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Alireza Salehi Golsefidy · Srivatsa Srinivas

Random walks on direct products of groups

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Abstract. Suppose $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ is generated by a symmetric set S of cardinality n where p is a prime number. Suppose the Cheeger constants of the Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$ with respect to $\pi_L(S)$ and $\pi_R(S)$ are at least c_0, where π_L and π_R are the projections to the left and the right components of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$, respectively. Then the Cheeger constant of the Cayley graph of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ with respect to S is at least c where c is a positive number which only depends on n and c_0. This gives an affirmative answer to a question of Lindenstrauss and Varjú.

Keywords: spectral gap, random walk, expanders, finite groups.

1. Introduction and statement of main results

1.1. Background and main results

Let $X^{(i)}$, $i \in \mathbb{N}$, be independent identically distributed (i.i.d.) random variables into a finite group G, with distribution μ. An *l-step random walk on G with distribution μ* is the random variable $X_\ell = X^{(\ell)} X^{(\ell-1)} \cdots X^{(1)}$. Given independent random variables X, Y into G with distribution μ and ν, respectively, the distribution of XY is the convolution

$$\mu * \nu(y) := \sum_{x \in G} \mu(x)\nu(x^{-1}y)$$

of μ and ν. Similarly $f * g$ can be defined for any $f, g \in L^2(G)$. Now the distribution of X_ℓ is

$$\mu^{*(\ell)} := \underbrace{\mu * \cdots * \mu}_{\ell}.$$

We say that a measure μ on G is *symmetric* if $\mu(x) = \mu(x^{-1})$ for every $x \in G$. In this work, we assume that μ is symmetric. We note that μ induces an operator $T_\mu : L^2(G) \to L^2(G)$ given by

$$T_\mu(f) = \mu * f.$$

Alireza Salehi Golsefidy: Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA; golsefidy@ucsd.edu

Srivatsa Srinivas: Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA; scsriniv@ucsd.edu

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Moreover, G acts on $L^2(G)$ in the following manner:

$$(x \cdot f)(x') = f(x^{-1} \cdot x'),$$

and so $L^2(G)$ is a $\mathbb{C}[G]$-module. We can identify $L^2(G)$ with $\mathbb{C}[G]$ by sending f to $\sum_{x \in G} f(x)x$, and via this identification $f * g$ is sent to fg, the product of f and g in $\mathbb{C}[G]$. Let $L^2(G)^\circ$ be the space orthogonal to the constants in $L^2(G)$. Since $L^2(G)^\circ$ is a $\mathbb{C}[G]$-submodule of $L^2(G)$, T_μ sends $L^2(G)^\circ$ to itself. We note that for every $f \in L^2(G)$ we have

$$\|T_\mu(f)\|_2 = \Big\| \sum_{x \in G} \mu(x) x \cdot f \Big\|_2 \leq \sum_{x \in G} \mu(x) \|x \cdot f\|_2 = \|f\|_2.$$

We define the *spectral gap* of μ to be

$$\lambda(\mu) := \|T_\mu|_{L^2(G)^\circ}\|_{\mathrm{op}},$$

and inspired by the definition of Lyapunov exponent, we let

$$\mathscr{L}(\mu) = -\log \lambda(\mu),$$

where log denotes the base 2 logarithm.

Notice that the chosen name, spectral gap, might be a bit misleading. Since μ is a symmetric measure, T_μ is a self-adjoint averaging operator, so the spectrum of T_μ consists of real numbers in $[-1, 1]$. The absolute values of the eigenvalues of T_μ give us $|G|$ numbers (with multiplicity) in the interval $[0, 1]$. The second largest number in this list is $\lambda(\mu)$, and the gap between $\lambda(\mu)$ and 1 is what we would like to control; clearly, $\mathscr{L}(\mu)$ gives us a way to measure this gap. In the literature, a family $\{\mu_i\}_i$ of probability measures is said to have *the spectral gap property* if $\sup_i \lambda(\mu_i) < 1$, and so despite this caveat, we still use $\lambda(\mu)$ to denote the spectral gap of μ.

Note that $\mathscr{L}(\mu) > 0$ implies that the support of μ generates G. If X is a random variable with values in G and distribution μ, we let $\lambda(X) := \lambda(\mu)$ and $\mathscr{L}(X) := \mathscr{L}(\mu)$.

The spectral gap $\lambda(\mu)$ gives us a measurement of how fast the random walk is getting equidistributed in G (at least in L^2-norm). To formulate this, for every finite set A, we let μ_A be the probability counting measure on A. Since for every probability measure ν on G, the orthogonal projection of ν onto the constants $\mathbb{C}\mu_G \subset L^2(G)$ is μ_G, we have

$$\begin{aligned} \|\mu^{*(\ell)} - \mu_G\|_2^2 &= \|\mu^{*(\ell)} * (\mu_{\{1\}} - \mu_G)\|_2^2 = \|T_\mu^\ell(\mu_{\{1\}} - \mu_G)\|_2^2 \\ &\leq \lambda(\mu)^{2\ell} \|\mu_{\{1\}} - \mu_G\|_2^2 = 2^{-2\mathscr{L}(\mu)\ell} \|\mu_{\{1\}} - \mu_G\|_2^2. \end{aligned}$$

Because of this type of control on convergence to equidistribution, we are interested in finding a lower bound for $\mathscr{L}(\mu)$ independent of $|G|$.

In this work, we investigate random walks in $G \times G$. In a forthcoming joint work of the first author with Mallahi-Karai and Mohammadi [8], a result in the following framework is proved: if two compact groups G_1 and G_2 are *drastically different*, then for a probability measure μ on $G_1 \times G_2$ we have $\mathscr{L}(\mu) > 0$ if $\mathscr{L}(\pi_L[\mu]) > 0$ and $\mathscr{L}(\pi_R[\mu]) > 0$,

where $\pi_L : G_1 \times G_2 \to G_1$ and $\pi_R : G_1 \times G_2 \to G_2$ are the projection maps. One cannot expect a similar result when G_1 and G_2 have a non-trivial common (topological) quotient. For instance, consider the case $G_1 = G_2 = G$ and let μ be the probability Haar measure of the diagonal embedding $\Delta(G)$ of G in $G \times G$. Then clearly $\mathcal{L}(\pi_L[\mu])$ and $\mathcal{L}(\pi_R[\mu])$ are positive, but $\mathcal{L}(\mu) = 0$. In this example, however, the support of μ does not generate (a dense subgroup) of $G \times G$. What if we add this extra algebraic condition? This subtlety is highlighted by Lindenstrauss and Varjú [14, Open problem 1.4] in the form of the following question:

Question 1 (Lindenstrauss–Varjú). *Suppose S is a symmetric generating set of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$. Is it possible to estimate the spectral gap of μ_S in terms of the spectral gaps of the projections to the direct factors and $|S|$?*

In this article we give an affirmative answer to this question.

Theorem 1. *Let μ be a symmetric measure on $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ such that*

$$\mathcal{L}(\pi_L[\mu]), \mathcal{L}(\pi_R[\mu]) \geq c_0 > 0$$

and the minimum of μ on its support is α_0. If the support of μ generates the group $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ and $p \gg_{c_0,\alpha_0} 1$, then $\mathcal{L}(\mu) \gg_{c_0,\alpha_0} 1$, where the implied positive constant only depends on c_0 and α_0.

In our forthcoming article, we use the results of this work together with modular representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$, Bourgain–Katz–Tao's sum-product result on finite fields, an investigation of certain unipotent group schemes, Gowers's theory of quasi-random groups, and Bourgain–Gamburd's method of gaining entropy to study random walks on group extensions. In particular, we extend Theorem 1, and show that if $\mathbb{G}$ is a connected simply-connected perfect $\mathbb{Q}$-algebraic group and F is a finite field with large enough characteristic, then the spectral gap of a random walk on $\mathbb{G}(F)$ can be bounded by the spectral gap of the projection of the random walk on simple quotients of $\mathbb{G}(F)$. For the sake of clarity, highlighting our entropy inequality, and separating more algebraic tools, we have decided to consider this special case in a separate article.

1.2. Applications

Studying spectral gaps of random walks on compact groups is an extremely interesting subject with many applications. For instance, it has been used to give an *explicit construction of expander graphs*, or recently in *affine sieve* and *sieve in groups*. We refer the reader to the beautiful surveys by Lubotzky [17] and Kowalski [12] for details and many more applications. Here first we present a combinatorial interpretation of Theorem 1 as it is formulated in the abstract, and then give one application which is new in nature and it is a consequence of our main result.

1.2.1. Cheeger constant and expander graphs. One of the interesting applications of the study of spectral gaps of random walks on a family of finite groups is its connection

with the explicit construction of *expander graphs*. For a graph with finitely many vertices $\mathcal{G} = (V, E)$, the *Cheeger constant* of $\mathcal{G}$ is defined to be

$$e(\mathcal{G}) = \inf_{A \subset V, |A| < |V|/2} |\partial(A)|/|A|,$$

where $\partial(A)$ is the set of vertices that are not in A but connected to a vertex in A via an edge. This constant measures how connected the graph $\mathcal{G}$ is. A family $(\mathcal{G}_i)_{i \in \mathcal{I}}$ of finite graphs is called an *expander family* if there exist $k, c > 0$ such that for all $i \in \mathcal{I}$, the maximum degree of a vertex in $\mathcal{G}_i$ is at most k and $e(\mathcal{G}_i) > c$. Expander families have an interesting history and have found applications in various areas of computer science and mathematics (see [10, 13]). An expander family gives us a family of *sparse* (not many edges attached to a vertex) yet *highly connected* (expansion constant bounded below) graphs.

A result of Dodziuk [5] and Alon [1] (see [16, Proposition 4.2.4]) gives an isoperimetric inequality for regular graphs. In particular, this result implies that if $\mathcal{G}$ is the Cayley graph of a finite group with respect to a symmetric generating set S, then $\mathcal{L}(\mu_S)$ has a positive lower bound in terms of $|S|$ and the Cheeger constant of $\mathcal{G}$, and conversely the Cheeger constant of $\mathcal{G}$ has a positive lower bound in terms of $\mathcal{L}(\mu_S)$ and $|S|$. Hence Theorem 1 implies the following.

Theorem 2. *Suppose S is a symmetric generating set of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ where p is a prime number. Let π_L and π_R be the projections to the left and right components of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$, respectively. Suppose $|\pi_L(S)| = |\pi_R(S)| = |S|$. Let $\mathcal{G}_L$ and $\mathcal{G}_R$ be the Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$ with respect to $\pi_L(S)$ and $\pi_R(S)$, respectively. Suppose the Cheeger constants $e(\mathcal{G}_L)$ and $e(\mathcal{G}_R)$ are at least c_0. Then the Cheeger constant of the Cayley graph of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ with respect to S is at least a positive number which depends only on $|S|$ and c_0.*

Let us remark that the technical condition on the cardinality of $\pi_L(S)$ and $\pi_R(S)$ is not needed. In a forthcoming article on random walks on perfect groups we will show that if μ and μ' are two probability measures on a finite group that have the same support and the minimum of μ and μ' on their support is at least α_0, then $\mathcal{L}(\mu) \ll_{\alpha_0} \mathcal{L}(\mu') \ll_{\alpha_0} \mathcal{L}(\mu)$ (this is probably well-known to experts). This result allows us to remove the above-mentioned technical condition.

1.2.2. Representation and character varieties. Let $F_2 = \langle a, b \rangle$ be the free group freely generated by a and b. The SL_2-representation variety of F_2 is given by the functor

$$\mathrm{Rep}_2(A) := \mathrm{Hom}(F_2, \mathrm{SL}_2(A))$$

from the category of unital commutative rings to the category of sets. It is clear that $\mathrm{Rep}_2(A)$ can be identified with two copies of $\mathrm{SL}_2(A)$. The group SL_2 acts on Rep_2 by conjugation. The geometric quotient of Rep_2 by SL_2 is a variety and by works of Fricke it has an integral model which is denoted by Ch_2. In particular, for every $\rho \in \mathrm{Rep}_2(A)$, we get a point $[\rho] \in \mathrm{Ch}_2(A)$, and we have $[\rho_1] = [\rho_2]$ for $\rho_1, \rho_2 \in \mathrm{Rep}_2(A)$ if there is

$x \in \mathrm{SL}_2(A)$ such that $\rho_2(y) = x\rho_1(y)x^{-1}$ for all $y \in F_2$ (notice that the converse of this statement is not correct, but that is not related to our application). For all $w \in F_2$ and $[\rho] \in \mathrm{Ch}_2(A)$, we let

$$t_w([\rho]) := \mathrm{Tr}(\rho(w)).$$

By works of Fricke, we can view t_w as a regular function on Ch_2. For every positive δ, we let

$$\mathrm{Rep}_2(\mathbb{F}_p)_\delta := \{\rho \in \mathrm{Rep}_2(\mathbb{F}_p) \mid \mathcal{L}(\mu_{\{\rho(a)^{\pm 1}, \rho(b)^{\pm 1}\}}) \geq \delta\}.$$

Note that $\bigcup_{\delta>0} \mathrm{Rep}_2(\mathbb{F}_p)_\delta = \{\rho \in \mathrm{Rep}_2(\mathbb{F}_p) \mid \rho(F_2) = \mathrm{SL}_2(\mathbb{F}_p)\}$. We can show that many t_w's can distinguish two distinct points $[\rho_1]$ and $[\rho_2]$ of $\mathrm{Ch}_2(\mathbb{F}_p)$ if $\rho_1, \rho_2 \in \mathrm{Rep}_2(\mathbb{F}_p)_\delta$.

Corollary 3. *Suppose δ is a positive number and $\rho_1, \rho_2 \in \mathrm{Rep}_2(\mathbb{F}_p)_\delta$. Suppose a, b freely generate a free group F_2. Then there is $0 < \beta := \beta(\delta) < 1$ such that for every positive integer ℓ the following statements hold:*

(i) *If $\rho_1 \neq \rho_2$, then*

$$\mu^{*(\ell)}_{\{a^{\pm 1}, b^{\pm 1}\}}(\{w \in F_2 \mid \rho_1(w) = \rho_2(w)\}) \leq p^{-1} + \beta^\ell |\mathrm{SL}_2(\mathbb{F}_p)|.$$

(ii) *If $[\rho_1] \neq [\rho_2]$, then either*

$$\mu^{*(\ell)}_{\{a^{\pm 1}, b^{\pm 1}\}}(\{w \in F_2 \mid t_w([\rho_1]) = t_w([\rho_2])\}) \leq 5p^{-1} + \beta^\ell |\mathrm{SL}_2(\mathbb{F}_p)|,$$

or there is an automorphism $\widehat{\phi}$ of $\mathrm{SL}_2(\mathbb{F}_p)$ such that for every $w \in F_2$, $\rho_2(w) = \pm\widehat{\phi}(\rho_1(w))$. In the latter case, $t_w([\rho_1]) = \pm t_w([\rho_2])$ for every $w \in F_2$.

It is not known whether there is a positive number δ_0 such that

$$\mathrm{Rep}_2(\mathbb{F}_p)_{\delta_0} = \{\rho \in \mathrm{Rep}_2(\mathbb{F}_p) \mid \rho(F_2) = \mathrm{SL}_2(\mathbb{F}_p)\}.$$

So we do not know if Corollary 3 holds for a fixed universal constant β and any surjective $\rho_1, \rho_2 \in \mathrm{Rep}_2(\mathbb{F}_p)$.

1.3. Proof strategy

Note that since the natural quotient map

$$\overline{\tau} : \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$$

has a kernel of cardinality at most 4, it is enough to prove a similar result to Theorem 1 for $\mathrm{PSL}_2(\mathbb{F}_p)$. Let us recall that for a random variable X with finite support and probability law μ, the *Rényi entropy* of X is

$$H_2(X) := -\log \|\mu\|_2^2.$$

Let $X = (X_L, X_R)$ be a random variable into $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$ with distribution μ. We assume that μ is symmetric. Suppose that

$$\mathcal{L}(\pi_R[\mu]), \mathcal{L}(\pi_L[\mu]) > c_0 > 0,$$

μ has minimum α_0 on its support, and the support of μ generates $\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)$. Let L_1 be a large natural number. Using the Bourgain–Gamburd method (see [3]), we deduce that the only obstacle to μ having a spectral gap depending on c_0 and L_1 is if there exist an automorphism ϕ of $\mathrm{PSL}_2(\mathbb{F}_p)$ and an integer $\ell \geq L_1 \log|\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)|$ such that

$$\mathbb{P}(X_\ell \in \Gamma_\phi) \geq |\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)|^{-1/26}, \tag{1}$$

where $\Gamma_\phi = \{(x,\phi(x)) \mid x \in \mathrm{PSL}_2(\mathbb{F}_p)\}$ is the graph of ϕ. Since the group of outer automorphisms of $\mathrm{PSL}_2(\mathbb{F}_p)$ has only two elements, we reduce the general case to the case where ϕ is inner, say that $\phi(x) = zxz^{-1}$. Notice $\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)$ acts on $\mathrm{PSL}_2(\mathbb{F}_p)$ by the left and right multiplications; that means $(u,v)\cdot x = uxv^{-1}$. The graph Γ_ϕ is the stabilizer subgroup of $\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)$ with respect to z^{-1}; that means

$$\Gamma_\phi = \{(u,v) \in \mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p) \mid (u,v)\cdot z^{-1} = z^{-1}\}.$$

Thus inequality (1) turns into

$$\mathbb{P}(X_\ell \cdot z^{-1} = z^{-1}) \geq |\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)|^{-1/26}. \tag{2}$$

We would like to show that (2) is impossible. Inequality (2) suggests that the Rényi entropy of $X_\ell \cdot z^{-1}$ should be *small*. This brings us to studying $H_2(X_\ell \cdot Y)$ where Y is a random variable on $\mathrm{PSL}_2(\mathbb{F}_p)$ that has small Rényi entropy. We note that $X_\ell \cdot Y = (X_L)_\ell Y (X_R)_\ell^{-1}$, and we know that $(X_L)_\ell$ and $(X_R)_\ell$ are almost equidistributed for $\ell \gg_{c_0} \log|\mathrm{PSL}_2(\mathbb{F}_p)|$. But we do not know how $(X_L)_\ell$ and $(X_R)_\ell$ are correlated. All we know is that the range of X generates $\mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)$. The following lemma is instrumental in resolving this issue.

Lemma 4. *Suppose G and H are two finite groups, and G acts on H. Let $X^{(1)}, X^{(2)}$ be i.i.d. random variables into a finite group G and $Y^{(1)}, Y^{(2)}$ be i.i.d. random variables into a finite group H. Then*

$$H_2((X^{(1)}\cdot Y^{(1)})^{-1}(X^{(2)}\cdot Y^{(2)})) \geq H_2((X^{(1)}\cdot Y^{(1)})^{-1}(X^{(1)}\cdot Y^{(2)})).$$

Lemma 4 is consistent with the general principle that *reducing the degree of freedom should decrease the Rényi entropy*. Applying Lemma 4 to our random variable X, we get

$$\begin{aligned} H_2((X^{(1)}\cdot Y^{(1)})^{-1}(X^{(2)}\cdot Y^{(2)})) &\geq H_2((X^{(1)}\cdot Y^{(1)})^{-1}(X^{(1)}\cdot Y^{(2)})) \\ &= H_2((X_L^{(1)}Y^{(1)}(X_R^{(1)})^{-1})^{-1}(X_L^{(1)}Y^{(2)}(X_R^{(1)})^{-1}) \\ &= H_2(X_R^{(1)}(Y^{(1)})^{-1}Y^{(2)}(X_R^{(1)})^{-1}). \end{aligned} \tag{3}$$

Based on (3), we get a lower bound for $H_2((X^{(1)}\cdot Y^{(1)})^{-1}(X^{(2)}\cdot Y^{(2)}))$ using conjugation by X_R. Since we have control on the spectral gap of X_R, after $\ell_0 := O_{c_0}(1)$ steps the random walk $(X_R)_{\ell_0}$ gets close to equidistribution. This implies that conjugation by $(X_R)_{\ell_0}$ spreads the weight almost equally in the conjugacy classes that intersect the range of $(Y^{(1)})^{-1}Y^{(2)}$. Since every conjugacy class of $\mathrm{PSL}_2(\mathbb{F}_p)$ except $\{1\}$ has at least p elements and Y has *small* Rényi entropy, we obtain the following dichotomy:

- Either Y is *almost* concentrated at one point or we *gain* Rényi entropy after conjugation by X_R.

By gaining Rényi entropy, we mean that for some $\varepsilon := \varepsilon(\alpha_0) > 0$ and $\ell_0 := O_{c_0,\alpha_0}(1)$,

$$H_2((X_R)_{\ell_0}(Y^{(1)})^{-1}Y^{(2)}(X_R)_{\ell_0}^{-1}) \geq H_2((Y^{(1)})^{-1}Y^{(2)}) + \varepsilon.$$

If Y is *almost* concentrated at one point, we use the assumption that the range of X generates $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$ to show the existence of $\varepsilon_1 := \varepsilon_1(\alpha_0) > 0$ such that

$$H_2(X \cdot Y) \geq H_2(Y) + \varepsilon_1.$$

Altogether we obtain the following lemma.

Lemma 5. *Let $G := \mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$. Suppose $X := (X_L, X_R)$ is a random variable with values in G and probability law μ. Set $\alpha_0 := \min\{\mathbb{P}(X = x) \mid x \in \mathrm{supp}(\mu)\}$ and suppose $\mathcal{L}(X_R) \geq c_0 > 0$. Suppose that the range of X generates G. Then there exist constants $L, C \gg_{c_0,\alpha_0} 1$ such that for every random variable Y on $\mathrm{PSL}_2(\mathbb{F}_p)$ and every $\ell \geq L \log |G|$,*

$$H_2(X_\ell \cdot Y) \geq \tfrac{1}{12} \log |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)| - C, \tag{4}$$

where $X \cdot Y := X_L Y X_R^{-1}$ and X_ℓ is the ℓ-step random walk with respect to μ.

For large enough p, (4) implies

$$H_2(X_\ell \cdot Y) \geq \frac{1}{12.5} \log |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)|.$$

Therefore for every $\ell \geq L \log |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)|$, we have

$$\mathbb{P}(X_\ell \cdot z^{-1} = z^{-1}) \leq e^{-\frac{1}{2}H_2(X_\ell \cdot z^{-1})} \leq |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)|^{-1/25}.$$

Thus Lemma 5 shows that (2) cannot hold, and this completes the proof.

2. Notation and preliminary results

2.1. Conventions

If $f : G \to V$ is a function from a finite group to a $\mathbb{C}$-vector space and μ is a measure on G, we define

$$\int_G f(x)\, d\mu(x) := \sum_{x \in G} \mu(x) f(x)$$

We endow $L^2(G)$ with the inner product

$$\langle f, g \rangle = \sum_{x \in G} f(x)\overline{g(x)}$$

where $f, g \in L^2(G)$. For $f \in L^2(G)$, $\check{f} \in L^2(G)$ is given by

$$\check{f}(x) = \overline{f(x^{-1})}.$$

Note that if X is a random variable with values in G and distribution μ, then the probability law of X^{-1} is $\check{\mu}$.

For a subset A of a finite group G and a positive integer k, we let

$$\textstyle\prod_k A := \{a_1 \cdots a_k \mid a_1, \ldots, a_k \in A\}.$$

2.2. *Basics of Fourier analysis for finite groups and quasi-random groups*

Suppose G is a finite group. Then $\widehat{G}$ denotes the set of irreducible unitary subrepresentations of the regular representation $L^2(G)$. For $f \in L^2(G)$, the Fourier inverse of f is defined as

$$\widehat{f}(\pi) := \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g)^*,$$

where $\pi(g)^*$ is the adjoint of $\pi(g)$. For $f_1, f_2 \in L^2(G)$, the convolution of f_1 and f_2 is defined with respect to the counting measure (and not the *probability* counting measure)

$$f_1 * f_2(x) = \sum_{x_1 x_2 = x} f_1(x_1) f_2(x_2),$$

and we have

$$\widehat{f_1 * f_2}(\pi) = |G| \widehat{f_2}(\pi) \widehat{f_1}(\pi) \tag{5}$$

for every $\pi \in \widehat{G}$. The Parseval theorem states that

$$\|f\|_2^2 = \sum_{\pi \in \widehat{G}} \dim \pi \, \|\widehat{f}(\pi)\|_{\mathrm{HS}}^2, \tag{6}$$

where $\|f\|_2^2 := \frac{1}{|G|} \sum_{g \in G} |f(g)|^2$ and the Hilbert–Schmidt norm $\|T\|_{\mathrm{HS}}^2$ is $\mathrm{Tr}(TT^*)$.

Let us recall Gowers's notion of quasi-randomness and its consequences for the study of random walks (see [9]).[1]

Definition. For $c \geq 0$, we say a finite group G is *c-quasi-random* if $\dim \pi \geq |G|^c$ for every non-trivial $\pi \in \widehat{G}$.

The following mixing inequality is one of the main properties of a quasi-random group (see [2, Theorem 2.1], [9], and [18, Lemma 6.1]).

Lemma 6. *Suppose c is a positive number and G is a c-quasi-random group. Then for all $f_1 \in L^2(G)$ and $f_2 \in L^2(G)^\circ$,*

$$\|f_1 * f_2\|_2 \leq |G|^{(1-c)/2} \|f_1\|_2 \|f_2\|_2.$$

[1] It should be pointed out that in [19], Sarnak and Xue had implicitly used the concept of quasi-randomness in order to prove a spectral gap property.

Proof. By the Parseval theorem (see (6)) and (5), we obtain

$$\begin{aligned}\|f_1 * f_2\|_2^2 &= \sum_{\pi\in\widehat{G}} \dim\pi\, \|\widehat{f_1 * f_2}(\pi)\|_{\mathrm{HS}}^2 = |G|^2 \sum_{\pi\in\widehat{G},\,\pi\neq 1} \dim\pi\, \|\widehat{f}_2(\pi)\widehat{f}_1(\pi)\|_{\mathrm{HS}}^2 \\ &\le |G|^{2-c}\Big(\sum_{\pi\in\widehat{G}} \dim\pi\, \|\widehat{f}_2(\pi)\|_{\mathrm{HS}}^2\Big)\Big(\sum_{\pi\in\widehat{G}} \dim\pi\, \|\widehat{f}_1(\pi)\|_{\mathrm{HS}}^2\Big) \\ &\le |G|^{2-c}\|f_1\|_2^2\|f_2\|_2^2.\end{aligned}$$

Hence $\|f_1 * f_2\|_2^2 \le |G|^{1-c}\|f_1\|_2^2\|f_2\|_2^2$, and the claim follows. ∎

As pointed out in the introduction, if X is a symmetric random variable with law μ, then $T_\mu : L^2(G) \to L^2(G), T_\mu(f) := f * \mu$, is a self-adjoint operator, and $\lambda(\mu)$ is the maximum of the absolute values of the eigenvalues of $T_\mu|_{L^2(G)^\circ} : L^2(G)^\circ \to L^2(G)^\circ$. Hence one can see that

$$\mathcal{L}(\mu^{(\ell)}) = \ell\mathcal{L}(\mu)$$

for every positive integer ℓ.

Lemma 7. *Suppose c is a positive number and G is a c-quasi-random group. Suppose X is a symmetric random variable with values in G. Then*

(i) $H_2(X) \ge (1 - c/2)\log|G|$ *implies* $\mathcal{L}(X) \ge (c/4)\log|G|$.

(ii) *Suppose X_ℓ is the random variable after ℓ-step random walk with respect to X. Suppose $C > 0$, and $H_2(X_\ell) \ge (1 - c/2)\log|G|$ for a positive integer $\ell \le C\log|G|$. Then $\mathcal{L}(X) \ge c/(4C)$.*

Proof. Suppose μ is the probability law of X; that means $\mu(x) := \mathbb{P}(X = x)$ for every $x \in G$. By Lemma 6, for every $f \in L^2(G)^\circ$,

$$\|\mu * f\|_2 \le |G|^{(1-c)/2}\|\mu\|_2\|f\|_2.$$

Hence

$$\mathcal{L}(\mu) \ge \frac{-1+c}{2}\log|G| + \frac{1}{2}H_2(X) \ge \frac{c}{4}\log|G|.$$

This implies (i). Next applying (i) for the random variable X_ℓ, we deduce

$$C\log|G|\,\mathcal{L}(\mu) \ge \ell\mathcal{L}(\mu) \ge \frac{c}{4}\log|G|,$$

and (ii) follows. ∎

2.3. Group action and convolution

When a finite group G acts on a finite set H, we write $G \curvearrowright H$. An action of G on H induces an action of G on $L^2(H)$ by

$$(x \cdot f)(y) = f(x^{-1} \cdot y)$$

for $x \in G$, $y \in H$ and $f \in L^2(H)$. This is a unitary action and this way we can view $L^2(H)$ as a $\mathbb{C}[G]$-module. Given $\mu \in L^2(G)$ and $f \in L^2(H)$, we define

$$\mu \boxtimes f = \sum_{x \in G} \mu(x)(x \cdot f) = \int_G (x \cdot f)\, d\mu(x).$$

Note that if X is a random variable with values in G and distribution μ, Y is a random variable with values in H and distribution η, and X and Y are independent, then the distribution of $X \cdot Y$ is $\mu \boxtimes \eta$. Notice that the group action properties give us the following relations. If $\mu, \nu \in L^2(G)$ and $f \in L^2(H)$ then

$$\mu \boxtimes (\nu \boxtimes f) = (\mu * \nu) \boxtimes f.$$

Moreover, $\boxtimes : L^2(G) \times L^2(H) \to L^2(H)$, $(\mu, f) \mapsto \mu \boxtimes f$, is a bilinear map. We call $\boxtimes$ the *convolution associated to* $G \curvearrowright H$. Notice that for the counting probability measure μ_G on G,

$$\mu_G \boxtimes \cdot : L^2(H) \to L^2(H), \quad f \mapsto \mu_G \boxtimes f,$$

is the orthogonal projection of $L^2(H)$ onto the space $L^2(H)^G$ of G-invariant functions in $L^2(H)$. Thus $f \mapsto f - \mu_G \boxtimes f$ is the orthogonal projection from $L^2(H)$ to the space $(L^2(H)^G)^\perp$ orthogonal to the space of G-invariant functions. We also observe that if $\mu_{\{1\}}$ is the point mass at the identity, then $f = \mu_{\{1\}} \boxtimes f$ for every $f \in L^2(H)$. Therefore the orthogonal projection from $L^2(H)$ to $(L^2(H)^G)^\perp$ is given by

$$f \mapsto (\mu_{\{1\}} - \mu_G) \boxtimes f. \tag{7}$$

Notice that every irreducible subrepresentation V of $(L^2(H)^G)^\perp$ is non-trivial, and so by Maschke's theorem [6, Theorem 4.1.1], there is a G-module isometric embedding $i : V \to L^2(G)^\circ$. Hence for every probability measure μ on G and $f \in V$, we have

$$\begin{aligned} \|\mu \boxtimes f\|_2 &= \|i_V(\mu \boxtimes f)\|_2 = \|\mu * i_V(f)\|_2 \\ &\le \lambda(\mu)\|i_V(f)\|_2 = \lambda(\mu)\|f\|_2. \end{aligned} \tag{8}$$

Since $(L^2(H)^G)^\perp$ is a direct sum of pairwise orthogonal irreducible subrepresentations, from (8) we infer that $\|\mu \boxtimes f\|_2 \le \lambda(\mu)\|f\|_2$ for every $f \in (L^2(H)^G)^\perp$. Combining this result with (7), we deduce that

$$\begin{aligned} \|(\mu - \mu_G) \boxtimes f\|_2 &= \|(\mu * (\mu_{\{1\}} - \mu_G)) \boxtimes f\|_2 = \|\mu \boxtimes ((\mu_{\{1\}} - \mu_G) \boxtimes f)\|_2 \\ &\le \lambda(\mu)\|(\mu_{\{1\}} - \mu_G) \boxtimes f\|_2 \le \lambda(\mu)\|f\|_2 \end{aligned} \tag{9}$$

for every $f \in L^2(H)$.

3. An inequality for the Rényi entropy of random variables

The aim of this section is to prove Lemma 4. Let us recall that in the setting of Lemma 4, we have two finite groups G and H, and G acts on H. There are i.i.d. random variables $X^{(1)}, X^{(2)}$ with values in G and distribution μ, and i.i.d. random variables $Y^{(1)}, Y^{(2)}$

with values in H and distribution η. In this section, we work with the (non-normalized) counting measure m_H on H.

We start with two sets of convolution identities. The first one is well-known and the second one is based on the fact that $*$ is bilinear.

Lemma 8. *In the above setting, for $f, g, h \in L^2(H)$, the following identities hold:*

(i) $f * (g * h) = (f * g) * h$, $\langle f * g, h\rangle = \langle f, h * \check{g}\rangle$ *and* $\langle f * g, h\rangle = \langle g, \check{f} * h\rangle$,

(ii) $(\mu \boxtimes f) * (\mu \boxtimes g) = \int_{G^2} (u \cdot f) * (v \cdot g)\, d(\mu^{\otimes 2})(u, v)$.

Proof. Both parts easily follow from switching the order of summations. Here we only discuss (ii), which follows from the fact that $\mu \boxtimes f = \int_G (x \cdot f) d\mu(x)$ and $*$ is bilinear:

$$\begin{aligned}(\mu \boxtimes f) * (\mu \boxtimes g) &= \left(\int_G (u \cdot f)\, d\mu(u)\right) * \left(\int_G (v \cdot g)\, d\mu(v)\right) \\ &= \int_{G^2} (u \cdot f) * (v \cdot g)\, d\mu^{\otimes 2}(u, v).\end{aligned}$$

This completes the proof. ∎

We will be working with a new norm on $L^2(H)$, which we denote by $|||\cdot|||$ and which can be viewed as a non-commutative version of Gowers's U^2-norm. We will show that this norm is preserved by *shifted-automorphism* group actions $G \curvearrowright H$.

Definition. Suppose G and H are two groups. An action $G \curvearrowright H$ is called a *shifted-automorphism group action* if there are a group homomorphism $\phi : G \to \mathrm{Aut}(H)$ and a function $c : G \to H$ such that $x \cdot y = c(x)(\phi(x))(y)$ for all $x \in G$ and $y \in H$.

Notice that for every group H, the group action $H \times H \curvearrowright H$ given by $(x_L, x_R) \cdot y := x_L y x_R^{-1}$ is a shifted-automorphism group action, because

$$x_L y x_R^{-1} = c(x_L, x_R)\phi(x_L, x_R)(y),$$

where $c(x_L, x_R) = x_L x_R^{-1}$ and $\phi : H \times H \to \mathrm{Aut}(H)$ is a group homomorphism given by

$$\phi(x_L, x_R)(y) := x_R y x_R^{-1}.$$

Lemma 9. *Suppose H is a finite group. Let $|||f||| := \|\check{f} * f\|_2^{1/2}$ for $f \in L^2(H)$. Then the following statements hold:*

(i) $|||\cdot|||$ *is a norm and* $\|f\|_2 \le |||f|||$ *for every non-negative* $f \in L^2(H)$.

(ii) *Suppose $G \curvearrowright H$ is a shifted-automorphism group action. Then for all $x \in G$ and $f \in L^2(H)$, we have $|||x \cdot f||| = |||f|||$.*

(iii) *Suppose $G \curvearrowright H$ is a shifted-automorphism group action and μ is a probability measure on G. Then $|||\mu \boxtimes f||| \le |||f|||$ for every $f \in L^2(H)$.*

Proof. (i) For every $g \in L^2(H)$, the convolution operator

$$T_g : L^2(H) \to L^2(H), \quad T_g(h) := g * h,$$

is an integral operator with kernel $K_g : H \times H \to \mathbb{C}$, $K_g(x, y) := g(xy^{-1})$. Therefore the Hilbert–Schmidt norm $\|T_g\|_{\mathrm{HS}}$ is equal to $\|K_g\|_2$ (see [4, Chapter II, Proposition 4.7]). Notice that

$$\|K_g\|_2^2 = \sum_{x,y\in H} |g(x^{-1}y)|^2 = |H|\,\|g\|_2^2,$$

and so $\|T_g\|_{\mathrm{HS}} = |H|^{1/2}\|g\|_2$. Hence

$$|||f||| = \|\check{f} * f\|_2^{1/2} = |H|^{-1/4}\|T_{\check{f}*f}\|_{\mathrm{HS}}^{1/2}. \tag{10}$$

By Lemma 8, we have $T_f^* = T_{\check{f}}$. Hence by (10), we obtain

$$|||f||| = |H|^{-1/4}\|T_f^* \circ T_f\|_{\mathrm{HS}}^{1/2}. \tag{11}$$

For an operator $T : L^2(H) \to L^2(H)$, let $|||T||| := \|T^* \circ T\|_{\mathrm{HS}}^{1/2}$. Notice that if $\sigma_1, \ldots, \sigma_n$ are the singular values of T, then

$$|||T||| = (\sigma_1^4 + \cdots + \sigma_n^4)^{1/4}. \tag{12}$$

By (12) and [11, Theorem 7.4.24], $|||\cdot|||$ is a unitarily invariant norm on $\mathrm{End}_{\mathbb{C}}(L^2(H))$. Therefore by (10) for all $f, g \in L^2(H)$, we have

$$|||f + g||| = |H|^{-1/4}|||T_f + T_g||| \le |H|^{-1/4}|||T_f||| + |H|^{-1/4}|||T_g||| = |||f||| + |||g|||.$$

For all $c \in \mathbb{C}$ and $f \in L^2(H)$, we clearly have $|||cf||| = |c|\,|||f|||$, and $|||f||| = 0$ implies $f = 0$. Hence $|||\cdot|||$ is a norm on $L^2(H)$. For two non-negative functions f and g, we have

$$\begin{aligned}\|f * g\|_2^2 &= \sum_x (f * g)(x)^2 = \sum_x \Big(\sum_{x_1x_2=x} f(x_1)g(x_2)\Big)^2 \\ &\ge \sum_x \sum_{x_1x_2=x} f(x_1)^2 g(x_2)^2 = \|f\|_2^2\|g\|_2^2,\end{aligned}$$

and so $\|f * g\|_2 \ge \|f\|_2\|g\|_2$. Therefore for a non-negative function f, we have

$$|||f||| = \|\check{f} * f\|_2^{1/2} \ge (\|\check{f}\|_2\|f\|_2)^{1/2} = \|f\|_2.$$

(ii) Notice that $\{\mu_{\{y\}}\}_{y\in H}$ is an orthonormal basis of $L^2(H)$, and for y, y' in H and $f \in L^2(H)$, the (y, y')-matrix entry of T_f is $f(yy'^{-1})$. Hence the (y, y')-matrix entry of $T_{x\cdot f}$ is

$$f(x^{-1}\cdot(yy'^{-1})) = f\big(c(x^{-1})(\phi(x^{-1}))(y)(\phi(x^{-1}))(y')^{-1}\big), \tag{13}$$

where $\phi : G \to \mathrm{Aut}(H)$ and $c : G \to H$ give us the shifted-automorphism group action $G \curvearrowright H$. By (13), the (y, y')-entry of $T_{x\cdot f}$ is equal to the $((\phi(x^{-1}))(y), (\phi(x^{-1}))(y'))$-matrix entry of $T_{c_x\cdot f}$ where $c_x := c(x^{-1})^{-1} \in H$ and $(c_x \cdot f)(y) = f(c_x^{-1}y)$ for every

$y \in H$. For every $x \in G$, let $\sigma(x) : L^2(H) \to L^2(H)$ be the unitary operation given by $(\sigma(x)(f))(y) := f(\phi(x)^{-1}(y))$. Then by the above discussion, we deduce that

$$T_{x\cdot f} = \sigma(x) \circ T_{c_x \cdot f} \circ \sigma(x)^{-1}. \tag{14}$$

For every $y \in H$, let $l(y) : L^2(H) \to L^2(H), (l(y))(f) := y \cdot f$. Then $l(y)$ is a unitary map and $T_{y\cdot f} = l(y) \circ T_f$. Hence by (14), we obtain

$$T_{x\cdot f} = \sigma(x) \circ l(c_x) \circ T_f \circ \sigma(x)^{-1}. \tag{15}$$

Therefore by [11, Theorem 7.4.24] and (15), we conclude that $|||T_{x\cdot f}||| = |||T_f|||$, and so

$$|||x \cdot f||| = |H|^{-1/4}|||T_{x\cdot f}||| = |H|^{-1/4}|||T_f||| = |||f|||.$$

(iii) For every probability measure μ on G, by the first two parts we have

$$|||\mu \boxtimes f||| \le \Big|\Big|\Big| \sum_{x\in G} \mu(x) x \cdot f \Big|\Big|\Big| \le \sum_{x\in G} \mu(x)|||x \cdot f||| = |||f|||.$$

This completes the proof. ∎

The following inequality plays an important role in proving Lemma 4.

Lemma 10. *We have* $|||\mu \boxtimes f|||^2 \le \| \int_G \widetilde{(u \cdot f)} * (u \cdot f)\, d\mu(u)\|_2$.

Proof. Observe that $\mu \boxtimes f = \int_G (x \cdot f)\, d\mu(x)$ and $\widetilde{\mu \boxtimes f} = \int_G \widetilde{x \cdot f}\, d\mu(x)$. Therefore similar to Lemma 8 (ii), we have

$$(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f}) = \int_{G^2} (u \cdot f) * (\widetilde{v \cdot f})\, d\mu^{\otimes 2}(u, v). \tag{16}$$

By (16) and the semilinearity of the dot product, we obtain

$$\begin{aligned} &\|(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f})\|_2^2 \\ &\quad = \int_{G^4} \langle (u_1 \cdot f) * (\widetilde{v_1 \cdot f}), (u_2 \cdot f) * (\widetilde{v_2 \cdot f})\rangle\, d\mu^{\otimes 4}(u_1, v_1, u_2, v_2). \end{aligned} \tag{17}$$

Let $c : G^4 \to \mathbb{C}$, $c(u_1, v_1, u_2, v_2) := \langle (u_1 \cdot f) * (\widetilde{v_1 \cdot f}), (u_2 \cdot f) * (\widetilde{v_2 \cdot f})\rangle$. Then by the Cauchy–Schwarz inequality, we have $\|c\|_1 \le \|c\|_2$ with respect to the probability measure $\mu^{\otimes 4}$, and so by (17), we deduce that

$$\|(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f})\|_2^2 \le \left(\int_{G^4} |c|^2\, d\mu^{\otimes 4}\right)^{1/2}. \tag{18}$$

Another application of the Cauchy–Schwarz inequality implies that

$$\begin{aligned} |c(u_1, v_1, u_2, v_2)|^2 &\le \|(u_1 \cdot f) * (\widetilde{v_1 \cdot f})\|_2^2 \|(u_2 \cdot f) * (\widetilde{v_2 \cdot f})\|_2^2 \\ &= c(u_1, v_1, u_1, v_1) c(u_2, v_2, u_2, v_2). \end{aligned} \tag{19}$$

By (18) and (19), we obtain

$$\|(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f})\|_2^2 \le \left(\int_{G^4} c(u_1, v_1, u_1, v_1)c(u_2, v_2, u_2, v_2)\, d\mu^{\otimes 4}\right)^{1/2} = \int_{G^2} c(u, v, u, v)\, d\mu^{\otimes 2}. \tag{20}$$

Notice that by Lemma 8 (i), for all $u, v \in G$ we have

$$\begin{aligned} c(u, v, u, v) &= \langle (u \cdot f) * (\widetilde{v \cdot f}), (u \cdot f) * (\widetilde{v \cdot f})\rangle \\ &= \langle (\widetilde{u \cdot f}) * (u \cdot f), (\widetilde{v \cdot f}) * (v \cdot f)\rangle. \end{aligned} \tag{21}$$

By (20), (21), and the semilinearity of the dot product, we deduce that

$$\begin{aligned} \|(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f})\|_2^2 &\le \int_{G^2} \langle (\widetilde{u \cdot f}) * (u \cdot f), (\widetilde{v \cdot f}) * (v \cdot f)\rangle\, d\mu^{\otimes 2}(u, v) \\ &= \left\langle \int_G (\widetilde{u \cdot f}) * (u \cdot f)\, d\mu(u), \int_G (\widetilde{v \cdot f}) * (v \cdot f)\, d\mu(v) \right\rangle. \end{aligned} \tag{22}$$

By (22), we conclude that $\|(\mu \boxtimes f) * (\widetilde{\mu \boxtimes f})\|_2 \le \| \int_G (\widetilde{u \cdot f}) * (u \cdot f)\, d\mu(u)\|_2$, which completes the proof. ∎

We finish this section by proving Lemma 4.

Proof of Lemma 4. Note that the law of $(X^{(1)} \cdot Y^{(1)})^{-1}(X^{(1)} \cdot Y^{(2)})$ is given by

$$\int_G (\widetilde{u \cdot \eta}) * (u \cdot \eta)\, d\mu(u),$$

and the law of $(X^{(1)} \cdot Y^{(1)})(X^{(2)} \cdot Y^{(2)})^{-1}$ is given by

$$(\mu \boxtimes \eta) * (\widetilde{\mu \boxtimes \eta}).$$

Therefore by Lemma 10 we conclude that

$$\begin{aligned} H_2((X^{(1)} \cdot Y^{(1)})^{-1}(X^{(2)} \cdot Y^{(2)})) &= H_2((X^{(1)} \cdot Y^{(1)})(X^{(2)} \cdot Y^{(2)})^{-1}) \\ &\ge H_2((X^{(1)} \cdot Y^{(1)})^{-1}(X^{(1)} \cdot Y^{(2)})), \end{aligned}$$

which finishes the proof. ∎

Corollary 11. *Suppose H is a finite group, μ is a probability measure on $H \times H$, and η is a probability measure on H. Consider the action*

$$H \times H \curvearrowright H \quad \text{given by} \quad (x_L, x_R) \cdot y := x_L y x_R^{-1}$$

and the conjugation action $H \curvearrowright H$. Accordingly define $\boxtimes$ from $L^2(H \times H) \times L^2(H)$ to $L^2(H)$ and $\boxtimes$ from $L^2(H) \times L^2(H)$ to $L^2(H)$. Then

$$\|\mu \boxtimes \eta\|^2 \le \|\pi_R[\mu] \boxtimes (\check{\eta} * \eta)\|_2,$$

where $\pi_R : H \times H \to H$ is the projection to the right component.

Proof. Suppose $X^{(1)} := (X_L^{(1)}, X_R^{(1)})$ and $X^{(2)} := (X_L^{(2)}, X_R^{(2)})$ are two independent random variables with probability law μ, and $Y^{(1)}$ and $Y^{(2)}$ are two independent random variables with probability law η. Applying Lemma 4 for the given group action $H \times H \curvearrowright H$, we obtain

$$H_2((X^{(1)} \cdot Y^{(1)})(X^{(2)} \cdot Y^{(2)})^{-1}) \geq H_2((X^{(1)} \cdot Y^{(1)})^{-1}(X^{(1)} \cdot Y^{(2)})). \tag{23}$$

Notice that $(X^{(1)} \cdot Y^{(1)})^{-1}(X^{(1)} \cdot Y^{(2)}) = X_R^{(1)} Y^{(1)^{-1}} Y^{(2)} X_R^{(1)^{-1}}$, and so the distribution of this random variable is given by

$$\pi_R[\mu] \boxtimes (\check{\eta} * \eta). \tag{24}$$

We also notice that the distribution of $(X^{(1)} \cdot Y^{(1)})(X^{(2)} \cdot Y^{(2)})^{-1}$ is given by $(\mu \boxtimes \eta) * \widetilde{(\mu \boxtimes \eta)}$. Hence by (23), we conclude that

$$\|(\mu \boxtimes \eta) * \widetilde{(\mu \boxtimes \eta)}\|_2 \leq \|\pi_R[\mu] \boxtimes (\check{\eta} * \eta)\|_2,$$

which completes the proof. ∎

4. An escaping lemma

The main goal of this section is to prove Lemma 5. Our approach is inspired by a method of Lindenstrauss and Varjú developed in [14]. In this section, we will be working with the action $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p) \curvearrowright \mathrm{PSL}_2(\mathbb{F}_p)$ given by $(u, v) \cdot x = uxv^{-1}$, and the conjugation action $\mathrm{PSL}_2(\mathbb{F}_p) \curvearrowright \mathrm{PSL}_2(\mathbb{F}_p)$ given by $x \cdot y = xyx^{-1}$. We reformulate the statement of Lemma 5 in terms of distributions.

Lemma 12. *Let μ be a measure on $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$. Suppose that the minimum of μ on its support is at least $\alpha_0 > 0$ and $\mathcal{L}(\pi_R[\mu]) \geq c_0 > 0$, where $\pi_R : \mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p)$ is the projection to the right component. Suppose that the support of μ generates $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$. Then there exist constants $L, C \gg_{c_0,\alpha_0} 1$ such that for every probability measure η on $\mathrm{PSL}_2(\mathbb{F}_p)$ and every integer $\ell \geq L \log p$,*

$$\|\mu^{*(\ell)} \boxtimes \eta\|_2 \leq C\,|\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)|^{-1/24}.$$

We start by proving that if $\check{\eta} * \eta$ is almost a point mass at the identity, then η is almost a point mass. Here, a measure is viewed as almost a point mass if a single point is responsible for a κ portion of its L^2-norm. This lemma is essentially proved in [14, Lemma 5].

Lemma 13. *Suppose η is a probability measure on a finite group H and $\|\eta\|_\infty \leq \kappa \|\eta\|_2$ for some $\kappa > 0$. Then*

$$\|\eta\|^2 \geq \sqrt{2 - \kappa^2}\, \check{\eta} * \eta(1).$$

Proof. Notice that

$$\check{\eta} * \eta(1) = \sum_{y \in H} \check{\eta}(y^{-1})\eta(y) = \sum_{y \in H} \eta(y)^2 = \|\eta\|_2^2. \tag{25}$$

Next by a simple computation we obtain

$$\sum_{y\in H\setminus\{1\}} \check{\eta}*\eta(y)^2 = \sum_{y\neq 1}\Big(\sum_{y_1y_2=y}\check{\eta}(y_1)\eta(y_2)\Big)^2 \geq \sum_{y\neq 1}\sum_{y_1y_2=y}\check{\eta}(y_1)^2\eta(y_2)^2$$
$$= \Big(\sum_{y_1}\check{\eta}(y_1)^2\Big)\Big(\sum_{y_2}\eta(y_2)^2\Big) - \sum_{y_2}\check{\eta}(y_2^{-1})^2\eta(y_2)^2 = \|\eta\|_2^4 - \|\eta\|_4^4. \quad (26)$$

On the other hand, because $\kappa\|\eta\|_2 \geq \|\eta\|_\infty$, we deduce that

$$\|\eta\|_4^4 = \sum_y \eta(y)^4 \leq \|\eta\|_\infty^2\|\eta\|_2^2 \leq \kappa^2\|\eta\|_2^4. \quad (27)$$

By (25)–(27), we obtain

$$\|\check{\eta}*\eta\|_2^2 \geq \check{\eta}*\eta(1)^2 + (\|\eta\|_2^4 - \|\eta\|_4^4) \geq \|\eta\|_2^4 + (\|\eta\|_2^4 - \kappa^2\|\eta\|_2^4)$$
$$= (2-\kappa^2)\check{\eta}*\eta(1)^2. \quad (28)$$

This completes the proof. ∎

Next we show that if Z is a random variable with values in $\mathrm{PSL}_2(\mathbb{F}_p)$ and uniform distribution and Y is an independent random variable with values in $\mathrm{PSL}_2(\mathbb{F}_p)$, then

$$H_2(ZYZ^{-1}) \geq \min\{-2\log(\mathbb{P}(Y=1)), \log p\} - 2\log 2.$$

Notice that since $\mathbb{P}(ZYZ^{-1}=1) = \mathbb{P}(Y=1)$, we have

$$H_2(ZYZ^{-1}) \leq -2\log(\mathbb{P}(Y=1)).$$

Similar to the previous lemmas in this section, we formulate the lemma in terms of distributions.

Lemma 14. *Let $H := \mathrm{PSL}_2(\mathbb{F}_p)$ and μ_H be the probability counting measure on H. Suppose η is a probability measure on H. Consider the conjugation action $H \curvearrowright H$ given by $z\cdot y := zyz^{-1}$, and let $\mu_H \boxtimes \eta$ be the convolution associated to the conjugation action. Then*

$$\|\mu_H \boxtimes \eta\|_2 \leq \eta(1) + p^{-1/2}.$$

Proof. For every $x \in H$, let $\mathrm{Cl}(x)$ be the conjugacy class of x. Since H is generated by its elements of order p, for every proper subgroup $K \subsetneq H$ we have $[H:K] \geq p$. Hence for every $x \in H$, either $|\mathrm{Cl}(x)| \geq p$ or $\mathrm{Cl}(x) = \{x\}$. The latter holds exactly when x is in the center $Z(H)$ of H. Notice that $Z(H) = \{1\}$, and so for every $x \in H\setminus\{1\}$,

$$|\mathrm{Cl}(x)| \geq p. \quad (29)$$

Recall that for every $A \subseteq H$, μ_A denotes the probability counting measure on A. Notice that for every $x \in H$, we have

$$\mu_H \boxtimes \mu_{\{x\}} = \mu_{\mathrm{Cl}(x)}.$$

Hence by the bilinearity of $\boxtimes$, we have

$$\mu_H \boxtimes \eta = \sum_{x\in H} \eta(x)\mu_{\mathrm{Cl}(x)} = \sum_{c\in \mathrm{Conj}(H)} \eta(c)\mu_c, \tag{30}$$

where $\mathrm{Conj}(H)$ is the set of all conjugacy classes of H. By (30), the triangle inequality, and (29), we conclude that

$$\begin{aligned}\|\mu_H \boxtimes \eta\|_2 \le& \sum_{c\in \mathrm{Conj}(H)} \eta(c)\|\mu_c\|_2 = \sum_{c\in \mathrm{Conj}(H)} \eta(c)|c|^{-1/2}\\ \le& \eta(1) + \eta(H\setminus\{1\})p^{-1/2}.\end{aligned}$$

This completes the proof. ∎

Before proving Lemma 5, we show a lemma on symmetric generating sets of a finite group.

Lemma 15. *Suppose S is a symmetric generating set of a group G. Suppose G acts on a set Ω and there are distinct points $x_1, x_2 \in \Omega$ such that $s\cdot x_1 = x_2$ for every $s \in S$. Then G has a subgroup of index 2.*

Proof. By assumption we have $s^{-1}\cdot x_2 = x_1$ for every $s \in S$. Because S is symmetric, we deduce that $s\cdot x_2 = x_1$ for every $s \in S$. Hence $(s_1 \dots s_k)\cdot x_1 \in \{x_1, x_2\}$ for all $s_1, \dots, s_k \in S$. On other hand, because S is a symmetric generating set of G, it follows that $G = \{1\} \cup \bigcup_{k=1}^{\infty} \prod_k S$; and so the G-orbit of x_1 has exactly two points. Therefore the stabilizer subgroup of G with respect to x_1 is of index 2. This completes the proof. ∎

Notice that $\mathrm{PSL}_2(\mathbb{F}_p)$ is generated by its p-elements, and so if $p > 2$ then $\mathrm{PSL}_2(\mathbb{F}_p)$ does not have a subgroup of order 2.

In the rest of this section, we prove Lemma 12, which is a reformulation of Lemma 5.

Proof of Lemma 12. Recall that α_0 is the minimum of μ on its support. Choose $0 < \kappa_0 < 1$ such that $\sqrt{\alpha_0^2 + (1-\alpha_0)^2} + \sqrt{1-\kappa_0^2} < 1$. Let η be the probability law of Y. We are going to consider two cases mostly depending on whether or not η is almost a point mass or not.

Case 1: $\|\eta\|_\infty/\|\eta\|_2 > \kappa_0$. In this case, there exists $x_0 \in \mathrm{PSL}_2(\mathbb{F}_p)$ such that $\eta(x_0)^2 > \kappa_0^2\|\eta\|_2^2$. Let $\eta_{x_0}^{\perp} := \eta\mathbb{1}_{\mathrm{PSL}_2(\mathbb{F}_p)\setminus\{x_0\}}$ where $\mathbb{1}_{\mathrm{PSL}_2(\mathbb{F}_p)\setminus\{x_0\}}$ is the characteristic function of $\mathrm{PSL}_2(\mathbb{F}_p)\setminus\{x_0\}$. Notice that

$$\eta = \eta(x_0)\mu_{\{x_0\}} + \eta_{x_0}^{\perp} \quad \text{and} \quad \mu_{\{x_0\}} \perp \eta_{x_0}^{\perp}. \tag{31}$$

By (31), we have $\|\eta\|_2^2 = \eta(x_0)^2 + \|\eta_{x_0}^{\perp}\|_2^2$, and so

$$\|\eta_{x_0}^{\perp}\|_2^2 < (1-\kappa_0^2)\|\eta\|_2^2. \tag{32}$$

Moreover, by (31), we obtain $\mu\boxtimes\eta = \eta(x_0)\,\mu\boxtimes\mu_{\{x_0\}} + \mu\boxtimes\eta_{x_0}^{\perp}$. Notice that

$$\mu\boxtimes\mu_{\{x_0\}} = \sum_{y\in \mathrm{PSL}_2(\mathbb{F}_p)\times\mathrm{PSL}_2(\mathbb{F}_p)} \mu(y)\mu_{\{y\cdot x_0\}} = \sum_{yG_{x_0}\in G/G_{x_0}} \mu(yG_{x_0})\mu_{y\cdot x_0}, \tag{33}$$

where $G := \mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$ and G_{x_0} is the stabilizer subgroup of G with respect to x_0. Since the support of μ is a symmetric generating set of $\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)$ and the minimum of μ on its support is α_0, by Lemma 15 and (33) we conclude that

$$\|\mu \boxtimes \mu_{\{x_0\}}\|_2 \le (\alpha_0^2 + (1-\alpha_0)^2)^{1/2}. \tag{34}$$

Therefore by the triangle inequality, (34), and (32), we conclude that

$$\begin{aligned}\|\mu \boxtimes \eta\|_2 &\le \eta(x_0)\|\mu \boxtimes \mu_{\{x_0\}}\|_2 + \|\mu \boxtimes \eta^{\perp}_{x_0}\|_2 \\ &\le \eta(x_0)(\alpha_0^2 + (1-\alpha_0)^2)^{1/2} + \|\eta^{\perp}_{x_0}\|_2 \\ &\le \big((\alpha_0^2 + (1-\alpha_0)^2)^{1/2} + (1-\kappa_0^2)^{1/2}\big)\|\eta\|_2. \end{aligned} \tag{35}$$

Case 2: $\|\eta\|_\infty/\|\eta\|_2 \le \kappa_0$. Choose $0 < \kappa_1 < 1$ such that $(2-\kappa_0^2)^{-1/2} + 2\kappa_1 < 1$. Since $\mathcal{L}(\pi_R[\mu]) > c_0$, there is a positive integer ℓ_0 which is bounded by a function of c_0 (and κ_1) such that $\lambda(\pi_R[\mu]^{*(\ell_0)}) < \kappa_1$. Set $H := \mathrm{PSL}_2(\mathbb{F}_p)$ and $\nu := \mu^{*(\ell_0)}$, and so $\lambda(\pi_R[\nu]) < \kappa_1$. Then by Corollary 11,

$$\begin{aligned}|||\nu \boxtimes \eta|||^2 &\le \|\pi_R[\nu] \boxtimes (\check{\eta} * \eta)\|_2 \\ &\le \|(\pi_R[\nu] - \mu_H) \boxtimes (\check{\eta} * \eta)\|_2 + \|\mu_H \boxtimes (\check{\eta} * \eta)\|_2. \end{aligned} \tag{36}$$

By (36), (9), and Lemma 14, we deduce that

$$|||\nu \boxtimes \eta|||^2 \le \kappa_1 |||\eta|||^2 + \check{\eta} * \eta(1) + p^{-1/2}. \tag{37}$$

By (37) and Lemma 13, we have

$$|||\nu \boxtimes \eta|||^2 \le \kappa_1 |||\eta|||^2 + (2-\kappa_0^2)^{-1/2}|||\eta|||^2 + p^{-1/2}. \tag{38}$$

Next we combine the inequalities in (35) and (38). Suppose β is a positive number less than 1 which is more than

$$\max\{(2-\kappa_0^2)^{-1/2} + 2\kappa_1, (\alpha_0^2 + (1-\alpha_0)^2)^{1/2} + (1-\kappa_0^2)^{1/2}\}.$$

Then by (35) and (38), at least one of the following three inequalities holds:

$$\|\nu \boxtimes \eta\|_2 \le \beta\|\eta\|_2, \quad \text{or} \quad |||\nu \boxtimes \eta|||^2 \le \beta|||\eta|||^2, \quad \text{or} \quad |||\eta|||^2 \le \kappa_1^{-1} p^{-1/2}. \tag{39}$$

Applying (39) repeatedly, by Lemma 9, we conclude that for every integer $\ell \ge 2\log p/(-\log\beta)$ at least one of the following three inequalities holds:

$$\begin{aligned}\|\nu^{*(\ell)} \boxtimes \eta\|_2 &< \beta^{\ell/2} \le 1/p, \quad \text{or} \\ |||\nu^{*(\ell)} \boxtimes \eta|||^2 &< \beta^{\ell/2} \le 1/p, \quad \text{or} \\ |||\nu^{*(\ell)} \boxtimes \eta|||^2 &< \kappa_1^{-1} p^{-1/2}. \end{aligned} \tag{40}$$

By Lemma 9 (i) and (40), for every $\ell \ge 2\log p/(-\log\beta)$,

$$\|\nu^{*(\ell)} \boxtimes \eta\|_2 \le \kappa_1^{-1/2} p^{-1/4}.$$

Since $|\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)| \le p^6$, we conclude that

$$\|\nu^{*(\ell)} \boxtimes \eta\|_2^2 \le \kappa_1^{-1} |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)|^{-1/12},$$

which completes the proof. ∎

5. Proving the main result

In this section, we will be using the Bourgain–Gamburd method to prove Theorem 1 based on Lemma 5. Bourgain and Gamburd in their seminal work [3] laid out a blueprint for finding a bound for the spectral gap of a random walk in a (single scale) finite group. One of the important results that they proved is the following proposition (see [3], [20], [15], [18]).

Proposition 16. *Let G be a finite group. Suppose X and Y are two independent random variables with values in G, and $K \geq 2$. If*

$$H_2(XY) \leq \frac{H_2(X) + H_2(Y)}{2} + \log K,$$

then there exist $A \subseteq G$ and a universal fixed positive number R with the following properties:

(i) (Approximate structure) *A is a K^R-approximate subgroup; by this we mean A is symmetric, $1 \in A$, and there is a subset B of $A \cdot A$ such that $|B| \leq K^R$ and $A \cdot A \subseteq B \cdot A \cap A \cdot B$.*

(ii) (Controlling the size) $\big|\log|A| - H_2(X)\big| \leq R \log K$.

(iii) (Almost equidistribution) *For every $a \in A$,*

$$\mathbb{P}(X'X = a) \geq \frac{1}{K^R|A|},$$

where X' is an independent random variable that has the same distribution as X^{-1}.

In this section, using Proposition 16, we will first prove the following lemma.

Lemma 17. *Let $\varepsilon > 0$ and $H := \mathrm{SL}_2(\mathbb{F}_p)$. Suppose $Y := (Y_L, Y_R)$ is a random variable with values in $H \times H$, distribution $\mathcal{P}$, and the following properties:*

- (Close to a coupling) *For every $y \in H$, $\mathbb{P}(Y_L = y) \leq 2|H|^{-1}$ and $\mathbb{P}(Y_R = y) \leq 2|H|^{-1}$.*
- (Room for improvement) $H_2(Y) \leq (1 - \varepsilon)\log|H \times H|$.

Then there is a positive number γ_0 depending on ε and a universal positive constant R such that for every positive $\gamma \leq \gamma_0$ at least one of the following statements hold:

(i) (Exceptional cases) $|H|^{R\gamma} \leq 2$.

(ii) (Gaining entropy) $H_2(Y_2) \geq H_2(Y) + \gamma \log|H \times H|$ *where Y_2 is the 2-step random walk with respect to $\mathcal{P}$.*

(iii) (Graph of an automorphism) $\mathbb{P}(\overline{\iota}(Y_2) \in \Gamma_\phi) \geq |H \times H|^{-R\gamma}$ *for some automorphism ϕ of $\mathrm{PSL}_2(\mathbb{F}_p)$, where*

$$\overline{\iota} : H \times H \to \mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p), \quad \overline{\iota}(x_L, x_R) := (\{\pm x_L\}, \{\pm x_R\}),$$

and $\Gamma_\phi := \{(x, \phi(x)) \mid x \in \mathrm{PSL}_2(\mathbb{F}_p)\}$.

Theorem 1 will be proved using Lemmas 17, 5, and 7.

Proof of Lemma 17. Suppose $\gamma \le \gamma_0$, where γ_0 is a sufficiently small positive number to be specified later. Assume that we are not in the *exceptional cases* and we do not *gain enough entropy*; that means $|H|^{R\gamma} > 2$ and $H_2(Y_2) < H_2(Y) + \gamma \log |H \times H|$. Then by Proposition 16, there is an $|H \times H|^{R\gamma}$-approximate subgroup A such that

$$\big|\log |A| - H_2(Y)\big| \le R\gamma \log |H \times H| \quad \text{and} \quad \mathbb{P}(Y_2 \in A) \ge |H \times H|^{-R\gamma}. \tag{41}$$

Notice that by the *close to a coupling* condition, we have

$$\mathbb{P}((Y_L)_2 \in \pi_L(A)) \le 2|\pi_L(A)|\,|H|^{-1}.$$

Therefore,

$$|H \times H|^{-R\gamma} \le \mathbb{P}(Y_2 \in A) \le \mathbb{P}((Y_L)_2 \in \pi_L(A)) \le 2|\pi_L(A)|\,|H|^{-1}.$$

A similar result holds for the projection to the right copy of H. Hence

$$|\pi_L(A)| \ge |H|^{1-3R\gamma} \quad \text{and} \quad |\pi_R(A)| \ge |H|^{1-3R\gamma}. \tag{42}$$

Notice that by a result of Frobenius [7], the degree of every non-trivial irreducible representation of $\mathrm{SL}_2(\mathbb{F}_p)$ is at least $(p-1)/2$, and so there exists $c_1 > 0$ such that for all primes $p \ge 5$, $\mathrm{SL}_2(\mathbb{F}_p)$ is c_1-quasi-random. Therefore by a result of Gowers (see [2, 9]) and (42), for $\gamma < c_1/(9R)$, we have

$$\textstyle\prod_3 \pi_L(A) = \mathrm{SL}_2(\mathbb{F}_p) \quad \text{and} \quad \prod_3 \pi_R(A) = \mathrm{SL}_2(\mathbb{F}_p). \tag{43}$$

Claim 1. *In the above setting, for a small enough γ_0 depending only on ε, we have*

$$\textstyle\prod_9 A \cap (H \times \{\pm 1\}) \subseteq \{(\pm 1, \pm 1)\} \quad \textit{and} \quad \prod_9 A \cap (\{\pm 1\} \times H) \subseteq \{(\pm 1, \pm 1)\}.$$

Proof of Claim 1. It is clear that by symmetry it is enough to prove only one of the inclusions. Suppose to the contrary that $(x, e) \in \prod_9 A$ for some $x \in \mathrm{SL}_2(\mathbb{F}_p) \setminus \{\pm 1\}$ and $e \in \{\pm 1\}$. Since $\pi_L(\prod_3 A) = \mathrm{SL}_2(\mathbb{F}_p)$, we obtain

$$\mathrm{Cl}(x) \times \{e\} \subseteq \textstyle\prod_{15} A. \tag{44}$$

By [21, Theorems 2.2, 2.3], we have $\prod_3 \mathrm{Cl}(x) = \mathrm{SL}_2(\mathbb{F}_p)$ for every prime $p \ge 5$. Therefore, by (44), $\mathrm{SL}_2(\mathbb{F}_p) \times \{1\} \subseteq \prod_{90} A$. Hence by (43),

$$\textstyle\prod_{93} A = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p). \tag{45}$$

Because A is an $|H \times H|^{R\gamma}$-approximate subgroup,

$$|\textstyle\prod_{93} A| \le |H \times H|^{92R\gamma}|A|. \tag{46}$$

Thus by (45) and (46), we conclude that

$$|A| \ge |H \times H|^{1-92R\gamma}. \tag{47}$$

By the condition on the *room for improvement* and (41), we deduce that

$$|A| \le |H \times H|^{1-\varepsilon+R\gamma}. \tag{48}$$

By (47) and (48), we reach a contradiction if $\gamma < \varepsilon/(93R)$. This completes the proof of Claim 1.

Claim 2. *Let* $\iota : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p), \iota(x) := \{\pm x\}$ *and* $\overline{A} := \iota(A)$. *Then there is an automorphism* $\phi : \mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p)$ *such that* $\prod_3 \overline{A} = \Gamma_\phi$.

Proof of Claim 2. By (43), for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$, there is $\widetilde{\phi}(x) \in \mathrm{SL}_2(\mathbb{F}_p)$ such that $(x, \widetilde{\phi}(x))$ is in $\prod_3 A$. Since A is a symmetric set, for all $x, y \in \mathrm{SL}_2(\mathbb{F}_p)$ we obtain

$$(1, \widetilde{\phi}(x)\widetilde{\phi}(y)\widetilde{\phi}(xy)^{-1}) = (x, \widetilde{\phi}(x))(y, \widetilde{\phi}(y))(xy, \widetilde{\phi}(xy))^{-1} \in \prod_9 A. \tag{49}$$

By (49) and Claim 1, we deduce that $\widetilde{\phi}(xy) = \pm\widetilde{\phi}(x)\widetilde{\phi}(y)$ for all x, y in $\mathrm{SL}_2(\mathbb{F}_p)$. Notice that for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$, $(x, y) \in \prod_3 A$ implies that $(1, \widetilde{\phi}(x)y^{-1}) \in \prod_9 A$. Hence by Claim 1, we have $y = \pm\widetilde{\phi}(x)$. Let

$$\phi : \mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PSL}_2(\mathbb{F}_p), \quad \phi(\{\pm x\}) := \{\pm\widetilde{\phi}(x)\}.$$

Notice that because $\widetilde{\phi}(xy) = \pm\widetilde{\phi}(x)\widetilde{\phi}(y)$, ϕ is a well-defined group homomorphism. By (43), we deduce that ϕ is surjective, and Claim 1 implies that it is injective. Altogether, ϕ is an automorphism of $\mathrm{PSL}_2(\mathbb{F}_p)$ and $\iota(\prod_3 A) = \Gamma_\phi$. This completes the proof of Claim 2.

By Claim 2 and (41), we conclude that $\mathbb{P}(\overline{\iota}(Y_2) \in \Gamma_\phi) \ge |H \times H|^{-R\gamma}$, which completes the proof. ∎

Proof of Theorem 1. Because $\mathcal{L}(\pi_L[\mu])$ and $\mathcal{L}(\pi_R[\mu])$ are at least c_0, by the Cauchy–Schwarz inequality and (9), for $\ell_0 \ge (2\log|\mathrm{SL}_2(\mathbb{F}_p)|)/c_0$, we have

$$\begin{aligned}\|\pi_L[\mu]^{*(\ell_0)} - \mu_H\|_\infty &= \|((\pi_L[\mu]^{*(\ell_0)} - \mu_H) * \mu_{\{1\}}) * \mu_{\{1\}}\|_\infty \\ &\le \|(\pi_L[\mu]^{*(\ell_0)} - \mu_H) * \mu_{\{1\}}\|_2 \le |H|^{-2}.\end{aligned} \tag{50}$$

By (50) and its counterpart for $\pi_R[\mu]$, for every $x \in H$ we obtain

$$\begin{aligned}\left|\pi_L[\mu^{*(\ell_0)}](x) - |H|^{-1}\right| &\le 2|H|^{-1}, \\ \left|\pi_R[\mu^{*(\ell_0)}](x) - |H|^{-1}\right| &\le 2|H|^{-1}.\end{aligned} \tag{51}$$

Suppose ℓ is a positive integer which is at least ℓ_0 and will be specified later. Let Y be a random variable with values in $H \times H$ and distribution $\mu^{*(\ell)}$.

As has been mentioned earlier, by a result of Frobenius every non-trivial representation of $\mathrm{SL}_2(\mathbb{F}_p)$ is of dimension at least $(p-1)/2$. Hence there is a $c_2 > 0$ such that $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ is a c_2-quasi-random group.

By Lemma 17, there is a $\gamma_0 > 0$ depending on c_2 and a universal positive constant R such that for every positive number $\gamma \le \gamma_0$, one of the following statements holds: either $|H|^{R\gamma} \le 2$, or

$$H_2(Y) > (1 - c_2/2)\log|H \times H|, \tag{52}$$

or there is an automorphism ϕ of $\mathrm{PSL}_2(\mathbb{F}_p)$ such that

$$\mathbb{P}(\iota(Y_2) \in \Gamma_\phi) \geq |H \times H|^{-R\gamma}, \tag{53}$$

where Γ_ϕ is the graph of ϕ, or

$$H_2(Y_2) \geq H_2(Y) + \gamma \log |H \times H|. \tag{54}$$

Claim 1. *There are positive numbers p_0, γ_1, and L' all depending only on c_0 and α_0 such that (53) does not hold for any automorphism ϕ, positive integer $\ell \geq L' \log |H \times H|$, positive number $\gamma = \gamma_1$, and prime $p \geq p_0$.*

Proof of Claim 1. Suppose to the contrary that (53) holds for an automorphism ϕ of $\mathrm{PSL}_2(\mathbb{F}_p)$. Let $X := (X_L, X_R)$ be a random variable with values in $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ and distribution μ. Let $\bar{X} := (\iota(X_L), \phi^{-1}(\iota(X_R)))$. Then by Lemma 5, there exist constants $L, C \gg_{c_0,\alpha_0} 1$ such that for every integer $\ell \geq L \log |H \times H|$ we have

$$H_2(\bar{X}_\ell \cdot Z) \geq \tfrac{1}{12} \log |\mathrm{PSL}_2(\mathbb{F}_p) \times \mathrm{PSL}_2(\mathbb{F}_p)| - C \tag{55}$$

where Z is a random variable with values in $\mathrm{PSL}_2(\mathbb{F}_p)$ and distribution $\mu_{\{1\}}$. Notice that

$$\begin{aligned} \mathbb{P}(\bar{X}_\ell \cdot Z = 1) &= \mathbb{P}\big(\iota((X_L)_\ell) = \phi^{-1}(\iota((X_R)_\ell))\big) = \mathbb{P}(\bar{\iota}(X_\ell) \in \Gamma_\phi) \\ &\geq |H \times H|^{-R\gamma_0}. \end{aligned} \tag{56}$$

By (56), we conclude that

$$H_2(\bar{X}_\ell \cdot Z) \leq R\gamma_0 \log |H \times H|. \tag{57}$$

By (55) and (57), we reach a contradiction if $p \gg_{c_0,\alpha_0} 1$. This completes proof of Claim 1.

Claim 2. *Suppose ℓ_1 is an integer in the interval*

$$[L' \log |H \times H|, 2L' \log |H \times H|].$$

Let X be a random variable with distribution μ. Then

$$H_2(X_\ell) > (1 - c_2/2) \log |H \times H|$$

for every integer $\ell \geq 2^{\gamma_1^{-1}} \ell_1$.

Proof of Claim 2. Let $Y := X_{\ell_1}$, and consider the sequence $\{H_2(Y_{2^k})\}_k$ of Rényi entropies. By Claim 1, for every positive integer k, either

$$\begin{aligned} H_2(Y_{2^k}) &> (1 - c_2/2) \log |H \times H|, \quad \text{or} \\ H_2(Y_{2^{k+1}}) &\geq H_2(Y_{2^k}) + \gamma_1 \log(|H \times H|). \end{aligned}$$

Hence $H_2(Y_{2^k}) > (1 - c_2/2) \log(|H \times H|)$ if k is an integer greater than γ_1^{-1}. This completes the proof of Claim 2.

By Claim 2 and Lemma 7, we conclude that either $p \ll_{c_0,\alpha_0} 1$ or

$$\mathcal{L}(\mu) \geq \frac{c_2}{2^{\gamma_1^{-1}+1} L'} \gg_{\alpha_0,c_0} 1.$$

This completes the proof of the main theorem. ∎

6. Proof of Corollary 3

We can and will assume that $p \geq 5$. Suppose ρ_1 and ρ_2 are two distinct points of $\mathrm{Rep}_2(\mathbb{F}_p)_\delta$. Let

$$\Omega := \{(\rho_1(a), \rho_2(a))^{\pm 1}, (\rho_1(b), \rho_2(b))^{\pm 1}\}.$$

Let H be the group generated by Ω. Notice that $\pi_L(H) = \mathrm{Im}(\rho_1) = \mathrm{SL}_2(\mathbb{F}_p)$ and $\pi_R(H) = \mathrm{Im}(\rho_2) = \mathrm{SL}_2(\mathbb{F}_p)$. Therefore for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$, there is $\widetilde{\phi}(x) \in \mathrm{SL}_2(\mathbb{F}_p)$ such that $(x, \widetilde{\phi}(x)) \in H$. Notice that if $(y, 1) \in H$, then for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$,

$$(xyx^{-1}, 1) = (x, \widetilde{\phi}(x))(y, 1)(x, \widetilde{\phi}(x))^{-1} \in H.$$

Therefore $\pi_L(H \cap \ker \pi_R)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$, and so it is either $\mathrm{SL}_2(\mathbb{F}_p)$ or central. Similarly $\pi_R(H \cap \ker \pi_L)$ is either $\mathrm{SL}_2(\mathbb{F}_p)$ or central. Altogether, either $H = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ or $\bar{\iota}(H) = \Gamma_\phi$ where Γ_ϕ is the graph of an automorphism ϕ of $\mathrm{PSL}_2(\mathbb{F}_p)$. Notice that by [22, Theorem 3.2], there is an automorphism $\widehat{\phi}$ of $\mathrm{SL}_2(\mathbb{F}_p)$ such that $\phi(\{\pm x\}) = \{\pm\widehat{\phi}(x)\}$ for every $x \in \mathrm{SL}_2(\mathbb{F}_p)$.

Let μ be the probability counting measure on Ω. Since $\rho_1, \rho_2 \in \mathrm{Rep}_2(\mathbb{F}_p)_\delta$, $\mathfrak{L}(\pi_L[\mu])$ and $\mathfrak{L}(\pi_R[\mu])$ are at least δ. Hence if $H = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$, then by Theorem 1, $\lambda(\mu) \leq \lambda_0$ for some positive number λ_0 which is less than 1 and only depends on δ. Then for every positive integer ℓ, we have

$$\begin{aligned}
&\mu^{*(\ell)}_{\{a^\pm, b^\pm\}}(\{w \in F_2 \mid \rho_1(w) = \rho_2(w)\}) \\
&\qquad \leq |\mathrm{SL}_2(\mathbb{F}_p)|^{-1} + |(\mu_\Omega^{*(\ell)} - \mu_H)(\{(x, x) \mid x \in \mathrm{SL}_2(\mathbb{F}_p)\})| \\
&\qquad \leq |\mathrm{SL}_2(\mathbb{F}_p)|^{-1} + \lambda_0^\ell |\mathrm{SL}_2(\mathbb{F}_p)|.
\end{aligned}$$

If $\bar{\iota}(H) = \Gamma_\phi$, then for every $w \in F_2$ we have $\iota(\rho_2(w)) = \phi(\iota(\rho_1(w)))$. This means that for every $w \in F_2$ we have $\rho_2(w) = \pm\widehat{\phi}(\rho_1(w))$. Hence

$$\begin{aligned}
&\mu^{*(\ell)}_{\{a^\pm, b^\pm\}}(\{w \in F_2 \mid \rho_1(w) = \rho_2(w)\}) \\
&\qquad \leq \mu_\Omega^{*(\ell)}(\{(x, \pm\widehat{\phi}(x)) \mid \pm\widehat{\phi}(x) = x\}) \\
&\qquad \leq \mu_{\pi_L(\Omega)}^{*(\ell)}(\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \phi(x) = \pm x\}) \\
&\qquad \leq \frac{|\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \phi(x) = \pm x\}|}{|\mathrm{SL}_2(\mathbb{F}_p)|} + 2^{-\delta\ell}|\mathrm{SL}_2(\mathbb{F}_p)|.
\end{aligned}$$

Notice that since $\rho_1 \neq \rho_2$, $\widehat{\phi}$ is a non-trivial automorphism of $\mathrm{SL}_2(\mathbb{F}_p)$. Therefore

$$\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \phi(x) = \pm x\}$$

is a proper subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$. Hence we conclude that

$$\mu^{*(\ell)}_{\{a^\pm, b^\pm\}}(\{w \in F_2 \mid \rho_1(w) = \rho_2(w)\}) \leq p^{-1} + 2^{-\delta\ell}|\mathrm{SL}_2(\mathbb{F}_p)|.$$

Suppose $[\rho_1] \neq [\rho_2]$. In the above setting, if $H = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$, then by a similar argument we have

$$\begin{aligned}\mu^{*(\ell)}_{\{a^{\pm 1},b^{\pm 1}\}}(\{w \in F_2 \mid t_w([\rho_1]) = t_w([\rho_2])\}) &\\ \leq \mu^{*(\ell)}_{\Omega}(\{(x_1,x_2) \mid \mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)\}) &\\ \leq \frac{|\{(x_1,x_2) \mid \mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)\}|}{|\mathrm{SL}_2(\mathbb{F}_p)|^2} + \lambda_0^{\ell}|\mathrm{SL}_2(\mathbb{F}_p)|. &\quad (58)\end{aligned}$$

Notice that for a given $a \in \mathbb{F}_p$, we have

$$\begin{aligned}&|\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \mathrm{Tr}(x) = a\}| \\ &\qquad = |\{(x_{11}, x_{12}, a - x_{11}, x_{21}) \in \mathbb{F}_p^4 \mid x_{11}(a - x_{11}) - x_{12}x_{21} = 1\}|,\end{aligned}$$

and for given x_{12}, x_{21} there are at most two choices for x_{11}. Hence

$$|\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \mathrm{Tr}(x) = a\}| \leq 2p^2 \tag{59}$$

for every $a \in \mathbb{F}_p$. This implies that

$$|\{(x_1,x_2) \mid \mathrm{Tr}(x_1) = \mathrm{Tr}(x_2)\}| = \sum_{a \in \mathbb{F}_p} |\{x \in \mathrm{SL}_2(\mathbb{F}_p) \mid \mathrm{Tr}(x) = a\}|^2 \leq 4p^5. \tag{60}$$

By (58) and (60), for $p \geq 5$, we obtain

$$\begin{aligned}\mu^{*(\ell)}_{\{a^{\pm},b^{\pm}\}}(\{w \in F_2 \mid \rho_1(w) = \rho_2(w)\}) &\leq \frac{4}{p}(1 - p^{-2})^{-2} + \lambda_0^{\ell}|\mathrm{SL}_2(\mathbb{F}_p)| \\ &< \frac{5}{p} + \lambda_0^{\ell}|\mathrm{SL}_2(\mathbb{F}_p)|.\end{aligned}$$

In the above setting, if $\overline{\iota}(H) = \Gamma_\phi$, then as discussed above, for some automorphism $\widehat{\phi}$ of $\mathrm{SL}_2(\mathbb{F}_p)$ we have $\rho_2(w) = \pm\widehat{\phi}(\rho_1(w))$. Hence by [22, Theorem 3.2], $\mathrm{Tr}(\rho_2(w)) = \pm\,\mathrm{Tr}(\rho_1(w))$ for every $w \in F_2$. This completes the proof.

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