi_j(x^j+r_jy)}{r_j^2}$, $N_j(y)=\frac{N(x^j+r_jy)}{r_j^2}$ and $p_j(y)=\mathcal{D}^2h(x^j)y\cdot y$, we obtain

$$|v_j(y)-N_j(y)-p_j(y)|\le Cr_j|y|^3 \quad \text{in } B_{\frac{1}{2}}(x^j).$$

Thus,

$$v_j-\Pi(v_j)=N_j-\Pi(N_j)+o(1),\quad j\to\infty.$$

Therefore, N_j converges locally to N_0, where

$$-\Delta N_0=\mathbf{I}_C-2\Omega,\quad N_0(0)=\nabla N_0(0)=0 \quad\text{and}\quad N_0-\Pi(N_0)=\tilde{z}.$$

Then it suffices to show that $N_0(y)=o(|y|^3)$ as $|y|\to\infty$ since one would conclude $\tilde{z}=N_0-\Pi(N_0)=z$ due to the uniqueness result given in Lemma 3.7. Since $\mathcal{D}^2N_0\in\mathrm{BMO}$ we have

$$\int_{B_1}\left|\frac{\mathcal{D}^2(N_0(Ry))-\overline{\mathcal{D}^2(N_0(R\cdot))}}{\sup_{B_R}|\mathcal{D}^2N_0|}\right|^2dy\le C\frac{R^4}{\sup_{B_R}|\mathcal{D}^2N_0|^2}$$

for all $R\in(0,+\infty)$, where $\overline{\mathcal{D}^2(N_0(R\cdot))}$ denotes the mean value of $\mathcal{D}^2(N_0(R\cdot))$ on B_1. Therefore, we have that a subsequence $\frac{N_0(R_k\cdot)}{\sup_{B_{R_k}}|\mathcal{D}^2N_0|}$ converges to a degree-two homogeneous harmonic polynomial as $R_k\to\infty$. Assume

$$\limsup_{|y|\to\infty}\frac{|N_0(y)|}{|y|^3}>0.$$

Since $\Delta(N_0 - z) = 0$ and

$$\limsup_{|y|\to\infty} \frac{|N_0(y) - z(y)|}{|y|^3} > 0,$$

$N_0 - z$ must be a harmonic polynomial of degree $m \geq 3$ which is a contradiction. ■

Based on Lemma 3.9, Lemma 3.6 is proved by the same approach as [1, Theorem A], hence we omit it here.

Now we are in the position to prove the main theorem.

Proof of Theorem 1.1. Employ Theorem 3.2 by letting $\lambda_1 = 1 - 2\Omega$ and $\lambda_2 = 2\Omega$. Suppose that there is no stagnation point on the boundary, namely $|\nabla\psi| \neq 0$ on ∂D. Since ∂D is a connected component of level set $\{\psi = 0\}$ and $\psi \in C^{1,\alpha}$ due to the standard elliptic regularity theorem, ∂D is a $C^{1,\alpha}$ curve, which is indeed locally real-analytic due to Theorem 3.2. Since the curve ∂D is real-analytic hence C^∞, ∂D is a C^∞ smooth manifold due to the fundamental theorem concerning partition of unity.

For the singular set, by Proposition 3.4 we have that only type I singular points can occur. Moreover, suppose $x^0 \in S^\psi \subset \partial D$, i.e., $\nabla\psi(x_0) = 0$. By Theorem 3.2 and Lemma 3.6, there exists a small parameter $r_0 > 0$ such that $\{\psi = 0\} \cap B_{r_0}(x_0)$ consists of two C^1 curves intersecting each other at right angles at x^0, meaning that x^0 must be an isolated singular point.

In the last part, we shall show that the singular set is finite, or equivalently the isolatedness of singular points since ∂D is rectifiable. Assume the conclusion fails, i.e., the singular points are not isolated. There would be a sequence of singular points

$$\{x_j\}_{j=1}^{\infty} \in S^\psi := \big\{x \in \partial D \;\big|\; \psi(x_j) = |\nabla\psi(x_j)| = 0\big\},$$

converging to $x^0 \in \partial D$ satisfying $\nabla\psi(x^0) = 0$ by taking a subsequence since S^ψ is a closed set.

We note that x^0 must be a singular point, i.e., for any $r > 0$, $\partial D \cap B_r(x^0)$ cannot be represented as the graph of a smooth function. Indeed, Theorem 3.2 establishes the classification for all critical points on ∂D satisfying $\nabla\psi(x^0) = 0$. By Proposition 3.4, only blow-up limit I can occur. Then Lemma 3.6 implies that there exists $r_0 > 0$ such that $\partial D \cap B_{r_0}(x^0)$ is a crossing of two C^1 curves thus $x_j \notin \partial D \cap B_{r_0}(x^0)$. Therefore, x^0 is a singular point on the boundary, and $x_j \notin \cap B_{r_0}(x^0)$, which contradicts the assumption $x_j \to x^0$. The proof is then complete. ■

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References

[1] J. Andersson, H. Shahgholian, and G. S. Weiss, Uniform regularity close to cross singularities in an unstable free boundary problem. *Comm. Math. Phys.* **296** (2010), no. 1, 251–270 Zbl 1197.35088 MR 2606634

[2] J. Andersson, H. Shahgholian, and G. S. Weiss, On the singularities of a free boundary through Fourier expansion. *Invent. Math.* **187** (2012), no. 3, 535–587 Zbl 1234.35318 MR 2891877

[3] A. L. Bertozzi and P. Constantin, Global regularity for vortex patches. *Comm. Math. Phys.* **152** (1993), no. 1, 19–28 Zbl 0771.76014 MR 1207667

[4] J. Burbea, Motions of vortex patches. *Lett. Math. Phys.* **6** (1982), no. 1, 1–16 Zbl 0484.76031 MR 0646163

[5] L. A. Caffarelli, The regularity of free boundaries in higher dimensions. *Acta Math.* **139** (1977), no. 3-4, 155–184 Zbl 0386.35046 MR 0454350

[6] A. Castro, D. Córdoba, and J. Gómez-Serrano, Uniformly rotating analytic global patch solutions for active scalars. *Ann. PDE* **2** (2016), no. 1, article no. 1 Zbl 1397.35020 MR 3462104

[7] J.-Y. Chemin, Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)* **26** (1993), no. 4, 517–542 Zbl 0779.76011 MR 1235440

[8] J.-Y. Chemin, *Perfect incompressible fluids*. Oxford Lecture Ser. Math. Appl. 14, The Clarendon Press, Oxford University Press, New York, 1998 Zbl 0927.76002 MR 1688875

[9] F. de la Hoz, T. Hmidi, J. Mateu, and J. Verdera, Doubly connected V-states for the planar Euler equations. *SIAM J. Math. Anal.* **48** (2016), no. 3, 1892–1928 Zbl 1342.35239 MR 3507551

[10] T. M. Elgindi and I.-J. Jeong, On singular vortex patches, I: Well-posedness issues. *Mem. Amer. Math. Soc.* **283** (2023), no. 1400, v+89 Zbl 1508.76028 MR 4537304

[11] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*. Appl. Math. Sci. 159, Springer, New York, 2004 Zbl 1059.65103 MR 2050138

[12] B. Fan, Y. Wang, and W. Zhan, Radial symmetry of stationary and uniformly-rotating solutions to the 2D Euler equation in a disc. 2024, arXiv:2506.05034v1

[13] A. Figalli, X. Ros-Oton, and J. Serra, Generic regularity of free boundaries for the obstacle problem. *Publ. Math. Inst. Hautes Études Sci.* **132** (2020), 181–292 Zbl 1456.35234 MR 4179834

[14] A. Figalli and J. Serra, On the fine structure of the free boundary for the classical obstacle problem. *Invent. Math.* **215** (2019), no. 1, 311–366 Zbl 1408.35228 MR 3904453

[15] C. García and S. V. Haziot, Global bifurcation for corotating and counter-rotating vortex pairs. *Comm. Math. Phys.* **402** (2023), no. 2, 1167–1204 Zbl 1529.35353 MR 4627319

[16] J. Gómez-Serrano, J. Park, J. Shi, and Y. Yao, Symmetry in stationary and uniformly rotating solutions of active scalar equations. *Duke Math. J.* **170** (2021), no. 13, 2957–3038 Zbl 1478.35013 MR 4312192

[17] Z. Hassainia, N. Masmoudi, and M. H. Wheeler, Global bifurcation of rotating vortex patches. *Comm. Pure Appl. Math.* **73** (2020), no. 9, 1933–1980 Zbl 1452.76038 MR 4156612

[18] T. Hmidi and J. Mateu, Bifurcation of rotating patches from Kirchhoff vortices. *Discrete Contin. Dyn. Syst.* **36** (2016), no. 10, 5401–5422 Zbl 1351.35116 MR 3543554

[19] T. Hmidi and J. Mateu, Degenerate bifurcation of the rotating patches. *Adv. Math.* **302** (2016), 799–850 Zbl 1352.35101 MR 3545942

[20] T. Hmidi, J. Mateu, and J. Verdera, Boundary regularity of rotating vortex patches. *Arch. Ration. Mech. Anal.* **209** (2013), no. 1, 171–208 Zbl 1286.35201 MR 3054601

[21] V. I. Judovič, Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat i Mat. Fiz.* **3** (1963), 1032–1066 Zbl 0129.19402 MR 0158189

[22] L. Karp and A. S. Margulis, Newtonian potential theory for unbounded sources and applications to free boundary problems. *J. Anal. Math.* **70** (1996), 1–63 Zbl 0876.35136 MR 1444257

[23] D. Kinderlehrer, L. Nirenberg, and J. Spruck, Regularity in elliptic free boundary problems. *J. Analyse Math.* **34** (1978), 86–119 (1979) Zbl 0402.35045 MR 0531272

[24] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*. Pure Appl. Math. 88, Academic Press, Harcourt Brace Jovanovich, Publishers, New York-London, 1980 Zbl 0457.35001 MR 0567696

[25] A. Kiselev and C. Li, Global regularity and fast small-scale formation for Euler patch equation in a smooth domain. *Comm. Partial Differential Equations* **44** (2019), no. 4, 279–308 Zbl 1416.35195 MR 3941226

[26] A. Kiselev and X. Luo, Illposedness of C^2 vortex patches. *Arch. Ration. Mech. Anal.* **247** (2023), no. 3, article no. 57 Zbl 1516.35316 MR 4597611

[27] A. Kiselev and V. Šverák, Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Ann. of Math. (2)* **180** (2014), no. 3, 1205–1220 Zbl 1304.35521 MR 3245016

[28] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*. Appl. Math. Sci. 96, Springer, New York, 1994 Zbl 0789.76002 MR 1245492

[29] R. Monneau and G. S. Weiss, An unstable elliptic free boundary problem arising in solid combustion. *Duke Math. J.* **136** (2007), no. 2, 321–341 Zbl 1119.35123 MR 2286633

[30] E. A. Overman, II, Steady-state solutions of the Euler equations in two dimensions. II. Local analysis of limiting V-states. *SIAM J. Appl. Math.* **46** (1986), no. 5, 765–800 Zbl 0608.76018 MR 0858995

[31] J. Park, Quantitative estimates for uniformly-rotating vortex patches. *Adv. Math.* **411** (2022), no. part A, article no. 108779 Zbl 1504.35265 MR 4515733

[32] P. Serfati, Une preuve directe d'existence globale des vortex patches 2D. *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), no. 6, 515–518 Zbl 0803.76022 MR 1270072

[33] B. Turkington, On steady vortex flow in two dimensions. I, II. *Comm. Partial Differential Equations* **8** (1983), no. 9, 999–1030, 1031–1071 Zbl 0523.76014 Zbl 0523.76015 MR 0702729

[34] J. Verdera, The global regularity of vortex patches revisited. *Adv. Math.* **416** (2023), article no. 108917 Zbl 1508.35050 MR 4549427

[35] H. M. Wu, E. A. Overman, II, and N. J. Zabusky, Steady-state solutions of the Euler equations in two dimensions: Rotating and translating V-states with limiting cases. I. Numerical algorithms and results. *J. Comput. Phys.* **53** (1984), no. 1, 42–71 Zbl 0524.76029 MR 0734586

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