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Scattering problem for Vlasov-type equations on the d-dimensional torus with Gevrey data

Dario Benedetto, Emanuele Caglioti, Antoine Gagnebin, Mikaela Iacobelli, and Stefano Rossi

Abstract. In this article, we consider Vlasov-type equations describing the evolution of single-species-type plasmas, such as those composed of electrons (Vlasov–Poisson) or ions (screened Vlasov–Poisson/Vlasov–Poisson with massless electrons). We solve the final data problem on the torus $\mathbb{T}^d$, $d \geq 1$, by considering asymptotic states of regularity Gevrey-$\frac{1}{\gamma}$ with $\gamma > \frac{1}{3}$, small perturbations of homogeneous equilibria satisfying the Penrose stability condition. This extends to the Gevrey perturbative case, and to higher dimensions, the scattering result in analytic regularity obtained by Caglioti and Maffei [J. Statist. Phys. 92 (1998), 301–323] and answers an open question raised in Bedrossian (2022).

1. Introduction

In this paper, we investigate the asymptotic behavior of systems modeling dilute, collisionless, non-relativistic plasmas in a periodic domain.

In these systems, negatively charged electrons and positively charged ions have significantly different masses – the mass ratio being of order 10^3; see e.g. [13]. This dissimilarity leads to a separation in the relevant timescales of evolution for the different species. Consequently, it is reasonable to describe these species' evolution separately and use different models for their dynamics. We englobe such models in a unique framework given by a general Vlasov-type equation:

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - E[f](t,x) \cdot \nabla_v f(t,x,v) = 0, \\ E[f](t,x) = -\nabla V(t,x), \\ -\Delta V(t,x) + \beta V(t,x) + h(V)(t,x) = \int_{\mathbb{R}^d} f(t,x,v) dv - 1. \end{cases} \tag{1.1}$$

Here, we focus on the periodic setting $x \in \mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. The unknown $f(t,x,v)$ is the distribution function of particles of a given species at time t, position x, and velocity v,

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where $(t, x, v) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$. We denote by $E[f]$ the mean-field force generated by the distribution of particles.

We will consider different choices of parameters $\beta \geq 0$ and analytic functions h. In this way, one could study several models for plasma physics systems. For example, the main kinetic equation used to model plasmas is the so-called Vlasov–Poisson (VP) system, which corresponds to the case $\beta = h = 0$. It is a well-known model in kinetic theory, which describes the motion of electrons (i.e., $f(t, x, v)$ is the distribution function for the electrons) in plasmas when we neglect collisions and magnetic effects and consider the ions as a stationary background.

The (VP) system has been extensively studied over the past decades, and a vast amount of literature on results is already available on the existence of global classical and weak solutions with various conditions on the initial data. We mention the early work of Kurth [40] for a local-in-time existence result for the (VP) system in dimension three. For global well-posedness of classical solutions on the whole space, we refer to the work of Iordanskiĭ [39] in dimension one, Ukai and Okabe [54] in dimension two, and Lions and Perthame [44], Pfaffelmoser [52] in the three-dimensional case (see also [3,53]). For weak solutions on the whole space, we refer to the work of Arsenev [1] (see also [2, 4, 30]), while Batt and Rein [6] proved the global well-posedness on the torus (see also [16,49]). Finally, Loeper [45] made a significant improvement to the uniqueness theory (see also [29, 34, 47]).

While the (VP) system describes the evolution of electrons, on the ion timescale, electrons move faster than ions, and it becomes relevant to consider electron–electron collisions in the model. Thus, it is natural to assume that the electron distribution is already in thermal equilibrium on the ion timescale, and the electron density is given by a Maxwell–Boltzmann distribution. The resulting model under these assumptions is the Vlasov–Poisson system for massless electrons (VPME), sometimes called ionic Vlasov–Poisson. This model can be obtained rigorously as the mass ratio between ions and electrons becomes small. We refer to [5] for a mathematical proof of the above result and to [24] for a more thorough introduction to this model. The (VPME) system consists of a Vlasov equation coupled to a non-linear Poisson equation that describes how the fluctuation between the ion distribution and the Maxwell–Boltzmann electron distribution generates the electric potential. In our notation, (VPME) can be obtained by setting $\beta = 1$ and $h(U) = e^U - 1 - U$.

The (VPME) system has received less mathematical attention than the (VP) system due to the additional difficulty in the non-linear Poisson coupling. Bouchut [12] was the first to construct global weak solutions in the whole space in dimension three. In the one-dimensional setting, weak solutions were constructed globally in time by Han-Kwan and Iacobelli [27]. Griffin-Pickering and Iacobelli have obtained the first result of global well-posedness in the case of the whole space in dimension three [22] and of the torus in dimensions two and three [23]. Moreover, Cesbron and Iacobelli showed the well-posedness of (VPME) in bounded domains [15]. We refer to [24] for an exhaustive review of the global well-posedness theory of (VPME).

In the setting of our system (1.1), another important example of a plasma physics model is the screened (VP) system. This model can be obtained by choosing $\beta = 1$ and $h(U) = 0$, and corresponds to a first-order approximation of the exponential term e^U by $1 + U$ in the Poisson coupling of the (VPME) system. Physically speaking, it can be shown that this approximation holds true when the electric energy of the electrons is much smaller than the kinetic energy. The weak global existence of this system is explored in [25, Theorem 2.1]. Readers can refer to [26, Section 1.1 and 1.2] for a more in-depth understanding of this system.

The analysis we conduct here is focused on the study of the system (1.1) in a general framework that covers the three plasma models mentioned earlier. To be more specific, we assume $\beta \geq 0$, and we consider $h: (-R, R) \to \mathbb{R}$ to be analytic, with an analyticity radius denoted by R, and it satisfies the condition $h(x) = O(x^2)$ as x approaches zero. It is worth noting that in the (VP), (VPME), or screened (VP) system, we have $R = \infty$.

In this work, we are interested in the long-time behavior of solutions of Vlasov-type equations on the periodic domain $\mathbb{T}^d$. In 1946, Landau [41] observed that the Vlasov–Poisson equation (i.e., (1.1) with $\beta = 0$ and $h = 0$) linearized around a Maxwellian equilibrium is exactly solvable and that, for analytic initial data, the electric field decays exponentially fast to zero, so the flow governed by the mean-field force is asymptotically free. This electric field decay was experimentally observed in plasmas only 18 years later by Malmberg and Wharton [46].

Today, what is now called Landau damping is a well-known collisionless relaxation phenomenon studied in plasma physics literature, where several works analyze and discuss Landau's pioneering result [41]. We mention the work of Penrose [51], who improved Landau's result for general analytic spatially homogeneous equilibria. Nevertheless, the extension from the linear to the true non-linear case has proved to be particularly difficult for the mathematical theory.

The first one-dimensional non-linear result was that of Caglioti and [14]. They solved the final data problem in analytic regularity, proving the existence of solutions for large times in a non-perturbative regime using a Lagrangian approach and fixed point techniques. Subsequently, Hwang and Velázquez [33] gave a proof with less restrictive hypotheses. Concerning this approach, see also the work of Gagnebin [19] who extended the work of Caglioti and Maffei [14] to the one-dimensional ions dynamic (VPME) – i.e., (1.1) with $\beta = 1$, and $h(U) = e^U - 1 - U$.

The solution to the Landau damping for the Vlasov–Poisson system in arbitrary dimension was finally solved in 2011 in the work of Mouhot and Villani [48]. They treated the perturbative regime around homogeneous Penrose-stable equilibria considering analytic and Gevrey initial data, with a Gevrey index close to 1. The proof, which uses a Newton scheme combining both an Eulerian and a Lagrangian approach, was subsequently simplified and extended to Gevrey $\gamma > \frac{1}{3}$ regular initial data by Bedrossian, Masmoudi, and Mouhot [9], using paraproduct decomposition techniques instead of the Newton scheme (it is worth mentioning that their results also include the screened (VP) case - i.e., (1.1) with $\beta = 1$ and $h = 0$). Recently, Grenier, Nguyen, and Rodnianski further simplified the proof in [21] using a different functional setting, better characterizing the invertibility of the linearized term and simplifying the non-linear analysis. Concerning the ion dynamics, we mention the recent work by Gagnebin and Iacobelli [20], where they solve the Cauchy problem, proving Landau damping for the (VPME) system on the torus $\mathbb{T}^d$, also for $\gamma > \frac{1}{3}$.

Bedrossian [7] proved that Mouhot–Villani's proof cannot be extended to high Sobolev spaces in the case of gravitational interactions by showing inflation of Sobolev norms for solutions that exhibit arbitrarily many isolated plasma echoes. In addition, Lin and Zeng [42] and [43] showed the existence of periodic traveling BGK waves within any small neighborhood in H^σ with $\sigma < \frac{3}{2}$ of a general homogeneous equilibrium, thus proving that Landau damping cannot hold for small Sobolev regularity. The problem is different in other related equations, such as the Vlasov–HMF equation, where this phenomenon is absent (for the Vlasov–HMF case, see the work by Faou and Rousset [17] for Landau damping in Sobolev spaces and the recent one by Benedetto, Caglioti, and Rossi [11], where the final data and initial data Landau damping problems with analytic regularity are compared).

While in the periodic case on $\mathbb{T}^d$, the phase mixing effect holds, allowing one to trade regularity for decay, in the whole space $\mathbb{R}^d$ the problem is radically different. In particular, due to the dispersive mechanism, the decay of the electric field is only algebraic, moreover, in general, the Penrose stability condition does not hold. In this regard, see the work by Ionescu, Pausader, Wang, and Widmayer [35] and by Flynn, Ouyang, Pausader, and Widmayer [18] for the dynamics around vacuum for (VP) system, the work by Pausader and Widmayer [50] for the stability of a charged particle, and the works by Ionescu, Pausader, Wang, and Widmayer [36, 37] for the stability of the Poisson equilibrium. In the case of the screened (VP) system on the whole space, the Penrose stability condition holds and this allows one to prove Landau damping even with low Sobolev regularity; see the work by Bedrossian, Masmoudi, and Mouhot [10] and by Han-Kwan, Nguyen, and Rousset [28] for the $d = 3$ case, and the works by Huang, Nguyen, and Xu [31, 32].

Let us consider solutions of (1.1) of the form

$$f(t,x,v) = \mu(v) + r(t,x,v), \tag{1.2}$$

where r is the perturbation and μ an analytic homogeneous equilibrium satisfying a suitable stability condition (see (1.11)). The goal of this work is to study the final data problem where an asymptotic datum $g_\infty(x,v)$ with Gevrey regularity is given (see (1.12) for more details). By inserting (1.2) into our system (1.1), we find that the perturbation $r(t,x,v)$ verifies the equation

$$\begin{cases} \partial_t r(t,x,v) + v\cdot\nabla_x r(t,x,v) - E[r](t,x)\cdot\nabla_v(\mu(v)+r(t,x,v)) = 0, \\ E[r](t,x) = -\nabla U(t,x), \\ -\Delta U(t,x) + \beta U(t,x) + h(U(t,x)) = \displaystyle\int_{\mathbb{R}^d} r(t,x,v)\,dv. \end{cases} \tag{1.3}$$

We prove the existence of perturbative solutions $r(t,x,v)$ such that the electric field asymptotically vanishes and it holds that

$$\|r(t,x+vt,v)-g_\infty(x,v)\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\to 0,\quad \text{as } t\to+\infty, \tag{1.4}$$

so that

$$\|f(t,x+vt,v)-f_\infty(x,v)\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\to 0,\quad \text{as } t\to+\infty,$$

where $f_\infty(x,v):=\mu(v)+g_\infty(x,v)$. This answers the question about the scattering map addressed by Bedrossian in his review on Landau damping [8].

While completing this work, we learned that Ionescu, Pausader, Wang, and Widmayer were independently working on the solution of the Landau damping and the final data problem in the case of the (VP) system with critical regularity $\gamma=\frac{1}{3}$, thus constructing the scattering map for the problem and obtaining results similar to ours. These results were published in the very recent preprint [38]. Compared with them, in our work, we do not treat the case $\gamma=\frac{1}{3}$, but we deal with a more general Vlasov-type equation allowing a non-linearity in the Poisson coupling and therefore including the (VPME) system.

1.1. Notation

Let $(k,\eta)\in\mathbb{Z}^d\times\mathbb{R}^d$. We denote the Fourier coefficient of a given function $\rho:\mathbb{R}_t\times\mathbb{T}^d_x\to[0,+\infty)$ by

$$\hat\rho_t(k)\equiv\hat\rho(t,k):=\int_{\mathbb{T}^d}\rho(t,x)e^{-ik\cdot x}\,dx,$$

so that we have the usual Fourier inversion formula

$$\rho(t,x)=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\hat\rho_t(k)e^{ik\cdot x}.$$

Analogously, we write the Fourier transform of a given distribution function $f:\mathbb{R}_t\times\mathbb{T}^d_x\times\mathbb{R}^d_v\to[0,+\infty)$ as

$$\hat f_t(k,\eta)\equiv\hat f(t,k,\eta):=\int_{\mathbb{T}^d\times\mathbb{R}^d}f(t,x,v)e^{-ik\cdot x}e^{-i\eta\cdot v}\,dx\,dv,$$

with the inversion formula

$$f(t,x,v)=\frac{1}{(2\pi)^{2d}}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\hat f_t(k,\eta)e^{ik\cdot x}e^{i\eta\cdot v}\,d\eta.$$

For a function defined on the phase space $\mathbb{T}^d_x\times\mathbb{R}^d_v$, we define the Fourier multiplier operator $A_z(\nabla_x,\nabla_v)$ as

$$\widehat{(A_z(\nabla_x,\nabla_v)f)}(k,\eta)\equiv\widehat{A_zf}(k,\eta):=A_z(k,\eta)\hat f(k,\eta), \tag{1.5}$$

where $A_z(k,\eta) := e^{z\langle k,\eta\rangle^\gamma}\langle k,\eta\rangle^\sigma$, with Gevrey index $\gamma \in (0,1]$, Sobolev correction $\sigma > 0$, and Gevrey regularity radius $z > 0$. Here, $\langle k,\eta\rangle := \sqrt{1+|k|^2+|\eta|^2}$ denotes the Japanese bracket. In the following, we also use the Fourier multiplier

$$\widehat{(B_z(\nabla_x,\nabla_v)f)}(k,\eta) \equiv \widehat{B_z f}(k,\eta) := \langle k,\eta\rangle A_z(k,\eta)\hat{f}(k,\eta). \tag{1.6}$$

For a time-dependent function on the physical space $\mathbb{T}^d$, we define the analogous multiplier by setting $\eta = kt$ in (1.5), with $k \in \mathbb{Z}^d$ and $t \in \mathbb{R}$:

$$\widehat{(A_z(\nabla_x, t\nabla_x)\rho)}(t,k) \equiv \widehat{A_z\rho}(t,k) := A_z(k,kt)\widehat{\rho_t}(k).$$

To quantify the regularity of the distribution function and the decay of the electric field, we use generalized Gevrey-γ^{-1} L^2 norms for our main unknowns. More precisely, given a Gevrey index $\gamma \in (0,1]$, Sobolev exponent $\sigma > 0$, and a function $\rho\colon \mathbb{R}_t \times \mathbb{T}^d_x \to [0,+\infty)$, we introduce

$$\|A_t(\nabla_x, t\nabla_x)\rho_t\|^2_{L^2_x} = \sum_{k\in\mathbb{Z}^d} e^{2\lambda(t)\langle k,kt\rangle^\gamma}|\hat{\rho}_t(k)|^2\langle k,kt\rangle^{2\sigma},$$

where, with a little abuse of notation, we define

$$A_t(k,\eta) := e^{\lambda(t)\langle k,\eta\rangle^\gamma}\langle k,\eta\rangle^\sigma, \tag{1.7}$$

with

$$\lambda(t) := \lambda_\infty - C\langle t\rangle^{-\delta}, \tag{1.8}$$

where $\lambda_\infty > 0$ and $\delta \ll 1$, $C > 0$ such that $\lambda(0) > 0$. Throughout the paper, without risk of ambiguity, we will sometimes simplify the notation by writing $A\rho(t)$ instead of $A_t\rho(t)$, and similarly for the multiplier B.

For a time-dependent function on the phase space $f\colon \mathbb{R}_t \times \mathbb{T}^d_x \times \mathbb{R}^d_v \to \mathbb{R}_+$, $j \in \mathbb{N}^d$ a multi-index, and $M > \frac{d}{2}$, we introduce

$$\begin{aligned}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)f_t\|^2_{L^2_{x,v}} &\approx \sum_{|j|\le M}\|B_t(k,\eta)\partial^j_\eta \hat{f}_t\|^2_{L^2_{k,\eta}} \\ &= \sum_{|j|\le M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d} e^{2\lambda(t)\langle k,\eta\rangle^\gamma}|\partial^j_\eta \hat{f}_t(k,\eta)|^2\langle k,\eta\rangle^{2\sigma+2}\,d\eta.\end{aligned} \tag{1.9}$$

See (3.1) and (3.2) for a justification of the equivalence in (1.9). In this paper, C denotes a generic constant that may change from line to line. Moreover, we define $\mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{0\}$ and $\mathbb{R}_+ := [0,+\infty)$. We say that $A \approx B$ if $C^{-1}B \le A \le CB$, for some $C > 0$.

1.2. Main result

Before stating our main Theorem 1.1, we introduce the technical assumptions satisfied by the homogeneous equilibrium $\mu(v)$:

(H1) μ is real analytic and satisfies for $M > \frac{d}{2}$, $\lambda > \lambda_\infty > 0$, and for all multi-index $j \in \mathbb{N}^d$,

$$\sum_{|j| \le M} \|e^{\lambda \langle \eta \rangle} |\partial_\eta^j \hat{\mu}(\eta)|\|_{L^\infty_\eta} < \infty, \tag{1.10}$$

where λ_∞ is introduced in (1.8);

(H2) μ satisfies the following Penrose stability condition: there exists a small positive constant κ_0 such that

$$\inf_{k \in \mathbb{Z}^d_*; \Re\tau \ge 0} \left| 1 + \frac{|k|^2}{\beta + |k|^2} \int_0^{+\infty} t \hat{\mu}(kt) e^{-\tau t}\, dt \right| \ge \kappa_0 > 0, \tag{1.11}$$

where $\tau \in \mathbb{C}$ and $\Re\tau$ is the real part of τ;

(H3) $\int_{\mathbb{R}^d} \mu(v)\, dv = 1$.

Theorem 1.1. *Let us consider the system (1.3) with $\beta \ge 0$ a non-negative constant and $h: \mathbb{R} \to \mathbb{R}$ an analytic function such that $h(x) = \mathcal{O}(x^2)$ when x goes to zero. Let μ be a homogeneous equilibrium that satisfies the hypotheses (H1)–(H3) with $\lambda > 0$ as in (1.10) and let g_∞ be a Gevrey function of mean zero such that for $\sigma > d + 5$, $M > \frac{d}{2}$, $b > \frac{5}{2}$, and $\gamma \in (\frac{1}{3}, 1)$,*

$$\|\langle v \rangle^M e^{\lambda \langle \nabla_x, \nabla_v \rangle^\gamma} \langle \nabla_x, \nabla_v \rangle^{\sigma+b} g_\infty\|_{L^2_{x,v}} < \varepsilon. \tag{1.12}$$

Then, for sufficiently small $\varepsilon > 0$, there exists a unique Gevrey solution r of (1.3) such that the electric field $E[r]$ decays exponentially fast to zero as time goes to $+\infty$, in the sense that

$$\|A_{\bar{\lambda}}(\nabla_x, t\nabla_x) E[r](t)\|^2_{L^2_x} \le C\varepsilon e^{-C\langle t \rangle^\gamma} \tag{1.13}$$

where $C > 0$ is a constant and $\bar{\lambda} < \lambda(0)$ in (1.8). Moreover, we have

$$\lim_{t \to +\infty} \|r(t, x + vt, v) - g_\infty(x, v)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} = 0.$$

Remarks and novelty. We present here some remarks and discuss the novelties of our approach.

Remark 1.2. The result in Theorem 1.1 gives an affirmative answer to the problem posed by Bedrossian in the recent review [8, Section 6] about the existence and injectivity of the wave operator map $\mathcal{W}(g_\infty) = g_0$, which associates the initial data to the asymptotic data.

Remark 1.3. Combining Theorem 1.1 with the Landau damping for the forward problem proved in [20, Theorem 1.1] guarantees the construction of the scattering map $\mathcal{S}(g_{-\infty}) = g_{+\infty}$ that maps the state at $-\infty$ to the asymptotic state at $+\infty$.

Let us introduce the new unknown $g_t(x, v) := r(t, x + vt, v)$. Since

$$\partial_t g = \partial_t r + v \cdot \nabla_x r, \quad \nabla_x g = \nabla_x r,$$

it follows that the profile $g_t(x,v)$ verifies the equation

$$\begin{cases} \partial_t g_t(x,v) + E_t(x+vt)\cdot\nabla_v\mu(v) = -E_t(x+vt)\cdot(\nabla_v - t\nabla_x)g_t(x,v), \\ E_t(x) = -\nabla U_t(x), \\ -\Delta U_t(x) + \beta U_t(x) + h(U_t(x)) = \int_{\mathbb{R}^d} g_t(x-vt,v)\,dv. \end{cases} \tag{1.14}$$

Moreover, we can rewrite (1.4) as

$$\|g_t(x,v) - g_\infty(x,v)\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \to 0, \quad \text{as } t\to+\infty, \tag{1.15}$$

where the asymptotic datum g_∞ will be assumed to be a mean-zero function with Gevrey regularity as in (1.12).

Observe that the system (1.14) is composed of two non-linear equations, one non-linear transport-type PDE for g, and a non-linear elliptic equation for U. We treat the non-linearity for the Poisson equation by introducing the modified density

$$\varrho_t(x) := (\beta - \Delta)U_t(x). \tag{1.16}$$

It holds that $\varrho_t(x)$ satisfies a closed equation; see (2.1). We use this fact to study the coupled system given by (g,ϱ), deriving global-in-time a priori estimates for it. We will see that assumptions (H1) and (H2) on the homogeneous equilibrium μ are necessary to invert the linear term in equation (2.1). Moreover, the convolution operator for the linearized term is different from the one in the Cauchy problem and requires the introduction of the *two-sided Laplace transform* instead of the one-sided one. Regarding the a priori estimates for the non-linearities, we study the system (g,ϱ) by following the approach introduced in the recent review by Bedrossian [8], recovering, also in this scattering problem, the Gevrey threshold for $\gamma > \frac{1}{3}$. Technically, the analysis of the final data problem requires solving the equation (1.14) backwards in time, from infinity to time zero. This involves the use of a regularity parameter in (1.8), which is increasing in time instead of a decreasing one as in the forward problem. Moreover, dealing with general Vlasov-type equations, we also need to study further non-linearity in the elliptic equation, which is treated by means of elliptic PDE techniques. In Proposition 4.2, we also prove a result of independent interest on the existence of solutions for the elliptic equation in (1.14) with small source terms in Gevrey spaces.

After obtaining the a priori estimates for the modified density ϱ_t, as stated in Lemma 2.6 and Proposition 2.7, along with the a priori bounds on g_t from Proposition 3.1, we establish the proof of the main theorem by employing a fixed-point argument. This argument is applied to a contractive mapping defined by a suitable linear PDE, where the contractivity will result from the application of the a priori estimates obtained.

The structure of the paper is the following: In Section 2 we focus on the equation verified by the modified density $\varrho_t(x)$ defined in (1.16). After writing down the Volterra equation verified by ϱ, in Section 2.1 we invert the linear term by using the Penrose stability condition (1.11) and we study the decay property of the related resolvent kernel. This allows us to give a priori estimates on $\varrho_t(x)$ by focusing only on the non-linear term. We do this in Section 2.2. In Section 3, we perform the a priori estimates on the distribution function g. Finally, in Section 4, we state a general well-posedness result with Gevrey data for the non-linear Poisson equation in (1.14) and construct solutions to the final data problem.

2. A priori estimates on the modified density ϱ

As mentioned in the introduction, we consider the system (g, ϱ) where ϱ is the modified density defined as $\varrho_t(x) := -\Delta_x U_t(x) + \beta U_t(x)$ so that $\int_{\mathbb{R}^d} g(t, x - vt, v)\, dv = \varrho_t(x) + h(U_t(x))$.

In this section, we obtain L^2-in-time estimates in Gevrey regularity for the modified density ϱ. We first write down the equation verified by the Fourier coefficients of $\varrho_t(x)$ to carry out this analysis.

Lemma 2.1. *Let μ be an analytic homogeneous equilibrium. Let g be the unique solution to the problem* (1.14) *with asymptotic condition* (1.15)*. Then we have the following equation for $\hat{\varrho}_t(k)$:*

$$\hat{\varrho}_t(k) + \frac{|k|^2}{\beta + |k|^2} \int_t^{+\infty} \hat{\varrho}_s(k)(s-t)\hat{\mu}(-k(s-t))\, ds = \widehat{S}_t(k), \quad k \in \mathbb{Z}^d, \tag{2.1}$$

where the source term $\widehat{S}_t(k)$ is given by

$$\begin{aligned} \widehat{S}_t(k) &:= \hat{g}_\infty(k, kt) - \widehat{h(U_t)}(k) \\ &\quad - \sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} (s-t) \frac{k \cdot \ell}{\beta + |\ell|^2} \hat{\varrho}_s(\ell) \hat{g}_s(k - \ell, kt - \ell s)\, ds. \end{aligned} \tag{2.2}$$

Proof. We start by taking the Fourier transform with respect to $k \in \mathbb{Z}^d$ and $\eta \in \mathbb{R}^d$ of (1.14). So using the properties of the Fourier transform we have

$$\begin{aligned} \partial_t \hat{g}_t(k, \eta) &= -\widehat{E}_t(k) \cdot \widehat{\nabla_v \mu}(\eta - kt) - i \sum_{\ell \in \mathbb{Z}^d_*} (\eta - kt) \cdot \widehat{E}_t(\ell) \hat{g}_t(k - \ell, \eta - \ell t) \\ &= -i(\eta - kt) \cdot \widehat{E}_t(k) \hat{\mu}(\eta - kt) - i \sum_{\ell \in \mathbb{Z}^d_*} (\eta - kt) \cdot \widehat{E}_t(\ell) \hat{g}_t(k - \ell, \eta - \ell t). \end{aligned}$$

Since $\widehat{E}_t(k) = -ik\widehat{U}_t(k)$ we get

$$\begin{aligned} \partial_t \hat{g}_t(k, \eta) &= -[(\eta - kt) \cdot k] \widehat{U}_t(k) \hat{\mu}(\eta - kt) \\ &\quad - \sum_{\ell \in \mathbb{Z}^d_*} [(\eta - kt) \cdot \ell] \widehat{U}_t(\ell) \hat{g}_t(k - \ell, \eta - \ell t). \end{aligned} \tag{2.3}$$

Integrating (2.3) in time, by the asymptotic condition (1.15) we obtain

$$\hat{g}_t(k,\eta) = \hat{g}_\infty(k,\eta) + \int_t^{+\infty} [(\eta - ks)\cdot k]\widehat{U}_s(k)\hat{\mu}(\eta - ks)\,ds + \sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} [(\eta - ks)\cdot \ell]\widehat{U}_s(\ell)\hat{g}_s(k-\ell,\eta-\ell s)\,ds.$$

We now consider the Fourier modes of g with $\eta = kt$:

$$\hat{g}_t(k,kt) = \hat{g}_\infty(k,kt) - \int_t^{+\infty} (s-t)|k|^2\widehat{U}_s(k)\hat{\mu}(-k(s-t))\,ds - \sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} (s-t)k\cdot\ell\widehat{U}_s(\ell)\hat{g}_s(k-\ell,\eta-\ell s)\,ds. \tag{2.4}$$

Taking the Fourier transform of the Poisson coupling in (1.14), and noting that

$$\widehat{\int_{\mathbb{R}^d} g(t,x-vt,v)\,dv}(k) = \hat{g}_t(k,kt),$$

we obtain

$$\hat{g}_t(k,kt) = -\widehat{\Delta_x U_t}(k) + \beta\widehat{U}_t(k) + \widehat{h(U_t)}(k) = \hat{\varrho}_t(k) + \widehat{h(U_t)}(k). \tag{2.5}$$

Then, plugging equation (2.5) into (2.4), we obtain the closed equation in (2.1). ∎

Note that, since g_∞ has mean zero, $\hat{\varrho}_t(0) = -\widehat{h(U_t)}(0)$. Hence, for $k\in\mathbb{Z}^d_*$ and considering a given source term $\widehat{S}_t(k)$, equation (2.1) is an integral Volterra-type equation for $\hat{\varrho}_t(k)$. In the following subsection, we focus on how to invert this linear term.

2.1. Invertibility of the linear term

We start by introducing the one-sided and two-sided Laplace transforms. Let $\phi\colon\mathbb{R}\to\mathbb{R}$ be of exponential type with parameter $c>0$, i.e., there exists $c>0$ such that $|\phi(t)|\le Ce^{-c|t|}$, for a constant $C>0$. Then the one-sided Laplace transform of ϕ is given by

$$\mathcal{L}[\phi_t](\tau) := \int_0^{+\infty} e^{-\tau t}\phi(t)\,dt, \quad \tau\in\mathbb{C},$$

while the two-sided Laplace transform is given by

$$\mathcal{B}[\phi_t](\tau) := \int_{-\infty}^{+\infty} e^{-\tau t}\phi(t)\,dt,$$

both well defined for $|\Re\tau| < c$.

To invert the linear term of (2.1), we state the following lemma.

Lemma 2.2. *Let ψ and ϕ be two real-valued functions of exponential type with parameter $c > 0$. Then for $|\Re\tau| < c$, we have*

$$\mathcal{B}\left[\int_t^{+\infty} \psi(s)\phi(s-t)\,ds\right](\tau) = \mathcal{B}[\psi_t](\tau)\mathcal{L}[\phi_t](-\tau). \tag{2.6}$$

Proof. We have

$$\begin{aligned}\mathcal{B}\left[\int_t^{+\infty} \psi(s)\phi(s-t)\,ds\right](\tau) &= \int_{-\infty}^{+\infty}\int_t^{+\infty} \psi(s)\phi(s-t)e^{-\tau t}\,ds\,dt \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{s} \psi(s)\phi(s-t)e^{-\tau s}e^{\tau(s-t)}\,dt\,ds \\ &= \int_{-\infty}^{+\infty} \psi(s)e^{-\tau s}\,ds \int_0^{+\infty} \phi(y)e^{\tau y}\,dy \\ &= \mathcal{B}[\psi_t](\tau)\mathcal{L}[\phi_t](-\tau),\end{aligned}$$

where we used Fubini's theorem and the change of variable $y = s - t$. ∎

2.1.1. Resolvent estimates. In the following, we assume $k \in \mathbb{Z}^d_*$. Using the one-sided and two-sided Laplace transforms, we will convert the integral Volterra-type equation (2.1) for $\hat{\varrho}_t(k)$ into an algebraic equation. By taking the two-sided Laplace transform of (2.1), thanks to (2.6), we get

$$\mathcal{B}[\hat{\varrho}_t(k)](\tau) + \frac{|k|^2}{\beta+|k|^2}\mathcal{L}[t\hat{\mu}(-kt)](-\tau)\mathcal{B}[\hat{\varrho}_t(k)](\tau) = \mathcal{B}[\widehat{S}_t(k)](\tau),$$

which gives the solution

$$\mathcal{B}[\hat{\varrho}_t(k)](\tau) = \frac{\mathcal{B}[\widehat{S}_t(k)](\tau)}{1 + \frac{|k|^2}{\beta+|k|^2}\mathcal{L}[t\hat{\mu}(-kt)](-\tau)}. \tag{2.7}$$

By the Penrose stability condition (1.11), the denominator of (2.7) never vanishes. Indeed, note that the requirement

$$\inf_{k\in\mathbb{Z}^d_*;\Re\tau\le 0}\left|1 + \frac{|k|^2}{\beta+|k|^2}\int_0^{+\infty} t\hat{\mu}(-kt)e^{\tau t}\,dt\right| \ge \kappa_0 > 0$$

needed in (2.7) is implied by (1.11). Next we define the kernel $\widetilde{K}_k(\tau)$, as

$$\widetilde{K}_k(\tau) := -\frac{\frac{|k|^2}{\beta+|k|^2}\mathcal{L}[t\hat{\mu}(-kt)](\tau)}{1 + \frac{|k|^2}{\beta+|k|^2}\mathcal{L}[t\hat{\mu}(-kt)](\tau)}. \tag{2.8}$$

Therefore, (2.7) becomes

$$\mathcal{B}[\hat{\varrho}_t(k)](\tau) = \mathcal{B}[\widehat{S}_t(k)](\tau) + \widetilde{K}_k(-\tau)\mathcal{B}[\widehat{S}_t(k)](\tau). \tag{2.9}$$

Thanks to [21, Lemma 3.2], we have the following property of the resolvent kernel.

Lemma 2.3. *Let μ be an analytic homogeneous equilibrium satisfying assumptions (H1)–(H3) with $\lambda > 0$ as in (1.10). Then there is a positive constant $\lambda_1 < \frac{1}{2}\lambda$ so that the function $\widetilde{K}_k(\tau)$ is an analytic function in $\{\Re\tau \geq -\lambda_1|k|\}$. Moreover, there exists a constant C such that*

$$|\widetilde{K}_k(\tau)| \leq \frac{C}{1+|k|^2+|\Im\tau|^2}, \tag{2.10}$$

for $k \neq 0$ and $\Re\tau = -\lambda_1|k|$.

Note, by sending $k \mapsto -k$ in (2.8), our resolvent kernel $\widetilde{K}$ is equal, up to the constant $|k|^2(\beta+|k|^2)^{-1}$ to the one in [21, equation (3.6)]. Since the estimates on this kernel depend only on the modulus of k, the proof of Lemma 2.3 is analogous to the one in [21, Lemma 3.2] and is therefore omitted.

2.1.2. Inverse of the resolvent kernel. First, we recall the definition of the inverse Laplace transform for the function $\widetilde{K}_k(\tau)$ defined in the region $\{\Re\tau \geq -\lambda_1|k|\}$,

$$\widehat{K}_k(t) \equiv \mathcal{L}^{-1}[\widetilde{K}_k(\tau)](t) := \frac{1}{2\pi i}\int_{\{\Re\tau=\gamma_0\}} \widetilde{K}_k(\tau)e^{\tau t}\,d\tau, \tag{2.11}$$

for some constant $\gamma_0 > 0$. Then, we have the following property.

Lemma 2.4 ([21, Proposition 3.3]). *The function $\widehat{K}_k(t)$ satisfies*

$$|\widehat{K}_k(t)| \leq Ce^{-\lambda_1|k|t}, \tag{2.12}$$

where the constant λ_1 comes from Lemma 2.3.

Proof. By Lemma 2.3, we know that $\widetilde{K}_k(\tau)$ is analytic in the region $\{\Re\tau \geq -\lambda_1|k|\}$; therefore we have by the Cauchy theory on a complex contour integral,

$$\widehat{K}_k(t) = \frac{1}{2\pi i}\int_{\{\Re\tau=\gamma_0\}} \widetilde{K}_k(\tau)e^{\tau t}\,d\tau = \frac{1}{2\pi i}\int_{\{\Re\tau=-\lambda_1|k|\}} \widetilde{K}_k(\tau)e^{\tau t}\,d\tau.$$

That is, we can deform the complex contour of integration from $\{\Re\tau = \gamma_0\}$ into $\{\Re\tau = -\lambda_1|k|\}$. Then using (2.10) we get

$$\begin{aligned}
|\widehat{K}_k(t)| &\leq \frac{1}{2\pi}\int_{\{\Re\tau=-\lambda_1|k|\}} |\widetilde{K}_k(\tau)e^{\tau t}|\,d\tau \\
&\leq \frac{1}{2\pi}\int_{\{\Re\tau=-\lambda_1|k|\}} \frac{C}{1+|k|^2+|\Im\tau|^2}e^{-\lambda_1|k|t}\,d\tau \\
&\leq \frac{C}{2\pi}e^{-\lambda_1|k|t}\int_{-\infty}^{\infty}\frac{1}{1+|k|^2+|y|^2}\,dy \leq Ce^{-\lambda_1|k|t}.
\end{aligned}$$
∎

Next, taking the inverse two-sided Laplace transform of (2.9), we have the following proposition.

Proposition 2.5. *Let μ be an analytic homogeneous equilibrium satisfying assumptions (H1)–(H3). Let $\hat{\varrho}_t(k)$ be a solution of (2.1) and let $\widehat{K}_k(t)$ be given by (2.11). Then we can express $\hat{\varrho}_t(k)$ as*

$$\hat{\varrho}_t(k) = \widehat{S}_t(k) + \int_t^{+\infty} \widehat{K}_k(s-t)\widehat{S}_s(k)\,ds, \tag{2.13}$$

where $\widehat{S}_t(k)$ is given by (2.2) and $\widehat{K}_t(k)$ satisfies (2.12).

Proof. With (2.11), we can rewrite (2.9) as

$$\begin{aligned}\mathcal{B}[\hat{\varrho}_t(k)](\tau) &= \mathcal{B}[\widehat{S}_t(k)](\tau) + \widetilde{K}_k(-\tau)\mathcal{B}[\widehat{S}_t(k)](\tau)\\ &= \mathcal{B}[\widehat{S}_t(k)](\tau) + \mathcal{L}[\widehat{K}_k(t)](-\tau)\mathcal{B}[\widehat{S}_t(k)](\tau).\end{aligned}$$

Then we take the inverse two-sided Laplace transform, and we recall the convolution property (2.6). ■

2.2. Estimates on the modified density ϱ

Thanks to the previous Proposition 2.5, to estimate ϱ_t we just need to study the source term in (2.2). Indeed, recalling the notation for the weight $A_t(k,\eta) := e^{\lambda(t)\langle k,\eta\rangle^{\gamma}}\langle k,\eta\rangle^{\sigma}$, where $k \in \mathbb{Z}^d$, $\eta \in \mathbb{R}^d$, and $\lambda(t)$ as in (1.8), we can prove the following lemma.

Lemma 2.6. *Assume $\hat{\varrho}_t(k)$ satisfies (2.13). Under the hypothesis of Proposition 2.5, we have*

$$\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \le C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)S_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)},$$

for $b > 0$ and some constant $C > 0$.

Proof. Starting from (2.13), we multiply by $\langle t\rangle^b A_t(k,kt)$ on both sides and consider the $L^2_t(\mathbb{R}_+)L^2_k(\mathbb{Z}^d_*)$ norm. Therefore, we get

$$\begin{aligned}\|\langle t\rangle^b A_t(k,kt)\hat{\varrho}_t(k)\|_{L^2_t L^2_k} &\le \|\langle t\rangle^b A_t(k,kt)\widehat{S}_t(k)\|_{L^2_t L^2_k}\\ &\quad + \left\|\langle t\rangle^b A_t(k,kt)\int_t^{+\infty}\widehat{K}_k(s-t)\widehat{S}_s(k)\,ds\right\|_{L^2_t L^2_k},\end{aligned}$$

where $\widehat{K}_k(t)$ is given by (2.11). We only need to control the last norm on the right-hand side: by using $\lambda(t) \le \lambda(s)$ and $A_t(k,kt) \le A_s(k,ks)$ if $t \le s$, we obtain

$$\begin{aligned}&\left\|\langle t\rangle^b A_t(k,kt)\int_t^{+\infty}\widehat{K}_k(s-t)\widehat{S}_s(k)\,ds\right\|_{L^2_t L^2_k}\\ &\qquad\le \left\|\int_t^{+\infty}\widehat{K}_k(s-t)\langle s\rangle^b A_s(k,ks)\widehat{S}_s(k)\,ds\right\|_{L^2_t L^2_k}.\end{aligned}$$

Let us define, for a fixed $k \in \mathbb{Z}^d_*$, the operator

$$\mathcal{P}_k(\phi) := \int_t^{+\infty} \widehat{K}_k(s-t)\phi(s)\, ds.$$

We show that $\|\mathcal{P}_k\|_{L^2_t(\mathbb{R}_+)\to L^2_t(\mathbb{R}_+)} \leq C$, for some constant $C > 0$ uniform in $k \in \mathbb{Z}^d_*$. We have

$$\|\mathcal{P}_k(\phi)\|_{L^\infty_t(\mathbb{R}_+)} \leq \frac{C}{\lambda_1|k|}\|\phi\|_{L^\infty_t(\mathbb{R}_+)}, \quad \|\mathcal{P}_k(\phi)\|_{L^1_t(\mathbb{R}_+)} \leq \frac{C}{\lambda_1|k|}\|\phi\|_{L^1_t(\mathbb{R}_+)},$$

where we used the decay property in (2.12) and Fubini's theorem. Therefore, by the Riesz–Thorin interpolation theorem, we get that $\mathcal{P}_k$ is bounded in $L^2_t(\mathbb{R}_+)$. Hence, by taking the $L^2_k(\mathbb{Z}^d_*)$ and using Fubini's theorem again, we obtain

$$\left\| \langle t\rangle^b A_t(k,kt) \int_t^{+\infty} \widehat{K}_k(s-t)\widehat{S}_s(k)\, ds \right\|_{L^2_t L^2_k} \leq \frac{C}{\lambda_1}\|\langle t\rangle^b A_t(k,kt)\widehat{S}_t(k)\|_{L^2_t L^2_k}.$$

Finally, noticing that $\hat{\varrho}_t(0) = -\widehat{h(U_t)}(0)$, we get the result. ∎

We now pass to estimating the source term in (2.13). This contains the non-linear integral term where the echoes mechanism appears and where we recover the Gevrey limitation $\gamma > \frac{1}{3}$. We carry this analysis in the following proposition.

Proposition 2.7. *Let S_t be the inverse Fourier transform of* (2.2) *and g_∞ be a Gevrey function of mean zero satisfying* (1.12). *Then for $\sigma > d$, $M > \frac{d}{2}$, $b > 0$, we have*

$$\begin{aligned}
&\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)S_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \\
&\quad \leq C\|\langle v\rangle^M e^{\lambda\langle\nabla_x,\nabla_v\rangle^\gamma}\langle\nabla_x,\nabla_v\rangle^{\sigma+b} g_\infty\|_{L^2_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)} \\
&\qquad + \|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)h(U_t)\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \\
&\qquad + C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)},
\end{aligned}$$

for some constant $C > 0$, where $B_t(\nabla_x,\nabla_v)$ is defined in (1.6) *and $\lambda > 0$ as in* (1.10).

To prove this result, we need some useful Gevrey inequalities, which we summarize in the following lemma.

Lemma 2.8 (See e.g. [9, Lemma 3.2]). *Let $x, y \geq 0$ and $\gamma \in (0,1)$.*

(i) *The triangle inequality*

$$C(\langle x\rangle^\gamma + \langle y\rangle^\gamma) \leq \langle x+y\rangle^\gamma \tag{2.14}$$

holds for some $C = C(\gamma) > 0$ depending only on γ.

(ii) *There exists a constant $C = C(\gamma)$ such that*

$$|\langle x\rangle^\gamma - \langle y\rangle^\gamma| \leq C\frac{\langle x-y\rangle}{\langle x\rangle^{1-\gamma} + \langle y\rangle^{1-\gamma}}. \tag{2.15}$$

(iii) *If* $|x-y| \leq \frac{x}{K}$ *for some* $K > 1$, *then we have the following inequality:*

$$|\langle x\rangle^{\gamma} - \langle y\rangle^{\gamma}| \leq \frac{\gamma}{(K-1)^{1-\gamma}} \langle x-y\rangle^{\gamma}. \tag{2.16}$$

(iv) *If* $y \leq x \leq Ky$ *for some* $K > 0$, *then*

$$\langle x+y\rangle^{\gamma} \leq \Big(\frac{K}{1+K}\Big)^{1-\gamma} (\langle x\rangle^{\gamma} + \langle y\rangle^{\gamma}). \tag{2.17}$$

In particular, the improved inequalities for $\gamma < 1$ given in (2.15), (2.16), and (2.17) are important in the following. Indeed, note that these estimates deteriorate when $\gamma = 1$. This implies a gain of decay in the estimates that is absent if $\gamma = 1$.

Proof of Proposition 2.7. Recall that

$$\hat{S}_t(k) := \hat{g}_{\infty}(k,kt) - \widehat{h(U_t)}(k) - \sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} (s-t)\frac{k\cdot\ell}{\beta+|\ell|^2}\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell,kt-\ell s)\,ds.$$

By the triangle inequality, we treat each term separately.

Estimate on g_∞*: Asymptotic term.* First, we show the bound on the asymptotic datum g_∞ with Gevrey regularity parameter λ. Using that $\int g_\infty(x,v)\,dx\,dv = 0$ and $\lambda > \lambda(t)$, we get

$$\begin{aligned}
&\|\langle t\rangle^{b} A_t(\nabla_x, t\nabla_x) g_\infty\|^2_{L^2_t L^2_x} \\
&\quad = \int_0^{+\infty} \sum_{k\in\mathbb{Z}^d} \langle t\rangle^{2b} A_t^2(k,kt)|\hat{g}_\infty(k,kt)|^2\,dt \\
&\quad \leq \sum_{k\in\mathbb{Z}^d_*} \int_0^{+\infty} \langle k,kt\rangle^{2b} A_t^2(k,kt)|\hat{g}_\infty(k,kt)|^2\,dt \\
&\quad \leq \sum_{k\in\mathbb{Z}^d_*} \int_0^{+\infty} \Big\langle k, t\frac{k}{|k|}\Big\rangle^{2b} e^{2\lambda\langle k,\frac{k}{|k|}t\rangle^{\gamma}} \Big\langle k,\frac{k}{|k|}t\Big\rangle^{2\sigma} \Big|\hat{g}_\infty\Big(k,\frac{k}{|k|}t\Big)\Big|^2\,dt \\
&\quad \leq \sum_{k\in\mathbb{Z}^d_*} \|e^{\lambda\langle k,\eta\rangle^{\gamma}}\langle k,\eta\rangle^{\sigma+b}\hat{g}_\infty(k,\eta)\|^2_{L^2_\eta(\mathbb{R}\frac{k}{|k|})} \\
&\quad \leq C\sum_{k\in\mathbb{Z}^d_*}\sum_{0\leq j\leq M} \|D^j_\eta(e^{\lambda\langle k,\eta\rangle^{\gamma}}\langle k,\eta\rangle^{\sigma+b}\hat{g}_\infty(k,\eta))\|^2_{L^2_\eta(\mathbb{R}^d)} \\
&\quad \leq C\|\langle v\rangle^{M} e^{\lambda\langle\nabla_x,\nabla_v\rangle^{\gamma}}\langle\nabla_x,\nabla_v\rangle^{\sigma+b} g_\infty\|^2_{L^2_{x,v}},
\end{aligned} \tag{2.18}$$

where in the fourth inequality, we used the Sobolev trace lemma with $M > \frac{d}{2}$ to bound the L^2 integral on the line.

Estimate on $h(U_t)$. The $L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)$ norm of the second term is exactly

$$\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)h(U_t)\|_{L^2_t L^2_x}; \tag{2.19}$$

we refer to Section 4 for the treatment of this non-linearity.

Therefore, it remains to treat the non-linear contribution

$$T_{\mathrm{NL}} := \sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}(s-t)\frac{k\cdot\ell}{\beta+|\ell|^2}\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell, kt-\ell s)\,ds.$$

Estimate on T_{NL}: Non-linear term. The idea is to subdivide the analysis of this term into different regions of frequency. We use the following splitting:

$$1 = \mathbb{1}_{|\ell,\ell s|\le\frac{1}{2}|k-\ell,kt-\ell s|} + \mathbb{1}_{|k-\ell,kt-\ell s|\le\frac{1}{2}|\ell,\ell s|} + \mathbb{1}_{|k-\ell,kt-\ell s|\approx|\ell,\ell s|}. \tag{2.20}$$

Therefore, we can rewrite T_{NL} as

$$\begin{aligned}
&\langle t\rangle^b A_t(k,kt)T_{\mathrm{NL}}\\
&= \langle t\rangle^b A_t(k,kt)\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathbb{1}_{|\ell,\ell s|\le\frac{1}{2}|k-\ell,kt-\ell s|}\frac{\ell\cdot k}{\beta+|\ell|^2}(s-t)\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell,kt-\ell s)\,ds\\
&\quad + \langle t\rangle^b A_t(k,kt)\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathbb{1}_{|k-\ell,kt-\ell s|\le\frac{1}{2}|\ell,\ell s|}\frac{\ell\cdot k}{\beta+|\ell|^2}(s-t)\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell,kt-\ell s)\,ds\\
&\quad + \langle t\rangle^b A_t(k,kt)\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathbb{1}_{|\ell,\ell s|\approx|k-\ell,kt-\ell s|}\frac{\ell\cdot k}{\beta+|\ell|^2}(s-t)\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell,kt-\ell s)\,ds\\
&=: T_{\mathrm{LH}} + T_{\mathrm{HL}} + T_{\mathrm{HH}}.
\end{aligned} \tag{2.21}$$

Non-linear estimate on T_{LH}. We first estimate T_{LH}. In this region $|\ell,\ell s|\le\frac{1}{2}|k-\ell, kt-\ell s|$, and we have

$$\langle k,kt\rangle \le \langle \ell,\ell s\rangle + \langle k-\ell,kt-\ell s\rangle \le 2\langle k-\ell,kt-\ell s\rangle.$$

Therefore,

$$\begin{aligned}
A_t(k,kt) &\le 2^\sigma e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}e^{\lambda(s)\langle k,kt\rangle^\gamma}\langle k-\ell,kt-\ell s\rangle^\sigma\\
&\le 2^\sigma e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}e^{c\lambda(s)\langle \ell,\ell s\rangle^\gamma}e^{\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}\langle k-\ell,kt-\ell s\rangle^\sigma,
\end{aligned} \tag{2.22}$$

where we used (2.16) to get

$$|\langle k,kt\rangle^\gamma - \langle k-\ell,kt-\ell s\rangle^\gamma| \le c\langle \ell,\ell s\rangle^\gamma,$$

with $c < 1$. Thus, using (2.22), we get

$$|T_{\mathrm{LH}}| \le \langle t\rangle^b A_t(k,kt) \sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \frac{|\ell\cdot k|}{\beta+|\ell|^2} \times (s-t)|\hat{\varrho}_s(\ell)||\hat{g}_s(k-\ell,kt-\ell s)|\,ds$$
$$\le C\sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \langle t\rangle^b e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} e^{c\lambda(s)\langle \ell,\ell s\rangle^\gamma} \frac{|\ell\cdot k|}{\beta+|\ell|^2} \times (s-t)|\hat{\varrho}_s(\ell)||\widehat{Ag}_s(k-\ell,kt-\ell s)|\,ds.$$

In this case, we can bound $e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} \le 1$, therefore

$$|T_{\mathrm{LH}}| \le C\sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \langle t\rangle^b e^{c\lambda(s)\langle \ell,\ell s\rangle^\gamma} \frac{|\ell\cdot k|}{\beta+|\ell|^2} \times (s-t)|\hat{\varrho}_s(\ell)||\widehat{Ag}_s(k-\ell,kt-\ell s)|\,ds.$$

Thus, adding the weight A to ϱ and using that

$$|k|(s-t) = |ks-\ell s+\ell s-kt| \le \langle s\rangle|k-\ell| + |\ell s-kt| \le \langle s\rangle\langle k-\ell,kt-\ell s\rangle, \quad (2.23)$$

we have

$$|T_{\mathrm{LH}}| \le C\sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \frac{\langle t\rangle^b}{\langle \ell,\ell s\rangle^\sigma} e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma} \times \frac{\langle s\rangle}{|\ell|}|\widehat{A\varrho}_s(\ell)|\langle k-\ell,kt-\ell s\rangle|\widehat{Ag}_s(k-\ell,kt-\ell s)|\,ds$$
$$\le C\sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \frac{\langle s\rangle^{b+1}}{\langle \ell,\ell s\rangle^\sigma} e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma} \times |\widehat{A\varrho}_s(\ell)||\widehat{Bg}_s(k-\ell,kt-\ell s)|\,ds. \quad (2.24)$$

Therefore, taking the $L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)$ norm and using Cauchy–Schwarz in ℓ and s, we get

$$\|T_{\mathrm{LH}}\|^2_{L^2_tL^2_x} \le C\int_0^{+\infty}\sum_{k\in\mathbb{Z}^d}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\le \frac12|k-\ell,kt-\ell s|} \frac{\langle s\rangle^{b+1}}{\langle \ell,\ell s\rangle^\sigma} \times e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma}|\widehat{A\varrho}_s(\ell)||\widehat{Bg}_s(k-\ell,kt-\ell s)|\,ds\Bigg)^2 dt$$

$$\leq C \int_0^{+\infty} \sum_{k \in \mathbb{Z}^d} \bigg(\sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell, \ell s| \leq \frac{1}{2}|k-\ell, kt-\ell s|} \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}} e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}} |\widehat{A\varrho_s}(\ell)|\, ds \bigg)$$

$$\times \bigg(\sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell, \ell s| \leq \frac{1}{2}|k-\ell, kt-\ell s|} \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}}$$

$$\times e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}} |\widehat{A\varrho_s}(\ell)| |\widehat{Bg}_s(k-\ell, kt-\ell s)|^2\, ds \bigg)\, dt.$$

We treat the first parenthesis using Cauchy–Schwarz again:

$$\sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell, \ell s| \leq \frac{1}{2}|k-\ell, kt-\ell s|} \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}} e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}} |\widehat{A\varrho_s}(\ell)|\, ds$$

$$\leq \bigg(\sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \langle s \rangle^{2b} |\widehat{A\varrho_s}(\ell)|^2\, ds \bigg)^{\frac{1}{2}} \bigg(\sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \frac{\langle s \rangle^2}{\langle \ell, \ell s \rangle^{2\sigma}} e^{-2(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}}\, ds \bigg)^{\frac{1}{2}}$$

$$\leq \bigg(\int_0^{+\infty} \sum_{\ell \in \mathbb{Z}^d_*} \langle s \rangle^{2b} |\widehat{A\varrho_s}(\ell)|^2\, ds \bigg)^{\frac{1}{2}} \bigg(\sum_{\ell \in \mathbb{Z}^d_*} \frac{1}{\langle \ell \rangle^{2\sigma}} \int_t^{+\infty} \langle s \rangle^2 e^{-2(1-c)\lambda(0)\langle s\rangle^{\gamma}}\, ds \bigg)^{\frac{1}{2}}$$

$$\leq C \| \langle t \rangle^b A_t(\nabla_x, t\nabla_x) \varrho_t \|_{L^2_t L^2_x},$$

where we used that the second term is bounded by a constant if $\sigma > \frac{d}{2}$ and recall $\lambda(s) > \lambda(0)$. Thus, by Fubini's theorem,

$$\begin{aligned} \|T_{\mathrm{LH}}\|^2_{L^2_t L^2_x} &\leq C \| \langle t \rangle^b A_t(\nabla_x, t\nabla_x) \varrho_t \|_{L^2_t L^2_x} \\ &\quad \times \int_0^{+\infty} \int_t^{+\infty} \sum_{\ell \in \mathbb{Z}^d_*} \sum_{k \in \mathbb{Z}^d} \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}} e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}} \\ &\qquad \times |\widehat{A\varrho_s}(\ell)| |\widehat{Bg}_s(k-\ell, kt-\ell s)|^2\, ds\, dt \\ &\leq C \| \langle t \rangle^b A_t(\nabla_x, t\nabla_x) \varrho_t \|_{L^2_t L^2_x} \\ &\quad \times \int_0^{+\infty} \int_t^{+\infty} \bigg(\sum_{\ell \in \mathbb{Z}^d_*} |\widehat{A\varrho_s}(\ell)| \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}} e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}} \bigg) \\ &\qquad \times \bigg(\sum_{k \in \mathbb{Z}^d} \sup_{\eta} |\widehat{Bg}_s(k, \eta)|^2 \bigg)\, ds\, dt \\ &\leq C \| \langle t \rangle^b A_t(\nabla_x, t\nabla_x) \varrho_t \|_{L^2_t L^2_x} \sup_{t \in \mathbb{R}_+} \| \langle v \rangle^M B_t(\nabla_x, \nabla_v) g_t \|^2_{L^2_{x,v}} \\ &\quad \times \int_0^{+\infty} \int_t^{+\infty} \sum_{\ell \in \mathbb{Z}^d_*} |\widehat{A\varrho_s}(\ell)| \frac{\langle s \rangle^{b+1}}{\langle \ell, \ell s \rangle^{\sigma}} e^{-(1-c)\lambda(s)\langle \ell, \ell s\rangle^{\gamma}}\, ds\, dt. \end{aligned}$$

The bound, in the penultimate inequality on $\widehat{Bg}$, is obtained using the L^∞ Sobolev embedding with $M > \frac{d}{2}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\sup_{\eta\in\mathbb{R}^d}|\widehat{Bg}_s(k,\eta)|^2 &\le C\sum_{k\in\mathbb{Z}^d}\sum_{|j|\le M}\|\partial_\eta^j(\widehat{Bg}_s(k,\eta))\|^2_{L^2_\eta}\\ &\le C\|\langle v\rangle^M B_s(\nabla_x,\nabla_v)g_s\|^2_{L^2_{x,v}}.\end{aligned} \tag{2.25}$$

By using that $\lambda(s) > \lambda(0)$ and $s > t$, we get

$$\begin{aligned}\|T_{\mathrm{LH}}\|^2_{L^2_tL^2_x} &\le C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_tL^2_x}\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}}\\ &\quad\times\int_0^{+\infty}e^{-\frac{(1-c)}{2}\lambda(0)\langle t\rangle^\gamma}\int_t^{+\infty}\sum_{\ell\in\mathbb{Z}^d_*}|\widehat{A\varrho}_s(\ell)|\frac{\langle s\rangle^{b+1}}{\langle \ell,\ell s\rangle^\sigma}e^{-\frac{(1-c)}{2}\lambda(s)\langle \ell,\ell s\rangle^\gamma}\,ds\,dt.\end{aligned}$$

Then, by Cauchy–Schwarz in s and ℓ we have

$$\begin{aligned}\|T_{\mathrm{LH}}\|^2_{L^2_tL^2_x} &\le C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_tL^2_x}\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}}\\ &\quad\times\int_0^{+\infty}e^{-\frac{(1-c)}{2}\lambda(0)\langle t\rangle^\gamma}\left(\int_t^{+\infty}\sum_{\ell\in\mathbb{Z}^d_*}\langle s\rangle^{2b}|\widehat{A\varrho}_s(\ell)|^2\,ds\right)^{\frac12}\\ &\quad\times\left(\int_t^{+\infty}\sum_{\ell\in\mathbb{Z}^d_*}\frac{\langle s\rangle^2}{\langle \ell,\ell s\rangle^{2\sigma}}e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma}\,ds\right)^{\frac12}dt\\ &\le C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|^2_{L^2_tL^2_x}\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}},\end{aligned} \tag{2.26}$$

where we used that the integral in s and sum over ℓ is finite for $\sigma > \frac{d}{2}$, and the integral in time is bounded thanks to the exponential decay in t.

Non-linear estimate on T_{HL}. We now turn to the analysis of the term T_{HL} in (2.21):

$$\begin{aligned}T_{\mathrm{HL}} := \langle t\rangle^b A_t(k,kt)\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathbb{1}_{|k-\ell,kt-\ell s|\le\frac12|\ell,\ell s|}\frac{\ell\cdot k}{\beta+|\ell|^2}(s-t)\hat{\varrho}_s(\ell)\\ \times\hat{g}_s(k-\ell,kt-\ell s)\,ds.\end{aligned}$$

In this region $|k-\ell,kt-\ell s| \le \frac12|\ell,\ell s|$, and we have

$$\langle k,kt\rangle \le \langle \ell,\ell s\rangle + \langle k-\ell,kt-\ell s\rangle \le 2\langle \ell,\ell s\rangle.$$

Therefore,

$$\begin{aligned}A_t(k,kt) &\le 2^\sigma e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}e^{\lambda(s)\langle k,kt\rangle^\gamma}\langle \ell,\ell s\rangle^\sigma\\ &\le 2^\sigma e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}e^{c\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}e^{\lambda(s)\langle \ell,\ell s\rangle^\gamma}\langle \ell,\ell s\rangle^\sigma,\end{aligned} \tag{2.27}$$

where we used (2.16) to get

$$|\langle k, kt\rangle^\gamma - \langle \ell, \ell s\rangle^\gamma| \le c\langle k-\ell, kt-\ell s\rangle^\gamma,$$

with $c < 1$. Hence, by (2.27),

$$\begin{aligned}|T_{\mathrm{HL}}| &\le C \sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|k-\ell,kt-\ell s|\le\frac{1}{2}|\ell,\ell s|} \langle t\rangle^b e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} e^{c\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma} \\ &\quad\times \frac{|\ell\cdot k|}{\beta+|\ell|^2}(s-t)|\widehat{A\varrho_s}(\ell)||\hat{g}_s(k-\ell,kt-\ell s)|\,ds \\ &\le C \sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|k-\ell,kt-\ell s|\le\frac{1}{2}|\ell,\ell s|} \langle t\rangle^b e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma} \frac{e^{-(1-c)\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}}{\langle k-\ell,kt-\ell s\rangle^{\sigma+1}} \\ &\quad\times \frac{|\ell\cdot k|}{\beta+|\ell|^2}(s-t)\,|\widehat{A\varrho_s}(\ell)||\widehat{Bg_s}(k-\ell,kt-\ell s)|\,ds.\end{aligned}$$

Next, similarly to (2.25), we have

$$|\widehat{Bg_s}(k-\ell,kt-\ell s)| \le \sum_{\ell\in\mathbb{Z}^d_*}\sup_{\eta\in\mathbb{R}^d} |\widehat{Bg_s}(k-\ell,\eta)| \le C\|\langle v\rangle^M B_s(\nabla_x,\nabla_v)g_s\|_{L^2_{x,v}}.$$

Therefore,

$$|T_{\mathrm{HL}}| \le C \sup_{t\in\mathbb{R}_+} \|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}} \sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} \mathcal{K}(t,s,k,\ell)\langle s\rangle^b|\widehat{A\varrho_s}(\ell)|\,ds,$$

where

$$\begin{aligned}\mathcal{K}(t,s,k,\ell) &:= \frac{\langle t\rangle^b}{\langle s\rangle^b}\frac{|\ell\cdot k|}{\beta+|\ell|^2}(s-t) \\ &\quad\times e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}\frac{e^{-(1-c)\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}}{\langle k-\ell,kt-\ell s\rangle^{\sigma+1}}\mathbb{1}_{|k-\ell,kt-\ell s|\le\frac{1}{2}|\ell,\ell s|}.\end{aligned} \tag{2.28}$$

By taking the $L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)$ norm, we get by Cauchy–Schwarz in s and ℓ,

$$\begin{aligned}\|T_{\mathrm{HL}}\|^2_{L^2_tL^2_x} &\le C\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}} \\ &\quad\times\int_0^{+\infty}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathcal{K}(t,s,k,\ell)\langle s\rangle^b|\widehat{A\varrho_s}(\ell)|\,ds\Big)^2 dt \\ &\le C\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}} \\ &\quad\times\int_0^{+\infty}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathcal{K}(t,s,k,\ell)\langle s\rangle^{2b}|\widehat{A\varrho_s}(\ell)|^2\,ds\Big) \\ &\quad\times\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathcal{K}(t,s,k,\ell)\,ds\Big)dt.\end{aligned}$$

Then, by Fubini's theorem,

$$
\begin{aligned}
&\|T_{\mathrm{HL}}\|^2_{L^2_t L^2_x} \qquad (2.29)\\
&\leq C \sup_{t\in\mathbb{R}_+} \|\langle v\rangle^M B_t(\nabla_x,\nabla_v) g_t\|^2_{L^2_{x,v}} \sup_{k\in\mathbb{Z}^d_*, t\in\mathbb{R}_+} \Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} \mathcal{K}(t,s,k,\ell)\,ds\Big)\\
&\quad\times \int_0^{+\infty}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathcal{K}(t,s,k,\ell)\langle s\rangle^{2b}|\widehat{A\varrho_s}(\ell)|^2\,ds\Big)dt\\
&\leq C \sup_{t\in\mathbb{R}_+} \|\langle v\rangle^M B_t(\nabla_x,\nabla_v) g_t\|^2_{L^2_{x,v}} \sup_{k\in\mathbb{Z}^d_*, t\in\mathbb{R}_+} \Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty} \mathcal{K}(t,s,k,\ell)\,ds\Big)\\
&\quad\times \int_0^{+\infty}\sum_{\ell\in\mathbb{Z}^d_*}\langle s\rangle^{2b}|\widehat{A\varrho_s}(\ell)|^2\Big(\sum_{k\in\mathbb{Z}^d}\int_0^{s}\mathcal{K}(t,s,k,\ell)\,dt\Big)ds\\
&\leq C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|^2_{L^2_t L^2_x}\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v) g_t\|^2_{L^2_{x,v}}\\
&\quad\times \sup_{k\in\mathbb{Z}^d_*, t\in\mathbb{R}_+}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_t^{+\infty}\mathcal{K}(t,s,k,\ell)\,ds\Big)\sup_{\ell\in\mathbb{Z}^d_*, s\in\mathbb{R}_+}\Big(\sum_{k\in\mathbb{Z}^d_*}\int_0^{s}\mathcal{K}(t,s,k,\ell)\,dt\Big).
\end{aligned}
$$

Therefore, the analysis of the term T_{HL} reduces to estimating the integral quantities of the kernel $\mathcal{K}$ in the last line of (2.29). Let us start with the first one. Note that we have for all time $s \geq t$,

$$
\lambda(s)-\lambda(t) = \frac{C}{\langle t\rangle^\delta} - \frac{C}{\langle s\rangle^\delta} \geq C\frac{|s-t|}{\langle s\rangle\langle t\rangle^\delta}. \tag{2.30}
$$

This implies we get enough exponential decay in (2.28) when $s \geq 2t$. Hence, we split the time integral into two regions: $t \leq s \leq 2t$ and $s \geq 2t$. For $s \geq 2t$, we get from (2.30),

$$
\lambda(s)-\lambda(t) \geq \lambda(2t)-\lambda(t) \geq C\frac{|t|}{\langle 2t\rangle\langle t\rangle^\delta} \geq C\langle s\rangle^{-\delta}. \tag{2.31}
$$

Therefore,

$$
\begin{aligned}
\mathcal{I}_1 &:= \sup_{k,t}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_{2t}^{+\infty}\mathcal{K}(t,s,k,\ell)\,ds\Big)\\
&\leq \sup_{k,t}\Big(\sum_{\ell\in\mathbb{Z}^d_*}\int_{2t}^{+\infty}\frac{\langle t\rangle^b}{\langle s\rangle^b}\frac{|\ell\cdot k|}{\beta+|\ell|^2}(s-t)\\
&\qquad\qquad \times e^{-C\langle s\rangle^{-\delta}\langle k,kt\rangle^\gamma}\frac{e^{-(1-c)\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}}{\langle k-\ell,kt-\ell s\rangle^{\sigma+1}}\mathbb{1}_{|k-\ell,kt-\ell s|\leq\frac{1}{2}|\ell,\ell s|}\,ds\Big).
\end{aligned}
$$

In the region $|k-\ell, kt-\ell s| \leq \frac{1}{2}|\ell,\ell s|$, we have

$$
\langle \ell,\ell s\rangle \leq \langle k,kt\rangle + \langle k-\ell,kt-\ell s\rangle \leq \langle k,kt\rangle + \frac{1}{2}\langle \ell,\ell s\rangle \leq 2\langle k,kt\rangle. \tag{2.32}
$$

Hence, by (2.23) and (2.32), we get for δ sufficiently small and $\sigma > d$,

$$\begin{aligned}\mathcal{I}_1 &\le \sup_{k,t}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*}\int_{2t}^{+\infty}\frac{\langle t\rangle^b}{\langle s\rangle^b}\langle s\rangle e^{-C\langle s\rangle^{-\delta}\langle \ell,\ell s\rangle^\gamma}\frac{e^{-(1-c)\lambda(s)\langle k-\ell,kt-\ell s\rangle^\gamma}}{\langle k-\ell,kt-\ell s\rangle^\sigma}\,ds\Bigg)\\ &\le \sup_{k,t}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*}\frac{1}{\langle k-\ell\rangle^\sigma}\int_{2t}^{+\infty}\langle s\rangle e^{-C\langle s\rangle^{\gamma-\delta}}\,ds\Bigg)\\ &\le C\sup_{k,t}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*}\frac{1}{\langle k-\ell\rangle^\sigma}\Bigg)\le C.\end{aligned}$$

Now, let us treat the case $t \le s \le 2t$. If $\ell = k$, we have

$$\begin{aligned}&\sup_{k,t}\Bigg(\int_t^{2t}\mathcal{K}(t,s,k,k)\,ds\Bigg)\\ &\qquad\le \sup_{k,t}\Bigg(\int_t^{2t}(s-t)e^{-(\lambda(s)-\lambda(t))\langle k,kt\rangle^\gamma}\frac{e^{-(1-c)\lambda(s)\langle k(t-s)\rangle^\gamma}}{\langle k(t-s)\rangle^{\sigma+1}}\,ds\Bigg)\\ &\qquad\le \sup_{k,t}\Bigg(\int_t^{2t}e^{-C\langle t-s\rangle^\gamma}\,ds\Bigg)\le C,\end{aligned}$$

where we used that $(1-c)\lambda(s) \ge C$ for some constant $C > 0$. Therefore, we only have to look at the sum over ℓ when $\ell \ne k$. By (2.23), the Gevrey inequality in (2.14), and following the argument [9, p. 62], we have

$$\begin{aligned}\mathcal{I}_2 &:= \sup_{k,t}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*,\ell\ne k}\int_t^{2t}\mathcal{K}(t,s,k,\ell)\,ds\Bigg) &(2.33)\\ &\le C\sup_{k,t}\Bigg(\sum_{\ell\in\mathbb{Z}^d_*}\sum_{j:\ell_j\ne k_j}\int_t^{2t}\frac{\langle s\rangle}{|\ell|}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\\ &\qquad\times\prod_{i\ne j}^d e^{-C\langle k_i-\ell_i\rangle^\gamma}\mathbb{1}_{\ell\ne k}\,ds\Bigg)\\ &\le C\sup_{k,t}\Bigg(\sum_{j=1}^d\sum_{\ell_j\in\mathbb{Z}}\int_t^{2t}\frac{\langle s\rangle}{\langle \ell_j\rangle}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\ne k_j}\,ds\Bigg).\end{aligned}$$

In this way, we can reduce our analysis to the one-dimensional case. Note that if either k_j or ℓ_j is zero, we get exponential decay in the kernel as in the previous case, hence we can assume that both k_j and ℓ_j are different from zero.

We split the support of the last integral between $|k_jt-\ell_js| \le \frac{1}{100}\langle t\rangle$ and $|k_jt-\ell_js| \ge \frac{1}{100}\langle t\rangle$. Let us first treat the easy case $|k_jt-\ell_js| \ge \frac{1}{100}\langle t\rangle$, where we can get enough exponential decay in (2.28). Since $t \le s \le 2t$, we get $|k_jt-\ell_js| \ge \frac{1}{200}\langle s\rangle$. Therefore, using (2.23), we obtain

$$\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_t^{2t}\frac{\langle s\rangle}{\langle \ell_j\rangle}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^{\gamma}}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\gamma}}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\sigma}}\mathbb{1}_{\ell_j\neq k_j}\,ds\Big)$$
$$\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_t^{2t}\langle s\rangle\frac{e^{-C|k_jt-\ell_js|^{\gamma}}}{\langle k_j-\ell_j\rangle^{\sigma}}\,ds\Big)$$
$$\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\frac{1}{\langle k_j-\ell_j\rangle^{\sigma}}\int_t^{2t}\langle s\rangle e^{-\frac{C}{200}\langle s\rangle^{\gamma}}\,ds\Big)\le C.$$

The second case $|k_jt-\ell_js|\le\frac{1}{100}\langle t\rangle$ is more challenging, and it is the part of the proof where we need the requirement on the Gevrey regularity assumption $\gamma\in(\frac{1}{3},1)$. If k_j and ℓ_j have opposite signs, then $|k_jt-\ell_js|\ge|\ell_js|$ and we get sufficient exponential decay in (2.28) as before, hence it is sufficient to assume that $k_j\ell_j>0$.

Analogously, if $|k_j|<|\ell_j|$ then $|k_jt-\ell_js|\ge(|\ell_j|-|k_j|)s$, hence we only have to consider the case $|k_j|>|\ell_j|$. By (2.30), we have

$$\lambda(s)-\lambda(t)\ge C\frac{|t-s|}{\langle s\rangle\langle t\rangle^{\delta}}=\frac{C}{\langle s\rangle\langle t\rangle^{\delta}}\Big(s-\frac{k_jt}{\ell_j}\Big)-\frac{C}{\langle s\rangle\langle t\rangle^{\delta}}\Big(t-\frac{k_jt}{\ell_j}\Big)\ge\frac{\langle t\rangle^{1-\delta}}{\langle s\rangle}\frac{1}{|\ell_j|}. \tag{2.34}$$

Hence, using (2.34) in (2.33), we get for some $\alpha>0$,

$$\mathcal{I}_2\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_t^{2t}\frac{\langle s\rangle}{|\ell_j|}e^{-\frac{\langle t\rangle^{1-\delta}}{\langle s\rangle}\frac{1}{|\ell_j|}\langle k_j,k_jt\rangle^{\gamma}}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\gamma}}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\sigma}}\mathbb{1}_{\ell_j\neq k_j}\,ds\Big)$$
$$\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_t^{2t}\frac{\langle s\rangle}{|\ell_j|}\frac{\langle s\rangle^{\alpha}|\ell_j|^{\alpha}}{\langle t\rangle^{(1-\delta)\alpha}\langle k_j,k_jt\rangle^{\gamma\alpha}}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\gamma}}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\sigma}}\mathbb{1}_{\ell_j\neq k_j}\,ds\Big)$$
$$\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_t^{2t}\langle t\rangle^{1+\alpha-(1-\delta)\alpha-\gamma\alpha}|\ell_j|^{\alpha-1-\gamma\alpha}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\gamma}}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^{\sigma}}\mathbb{1}_{\ell_j\neq k_j}\,ds\Big),$$

where we used that $|k_j|>|\ell_j|$, $\langle\ell_j,\ell_js\rangle\ge|\ell_js|$, and $t\le s\le 2t$. Choosing α such that $1+\alpha-(1-\delta)\alpha-\gamma\alpha=0$, i.e., $\alpha=(\gamma-\delta)^{-1}$, and doing a change of variable $\tau=\ell_js$, we get

$$\mathcal{I}_2\le C\sup_{k,t}\Big(\sum_{j=1}^{d}\sum_{\ell_j\in\mathbb{Z}_*}\int_0^{+\infty}|\ell_j|^{\alpha-1-\gamma\alpha}\frac{1}{|\ell_j|}\frac{e^{-C\langle k_j-\ell_j,k_jt-\tau\rangle^{\gamma}}}{\langle k_j-\ell_j,k_jt-\tau\rangle^{\sigma}}\,d\tau\Big).$$

In order to be able to sum over ℓ, we need $\alpha-1-\gamma\alpha<1$. Therefore,

$$\frac{1}{\gamma-\delta}-1-\frac{\gamma}{\gamma-\delta}<1\quad\Longrightarrow\quad\frac{1+2\delta}{3}<\gamma.$$

This inequality gives us the Gevrey regularity threshold that we have to impose, which is $\gamma \in (\frac{1}{3}, 1)$. Finally,

$$\begin{aligned}\mathcal{I}_2 &\leq C \sup_{k,t} \left(\sum_{j=1}^{d} \sum_{\ell_j \in \mathbb{Z}_*} \int_0^{+\infty} \frac{e^{-C\langle k_j - \ell_j, k_j t - \tau\rangle^\gamma}}{\langle k_j - \ell_j, k_j t - \tau\rangle^\sigma} \, d\tau \right) \\ &\leq C \sup_{k,t} \left(\sum_{j=1}^{d} \sum_{\ell_j \in \mathbb{Z}_*} \langle k_j - \ell_j\rangle^{-\sigma} \int_0^{+\infty} e^{-C\langle k_j t - \tau\rangle^\gamma} \, d\tau \right) \leq C.\end{aligned}$$

Next, we turn to the estimate of the second integral kernel in (2.29). In this case, we have $0 \leq t \leq s$, and therefore (2.30) holds as well. Again, we split the time integral into two regions: $0 \leq t \leq \frac{s}{2}$ and $\frac{s}{2} \leq t \leq s$. As before, we get enough exponential decay for $0 \leq t \leq \frac{s}{2}$, so let us first treat this case. By (2.31), we get

$$\begin{aligned}\mathcal{I}_3 &:= \sup_{\ell,s} \left(\sum_{k \in \mathbb{Z}_*^d} \int_0^{\frac{s}{2}} \mathcal{K}(t,s,k,\ell) \, dt \right) \\ &\leq \sup_{\ell,s} \left(\sum_{k \in \mathbb{Z}_*^d} \int_0^{\frac{s}{2}} \frac{\langle t\rangle^b}{\langle s\rangle^b} \frac{|\ell \cdot k|}{\beta + |\ell|^2} (s-t) e^{-C\langle s\rangle^{-\delta}\langle k, kt\rangle^\gamma} \right. \\ &\qquad\qquad \left. \times \frac{e^{-(1-c)\lambda(s)\langle k-\ell, kt-\ell s\rangle^\gamma}}{\langle k-\ell, kt-\ell s\rangle^{\sigma+1}} \mathbb{1}_{|k-\ell, kt-\ell s| \leq \frac{1}{2}|\ell, \ell s|} \, dt \right).\end{aligned}$$

Then, by (2.23) and (2.32), we obtain

$$\begin{aligned}\mathcal{I}_3 &\leq \sup_{\ell,s} \left(\sum_{k \in \mathbb{Z}_*^d} \int_0^{\frac{s}{2}} \frac{\langle t\rangle^b}{\langle s\rangle^b} \langle s\rangle e^{-C\langle s\rangle^{-\delta}\langle \ell, \ell s\rangle^\gamma} \frac{e^{-(1-c)\lambda(s)\langle k-\ell, kt-\ell s\rangle^\gamma}}{\langle k-\ell, kt-\ell s\rangle^\sigma} \, dt \right) \\ &\leq \sup_{\ell,s} \left(\sum_{k \in \mathbb{Z}_*^d} \frac{1}{\langle k-\ell\rangle^\sigma} \int_0^{\frac{s}{2}} \langle s\rangle e^{-C\langle s\rangle^{\gamma-\delta}} \, dt \right) \\ &\leq C \sup_{\ell,s} \left(\sum_{k \in \mathbb{Z}_*^d} \frac{1}{\langle k-\ell\rangle^\sigma} \right) \leq C.\end{aligned}$$

Then, let us treat the case $\frac{s}{2} \leq t \leq s$. If $k = \ell$, we have

$$\begin{aligned}\sup_{\ell,s} \left(\int_{\frac{s}{2}}^{s} \mathcal{K}(t,s,\ell,\ell) \, dt \right) &\leq \sup_{\ell,s} \left(\int_{\frac{s}{2}}^{s} (s-t) e^{-(\lambda(s)-\lambda(t))\langle \ell, \ell t\rangle^\gamma} \frac{e^{-(1-c)\lambda(s)\langle \ell(t-s)\rangle^\gamma}}{\langle \ell(t-s)\rangle^{\sigma+1}} \, dt \right) \\ &\leq \sup_{\ell,s} \left(\int_{\frac{s}{2}}^{s} e^{-C\langle t-s\rangle^\gamma} \, dt \right) \leq C,\end{aligned}$$

where we used that $(1-c)\lambda(s) \geq C$ for some constant $C > 0$. Therefore, we only have to consider the sum over k when $k \neq \ell$. Similarly to (2.33) we have using (2.23), the Gevrey inequality in (2.14), and following the argument [9, p. 62]:

$$\mathcal{I}_4 := \sup_{\ell,s}\Bigg(\sum_{k\in\mathbb{Z}^d_*}\int_{\frac{s}{2}}^{s}\mathcal{K}(t,s,k,\ell)\,dt\Bigg) \tag{2.35}$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{k\in\mathbb{Z}^d_*}\sum_{j:k_j\neq\ell_j}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle}{|\ell|}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\times\prod_{i\neq j}^{d}e^{-C\langle k_i-\ell_i\rangle^\gamma}\mathbb{1}_{\ell\neq k}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle}{\langle \ell_j\rangle}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg).$$

In this way, we can reduce our analysis to the one-dimensional case as before. Note that again, if either k_j or ℓ_j is zero, we get exponential decay in the kernel as in the previous case; hence, we can assume that both k_j and ℓ_j are different from zero.

Similarly to $\mathcal{I}_2$, we split the support of the integral between $|k_jt-\ell_js|\leq\frac{1}{100}\langle s\rangle$ and $|k_jt-\ell_js|\geq\frac{1}{100}\langle s\rangle$. We first treat the latter case. Starting from (2.35), we have

$$C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle}{\langle \ell_j\rangle}e^{-(\lambda(s)-\lambda(t))\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\int_{\frac{s}{2}}^{s}\langle s\rangle\frac{e^{-C|k_jt-\ell_js|^\gamma}}{\langle k_j-\ell_j\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\frac{1}{\langle k_j-\ell_j\rangle^\sigma}\int_{\frac{s}{2}}^{s}\langle s\rangle e^{-\frac{C}{100}\langle s\rangle^\gamma}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\frac{1}{\langle k_j-\ell_j\rangle^\sigma}\langle s\rangle^2e^{-\frac{C}{100}\langle s\rangle^\gamma}\Bigg)\leq C.$$

Let us now estimate the case $|k_jt-\ell_js|\leq\frac{1}{100}\langle s\rangle$. By the same argument as for $\mathcal{I}_2$, we only have to consider the case $|k_j|>|\ell_j|$. Hence, by using (2.34) in (2.35), we obtain for some $\alpha>0$,

$$\mathcal{I}_4\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle}{|\ell_j|}e^{-\langle t\rangle^{1-\delta}\langle s\rangle^{-1}|\ell_j|^{-1}\langle k_j,k_jt\rangle^\gamma}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle}{|\ell_j|}\frac{|\ell_j|^\alpha\langle s\rangle^\alpha}{\langle t\rangle^{(1-\delta)\alpha}\langle k_j,k_jt\rangle^{\gamma\alpha}}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg)$$

$$\leq C\sup_{\ell,s}\Bigg(\sum_{j=1}^{d}\sum_{k_j\in\mathbb{Z}_*}\int_{\frac{s}{2}}^{s}\frac{\langle s\rangle^{1-(1-\delta)\alpha-\gamma\alpha+\alpha}}{|\ell_j|^{1-\alpha+\gamma\alpha}}\frac{e^{-C\langle k_j-\ell_j,k_jt-\ell_js\rangle^\gamma}}{\langle k_j-\ell_j,k_jt-\ell_js\rangle^\sigma}\mathbb{1}_{\ell_j\neq k_j}\,dt\Bigg),$$

where we used $\langle k_j, k_j t\rangle \ge |k_j t|$, $t \ge \frac{s}{2}$, and $|k_j| > |\ell_j|$. Then we choose α such that $1-(1-\delta)\alpha - \gamma\alpha + \alpha = 0$, i.e., $\alpha = (\gamma-\delta)^{-1}$, and we do the change of variable $\tau = k_j t$. Therefore,

$$\mathcal{I}_4 \le C \sup_{\ell,s} \Bigg(\sum_{j=1}^{d} \sum_{k_j \in \mathbb{Z}_*} \int_0^{+\infty} |\ell_j|^{\alpha-\gamma\alpha-1} \frac{1}{|k_j|} \frac{e^{-C\langle k_j-\ell_j, \tau-\ell_j s\rangle^\gamma}}{\langle k_j-\ell_j, \tau-\ell_j s\rangle^\sigma} \, d\tau \Bigg)$$
$$\le C \sup_{\ell,s} \Bigg(\sum_{j=1}^{d} \sum_{k_j \in \mathbb{Z}_*} \int_0^{+\infty} |\ell_j|^{\alpha-\gamma\alpha-1} \frac{1}{|\ell_j|} \frac{e^{-C\langle k_j-\ell_j, \tau-\ell_j s\rangle^\gamma}}{\langle k_j-\ell_j, \tau-\ell_j s\rangle^\sigma} \, d\tau \Bigg).$$

In order to be able to sum over k uniformly in ℓ, we need $\alpha - 1 - \gamma\alpha < 1$. Therefore,

$$\frac{1}{\gamma-\delta} - 1 - \frac{\gamma}{\gamma-\delta} < 1 \quad \Longrightarrow \quad \frac{1+2\delta}{3} < \gamma.$$

Finally,

$$\mathcal{I}_4 \le C \sup_{\ell,s} \Bigg(\sum_{j=1}^{d} \sum_{k_j \in \mathbb{Z}_*} \int_0^{+\infty} \frac{e^{-C\langle k_j-\ell_j, \tau-\ell_j s\rangle^\gamma}}{\langle k_j-\ell_j, \tau-\ell_j s\rangle^\sigma} \, d\tau \Bigg)$$
$$\le C \sup_{\ell,s} \Bigg(\sum_{j=1}^{d} \sum_{k_j \in \mathbb{Z}_*} \langle k_j-\ell_j\rangle^{-\sigma} \int_0^{+\infty} e^{-C\langle \tau-\ell_j s\rangle^\gamma} \, d\tau \Bigg) \le C,$$

which concludes the analysis of the second kernel.

Non-linear estimate on T_{HH}. Finally, we estimate the last term of (2.21),

$$T_{\mathrm{HH}} := \langle t\rangle^b A_t(k,kt) \sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s| \approx |k-\ell, kt-\ell s|} \frac{\ell \cdot k}{|\ell|^2} (s-t) \hat{\varrho}_s(\ell) \hat{g}_s(k-\ell, kt-\ell s) \, ds.$$

In this region, we have $\frac{1}{2}|\ell,\ell s| \le |k-\ell, kt-\ell s| \le 2|\ell,\ell s|$. Therefore, we can apply inequality (2.17),

$$\langle k,kt\rangle^\gamma = \langle k-\ell+\ell, kt-\ell s+\ell s\rangle^\gamma \le c(\langle k-\ell, kt-\ell s\rangle^\gamma + \langle \ell,\ell s\rangle^\gamma),$$

where $c = c(\gamma) \in (0,1)$. This allows us to bound

$$A_t(k,kt) \le \langle k,kt\rangle^\sigma e^{c\lambda(t)\langle k-\ell, kt-\ell s\rangle^\gamma} e^{c\lambda(t)\langle \ell,\ell t\rangle^\gamma}$$
$$\le C\langle k-\ell, kt-\ell s\rangle^\sigma \langle \ell,\ell s\rangle^\sigma e^{c\lambda(s)\langle k-\ell, kt-\ell s\rangle^\gamma} e^{c\lambda(s)\langle \ell,\ell s\rangle^\gamma},$$

where we used $s > t$ and $\lambda(t)$ is an increasing function. Hence,

$$|T_{\mathrm{HH}}| \le C \sum_{\ell \in \mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s| \approx |k-\ell, kt-\ell s|} e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma} e^{-(1-c)\lambda(s)\langle k-\ell, \eta-\ell s\rangle^\gamma}$$
$$\times \langle s\rangle^b \frac{|\ell \cdot k|}{\beta+|\ell|^2} |s-t| |\widehat{A\varrho}_s(\ell)| |\widehat{Ag}_s(k-\ell, kt-\ell s)| \, ds.$$

Using (2.23) we can bound $\frac{|\ell\cdot k|}{\beta+|\ell|^2}|s-t| \le \langle s\rangle\langle k-\ell,\eta-\ell s\rangle$. Therefore,

$$|T_{\mathrm{HH}}| \le C \sum_{\ell\in\mathbb{Z}^d_*} \int_t^{+\infty} \mathbb{1}_{|\ell,\ell s|\approx|k-\ell,kt-\ell s|} e^{-(1-c)\lambda(s)\langle \ell,\ell s\rangle^\gamma} e^{-(1-c)\lambda(s)\langle k-\ell,\eta-\ell s\rangle^\gamma} \times \langle s\rangle^{b+1}|\widehat{A\varrho_s}(\ell)||\widehat{Bg_s}(k-\ell,kt-\ell s)|\,ds.$$

By taking the $L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)$ norm and proceeding as for the term T_{LH} after estimate (2.24), we get the final estimate thanks to the two exponential decays in the integral. That is,

$$\|T_{\mathrm{HH}}\|^2_{L^2_tL^2_x} \le C\|\langle t\rangle^b A_t(\nabla_x,t\nabla_x)\varrho_t\|^2_{L^2_tL^2_x} \sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|^2_{L^2_{x,v}}. \tag{2.36}$$

Hence, by collecting (2.18), (2.19), (2.26), (2.29), and (2.36) we get the result of Proposition 2.7. ■

3. A priori estimates on the distribution function g

In this section, we give a priori estimates for the distribution function g. Inserting the expression for the modified density $\hat{\varrho}_t(k) = (|k|^2+\beta)\hat{U}_t(k)$ in (2.3), we get the following equation for $\hat{g}_t(k,\eta)$:

$$\partial_t\hat{g}_t(k,\eta) = -\frac{[(\eta-kt)\cdot k]}{|k|^2+\beta}\hat{\varrho}_t(k)\hat{\mu}(\eta-kt) - \sum_{\ell\in\mathbb{Z}^d}\frac{[(\eta-kt)\cdot\ell]}{|\ell|^2+\beta}\hat{\varrho}_t(\ell)\hat{g}_t(k-\ell,\eta-\ell t).$$

We consider energy estimates by using the L^2 norm introduced in (1.9). Recalling the notation $B_t(k,\eta) = e^{\lambda(t)\langle k,\eta\rangle^\gamma}\langle k,\eta\rangle^{\sigma+1}$, note that for any multi-index $j\in\mathbb{N}^d$ and $\lambda(t)\in[0,\lambda]$, there exists a constant $C = C(|j|,\lambda)$ such that

$$|\partial^j_\eta B_t(k,\eta)| \le C\frac{1}{\langle k,\eta\rangle^{|j|(1-\gamma)}}B_t(k,\eta).$$

Thanks to the previous inequality, we get the bounds

$$\begin{aligned}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}} &\le C\sum_{|j|\le M}\|B_t(k,\eta)\partial^j_\eta\hat{g}_t\|_{L^2_{k,\eta}}\\ &\le C\sum_{|j|\le M}\|B_t(\nabla_x,\nabla_v)(v^jg_t)\|_{L^2_{x,v}}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\sum_{|j|\le M}\|B_t(\nabla_x,\nabla_v)(v^jg_t)\|_{L^2_{x,v}} &\le \sum_{|j|\le M}\|\partial^j_\eta(B_t(k,\eta)\hat{g}_t)\|_{L^2_{k,\eta}}\\ &\le C\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}}.\end{aligned} \tag{3.2}$$

Therefore,

$$\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}} \approx \sum_{|j|\le M} \|B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}}, \tag{3.3}$$

and it suffices to get a priori estimates on the second norm.

Proposition 3.1 (A priori estimates on the distribution function). *Let g be the solution of the Vlasov equation (1.14). Let μ be an analytic homogeneous equilibrium satisfying assumptions (H1)–(H3) with $\lambda > 0$ as in (1.10) and let g_∞ be a Gevrey function of mean zero satisfying (1.12). Then for $\sigma > d+5$, $M > \frac{d}{2}$, it holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{0\le|j|\le M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}\\
&\quad+(C\langle t\rangle^{-2}\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}-\dot\lambda(t))\sum_{0\le|j|\le M}\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}}\\
&\ge -C\langle t\rangle^2\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\sum_{0\le|j|\le M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}\\
&\quad-C\langle t\rangle\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\sum_{0\le|j|\le M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}},
\end{aligned}\tag{3.4}$$

for some constant $C > 0$.

The following two lemmas are used several times in the proof of Proposition 3.1.

Lemma 3.2 ([9, Lemma 3.1]). *The two inequalities listed below hold:*

(i) *Let $\phi,\varphi\in L^2(\mathbb{Z}^d\times\mathbb{R}^d)$ and $\langle k\rangle^\theta\psi\in L^2(\mathbb{Z}^d)$ for $\theta > \frac{d}{2}$. Then, for any $t\in\mathbb{R}$, we have*

$$\left|\sum_{k,\ell\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\phi(k,\eta)\psi(\ell)\varphi(k-\ell,\eta-\ell t)\,d\eta\right| \le C\|\phi\|_{L^2_{k,\eta}}\|\langle k\rangle^\theta\psi\|_{L^2_k}\|\varphi\|_{L^2_{k,\eta}}. \tag{3.5}$$

(ii) *Let $\phi,\langle k\rangle^\theta\varphi\in L^2(\mathbb{Z}^d\times\mathbb{R}^d)$ and $\psi\in L^2(\mathbb{Z}^d)$ for $\theta > \frac{d}{2}$. Then, for any $t\in\mathbb{R}$, we have*

$$\left|\sum_{k,\ell\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\phi(k,\eta)\psi(\ell)\varphi(k-\ell,\eta-\ell t)\,d\eta\right| \le C\|\phi\|_{L^2_{k,\eta}}\|\psi\|_{L^2_k}\|\langle k\rangle^\theta\varphi\|_{L^2_{k,\eta}}. \tag{3.6}$$

The proof of Lemma 3.2 follows by applying the Cauchy–Schwarz and Young inequalities; see e.g. [9, Lemma 3.1]. By (2.15) and (2.16), the following result also holds.

Lemma 3.3. *Let $x, y \ge 0$ and $\gamma\in(0,1)$ such that $|x-y| < \frac{x}{2}$. Then there exists a $c\in(0,1)$ such that*

$$|e^{\lambda x^\gamma}-e^{\lambda y^\gamma}| \le \lambda C\frac{|x-y|}{x^{1-\gamma}+y^{1-\gamma}}e^{c\lambda|x-y|^\gamma}e^{\lambda x^\gamma}.$$

Proof of Proposition 3.1. We compute the time derivative of the $L^2_{x,v}$ norm in (3.3). By the product rule, we get,

$$\frac{1}{2}\frac{d}{dt}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} = \dot{\lambda}(t)\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}} + \sum_{k\in\mathbb{Z}^d}\Re\int_{\mathbb{R}^d}\overline{B_t(k,\eta)\partial^j_\eta\hat{g}_t(k,\eta)}B_t(k,\eta)\partial_t\partial^j_\eta\hat{g}_t(k,\eta)\,d\eta.$$

Recall that the Fourier transform of the Vlasov equation (1.14) is given by

$$\partial_t\hat{g}_t(k,\eta) = -\widehat{E}_t(k)\cdot\widehat{\nabla_v\mu}(\eta-kt) - i\sum_{\ell\in\mathbb{Z}^d}(\eta-kt)\cdot\widehat{E}_t(\ell)\hat{g}_t(k-\ell,\eta-\ell t).$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} - \dot{\lambda}(t)\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}} = \mathcal{T}_{\mathrm{L}} + \mathcal{T}_{\mathrm{NL}}, \tag{3.7}$$

where

$$\mathcal{T}_{\mathrm{L}} := -\sum_{k\in\mathbb{Z}^d}\Re\int_{\mathbb{R}^d}\overline{B_t(k,\eta)\partial^j_\eta\hat{g}_t(k,\eta)}B_t(k,\eta)\widehat{E}_t(k)\cdot\partial^j_\eta\widehat{\nabla_v\mu}(\eta-kt)\,d\eta, \tag{3.8}$$

$$\mathcal{T}_{\mathrm{NL}} := -\sum_{k\in\mathbb{Z}^d}\Re i\int_{\mathbb{R}^d}\overline{B_t(k,\eta)\partial^j_\eta\hat{g}_t(k,\eta)}B_t(k,\eta) \times\partial^j_\eta\Bigg(\sum_{\ell\in\mathbb{Z}^d}(\eta-kt)\cdot\widehat{E}_t(\ell)\hat{g}_t(k-\ell,\eta-\ell t)\Bigg)\,d\eta. \tag{3.9}$$

Using the bound

$$\begin{aligned}|\mathcal{T}_{\mathrm{L}}+\mathcal{T}_{\mathrm{NL}}| \le{}& C\langle t\rangle\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}}\\ &+ C\langle t\rangle^{-2}\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}\\ &+ C\langle t\rangle\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}\end{aligned} \tag{3.10}$$

in (3.7), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} - \dot{\lambda}(t)\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}}\\ &\quad\ge -|\mathcal{T}_{\mathrm{L}}+\mathcal{T}_{\mathrm{NL}}|\\ &\quad\ge -C\langle t\rangle\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}}\\ &\qquad - C\langle t\rangle^{-2}\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}\\ &\qquad - C\langle t\rangle\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}.\end{aligned}$$

By summing over j, this shows estimate (3.4) of Proposition 3.1 assuming that we have (3.10). So let us prove that estimate (3.10) holds.

Estimate on $\mathcal{T}_{\mathrm{L}}$: Linear term. We start by studying the linear term (3.8) involving the homogeneous equilibrium μ. Using that $B_t(k,\eta) \le B_t(\eta - kt)B_t(k,kt)$, where $B_t(\eta) = e^{\lambda(t)\langle\eta\rangle}\langle\eta\rangle^{\sigma+1}$, and recalling the relation $\widehat{E}_t(k) = -ik\widehat{U}_t(k)$ and the definition of the modified density $\hat{\varrho}_t(k) = (|k|^2+\beta)\widehat{U}_t(k)$, we obtain

$$|\mathcal{T}_L| \le \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d} B_t(k,kt)\frac{|k|}{|k|^2+\beta}|\hat{\varrho}_t(k)|B_t(\eta-kt)|\partial_\eta^j\widehat{\nabla_v\mu}(\eta-kt)| \times B_t(k,\eta)|\partial_\eta^j\hat{g}_t(k,\eta)|\,d\eta$$
$$\le \langle t\rangle\sum_{k\in\mathbb{Z}^d_*}|\widehat{A\varrho}_t(k)|\int_{\mathbb{R}^d} B_t(\eta-kt)|\partial_\eta^j\widehat{\nabla_v\mu}(\eta-kt)|B_t(k,\eta)|\partial_\eta^j\hat{g}_t(k,\eta)|\,d\eta,$$

where in the last inequality, we used the fact that $|k|^{-1}B_t(k,kt) \le \langle t\rangle A_t(k,kt)$, for $k \neq 0$. By applying Cauchy–Schwarz in η we obtain

$$|\mathcal{T}_L| \le \langle t\rangle\sum_{k\in\mathbb{Z}^d_*}|\widehat{A\varrho}_t(k)|\left(\int_{\mathbb{R}^d} B_t^2(\eta-kt)|\partial_\eta^j\widehat{\nabla_v\mu}(\eta-kt)|^2\,d\eta\right)^{\frac{1}{2}} \times\left(\int_{\mathbb{R}^d} B_t^2(k,\eta)|\partial_\eta^j\hat{g}_t(k,\eta)|^2\,d\eta\right)^{\frac{1}{2}}.$$

Using the assumption (1.10) on the equilibrium, we can bound the first integral as follows:

$$\int_{\mathbb{R}^d} B_t^2(\eta-kt)|\partial_\eta^j\widehat{\nabla_v\mu}(\eta-kt)|^2\,d\eta = \int_{\mathbb{R}^d} B_t^2(\eta-kt)|\partial_\eta^j((\eta-kt)\hat{\mu}(\eta-kt))|^2\,d\eta$$
$$\le C\int_{\mathbb{R}^d} B_t^2(\eta-kt)\langle\eta-kt\rangle^2(|\partial_\eta^j\hat{\mu}(\eta-kt)|^2+|\partial_\eta^{j-1}\hat{\mu}(\eta-kt)|^2)\,d\eta$$
$$\le C\left(\sum_{|p|\le M}\|e^{\lambda\langle\eta\rangle}|\partial_\eta^p\hat{\mu}(\eta)|\|^2_{L^\infty_\eta}\right)\int_{\mathbb{R}^d} B_t^2(\eta-kt)\langle\eta-kt\rangle^2 e^{-2\lambda\langle\eta-kt\rangle}\,d\eta$$
$$\le C(\lambda,\sigma)\left(\sum_{|p|\le M}\|e^{\lambda\langle\eta\rangle}|\partial_\eta^p\hat{\mu}(\eta)|\|^2_{L^\infty_\eta}\right),$$

where we recall that $\lambda > \lambda(t)$. Hence, applying Cauchy–Schwarz in k we obtain

$$|\mathcal{T}_L| \le C\langle t\rangle\left(\sum_{|p|\le M}\|e^{\lambda\langle\eta\rangle}|\partial_\eta^p\hat{\mu}(\eta)|\|_{L^\infty_\eta}\right)\left(\sum_{k\in\mathbb{Z}^d}|\widehat{A\varrho}_t(k)|^2\right)^{\frac{1}{2}} \times\left(\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d} B_t^2(k,\eta)|\partial_\eta^j\hat{g}_t(k,\eta)|^2\,d\eta\right)^{\frac{1}{2}}$$
$$= C\langle t\rangle\left(\sum_{|p|\le M}\|e^{\lambda\langle\eta\rangle}|\partial_\eta^p\hat{\mu}(\eta)|\|_{L^\infty_\eta}\right)\|A_t(\nabla_x,t\nabla_x)\varrho_t\|_{L^2_x}\|B_t(\nabla_x,\nabla_v)(v^jg_t)\|_{L^2_{x,v}}. \tag{3.11}$$

Estimate on $\mathcal{T}_{\mathrm{NL}}$: Non-linear term. We now treat the non-linear term $\mathcal{T}_{\mathrm{NL}}$, (3.9). Introducing commutators in the η-derivatives, we get $\mathcal{T}_{\mathrm{NL}} = \mathcal{T}_{\mathrm{NL}}^{(1)} + \mathcal{T}_{\mathrm{NL}}^{(2)}$, where

$$\mathcal{T}_{\mathrm{NL}}^{(1)} := -\sum_{k,\ell\in\mathbb{Z}^d} \Re i \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)} B_t(k,\eta)(\eta-kt)\cdot \widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\,d\eta,$$

$$\begin{aligned}\mathcal{T}_{\mathrm{NL}}^{(2)} := &-\sum_{k,\ell\in\mathbb{Z}^d} \Re i \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)} B_t(k,\eta)\partial_\eta^j((\eta-kt)\cdot \widehat{E}_t(\ell)\hat{g}_t(k-\ell,\eta-\ell t))\\ &+\sum_{k,\ell\in\mathbb{Z}^d} \Re i \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)} B_t(k,\eta)(\eta-kt)\cdot \widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\,d\eta.\end{aligned}$$

Non-linear estimate on $\mathcal{T}_{\mathrm{NL}}^{(1)}$. We introduce a commutator in the weight $B_t(k,\eta)$ which reflects the Hamiltonian structure of the equation:

$$\begin{aligned}\mathcal{T}_{\mathrm{NL}}^{(1)} = &-\Re i \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)}(B_t(k,\eta) - B_t(k-\ell,\eta-\ell t))\\ &\qquad\times(\eta-kt)\cdot \widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\,d\eta\\ &-\Re i \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)} B_t(k-\ell,\eta-\ell t)\\ &\qquad\times(\eta-kt)\cdot \widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\,d\eta.\end{aligned}\tag{3.12}$$

We can show that the last term in (3.12) is zero: indeed by the change of variables $\ell\to-\ell$ and $k\to k-\ell$, we get

$$\begin{aligned}&-\Re \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)} B_t(k-\ell,\eta-\ell t)(\eta-kt)\cdot i\,\widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\,d\eta\\ &\quad= -\Re \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{B_t(k-\ell,\eta)\partial_\eta^j \hat{g}_t(k-\ell,\eta)} B_t(k,\eta+\ell t)\\ &\qquad\qquad\times(\eta-(k-\ell)t)\cdot i\,\widehat{E}_t(-\ell)\partial_\eta^j \hat{g}_t(k,\eta+\ell t)\,d\eta\\ &\quad= \Re \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} B_t(k-\ell,\eta-\ell t)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)\\ &\qquad\qquad\times\overline{B_t(k,\eta)(\eta-kt)\cdot i\,\widehat{E}_t(\ell)\partial_\eta^j \hat{g}_t(k,\eta)}\,d\eta = 0,\end{aligned}$$

where we applied the change of variable $\eta\mapsto\eta-\ell t$ and the fact that $i\,\widehat{E}_t(-\ell) = -\overline{i\,\widehat{E}_t(\ell)}$ for the second equality. We deduce that the expression is zero, noting that $\Re(z_1\overline{z_2}) = \Re(\overline{z_1}z_2)$ for $z_1, z_2\in\mathbb{C}$.

Next, let us treat the first term of (3.12). We use the usual frequency decomposition as in (2.20):

$$\Big|\Re i \sum_{k,\ell\in\mathbb{Z}^d}\int_{\mathbb{R}^d} \overline{B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)}(B_t(k,\eta)-B_t(k-\ell,\eta-\ell t)) \times (\eta-kt)\cdot \widehat{E}_t(\ell)\partial_\eta^j\hat{g}_t(k-\ell,\eta-\ell t)\,d\eta\Big|$$

$$\le \sum_{k,\ell\in\mathbb{Z}^d}\int_{\mathbb{R}^d}(\mathbb{1}_{|\ell,\ell t|\le\frac{1}{2}|k-\ell,\eta-\ell t|}+\mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|}+\mathbb{1}_{|\ell,\ell t|\approx|k-\ell,\eta-\ell t|}) \times|B_t(k,\eta)\partial_\eta^j\hat{g}_t(k,\eta)||B_t(k,\eta)-B_t(k-\ell,\eta-\ell t)||\eta-kt| \times|\widehat{E}_t(\ell)||\partial_\eta^j\hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta$$

$$=: \mathcal{T}_{\mathrm{LH}}+\mathcal{T}_{\mathrm{HL}}+\mathcal{T}_{\mathrm{HH}}.$$

Term $\mathcal{T}_{\mathrm{LH}}$. Since $|\ell,\ell t|\le\frac{1}{2}|k-\ell,\eta-\ell t|$, we get, by the triangle inequality,

$$\langle k,\eta\rangle\le\langle\ell,\ell t\rangle+\langle k-\ell,\eta-\ell t\rangle\le 2\langle k-\ell,\eta-\ell t\rangle.$$

Moreover, we have

$$|\langle k-\ell,\eta-\ell t\rangle-\langle k,\eta\rangle|\le\langle\ell,\ell t\rangle\le\frac{\langle k-\ell,\eta-\ell t\rangle}{2}.$$

As a consequence, we can apply Lemma 3.3 with $x=\langle k-\ell,\eta-\ell t\rangle$ and $y=\langle k,\eta\rangle$ to bound the difference of the weight as follows:

$$\begin{aligned}|B_t(k,\eta)-B_t(k-\ell,\eta-\ell t)|&\\ &=|e^{\lambda(t)\langle k,\eta\rangle^\gamma}\langle k,\eta\rangle^{\sigma+1}-e^{\lambda(t)\langle k-\ell,\eta-\ell t\rangle^\gamma}\langle k-\ell,\eta-\ell t\rangle^{\sigma+1}|\\ &\le C\langle k-\ell,\eta-\ell t\rangle^{\sigma+1}|e^{\lambda(t)\langle k,\eta\rangle^\gamma}-e^{\lambda(t)\langle k-\ell,\eta-\ell t\rangle^\gamma}|\\ &\le C\lambda(t)\langle k-\ell,\eta-\ell t\rangle^{\sigma+1}\frac{|\langle k-\ell,\eta-\ell t\rangle-\langle k,\eta\rangle|}{\langle k-\ell,\eta-\ell t\rangle^{1-\gamma}+\langle k,\eta\rangle^{1-\gamma}}\\ &\quad\times e^{c\lambda(t)|\langle k-\ell,\eta-\ell t\rangle-\langle k,\eta\rangle|^\gamma}e^{\lambda(t)\langle k-\ell,\eta-\ell t\rangle^\gamma},\end{aligned}$$

where $c\in(0,1)$. Then, using that $\lambda(t)\le\lambda$ and $|\langle k-\ell,\eta-\ell t\rangle-\langle k,\eta\rangle|\le\langle\ell,\ell t\rangle$, we obtain

$$|B_t(k,\eta)-B_t(k-\ell,\eta-\ell t)| \le C\lambda\frac{\langle\ell,\ell t\rangle}{\langle k-\ell,\eta-\ell t\rangle^{1-\gamma}+\langle k,\eta\rangle^{1-\gamma}}e^{c\lambda(t)\langle\ell,\ell t\rangle^\gamma}B_t(k-\ell,\eta-\ell t).$$

Hence,

$$\mathcal{T}_{\mathrm{LH}}\le C\sum_{k,\ell\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\mathbb{1}_{|\ell,\ell t|\le\frac{1}{2}|k-\ell,\eta-\ell t|}\frac{\langle\ell,\ell t\rangle}{\langle k-\ell,\eta-\ell t\rangle^{1-\gamma}+\langle k,\eta\rangle^{1-\gamma}}e^{c\lambda(t)\langle\ell,\ell t\rangle^\gamma} \times B_t(k-\ell,\eta-\ell t)|\eta-kt||\widehat{E}_t(\ell)||\partial_\eta^j\hat{g}_t(k-\ell,\eta-\ell t)||B_t(k,\eta)\partial_\eta^j\hat{g}_t(k,\eta)|\,d\eta.$$

Since we have $\langle \ell, \ell t\rangle |\widehat{E}_t(\ell)| \le C\langle t\rangle |\hat{\varrho}_t(\ell)|$, we obtain

$$\begin{aligned}\mathcal{T}_{\mathrm{LH}} \le C\langle t\rangle \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|\ell,\ell t|\le \frac{1}{2}|k-\ell,\eta-\ell t|} \frac{|\eta - kt|}{\langle k-\ell, \eta-\ell t\rangle^{1-\gamma} + \langle k,\eta\rangle^{1-\gamma}} e^{c\lambda(t)\langle \ell,\ell t\rangle^\gamma} \\ \times |\hat{\varrho}_t(\ell)| B_t(k-\ell,\eta-\ell t)|\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)| \\ \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\, d\eta.\end{aligned}$$

Since

$$\frac{|\eta - kt|}{\langle k-\ell, \eta-\ell t\rangle^{1-\gamma} + \langle k,\eta\rangle^{1-\gamma}} \le \langle t\rangle \langle k,\eta\rangle^{\frac{\gamma}{2}} \langle k-\ell,\eta-\ell t\rangle^{\frac{\gamma}{2}},$$

we get

$$\begin{aligned}\mathcal{T}_{\mathrm{LH}} \le C\langle t\rangle^{-2} \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|\ell,\ell t|\le \frac{1}{2}|k-\ell,\eta-\ell t|} e^{-(1-c)\lambda(t)\langle \ell,\ell t\rangle^\gamma} \langle t\rangle^4 \langle \ell,\ell t\rangle^{-\sigma} |\widehat{A\varrho}_t(\ell)| \\ \times \langle k-\ell,\eta-\ell t\rangle^{\frac{\gamma}{2}} |B_t(k-\ell,\eta-\ell t)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)| \\ \times \langle k,\eta\rangle^{\frac{\gamma}{2}} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\, d\eta.\end{aligned}$$

Hence, by (3.5) with $\theta = \frac{d}{2} + 1$, we get

$$\begin{aligned}\mathcal{T}_{\mathrm{LH}} &\le C\langle t\rangle^{-2} \|\langle \nabla_x\rangle^{-\sigma+\frac{d}{2}+5} A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|\langle \nabla_x,\nabla_v\rangle^{\gamma/2} B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} \\ &\le C\langle t\rangle^{-2} \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|\langle \nabla_x,\nabla_v\rangle^{\gamma/2} B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}, \qquad (3.13)\end{aligned}$$

thanks to the assumption $\sigma > 5 + \frac{d}{2}$.

Term $\mathcal{T}_{\mathrm{HL}}$. In this case, we do not need the difference inside the commutator, and we take the absolute values:

$$\begin{aligned}\mathcal{T}_{\mathrm{HL}} \le \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le \frac{1}{2}|\ell,\ell t|} (B_t(k,\eta) + B_t(k-\ell,\eta-\ell t)) \\ \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)||\eta - kt||\widehat{E}_t(\ell)||\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta \\ =: \mathcal{T}_{\mathrm{HL}}^{(1)} + \mathcal{T}_{\mathrm{HL}}^{(2)}.\end{aligned}$$

We estimate the first term as follows:

$$\begin{aligned}\mathcal{T}_{\mathrm{HL}}^{(1)} &:= \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le \frac{1}{2}|\ell,\ell t|} B_t(k,\eta)|B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)| \\ &\qquad \times |\eta - kt||\widehat{E}_t(\ell)||\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta \\ &\le C \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le \frac{1}{2}|\ell,\ell t|} (\langle \ell,\ell t\rangle^{\sigma+1} + \langle k-\ell,\eta-\ell t\rangle^{\sigma+1}) \\ &\qquad \times e^{\langle \ell,\ell t\rangle^\gamma} e^{\langle k-\ell,\eta-\ell t\rangle^\gamma} \\ &\qquad \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)||\eta - kt||\widehat{E}_t(\ell)||\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta,\end{aligned}$$

where we used the triangle inequality for $B_t(k,\eta)$. For $\langle \ell, \ell t\rangle^{\sigma+1}$, we use $\langle \ell, \ell t\rangle |\widehat{E}_t(\ell)| \le C\langle t\rangle |\hat{\varrho}_t(\ell)|$ and

$$|\eta - kt| \le |\eta - \ell t| + |\ell - k|\langle t\rangle \le \langle t\rangle\langle k-\ell, \eta-\ell t\rangle. \tag{3.14}$$

For $\langle k-\ell, \eta-\ell t\rangle^{\sigma+1}$, we use the same bound for the electric field and $|\eta - kt| \le \langle t\rangle\langle \ell, \ell t\rangle$ thanks to (3.14) and the region of the indicator function. Therefore,

$$\begin{aligned}\mathcal{T}_{\mathrm{HL}}^{(1)} \le C\langle t\rangle^2 \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\,|\widehat{A\varrho_t}(\ell)| \\ \times \langle k-\ell,\eta-\ell t\rangle e^{\langle k-\ell,\eta-\ell t\rangle^\gamma} |\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta \\ + C\langle t\rangle^2 \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\langle \ell,\ell t\rangle e^{\langle \ell,\ell t\rangle^\gamma} \\ \times |\hat{\varrho}_t(\ell)|\,|B_t(k-\ell,\eta-\ell t)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta.\end{aligned}$$

By using (3.6) with $\theta = \sigma$, we can bound the first term allowing extra spatial moment on $|\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|$, and for the second term we use (3.5) with $\theta = \sigma - 1$ allowing extra spatial moment on $|\hat{\varrho}_t(\ell)|$. Hence,

$$\mathcal{T}_{\mathrm{HL}}^{(1)} \le C\langle t\rangle^2 \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}. \tag{3.15}$$

The second term is easier to treat. We have

$$\begin{aligned}\mathcal{T}_{\mathrm{HL}}^{(2)} := \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\,|\eta - kt| \\ \times |\widehat{E}_t(\ell)|\,|B_t(k-\ell,\eta-\ell t)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta.\end{aligned}$$

Note that we have again $|\eta - kt| \le \langle t\rangle\langle \ell,\ell t\rangle$ thanks to (3.14) and the region of the indicator function. Therefore,

$$|\eta - kt|\,|\widehat{E}_t(\ell)| \le \langle t\rangle\langle \ell,\ell t\rangle |\widehat{E}_t(\ell)| \le C\langle t\rangle^2 |\hat{\varrho}_t(\ell)|.$$

Thus,

$$\begin{aligned}\mathcal{T}_{\mathrm{HL}}^{(2)} \le C\langle t\rangle^2 \sum_{k,\ell\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)|\,|\hat{\varrho}_t(\ell)| \\ \times |B_t(k-\ell,\eta-\ell t)\partial_\eta^j \hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta.\end{aligned}$$

Hence, by applying (3.5) with $\theta = \sigma$, we get

$$\mathcal{T}_{\mathrm{HL}}^{(2)} \le C\langle t\rangle^2 \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}. \tag{3.16}$$

Term $\mathcal{T}_{\mathrm{HH}}$. This one is estimated as $\mathcal{T}_{\mathrm{HL}}$. Note that in this region, we have $\frac{1}{2}|\ell, \ell t| \leq |k-\ell, \eta-\ell t| \leq 2|\ell, \ell t|$. Therefore, (3.14) also holds with a factor 2. That is,

$$|\eta - kt| \leq \langle t\rangle\langle k-\ell, \eta-\ell t\rangle \leq 2\langle t\rangle\langle \ell, \ell t\rangle.$$

Then, we can repeat the same analysis by splitting $\mathcal{T}_{\mathrm{HH}}$ into $\mathcal{T}_{\mathrm{HH}}^{(1)}$ and $\mathcal{T}_{\mathrm{HH}}^{(2)}$ as we did with $\mathcal{T}_{\mathrm{HL}}$. Hence,

$$\mathcal{T}_{\mathrm{HH}} \leq C\langle t\rangle^2 \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|B_t(\nabla_x, \nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}. \tag{3.17}$$

Non-linear estimate on $\mathcal{T}_{\mathrm{NL}}^{(2)}$. With the same techniques, we can treat the commutator term in the η-derivatives. In this case we have

$$\begin{aligned}
|\mathcal{T}_{\mathrm{NL}}^{(2)}| &\leq C \sum_{k,\ell\in\mathbb{Z}^d} \sum_{0\leq|p|<|j|} \int_{\mathbb{R}^d} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta) B_t(k,\eta)\widehat{E}_t(\ell)\partial_\eta^p \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta \\
&\leq C \sum_{k,\ell\in\mathbb{Z}^d} \sum_{0\leq|p|<|j|} \int_{\mathbb{R}^d} (\mathbb{1}_{|\ell,\ell t|\leq\frac{1}{2}|k-\ell,\eta-\ell t|} + \mathbb{1}_{|k-\ell,\eta-\ell t|\leq\frac{1}{2}|\ell,\ell t|} + \mathbb{1}_{|\ell,\ell t|\approx|k-\ell,\eta-\ell t|}) \\
&\qquad\qquad \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta) B_t(k,\eta)\widehat{E}_t(\ell)\partial_\eta^p \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta.
\end{aligned}$$

We bound the first term with $\mathbb{1}_{|\ell,\ell t|\leq\frac{1}{2}|k-\ell,\eta-\ell t|}$. By using the triangle inequality,

$$\langle k,\eta\rangle \leq \langle \ell,\ell t\rangle + \langle k-\ell,\eta-\ell t\rangle \leq 2\langle k-\ell,\eta-\ell t\rangle$$

and

$$B_t(k,\eta) \leq C\langle k-\ell,\eta-\ell t\rangle^{\sigma+1} e^{\lambda(t)\langle k-\ell,\eta-\ell t\rangle^\gamma} e^{\lambda(t)\langle \ell,\ell t\rangle^\gamma}.$$

Therefore, using $|\widehat{E}_t(\ell)| \leq |\hat{\varrho}_t(\ell)|$, we get

$$\begin{aligned}
&C \sum_{k,\ell\in\mathbb{Z}^d} \sum_{0\leq|p|<|j|} \int_{\mathbb{R}^d} \mathbb{1}_{|\ell,\ell t|\leq\frac{1}{2}|k-\ell,\eta-\ell t|} \\
&\qquad\qquad \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta) B_t(k,\eta)\widehat{E}_t(\ell)\partial_\eta^p \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta \\
&\quad\leq C \sum_{k,\ell\in\mathbb{Z}^d} \sum_{0\leq|p|<|j|} \int_{\mathbb{R}^d} \mathbb{1}_{|\ell,\ell t|\leq\frac{1}{2}|k-\ell,\eta-\ell t|} |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta)| e^{\lambda(t)\langle \ell,\ell t\rangle^\gamma} \\
&\qquad\qquad \times |\hat{\varrho}_t(\ell)| |B_t(k-\ell,\eta-\ell t)| |\partial_\eta^p \hat{g}_t(k-\ell,\eta-\ell t)|\, d\eta \\
&\quad\leq C \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}},
\end{aligned} \tag{3.18}$$

where we used (3.5) with $\theta = \sigma$ for the last inequality.

The second term with $\mathbb{1}_{|k-\ell,\eta-\ell t|\leq\frac{1}{2}|\ell,\ell t|}$ can be treated exactly as $\mathcal{T}_{\mathrm{HL}}^{(1)}$ above. Note, however, that in this case, we do not have the factor $|\eta - kt|$ and therefore only have one power of $\langle t\rangle$ and not $\langle t\rangle^2$ as before. That is,

$$C \sum_{k,\ell\in\mathbb{Z}^d} \sum_{0\le|p|<|j|} \int_{\mathbb{R}^d} \mathbb{1}_{|k-\ell,\eta-\ell t|\le\frac{1}{2}|\ell,\ell t|} \times |B_t(k,\eta)\partial_\eta^j \hat{g}_t(k,\eta) B_t(k,\eta)\widehat{E}_t(\ell)\partial_\eta^p \hat{g}_t(k-\ell,\eta-\ell t)|\,d\eta$$
$$\le C\langle t\rangle \|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} \|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}}. \tag{3.19}$$

Using the same argument as before, the term with $\mathbb{1}_{|\ell,\ell t|\approx|k-\ell,\eta-\ell t|}$ is treated similarly.

Hence, by collecting the estimates (3.11), (3.13), (3.15), (3.16), (3.17), (3.18), and (3.19) we get estimate (3.10). ■

4. Construction of solutions: Proof of Theorem 1.1

In this section, we conclude our analysis, constructing a solution to the Vlasov-type system (1.14) with given asymptotic state g_∞ as in (1.12).

Prior to this, we state two useful results to treat the non-linearity inside the Poisson coupling in (1.14).

Lemma 4.1. *Let $h:\mathbb{R}\to\mathbb{R}$ be an analytic function with analyticity radius $R>0$,*

$$h(y) = \sum_{n\ge 2} a_n y^n, \quad \{a_n\}\subset\mathbb{R},$$

and we define $\tilde{h}(y) := \sum_{n\ge2}|a_n|y^n$. Let $\omega:\mathbb{T}^d\to\mathbb{R}$ be a Gevrey function such that

$$\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x} < R,$$

where $A_z(\nabla_x, t\nabla_x)$ is defined in (1.7) with $\sigma > d/2$. Then

$$\|A_z(\nabla_x, t\nabla_x)h(\omega)\|_{L^2_x} \le \tilde{h}(C\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x}),$$

for a constant C.

Proof. Let us first prove the inequality in the case $h(y)=y^2$:

$$\|A_z(\nabla_x, t\nabla_x)\omega^2\|_{L^2_x} \le C\|A_z(\nabla_x, t\nabla_x)\omega\|^2_{L^2_x}. \tag{4.1}$$

By direct computation, we have

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}^d} A_z^2(k,kt)|\widehat{\omega^2}(k)|^2\Big)^{\frac12} &= \Big(\sum_{k\in\mathbb{Z}^d} A_z^2(k,kt)\Big|\sum_{\ell\in\mathbb{Z}^d}\hat{\omega}(k-\ell)\hat{\omega}(\ell)\Big|^2\Big)^{\frac12}\\ &= \Big(\sum_{k\in\mathbb{Z}^d}\Big|\sum_{\ell\in\mathbb{Z}^d} A_z(k-\ell,(k-\ell)t)\hat{\omega}(k-\ell)A_z(\ell,\ell t)\\ &\qquad\times\hat{\omega}(\ell)\frac{A_z(k,kt)}{A_z(\ell,\ell t)A_z(k-\ell,(k-\ell)t)}\Big|^2\Big)^{\frac12}.\end{aligned}$$

We estimate the sum over ℓ by using Cauchy–Schwarz and the following claim:

$$S := \sum_{\ell \in \mathbb{Z}^d} A_z^2(k, kt) A_z^{-2}(\ell, \ell t) A_z^{-2}(k-\ell, (k-\ell)t) \leq C, \tag{4.2}$$

which follows by the inequalities

$$e^{z\langle k,kt\rangle^\gamma} \leq e^{z\langle \ell,\ell t\rangle^\gamma} e^{z\langle k-\ell,(k-\ell)t\rangle^\gamma}, \quad \langle k, kt\rangle^\sigma \leq C(\langle l, lt\rangle^\sigma + \langle k-l, (k-l)t\rangle^\sigma),$$

and that, for $\sigma > \frac{d}{2}$,

$$\begin{aligned} S &\leq \sum_{\ell \in \mathbb{Z}^d} \frac{\langle k, kt\rangle^{2\sigma}}{\langle \ell, \ell t\rangle^{2\sigma} \langle k-\ell, (k-\ell)t\rangle^{2\sigma}} \\ &\leq C \sum_{\ell \in \mathbb{Z}^d} \Big(\frac{1}{\langle \ell, \ell t\rangle^{2\sigma}} + \frac{1}{\langle k-\ell, (k-\ell)t\rangle^{2\sigma}}\Big) < +\infty. \end{aligned}$$

Therefore, by (4.2),

$$\begin{aligned} &\|A_z(\nabla_x, t\nabla_x)\omega^2\|_{L^2_x} \\ &\quad \leq C\Big(\sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d} A_z^2(k-\ell,(k-\ell)t)|\widehat{\omega}(k-\ell)|^2 A_z^2(\ell,\ell t)|\widehat{\omega}(\ell)|^2\Big)^{\frac{1}{2}} \\ &\quad = C\Big(\sum_{\ell\in\mathbb{Z}^d} A_z^2(\ell,\ell t)|\widehat{\omega}(\ell)|^2\Big(\sum_{k\in\mathbb{Z}^d} A_z^2(k-\ell,(k-\ell)t)|\widehat{\omega}(k-\ell)|^2\Big)\Big)^{\frac{1}{2}} \\ &\quad = C\sum_{k\in\mathbb{Z}^d} A_z^2(k,kt)|\widehat{\omega}(k)|^2 = C\|A_z(\nabla_x, t\nabla_x)\omega\|^2_{L^2_x}. \end{aligned}$$

Iterating (4.1) for $h(y) = y^n$ with $n \in \mathbb{N}_{\geq 2}$, we get

$$\|A_z(\nabla_x, t\nabla_x)\omega^n\|_{L^2_x} \leq C^{n-1}\|A_z(\nabla_x, t\nabla_x)\omega\|^n_{L^2_x}. \tag{4.3}$$

Using that $h(y) = \sum_{n\geq 2} a_n y^n$, we conclude that

$$\|A_z(\nabla_x, t\nabla_x)h(\omega)\|_{L^2_x} \leq \sum_{n\geq 2}|a_n|\|A_z(\nabla_x, t\nabla_x)\omega^n\|_{L^2_x} \leq \tilde{h}(C\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x}),$$

where in the last inequality, we used (4.3). Note that the right-hand side in the last formula is well defined thanks to the assumption $\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x} < RC^{-1}$. ∎

To prove Theorem 1.1, we also need a well-posedness result in the Gevrey setting for the non-linear equation

$$-\Delta u(x) + \beta u(x) + h(u) = q(x), \quad x \in \mathbb{R}^d, \ \beta \geq 0,$$

with $q(x)$ a given source term with small Gevrey norm. For our purposes, it is convenient to reformulate the problem in terms of the modified density $\varrho := (\beta - \Delta)u$ and study the equivalent equation

$$\varrho(x) = q(x) - h((\beta - \Delta)^{-1}\varrho). \tag{4.4}$$

We denote by $\mathcal{B}_r$ the closed ball of Gevrey functions ω such that $\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x} \le r$.

Proposition 4.2. *Let $h(y) = \sum_{n\ge 2} a_n y^n$ be an analytic function with analyticity radius $R > 0$ and consider a Gevrey source term $q \in \mathcal{B}_\varepsilon$, for $\varepsilon, z > 0$ and $\sigma > \frac{d}{2}$. If ε is sufficiently small, there exists $r(\varepsilon)$ such that $r(\varepsilon) = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ and so that (4.4) admits a unique Gevrey solution $\varrho \in \mathcal{B}_{r(\varepsilon)}$.*

Proof. For a given $q \in \mathcal{B}_\varepsilon$, we define the map

$$\mathcal{M}_q : \mathcal{B}_r \to \mathcal{B}_r, \quad \omega \mapsto \mathcal{M}_q(\omega) := q(x) - h((\beta - \Delta)^{-1}\omega).$$

This map is well defined if ε and r are sufficiently small. Indeed,

$$\begin{aligned}\|A_z(\nabla_x, t\nabla_x)\mathcal{M}_q(\omega)\|_{L^2_x} &\le \|A_z(\nabla_x, t\nabla_x)q\|_{L^2_x} + \|A_z(\nabla_x, t\nabla_x)h((\beta - \Delta)^{-1}\omega)\|_{L^2_x} \\ &\le \varepsilon + \tilde{h}(C\|A_z(\nabla_x, t\nabla_x)((\beta - \Delta)^{-1}\omega)\|_{L^2_x}),\end{aligned}$$

where we used that $q \in \mathcal{B}_\varepsilon$ and Lemma 4.1 to take h out of the L^2_x norm. For the second term, note that on the Fourier side,

$$|\widehat{((\beta - \Delta)^{-1}\omega)}(k)| = \left|\frac{1}{\beta + |k|^2}\hat{\omega}(k)\right| \le |\hat{\omega}(k)|, \tag{4.5}$$

so we get

$$\|A_z(\nabla_x, t\nabla_x)\mathcal{M}_q(\omega)\|_{L^2_x} \le \varepsilon + \tilde{h}(C\|A_z(\nabla_x, t\nabla_x)\omega\|_{L^2_x}) \le \varepsilon + Cr^2,$$

where we used that $\omega \in \mathcal{B}_r$, and $\tilde{h}(x) = \mathcal{O}(x^2)$ for x small. It follows that,

$$\|A_z(\nabla_x, t\nabla_x)\mathcal{M}_q(\omega)\|_{L^2_x} \le r,$$

provided ε is sufficiently small and $r \in (r_-(\varepsilon), r_+(\varepsilon))$, $r_\pm(\varepsilon) := (2C)^{-1}(1 \pm \sqrt{1 - 4C\varepsilon})$.

We now want to show that $\mathcal{M}_q$ is a contraction with respect to the previously defined Gevrey norm so that, by a fixed point argument, we get the existence of a unique solution of (4.4) inside $\mathcal{B}_r$.

Let $\omega_1, \omega_2 \in \mathcal{B}_r$. Then

$$\begin{aligned}&\|A_z(\nabla_x, t\nabla_x)[\mathcal{M}_q(\omega_1) - \mathcal{M}_q(\omega_2)]\|_{L^2_x} \\ &\quad\le \|A_z(\nabla_x, t\nabla_x)[h((\beta - \Delta)^{-1}\omega_1) - h((\beta - \Delta)^{-1}\omega_2)]\|_{L^2_x} \\ &\quad\le \sup_{|\xi|\le r} |h'(\xi)| \|A_z(\nabla_x, t\nabla_x)[(\beta - \Delta)^{-1}(\omega_1 - \omega_2)]\|_{L^2_x}.\end{aligned}$$

Then, using (4.5) and $h'(x) = \mathcal{O}(x)$ for x small, we have

$$\|A_z(\nabla_x, t\nabla_x)[\mathcal{M}_q(\omega_1) - \mathcal{M}_q(\omega_2)]\|_{L^2_x} \leq Cr\|A_z(\nabla_x, t\nabla_x)(\omega_1 - \omega_2)\|_{L^2_x}.$$

Therefore, for $r(\varepsilon) \in (r_-(\varepsilon), r_+(\varepsilon))$ sufficiently small, there exists a unique $\varrho \in \mathcal{B}_{r(\varepsilon)}$ such that

$$\varrho = q(x) - h((\beta - \Delta)^{-1}\varrho).$$

■

We have now completed the preliminary work required to prove Theorem 1.1. Before starting the proof, in the following proposition we recollect the a priori estimates for the coupled system given by (ϱ_t, g_t) obtained combining Lemma 2.6 and Proposition 2.7 for the modified density ϱ_t and Proposition 3.1 for the distribution function g_t.

Proposition 4.3 (A priori estimates for the coupled system (ϱ_t, g_t)). *Let μ be an analytic homogeneous equilibrium satisfying assumptions (H1)–(H3) with $\lambda > 0$ as in (1.10) and let g_∞ be a Gevrey function of mean zero satisfying (1.12).*

For $\sigma > d + 5$, $b > \frac{5}{2}$, and $M > \frac{d}{2}$, the following a priori estimates hold for the coupled system (ϱ_t, g_t) solution to (1.14) with $\varrho_t = (\beta - \Delta)U_t$:

(1) *A priori estimate for ϱ_t:*

$$\begin{aligned}
&\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \\
&\quad\leq C\|\langle v\rangle^M e^{\lambda\langle\nabla_x,\nabla_v\rangle^\gamma}\langle\nabla_x,\nabla_v\rangle^{\sigma+b} g_\infty\|_{L^2_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)} \\
&\qquad+\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)h(U_t)\|_{L^2_tL^2_x} \\
&\qquad+C\|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_t(\mathbb{R}_+)L^2_x(\mathbb{T}^d)} \\
&\qquad\quad\times\sup_{t\in\mathbb{R}_+}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)g_t\|_{L^2_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)}.
\end{aligned}\tag{4.6}$$

(2) *A priori estimate for g_t:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{0\leq|j|\leq M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} \\
&\quad+(C\langle t\rangle^{-2}\|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x} - \dot{\lambda}(t)) \\
&\qquad\times\sum_{0\leq|j|\leq M}\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2}B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}} \\
&\geq -C\langle t\rangle^2\|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x}\sum_{0\leq|j|\leq M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|^2_{L^2_{x,v}} \\
&\quad-C\langle t\rangle\|A_t(\nabla_x, t\nabla_x)\varrho_t\|_{L^2_x}\sum_{0\leq|j|\leq M}\|B_t(\nabla_x,\nabla_v)(v^j g_t)\|_{L^2_{x,v}},
\end{aligned}\tag{4.7}$$

where $C > 0$ is a generic constant and A_t, B_t are the multipliers defined as in (1.5), (1.6) with the time-dependent parameter $\lambda(t)$ defined in (1.8).

Proof of Theorem 1.1. Let $X := C([0,+\infty); \mathcal{G}^{\bar{\lambda}})$ be the space of continuous-in-time Gevrey functions in the norm given by the supremum in time of (1.9) with Gevrey parameter $\bar{\lambda} < \lambda(0)$.

We consider the closed subset $\mathcal{C}_\varepsilon \subset X$: the set of Gevrey functions ϕ in X such that

$$N[\phi] := N_1[\phi] + N_2[q_\phi] \le C\varepsilon,$$
$$N_1[\phi] := \sup_{t\in[0,+\infty)} \|\langle v\rangle^M B_t(\nabla_x,\nabla_v)\phi_t\|_{L^2_{x,v}}, \quad N_2[q_\phi] := \|\langle t\rangle^b A_t(\nabla_x, t\nabla_x) q_\phi\|_{L^2_t L^2_x},$$

where $q_\phi(t,x) := \int \phi(t, x - tv, v)\,dv$. Note that thanks to the Sobolev embedding on the Fourier side,

$$\left\| A_t(\nabla_x, t\nabla_x) \int \phi(t, x - tv, v) dv \right\|_{L^2_x} \le N_1[\phi] \le C\varepsilon. \tag{4.8}$$

We introduce the map

$$\mathcal{F} : \mathcal{C}_\varepsilon \to \mathcal{C}_\varepsilon, \quad \{\phi_t\}_{t\in[0,+\infty)} \mapsto \mathcal{F}(\{\phi_t\}) := \{\psi_t\}_{t\in[0,+\infty)},$$

where ψ is the unique solution to the linear problem

$$\begin{cases} \partial_t \psi(t,x,v) + E_\phi(t, x+vt)\cdot(\nabla_v - t\nabla_x)\psi(t,x,v) = -E_\psi(t,x+vt)\cdot\nabla_v\mu(v), \\ E_\phi(t,x) = -\nabla u_\phi(t,x), \\ -\Delta u_\phi(t,x) + \beta u_\phi(t,x) + h(u_\phi)(t,x) = q_\phi(t,x), \\ E_\psi(t,x) = -\nabla u_\psi(t,x), \\ -\Delta u_\psi(t,x) + \beta u_\psi(t,x) + h(u_\phi)(t,x) = q_\psi(t,x), \\ \lim_{t\to\infty} \|\psi_t - g_\infty\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)} = 0, \end{cases} \tag{4.9}$$

where g_∞ is a Gevrey function with parameter $\lambda > \lambda(t)$ such that $\int_{\mathbb{T}^d\times\mathbb{R}^d} g_\infty(x,v)\,dx\,dv = 0$ and satisfying (1.12).

The map is well defined, provided that ε is sufficiently small, thanks to Proposition 4.2 and the a priori estimates recalled in Proposition 4.3. Indeed, starting from the a priori estimate (4.7) for the linear system (4.9) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \sum_{0\le|j|\le M} \|B_t(\nabla_x,\nabla_v)(v^j\psi_t)\|^2_{L^2_{x,v}} \\ &\qquad + (C\langle t\rangle^{-2}\|A_t(\nabla_x,t\nabla_x)\varrho_\phi\|_{L^2_x} - \dot{\lambda}(t))\|\langle\nabla_x,\nabla_v\rangle^{\gamma/2} B_t(\nabla_x,\nabla_v)(v^j\psi_t)\|_{L^2_{x,v}} \\ &\quad \ge -C\langle t\rangle^2 \|A_t(\nabla_x,t\nabla_x)\varrho_\phi\|_{L^2_x} \sum_{0\le|j|\le M} \|B_t(\nabla_x,\nabla_v)(v^j\psi_t)\|^2_{L^2_{x,v}} \\ &\qquad - C\langle t\rangle \|A_t(\nabla_x,t\nabla_x)\varrho_\psi\|_{L^2_x} \sum_{0\le|j|\le M} \|B_t(\nabla_x,\nabla_v)(v^j\psi_t)\|_{L^2_{x,v}}, \end{aligned} \tag{4.10}$$

where ϱ_ϕ is the solution of $\varrho_\phi = q_\phi - h((\beta-\Delta)^{-1}\varrho_\phi)$ and $\varrho_\psi := q_\psi - h((\beta-\Delta)^{-1}\varrho_\phi)$. Note that, by Proposition 4.2 and thanks to (4.8), ϱ_ϕ is well defined and that

$$\|A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_x} \le C\varepsilon.$$

On the other hand, the non-linearity in the definition of ϱ_ψ is given by the modified density ϱ_ϕ. This is necessary to treat the non-linearity h as a perturbative term of $O(\varepsilon^2)$, as shown in (4.12) and (4.13). In other words, the contractive map $\mathcal{F}$ is defined in such a way that both non-linearities in the Vlasov-type equation and the Poisson-type equation are replaced by the data generated by ϕ.

Moreover, note that in (4.10) the term $E_\psi(t, x+vt)\cdot\nabla_v\mu(v)$ corresponds to the linear term in the a priori estimate (4.7) and gives the term with the $\langle t\rangle$ in (4.10), while $E_\phi(t, x+vt)\cdot(\nabla_v - t\nabla_x)\psi(t,x,v)$ corresponds to the non-linear term in the a priori estimate (4.7) and gives the terms with $\langle t\rangle^{-2}$ and $\langle t\rangle^2$ in (4.10). Then, by using the equivalence of norm (3.3), and the fact that

$$(C\langle t\rangle^{-2}\|A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_x} - \dot{\lambda}(t)) < 0,$$

for $\delta \ll 1$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)\psi_t\|_{L^2_{x,v}} \ge{}& -C\langle t\rangle^2\|A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_x}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)v^j\psi_t\|_{L^2_{x,v}}\\ &- C\langle t\rangle\|A_t(\nabla_x, t\nabla_x)\varrho_\psi\|_{L^2_x}.\end{aligned}$$

Integrating from t to $+\infty$ and rearranging terms, we get

$$\begin{aligned}\|\langle v\rangle^M B_t(\nabla_x,\nabla_v)\psi_t\|_{L^2_{x,v}} \le{}& \|\langle v\rangle^M B_{\lambda_\infty}(\nabla_x,\nabla_v)g_\infty\|_{L^2_{x,v}}\\ &+ C\int_t^{+\infty}\langle s\rangle^2\|A_s(\nabla_x, s\nabla_x)\varrho_\phi\|_{L^2_x}\|\langle v\rangle^M B_s(\nabla_x,\nabla_v)\psi_s\|_{L^2_{x,v}}\,ds\\ &+ C\int_t^{+\infty}\langle s\rangle\|A_s(\nabla_x, s\nabla_x)\varrho_\psi\|_{L^2_x}\,ds.\end{aligned}$$

Next, by using Cauchy–Schwarz in time with $b > \frac{5}{2}$, we have

$$N_1[\psi] \le C\|g_\infty\|_{\mathcal{G}} + CN_2[\varrho_\phi]N_1[\psi] + CN_2[\varrho_\psi], \tag{4.11}$$

where $\|g_\infty\|_{\mathcal{G}} := \|\langle v\rangle^M e^{\lambda\langle\nabla_x,\nabla_v\rangle^\gamma}\langle\nabla_x,\nabla_v\rangle^{\sigma+b}g_\infty\|_{L^2_{x,v}}$. By inequality (4.6), we obtain

$$N_2[\varrho_\psi] \le C\|g_\infty\|_{\mathcal{G}} + CN_2[h((\beta-\Delta)^{-1}\varrho_\phi)] + CN_2[\varrho_\phi]N_1[\psi]. \tag{4.12}$$

Note, thanks to Lemma 4.1,

$$\begin{aligned}N_2[h((\beta-\Delta)^{-1}\varrho_\phi)] &= \Big\|\langle t\rangle^b\|A_t(\nabla_x, t\nabla_x)h((\beta-\Delta)^{-1}\varrho_\phi)\|_{L^2_x}\Big\|_{L^2_t}\\ &\le \Big\|\langle t\rangle^b\tilde{h}(\|A_t(\nabla_x, t\nabla_x)((\beta-\Delta)^{-1}\varrho_\phi)\|_{L^2_x})\Big\|_{L^2_t}\\ &\le \Big\|\langle t\rangle^b\tilde{h}(\|A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_x})\Big\|_{L^2_t}.\end{aligned}$$

Then, by linearization of $\tilde{h}(x)$, we get

$$N_2[h((\beta-\Delta)^{-1}\varrho_\phi)] \le C\varepsilon \big\| \langle t\rangle^b \| A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_x}\big\|_{L^2_t} \\ = C\varepsilon \|\langle t\rangle^b A_t(\nabla_x, t\nabla_x)\varrho_\phi\|_{L^2_t L^2_x} \le C\varepsilon^2, \tag{4.13}$$

since by assumption $\phi \in \mathscr{C}_\varepsilon$. For $N_2[\varrho_\phi]$ we have

$$N_2[\varrho_\phi] \le N_2[q_\phi] + N_2[h((\beta-\Delta)^{-1}\varrho_\phi)] \le C\varepsilon + C\varepsilon^2, \tag{4.14}$$

where we used (4.13) and $\phi \in \mathscr{C}_\varepsilon$. Therefore, by assumption (1.12) and inequalities (4.13) and (4.14) in (4.12), we get

$$N_2[\varrho_\psi] \le C\varepsilon + C\varepsilon N_1[\psi]. \tag{4.15}$$

Finally, using (1.12), (4.14), and (4.15) in (4.11), we obtain

$$N_1[\psi] \le C\varepsilon + C\varepsilon N_1[\psi]. \tag{4.16}$$

Collecting the inequalities in (4.15) and (4.16), we get that $\psi \in \mathscr{C}_\varepsilon$, provided that ε is sufficiently small.

To construct solutions to the non-linear problem with asymptotic state g_∞, we want to prove that $\mathscr{F}$ is contractive in the norm $\mathcal{N}[\cdot]$ defined as $N[\cdot]$ but with a different $\tilde{\lambda}(t) = \tilde{\lambda}_\infty - C\langle t\rangle^{-\delta}$, $\tilde{\lambda}_\infty < \lambda_\infty$. Note that we have the equation

$$\partial_t(\psi^{(2)} - \psi^{(1)}) + ((E_{\phi^{(2)}} - E_{\phi^{(1)}})(x+vt)) \cdot (\nabla_v - t\nabla_x)\psi^{(2)} \\ + E_{\phi^{(1)}}(t, x+vt) \cdot (\nabla_v - t\nabla_x)(\psi^{(2)} - \psi^{(1)}) = -(E_{\psi^{(2)}} - E_{\psi^{(1)}}) \cdot \nabla_v \mu,$$

and that

$$\varrho_{\phi^{(2)}} - \varrho_{\phi^{(1)}} = \int [\phi^{(2)}(t, x-tv, v) - \phi^{(1)}(t, x-tv, v)]\, dv \\ - h((\beta-\Delta)^{-1}\varrho_{\phi^{(2)}}) + h((\beta-\Delta)^{-1}\varrho_{\phi^{(1)}}).$$

By the a priori estimate in (4.7), we get

$$\mathcal{N}_1[\psi^{(2)} - \psi^{(1)}] \le C\mathcal{N}_2[\varrho_{\phi^{(2)}} - \varrho_{\phi^{(1)}}] N_1[\psi^{(2)}] + C N_2[\varrho_{\phi^{(1)}}]\mathcal{N}_1[\psi^{(2)} - \psi^{(1)}] \\ + C\mathcal{N}_2[\varrho_{\psi^{(2)}} - \varrho_{\psi^{(1)}}],$$

where $\mathcal{N}_1[\cdot]$ and $\mathcal{N}_2[\cdot]$ are defined as $N_1[\cdot]$ and $N_2[\cdot]$ but with $\tilde{\lambda}(t)$. Since, by Proposition 2.7,

$$\mathcal{N}_2[\varrho_{\psi^{(2)}} - \varrho_{\psi^{(1)}}] \le C\mathcal{N}_2[h((\beta-\Delta)^{-1}\varrho_{\phi^{(2)}}) - h((\beta-\Delta)^{-1}\varrho_{\phi^{(1)}})] \\ + C\mathcal{N}_2[\varrho_{\phi^{(1)}} - \varrho_{\phi^{(2)}}] N_1[\psi^{(2)}] \\ + C N_2[\varrho_{\phi^{(1)}}]\mathcal{N}_1[\psi^{(2)} - \psi^{(1)}],$$

using that $N_1[\psi^{(2)}] + N_2[\varrho_{\phi^{(1)}}] \leq C\varepsilon$, we conclude that

$$\mathcal{N}[\psi^{(2)} - \psi^{(1)}] \leq C\varepsilon\mathcal{N}[\phi^{(2)} - \phi^{(1)}],$$

proving the contraction property for small ε and the global existence of the solution $g_t(x, v)$ of (1.14) such that $\mathcal{N}[g] \leq C\varepsilon$. Finally, note that for $\bar{\lambda} < \tilde{\lambda}(0)$ we recover the decay property of the electric field (1.13) by the previous estimates

$$\begin{aligned}\|A_{\bar{\lambda}}(\nabla_x, t\nabla_x)E[r](t)\|^2_{L^2_x} &\leq \sum_{k\in\mathbb{Z}^d} e^{-2(\lambda(t)-\bar{\lambda})\langle k.kt\rangle^\gamma} e^{2\lambda(t)\langle k.kt\rangle^\gamma} \langle k, kt\rangle^{2\sigma} |\hat{\varrho}_t(k)|^2 \\ &\leq C\varepsilon e^{-C\langle t\rangle^\gamma},\end{aligned}$$

and, using that

$$\begin{aligned}\hat{g}_t(k,\eta) = \hat{g}_\infty(k,\eta) &+ \int_t^{+\infty} \frac{[(\eta - ks)\cdot k]}{|k|^2+\beta}\hat{\varrho}_s(k)\hat{\mu}(\eta - ks)\,ds \\ &+ \int_t^{+\infty} \sum_{\ell\in\mathbb{Z}^d} \frac{[(\eta - ks)\cdot \ell]}{|\ell|^2+\beta}\hat{\varrho}_s(\ell)\hat{g}_s(k-\ell, \eta - \ell s)\,ds,\end{aligned}$$

we also get the convergence of g_t towards the asymptotic condition g_∞:

$$\|g_t(x,v) - g_\infty(x,v)\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \leq \|A_{\bar{\lambda}}(\nabla_x, \nabla_v)(g_t - g_\infty)\|_{L^2_{x,v}} \leq C\varepsilon e^{-C\langle t\rangle^\gamma}. \quad \blacksquare$$

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Dario Benedetto
Dipartimento di Matematica, Università di Roma “La Sapienza”, Piazzale A. Moro 2, 00185 Roma, Italy; benedetto@mat.uniroma1.it

Emanuele Caglioti
Dipartimento di Matematica, Università di Roma “La Sapienza”, Piazzale A. Moro 2, 00185 Roma, Italy; caglioti@mat.uniroma1.it

Antoine Gagnebin
Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland; antoine.gagnebin@math.ethz.ch

Mikaela Iacobelli
Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland;
mikaela.iacobelli@math.ethz.ch

Stefano Rossi
Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland;
stefano.rossi@math.ethz.ch