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Khovanov-type homologies of null homologous links in $\mathbb{RP}^3$

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Abstract. Let L be a null homologous link in $\mathbb{RP}^3$. We define Khovanov-type homologies of L which depend on an extra input $\alpha = (V_0, V_1, f, g)$ consisting of two graded vectors spaces and two maps between them. With some specific choice of $\alpha = \alpha_{\mathrm{APS}}$, we recover the categorification of the Kauffman bracket due to Asaeda–Przytycki–Sikora. With another choice of $\alpha = \alpha_{\mathrm{HF}}$, we construct a spectral sequence from our theory converging to the Heegaard–Floer homology of the even branched double cover of $\mathbb{RP}^3$.

1. Introduction

In [1], Asaeda, Przytycki, and Sikora extended the original construction of Khovanov homology in [7] to links in interval-bundles over surfaces, categorifying the Kauffman bracket. In particular, their construction gives a homology theory for links in $\mathbb{RP}^3$, by viewing $\mathbb{RP}^3 \backslash \{*\}$ as the twisted I-bundle over $\mathbb{RP}^2$. For I-bundles over non-orientable surfaces like $\mathbb{RP}^2$, their theory was defined with $\mathbb{F}_2$ coefficients. In [4], Gabrovšek extended the definition for links in $\mathbb{RP}^3$ to $\mathbb{Z}$ coefficients by fixing a sign convention.

In the first half of this paper, we are going to generalize the construction in [1] to get a family of Khovanov-type link homologies $\widetilde{Kh}^{\alpha}(L)$ for null homologous links in $\mathbb{RP}^3$ with $\mathbb{F}_2$ coefficients. All links are assumed to be oriented in the paper. Our homology theory depends on an extra input $\alpha = (V_0, V_1, f, g)$ called a *dyad*, consisting of two graded vector spaces V_0, V_1 and maps $f: V_0 \to V_1$, $g: V_1 \to V_0$ between them such that $f \circ g = 0$, $g \circ f = 0$.

Theorem 1.1. *For each dyad α, the homology $\widetilde{Kh}^{\alpha}(L)$ is an invariant of null homologous links in $\mathbb{RP}^3$.*

With a specific choice of the dyad $\alpha_{\mathrm{APS}} = (V, V, 0, 0)$, where $V = \langle v_+, v_- \rangle$, we recover a reduced version of the homology defined in [1] for null homologous links in $\mathbb{RP}^3$. The novelty of our construction is that we associate to each smoothing L_s

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of the link L an extra parameter $e_s(P) \in \{0, 1\}$, and the vector space we associate to the smoothing L_s in the chain complex $\widetilde{CKh}^\alpha(L)$ will be different depending on the value of $e_s(P)$. Here, P is a point in the complement of the link projection in $\mathbb{RP}^2$, and $e_s(P)$ counts the number of circles mod 2 in the smoothing L_s which encircles P. The Euler characteristic of the homology theory $\widetilde{Kh}^\alpha(L)$ is a linear combination of the even and odd Jones polynomials of L, with coefficients given by the graded dimension of V_0 and V_1, respectively. Here, the even (resp., odd) Jones polynomials of L are defined using the usual skein relation for Jones polynomial, with different initial values for unlinks. The sum of them is the usual Kauffman bracket for null homologous links in $\mathbb{RP}^3$ up to some shifts. We will also introduce an unreduced version $Kh^\alpha(L)$ of the homology, and discuss briefly what happens to other links in $\mathbb{RP}^3$ which are non-trivial in $H_1(\mathbb{RP}^3, \mathbb{Z})$.

In the second half of the paper, we will relate the Heegaard–Floer homology of a branched double cover of $\mathbb{RP}^3$ over a null homologous link L to the Khovanov-type homology $\widetilde{Kh}^{\alpha_{\mathrm{HF}}}(m(L))$ for another choice of the dyad $\alpha_{\mathrm{HF}} = (W, \overline{V}, f, g)$. In this case, $W = \langle a, b, c, d\rangle$, $\overline{V} = \langle \bar{v}_+, \bar{v}_-\rangle$, and

$$\begin{aligned} f(a) &= f(d) = 0, & f(b) &= f(c) = \bar{v}_-, \\ g(\bar{v}_-) &= 0, & g(\bar{v}_+) &= b + c. \end{aligned}$$

For a link L in S^3, we can form the branched double cover $\Sigma(S^3, L)$ of S^3. In [12], Ozsváth and Szabó defined a spectral sequence which converges to $\widehat{HF}(\Sigma(S^3, L))$ with $\mathbb{F}_2$ coefficients. The E^2 term of this spectral sequence gives the reduced Khovanov homology of the mirror $m(L)$ of L. We consider an extension of this construction for null homologous links in $\mathbb{RP}^3$, and obtain the following result.

Theorem 1.2. *Let L be a null homologous link in $\mathbb{RP}^3$. There is a spectral sequence whose E^2 term consists of the Khovanov-type homology $\widetilde{Kh}^{\alpha_{\mathrm{HF}}}(m(L))$ of the mirror of L with the dyad $\alpha_{\mathrm{HF}} = (W, \overline{V}, f, g)$, which converges to the Heegaard–Floer homology $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, L))$ of the even branched double cover $\Sigma_0(\mathbb{RP}^3, L)$ of $\mathbb{RP}^3$.*

We only consider null homologous links in $\mathbb{RP}^3$ because the branched double cover $\Sigma(\mathbb{RP}^3, L)$ only exists for those links. What is more, there are two branched double covers for each null homologous link L, and we will make a specific choice, called the even branched cover $\Sigma_0(\mathbb{RP}^3, L)$. The construction of the spectral sequence is essentially the same as the one in [12], with some particular treatment of the cobordism corresponding to $1 \to 1$ bifurcation, which is special for link projections to $\mathbb{RP}^2$. Another difference is that when defining the spectral sequence, we need to use both of the branched double covers. For each smoothing L_s of L, we will use the even branched double cover $\Sigma_0(\mathbb{RP}^3, L_s)$ if the extra parameter $e_s(P)$ we introduced earlier equals 0, and the odd branched double cover $\Sigma_1(\mathbb{RP}^3, L_s)$ if $e_s(P) = 1$. This is the reason we want to introduce the extra input $\alpha = (V_0, V_1, f, g)$ into our homology theory, where V_0, V_1 basically correspond to the two different branched double covers, and f, g correspond to the induced maps on $\widehat{HF}$ between the two branched double covers by performing surgeries associated to $1 \to 1$ bifurcations.

Here is the organization of this paper. In Section 2, we define the Khovanov-type homology $\widetilde{Kh}^\alpha(L)$ combinatorially. For each null homologous link projection L, an arbitrary point P in the complement of L in $\mathbb{RP}^2$ and each dyad α, we define the underlying vector space of a chain complex $\widetilde{CKh}^{P,\alpha}(L)$ in Section 2.1. The differential is presented in Section 2.2. In Section 2.3, we fix a canonical choice of the point P, and show that the homology $\widetilde{Kh}^\alpha(L)$ is an invariant of null homologous links in $\mathbb{RP}^3$. In Section 2.4, we express the Euler characteristic of $\widetilde{Kh}^\alpha(L)$ through skein relations. Section 2.5 briefly describes the unreduced versions $CKh^\alpha(L)$ and $Kh^\alpha(L)$. We give an example calculation of $\widetilde{Kh}^\alpha(L)$ for a specific link projection L with some specific choices of α in Section 2.6. Finally, in Section 2.7, we discuss the situation for other links in $\mathbb{RP}^3$. In Section 3, we give the construction of the spectral sequence converging to $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, L))$. We discuss the branched double cover $\Sigma(\mathbb{RP}^3, L)$ of $\mathbb{RP}^3$ in Section 3.1. Then, we quickly review the construction in [12] in Section 3.2. In Section 3.3, we compute the E^2 term of our spectral sequence, and show that it is equal to $\widetilde{Kh}^{\alpha_{\mathrm{HF}}}(m(L))$.

2. Definition of the homology

An oriented link K in $\mathbb{RP}^3$ is *null homologous* if $[K] = 0$ in $H_1(\mathbb{RP}^3, \mathbb{Z})$. Note that a null homologous link could have an even number of components which are non-trivial in $H_1(\mathbb{RP}^3, \mathbb{Z})$. Given a null homologous link K in $\mathbb{RP}^3$, we consider its projection L to $\mathbb{RP}^2$, by identifying $\mathbb{RP}^3 \backslash \{*\}$ with $\mathbb{RP}^2 \tilde{\times} I$, the twisted I-bundle over $\mathbb{RP}^2$. We will associate a Khovanov-type chain complex to the link projection L, and show its homology is an oriented link invariant for null homologous links in $\mathbb{RP}^3$.

First, we introduce some basic algebra notions. All vector spaces in this paper are over $\mathbb{F}_2$ unless stated otherwise. Let V be the graded vector space spanned by v_+ and v_-, with quantum gradings $q\deg(v_+) = 1$ and $q\deg(v_-) = -1$. As in the usual definition of Khovanov homology, V has the structure of a Frobenius algebra, with multiplication $m : V \otimes V \mapsto V$ such that

$$m(v_+ \otimes v_+) = v_+, \quad m(v_+ \otimes v_-) = m(v_- \otimes v_+) = v_-, \quad m(v_- \otimes v_-) = 0,$$

and comultiplication $\Delta : V \mapsto V \otimes V$ such that

$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+, \quad \Delta(v_-) = v_- \otimes v_-.$$

Note that both m and Δ change the quantum degree by -1.

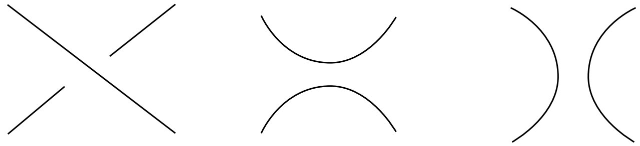


Figure 1. Smoothings.

Definition 2.1. A *dyad* is a tuple $\alpha = (V_0, V_1, f, g)$, where V_0 and V_1 are graded vector spaces, $f : V_0 \mapsto V_1$ and $g : V_1 \mapsto V_0$ are linear maps, both of quantum degree -1, and $f \circ g = 0$, $g \circ f = 0$. The *dual dyad* α^* of a dyad $\alpha = (V_0, V_1, f, g)$ is $\alpha^* = (V_1, V_0, g, f)$, obtained by switching V_1 with V_0 and g with f.

For each dyad $\alpha = (V_0, V_1, f, g)$, we give each V_i a trivial right-bimodule structure over V, with multiplication $m : V_i \otimes V \mapsto V_i$ such that

$$m(y \otimes v_+) = y, \quad m(y \otimes v_-) = 0 \quad \forall y \in V_i,\ i = 1, 2,$$

and comultiplication $\Delta : V_i \mapsto V_i \otimes V$ such that

$$\Delta(y) = y \otimes v_- \quad \forall y \in V_i,\ i = 1, 2.$$

Again, both m and Δ have quantum degree -1.

2.1. The cube of resolutions

Label the crossings of a null homologous link projection L in $\mathbb{RP}^2$ from 1 to n. A *state* $s \in \{0, 1\}^n$ is a choice of 0 or 1 for each crossing. Given a state s, we form a smoothing of L according to the rule in Figure 1. Then, we associate a vector space to each smoothing L_s as follows.

Define L_s as the link diagram obtained from L by smoothing according to the state s. For each s, the link L_s is a disjoint union of k_s embedded circles in $\mathbb{RP}^2$. Now, we show each circle in L_s is trivial in $H_1(\mathbb{RP}^2, \mathbb{Z})$, hence divides $\mathbb{RP}^2$ into a disk and a Möbius band.

Lemma 2.2. *For the link projection L of a null homologous link K in $\mathbb{RP}^3$, every smoothing L_s is null homologous as well. In particular, each circle S^1 in L_s is trivial in $H_1(\mathbb{RP}^2, \mathbb{Z})$, and divides $\mathbb{RP}^2$ into a disk and a Möbius band.*

Proof. We draw $\mathbb{RP}^2$ as a disk with half of the boundary identified with the other half in opposite direction. For the projection L in $\mathbb{RP}^2$, consider the number of intersection points of it with half of the boundary of the disk. L represents the generator of

$H_1(\mathbb{RP}^2, \mathbb{Z}) = \mathbb{Z}/2$ if the intersection number is odd, and is null homologous if the intersection number is even. Note that the intersection number is unchanged during the smoothing procedure, so L_s has an even number of intersections for each state s, as we start with a null homologous link projection L. Hence, the smoothing L_s is null homologous for every s.

Now, we prove each circle in L_s is null homologous by contradiction. Suppose there is an embedded circle in L_s which represents the generator of $H_1(\mathbb{RP}^2, \mathbb{Z})$, then cutting $\mathbb{RP}^2$ along this circle we obtain a disk D^2. As L_s is a disjoint union of circles, there cannot be another circle in L_s representing the generator of $H_1(\mathbb{RP}^2, \mathbb{Z})$. Since L_s is null homologous, we get a contradiction. Hence, each circle in L_s is null homologous, and divides $\mathbb{RP}^2$ into a disk and Möbius band. ∎

Pick a point P in the complement of the link projection L in $\mathbb{RP}^2$ such that P lies in the complement of each smoothing L_s as well.

Definition 2.3. For each null homologous circle S^1 in $\mathbb{RP}^2$, we say P is *encircled* by S^1 if P lies in the disk bounded by S^1. Define the *encircling number* $e_s(P) \in \{0, 1\}$ as the number of circles in L_s encircling P mod 2.

We will associate different vector spaces $\widetilde{CKh}_s^{P,\alpha}(L)$ to s depending on the value of $e_s(P)$. For a given dyad $\alpha = (V_0, V_1, f, g)$, define

$$\widetilde{CKh}_s^{P,\alpha}(L) = V_{e_s(P)} \otimes V^{\otimes(k_s-1)},$$

where k_s is the total number of circles in the smoothing L_s.

Now, we apply the usual flattening of cube operation to define a chain complex $\widetilde{CKh}_\bullet^{P,\alpha}(L)$.

Definition 2.4. The *Khovanov-type chain complex* $\widetilde{CKh}_\bullet^{P,\alpha}(L)$ of a null homologous link projection L in $\mathbb{RP}^2$, with a point P in the complement of L and a dyad α, is

$$\widetilde{CKh}_i^{P,\alpha}(L) = \bigoplus_{\substack{s\in\{0,1\}^n \\ \#1(s)=i+n_-}} \widetilde{CKh}_s^{P,\alpha}(L)\{i + n_+ - n_-\}, \tag{2.1}$$

where $\#1(s)$ is the number of 1s in the state s, n_+ and n_- are the numbers of positive and negative crossings in L, respectively, and $\widetilde{CKh}_s^{P,\alpha}(L)\{i + n_+ - n_-\}$ is the vector space obtained from $\widetilde{CKh}_s^{P,\alpha}(L)$ with quantum degree shifted up by $i + n_+ - n_-$. The differential will be defined in the next section. The homology of $\widetilde{CKh}_\bullet^{P,\alpha}(L)$ is denoted as $\widetilde{Kh}_\bullet^{P,\alpha}(L)$.

Note that we make a shift in the homology degree by $-n_-$ as well by letting $\#1(s) = i + n_-$, as in the usual Khovanov homology.

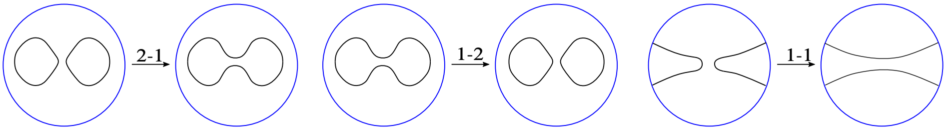


Figure 2. Bifurcations.

2.2. The differential

Choose a marked point M on the link projection L. The differential map depends on the choice of M, but we will show later that the homology does not depend on it. As in the usual Khovanov homology, the differential $d : \widetilde{CKh}_i^{P,\alpha}(L) \mapsto \widetilde{CKh}_{i+1}^{P,\alpha}(L)$ is given by a summation of maps over edges in the cube of resolutions. Each edge corresponds to changing the smoothing from 0 to 1 at one crossing, by our definition of the homology degree $i = \#1(s) - n_-$. There are three cases, as shown in Figure 2:

(1) $2 \to 1$ bifurcation, where two circles in L_s merge into a circle in $L_{s'}$;

(2) $1 \to 2$ bifurcation, where a circle in L_s splits into two circles in $L_{s'}$;

(3) $1 \to 1$ bifurcation, where a circle in L_s twists into a new circle in $L_{s'}$.

The first two bifurcations are similar to the corresponding ones for link projections in $\mathbb{R}^2$, while the third one only appears in non-orientable surfaces, e.g., $\mathbb{RP}^2$.

First, we study the change of $e_s(P)$ under different bifurcations, which will determine the domain and codomain of the chain map on each edge. Recall $e_s(P)$ is the number of circles in L_s encircling P mod 2.

Lemma 2.5. *We have* $e_{s'}(P) = e_s(P)$ *for the* $2 \to 1$ *and* $1 \to 2$ *bifurcations, while* $e_{s'}(P) = e_s(P) + 1$ *mod* 2 *for the* $1 \to 1$ *bifurcation.*

Proof. See Figure 3 for an illustration of the situation.

Let us consider the $2 \to 1$ bifurcation first. For a circle S^1 not involved in the bifurcation, it is not changed in the process at all, so the encircling state of P with it is not changed as well. So, we only need to consider the relation of P with the two merging circles. P could be encircled by 0, 1 or 2 of them. If P is encircled by none of them, then P is not encircled by the new circle as well, so $e_{s'}(P) = e_s(P)$. If P is encircled by one of them, then P is encircled by the new circle, so $e_{s'}(P) = e_s(P)$. If P is encircled by two of them, then one disk lies in the other disk, and the disk bounded by the new circle is the complement of the smaller disk in the larger one. Therefore, P is not encircled by the new circle, and $e_{s'}(P) = e_s(P)$.

The case for the $1 \to 2$ bifurcation is similar, except we reverse the arrow in the process. Hence, we get the same conclusion that $e_{s'}(P) = e_s(P)$.

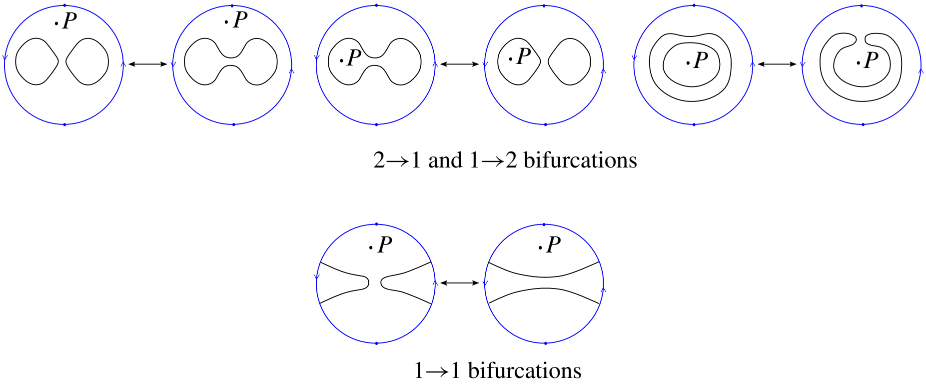


Figure 3. Changes of $e_s(P)$ under bifurcations.

For the $1 \to 1$ bifurcation, the disk part and the Möbius band part of the involved circle are switched, so $e_{s'}(P) = e_s(P) + 1 \bmod 2$. ∎

Now, we define the differential map $d : \widetilde{CKh}_s^{P,\alpha}(L) \mapsto \widetilde{CKh}_{s'}^{P,\alpha}(L)$ in each case. Denote $i = e_s(P)$ for simplicity of notation. We identify each circle in L_s with a factor in the tensor product

$$\widetilde{CKh}_s^{P,\alpha}(L) = V_i \otimes V^{\otimes(k_s-1)},$$

by assigning V_i to the circle in L_s with the marked point M, and V to each of the rest circles. For the $2 \to 1$ and $1 \to 2$ bifurcations, d is identity on tensor factors corresponding to circles not involved, and we specify what d does to the tensor factors corresponding to the involved circles as follows.

(1) $2 \to 1$ *bifurcation*. By the above lemma, $e_s(P) = e_{s'}(P)$. The number of circles is decreased by 1, so d is a map from $V_i \otimes V^{\otimes(k_s-1)}$ to $V_i \otimes V^{\otimes(k_s-2)}$.

(a) Suppose the marked point M lies on one of the two circles involved in the bifurcation. The differential is multiplying V by the trivial V-module V_i. Specifically,

$$d = m \otimes \mathrm{id} : (V_i \otimes V) \otimes V^{\otimes(k_s-2)} \mapsto V_i \otimes V^{\otimes(k_s-2)},$$

where $m : V_i \otimes V \mapsto V_i$ is the multiplication defined previously, with v_+ acting by identity, and v_- acting by zero.

(b) Suppose the marked point M does not lie on either of the two circles involved in the bifurcation. Then, the differential works like the multiplication $m : V \otimes V \mapsto V$ in the usual Khovanov homology.

(2) $1 \to 2$ *bifurcation*. Again, $e_s(P) = e_{s'}(P)$ and the number of circles is increased by 1, so d is a map from $V_i \otimes V^{\otimes(k_s-1)}$ to $V_i \otimes V^{\otimes k_s}$.

(a) Suppose the marked point M lies on the circle involved in the bifurcation. The differential is the comultiplication on the trivial V-comodule V_i. Specifically,

$$d = \Delta \otimes \mathrm{id} : V_i \otimes V^{\otimes(k_s-1)} \mapsto (V_i \otimes V) \otimes V^{\otimes(k_s-1)},$$

where $\Delta : V_i \mapsto V_i \otimes V$ is the comultiplication $\Delta(y) = y \otimes v_-$ for any $y \in V_i$.

(b) Suppose the marked point M does not lie on the circle involved in the bifurcation. The differential works like the comultiplication $\Delta : V \mapsto V \otimes V$ in the usual Khovanov homology.

(3) $1 \to 1$ *bifurcation*. For the $1 \to 1$ bifurcation, we have $e_{s'}(P) = e_s(P) + 1 \mod 2$, and the number of circles is unchanged, so d is a map from $V_i \otimes V^{\otimes(k_s-1)}$ to $V_{i+1} \otimes V^{\otimes(k_s-1)}$. This time, the differential map is the same no matter whether M lies on the involved circle or not. We change V_i to V_{i+1} by the maps f or g in the dyad $\alpha = (V_0, V_1, f, g)$.

(a) If $e_s(P) = 0$, then $d = f \otimes \mathrm{id} : V_0 \otimes V^{\otimes(k_s-1)} \mapsto V_1 \otimes V^{\otimes(k_s-1)}$.

(b) If $e_s(P) = 1$, then $d = g \otimes \mathrm{id} : V_1 \otimes V^{\otimes(k_s-1)} \mapsto V_0 \otimes V^{\otimes(k_s-1)}$.

Observe that the map $d : \widetilde{CKh}_s^{P,\alpha}(L) \mapsto \widetilde{CKh}_{s'}^{P,\alpha}(L)$ lowers the quantum degree by 1 for all the above cases. (Recall that in the definition of a dyad, f and g are required to have quantum grading -1.) Therefore, after the shift in the definition (2.1), the chain map preserves the quantum degree.

Now, it is time to check this definition does give a chain complex.

Proposition 2.6. *We have* $d^2 = 0$.

Proof. It is enough to show that each square of the resolution cube commutes. (As we are working over $\mathbb{F}_2$, commuting is the same as anticommuting.) Hence, it is enough to consider link projections with two crossings in $\mathbb{RP}^2$, and ignore the remaining part of the projection. A little caution needs to be taken in terms of the location of the marked point M. It could appear on the link projection with two crossings, or it could lie on the neglected part.

First, we quote the following result from [4]: up to symmetries, there are exactly seven singular graphs in $\mathbb{RP}^2$ with 2 singular points, as shown in Figure 4. (Thanks are due to Mike Willis, who pointed to the author that in [4] Gabrovšek missed the singular graph (g) listed in Figure 4.)

Now, we replace each singular point by a positive or negative crossing to get a link with two crossings. For diagrams (c), (d), and (g), the corresponding links are not null homologous, so we omit them in our discussion. For (e) and (f), the corresponding

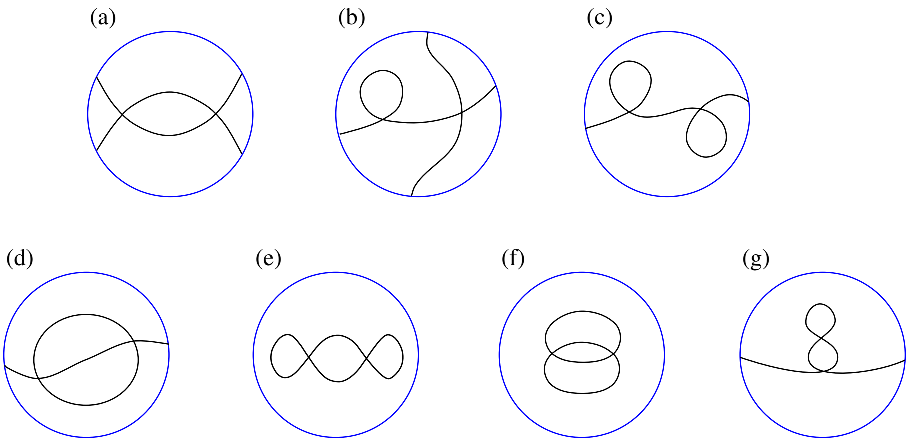


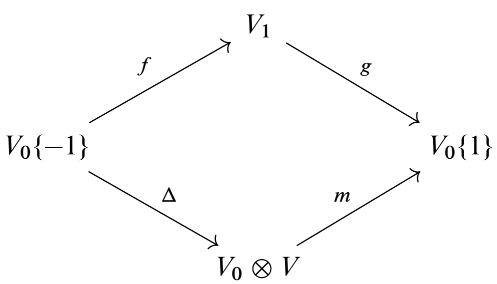
Figure 4. Singular graphs in $\mathbb{RP}^2$ with 2 singular points.

links are affine (they lie in a disk inside $\mathbb{RP}^2$), so there are no $1 \to 1$ bifurcations. Then, the differential behaves like the reduced Khovanov homology if the marked point M is presented, or the usual Khovanov homology if the marked point is not presented. So, we have $d^2 = 0$ in these two cases as well. For the rest two diagrams (a) and (b), we will check its commutativity by hand. We need to compute different cases depending on whether the crossings are positive or negative, where the point P is and whether the marked point M is present for each links. One example will be presented here and the rest are left as exercises.

Consider the link projection in Figure 5 and its resolutions in Figure 6.

(1) Suppose the marked point M is present.

(a) The cube of resolutions is drawn in 1(a) of Figure 6, with a choice of the extra point P.



The corresponding vector spaces and maps between them are as shown above. Both of the compositions $m \circ \Delta$, $g \circ f$ are equal to 0, so it commutes.

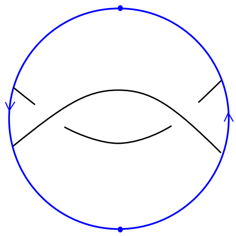


Figure 5. One example of link projection with two crossings.

(b) We have the same cube of resolutions as before, but with another choice of P in 1(b) of Figure 6.

$$\begin{array}{ccccc} & \xrightarrow{g} & V_0 & \xrightarrow{f} & \\ V_1\{-1\} & & & & V_1\{1\} \\ & \xrightarrow{\Delta} & V_1 \otimes V & \xrightarrow{m} & \end{array}$$

Again, we have $m \circ \Delta = f \circ g = 0$.

(2) Suppose the marked point M is not present.

(a) Even though the circle with marked point M is not involved, we draw it here, as the $1 \to 1$ bifurcation induces non-trivial map on the tensor factor corresponding to the marked circle. The cube of resolutions is shown in 2(a) of Figure 6.

$$\begin{array}{ccccc} & \xrightarrow{f \otimes \mathrm{id}_V} & V_1 \otimes V & \xrightarrow{g \otimes \mathrm{id}_V} & \\ V_0 \otimes V\{-1\} & & & & V_0 \otimes V\{1\} \\ & \xrightarrow{\mathrm{id}_{V_0} \otimes \Delta} & V_0 \otimes V \otimes V & \xrightarrow{\mathrm{id}_{V_0} \otimes m} & \end{array}$$

Again, both the compositions $m \circ \Delta$ and $g \circ f$ are equal to 0. This time, we use the fact that we are working over $\mathbb{F}_2$, as $m \circ \Delta(v_+) = m(v_- \otimes v_+ + v_+ \otimes v_-) = 2v_- = 0$.

(b) We change the position of the point P to get the cube of resolutions in 2(b) of Figure 6.

$$\begin{array}{ccccc} & \xrightarrow{g \otimes \mathrm{id}_V} & V_0 \otimes V & \xrightarrow{f \otimes \mathrm{id}_V} & \\ V_1 \otimes V\{-1\} & & & & V_1 \otimes V\{1\} \\ & \xrightarrow{\mathrm{id}_{V_1} \otimes \Delta} & V_1 \otimes V \otimes V & \xrightarrow{\mathrm{id}_{V_1} \otimes m} & \end{array}$$

Again, this commutes because $f \circ g = 0$ and $m \circ \Delta = 0$. ∎

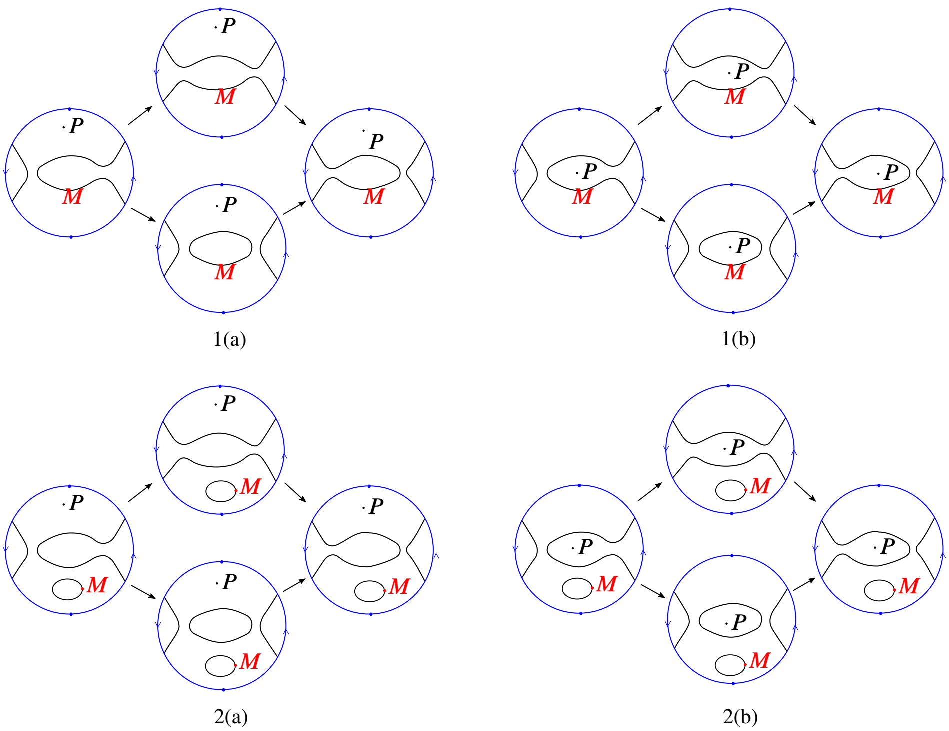


Figure 6. The cubes of resolutions corresponding to different P and M.

This finishes the definition of the chain complex $\widetilde{CKh}_{\bullet}^{P,\alpha}(L)$ in Definition 2.4.

2.3. Well-definedness of the homology

In the last section, we got a chain complex $\widetilde{CKh}_{\bullet}^{P,\alpha}(L)$ for a link projection L with a marked point M on the projection L, a dyad α and another point P outside the projection L. In this section, we will show that the homology $\widetilde{Kh}_{\bullet}^{P,\alpha}(L)$ is well-defined in the following sense: it is invariant under Reidemeister moves, and it does not depend on the choice of the marked point M. $\widetilde{Kh}_{\bullet}^{P,\alpha}(L)$ does depend on the choice of the point P, but we will give a canonical choice of the point P for a given null homologous link L. Therefore, we get a well-defined homology $\widetilde{Kh}_{\bullet}^{\alpha}(L)$ which only depends on the null homologous link L and the dyad α.

We start with the dependence on P. The underlying vector space of the chain complex $\widetilde{CKh}_{\bullet}^{P,\alpha}(L)$ depends on the choice of P. To express the dependence in a succinct way, we take a detour into the discussion of Seifert surfaces and linking numbers for null homologous links in $\mathbb{RP}^3$ first.

Definition 2.7. For a null homologous link K in $\mathbb{RP}^3$, a *Seifert surface* F of K is a connected compact oriented surface contained in $\mathbb{RP}^3$ such that K is its oriented boundary.

Note that the usual method to construct Seifert surfaces for links in $\mathbb{R}^3$, see, for example, [8, Theorem 2.2], works verbatim for null homologous links in $\mathbb{RP}^3$ as well.

Definition 2.8. Given a null homologous link K and an oriented link K' in $\mathbb{RP}^3$, we define the *linking number* $\operatorname{lk}(K, K')$ of K and K' as the intersection number $\langle F, K'\rangle$ of a Seifert surface F of K with K'.

Note that the linking number $\operatorname{lk}(K, K')$ does not depend on the choice of the Seifert surface F of K, as $H_2(\mathbb{RP}^3, \mathbb{Z}) = 0$. It does depend on the orientation of $\mathbb{RP}^3$, K and K', but if we only care about the parity of $\operatorname{lk}(K, K')$, which will be our main concern, then it does not matter how we choose the orientations. The orientation of K will be fixed by the given orientation of the null homologous link, and we will not specify the orientation of K' nor $\mathbb{RP}^3$ in the following discussion.

Let us go back to the dependence of $\widetilde{CKh}^{P,\alpha}_{\bullet}(L)$ on P. Define C_P as the union of the fiber over P in the twisted I-bundle $\mathbb{RP}^2\tilde{\times}I$ with the deleted point $*$ in $\mathbb{RP}^3\backslash\{*\} = \mathbb{RP}^2\tilde{\times}I$. So, C_P is a circle in $\mathbb{RP}^3$. We can divide the complement of L in $\mathbb{RP}^2$ into two regions $\mathbb{RP}^2\backslash L = R_0 \sqcup R_1$ such that

$$R_i = \big\{P \in \mathbb{RP}^2\backslash L \mid \operatorname{lk}(L, C_P) = i \bmod 2\big\}. \tag{2.2}$$

We will show that $\widetilde{CKh}^{P,\alpha}_{\bullet}(L)$ depends on P only through the parity of $\operatorname{lk}(L, C_P)$.

Proposition 2.9. *For two points P, Q in the complement of L in $\mathbb{RP}^2$,*

$$\widetilde{CKh}^{Q,\alpha}_{\bullet}(L) = \begin{cases} \widetilde{CKh}^{P,\alpha}_{\bullet}(L) & \text{if } \operatorname{lk}(L, C_Q) = \operatorname{lk}(L, C_P) \bmod 2, \\ \widetilde{CKh}^{P,\alpha^*}_{\bullet}(L) & \text{if } \operatorname{lk}(L, C_Q) = \operatorname{lk}(L, C_P) + 1 \bmod 2, \end{cases}$$

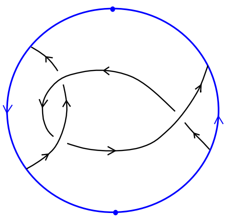
where $\alpha^* = (V_1, V_0, g, f)$ is the dual dyad of $\alpha = (V_0, V_1, f, g)$ as defined in Definition 2.1.

Proof. Choose an embedded path γ in $\mathbb{RP}^2$ connecting P and Q and avoiding the double points of L. Since L is null homologous in $\mathbb{RP}^2$, the number of intersection points $\langle \gamma, L\rangle$ between γ and L mod 2 does not depend on the choice of γ. From the construction of Seifert surfaces of L, it is clear that

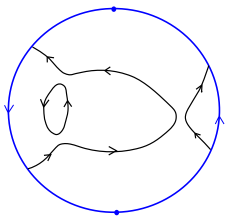
$$\operatorname{lk}(L, C_Q) = \operatorname{lk}(L, C_P) + \langle \gamma, L\rangle \bmod 2.$$

We separate into two cases depending on the parity of $\langle \gamma, L\rangle$.

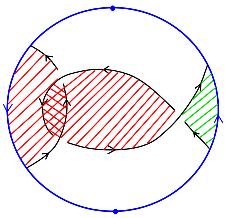
(1) If $\langle \gamma, L\rangle$ is even, then for any given state $s \in \{0, 1\}^n$, the intersection number $\langle \gamma, L_s\rangle$ of γ and the smoothing L_s is even as well, as γ avoids the double points of L.



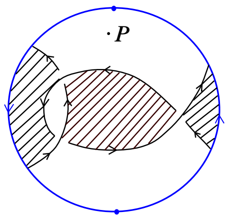
The knot projection D of a null homologous knot L



The orientation-preserving resolution of D



A Seifert surface of L, where the orientation is such that in the red region, the normal vector is pointing upwards, and in the green region, it is pointing downwards.



The division of the complement of D into R_0 and R_1, where the blank region is R_0, and the shaded region is R_1, and a choice of the point P.

Figure 7. An illustration of a null homologous knot in $\mathbb{RP}^3$, with a Seifert surface and the decomposition $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$.

For each circle c in L_s, if γ intersects c an even number of times, then either both P and Q are encircled by c, or neither P nor Q is encircled by c. If γ intersects c odd times, then exactly one of P and Q is encircled by c. Since $\langle \gamma, L_s \rangle$ is even, there are even numbers of circles in L_s which intersect γ odd times, so the numbers of circles encircling P and Q in the smoothing L_s are the same mod 2, i.e., $e_s(P) = e_s(Q)$ for any state s. Therefore, the two chain complexes $\widetilde{CKh}_{\bullet}^{P,\alpha}(L)$ and $\widetilde{CKh}_{\bullet}^{Q,\alpha}(L)$ are exactly the same.

(2) If $\langle \gamma, L \rangle$ is odd, then by the previous argument, we get $e_s(Q) = e_s(P) + 1$ mod 2 for any state s. By checking the definition of the chain complex, we find it is equivalent to switching the role of V_0 with V_1 and f with g. Therefore, we have $\widetilde{CKh}_{\bullet}^{Q,\alpha^*}(L) = \widetilde{CKh}_{\bullet}^{P,\alpha}(L)$, where $\alpha^* = (V_1, V_0, g, f)$ is the dual dyad of $\alpha = (V_0, V_1, f, g)$. ■

See Figure 7 for an example of the division $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$. From now on, we will always choose P to lie in the region R_0, and drop P from the notation $\widetilde{CKh}_{\bullet}^{P,\alpha}(L)$ and $\widetilde{Kh}_{\bullet}^{P,\alpha}(L)$.

Remark. If we change the orientation of some components of the null homologous link K, it will change the orientation-preserving smoothing $\widehat{L}$, and hence, the division $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$. For this reason, we need to fix an orientation on the null homologous link K.

Now, we will move on to prove invariance under different choices of the marked point M.

Proposition 2.10. *The homology $\widetilde{Kh}^{\alpha}_{\bullet}(L)$ does not depend on the choice of marked point M on L.*

Proof. Our proof is inspired by the discussion relating the Heegaard–Floer homology of the branched double cover of S^3 branched over K and the Khovanov homology of it in [12]. See [13] as well. We define an automorphism $\phi_s : \widetilde{CKh}^{\alpha}_{s}(L) \mapsto \widetilde{CKh}^{\alpha}_{s}(L)$ induced by the change of marked point M for each state s, and check it commutes with the differential d. Therefore, we obtain a chain automorphism

$$\widetilde{CKh}^{\alpha}_{\bullet}(L) \mapsto \widetilde{CKh}^{\alpha}_{\bullet}(L),$$

which induces an isomorphism on $\widetilde{Kh}^{\alpha}_{\bullet}(L)$.

Let us begin to define ϕ_s. For a given state s, there is a marked circle in the smoothing L_s. Label the marked circle by 0, and the rest of the circles in the smoothing L_s from 1 to $k_s - 1$. Suppose we change the position of the marked point M to the circle labeled 1. Give $V^{\otimes(k_s-1)}$ an algebra structure as the quotient of the polynomial algebra over $\mathbb{F}_2$ generated by S_i, quotienting out by the relations $S_i^2 = 0$ for $i = 1, \dots, k_s - 1$, where $S_i = v_+ \otimes v_+ \otimes \cdots \otimes v_- \otimes v_+ \otimes \cdots \otimes v_+$ with v_- at the ith component. Define $\eta_s : V^{\otimes(k_s-1)} \mapsto V^{\otimes(k_s-1)}$ to be the algebra automorphism such that

$$\eta_s(S_1) = S_1,$$
$$\eta_s(S_i) = S_1 + S_i \quad \text{for } 2 \le i \le k_s - 1$$

and extend multiplicatively.

In this formalism, merging of two circles labeled i and j corresponds to further quotienting out the relation $S_i = S_j$, and splitting of a circle labeled i corresponds to product with $S_i + S_{k_s}$, where k_s is the label of the new circle. Merging of a circle labeled i with the marked circle corresponds to dividing out by the relation $S_i = 0$, and splitting of the marked circle corresponds to product with S_{k_s} with k_s the label of the new circle. See [12, Sections 5 and 6] for more details.

Define $\phi_s : V_i \otimes V^{\otimes(k_s-1)} \mapsto V_i \otimes V^{\otimes(k_s-1)}$ as

$$\phi_s = \mathrm{id}_{V_i} \otimes \eta_s, \quad \text{where } i = e_s(P).$$

Now, we need to check the following diagram commutes, where d_1, d_2 are the corresponding edge maps, labeled differently for later calculation:

$$\begin{array}{ccc} \widetilde{CKh}^{\alpha}_{s}(L) & \xrightarrow{d_1} & \widetilde{CKh}^{\alpha}_{s'}(L) \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_{s'}} \\ \widetilde{CKh}^{\alpha}_{s}(L) & \xrightarrow{d_2} & \widetilde{CKh}^{\alpha}_{s'}(L) \end{array}$$

We discuss the cases when d_1, d_2 are given by the $2 \to 1$ bifurcation, the $1 \to 2$ bifurcation or the $1 \to 1$ bifurcation separately.

The $1 \to 1$ bifurcation is the easiest case, as d acts only on the V_i component of the tensor product, so it commutes with the change of variables in the $V^{\otimes(k_s-1)}$ component.

For the $2 \to 1$ bifurcation, it separates into cases depending on whether the marked circles are involved or not. Note that all the maps in the square are algebra homomorphisms on the tensor component $V^{\otimes(k_s-1)}$ and identity on the V_i component, so it is enough to check it on the generators S_i. We check different cases as follows.

(1) Suppose the bifurcation merges two circles different from the circle 0 and 1, say circle 2 and circle 3. Then, both η_s and $\eta_{s'}$ send S_1 to S_1 and S_i to $S_i + S_1$ for $i \neq 1$, while both d_1 and d_2 quotient out the relation $S_2 = S_3$. Then, it is obvious the above square commutes.

(2) Suppose the bifurcation merges the circle 0 with a circle other than circle 1, say circle 2. Then, η_s and $\eta_{s'}$ behave the same as in the first case, while d_1 quotients out $S_2 = 0$, and d_2 quotients out $S_1 = S_2$. Hence,

$$\begin{aligned} &\eta_{s'} \circ d_1(S_1) = \eta_{s'}(S_1) = S_1 = d_2(S_1) = d_2 \circ \eta_s(S_1), \\ &\eta_{s'} \circ d_1(S_2) = \eta_{s'}(0) = 0 = d_2(S_1 + S_2) = d_2 \circ \eta_s(S_2), \\ &\eta_{s'} \circ d_1(S_i) = \eta_{s'}(S_i) = S_1 + S_i = d_2(S_1 + S_i) = d_2 \circ \eta_s(S_i) \quad \text{for } i > 2. \end{aligned}$$

If the bifurcation merges circle 1 with a circle other than circle 0, then we get a similar diagram, except the arrows ϕ_s and $\phi_{s'}$ are reversed. We can check its commutativity by a similar calculation.

(3) Suppose the bifurcation merges circle 0 and circle 1. η_s is the same as above, while η'_s is the identity, as we move M on the same circle. d_1, d_2 both quotient out the relation $S_1 = 0$. Hence,

$$\begin{aligned} &\eta_{s'} \circ d_1(S_1) = \eta_{s'}(0) = 0 = d_2(S_1) = d_2 \circ \eta_s(S_1), \\ &\eta_{s'} \circ d_1(S_i) = \eta_{s'}(S_i) = S_i = d_2(S_1 + S_i) = d_2 \circ \eta_s(S_i) \quad \text{for } i \geq 2. \end{aligned}$$

The analysis for the $1 \to 2$ bifurcation is similar. Again, we divide into cases depending on whether the marked circle is involved or not.

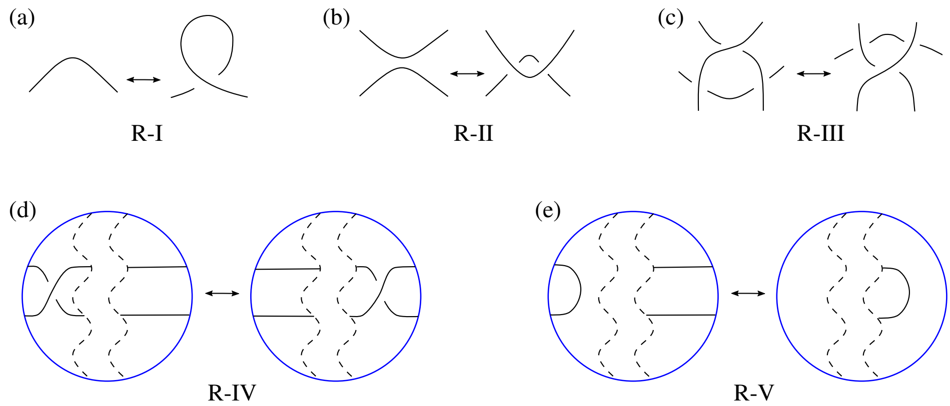


Figure 8. Reidemeister moves in $\mathbb{RP}^3$.

(1) Suppose the bifurcation splits a circle other than circle 0 and circle 1, say circle 2, into two new circles labeled 2 and k. Then, both d_1 and d_2 are multiplying $S_2 + S_k$. The change of variables map $\phi_{s'}$ sends $S_2 + S_k$ to $S_2 + S_1 + S_k + S_1$, which equal to $S_2 + S_k$ as we are working over $\mathbb{F}_2$. So, the square commutes.

(2) Suppose the bifurcation splits the circle 0. Then, d_1 is multiplying S_k, and d_2 is multiplying $S_1 + S_k$. As $\phi_{s'}(S_k) = S_1 + S_k$, the square commutes.

(3) Suppose the bifurcation splits the circle 1. Then, d_1 is multiplying $S_1 + S_k$ and d_2 is multiplying S_k. Then, the square commutes as well, because $\phi_{s'}(S_k + S_1) = S_1 + S_k + S_1 = S_k$, as we are working over $\mathbb{F}_2$. ∎

Remark. The choice of the change of variable map η_s for moving the marked point will become natural when we discuss the relation of this homology with the Heegaard–Floer homology of a branched double cover of $\mathbb{RP}^3$ branched over K.

Now, we discuss the invariance of $\widetilde{Kh}^{\alpha}_{\bullet}(L)$ under Reidemeister moves. First, there are 5 Reidemeister moves in $\mathbb{RP}^2$, the three usual ones and two additional ones that act across the boundary of the 2-disk. They are drawn in Figure 8. Two links are ambient isotopic in $\mathbb{RP}^3$ if a link projection of one link can be transformed to a link projection of the other by a finite sequence of the moves R-I to R-V. See [3] for further discussion on this.

The decomposition $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$ described before Proposition 2.9 is changed under Reidemeister moves such that the region in $\mathbb{RP}^2$ swept by the moves is changed from R_i to R_{i+1}. We will pick a point P which lies in R_0 for both of the link projections before and after the Reidemeister move. It could be achieved by choosing P which lies in R_0 for the link projection before the Reidemeister move such that the

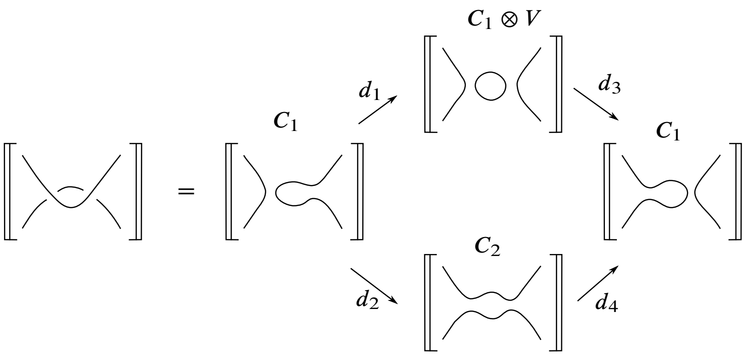


Figure 9. Invariance under RII.

Reidemeister move does not cross it. So, we can assume the Reidemeister moves does not cross the point P.

Proposition 2.11. *The homology $\widetilde{Kh}^{\alpha}_{\bullet}(L)$ is invariant under Reidemeister moves, so it is an invariant of null homologous links in $\mathbb{RP}^3$.*

Proof. Note that R-IV and R-V do not change the chain complex $\widetilde{CKh}^{\alpha}_{\bullet}(L)$, so the homology is of course invariant. We are left with Reidemeister moves R-I, R-II and R-III. For them, the proof of invariance for the usual Khovanov homology, as described in [2, Section 3.5], works with a slight change. We borrow the notation $[\![L]\!]$ for $\widetilde{CKh}^{\alpha}_{\bullet}(L)$ from [2] as well, and we only draw the part of L which is changed under the Reidemeister moves.

For R-I, we can assume the marked point M does not lie on the moving part of the link under R-I, as we can move M by a change of variable described in Proposition 2.10 if necessary. Then, the proof of the invariance of the Khovanov homology under R-I works here as well with no change. Note that we have done the shift in the homological degree implicitly in the definition of $\widetilde{CKh}^{\alpha}_{i}(L)$ in Definition 2.4, by letting $\#1(s) = i + n_-$. The shift in the quantum grading is the same as in [2] as well, which is $\#1(s) + n_+ - 2n_- = i + n_+ - n_-$. The reason we are doing these shifts is precisely to make the homology $\widetilde{Kh}^{\alpha}_{\bullet}(L)$ invariant under R-I.

For R-II, again, we can assume the marked point M does not lie on the moving part of the link under R-II. Consider the chain complex in Figure 9, where $d_1 : C_1 \mapsto C_1 \otimes V$ is the chain map given by comultiplication Δ, and $d_3 : C_1 \otimes V \mapsto C_1$ is the multiplication m. Note that d_3 is an isomorphism on the subchain C' of $C_1 \otimes V$, which takes value v_+ in the factor V. So, we can quotient out the subcomplex $\{C' \mapsto C_1\}$ without changing the homology. Now, d_1 induces an isomorphism between C_1 and $(C_1 \otimes V)/C'$. Then, we further quotient out the subcomplex generated by C_1. This final quotient is isomorphic to C_2, as in the proof for the usual

Khovanov homology. Observe that in the proof, we only used the properties of maps d_1 and d_3, which correspond to splitting and merging a circle, respectively, and are entirely similar to the corresponding maps in the usual Khovanov chain complex. The map d_2 and d_4 might be different from the corresponding maps in the usual Khovanov chain complex, as some $1 \to 1$ bifurcations could happen, but we do not need to use the properties of d_2 and d_4 in the proof.

For R-III, we can assume the marked point does no lie on the moving part of R-III as well. Then, the usual proof of invariance under R-III works here as well, for the similar reason as in the proof of invariance under R-II. Check Section 3.5 of [2] for more details. ■

Therefore, we obtain the following theorem.

Theorem 2.12. *For each dyad* α*,* $\widetilde{Kh}^{\alpha}(L)$ *is an invariant of null homologous links in* $\mathbb{RP}^3$.

2.4. The Euler characteristic

The Euler characteristic of the usual Khovanov homology for links in S^3 gives the unnormalized Jones polynomial of the link. For (not necessarily null homologous) framed links in $\mathbb{RP}^3$, [1, 4] construct some Khovanov-type homology whose Euler characteristic gives the Kauffman bracket $\langle L \rangle$ of the framed link L, which is an element in the skein module $S(\mathbb{RP}^3)$ of $\mathbb{RP}^3$. $S(\mathbb{RP}^3)$ is the free $\mathbb{Z}[A^{\pm 1}]$-module over two generators. See the discussion of the skein module of lens space in [6]. In this paper, we use another convention such that $q = -A^2$. Our homology theory depends on the input $\alpha = (V_0, V_1, f, g)$, and its Euler characteristic will be a linear combination of the graded dimension of V_0 and V_1. Let us begin with some definitions.

Definition 2.13. For a graded vector space $W = \bigoplus_m W_m$ with homogeneous components $\{W_m\}$, the *graded dimension* of W is the Laurent polynomial $q\dim(W) = \sum_m q^m \dim(W_m)$. The *Euler characteristic* $\chi(C)$ of a chain complex $C = \bigoplus_{i,m} C_{i,m}$ of graded vector space is the alternating sum of the graded dimensions of its homology groups, where i is the homology grading, and m is the quantum grading,

$$\chi(C) = \sum_{i,m} (-1)^i q^m dim(H_{i,m}).$$

Note that $\chi(C)$ is the same as the alternating sum of the graded dimensions of its chain groups if the differential d preserves the quantum grading, and all the chain groups are finite dimensional.

Now, we define two variations of the Kauffman bracket of null homologous link projection in $\mathbb{RP}^2$. Fix a null homologous link K in $\mathbb{RP}^3$ and its projection L to $\mathbb{RP}^2$. Choose a point P in the complement of L in $\mathbb{RP}^2$.

Definition 2.14. Define the *even Kauffman bracket* $\langle L\rangle_0^P$ of a null homologous link projection L in $\mathbb{RP}^2$ avoiding P by the following rules.

(1) The skein relation: $\langle \times \rangle_0^P = \langle \asymp \rangle_0^P - q\langle\,)(\,\rangle_0^P$.

(2) If L is a disjoint union of k (necessarily null homologous) circles, then

$$\langle L\rangle_0^P = \begin{cases} (q+q^{-1})^{k-1} & \text{if } P \text{ is encircled by an even number of circles in } L, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define the *odd Kauffman bracket* $\langle L\rangle_1^P$ of a null homologous link projection L in $\mathbb{RP}^2$ avoiding P by the following rules.

(1) The skein relation: $\langle \times \rangle_1^P = \langle \asymp \rangle_1^P - q\langle\,)(\,\rangle_1^P$.

(2) If L is a disjoint union of k (necessarily null homologous) circles, then

$$\langle L\rangle_1^P = \begin{cases} (q+q^{-1})^{k-1} & \text{if } P \text{ is encircled by an odd number of circles in } L, \\ 0 & \text{otherwise.} \end{cases}$$

As in the case of $\widetilde{CKh}_\bullet^{P,\alpha}(L)$, the Kauffman bracket $\langle L\rangle_i^P$ depends on P only through the parity of the linking number $\mathrm{lk}(L, C_P)$ defined in Definition 2.8, so for a given null homologous link projection L, we pick P such that $\mathrm{lk}(L, C_P)$ is even, and drop P from the notation $\langle L\rangle_i^P$ for $i = 0, 1$.

Definition 2.15. Define the even/odd Jones polynomial of L by the expression $J_i(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L\rangle_i$ for $i = 0, 1$, respectively.

It is easy to see that $J_i(L)$ is invariant under Reidemeister moves in $\mathbb{RP}^2$, so it is an invariant for null homologous links in $\mathbb{RP}^3$.

From our definition of the chain complex $\widetilde{CKh}_\bullet^\alpha(L)$, we have the following description of its Euler characteristics.

Proposition 2.16. *For a given null homologous link projection L and a dyad $\alpha = (V_0, V_1, f, g)$, we have $\chi(\widetilde{CKh}_\bullet^\alpha(L)) = q\dim(V_0)J_0(L) + q\dim(V_1)J_1(L)$.*

Proof. By the definition of the chain complex $\widetilde{CKh}_\bullet^\alpha(L)$, both the left-hand side and the right-hand side of the equation satisfy the same skein relation relating link projections $\times$, $\asymp$ and $)($, so it is enough to verify the equation when L is a disjoint union of circles, which again follows from the definition of $\widetilde{CKh}_\bullet^\alpha(L)$. ∎

Note that the sum of the even and odd Kauffman bracket recovers the usual Kauffman bracket for null homologous link projections in $\mathbb{RP}^2$:

$$\langle L\rangle = \langle L\rangle_0 + \langle L\rangle_1,$$

so they are some refinements of the usual Kauffman bracket for null homologous link projections in $\mathbb{RP}^2$.

Another observation is that the Euler characteristic $\chi(\widetilde{CKh}^{\alpha}_{\bullet}(L))$ does not depend on the maps $f : V_0 \mapsto V_1$ and $g : V_1 \mapsto V_0$ in α. Therefore, by changing f, g while keeping V_0 and V_1 fixed, we get different homology theories categorifying the same Jones polynomial.

2.5. An unreduced version of the chain complex

In defining the chain complex $\widetilde{CKh}^{\alpha}_{\bullet}(L)$, we choose a marked point M on L, assign the vector space V_i to the marked circle in each smoothing L_s and assign V to each of the other circles in L_s. This resembles the usual definition of the reduced Khovanov homology for links in S^3, where we assign the base field $\mathbb{F}_2$ to the marked circle and V to each of the other circles. The difference with the usual one occurs when there is a $1 \to 1$ bifurcation; then, we change the vector space associated to the marked circle from V_i to V_{i+1}.

As in the usual Khovanov homology, we can define an unreduced version of the chain complex as well, denoted as $CKh^{\alpha}_{\bullet}(L)$. Pick a point P in the preferred region R_0 of the complement of L. (See the discussion before Proposition 2.9.) Now, for each state s, we associate the vector space $V_{e_s(P)} \otimes V^{\otimes k_s}$ to it, which assigns V to each of the circles in the smoothing L_s, and treat $V_{e_s(P)}$ as a background component. For the differential, we use the usual maps $m : V \otimes V \mapsto V$ and $\Delta : V \mapsto V \otimes V$ for the $2 \to 1$ bifurcation and the $1 \to 2$ bifurcation, respectively, as in the usual Khovanov homology. For the $1 \to 1$ bifurcation, we use maps $f : V_0 \mapsto V_1$ and $g : V_1 \mapsto V_0$ acting on the background component V_i. The proofs of $d^2 = 0$ and the invariance of the homology $Kh^{\alpha}_{\bullet}(L)$ under Reidemeister moves are almost the same as we have presented in the previous section (actually slightly easier because we do not need to divide into cases according to where the marked point is). The Euler characteristic of $CKh^{\alpha}_{\bullet}(L)$ is then related to the unnormalized Jones polynomial,

$$\chi(CKh^{\alpha}_{\bullet}(L)) = (q + q^{-1})\chi(\widetilde{CKh}^{\alpha}_{\bullet}(L)).$$

Remark. It might have been more natural to start with the unreduced version instead of the reduced version. The reason we chose to present the reduced version first is because the reduced one is what we obtained in the computation of the Heegaard–Floer homology of branched double covers of $\mathbb{RP}^3$ branched over a link. Another reason is that the proofs for the unreduced version are contained in the proofs for the reduced version.

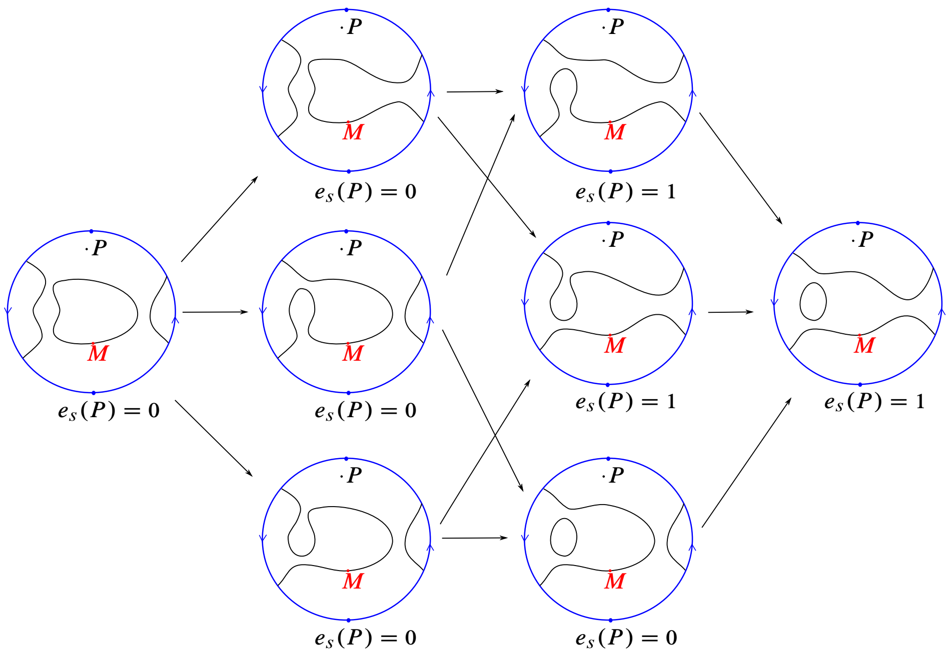


Figure 10. The cube of resolutions.

2.6. Some example calculations

In this section, we present some computations of $\widetilde{Kh}^{\alpha}_{\bullet}(K)$ for some specific choices of the dyad $\alpha = (V_0, V_1, f, g)$ and the null homologous knot K in $\mathbb{RP}^3$ drawn in Figure 7. The cube of resolution is drawn in Figure 10.

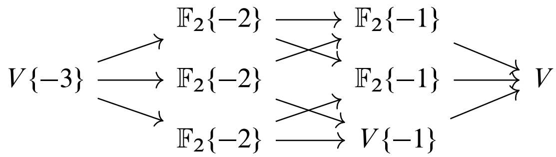
To save some space when writing down the homology, we express the Khovanov-type homology $\widetilde{Kh}^{\alpha}_{\bullet}(K)$ by its graded Poincaré polynomial $p(\widetilde{Kh}^{\alpha}_{\bullet}(K))$, which is

$$p(\widetilde{Kh}^{\alpha}_{\bullet}(K)) = \sum_{i,m} t^i q^m \dim(\widetilde{Kh}^{\alpha}_{i,m}(K)).$$

The Euler characteristic is obtained by taking $t = -1$.

(1) $\alpha_{\mathrm{APS}} = (\mathbb{F}_2, \mathbb{F}_2, 0, 0)$. For this choice of dyad, the unreduced chain complex $CKh^{\alpha_{\mathrm{APS}}}_{\bullet}$ recovers the chain complex defined in [1, 4] for null homologous links in $\mathbb{RP}^3$ with coefficients $\mathbb{F}_2$. Their homology theories work for all links in $\mathbb{RP}^3$; in addition to the null homologous ones [1] proposes a $\mathbb{F}_2$ version, and [4] fixes some choice of signs so that it works over $\mathbb{Z}$. Note that since we have the symmetry $\alpha_{\mathrm{APS}} = \alpha^*_{\mathrm{APS}}$, we can pick the point P anywhere in the complement of L in $\mathbb{RP}^2$, and it will give the same chain complex, so P

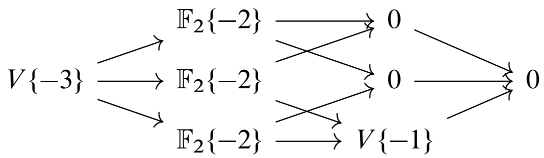
plays no role in their homology,



The chain complex is shown above, where the maps $V \mapsto \mathbb{F}_2$ are multiplication of V on the trivial V-module $\mathbb{F}_2$, the maps are $\mathbb{F}_2 \mapsto V$ are comultiplication of the trivial V-comodule $\mathbb{F}_2$, and the maps $\mathbb{F}_2 \mapsto \mathbb{F}_2$, $V \mapsto V$ are 0. The Poincaré polynomial is

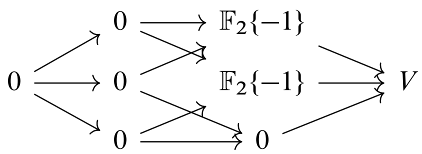
$$p(\widetilde{Kh}_{\bullet}^{\alpha_{\mathrm{APS}}}) = t^{-2}q^{-4} + t^{-1}q^{-2} + q^{-1} + 1 + tq.$$

(2) $\alpha_0 = (\mathbb{F}_2, 0, 0, 0)$ and its dual $\alpha_1 = \alpha_0^* = (0, \mathbb{F}_2, 0, 0)$. The corresponding chain complexes $\widetilde{CKh}_{\bullet}^{\alpha_i}(L)$ are subcomplexes of $\widetilde{CKh}_{\bullet}^{\alpha_{\mathrm{APS}}}(L)$, consisting of those states s such that $e_s(P) = i$ for $i = 0, 1$. Their Euler characteristics are equal to the even/odd Jones polynomials $\chi(CKh_{\bullet}^{\alpha_i}(L)) = J_i(L)$ for $i = 0, 1$, respectively. In some sense, all other chain complexes for other choices of α are linear combinations of these two chain complexes.



The chain complex for α_0 is shown above, where the arrows represent the same maps as in the case of α_{APS}. The Poincaré polynomial is

$$p(\widetilde{Kh}_{\bullet}^{\alpha_0}) = t^{-2}q^{-4} + t^{-1}q^{-2} + 1.$$



The chain complex for α_1 is as above. Note that

$$\widetilde{CKh}_{\bullet}^{\alpha_{\mathrm{APS}}}(L) = \widetilde{CKh}_{\bullet}^{\alpha_0}(L) \oplus \widetilde{CKh}_{\bullet}^{\alpha_1}(L),$$

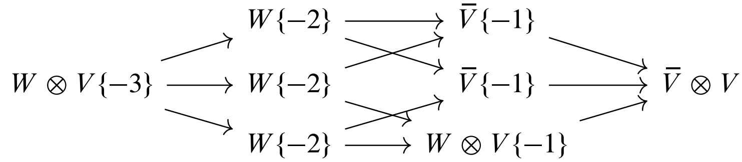
as the maps $f = g = 0$. The Poincaré polynomial is

$$p(\widetilde{Kh}_{\bullet}^{\alpha_1}) = q^{-1} + tq.$$

(3) $\alpha_{\mathrm{HF}} = (W, \overline{V}, f, g)$. $W = \langle a, b, c, d \rangle$ is the span of four elements a, b, c, d, with $q\deg(a) = 1$, $q\deg(b) = q\deg(c) = 0$ and $q\deg(d) = -1$. $\overline{V} = \langle \bar{v}_+, \bar{v}_- \rangle$ is the span of two elements $\bar{v}_+, \bar{v}_-$, with $q\deg(\bar{v}_+) = 1$ and $q\deg(\bar{v}_-) = -1$. f and g are defined as follows:

$$\begin{aligned} f(a) &= f(d) = 0, & f(b) &= f(c) = \bar{v}_-, \\ g(\bar{v}_-) &= 0, & g(\bar{v}_+) &= b + c. \end{aligned}$$

This one will appear later in the discussion of the Heegaard–Floer homology of branched double covers of $\mathbb{RP}^3$.



The chain complex is shown above, with the obvious maps on each arrow. The Poincaré polynomial is

$$p(\widetilde{Kh}_{\bullet}^{\alpha_{\mathrm{HF}}}) = \\ t^{-2}(q^{-5} + 2q^{-4} + q^{-3}) + t^{-1}(q^{-3} + q^{-2} + q^{-1}) + q^{-1} + 2 + q + tq^2.$$

In contrast, if we consider $\alpha'_{\mathrm{HF}} = (W, \overline{V}, 0, 0)$, changing both f and g to 0, then Euler characteristic stays the same, while the Poincaré polynomial becomes

$$\begin{aligned} p(\widetilde{Kh}_{\bullet}^{\alpha'_{\mathrm{HF}}}) = t^{-2}(q^{-5} + 2q^{-4} + q^{-3}) + t^{-1}(q^{-3} + 2q^{-2} + q^{-1}) \\ + q^{-2} + q^{-1} + 3 + q + t(1 + q^2). \end{aligned}$$

2.7. Other links in $\mathbb{RP}^3$

We discuss briefly what happens to other links in $\mathbb{RP}^3$, those that are non-vanishing in $H_1(\mathbb{RP}^3, \mathbb{Z})$, if we perform a similar construction in this subsection. Suppose K is an oriented link in $\mathbb{RP}^3$ such that $[K] \neq 0$ in $H_1(\mathbb{RP}^3, \mathbb{Z})$. Consider its link projection L in $\mathbb{RP}^2$ with n crossings. Now, for each $s \in \{0, 1\}^n$, the smoothing L_s of L is a disjoint union of several null homologous circles and one special circle which generates $H_1(\mathbb{RP}^2, \mathbb{Z})$. Each null homologous circle divides $\mathbb{RP}^2$ into a disk and Möbius band as before, while cutting $\mathbb{RP}^2$ along the special circle gives a disk. We can still define $e_s(P) \in \{0, 1\}$ by picking a point P in the complement of L in $\mathbb{RP}^2$ and count the number of null homologous circles encircling P mod 2. We can assign V to each trivial circle, and $V_{e_s(P)}$ to the special circle.

In terms of bifurcations, there is no $1 \to 1$ bifurcation because of the presence of the special circle. On the other hand, there are two kinds of $2 \to 1$ bifurcations, which correspond to merging of two trivial circles and merging of a trivial circle with the special circle, respectively. If the $2 \to 1$ bifurcation from L_s to $L_{s'}$ merges two trivial circles, then $e_s(P) = e_{s'}(P)$ as before, and the corresponding differential map is the same as the multiplication $m : V \otimes V \to V$ in the usual Khovanov homology. If the $2 \to 1$ bifurcation merges a trivial circle with the special circle, then the relation between $e_s(P)$ and $e_{s'}(P)$ depends on the specific position of P and the trivial circle P.

(1) If P is not encircled by the trivial circle, then $e_{s'}(P) = e_s(P)$, and the corresponding differential map is the multiplication $m : V_{e_s(P)} \otimes V \to V_{e_s(P)}$, where $V_{e_s(P)}$ has the structure of a trivial V-module as before.

(2) If P is encircled by the trivial circle, then $e_{s'}(P) = e_s(P) + 1 \mod 2$. We denote the corresponding differential map by $f_m : V_0 \otimes V \to V_1$ if $e_s(P) = 0$, and by $g_m : V_1 \otimes V \to V_0$ if $e_s(P) = 1$.

The situation of $1 \to 2$ bifurcations is similar. If the $1 \to 2$ bifurcation splits a trivial circle, then $e_s(P) = e_{s'}(P)$, and the differential map is the same as the comultiplication $\Delta : V \to V \otimes V$ in the usual Khovanov homology. If the $1 \to 2$ bifurcation splits the special circle into a trivial circle and the new special circle, then again we divide into two cases.

(1) If P is not encircled by the trivial circle, then $e_s(P) = e_{s'}(P)$, and the corresponding differential map is the comultiplication $\Delta : V_{e_s(P)} \to V_{e_s(P)} \otimes V$, where $V_{e_s(P)}$ has the structure of a trivial V-comodule.

(2) If P is encircled by the trivial circle, then $e_{s'}(P) = e_s(P) + 1 \mod 2$. We denote the corresponding differential map by $f_\Delta : V_0 \to V_1 \otimes V$ if $e_s(P) = 0$, and by $g_\Delta : V_1 \to V_0 \otimes V$ if $e_s(P) = 1$.

Then, to show $d^2 = 0$, we use a similar strategy by considering singular graphs in $\mathbb{RP}^2$ with two singular points as in Figure 4. This time configurations (a) and (b) are impossible. Configurations (e) and (f) are affine as before, so we are left with configurations (c), (d), and (g). They put the requirements

$$g_m \circ f_\Delta = 0, \quad f_m \circ g_\Delta = 0, \quad f_\Delta \circ g_m = 0, \quad g_\Delta \circ f_m = 0$$

on the differential maps. One way to achieve these requirements is starting with a dyad $\alpha = (V_0, V_1, f, g)$, and let

$$f_m = m \circ (f \otimes \mathrm{id}_V), \quad g_m = m \circ (g \otimes \mathrm{id}_V), \quad f_\Delta = \Delta \circ f, \quad g_\Delta = \Delta \circ g.$$

It can be shown that the homology $\widetilde{Kh}^\alpha(L)$ of the chain complex is invariant under Reidemeister moves not crossing the point P as before. Hence, we have the following theorem. Recall that C_P is the union of the point $*$ in $\mathbb{RP}^3\backslash\{*\} = \mathbb{RP}^2\tilde{\times} I$ and the fiber over P in the twisted I-bundle over P.

Theorem 2.17. *$\widetilde{Kh}^{\alpha}(L)$ is a link invariant for L considered as a link inside $\mathbb{RP}^3\backslash C_P$.*

The issue is that the homology depends on the choice of P in a subtle way. As L is non-trivial in $H_1(\mathbb{RP}^3,\mathbb{Z})$, it does not bound a Seifert surface, and we no longer have the subdivision $\mathbb{RP}^2\backslash L = R_0 \sqcup R_1$ as before. We cannot make a canonical choice of P for a link projection, so the homology is only defined for a link with a choice of an extra point P outside its link projection in $\mathbb{RP}^2$.

Remark. The space $\mathbb{RP}^3\backslash C_P$ is homeomorphic to the solid torus, which is the twisted I-bundle over a Möbius band. We can also view the solid torus as the trivial I-bundle over an annulus and consider the link projection to the annulus. This leads to another invariant called annular Khovanov homology. See [1, 5, 14].

3. Heegaard–Floer homology of branched double covers of $\mathbb{RP}^3$

In [12], Ozsváth and Szabó introduced a spectral sequence associated to a link $L \subset S^3$ converging to the Heegaard–Floer homology $\widehat{HF}(\Sigma(S^3, L), \mathbb{F}_2)$ of the branched double cover $\Sigma(S^3, L)$, whose E^2 page consists of the reduced Khovanov homology of the mirror $m(L)$ of L. In this section, we are going to extend this construction to null homologous links in $\mathbb{RP}^3$. In this section, we will obtain a spectral sequence converging to the Heegaard–Floer homology $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, L), \mathbb{F}_2)$ of the even branched double cover $\Sigma_0(\mathbb{RP}^3, L)$ of $\mathbb{RP}^3$, whose E^2 page consists of the Khovanov-type homology $\widetilde{Kh}_{\bullet}^{\alpha_{\mathrm{HF}}}(m(L))$ of the mirror $m(L)$ of L, with the dyad $\alpha_{\mathrm{HF}} = (W, \bar{V}, f, g)$ introduced in the example calculation (3) in Section 2.6.

3.1. Branched double covers of $\mathbb{RP}^3$

For a link K in a 3-manifold M, branched double covers $\Sigma_h(M, K)$ of M branched over K are classified by the set of maps

$$\{h : \pi_1(M\backslash K) \mapsto \mathbb{F}_2 \mid h([m_i]) = 1\},$$

where m_i is the meridian of the ith component of K. It is the same as the set of maps

$$\{h : H_1(M\backslash K, \mathbb{Z}) \mapsto \mathbb{F}_2 \mid h([m_i]) = 1\},$$

as $\mathbb{F}_2$ is abelian. For the purpose of this paper, we will discuss branched double covers $\Sigma(\mathbb{RP}^3, K)$ of $\mathbb{RP}^3$ when K is a null homologous link. Compute $H_1(\mathbb{RP}^3\backslash K, \mathbb{Z})$ for such K first.

Lemma 3.1. *Let K be a null homologous link in $\mathbb{RP}^3$ with n component. Then, we have*

$$H_1(\mathbb{RP}^3\backslash K, \mathbb{Z}) = \begin{cases} \mathbb{Z}^n \oplus \mathbb{Z}/2 & \text{if each component of } K \text{ is null homologous,} \\ \mathbb{Z}^n & \text{otherwise.} \end{cases}$$

Define $M = \langle [m_1], \ldots, [m_n] \rangle$ as the submodule of $H_1(\mathbb{RP}^3\backslash K, \mathbb{Z})$ generated by the meridians of each component of K. Then, in both cases, M is a submodule of index 2 in $H_1(\mathbb{RP}^3\backslash K, \mathbb{Z})$. In particular, there are two branched double covers $\Sigma_h(\mathbb{RP}^3, K)$, determined by $h([l])$ for some $[l] \in H_1(\mathbb{RP}^3\backslash L, \mathbb{Z})$ which is not in M.

Proof. Consider the Heegaard splitting $\mathbb{RP}^3 = U_1 \cup_f U_2$, where each U_i is a solid torus, and $f : \partial U_1 \mapsto \partial U_2$ is the map sending the meridian μ_1 of U_1 to $-\mu_2 + 2l_2$, where μ_2 and l_2 are the meridian and longitude of U_2, respectively. By isotopy, we can assume that K lies entirely in U_2. Now, we can construct $\mathbb{RP}^3\backslash K$ by gluing U_1 to $U_2\backslash K$ along f. For $U_2\backslash K$, we compute its homology in a way similar to computing the Wirtinger presentation of $\pi_1(S^3\backslash K)$. View $U_2 = A \times I$ as the trivial I-bundle over an annulus A, and consider the link projection of K to A. We can build $U_2\backslash K$ starting from $A \times I$, gluing a tube $S^1 \times I$ for each arc in the projection, then a disk for each crossing in the projection, and finally, a B^3. See, for example, [8, Chapter 11] for a detailed description. The only difference is that we start with a solid torus $A \times I$ instead of a ball B^3. From this cell decomposition, we have $H_1(U_2\backslash K, \mathbb{Z}) = \mathbb{Z}^{n+1} = \langle [m_1], [m_2], \ldots, [m_n], [l_2] \rangle$, generated by the meridians m_i of each component of L and the longitude l_2 of U_2. Now, gluing U_1 along f adds the relation $-[\mu_2] + 2[l_2]$ to $H_1(U_2\backslash K, \mathbb{Z})$, so we need to express $[\mu_2]$ in terms of $[m_i]$. For each component K_i of K, we have $[K_i] = c_i[l_2]$ in $H_1(U_2, \mathbb{Z})$ for some c_i. View $U_2 = D^2 \times S^1$ as the trivial disk bundle over S^1; then, the algebraic intersection number of K_i with a generic fiber D^2 is c_i. So, the punctured disk $D^2\backslash(D^2 \cap K)$ gives a relation

$$[\mu_2] = \sum_{i=1}^{n} c_i[m_i]$$

in $H_1(U_2\backslash K, \mathbb{Z})$, and

$$H_1(\mathbb{RP}^3\backslash K, \mathbb{Z}) = \langle [m_1], [m_2], \ldots, [m_n], [l_2] \rangle \Big/ \Big\langle -\sum_{i=1}^{n} c_i[m_i] + 2[l_2] \Big\rangle.$$

Now, we discuss the cases whether each component of K is null homologous or not.

(1) Suppose each component K_i of K is null homologous; then, all the c_i's are even integers, so

$$H_1(\mathbb{RP}^3\backslash K, \mathbb{Z}) = \mathbb{Z}^n \oplus \mathbb{Z}_2,$$

and $M = \langle [m_1], [m_2], \ldots, [m_n] \rangle$ is a submodule of index 2.

(2) Suppose there is some component of L which generates $H_1(\mathbb{RP}^3, \mathbb{Z})$; then, there is an even number of i such that c_i is odd, so

$$H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z}) = \mathbb{Z}^n$$

by some change of variables, and M is again a submodule of index 2.

From the description of $H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z})$, it is easy to see there are two maps $h : H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z}) \mapsto \mathbb{F}_2$ in each case that send all the meridians $[m_i]$ to 1, so there are two branched double covers $\Sigma_h(\mathbb{RP}^3, K)$ in each case. If we take some circle $l \in \mathbb{RP}^3 \backslash K$ such that $[l] \in H_1(\mathbb{RP}^3 \backslash K)$ is not in the submodule M (i.e., l will not be null homologous in $\mathbb{RP}^3$ if we fill in K), then the map $h : H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z}) \mapsto \mathbb{F}_2$, hence the branched double cover $\Sigma_h(\mathbb{RP}^3, K)$, is determined by $h([l])$, as $[l]$ together with the $[m_i]$'s generate $H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z})$. ■

Remark. If K is an oriented link in $\mathbb{RP}^3$ which generates $H_1(\mathbb{RP}^3, \mathbb{Z})$, then we have $[\mu_2] = \sum_i c_i[m_i]$ such that $\sum_i c_i$ is odd. If $h : H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z}) \mapsto \mathbb{F}_2$ were a map such that $h([m_i]) = 1$ for all i, then we would have

$$2h([l_2]) = h(2[l_2]) = h\left(\sum_{i=1}^{n} c_i[m_i]\right) = \sum_{i=1}^{n} c_i,$$

a contradiction. So, there is no branched double cover of $\mathbb{RP}^3$ over K if $[K]$ generates $H_1(\mathbb{RP}^3, \mathbb{Z})$.

Denote the projection from the branched double cover associated to the map $h : H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z}) \mapsto \mathbb{F}_2$ to $\mathbb{RP}^3$ by

$$p_h : \Sigma_h(\mathbb{RP}^3, K) \mapsto \mathbb{RP}^3.$$

The condition $h([l]) = 0$ is equivalent to requiring the preimage $p_h^{-1}(l)$ is a disjoint union of two circles instead of one large circle.

Definition 3.2. Let K be a null homologous link in $\mathbb{RP}^3$, and l be a circle in $\mathbb{RP}^3 \backslash K$ such that $[l]$ is not in the submodule $M \subset H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z})$ generated by the meridians of each component of K. We define $\Sigma(\mathbb{RP}^3, K, l)$ as the branched double cover $\Sigma_h(\mathbb{RP}^3, K)$ such that $h([l]) = 0$.

For a null homologous link K in $\mathbb{RP}^3$, recall we divided the complement of its projection L in $\mathbb{RP}^2$ into two regions $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$, picked a point P in $\mathbb{RP}^2 \backslash L$, and defined a circle C_P associated to it. See the discussion below Definition 2.8 and equation (2.2). C_P is such a circle that $[C_P] \in H_1(\mathbb{RP}^3 \backslash K, \mathbb{Z})$ is not in the submodule M.

Definition 3.3. For a null homologous link K in $\mathbb{RP}^3$, we will call $\Sigma(\mathbb{RP}^3, K, C_P)$ the *even branched double cover* of $\mathbb{RP}^3$ over K if $P \in R_0$, denoted by $\Sigma_0(\mathbb{RP}^3, K)$, and the *odd branched double cover* if $P \in R_1$, denoted by $\Sigma_1(\mathbb{RP}^3, K)$.

Note that it is well defined, as for a different choice $P' \in R_0$, $[C_P] - [C'_P]$ is a sum of an even number of meridians $[m_i]$ in $H_1(\mathbb{RP}^3\backslash K, \mathbb{Z})$ by the definition of R_0, so $h([C_P]) = h([C'_P])$.

Now, we describe all the other branched double covers for the smoothings L_s of the link projection L of K.

Lemma 3.4. *Let $s \in \{0,1\}^n$ be a state for the link projection L with n crossings, and L_s be the corresponding smoothing of L. Then, we have the branched double cover*

$$\Sigma(\mathbb{RP}^3, L_s, C_P) = \begin{cases} (\mathbb{RP}^3\#\mathbb{RP}^3)\#(S^1\times S^2)^{\#(k_s-1)} & \text{if } e_s(P)=0,\\ (S^1\times S^2)^{\#k_s} & \text{if } e_s(P)=1,\end{cases}$$

where k_s is the number of circles in L_s, and $e_s(P)$ is the number of circles in L_s encircling P mod 2.

Proof. If $e_s(P) = 0$, then pick a point $P' \in \mathbb{RP}^2\backslash L_s$ which is encircled by none of the circles in L_s, so $e_s(P') = 0$ as well. Consider a path γ in $\mathbb{RP}^2$ connecting P and P' such that γ intersects L_s transversely. Let c_i be the algebraic intersection number of γ with the ith circle in L_s; then,

$$[C_{P'}] = [C_P] + \sum_{i=1}^{k_s} c_i[m_i]$$

in $H_1(\mathbb{RP}^3\backslash L_s, \mathbb{Z})$. Note that $e_s(P') - e_s(P) = \sum_i c_i$ mod 2 by the definition of e_s, so $\sum_i c_i = 0$ mod 2. This implies $h([C_{P'}]) = h([C_P]) = 0$ as $h([m_i]) = 1$, so $\Sigma(\mathbb{RP}^3, L_s, C_P) = \Sigma(\mathbb{RP}^3, L_s, C_{P'})$, and we will look for $\Sigma(\mathbb{RP}^3, L_s, C_{P'})$.

Now, we describe the branched double cover $\Sigma(\mathbb{RP}^3, L_s, C_{P'})$ by induction on k_s. When $k_s = 0$, the link L_s is empty, and $\Sigma(\mathbb{RP}^3, \emptyset, C_{P'})$ is the double cover $\mathbb{RP}^3 \sqcup \mathbb{RP}^3$, as the preimage $p_h^{-1}(C_{P'})$ is a disjoint union of two circles. When $k_s = 1$, because P' is not encircled by this circle, we can pick some ball $B^3 \subset \mathbb{RP}^3\backslash C_{P'}$ containing this circle. The effect of adding one circle to the branching locus is the same as replacing the preimage $p_h^{-1}(B^3) = B^3 \sqcup B^3$ by the branched double cover of B^3 branched over the circle, which is $B^1 \times S^2$. It is the same as doing a 0-surgery (in the sense of replacing $S^0 \times B^3$ by $B^1 \times S^2$) to $\mathbb{RP}^3 \sqcup \mathbb{RP}^3$ with one B^3 in each copy of $\mathbb{RP}^3$. So, we get $\Sigma(\mathbb{RP}^3, L_s, C_{P'}) = \mathbb{RP}^3\#\mathbb{RP}^3$ for $k_s = 1$. Note that, in the process of 0-surgery, we did not change the preimage of $C_{P'}$ at all, which stays as a disjoint union of two circles. For each of the remaining circles in L_s, we apply a similar 0-surgery avoiding the preimage of $C_{P'}$. For connected 3-manifolds, doing

0-surgery is the same as taking connected sum with a copy of $S^1 \times S^2$, so we obtain $\Sigma(\mathbb{RP}^3, L_s, C_{P'}) = (\mathbb{RP}^3 \# \mathbb{RP}^3) \# (S^1 \times S^2)^{\#(k_s-1)}$.

If $e_s(P) = 1$, the proof is very similar. The only difference is that if we pick some $P' \in \mathbb{RP}^2 \backslash L$ encircled by none of the circles in L_s this time, then $e_s(P') - e_s(P) = 1$ mod 2, and $h([C'_P]) = 1$ instead of 0. Hence, we start with the other double cover S^3 of $\mathbb{RP}^3$ instead of $\mathbb{RP}^3 \sqcup \mathbb{RP}^3$ for the empty link, and we get $\Sigma(\mathbb{RP}^3, L_s, C_{P'}) = (S^1 \times S^2)^{\#k_s}$. ∎

Now, we review the relation between the branched double covers $\Sigma(\mathbb{RP}^3, L, C_P)$, $\Sigma(\mathbb{RP}^3, L_0, C_P)$ and $\Sigma(\mathbb{RP}^3, L_1, C_P)$, where L_0 (resp. L_1) are the link projections obtained from L by the 0 (resp., 1)-smoothings at one crossing, as introduced in [12, Section 2]. Note the convention they used for the 0 and 1-smoothings is the reverse of that in this paper.

Definition 3.5. Let M be an oriented 3-manifold with torus boundary and three simple, closed curves α, β, γ in ∂M with algebraic intersection numbers

$$\#(\alpha \cap \beta) = \#(\beta \cap \gamma) = \#(\gamma \cap \alpha) = -1.$$

A *triad* of 3-manifolds $(Y_\alpha, Y_\beta, Y_\gamma)$ is an ordered triple of 3-manifolds such that there exists an M and (α, β, γ) as above such that Y_i is obtained from M by attaching a solid torus along the boundary with the meridian mapped to i, for $i = \alpha, \beta, \gamma$, respectively.

The following lemma is shown as Proposition 2.1 in [12], except we switch 0 and 1-smoothings because of the different conventions.

Lemma 3.6. *We have the branched double covers* $(\Sigma(\mathbb{RP}^3, L, C_P), \Sigma(\mathbb{RP}^3, L_1, C_P), \Sigma(\mathbb{RP}^3, L_0, C_P))$ *form a triad.*

Proof. Consider some small ball B^3 containing the changed crossing of L in $\mathbb{RP}^3$. The branched double cover of S^2 branching over 4 points is a torus, and the branched double covers of B^3 branching over $\times$, $\asymp$, $)($ are all solid tori, with meridians obtained by double cover of one of the arcs pushed to the boundary of B^3. Therefore, they form a triad of 3-manifolds. Check [12] for details. ∎

We will give a more detailed account of the surgery from $\Sigma(\mathbb{RP}^3, L_1, C_P)$ to $\Sigma(\mathbb{RP}^3, L_0, C_P)$ in Section 3.3.

3.2. Link surgery spectral sequence of Heegaard–Floer Homology

In this subsection, we briefly review the construction of link surgeries spectral sequence in [12, Section 4], which is a generalization of the surgery long exact sequence associated to a triad of 3-manifolds $(Y_\alpha, Y_\beta, Y_\gamma)$:

$$\cdots \to \widehat{HF}(Y_\alpha) \to \widehat{HF}(Y_\beta) \to \widehat{HF}(Y_\gamma) \to \cdots .$$

For the definition and basic facts about Heegaard–Floer homology $\widehat{HF}$, see [11]. We will use Heegaard–Floer homology with coefficients $\mathbb{F}_2$ throughout the paper, so, we omit $\mathbb{F}_2$ from the notation.

Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an n-component, framed link (i.e., with a choice of longitude l_i for each component K_i) in a 3-manifold Y. A *multi-framing* is an n-tuple $I = (s_1, \ldots, s_n)$, where each $s_i \in \{\alpha, \beta, \gamma\}$. (In [12], the corresponding labels for s are $\{0, 1, \infty\}$. Unfortunately, 0 and 1-framed surgery correspond to the 1 and 0-smoothing, respectively, so we use $\{\alpha, \beta, \gamma\}$ instead.) For each multi-framing I, there is a three-manifold $Y(I)$, obtained from Y by performing s_i-framed surgery on the component K_i for $i = 1, \ldots, n$. Here, α means the l_i-framed surgery, β means the $(l_i + m_i)$-framed surgery where m_i is the meridian of K_i, and γ means no surgery, which is the same as the m_i-framed surgery.

We give the set $\{\alpha, \beta, \gamma\}^n$ the lexicographical order such that $\alpha < \beta < \gamma$. For $I = (s_i)_{i=1,\ldots,n}$ and $I' = (s'_i)_{i=1,\ldots,n} \in \{\alpha, \beta, \gamma\}^n$, we call I' an *immediate successor* of I if they are only different at one slot j such that $(s_j, s'_j) = (\alpha, \beta)$ or (β, γ).

For a sequence of multiframings $I^1 < \cdots < I^k$ such that each I^{i+1} is an immediate successor of I^i, there is an induced map on the Heegaard–Floer chain complex

$$D_{I^1 < \cdots < I^k} : \widehat{CF}(Y(I^1)) \mapsto \widehat{CF}(Y(I^k))$$

defined by counting holomorphic polygons.

Let $X = \bigoplus_{I \in \{\alpha,\beta,\gamma\}^n} \widehat{CF}(Y(I))$, endowed with a map $D : X \mapsto X$ defined by

$$D(\xi) = \sum_J \sum_{\{I = I^1 < \cdots < I^k = J\}} D_{I^1 < \cdots < I^k}(\xi)$$

summing over all sequences $I = I^1 < \cdots < I^k = J$ such that I^{i+1} is an immediate successor of I^i for $\xi \in \widehat{CF}(Y(I))$. We can filter X by the lexicographical order, and write $D = \sum_{i=0}^n d_i$, where each d_i is obtained by summing over sequences of length i and d_0 is the sum of differentials in the chain complex $\widehat{CF}(Y(I))$ for each I.

The main inputs from the theory of Heegaard–Floer homology are the following two propositions.

Proposition 3.7 ([12, Proposition 4.6]). *The map D satisfies $D^2 = 0$, so it is a differential map.*

Proposition 3.8 ([12, Theorem 4.7]). *Let K be a framed knot in a 3-manifold Y, and let*

$$\hat{f} : \widehat{CF}(Y_0(K)) \mapsto \widehat{CF}(Y_1(K))$$

denote the chain map induced by the cobordism of adding a 2-handle. Then, the chain complex $\widehat{CF}(Y)$ is quasi-isomorphic to the mapping cone of $\hat{f}$.

Then, by applying induction on n, together with an algebraic lemma on mapping cones (see [12, Lemma 4.4]), one can prove the following theorem.

Theorem 3.9 ([12, Theorem 4.1]). *There is a spectral sequence whose E^1 term is*

$$\bigoplus_{I\in\{\alpha,\beta\}^n} \widehat{HF}(Y(I)),$$

which converges to $\widehat{HF}(Y)$ such that the d_1 map is given by summing maps $\hat{f}_*$: $\widehat{HF}(Y(I)) \mapsto \widehat{HF}(Y(I'))$ for all pairs (I, I'), where I' is an immediate successor of I.

Now, we discuss the relation of this with the reduced Khovanov homology of links in S^3. Check Sections 5 and 6 in [12] for details. Consider the link projection $L \in \mathbb{R}^2$ of a link K in S^3 with n crossings. Let $Y = \Sigma(S^3, K)$ be the branched double cover of S^3 over K. For the ith crossing of L, pick the vertical arc in S^3 connecting the two double points, and let l_i be the preimage of this arc in Y. Each l is framed such that the 1-framed surgery of Y on l gives the branched double cover $\Sigma(S^3, K_0)$, where K_0 is obtained from K by changing the ith crossing with the 0-smoothing. For each state $I \in \{\alpha, \beta\}^n$, we consider the corresponding state $s = \phi(I) \in \{1, 0\}^n$, where ϕ sends α to 1 and β to 0. Then, the manifold $Y(I)$ obtained from Y by surgery according to the multiframing I is the same as the branched double cover $\Sigma(S^3, L_s)$ for the state $s = \phi(I)$, by the local analysis for small balls B^3 in S^3 containing the crossings as in Lemma 3.6.

Each L_s is unlink with k_s components, so the branched double cover $\Sigma(S^3, L_s) = Y(I)$ is a connected sum of $k_s - 1$ copies of $S^1 \times S^2$, and $\widehat{HF}(Y(I)) = V^{\otimes(k_s-1)}$, where $V = \langle v_+, v_-\rangle$ is the same as the vector space we associated to each circle in the Khovanov homology. Hence, we have an identification between $\widehat{HF}(Y(I))$ and $\widetilde{CKh}_{\phi(I)}(m(L))$, where $m(L)$ is the mirror of L. The mirror m appears as the map ϕ sends α to 1 and β to 0, so we turn all the crossings upside down to make it consistent with the usual convention for the Khovanov homology, which gives the Khovanov homology for $m(L)$ instead of L. This identification is natural in the sense that it turns the map d_1 in $\bigoplus_{I\in\{\alpha,\beta\}^n} \widehat{HF}(Y(I))$ to the differential d in $\widetilde{CKh}(m(L))$. Together with Theorem 3.9, we obtain the following result.

Theorem 3.10 ([12, Theorem 1.1]). *Let $L \subset S^3$ be a link. There is a spectral sequence whose E^2 terms consist of the reduced Khovanov homology of the mirror of L with coefficients in $\mathbb{F}_2$, which converges to $\widehat{HF}(\Sigma(S^3, L), \mathbb{F}_2)$.*

Our task in the next section is to generalize this result for null homologous links in $\mathbb{RP}^3$, where the reduced Khovanov homology is replaced by the Khovanov-type homology $\widetilde{Kh}^{\alpha_{\text{HF}}}(L)$ from Definition 2.4 such that α_{HF} is as in (3) in Section 2.6.

3.3. The differential d_1 in the spectral sequence for branched double covers of $\mathbb{RP}^3$

By the discussion in Lemma 3.4, we have an identification of $\widehat{HF}(Y(I))$ with $\widetilde{CKh}^{\alpha_{\text{HF}}}_{\phi(I)}(L)$ as vector spaces. We are left to compute d_1 in the spectral sequence to show it coincides with the corresponding edge maps in $\widetilde{CKh}^{\alpha_{\text{HF}}}$. For $1 \to 2$ and $2 \to 1$ bifurcations, the same proof in [12] for links in S^3 works, which we will review briefly. The case of $1 \to 1$ bifurcations needs a detailed description of the cobordism maps induced by the surgeries.

As discussed in Section 3.1, there are two branched double covers $\Sigma(\mathbb{RP}^3, K)$ for a null homologous link K. Let us fix which branched cover we will use first. For the link projection L of a null homologous link K in $\mathbb{RP}^3$, we will consider the even branched double cover $\Sigma(\mathbb{RP}^3, L, C_P)$ by picking $P \in R_0$, see Definition 3.3. For each smoothing L_s, we will consider the branched double cover $\Sigma(\mathbb{RP}^3, L_s, C_P)$ obtained from $\Sigma(\mathbb{RP}^3, L, C_P)$ by doing surgeries, as described in Lemma 3.6. Note that these branched double covers do not depend on the specific choice of P as long as $P \in R_0$. By Lemma 3.4, we have

$$\Sigma(\mathbb{RP}^3, L_s, C_P) = \begin{cases} (\mathbb{RP}^3\#\mathbb{RP}^3)\#(S^1 \times S^2)^{\#(k_s-1)} & \text{if } e_s(P) = 0, \\ (S^1 \times S^2)^{\#k_s} & \text{if } e_s(P) = 1, \end{cases}$$

where k_s is the number of circles in the smoothing L_s. Hence, the corresponding Heegaard–Floer homology is

$$\widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P)) = \begin{cases} W \otimes V^{\otimes(k_s-1)} & \text{if } e_s(P) = 0, \\ \bar{V} \otimes V^{\otimes(k_s-1)} & \text{if } e_s(P) = 1, \end{cases}$$

where $W = \widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3) = \langle a, b, c, d \rangle$, $V = \widehat{HF}(S^1 \times S^2) = \langle v_+, v_- \rangle$, and $\bar{V} = \widehat{HF}(S^1 \times S^2) = \langle \bar{v}_+, \bar{v}_- \rangle$. See, for example, [10, Sections 4 and 7] for the computation of $\widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3)$ and its absolute grading. Here, $\bar{V}$ and V are the same vector space. The reason we want to distinguish them will become clear in the proof of the next proposition. Note that the quantum gradings of the generators are twice their absolute gradings in Heegaard–Floer homology . Also, note that $\widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P))$ is exactly the vector space we associate to the state s in our chain complex $\widetilde{CKh}^{\alpha_{\text{HF}}}_s(L)$.

Proposition 3.11 ([12, Proposition 6.2]). *For the link projection L of a null homologous link in $\mathbb{RP}^3$ and a point $P \in R_0$, there is an isomorphism*

$$\Psi_s : \widetilde{CKh}_s^{\alpha_{\mathrm{HF}}}(L) \mapsto \widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P))$$

for each state s such that the following diagram commutes if $s \to s'$ is a $1 \to 2$ bifurcation or $2 \to 1$ bifurcation:

$$\begin{array}{ccc} \widetilde{CKh}_s^{\alpha_{\mathrm{HF}}}(L) & \xrightarrow{\quad d_{CKh} \quad} & \widetilde{CKh}_{s'}^{\alpha_{\mathrm{HF}}}(L) \\ \Big\downarrow \Psi_s & & \Big\downarrow \Psi_{s'} \\ \widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P)) & \xrightarrow{d_{\mathrm{HF}}} & \widehat{HF}(\Sigma(\mathbb{RP}^3, L_{s'}, C_P)) \end{array}$$

Here, d_{CKh} is the differential defined for the chain complex $\widetilde{CKh}_\bullet^{\alpha_{\mathrm{HF}}}(L)$ as in Section 2.2, and d_{HF} is the map induced on $\widehat{HF}$ by surgeries described in Lemma 3.6.

Proof. As in Section 5 of [12], we have described an algebra structure on $V^{\otimes(k_s-1)}$ as the quotient of the polynomial algebra over $\mathbb{F}_2$ generated by S_i, divided out by the relations $S_i^2 = 0$ for $i = 1, \dots, k_s - 1$, where $S_i = v_+ \otimes v_+ \otimes \cdots \otimes v_- \otimes v_+ \otimes \cdots \otimes v_+$ with v_- at the ith component in the proof of Proposition 2.10. So, $\widetilde{CKh}_s^{\alpha_{\mathrm{HF}}}(L)$ is the free module of this algebra generated by elements in V_i, where $V_i = W$ if $e_s(P) = 0$ and $V_i = \overline{V}$ if $e_s(P) = 1$. On the other hand, $\widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P))$ is also a free module over the algebra

$$\wedge^* H_1((S^1 \times S^2)^{\#(k_s-1)}, \mathbb{Z}_2) \cong V^{\otimes(k_s-1)},$$

generated by elements in V_i, where again $V_i = W = \widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3)$ if $e_s(P) = 0$, and $V_i = \overline{V} = \widehat{HF}(S^1 \times S^2)$ if $e_s(P) = 1$. Furthermore, we can identify generators of $H_1((S^1 \times S^2)^{\#(k_s-1)}, \mathbb{Z}_2)$ with circles in L_s as follows. Recall we have picked a point M on the link projection L when defining the differential. Label the circles in the smoothing L_s from 0 to $k_s - 1$ such that the circle with M is labeled 0. Let γ_i be the branched double cover of an arc from the circle 0 to the circle i in $\mathbb{RP}^2$ avoiding other circles. Then, $\{[\gamma_i]\}_{i=1}^{k_s-1}$ is a basis of $H_1((S^1 \times S^2)^{\#(k_s-1)}, \mathbb{Z}_2)$. We define

$$\xi_s : \wedge^* H_1((S^1 \times S^2)^{\#(k_s-1)}, \mathbb{Z}_2) \mapsto V^{\otimes(k_s-1)}$$

as the algebra automorphism such that $\xi_s([\gamma_i]) = S_i$, and Ψ_s the corresponding module isomorphism, which is identity on V_i.

Now, we show commutativity of the diagram for $1 \to 2$ bifurcations and $2 \to 1$ bifurcations. Note that d_{CKh} acts trivially on the V_i component in this case. On the

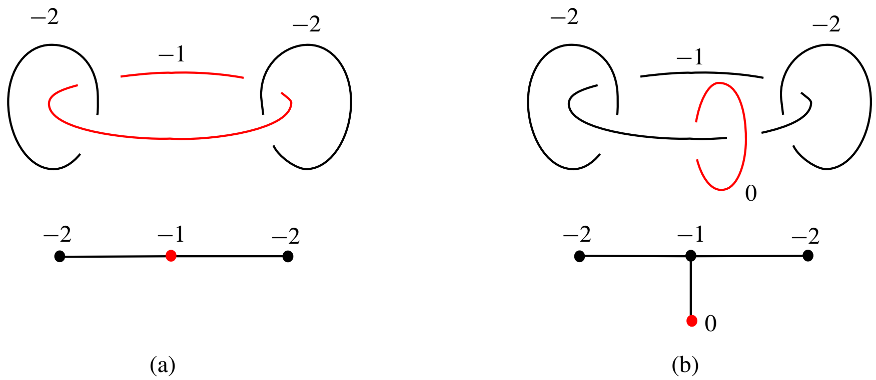


Figure 11. Kirby diagrams of surgeries associated to $1 \to 1$ bifurcations.

$V^{\otimes(k_s-1)}$ component, we have described the behavior d_{CKh} in this formalism in the proof of Proposition 2.10 as follows, which is from Section 5 of [12]:

$$d_{CKh} = \begin{cases} \text{quotienting out } S_i = S_j & \text{if } s \to s' \text{ merges circle } i \text{ with circle } j, \\ \text{wedging with } S_i + S_{k_s} & \text{if } s \to s' \text{ splits circle } i, \end{cases}$$

where k_s is the label of the newly created circle in the splitting case, and we denote $0 = S_0$ for the notational convenience. On the Heegaard–Floer homology side, we have exactly the same map via identifying $\widehat{HF}(\Sigma(\mathbb{RP}^3, L_s, C_P))$ as the free module over $V^{\otimes(k_s-1)}$ under the isomorphism ξ_s. See [12, Proposition 6.1] for a more detailed discussion of d_{HF}. Hence, the square commutes. ■

Remark. With the map ξ_s, it is natural to write down the change of variable map in Proposition 2.10 for changing the position of the marked point.

Now, we study the $1 \to 1$ bifurcation. We will draw specific Kirby diagrams for the surgeries, and compute the induced map on $\widehat{HF}$ using a proposition in [10].

If $s \to s'$ is a $1 \to 1$ bifurcation, then $e_{s'}(P) = e_s(p) + 1 \bmod 2$, and $k_s = k_{s'}$, so we switch from one branched double cover of $\mathbb{RP}^3$ over an unlink of k_s components to the other, while no change happens to the common $(S^1 \times S^2)^{\#(k_s-1)}$ component, nor the identification of generators of $H_1((S^1 \times S^2)^{\#(k_s-1)}, \mathbb{Z}_2)$ with the circles in the smoothing. So, it is enough to consider the case when $k_s = 1$.

Lemma 3.12. *The Kirby diagrams of the surgeries corresponding to the $1 \to 1$ bifurcation are shown in Figure 11, where (a) is the case when $e_s(P) = 0$ and (b) is the case when $e_s(P) = 1$, and the red curve is the one we are doing surgery on, with the framing specified by the number.*

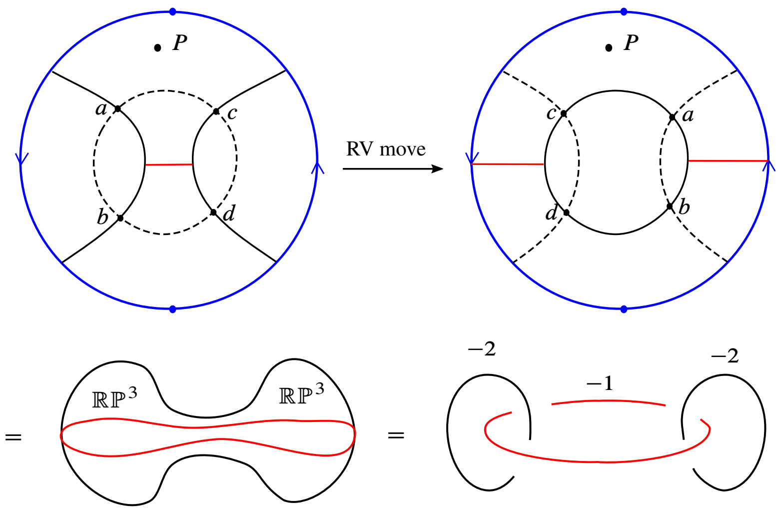


Figure 12. Explanation of Kirby diagram (a).

Proof. Suppose $e_s(P) = 0$; then, $e_{s'}(P) = 1$, and the surgery changes the branched double cover from $\mathbb{RP}^3\#\mathbb{RP}^3$ to $S^1 \times S^2$. The knot we are doing surgery along is the branched double cover of the red arc in the top two pictures of Figure 12, which could be seen as the red circle in the Kirby diagram of $\mathbb{RP}^3\#\mathbb{RP}^3$ in Figure 11 (a). To determine the framing, we can look at the 3-manifold when it is with framing $-n$. Such obtained 3-manifold is a lens space $L(p,q)$ with $p = 4n - 4$, $q = 2n - 1$, where p/q equal to the continued fraction $[2, n, 2]$. To get $S^1 \times S^2$, we have $p = 0$ and $n = 1$.

Now, if $e_s(P) = 1$, then $e_{s'}(P) = 0$ and the surgery changes the branched double cover from $S^1 \times S^2$ to $\mathbb{RP}^3\#\mathbb{RP}^3$. Note that the red arc in Figure 13 is isotopic to a pushoff of the arc ab, so the branched double cover of the red arc is isotopic to a meridian of the green curve, which is the knot we are doing surgery along to get $S^1 \times S^2$ from $\mathbb{RP}^3\#\mathbb{RP}^3$. To determine the framing, let us look at the following cobordism in Figure 14. The compositions of the top and bottom two arrows both represent the cobordism from $\mathbb{RP}^3\#\mathbb{RP}^3$ to itself by doing surgeries along the branched double cover of the red and green arcs. The difference is that we add the two 2-handles in different orders. It is clear that in the bottom-left cobordism from $\mathbb{RP}^3\#\mathbb{RP}^3$ to $\mathbb{RP}^3\#\mathbb{RP}^3\#(S^1 \times S^2)$, the framing of the red circle is 0. Therefore, the framing of the red circle is 0 in the top-right cobordism as well, which is the cobordism we are describing in Figure 13. ∎

Now, to compute the induced map on $\widehat{HF}$ by the surgeries, we quote the following proposition from [10].

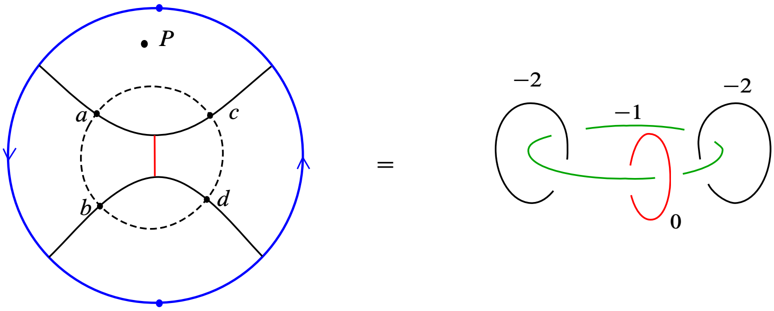


Figure 13. Explanation of Kirby diagram (b).

Proposition 3.13 ([10, Proposition 9.3]). *Let Y be a closed oriented 3-manifold, and $K \subset Y$ a framed knot such that the trace $Z(K)$ of the surgery along K has $b_2^-(Z(K)) = 0 = b_2^+(Z(K))$. Let $\mathfrak{s}$ be a* Spin^c *structure on $Z(K)$, whose restriction $\mathfrak{t}$ and $\mathfrak{p}$ to the boundary components Y and $Y(K)$, respectively, are torsion.*

(1) *If K represents a non-torsion class in $H_1(Y)$, and if $\widehat{HF}(Y,\mathfrak{t})$ is standard, then the induced map*

$$F_{Z(K),\mathfrak{s}} : \widehat{HF}(Y,\mathfrak{t}) \mapsto \widehat{HF}(Y(K),\mathfrak{p})$$

vanishes on the kernel of the action by $[K]$, inducing an isomorphism

$$\widehat{HF}(Y,\mathfrak{t})/\operatorname{Ker}[K] \cong \widehat{HF}(Y(K),\mathfrak{p}).$$

(2) *If K represents a torsion class in $H_1(Y)$ and $\widehat{HF}(Y(K),\mathfrak{p})$ is standard, then the map*

$$\widehat{HF}(Y,\mathfrak{t}) \mapsto \widehat{HF}(Y(K),\mathfrak{p})$$

induces an isomorphism

$$\widehat{HF}(Y,\mathfrak{t}) \cong Ker[L],$$

where $[L] \in H_1(Y(K))$ is represented by the core of the glued-in solid torus.

Here, $\widehat{HF}(Y,\mathfrak{t})$ is *standard* means that $\widehat{HF}(Y,\mathfrak{t}) \cong (\Lambda^b H^1(Y,\mathbb{Z})) \otimes \mathbb{F}_2$, where $b = b_1(Y)$. The original proposition is stated for HF^∞, but it holds for $\widehat{HF}$ as well. When $\widehat{HF}(Y,\mathfrak{t})$ is standard, we have $HF^\infty(Y,\mathfrak{t}) = \widehat{HF}(Y,\mathfrak{t}) \otimes \mathbb{Z}[U,U^{-1}]$, and the cobordism map is U-equivariant, so the statement for HF^∞ implies the one for $\widehat{HF}$.

Note that $\widehat{HF}(Y,\mathfrak{t})$ is standard for $Y = \mathbb{RP}^3\#\mathbb{RP}^3$ and $S^1 \times S^2$, and all torsion Spin^c structures of them. Now, we compute the induced map on $\widehat{HF}$ of the surgery

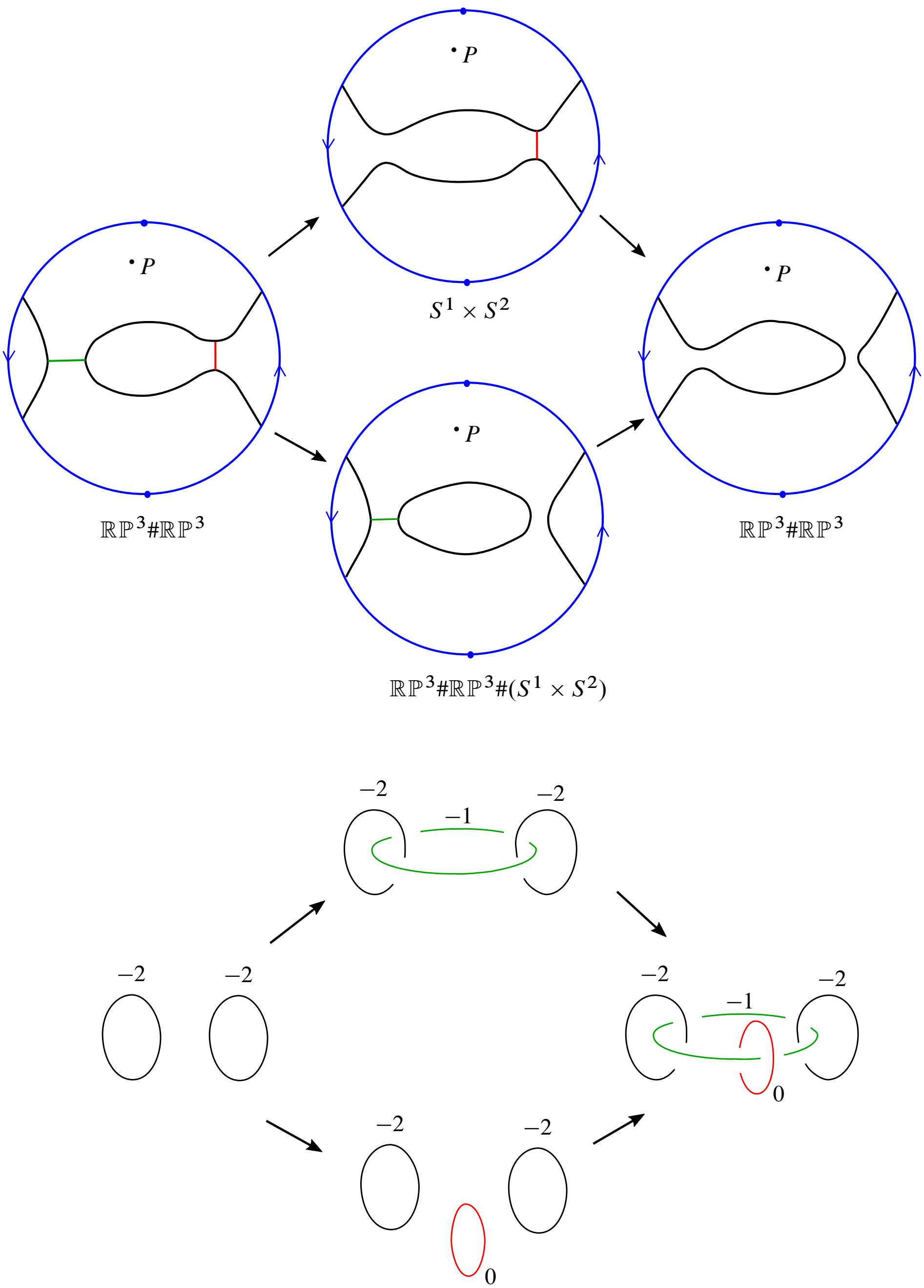


Figure 14. Two different decompositions of the cobordism from $\mathbb{RP}^3 \# \mathbb{RP}^3$ to itself.

corresponding to $1 \to 1$ bifurcations. Recall we have used the notation

$$\widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3) = W = \langle a, b, c, d \rangle,$$

where a, b, c, d are generators corresponding to different torsion Spin^c structures on $\mathbb{RP}^3 \# \mathbb{RP}^3$, with absolute gradings $1/2, 0, 0, -1/2$, respectively, and

$$\widehat{HF}(S^1 \times S^2) = \bar{V} = \langle \bar{v}_+, \bar{v}_- \rangle,$$

where $\bar{v}_+$ and $\bar{v}_-$ are generators corresponding to the torsion Spin^c structure $\mathfrak{s}_0$ on $S^1 \times S^2$, with absolute gradings $1/2, -1/2$, respectively.

Proposition 3.14. *For the cobordism Z_a associated to (a) in Figure 11, the induced map on $\widehat{HF}$ is*

$$\begin{aligned} &f = F_{Z_a} : \widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3) \mapsto \widehat{HF}(S^1 \times S^2), \\ &f(b) = f(c) = \bar{v}_-, \quad f(a) = f(d) = 0. \end{aligned}$$

For the cobordism Z_b associated to (b) in Figure 11, the induced map on $\widehat{HF}$ is

$$\begin{aligned} &g = F_{Z_b} : \widehat{HF}(S^1 \times S^2) \mapsto \widehat{HF}(\mathbb{RP}^3 \times \mathbb{RP}^3), \\ &g(\bar{v}_+) = b + c, \quad g(\bar{v}_-) = 0. \end{aligned}$$

Here, we are summing over all Spin^c *structures on the cobordism.*

Proof. Let us start with Z_a. Let us compute the degree shift of the cobordism map $F_{Z_a,\mathfrak{s}}$ on $\widehat{HF}$ first. We use the results of Section 11 in [9], where homology classes and Spin^c structures of the cobordism are described in terms of linking matrices.

Let

$$\Lambda = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

denote the linking matrix for (a) in Figure 11, and denote the ith column of Λ by Λ_i. Then, we have

$$H_2(Z_a, \mathbb{Z}) = \{u \in \mathbb{Z}^3 \mid \Lambda_1 \cdot u = \Lambda_2 \cdot u = 0\} = \langle (1,1,2) \rangle \cong \mathbb{Z},$$

and the intersection form $H_2(Z_a, \mathbb{Z}) \times H_2(Z_a, \mathbb{Z}) \mapsto \mathbb{Z}$ is represented by Λ restricted to the span of $(1, 1, 2)$, which is 0. Therefore, we have $b_2(Z_a) = 1$, and $b_2^+(Z_a) = b_2^-(Z_a) = 0$. Then, for any Spin^c structure s on Z_a, we have

$$\deg(F_{Z_a}, \mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(Z_a) - 3\sigma(Z_a)}{4} = \frac{0 - 2 - 0}{4} = -\frac{1}{2}.$$

So, non-trivial maps only happen from $\widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3, \mathfrak{t})$ to $\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the torsion Spin^c structure on $S^1 \times S^2$, and $\mathfrak{t}$ is a Spin^c structure on $\mathbb{RP}^3\#\mathbb{RP}^3$ such that the corresponding generator has absolute grading 0, i.e., generators b and c.

Consider the Spin^c structure $\mathfrak{t}$ corresponding to the generator b first, and let $\mathfrak{s}$ be the Spin^c structure on Z_a which restricts to $\mathfrak{t}$ and $\mathfrak{s}_0$ on each boundary component. Then, the pair $(Z_a, \mathfrak{s})$ satisfies the condition of Proposition 3.13, with $Y = \mathbb{RP}^3\#\mathbb{RP}^3$, and K the red curve in (a) of Figure 11. The knot K represents a torsion class in $H_1(\mathbb{RP}^3\#\mathbb{RP}^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let L be the core of the glued-in solid torus in $Y(K) = S^1 \times S^2$; then, $[L]$ is a generator of $H_1(S^1 \times S^2)$, and the action of $[L]$ on $\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0) = \langle \bar{v}_+, \bar{v} \rangle$ is such that

$$[L] \cdot \bar{v}_+ = \bar{v}_-, \quad [L] \cdot \bar{v}_- = 0.$$

So, $\mathrm{Ker}([L]) = \langle \bar{v}_- \rangle$, and $F_{Z_a,\mathfrak{s}} : \widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3, \mathfrak{t}) \mapsto \mathrm{Ker}[L]$ is an isomorphism sending b to $\bar{v}_-$ by Proposition 3.13. The case when $\mathfrak{t}$ corresponds to the generator c is similar, which is an isomorphism sending c to $\bar{v}_-$. Therefore, by summing up these two maps, we get the map

$$\begin{aligned} f &= F_{Z_a} : \widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3) \mapsto \widehat{HF}(S^1 \times S^2), \\ f(b) &= f(c) = \bar{v}_-, \quad f(a) = f(d) = 0 \end{aligned}$$

as stated in the proposition.

Now, we consider the cobordism Z_b as in (b) of Figure 11. This time, the linking matrix is

$$\Lambda = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let Λ_i be the ith column of Λ. Then, we have

$$H_2(Z_b, \mathbb{Z}) = \left\{ u \in \mathbb{Z}^4 | \Lambda_1 \cdot u = \Lambda_2 \cdot u = \Lambda_3 \cdot u = 0 \right\} = \langle (1,1,2,0) \rangle \cong \mathbb{Z},$$

and again, the intersection form is given by Λ restricted to the span of $(1, 1, 2, 0)$, which again vanishes. So, we get $b_2(Z_b) = 1$ and $b_2^+(Z_b) = b_2^-(Z_b) = 0$. For any Spin^c structure $\mathfrak{s}$ on Z_b, we again have $\deg(F_{Z_b,\mathfrak{s}}) = -1/2$. Therefore, non-trivial maps only happen from $\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0)$ to $\widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3, \mathfrak{t})$, where $\mathfrak{t}$ is again a Spin^c structure on $\mathbb{RP}^3\#\mathbb{RP}^3$ corresponding to the generator b or c of absolute grading 0.

Consider the Spin^c structure $\mathfrak{t}$ corresponding to the generator b first, and let $\mathfrak{s}$ be the Spin^c structure on Z_b which restricts to $\mathfrak{s}_0$ and $\mathfrak{t}$ on each boundary component. The pair $(Z_b, \mathfrak{s})$ satisfies the condition of Proposition 3.13 with $Y = S^1 \times S^2$

and K is the red curve in (b) of Figure 11. This time, $[K]$ is twice the generator of $H_1(S^1 \times S^2) = \mathbb{Z}$, which is non-torsion. According to Proposition 3.13, we have an isomorphism

$$F_{Z_b,\mathfrak{s}} : \widehat{HF}(S^1 \times S^2, \mathfrak{s}_0)/\operatorname{Ker}[K] \mapsto \widehat{HF}(\mathbb{RP}^3\#\mathbb{RP}^3, \mathfrak{t}),$$

where $\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0)/\operatorname{Ker}[K] = \langle \bar{v}_+ \rangle$. So, $F_{Z_b,\mathfrak{s}}$ sends $\bar{v}_+$ to b. The case when $\mathfrak{t}$ corresponds to the generator c is similar. By summing up these two maps, we get

$$g = F_{Z_b} : \widehat{HF}(S^1 \otimes S^2) \mapsto \widehat{HF}(\mathbb{RP}^3 \times \mathbb{RP}^3),$$
$$g(\bar{v}_+) = b + c, \quad g(\bar{v}_-) = 0.$$

■

Combining Propositions 3.11 and 3.14 and the description of the link spectral sequence of $\widehat{HF}$ relating to the cube of resolutions at the end of Section 3.2, we get the following theorem.

Theorem 3.15. *Let L be a null homologous link in $\mathbb{RP}^3$. There is a spectral sequence whose E^2 terms consist of the Khovanov-type homology $\widetilde{Kh}^{\alpha_{\mathrm{HF}}}(m(L))$ of the mirror of L with the dyad $\alpha_{\mathrm{HF}} = (W, \bar{V}, f, g)$ introduced in part (3) of Section 2.6, which converges to the Heegaard–Floer homology $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, L))$ of the even branched double cover $\Sigma_0(\mathbb{RP}^3, L)$ of $\mathbb{RP}^3$ branched over L.*

Proof. The proof is similar to that of Theorem 3.10. See the discussion at the end of Section 3.2. For each double point in a link projection of L to $\mathbb{RP}^2$, we associate a knot K_i in $\Sigma(\mathbb{RP}^3, L, C_P)$ to it, which is the branched double cover of a vertical arc connecting the two double points. Consider the link spectral sequence of $\widehat{HF}$ of the link in $\Sigma(\mathbb{RP}^3, L, C_P)$ consisting of all the K_i's. For each smoothing L_s of the link projection of L, we have the branched double cover $\Sigma(\mathbb{RP}^3, L_s, C_P)$ equal to $(\mathbb{RP}^3\#\mathbb{RP}^3)\#(S^1 \times S^2)^{\#(k_s-1)}$ if $e_s(P) = 0$, and equal to $(S^1 \times S^2)^{\#k_s}$ if $e_s(P) = 1$. Hence, the E^1 terms in the link surgery spectral sequence, which are the Heegaard–Floer homology of the corresponding branched double covers, are the same as $\widetilde{CKh}_{\alpha_{\mathrm{HF}}}(m(L))$ as vector spaces. The differential d_1 in the spectral sequence has been computed in Propositions 3.11 and 3.14, which is the same as the differential d in the chain complex $\widetilde{CKh}_{\alpha_{\mathrm{HF}}}(m(L))$. Therefore, E^2 terms of the link spectral sequence are the Khovanov-type homology $\widetilde{Kh}^{\alpha_{\mathrm{HF}}}(m(L))$. ■

Remark. There is another spectral sequence converging to the Heegaard–Floer homology of the odd branched double cover $\widehat{HF}(\Sigma_1(\mathbb{RP}^3, L))$, whose E^2 terms consist of $\widetilde{Kh}^{\alpha^*_{\mathrm{HF}}}(m(L))$, where $\alpha^*_{\mathrm{HF}} = (\bar{V}, W, g, f)$ is the dual dyad of α.

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