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Movie moves for framed foams from multijet transversality

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Abstract. We use multijet transversality techniques to give a presentation by generators and relations of the categories of framed tangled webs and foams.

1. Introduction

In a beautiful book[1], Carter and Saito [11] describe ways to represent surfaces embedded in 4-dimensional spaces and their isotopies. This can be understood as a higher-dimensional analog of Reidemeister's theorem, stating that generically a link can be represented by a planar diagram assembled from elementary pieces (namely, pieces of strands and crossings) and that an isotopy of the link translates into a succession of elementary moves. In the case of embedded surfaces (we will also refer to them as knotted surfaces), diagrams of links are replaced by "movies" (a sequence of link diagrams). An isotopy of the knotted surface then translates into a succession of elementary movie isotopies or "movie moves", for which Carter and Saito establish an exhaustive list.

The goal of the present paper is to extend this work to certain knotted singular CW complexes (namely, foams), which play a prominent role in recent developments of quantum knot homologies.

1.1. Functoriality of Khovanov homology

Khovanov's early definition of a homological lift of the Jones polynomial [20] extends to an invariant of knotted surfaces [21]. This is most easily seen under Bar-Natan's reformulation of Khovanov homology [2]. Indeed, in this latter version, the definition of the invariant makes use of cobordisms between (crossingless) curves. That way, an unknotted surface is naturally assigned a morphism, and elementary cobordism generators between links (namely, Reidemeister moves) are assigned morphisms from

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[1] Original sources should include [1, 9, 10, 17]

the very proof of the invariance. One then needs to check that these assignments do not depend on the chosen isotopy representative, and this is where Carter and Saito's movie moves come into play. Considering the images under Khovanov's process of all those moves, it turns out that the process is only independent on the choice up to a global sign. Functoriality holds over $\mathbb{Z}/2\mathbb{Z}$ but not over $\mathbb{Z}$ (see, for example, [2, page 1480]).

This functoriality defect was fixed first by Clark, Morrison and the second author [14] and Caprau [5], using cobordisms carrying special lines and adjoining a square root of -1 to the base ring, then by Blanchet [3] using singular cobordisms ($\mathfrak{gl}_2$ incarnations of foams) and working over the integers.

1.2. From knot cobordisms to tangled webs and foams

Blanchet's fix introduces the notion of foams as an intermediate object in Khovanov's process, allowing for a sign adjustment. That way, the homology of a knot remains unchanged, but the homomorphism associated to a classical cobordism gets adjusted by a sign, somehow related to the determinant representation of $\mathfrak{gl}_2$. Checking Carter–Saito's movie moves yields a functorial theory.

The appearance of the foams in the context of knot homologies goes back to the Khovanov–Rozansky's categorification [22, 23] (and Mackaay–Stosic–Vaz version of it [26]) of the Reshetikhin–Turaev's $\mathfrak{sl}_n$ invariants [33]. In these works, they appear as the most natural version of cobordisms between certain trivalent graphs called webs. These webs, as introduced by Kuperberg [24], can be understood as diagrammatic versions of the categories of representations that are used in the definition of Reshetikhin and Turaev's invariants. The interest in their study was reinforced by Cautis–Kamnitzer–Morrison's proof of a presentation for them [13]. An analogous presentation of the foam category was then provided by the first author with Rose [31]. Later, Robert–Wagner [34] gave a topological construction of the foam category using the technology of universal construction [4]. This latter formulation was then a key ingredient in the proof by Ehrig, Tubbenhauer, and Wedrich [16] of the functoriality of $\mathfrak{gl}_n$ link homologies.

However, having at hand this notion of webs, it is natural to consider tangled versions of these (that is, embeddings of webs in 3-space) and the definition of the Khovanov–Rozansky functors will extend (almost) for free. Such an extension finds further justification in skein approaches [28, 38] to generalizations of the invariants to 3-manifolds, an approach outlined by the first named author and Wedrich in [32]. However, while the definition of the extended version of the functors is given almost for free, the next question of functoriality confronts an unexpected issue: before the present paper, there did not exist a complete list of movie moves for foams suitable for functoriality proofs.

In this direction, the main reference is Carter's work [6], where he presents a list of moves for unframed embedded foams with no preferred vertical direction (see also the book of Carter and Kamada [8]). This work was related to several papers on knotted foams introduced in relation to quandles [7] or topics related to more classical topology [12].

We feel that there were thus two gaps in the literature, that this paper aims to fill: first, the vertical direction (because we want to be able to project down to link diagrams) and second, a notion of framing. This notion appears to us as necessary since naive attempts to prove functoriality in an unframed setting quickly run into contradictions (see [29, Section 2.3.2.2]). One might imagine that a closer analysis in the spirit of [15, 39] might avoid framing. In such a case, one can easily deduce from our main theorem 4.1 a complete list of framing-free moves.

It might be worth noting that, since this paper first appeared, the first named author used our main result to prove functoriality of Khovanov homology for webs, which was then used to prove skein positivity for surface [30].

1.3. Movie moves and multijet transversality

In order to organize the analysis of all possible movie moves, we have chosen to use the framework of multijet transversality (Section 2). This is related to Roseman's approach to higher knots [35], where transversality and Morse theoretic techniques are used to decompose isotopies of knots in higher dimensions. This entails working in the smooth setting. A framed foam is represented by a smooth map from a thickened abstract foam into $\mathbb{R}^4$. The space of all such smooth maps has a stratification, with strata corresponding to non-generic situations, such as double (or triple or quadruple) points in the projection to $\mathbb{R}^2$, framing vectors pointing the vertical direction, and so on. After a small perturbation, the original framed foam (and its associated multijets) can be made transverse to this stratification. The resulting transverse intersections lead to foam generators (caps, saddles, Reidemeister moves, and so on). Similarly, a 1-parameter family (isotopy) of foams can be perturbed to be transverse to the stratification, and the resulting transverse intersections correspond to movie moves for framed foams.

1.4. Results

In Section 2, we briefly recall the basics of multijet transversality.

In Section 3, we investigate the situation for framed webs and isotopies of them and give two versions of a presentation, using either half twists in Theorem 3.7 or full twists in Theorem 3.8. The first version might appeal more to a topologist, while the second one is designed for knotted web invariants.

In Section 4, we extend this analysis to the case of foams and isotopies of them. The main results of the paper are Theorems 4.1 and 4.2, which again are two versions of a full list of movie moves for framed foams between knotted framed tangled webs represented by diagrams.

2. Multijet transversality

We briefly review the basic settings of multijet transversality that will prove useful to us. (We rely on Golubitsky and Guillemin's textbook [19], in particular, Chapters 2.2 and 4.) Let X and Y be smooth manifolds. Denote by $J^k(X,Y)_{p,q}$ the set of equivalence classes of maps $f : X \to Y$ with $f(p) = q$, where the equivalence relation is that $f \sim_k g$ if f has kth order contact with g at p. This property is inductively defined as follows.

- If $k = 1$, $(df)_p = (dg)_p$.
- If $k > 1$, $(df)_p$ and $(dg)_p$ have $(k-1)$st order contact at every point in T_pX.

This amounts to asking that all partial derivatives of order up to k agree at p.

Then, one can form

$$J^k(X,Y) = \bigcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q},$$

the elements of which are called k-jets from X to Y. The set $J^k(X,Y)$ can be given the structure of a finite-dimensional smooth manifold in a natural way.

Given $f : X \to Y$ there is an associated k-jet $j^k f : X \to J^k(X,Y)$.

Now, consider

$$X^s = X \times \cdots \times X \quad \text{and} \quad X^{(s)} = \{(x_1,\ldots,x_s) \in X^s \mid \forall i,j, x_i \neq x_j\}.$$

One has source maps

$$\alpha : J^k(X,Y) \mapsto X, \quad \alpha^s : J^k(X,Y)^s \mapsto X^s,$$

and one can from the s-fold k-jet bundle

$$J^k_s(X,Y) = (\alpha^s)^{-1}(X^{(s)}).$$

Given $f : X \to Y$, there is an associated s-fold k-jet map $j^k_s f : X^{(s)} \to J^k_s(X,Y)$. The map $j^k_s f$ describes the behavior of f up to order k at s distinct points of X.

Our main tool is the following theorem of Mather (see [27, Proposition 3.3]) generalizing Thom's transversality theorems [36, 37] (see also [25] for a gentle introduction to the topic).

Theorem 2.1 (Multijet transversality theorem [19, Theorem 4.13]). *Let Z be a submanifold of $J_s^k(X,Y)$. Let*

$$T_Z = \{ f \in \mathcal{C}^\infty(X,Y) \mid j_s^k f \,\overline{\pitchfork}\, Z \}.$$

Then, T_Z is a residual subset of $\mathcal{C}^\infty(X,Y)$. Moreover, if Z is compact, then T_Z is open.

Above $\overline{\pitchfork}$ is the notation for transverse intersection, and residual means that it is the countable intersection of open dense subsets. In the case of a Baire space (which $\mathcal{C}^\infty(X,Y)$ is), this implies that it is dense.

The typical situation we will want to address using the above theorem is that of a mapping $f : F \mapsto \mathbb{R}^4$ of a (suitably modified) foam into $\mathbb{R}^4$. The submanifold Z will be given by some condition we wish to avoid or control, for example, having multiple points under the vertical projection. Then, we apply the theorem to claim that up to minor adjustment f can be made transverse to W, and then we go on to analyze what a local model is. Section 3.2.3, where we use this strategy to rederive the first Reidemeister move for framed knots, might be a good place where to find an illustration of this general process.

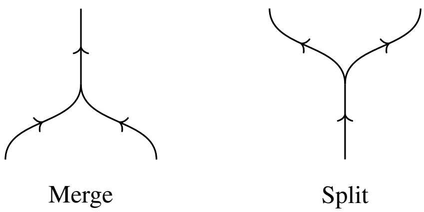
3. Reidemeister theorem for framed webs

In this section, we use multijet transversality to rederive the Reidemeister theorem for knotted webs, with a treatment for framing that will prove useful in the foam setting. The results are presented in Theorem 3.7 and Theorem 3.8.

3.1. Webs

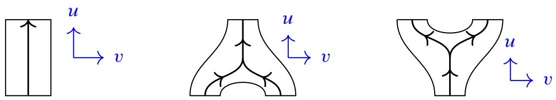
We first define webs as follows.

Definition 3.1. An abstract smooth web is a singular CW-complex of dimension 1 locally smoothly diffeomorphic to either an interval or smooth realizations of the following trivalent graphs:



In other words, restricting the chart maps to the 1-manifolds formed by selecting one of the two legs and the other strand forms a smooth 1-manifold. We write W_0 for the union of the 0-cells and W_1 for the union of open arcs given by the interiors of the 1-cells.

Then, one can turn an abstract web into a 2-manifold $\widetilde{W}$ by considering a 2-manifold with boundary with preferred web in it, locally smoothly diffeomorphic to one of the following pieces:



Throughout the first part of the paper, we will keep the convention that u is the tangent direction for the web, while v is the normal direction to W in $\widetilde{W}$.

Remark 3.2. Our asymmetric choice for trivalent vertices might sound surprising (for example, webs in [18] have rotation-invariant vertices), but it allows us to reduce the number of moves that will appear, in particular, for the tetrahedral vertices that will be introduced in the foam section. For example, the moves involving twists presented in [8, Chapter 10, Section 2] do not appear in our case, as they involve a symmetry of the trivalent vertex (the closest analogs in our context would be the last two relations in Theorem 4.1). Another reason for making such a choice is that it allows us to cover a web by smooth segments, or a foam by smooth disks. On a related note, one can erase some of the strands (or facets) and the remainder will still be a web (or foam). All of these are particularly useful when turning to functoriality proofs for Khovanov or Khovanov–Rozansky homologies (where labels of strands/facets break the symmetry anyway from the beginning).

We now consider smooth maps

$$f\colon \widetilde{W} \mapsto \mathbb{R}^3.$$

One recovers the usual notion of a tangled web by restricting f to W. Furthermore, we get a framing on $f(W)$ by looking at a section of the normal bundle of W in $\widetilde{W}$ (given by $\frac{\partial}{\partial v}$ in some parametric version for example).

Remark 3.3. Two notions of framed objects appear in the literature: either one considers a vector field (non-vanishing, with possibly extra conditions), or one considers actual ribbons. In what follows, we consider the first notion. However, f could produce the former or the latter, and several functions f could induce the same web with the same vector field. A deformation of f clearly induces a deformation of the associated framed web. Conversely, given a framed web, one can build an associated function f (extend f from W to $\widetilde{W}$ by going along the framing vector, drawing a ribbon of given length; by compactness one can find such a non-zero length so that the image is embedded). Then, an isotopy of webs comes from a deformation of functions.

We will denote by (x, y, z) coordinates in $\mathbb{R}^3$ with unit vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$. $\vec{z}$ stands for the vertical direction, and we have a preferred projection π onto

$$\mathbb{R}^2 = \{(x, y, 0)\}.$$

When we illustrate webs, the plane of the page coincides with this plane.

We first use transversality techniques to demonstrate that generically, compact webs, when mapped into $\mathbb{R}^3$, are embedded. In other words, this means that we can assume that f is injective on W. Consider Z the union of submanifolds[2] in J_0^2:

$$Z = \{(M_1, N_1, M_2, N_2) \mid M_1, M_2 \in W,\ N_1 = N_2 \in \mathbb{R}^3\big\} \subset \widetilde{W} \times \mathbb{R}^3 \times \widetilde{W} \times \mathbb{R}^3.$$

The set Z is of codimension 5 in the 10-dimensional product space. The codimension is computed as follows: $M_i \in W \subset \widetilde{W}$ gives two degrees of freedom. Since $N_1 = N_2$, then 3 coordinates are unnecessary. The graph

$$G = \big\{(M_1, f(M_1), M_2, f(M_2))\big\}$$

is a 4-dimensional subset of this same 10-dimensional space. Since $4 + 5 = 9 < 10$, generically, the graph G and the set Z do not intersect. So, by imposing the map f to produce an embedded web $f(W)$, we place ourselves in the generic situation.

For such an embedding, we may further impose the condition that df is of full rank on W so that the framing does not meet the strands. The condition to be avoided is expressed by (below $M_{r,s}(\mathbb{R})$ stands for $(k \times s)$-matrices with real coefficients)

$$\big\{(M, N, D) \mid M \in W, N \in \mathbb{R}^3, D \in M_{3,2}(\mathbb{R}), \mathrm{rank}(D) \leq 1\big\}.$$

The (3×2)-matrix expresses the differential df. If its rank is less than 2, then the framing direction is proportional to the tangent direction of W at the point M. Indeed, one of the two vectors can be chosen freely. Then, the other one has to be proportional to it, leaving us with only one choice (the proportionality factor) for three coordinates. The corresponding graph

$$\big\{(M, f(M), df_M) \mid M \in \widetilde{W}\big\}$$

is of dimension 2. Again, the condition on the rank is generic.

[2] As the web is not a manifold, we only get a union of submanifolds by considering all circles that cover the web. Indeed, at a point of W_0, one can choose two branches of the web so that there is a smooth flow through the vertex, producing a covering of W by means of several simple closed curves.

Definition 3.4. We say that a function f that is injective on W and with df having full rank on W represents a *framed web with generic projection* if the following conditions are fulfilled.

(1) $\pi \circ f$ has isolated double points, with both pre-images lying in W_1; the double points have no higher intersections.

(2) Such double points are transverse: the two vectors $\pi(\frac{\partial f}{\partial u})$ (at both pre-images of the double point) span $\mathbb{R}^2$.

(3) $\frac{\partial f}{\partial u}$ is not vertical on W (in other words, $\pi(\frac{\partial f}{\partial u}) \neq 0$ on W).

(4) Except at isolated (*half twist*) points, we have

$$\frac{\partial f}{\partial v} \notin \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right). \tag{3.1}$$

(5) The above half twist points are transverse:

$$\frac{\partial^2 f}{\partial u \partial v} \notin \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

This will ensure that the behavior of the (x, y) projection of the framing in the neighborhood of a half twist will be controlled by the $\partial u \partial v$ derivative.

We will see in Section 3.1.3 that a given embedding of a web can be approximated by one that satisfies these conditions.

Remark 3.5. Let us say a word about the framing convention. Above, we have considered that points are generic if the projection of the framing vector is not proportional to the tangent vector (and in particular non-zero). That way we isolate points where the framing goes over or under the strand. Later in the argument, we will want to turn the framing into a preferred position, namely, lying under π on the right-hand side of the strands (with respect to orientation). To do so, one can choose a rotation around $\frac{\partial f}{\partial u}$ that brings $\frac{\partial f}{\partial v}$ into the desired position (so that it projects to the right-hand side and maximizes the length of the projection). Our preferred choice of rotation is the one that does not make $\frac{\partial f}{\partial v}$ cross $\mathbb{R}\frac{\partial f}{\partial u} + \mathbb{R}^{-}\vec{z}$: any framing vector that is not pointing downward can be continuously brought to the right without pointing downward at any time. This allows to turn a framed web into one where the framing is on the right except at isolated points where the framing does a complete turn around a strand (we call this a twist). This process only requires that we isolate those points where the framing passes under a strand (because of the $\mathbb{R}^{-}\vec{z}$ condition). However, the process of bringing a framed web with generic projection to a framed web with right-handed framing introduces crossings (at trivalent points). We thus found it easier to isolate the cases where the framing passes under or over a strand (these situations are called half twists). This produces Theorems 3.7 and 4.1. At the very end, we will apply our preferred rotation and deduce classification results for right-sided framed webs: this yields Theorems 3.8 and 4.2.

Let us first find local models for generic points.

3.1.1. Local model for generic $M \in W_1$. Let $M \in W_1$ that is not a multiple point under π and so that condition (3) and condition (3.1) above hold.

Fix a local chart on $\widetilde{W}$ so that $M = 0$, and consider a neighborhood of M of the form $[-u_1, u_1] \times [-v_1, v_1]$. Since $\frac{\partial f}{\partial u}$ is not vertical, at least one of its x or y coordinates is non-zero. Up to rotation around a vertical axis in the target 3-space, one can assume that

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}, \qquad a > 0.$$

Then, upon post-action by a matrix in $GL_2 \subset GL_3$ (where GL_2 acts on the x and y coordinates), one can reduce df to be of the following kind:

$$df = \begin{pmatrix} a & 0 \\ 0 & c \\ b & d \end{pmatrix}.$$

Because of condition (3.1), we have $c \neq 0$.

Then, around M, one can write

$$f(\varepsilon_1, \varepsilon_2) = f(M) + \begin{pmatrix} a\varepsilon_1 \\ c\varepsilon_2 \\ b\varepsilon_1 + d\varepsilon_2 \end{pmatrix} + o(\varepsilon_1, \varepsilon_2).$$

Setting $\varepsilon_2 = 0$, this draws a portion of a line in the 3-space that projects into a portion of a line on the (x, y) plane (as $a \neq 0$). Now, we can look at the framing. Recall $(\frac{\partial f}{\partial v})_m = c\vec{y} + d\vec{z}$ with $c \neq 0$. Thus,

$$\left(\frac{\partial f}{\partial v}\right)_{(\varepsilon_1, 0)} = \begin{pmatrix} 0 \\ c \\ d \end{pmatrix} + o(\varepsilon_1).$$

This means that around M at first order, the projection of the framing can be simply obtained by transporting the framing at M. Depending on the sign of c, we get as generator

 or .

Above, we have indicated the framing by a red ribbon.

3.1.2. Local model for $M \in W_0$. Let us now look at $M \in W_0$ and run the same analysis. Conveniently, forgetting one of the two strands that arrive parallel brings us back to the previous situation, so we can make the same simplifications. Depending on the framing, this will bring us to one of the following situations:

We insist that the framing is parallel to the paper (it might appear to go into the paper in the picture above).

3.1.3. Conditions. Before looking for local models for double points and places where the assumption from equation (3.1) fails, let us first check that the assumptions we made on f to represent a framed web are reasonable.

Requiring $\pi \circ f$ to have double points corresponds to the union of submanifolds in $J_2^0(\widetilde{W}, \mathbb{R}^4)$ (we will often just write J_2^0):

$$Z = \big\{(M_1, N_1, M_2, N_2), M_1 \in W, M_2 \in W, \pi(N_1) = \pi(N_2)\big\}.$$

These manifolds are of codimension 4 (we get $1+1$ by restricting $M_i \in W$, and 2 from the equation $\pi(N_1) = \pi(N_2)$). Since the points $f(M_1)$ and $f(M_2)$ lie upon a vertical line that can move freely about the (x, y)-plane, the graph

$$\big\{(M_1, f(M_1), M_2, f(M_2))\big\}$$

is of dimension 4, so one can make f transverse to the submanifolds with isolated intersections.

Furthermore, if one requires M_1 or M_2 to lie on W_0, the codimension drops again by one. One then has codimension 5 submanifolds to avoid for a graph of dimension 4, and since $4 < 5$, transverse intersections are empty.

We argue that no higher multiplicities will generically occur. Indeed, in J_k^1, consider the singular set generated by the condition that k points from W project in the plane to the same point. All points should come from W, which contributes by k to the codimension of this set. Then, the image of the first point can move freely, and the $k-1$ other ones are constrained on a vertical line passing through the first one. For each of them, this contributes by 2 to the codimension of the set. This singular set thus has codimension

$$k + 2(k-1) = 3k - 2,$$

while the graph is of dimension $2k$. At $k = 2$, we have isolated singularities, at $k = 3$ we have a drop of 1 (so this will show up when passing to isotopies), and starting at $k = 4$, we fall into empty transverse intersections even when looking at 1-parameter families of functions.

We consider non-transverse double points on the projection in the context of J_2^1:

$$\Big\{(M_1, N_1, D_1, M_2, N_2, D_2), D_i \in M_{3,2}(\mathbb{R}) \mid M_1 \in W, M_2 \in W, \pi(N_1) = \pi(N_2), \begin{vmatrix} (D_1)_{1,1} & (D_2)_{1,1} \\ (D_1)_{2,1} & (D_2)_{2,1} \end{vmatrix} = 0\Big\}.$$

This is a union of submanifolds of codimensions 5, while the graph is still of dimension 4: transverse intersections are empty.

Now, let us work in J_1^1 and consider the following submanifolds, at which the tangent direction of W is vertical (condition (3) of Definition 3.4):

$$\Big\{(M, N, D), M \in W, N \in \mathbb{R}^3, D \in M_{3,2}(\mathbb{R}) \mid D = \begin{pmatrix} 0 & \cdot \\ 0 & \cdot \\ \cdot & \cdot \end{pmatrix}\Big\}.$$

The two 0's in the entries of the matrix specify that matrix as living in a 4-dimensional subset of the 6-dimensional space $M_{3,2}(\mathbb{R})$. This is thus a union of codimension 3 submanifolds, and the graph is of dimension 2. We again have empty transverse intersection.

We now turn our attention to the framing. Failure of the condition (3.1) can be written as follows:

$$df = \begin{pmatrix} a & \lambda a \\ b & \lambda b \\ c & \lambda c + \mu \end{pmatrix}.$$

One thus considers as singular set

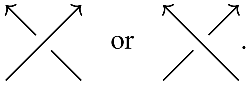
$$\Big\{(M, N, D), M \in W, N \in \mathbb{R}^3, D \in M_{3,2}(\mathbb{R}) \mid D = \begin{pmatrix} a & \lambda a \\ b & \lambda b \\ c & \lambda c+\mu \end{pmatrix}\Big\}.$$

Imposing that M lives on $W \subset \widetilde{W}$ counts for one in the codimension. D has 5 parameters that can be chosen freely for 6 entries, which also contributes by 1 to the codimension. One thus gets a singular set of codimension 2. Since this matches the dimension of the graph, one expects isolated transverse intersections. Notice that these intersections occur on W_1 (forcing them to be on W_0 drops the codimension by 1). Condition (5) also increases the codimension by 1, it holds generically for a web, and one will expect isolated points over time where it does not hold.

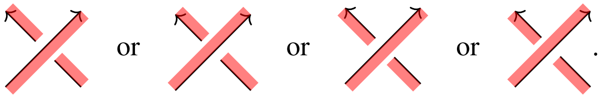
Furthermore, notice that imposing failure of the above condition on top of the double point case also drops the codimension by 1, and one can thus assume that these two situations arise at distinct places.

We will now go back to the above special situations and identify local models for them.

3.1.4. Multiple points. Let us first look at transverse double points in $\pi \circ f$. The situation does not involve trivalent vertices and is therefore part of classical knot theory: the local picture is determined by both derivatives (and transversality implies that they are not colinear) and looks like (the first one is a positive crossing, the second one a negative crossing)



As explained previously, the framing can be assumed to be non-singular under projection at all times, which we can emphasize as follows:



Mirroring left-to-right produces an analogous list of cases for the negative crossing.

3.1.5. Framing. We now turn our attention to the framing issue. Assume that at some point $M \in W_1$

$$(df)_M = \begin{pmatrix} a & \lambda a \\ b & \lambda b \\ c & \lambda c + \mu \end{pmatrix} \quad \text{with } \mu \neq 0.$$

Up to rotation in the x, y plane one can assume that $a > 0$ and $b = 0$, and up to adding to v a multiple of u, one reduces to

$$(df)_M = \begin{pmatrix} a & 0 \\ 0 & 0 \\ c & \mu \end{pmatrix}, \quad \text{where } a > 0, \ \mu \neq 0.$$

If we go to J_2^1 and consider the union of submanifolds corresponding to simultaneously requiring

- the failure of the framing condition (3.1);
- that the $\frac{\partial^2}{\partial u \partial v}$ entry of the matrix corresponding to the second derivative has zero coordinate in the y direction;

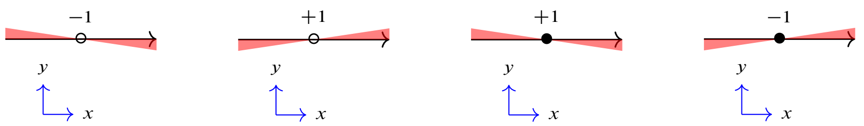
then the codimension increases again by one. This means that we can assume that the entry $\frac{\partial^2 f}{\partial u \partial v}(M) \neq \begin{pmatrix} \cdot \\ 0 \\ \cdot \end{pmatrix}$. Let us write

$$\frac{\partial^2 f}{\partial u \partial v}(M) \neq \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \beta \neq 0.$$

Up to symmetry, one can assume that $\beta > 0$. Then, by writing a Taylor expansion for $\frac{\partial f}{\partial v}$, one has

$$\left(\frac{\partial f}{\partial v}\right)_{(\varepsilon_1,0)} = \begin{pmatrix} \alpha\varepsilon_1 \\ \beta\varepsilon_1 \\ \mu + \gamma\varepsilon_1 \end{pmatrix} + o(\varepsilon_1).$$

This means that just before M on W, the framing was pointing to the right, while just after, it points to the left, passing at M under the strand if $\mu < 0$ and over the strand if $\mu > 0$. We make this into the first and third of the following generators (for more clarity, we indicate the framing). The other two generators correspond to $\beta < 0$.



$\mu < 0,\ \beta < 0$ $\mu < 0,\ \beta > 0$ $\mu > 0,\ \beta < 0$ $\mu > 0,\ \beta > 0$

We emphasize our notation: a $\circ$ sign means that the framing goes under the strand, while a $\bullet$ sign means that it goes over it. The sign indicated next to the framing change indicates whether the framing vector turns positively or negatively (with respect to the right-hand rule) around the tangent vector.

3.2. Isotopies of webs

Let us now consider isotopies of webs by looking at families

$$f : \widetilde{W} \times [0,1] \mapsto \mathbb{R}^3.$$

For $t \in [0,1]$, we denote by f_t the corresponding thickened web embedding.

Definition 3.6. We say that such a family is an isotopy if the following hold:

(1) $\forall t \in [0,1]$, f_t is injective on W;

(2) $\forall t \in [0,1]$, $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ is of rank 2 on W.

Both conditions fail at isolated times for generic families of functions. We thus only consider functions that have no such singular points, and notice that they are stable under small perturbations.

Looking again at the previous analysis, the codimensions remain unchanged but the dimension of the graph increases by one, so we have the following:

- isolated triple points;

- a 1-dimensional set of double points, that generically are transverse and lie on W_1; among which (both situations are mutually exclusive) are:
  - isolated non-transverse double points;
  - isolated double points from $W_0 \times W_1$;
- isolated points where $\frac{\partial f}{\partial u}$ is vertical on W;
- a 1-dimensional set of framing changes controlled by equation (3.1); among those are (situations are mutually exclusive):
  - isolated points where condition (5) does not hold;
  - isolated points on W_0;
  - isolated double points.

We will analyze these situations in the following paragraphs.

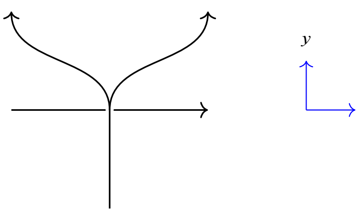
3.2.1. Triple points and non-transverse double points. Triple points may only involve points from W_1 and thus correspond to the third Reidemeister move, as usual, while non-transverse double points also involve only points from W_1 and yield the second Reidemeister move. We will investigate the other situations a little further.

3.2.2. Isolated double points involving a trivalent vertex. Let us start with isolated double points involving $M_1 \in W_1$ and $M_2 \in W_0$. We consider the case where M_2 is a split vertex (the analogous case of a merge vertex is omitted). We can assume that the projections of the derivatives in the u direction make the intersection transverse, so that, up to GL_2 action in the (x, y) plane in the target space, one has

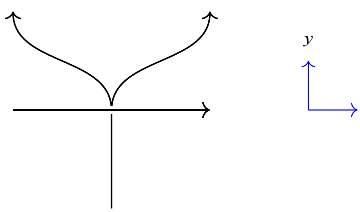
$$\left(\frac{\partial f}{\partial u}\right)_{M_1} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}, \quad a > 0 \quad ,$$

$$\left(\frac{\partial f}{\partial u}\right)_{M_2} = \begin{pmatrix} 0 \\ c \\ d \end{pmatrix}, \quad c \neq 0.$$

We consider the case when $c > 0$. The other one is symmetric. This controls the shape of the intersection as follows:



or



.

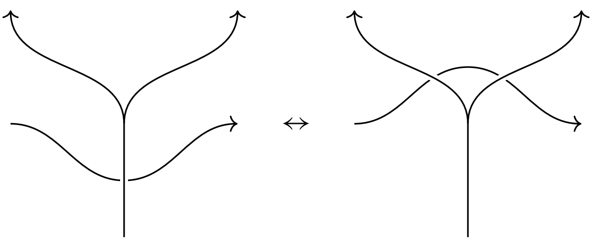
The shape just before or just after in the time direction will be controlled by $\frac{\partial f}{\partial t}$. Again by a codimension argument, we can assume that both vectors have non-zero y coordinates, denoted by η_1 for M_1 and η_2 for M_2. Then

$$f\left(M_1 + \begin{pmatrix} \varepsilon_1 \\ 0 \\ \delta_1 \end{pmatrix}\right) = f(M_1) + \varepsilon_1 \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} + \delta_1 \begin{pmatrix} \cdot \\ \eta_1 \\ \cdot \end{pmatrix}$$

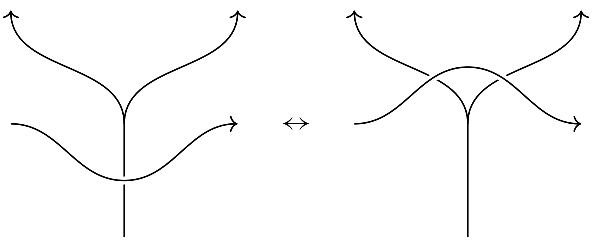
so that one sees a line moving north or south (depending on the sign of η_1) at pace $|\eta_1|$, and

$$f\left(M_2 + \begin{pmatrix} 0 \\ 0 \\ \delta_2 \end{pmatrix}\right) = f(M_2) + \delta_2 \begin{pmatrix} \cdot \\ \eta_2 \\ \cdot \end{pmatrix}$$

so that the trivalent point moves north or south (depending on the sign of η_2) at pace $|\eta_2|$ (the east/west component of the motion is not relevant). Altogether, depending on the sign of $\eta_2 - \eta_1$, one gets a move



or



We emphasize that no framing changes occur during the move.

3.2.3. Isolated points with vertical derivative. Now, we consider the case of an isolated point where the tangent vector is vertical. In the non-framed setting, this yields a Reidemeister 1 move. Here, we will have to be more careful about the framing. We consider

$$(u_0, v_0, t_0) \in W \times \{t_0\} \subset \widetilde{W} \times \{t_0\}$$

so that

$$\left(\frac{\partial f}{\partial u}\right)(u_0, v_0, t_0) = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} \quad (\alpha \neq 0).$$

Again by a codimension argument, one can assume that $\pi(\frac{\partial^2 f}{\partial u^2})$ is non-zero at (u_0, v_0, t_0), and up to rotation of the target space around a vertical axis we can assume that

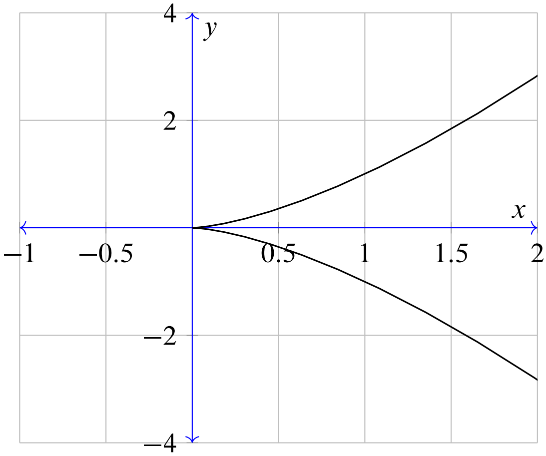
$$\frac{\partial^2 f}{\partial u^2} = \begin{pmatrix} \eta \\ 0 \\ . \end{pmatrix}, \quad \eta > 0.$$

Then, the y-coordinate in $\frac{\partial^3 f}{\partial u^3} = \zeta$ can be assumed to be non-zero, as well as the y-coordinate μ in $\frac{\partial^2 f}{\partial u \partial t}$.

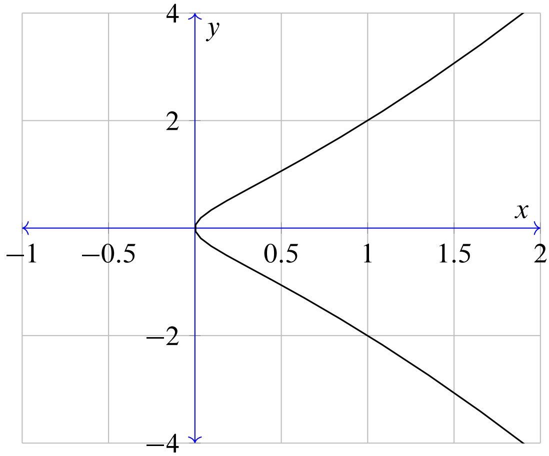
This brings us to the following situation (up to a drift proportional to δ that will not change the shape of the picture):

$$f(u_0 + \varepsilon, v_0, t_0 + \delta) \sim f(u_0, v_0, t_0) + \begin{pmatrix} \varepsilon^2 \eta \\ \varepsilon\delta\mu + \varepsilon^3 \zeta \\ \varepsilon\alpha \end{pmatrix}.$$

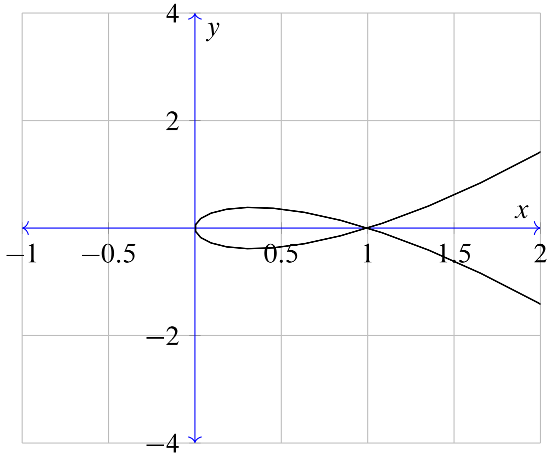
Assume that $\zeta > 0$ (the case of $\zeta < 0$ can be treated similarly). Then, at $\delta = 0$, the right-hand side of the equation above produces a curve with the following shape (we take the projection, and the example drawn below is for $\eta = \zeta = 1$):



Up to reversal of the time direction, we can assume that $\mu > 0$. Then, at positive δ, we have an additional component $\varepsilon\delta\mu$ in the y direction, that pushes up the part of the curve with $\varepsilon > 0$ and down the part of the curve with $\varepsilon < 0$: this has the result of smoothing the curl, as illustrated below ($\mu = 1$, $\delta = 1$).

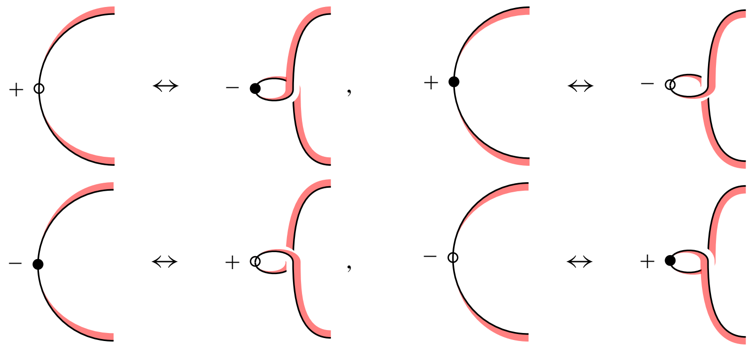


On the other hand, for $\delta < 0$, the upper part of the curve is pushed down and the bottom part is pushed up. For small ε, then $\varepsilon\delta \gg \varepsilon^3$ and the part in ε dominates in y coordinate. At larger ε we go back to the first picture, as ε^3 dominates $\varepsilon\delta$. Below we illustrate the case $\mu = 1$, $\delta = -1$.



Let us now focus on the framing. Recall that at (u_0, v_0, t_0) we have a framing vector that is not vertical. Thus, it projects onto a non-zero vector in the plane, and it determines the general shape of the framing around (u_0, v_0, t_0). In other words, the framing around (u_0, v_0, t_0) is always parallel to the one at (u_0, v_0, t_0) up to small perturbation. Assume that the framing vector has non-zero y-coordinate. Requiring this assumption to fail is a codimension 1 condition, and thus yields a non-generic situation. Then, one easily sees that the condition (3.1) will fail twice, one at $\delta < 0$ and once at $\delta > 0$, once passing over the derivative, once under the derivative (depending on the sign of α).

We get a move



(3.2)

3.2.4. Isolated framing changes with annihilation of second derivative. Let us now consider an isolated point $M \times \{t_0\} \in W_1 \times \{t_0\}$ of coordinates (u_0, v_0, t_0), where

$$(df)_M = \begin{pmatrix} a & \lambda a & \cdot \\ b & \lambda b & \cdot \\ c & \lambda c + \mu & \cdot \end{pmatrix}, \quad \mu \neq 0.$$

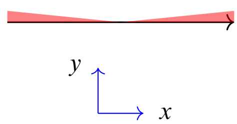
As before, we reduce the situation to

$$(df)_M = \begin{pmatrix} a & 0 & \cdot \\ 0 & 0 & \cdot \\ c & \mu & \cdot \end{pmatrix}, \quad \mu \neq 0.$$

We have furthermore assumed that $\frac{\partial^2 f}{\partial u \partial v}$ has zero y coordinate. We can assume that β', the y-coordinate of $\frac{\partial^3 f}{\partial u^2 \partial v}$ is non-zero, as the codimension again increases when requiring failure of this property. Let us assume, up to symmetry, that $\beta' > 0$. Then, one can write

$$\frac{\partial f_{t_0}}{\partial v}(u_0 + \varepsilon_1, v_0) = f(u_0, v_0, t_0) + \begin{pmatrix} \alpha\varepsilon_1 \\ \beta'\varepsilon_1^2 \\ \mu + \gamma\varepsilon_1 \end{pmatrix} + o(\varepsilon_1).$$

At $t = t_0$, this draws

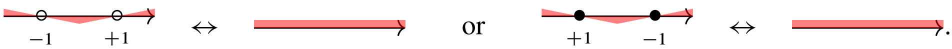


(3.3)

Let us now look at the time direction. We claim that $\nu = \frac{\partial^2 f}{\partial v \partial t}$ can be assumed to be non-zero, again by codimension considerations. Thus, one sees (up to a time drift in the x and z directions we have not included)

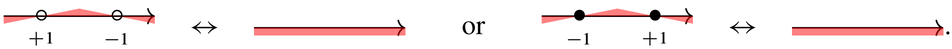
$$\frac{\partial f}{\partial v}(u_0+\varepsilon_1, v_0, t_0+\delta) \sim f(u_0, v_0, t_0) + \begin{pmatrix} \alpha\varepsilon_1 \\ \beta'\varepsilon_1^2 + \nu\delta \\ \mu + \gamma\varepsilon_1 \end{pmatrix}.$$

Depending on the sign of δ, one gets one or the other direction of the following moves (depending on the sign of μ):



(3.4)

With $\beta' < 0$, one gets



(3.5)

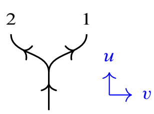
3.2.5. Isolated points on W_0 with framing breach. Assume that $M = (u_0, v_0, t_0) \in W_0 \times \{t_0\}$ with

$$(df)_M = \begin{pmatrix} a & \lambda a & \cdot \\ b & \lambda b & \cdot \\ c & \lambda c + \mu & \cdot \end{pmatrix}, \quad \mu \neq 0.$$

We again reduce to

$$(df)_M = \begin{pmatrix} a & 0 & \cdot \\ 0 & 0 & \cdot \\ c & \mu & \cdot \end{pmatrix}, \quad \mu \neq 0,\ a > 0.$$

We consider the case of split vertex (a merge vertex would be treated similarly), and for convenience we number the two outgoing strands 1 and 2 so that the v-coordinate on strand 1 is positive, while the v-coordinate on strand 2 is negative.



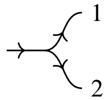
Denote by

$$\frac{\partial^2 f}{\partial u \partial v} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial t \partial v} = \begin{pmatrix} d_4 \\ d_5 \\ d_6 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial v^2} = \begin{pmatrix} d_7 \\ d_8 \\ d_9 \end{pmatrix}.$$

One can assume that d_2 and d_5 are non-zero. Let us first look at the shape of the web through time. Since at first order $\frac{\partial f}{\partial t}$ only produces a drift of the web through time, we can ignore its contribution. Below it is treated as zero, as well as all purely t higher derivatives. Similarly, the mixed derivative in $\partial u \partial t$ will not contribute to the change in shape of the web

$$\begin{aligned} f(u_0+\varepsilon_1, v_0+\varepsilon_2, t_0+\delta) &\sim f(u_0, v_0, t_0) \\ &+ \begin{pmatrix} a\varepsilon_1 + d_1\varepsilon_1\varepsilon_2 + d_4\varepsilon_2\delta + d_7\varepsilon_2^2 \\ d_2\varepsilon_1\varepsilon_2 + d_5\varepsilon_2\delta + d_8\varepsilon_2^2 \\ c\varepsilon_1 + \mu\varepsilon_2 + d_3\varepsilon_1\varepsilon_2 + d_6\varepsilon_2\delta + d_0\varepsilon_2^2 \end{pmatrix}. \end{aligned}$$

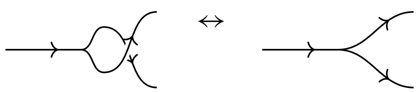
Let us analyze the expression in the y coordinate, and first consider $\delta = 0$. The term $d_8\varepsilon_2^2$ will not change the general shape of the image surface. The term $d_2\varepsilon_1\varepsilon_2$ determines whether the leg number 1 is sent to positive or negative y coordinates (resp., leg numbered 2 being sent to negative or positive coordinates). Assume $d_2 > 0$. Thus, at $\delta = 0$, we simply observe



.

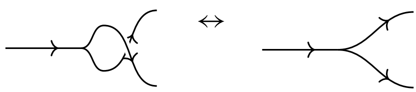
In the time direction we add a contribution of $d_5\varepsilon_2\delta$. Assume $d_5 > 0$ (up to time reversion one can reduce to this case). Then, the strand 1 is pushed higher in the y direction and the strand 2 is pushed lower in the y direction if $\delta > 0$, while for $\delta < 0$ the strand 1 is pushed lower in the y direction and the strand numbered 2 higher in the y direction. This phenomenon is predominant while $\varepsilon_1 \ll \delta$, but as ε_1 grows, $d_2\varepsilon_1\varepsilon_2$ controls the shape of the web. One thus reads (the first picture corresponds to $\mu > 0$, the second one to $\mu < 0$)

$\delta < 0$ $\qquad\qquad$ $\delta > 0$



or (3.6)

$\delta < 0$ $\qquad\qquad$ $\delta > 0$

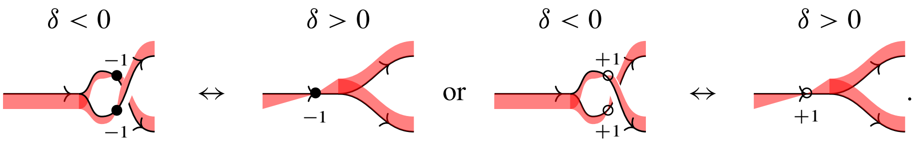


Indeed, the crossing sign is determined by $\mu\varepsilon_2$: the strand labeled 2 corresponds to negative values of ε_2, and thus the sign of $\mu < 0$ determines which of the strands is pushed up in the z direction.

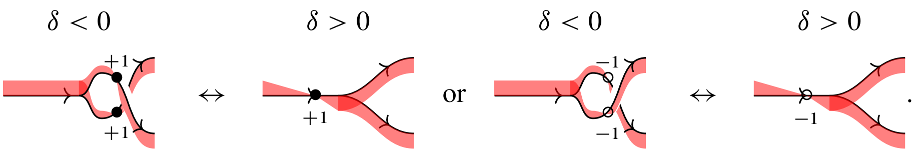
We now look at the framing vector and read

$$\left(\frac{\partial f}{\partial v}\right)(u_0+\varepsilon_2, v_0+\varepsilon_2, t_0+\delta) \sim \begin{pmatrix} d_1\varepsilon_1 + d_4\delta + d_7\varepsilon_2 \\ d_2\varepsilon_1 + d_5\delta + d_8\varepsilon_2 \\ \mu + d_3\varepsilon_1 + d_6\delta + d_9\varepsilon_2 \end{pmatrix}.$$

Recall that we assumed $d_5 > 0$, so along time the framing travels positively in the y direction. Furthermore, at $\delta = 0$, it is controlled by $d_2\varepsilon_1$, and we have already assumed that $d_2 > 0$. One thus has

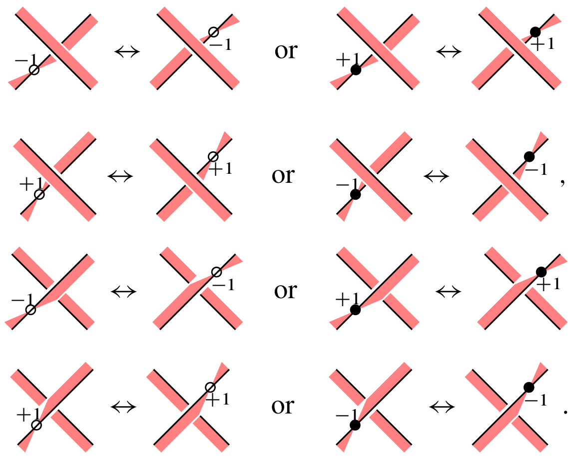


Similarly, one gets for $d_2 < 0$ the following moves:



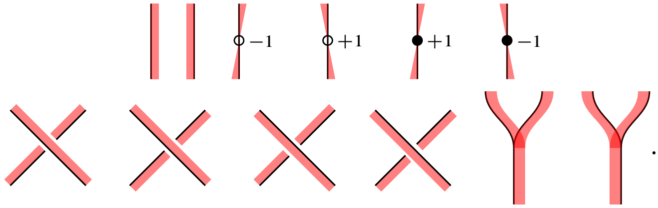
Note that the orientation of the strands plays no role, thus one gets the moves involving a merge vertex by simply reversing the orientations.

3.2.6. Isolated double points with framing breach. The analysis is similar to the previous one, but simpler. One gets (we have not shown orientation)

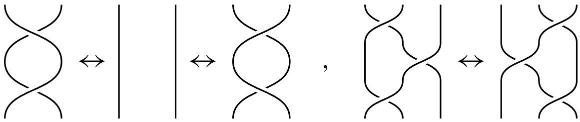


3.2.7. Statements. Wrapping up the previous analysis, one gets the following.

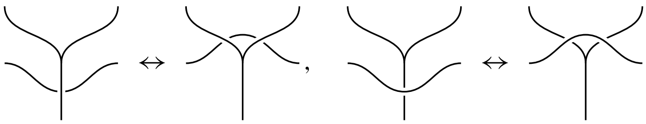
Theorem 3.7. *Oriented, framed tangled webs admit diagrams locally generated by the following pieces (orientations are to be added in any compatible way):*

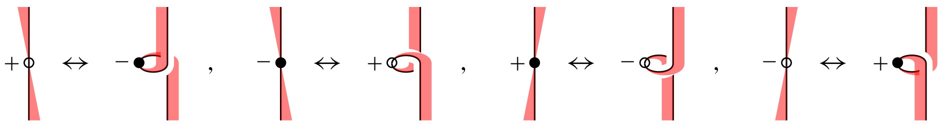


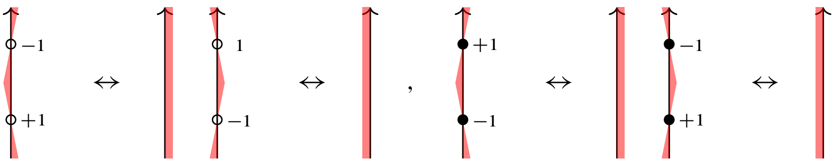
Isotopies of oriented, framed tangled webs induce planar isotopies together a finite number of moves from the following list (in pictures not containing framing information, this can be freely chosen without framing changes):

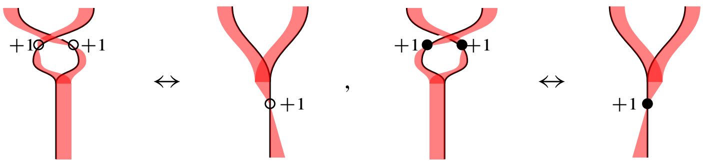


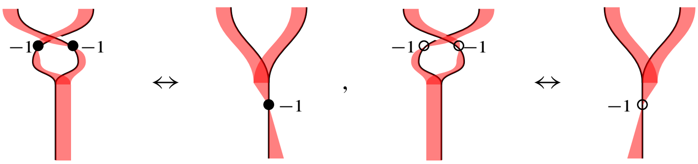
+ *other versions of the* R_3 *move;*









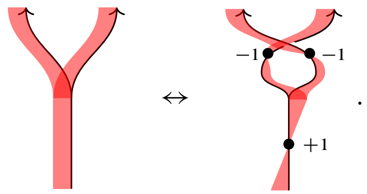


Proof. Going from the multi-jet bundle stratification to the statement of the theorem is a standard argument in geometric topology, but we will review the high-level structure of that argument here.

Start with an arbitrary framed tangled web. After a small isotopy, we may assume that the multi-jet map associated to the tangled web is transverse to the stratification (collection of submanifolds) described above (because the set of transverse maps is dense). As explained earlier in this section, near each transverse intersection point the corresponding tangled web projection looks like one of the generators listed in the first part of the theorem (strand, half twist, crossing, and trivalent vertex). This proves the first part of the theorem.

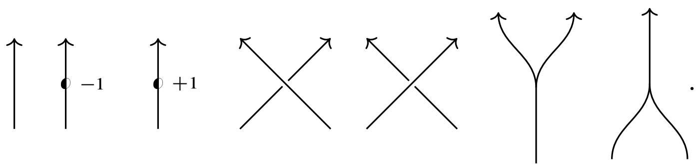
Now, consider an arbitrary isotopy of tangled webs, with the beginning and end of the isotopy already transverse (i.e., the multijet maps associated to the beginning and end of the isotopy are transverse to the jet-bundle stratification described above). Because the set of transverse maps is dense, we can make a small perturbation (second-order isotopy) to the isotopy such that the new isotopy is transverse. We now examine the possible transverse intersections. Each such intersection yields one of the framed Reidemeister moves listed in the second part of the theorem. ∎

Recall that away from the points where the framing points downward, we have a preferred isotopy that makes the framing flat and rightward-pointing with respect to the orientation of the web. Notice that in the case of a trivalent, this changes the shape of the web:

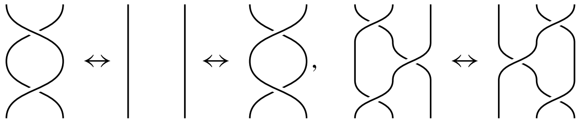


One can thus translate the previous theorem into the following one (where we do not show the framing anymore, as it is on the right-hand side of the strand at all times, except for the twists). For full twists, we use the symbol ◐.

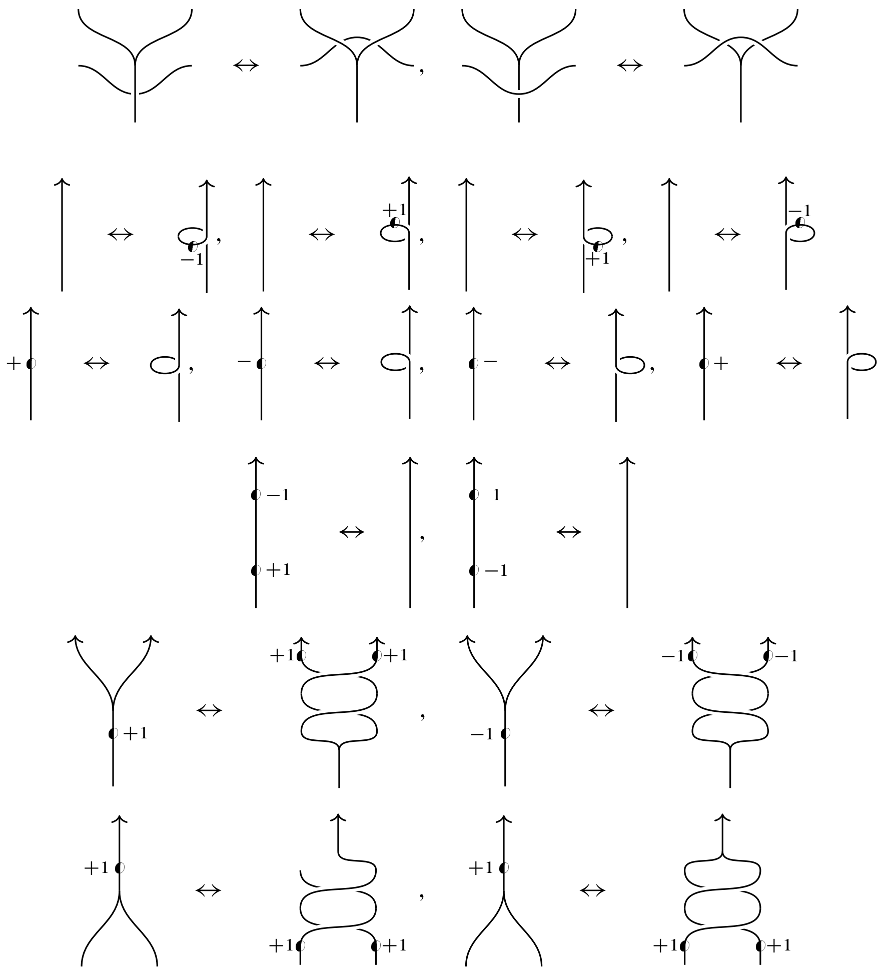
Theorem 3.8. *Oriented, framed tangled webs admit preferred diagrams locally generated by the following pieces:*



Isotopies of oriented, framed tangled webs induce planar isotopies together a finite number of moves from the following list (in the pictures that not containing framing *information, this can be freely chosen without framing changes):*



+ *other versions of the* R_3 *move;*

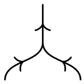
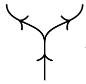


4. Movie moves for framed foams

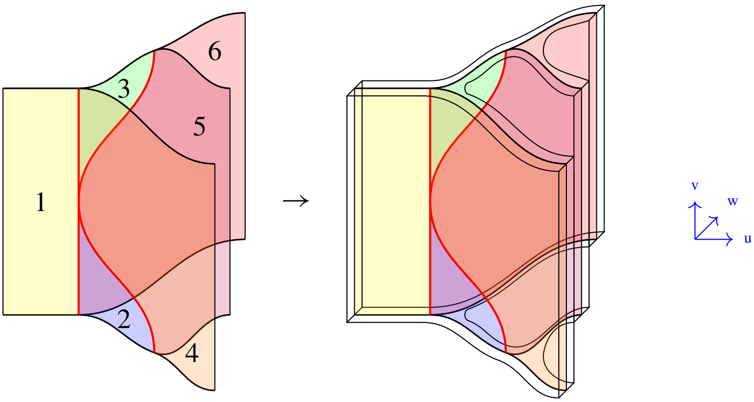
In this section, we will work on upgrading the Reidemeister theorem for framed, tangled webs one dimension higher. Natural cobordisms between webs are foams, and we will be looking for a presentation of the category of framed foams between tangled webs. The resulting theorem are Theorem 4.1 and Theorem 4.2. A complete list of moves (in the half-twist case) appears in Section 4.9. Throughout this section, we refer to moves in this list for illustration.

A foam F is a special kind of 2-dimensional CW complex. We denote by F^i the set of i-dimensional cells. $F^0 \cup F^1$ forms what we refer to as *seams*: for example, they are depicted in red in the left-hand side of equation 4.1 below. Just as in the previous section, we will encode the framing by considering a thickened version $N(F)$ containing a preferred copy $F \subset N(F)$. There should be an atlas making it locally smoothly diffeomorphic to one of the following elementary pieces.

- For $x \in F^2$, a local model is given by $\mathbb{D}^2 \times [0,1]$, with preferred copy $\mathbb{D}^2 \times \{\frac{1}{2}\}$.
- For $x \in F^1$, a local model for F is given by $Y \times [0,1]$, with Y a trivalent web:

$Y =$  or $Y =$  .

  Recalling from the previous section that Y can be turned into a 2-manifold with corners $\widetilde{Y}$, we define $N(F) = \widetilde{Y} \times [0,1]$ with preferred copy $Y \times [0,1]$.
- For $x \in F^0$, the local model is illustrated as follows:



(4.1)

  Above, we have indicated our preferred system of local coordinates (u, v, w) for the source foam. In the target space, as stated later, we will use letters x, y, z and s. More precisely, we want to have a singular surface that is combinatorially as shown above, and so that the following pieces assemble into a smooth rectangle:
  - facets 1, 2, and 4;
  - facets 1, 2, 3, and 5;
  - facets 1, 3, and 6.

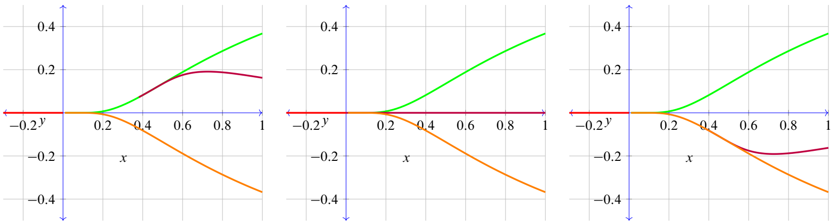
An explicit smooth realization of such a local model for F at a 6-valent point can be built as follows (but other, non-diffeomorphic ones can exist):

- $\{x \leq 0, y = 0, z\}$;

- $\{x > 0, y = e^{-\frac{1}{x}}, z\}$;
- $\{x > 0, y = -e^{-\frac{1}{x}}, z\}$;
-

$$x > e^{-\frac{1}{|z|}}, \quad y = \begin{cases} \exp\left(-\frac{1}{x}\right) - \exp\left(-\frac{1}{x - e^{-\frac{1}{z}}}\right) & \text{if } z > 0, \\ y = 0 & \text{if } z = 0, \\ -\exp\left(-\frac{1}{x}\right) + \exp\left(-\frac{1}{x - e^{\frac{1}{z}}}\right) & \text{if } z < 0. \end{cases}$$

Below we provide pictures of the slices at $z = 1$, $z = 0$ and $z = -1$.



Then, $N(F)$ is obtained by taking a neighborhood in the 3-ball.

We then consider smooth maps $f : N(F) \to \mathbb{R}^4$, and denoting by x, y, z, s the coordinates in $\mathbb{R}^4$, we consider the projection maps $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ that forgets the coordinate z. A framed foam in $\mathbb{R}^4$ is the image of F under such a map f that is injective on F, together with a non-tangent non-vanishing vector field, obtained by looking at the first derivative in the w direction (see the coordinate frames in 4.1).

We will also look at isotopies of foams, described by families f_t of such maps, $t \in [0, 1]$.

We require the map f (and all maps f_t) to respect the following conditions:

(1) f is injective when restricted to F;

(2) $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}\right)$ is full-rank on F.

Consider the first condition. The union of submanifolds where this is not met is

$$\left\{(M_1, N_1, M_2, N_1) \mid M_1, M_2 \in F, N_1 = N_2 \in \mathbb{R}^4\right\}.$$

Constraining M_1 and M_2 to live on $F \subset N(F)$ counts for 2 in the codimension. Then, since N_1 and N_2 live in a 4-dimensional space, their equality is a codimension 4 condition. Altogether, the singular set is a union of codimension 6 submanifolds in the 2-fold jet bundle. On the other hand, the graph formed by $\{M_1, f(M_1), M_2, f(M_2)\}$ is determined by M_1 and M_2 which are in the 3-dimensional space $N(F)$: the graph is 6-dimensional. One thus expects generically isolated intersections. We restrict our attention to maps that have no such intersections.

For the second condition, we have a set

$$\left\{(M, N, D) \mid M \in F, N \in \mathbb{R}^4, D \text{ is not full rank}\right\}.$$

The codimension is 1 for the restriction to F, and since D is a 4 by 3 matrix, requiring it not to be full rank amounts to freely choosing two columns, and asking the last one to be a linear combination of the first two ones. Since there are two parameters to choose for 4 entries, this is a codimension 2 condition. This all adds up to codimension 3, while the graph is of dimension 3 (4 if one adds the time parameter). As before, one expects generically isolated points where the condition is not fulfilled. We restrict out attention to maps that do not have such points.

Now, we will be looking for local models for generators of foams and foam isotopies, and relations between them.

As we go into this analysis, we will be using the following notations for derivatives:

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad \frac{\partial f}{\partial w} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Similarly, for second derivatives, we will use the following notations:

$$\frac{\partial^2 f}{\partial u^2} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial u \partial v} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

$$\frac{\partial^2 f}{\partial u \partial w} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial v^2} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix},$$

$$\frac{\partial^2 f}{\partial v \partial w} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial w^2} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}.$$

Higher derivatives will occasionally appear. We do not define global notations for them, and will choose local ones when required.

We will first list conditions that make the neighborhood of a given point trivial (meaning that it corresponds to an identity movie). Then, we will investigate the failure of each of these conditions. We consider successively points on F^2—the open 2-dimensional sheets of F, F^1—the open edges (seams) of F, or F^0—the vertices of F.

4.1. Conditions on the 2-dimensional locus

Given a point M on F^2, we can find a trivial neighborhood for the image $f(M)$ if

- $f(M)$ is not a multiple point under π;
- $(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ is not contained in the $s = 0$ sub-space of $\mathbb{R}^4$;
- $(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ does not contain the direction $\vec{z}$;
- $\frac{\partial f}{\partial w}$ is not contained in the space spanned by $\vec{z}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$.

  Indeed, consider $M \in F^2$ that has the previous properties. Then

$$(df)_M = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \star & \star & \cdot \end{pmatrix},$$

and we can assume that at least one of the entries marked with a star is non-zero (since the first two vectors are not contained in the $s = 0$ subspace). Up to reparametrization in u, v, we can reduce to

$$(df)_M = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}.$$

Now, since $(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ does not contain the direction $\vec{z}$, we can assume that

$$(df)_M \neq \begin{pmatrix} 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}.$$

Up to rotation in the x, y plane in the target space, we can assume that the differential is

$$(df)_M = \begin{pmatrix} > 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix},$$

and up to adding to the v coordinate a multiple of u, and to the w coordinates multiples of u and v, we reduce to

$$(df)_M = \begin{pmatrix} > 0 & 0 & 0 \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ 0 & b_4 & 0 \end{pmatrix}, \quad a_1 > 0,\ b_4 \neq 0.$$

The change of coordinate in w modifies the framing by a multiple of a vector that belongs to the tangent plane to F. This is of no effect from our perspective.

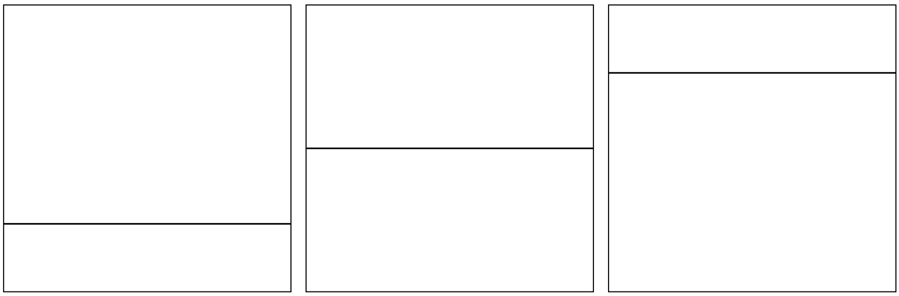
Furthermore, c_2 can be assumed to be non-zero because of the condition on $\frac{\partial f}{\partial w}$.

Then, one can run a Taylor expansion

$$f(m_1+\varepsilon_1, m_2+\varepsilon_2, m_3) = \begin{pmatrix} \varepsilon_1 a_1 \\ \varepsilon_2 b_2 \\ \varepsilon_1 a_3 + \varepsilon_2 b_3 \\ b_4 \varepsilon_2 \end{pmatrix},$$

$$\left(\frac{\partial f}{\partial w}\right)_{(m_1+\varepsilon_1, m_2+\varepsilon_2, m_3)} = \left(\frac{\partial f}{\partial w}\right)_{(m_1, m_2, m_3)} + o(\varepsilon_1, \varepsilon_2).$$

In terms of movies, one reads the following trivial movie.



We have not indicated the framing, which stays on the same side at all time and never vanishes.

4.2. Conditions on the 1-dimensional locus

Let us now look for a local model on F^1 in the easiest case. We will use the following conditions, for $M \in F^1$, with the local parametrization in the source space so that the direction v agrees with the direction of the seam:

- $f(M)$ is not a multiple point under π;
- $\frac{\partial f}{\partial v} \neq \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$;
- $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ does not contain the direction $\vec{z}$;
- $\frac{\partial f}{\partial v}$ is not contained in the space spanned by $\vec{z}$, $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$.

We use the condition on $\frac{\partial f}{\partial v}$ to add a multiple of v to u so that

$$(df)_M = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}.$$

This has the effect to not leave the seam strictly vertical, but makes it drift in the u direction.

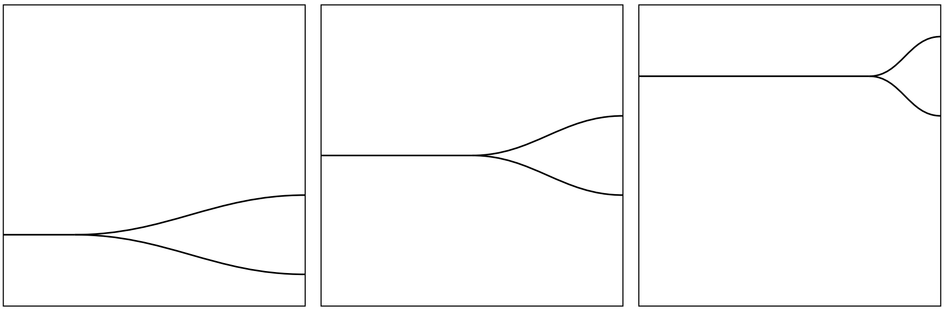
Then, the condition on the $\vec{z}$ direction implies that up to rotation in the x, y plane, we can restrict to

$$(df)_M = \begin{pmatrix} >0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}.$$

Then, we can use the condition on the framing and add to w multiples of u and v so that

$$(df)_M = \begin{pmatrix} >0 & \cdot & 0 \\ 0 & \cdot & \neq 0 \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & 0 \end{pmatrix}.$$

As before, we read from a Taylor expansion a trivial movie with non-vanishing framing in the projection.

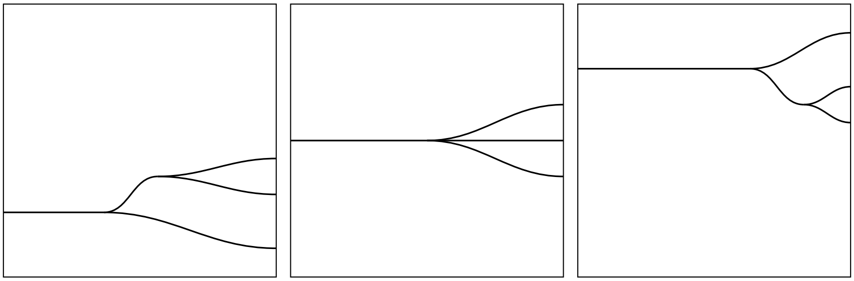


4.3. Conditions on the 0-dimensional locus

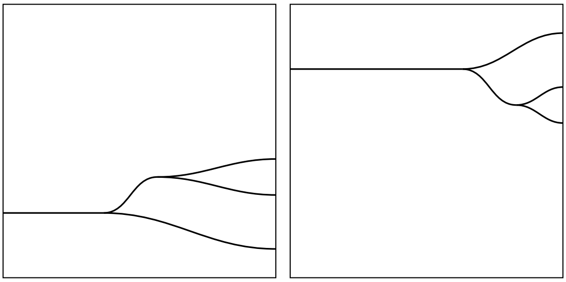
Finally, we focus on $M \in F^0$, with choice of coordinates as in equation (4.1). We assume the following:

- $f(M)$ is not a multiple point under π;
- $\frac{\partial f}{\partial v} \neq \begin{pmatrix} \cdot \\ \cdot \\ 0 \end{pmatrix}$;
- $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ does not contain the direction $\vec{z}$;
- $\frac{\partial f}{\partial w}$ is not contained in the space spanned by $\vec{z}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$.

We run the very same analysis as in the F^1 case, and obtain the following generator for a foam.



In general, the middle picture is omitted and we represent this generator as



Again, nothing interesting happens to the framing.

In the above, Sections 4.1, 4.2, and 4.3, we have identified the simplest local models for neighborhoods of points in an embedded foam: these are the identity cobordism over a segment, or over a trivalent vertex, or, in the last case, the associativity move depicted above. By doing so, we have listed sufficient conditions that ensure that we fall in such easy situations. We will now analyze all situations we have excluded in the above discussion, and look for other foam generators (movies) as well as generators for isotopies (movie moves).

4.4. Multiple points in projection

In J_k^0, we consider

$$\big\{(M_1, N_1, \ldots, M_k, N_k) \mid M_1 \in F, \ldots, M_k \in F, \pi(N_1) = \pi(N_2) = \cdots = \pi(N_k)\big\}.$$

The codimension is k times 1 for the restriction to F, and $(k-1)$ times 3 for the identification of three coordinates in N_i, $i > 1$, with those of N_1. In total we read $4k-3$. The dimension of the graph on the other hand is $3k$, or $3k+1$ if one considers a 1-parameter family of functions.

If $k > 3$ then $3k < 4k-3$, and if $k > 4$, $3k+1 < 4k-3$, so one only has to consider double or triple points for foams, and quadruple points for isotopies.

Double points in projection

We have a codimension 5 condition for a 6-dimensional graph, so we expect 1-dimensional generic intersections (2-dimensional through time). If instead of looking at multiple points from $F \times F$ one restricts to $F^1 \times F$ or $F \times F^1$, then one has isolated points (1-dimensional sets through time). For $F^1 \times F^1$, $F^2 \times F^0$ or $F^0 \times F^2$, the intersection is generically empty for a foam, and isolated for a time family. Finally, there are no intersections of the kind $F^0 \times F^0$, even through time.

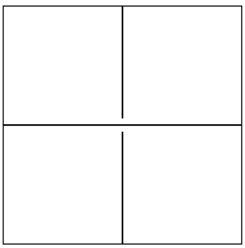
Let us start with double points $\pi(f(M_1)) = \pi(f(M_2))$ with M_1 and $M_2 \in F^2$. Asking the intersection to be transverse is equivalent to requiring that the following vectors generate a 3-dimensional space under π:

$$\left(\frac{\partial f}{\partial u}\right)_{M_1}, \quad \left(\frac{\partial f}{\partial v}\right)_{M_1}, \quad \left(\frac{\partial f}{\partial u}\right)_{M_2}, \quad \left(\frac{\partial f}{\partial u}\right)_{M_2}.$$

Failure of this requirement is a codimension 2 condition (up to permutation, one fixes the first two vectors, then the third and fourth ones are linear combinations of the first two ones: this takes two parameters for each of them, out of three coordinates). Since we already have restricted our attention to a 1-dimensional situation, this can be assumed for free for a foam. For a 1-parameter family there will be isolated points where this condition is not met.

Asking both $(\frac{\partial f}{\partial u})_{M_1}$ and $(\frac{\partial f}{\partial v})_{M_1}$ to have zero s-coordinate is a codimension 2 condition, so for a foam this can be assumed not to happen at a double point (there will be isolated points through time, yielding MM_{14} and MM_{15} as we shall see later). So, we can reparametrize at M_1 and M_2 so that $\frac{\partial f}{\partial u}$ has zero s-coordinate. It might happen (this is a codimension 1 condition) that $\pi((\frac{\partial f}{\partial u})_{M_1})$ and $\pi((\frac{\partial f}{\partial u})_{M_2})$ are colinear. We will analyze this situation later (this will give MM_3, MM_4 and MM_9) and assume for now that this is not the case. We will also assume that both u derivatives are not purely vertical. As this is a codimension 2 condition, this could happen for only one of the two vectors at isolated points through time. This will yield a version of the classical movie move MM_8, as we will see later. Up to reparametrization in the x, y variables, we can assume that $\pi((\frac{\partial f}{\partial u})_{M_1})$ is parallel to $\vec{x}$ and $\pi((\frac{\partial f}{\partial u})_{M_2})$ is parallel to $\vec{y}$.

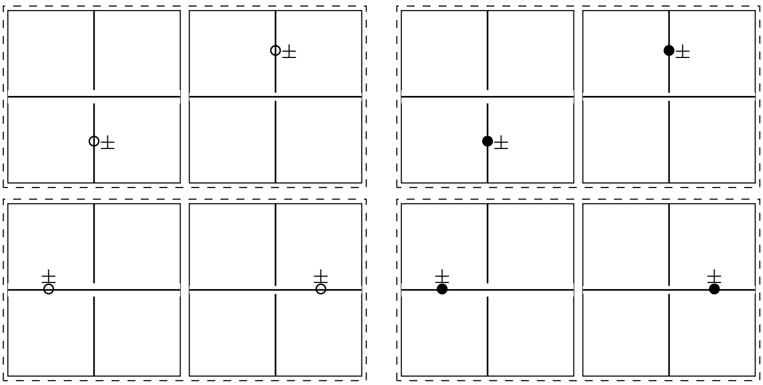
Then, one reads an identity movie over a crossing



.

Let us now look at the framing. If $\pi((\frac{\partial f}{\partial w})_{M_1})$ and $\pi((\frac{\partial f}{\partial w})_{M_2})$ lie outside of the space spanned by the other two tangent vectors and the vertical direction, then nothing happens. Requiring that the framing is vertical is a codimension 1 condition, so there are isolated points where this happens on one of the two strands. Through time, there also are isolated points where this happens on both strands at the same time.

Consider first the foam case, and assume that the framing vector projects onto the tangent vector to the strand parallel to the x direction (corresponding to M_1). Then, the shape of the framing on this strand will be determined by $(\frac{\partial^2 f}{\partial u \partial w})$ that can be assumed to have a non-zero y component (through time, this could be false). In the s direction, the framing is simply transported parallel if $(\frac{\partial^2 f}{\partial u \partial w})_y \neq 0$, while in the other case the control is ensured by $(\frac{\partial^2 f}{\partial v \partial w})$, assumed to have a non-zero y entry for codimension reason. Depending on signs, we are in one of the following situations (each generator can be read from left to right or right to left):



Now, assume that $(\frac{\partial f}{\partial u})_{M_1}$ and $(\frac{\partial f}{\partial u})_{M_2}$ are colinear. Again, this only happens at isolated points. Up to reparametrization, we assume that both are supported in the x direction. We have

$$\left(\frac{\partial f}{\partial u}\right)_{M_1} = \begin{pmatrix} a_1 \\ 0 \\ a_3 \\ 0 \end{pmatrix}, \quad \left(\frac{\partial f}{\partial u}\right)_{M_2} = \begin{pmatrix} a'_1 \\ 0 \\ a'_3 \\ 0 \end{pmatrix}.$$

We have

$$f(M_1 + (\varepsilon_1, \varepsilon_2)) \sim \begin{pmatrix} a_1\varepsilon_1 + b_1\varepsilon_2 \\ b_2\varepsilon_2 + d_2\varepsilon_1^2 + e_2\varepsilon_1\varepsilon_2 \\ * \\ b_4\varepsilon_2 + d_4\varepsilon_1^2 + e_4\varepsilon_1\varepsilon_2 \end{pmatrix}.$$

There is a similar expansion at M_2.

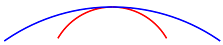
Focusing first on the $s = 0$ case, since $b_4 \neq 0$ one reads

$$\varepsilon_2 = -\frac{d_4}{b_4 + e_4\varepsilon_1} \sim -\frac{d_4}{b_4}\varepsilon_1^2.$$

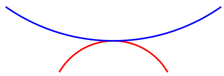
This justifies that the $s = 0$ slice is described by

$$\begin{pmatrix} a_1\varepsilon_1 \\ (d_2 - \frac{d_4}{b_4})\varepsilon_1^2 \\ * \\ 0 \end{pmatrix}.$$

Then, for codimension reason, we can assume that $d_2 - \frac{d_4}{b_4}$ and its M_2 analog are different (through time this will happen at isolated points, yielding one of the relations, the so-called movie move MM9, as we will see later). This determines a local model of the following kind (depending on the relative sign, and where we have colored the strands for better visibility):

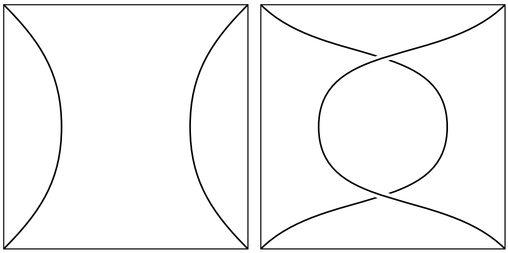


or



.

The behavior in the s direction is controlled by the y-coordinates b_2 of $\frac{\partial f}{\partial v}$ at M_1 and M_2. One can assume that the difference of the two entries is non-zero (but this will be zero at isolated points through time, yielding MM_3 and MM_4). In both cases, we arrive at the Reidemeister II generator (to be read in one or the other direction):



.

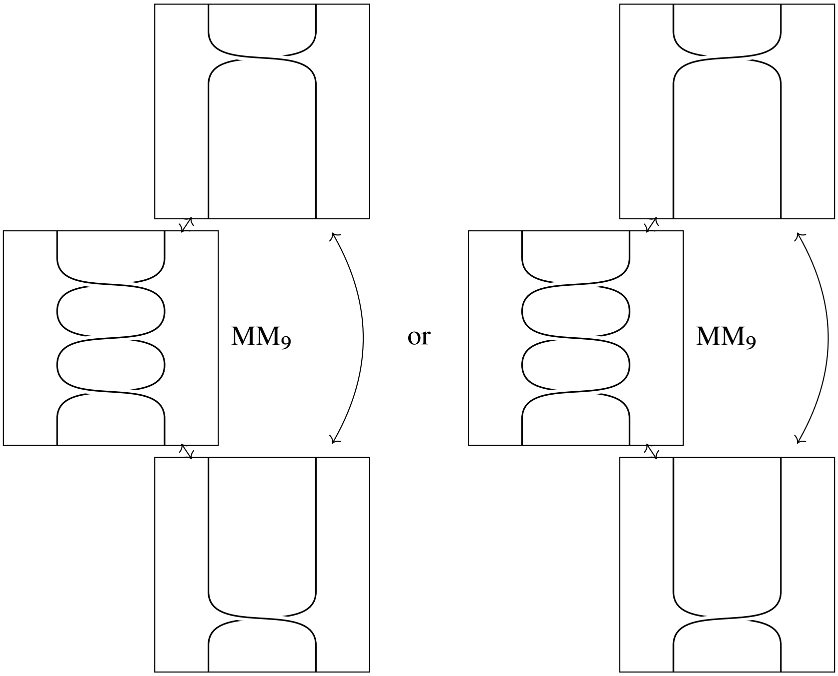
Let us now consider the case where the entries b_2 at M_1 and M_2 balance. Then (the prime stands for the entry corresponding to M_2)

$$(f(M_1 + \varepsilon_2) - f(M_2 + \varepsilon_2))_y \sim (g_2 - g_2')\varepsilon_2^2.$$

This yields the previously known movie moves (MM_3) and (MM_4) (for all displays of movie moves, see Section 4.9).

Now, if $d_2 - \frac{d_4}{b_4}$ and $d_2' - \frac{d_4'}{b_4'}$ are equal, then the difference in the y-coordinate is controlled by a term proportional to ε_1^3. This yields the classical movie move (MM_9).

Below we give its two versions, and later in the text when referring to movie moves from Section 4.9, we will assume that they come with all their symmetries.



By codimension count, we see that the framing can be assumed to be non-vertical at all time.

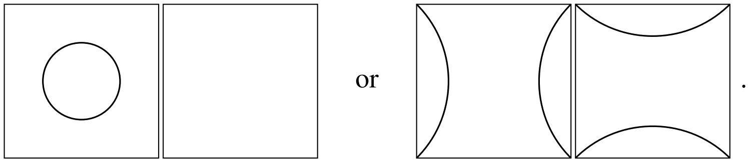
Going back to the Reidemeister II foam generator, one could see a framing change on one of the two strands. This yields the movie move (PF_1) and its analogs with other crossing and framing choices.

Going back to the case of a framing change going through a crossing, three things can happen through time: either only one change goes through but the vector $\frac{\partial^2 f}{\partial u \partial w}$ has zero y coordinate or the vector $\frac{\partial^2 f}{\partial v \partial w}$ has zero y coordinate, or a crossing change happens on each strand at the same time. In the first case we have the movie move (PF_6) (together with analogous versions with all crossings flipped).

In the second case, we get (PF_5) (again with its analogs with crossings or framing flipped).

In the case where both strands carry a framing change, then one obtains the movie move (PF_7) (and analogous ones including those where each strand carries a different kind of framing change).

Now, assuming that both $\left((\frac{\partial f}{\partial u})_{M_1}\right)$ and $\left((\frac{\partial f}{\partial v})_{M_1}\right)$ have zero s-coordinate amounts to saying that the neighborhood of M_1 is sent to (to be read in one or in the other direction) (see also the beginning of Section 4.5 for more details):



Superposed with an extra strand, one gets the classical movie move (MM_{14}), with the strand passing over or under the circle. Because of the codimension, one can assume that nothing occurs to the framing during this move. In the saddle case one gets the familiar movie move (MM_{15}), with the middle strand passing either over or under the saddle (again the framing remains constant).

Let us now argue about the case where the intersection is not transverse. Since $(\frac{\partial f}{\partial u})_{M_1}$ and $(\frac{\partial f}{\partial v})_{M_1}$ always form a 2-dimensional vector space (and the same holds for M_2), two things can happen for the intersection not to be transverse:

- both projections remain of dimension 2, and become equal. This is a codimension 2 condition, and it will recover the movie moves MM_3 and MM_4 already considered;
- one of the two projections is only of dimension 1, and included in the projection of the other tangent space. But this is a codimension 3 condition and thus cannot happen generically.

Finally, we consider the case where one of the u derivatives is purely vertical, say at M_1. One sees a generator from (3.2) happening on one of the two strands, superposed with an extra strand that intersects transversely (for codimension reasons). One then obtains the movie move (MM_8) (and its analogs with other choices for the crossings and twist sign).

Double points from $F^1 \times F^2$

Now, we consider double points from $F^1 \times F^2$. We have isolated points in a foam, and 1-dimensional sets in isotopies. Let us first establish a local model. We consider $M_1 \in F^1$ and $M_2 \in F^2$, and we suppose that the parametrization has been chosen so that the seam is supported in the v direction. By a codimension argument, one may assume that $(\frac{\partial f}{\partial v})_{M_1}$ has non-zero s coordinate (this will happen at isolated points through time). Then, upon reparametrizing in the u direction, one can assume that $(\frac{\partial f}{\partial u})_{M_1}$ has no s coordinate. Again because of the codimension, we can assume that $\pi((\frac{\partial f}{\partial u})_{M_1}) \neq 0$ (this is a codimension 2 condition, so this can be safely assumed even for isotopies), and up to rotation in the (x, y) plane that it is parallel to the x direction with positive coordinate.

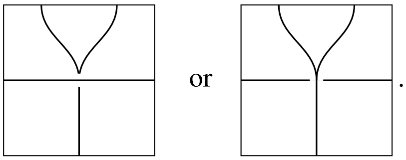
At this point, we have

$$(df)_{M_1} = \begin{pmatrix} >0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}.$$

Let us now look at M_2. As in the $F^2 \times F^2$ case, we reparametrize at M_2 so that $(\frac{\partial f}{\partial u})_{M_2}$ has zero s-coordinate and $(\frac{\partial f}{\partial v})_{M_2}$ has non-zero s coordinate. Then, we can assume (and this will fail through time at isolated points, see the move (OM_3)) that $\pi((\frac{\partial f}{\partial u})_{M_1})$ and $\pi((\frac{\partial f}{\partial u})_{M_2})$ are not colinear. We thus have

$$(df)_{M_1} = \begin{pmatrix} >0 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}, \quad (df)_{M_2} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \neq 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}.$$

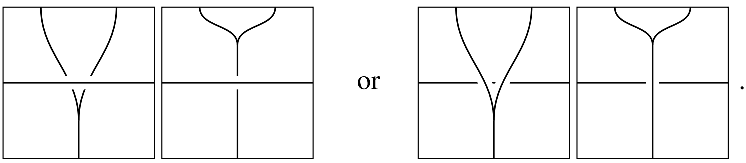
Since $(\frac{\partial f}{\partial v})_{M_2}$ has non-zero s coordinate, at $v = 0$ we read



The behavior as v varies is controlled by the difference of the y-coordinates in $(\frac{\partial f}{\partial v})_{M_1}$ and $(\frac{\partial f}{\partial v})_{M_2}$. Let us denote them as follows:

$$(df)_{M_1} = \begin{pmatrix} >0 & \cdot & \cdot \\ 0 & b_2 & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \neq 0 & \cdot \end{pmatrix}, \quad (df)_{M_2} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \neq 0 & b'_2 & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}.$$

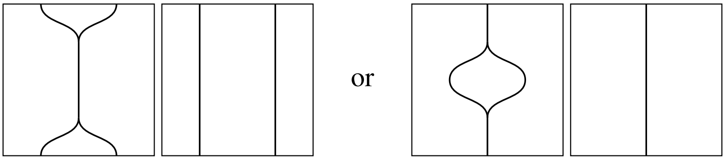
For a foam, one may assume that $b_2 - b'_2 \neq 0$ (but for 1-parameter families this will not hold at isolated points, see moves (I_1) and (I_2)). Then, one reads the following movie generator (reading from left to right corresponds to the case where $b'_2 > b_2$, the other case is obtained by reading in the reverse direction):



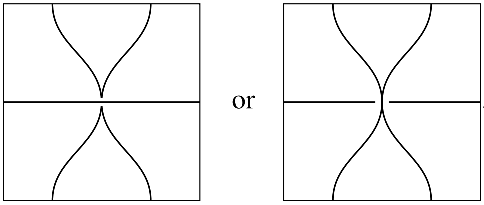
Again, the framing can be assumed not to be subject to any change (through time, this will yield moves from equations (PF_8) and (OM_4)).

Now, we add the time parameter and consider the failure of the conditions considered in the previous paragraph.

If $(\frac{\partial f}{\partial v})_{M_1}$ has zero s-coordinate, since this is a codimension 1 condition we are down to isolated points, and we can assume that the second derivative has non-zero s-coordinate. This implies that around M_1 we have



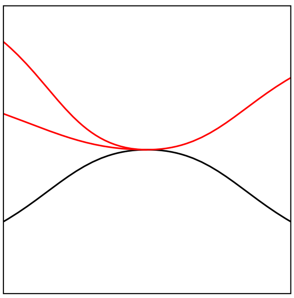
At the singularity we have the following situation (we only draw this in the first case, as the corresponding picture in the second case is not illuminating):



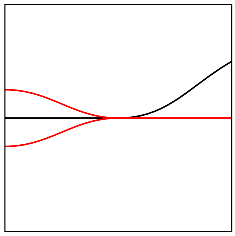
The evolution through time is controlled by the relative value of the y-coordinates of $(\frac{\partial f}{\partial t})_{M_1}$ and of $(\frac{\partial f}{\partial t})_{M_2}$. For codimension reason, the difference can be assumed to be non-zero, and we thus read analogs of the movie move (OM_1), which are the first genuinely foamy moves that we encounter. In the other case, we get the move (OM_2) (with the strand passing over or under the digon).

Again nothing happens to the framing.

Consider the case of non-transverse intersections. The case where $\pi((\frac{\partial f}{\partial u})_{M_1})$ and $\pi((\frac{\partial f}{\partial u})_{M_2})$ are colinear corresponds to a singular situation as shown below (we have colored the trivalent vertex for better visibility).



Indeed, one can compare the lines drawn by $f(M_1+(\varepsilon_1,0,0))$ and $f(M_2+(\varepsilon_1,0,0))$ as was done for the movie moves MM_3 and MM_4. Then, the strand goes away from the other line at pace proportional to ε_1^2, while the two legs of the web glue smoothly and thus diverge slower than ε_1^2. This excludes the following situation:



The shape before and after in the t direction will be controlled by the relative x coordinates of the t derivatives. The difference can be assumed to be non-zero, and we read the move (OM_3), with the strand passing over or under the trivalent vertex, and considered up to mirror image.

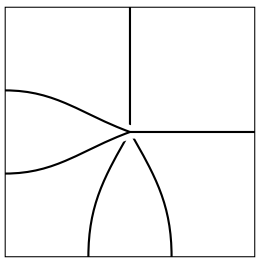
The case where the y-coordinates in the ∂v derivatives balance induces the invertibility moves (I_1) and (I_2).

It finally remains to analyze changes in framing. This can be supported on the strand or on the trivalent web, and we read the moves (PF_8) and (OM_4) (they are deduced from isotopies of webs by superposing an extra strand).

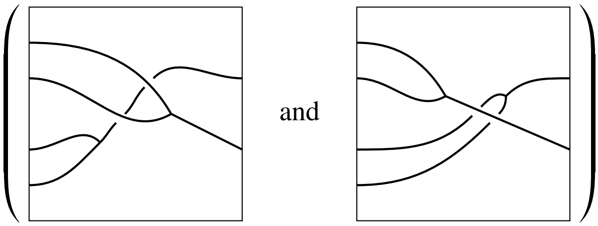
Double points in $F^1 \times F^1$

We are looking at a situation of codimension 7 (twice 1 for restriction to F, then twice 1 for restriction to F^1, and 3 for the equality of the x, y, s coordinates). Such a situation is thus not generic for a foam, but does happen at isolated points during a foam isotopy.

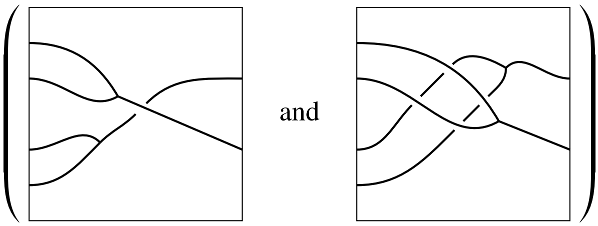
One may assume that one of the x or y entries of the $\frac{\partial f}{\partial u}$ derivatives at M_1 and M_2, the two points of interest, is non-zero. Up to rotation, the one at M_1 can be assumed to be parallel to the x direction. Then, the other one can be assumed to have a non-zero y coordinate: the intersection is transverse. One thus has a singular situation of the following kind:



Depending on the relative value of b_1 and b_2 (all of which can be assumed non-zero) at M_1 and M_2, the webs obtained for positive and negative values of v will be of the following kind:



or



Letting now t vary, depending on the relative value of the x and y entries of $\frac{\partial f}{\partial t}$ at M_1 and M_2 (and for codimension reason these values can be assumed to be different), one resolves the singularity by pushing one of the trivalent vertices in one direction for positive t and in the other direction for negative t, yielding the movie move (OM_6). For codimension reasons, the framing can be assumed not to change.

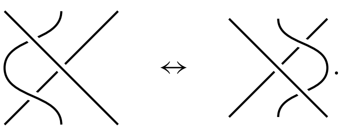
Double points in $F^0 \times F^2$

The analysis is similar to the one sketched previously and yields the movie move (OM_5) with the strand passing over or under the other web, and up to mirror symmetry.

This exhausts the analysis for double points.

Triple points

In the case of foams, we have isolated singularities, so no additional restriction can be imposed. Generically, triple points involve three points on F^2. The analysis thus parallels the classical situation, and we find as generators Reidemeister III moves:



These moves can occur with all possible orientations and compatible choices for crossings.

Adding up the time direction, one recovers the following classical movie moves: (MM_5), controlling the invertibility of the third Reidemeister move, and (MM_6), corresponding to two of the strands being non-transverse. These again should be considered with all possible versions of crossings.

It might also happen that the underlying surface is just a Reidemeister 3 generator moving along time without special feature, but crossing a frame change on one of the three strands. This yields the movie move (PF_9) (and analogs of it with other crossing conventions and with the half twist traveling on other strands).

Then, one can look at triple points involving one point on F^1 and two on F^2. This happens at isolated points, and no other degeneracy generically happens. One gets the move (OM_7) (with strands passing both over the trivalent vertex, or one over one under, or yet again both under the trivalent vertex), as in [6, page 3].

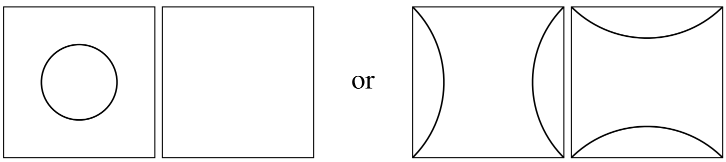
Quadruple points

Quadruple points only occur at isolated places, involving four points from F^2: this is covered by the classical link situation. One gets the classical movie move (MM_{10}). Again all versions of it should be considered (all orientations, and all compatible crossing flips).

This exhausts the classification of multiple points. We now consider the neighborhood of a single point.

4.5. Neighborhood of a point on the 2-dimensional locus

When framing is ignored, the analysis to be run here is classical and equivalent to Morse theory techniques. As foam generators, one gets birth or death cobordisms (as in the first picture below, read forward or backwards), and saddles (as in the second picture).



When adding the time dimension, one gets classical crossingless movie moves, which reduce to (MM_{11}), as well as (MM_{12}) and (MM_{13}) which will need additional attention to incorporate the framing data.

Recall from the previous paragraphs that Morse points occur when we assume that both $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ have vanishing s coordinates. This corresponds to a codimension

3 condition, while the graph of one function is of dimension 3. This thus occurs at isolated points. Denote by

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{pmatrix}.$$

By SL_2 action in the (x, y) plane one reduces as usual to (unless the two vectors are colinear, which will yield moves MM_{12} and MM_{13})

$$df_M = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & \pm 1 & c_2 \\ a_3 & b_3 & c_3 \\ 0 & 0 & c_4 \end{pmatrix}.$$

Up to symmetry, we will assume the entry b_2 to be 1. One can then write

$$f(u_0 + \varepsilon_1, v_0 + \varepsilon_2, w_0) = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ d_4\varepsilon_1^2 + e_4\varepsilon_1\varepsilon_2 + g_4\varepsilon_2^2 \end{pmatrix}.$$

By a codimension argument, one can assume that $\begin{vmatrix} d_4 & e_4 \\ e_4 & d_4 \end{vmatrix} \neq 0$ (this will fail through time at isolated points, see the movie move MM_{11}), yielding a cap, cup, or saddle generator.

Now, for the framing, since we have

$$df_M = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & \pm 1 & c_2 \\ a_3 & b_3 & c_3 \\ 0 & 0 & c_4 \end{pmatrix},$$

one can replace w by a linear combination of u, v and w to get

$$df_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ a_3 & b_3 & c_3 \\ 0 & 0 & c_4 \end{pmatrix}.$$

Through time it might happen that a framing change occurs at the Morse point ($c_4 = 0$), but for a generic foam $c_4 \neq 0$, and thus,

$$\frac{\partial f}{\partial w} \notin \mathrm{Span}_{\mathbb{R}} \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \vec{z} \right). \tag{4.2}$$

Since this is an open condition (as the three vectors span a space of dimension 3), it will remain true around M and thus the framing does not become singular on the generators.

Let us now go back to the case where $\begin{pmatrix} d_4 & e_4 \\ e_4 & d_4 \end{pmatrix}$ is of rank 1. One can reduce to the following situation:

$$f(u_0+\varepsilon_1, v_0+\varepsilon_2, w_0) = f(u_0, v_0, w_0) + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ d_4\varepsilon_1^2 + g_4\varepsilon_2^3 \end{pmatrix}, \quad d_4 \neq 0,\ g_4 \neq 0.$$

This yields the classical movie move (MM_{11}), with generically no framing change.

Let us now consider the case where the ∂u and ∂v derivatives have colinear x, y projections. Up to rotation in the x, y plane one can assume that

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a_1 \\ 0 \\ a_3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{pmatrix}, \quad b_1 > 0.$$

One can assume that $\begin{vmatrix} d_4 & e_4 \\ e_4 & g_4 \end{vmatrix} \neq 0$. This implies that there exists a change in coordinates u, v such that

$$\frac{\partial^2 f}{\partial u^2} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \pm 1 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial u \partial v} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ 0 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial v^2} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \pm 1 \end{pmatrix}.$$

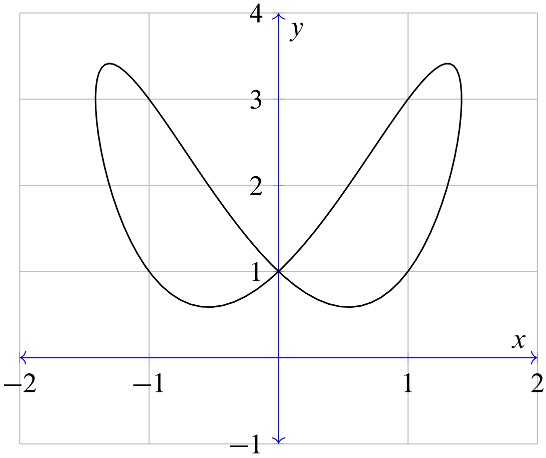
We first consider the case where the last line is $\varepsilon_1^2 + \varepsilon_2^2$. The case with two minus signs will follow by mirror. The mixed case will be considered later. This gives as Taylor expansion

$$f(u_0+\varepsilon_1, v_0+\varepsilon_2, w_0) = \begin{pmatrix} a_1\varepsilon_1 + b_1\varepsilon_2 \\ d_2\varepsilon_1^2 + e_2\varepsilon_1\varepsilon_2 + g_2\varepsilon_2^2 \\ a_3\varepsilon_1 + b_3\varepsilon_2 \\ \varepsilon_1^2 + \varepsilon_2^2 \end{pmatrix}.$$

Let us focus on the coordinates x, y and s, and fix a given value $s = r$. Since the three equations are homogeneous, the surface drawn is topologically just the cone on one of these curves, say for $r = 1$. At $r = 1$, we consider the set of points such that $\varepsilon_1^2 + \varepsilon_2^2 = 1$, which we can reparametrize as $\varepsilon_1 = \cos(\vartheta)$ and $\varepsilon_2 = \sin(\vartheta)$ for $\vartheta \in [-\pi, \pi]$. So, we care about the following parametric curve:

$$\big\{(a_1\cos(\vartheta) + b_1\sin(\vartheta), d_2\cos(\vartheta)^2 + e_2\cos(\vartheta)\sin(\vartheta) + g_2\sin(\vartheta)^2), \vartheta \in [-\pi, \pi]\big\}.$$

Below is an example with parameters $a_1 = b_1 = 1$ and $d_2 = 1$, $e_2 = 2$, $g_2 = 3$.



We want to claim that the above curve is representative of the generic situation. Let us go back to the curve

$$\left\{ \begin{pmatrix} a_1\varepsilon_1 + b_1\varepsilon_2 \\ d_2\varepsilon_1^2 + e_2\varepsilon_1\varepsilon_2 + g_2\varepsilon_2^2 \end{pmatrix}, \varepsilon_1^2 + \varepsilon_2^2 = 1 \right\}.$$

Up to changing coordinates

$$\varepsilon_1 = \frac{a_1\varepsilon_1' + b_1\varepsilon_2'}{\sqrt{a_1^2 + b_1^2}}$$

and

$$\varepsilon_2 = \frac{b_1\varepsilon_1' - a_1\varepsilon_2'}{\sqrt{a_1^2 + b_1^2}},$$

one reduces to

$$\left\{ \begin{pmatrix} \sqrt{a_1^2 + b_1^2}\varepsilon_1' \\ d_2'{\varepsilon_1'}^2 + e_2'\varepsilon_1'\varepsilon_2' + f_2'{\varepsilon_2'}^2 \end{pmatrix}, \ {\varepsilon_1'}^2 + {\varepsilon_2'}^2 = 1 \right\}.$$

Let us now look at multiple points: assume that (ξ_1, ξ_2) and (ξ_1', ξ_2') yield the same point. From the first coordinate, one reads that $\xi_1 = \xi_1'$, and thus from the equation ${\varepsilon_1'}^2 + {\varepsilon_2'}^2 = 1$ one gets that $\xi_2 = \pm\xi_2'$. Now, (ξ_1, ξ_2) and $(\xi_1, -\xi_2)$ produce the same point if $2e_2'\xi_1\xi_2 = 0$. Generically, $e_2' \neq 0$ and one gets $\xi_1 = 0$ and thus as only double points, $(0, 1)$ and $(0, -1)$. Thus, the curve is a circle with a single double point, the only topological solution being the one depicted above.

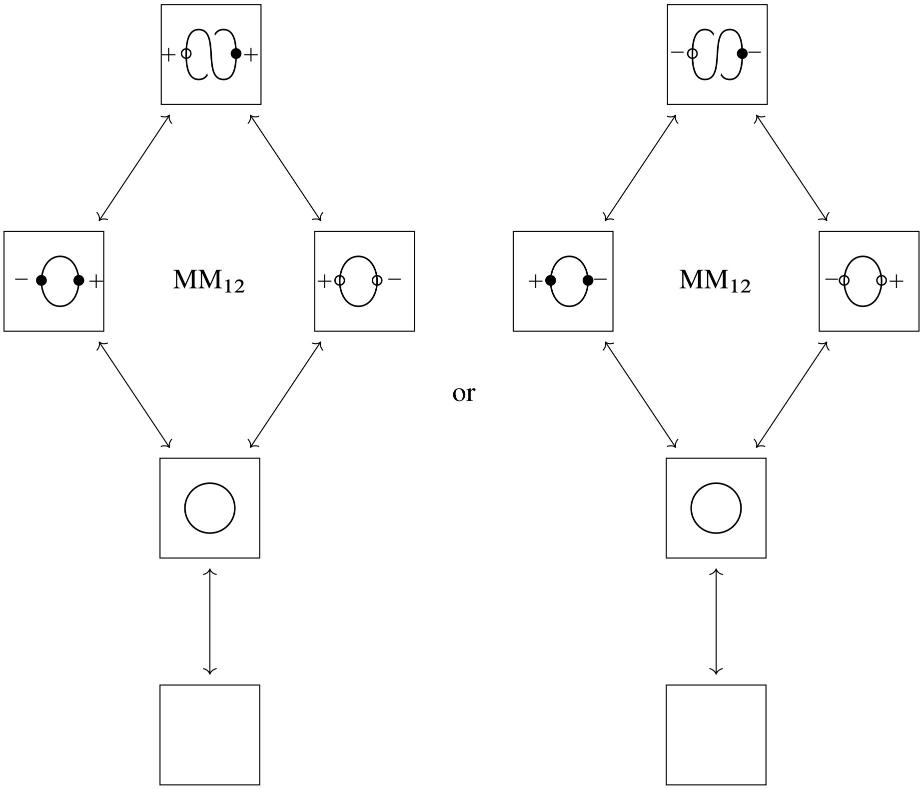
Adding up the time parameter, we will get a framed version of the classical move (MM_{12}). Let us focus on the framing data, and compare to the situation in equation (4.2). Now,

$$\mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \vec{z}\right)$$

is only two-dimensional but will become three-dimensional when perturbed. Although at the singular point the framing does not belong to this spanned space, this is not an open condition anymore. To see this, notice that one can only reduce to

$$\frac{\partial f}{\partial w} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

The non-vanishing of the y-coordinate will dictate the shape of the framing, and justifies the move (MM_{12}). Below we show two versions of it to illustrate the framing changes when changing the crossing.

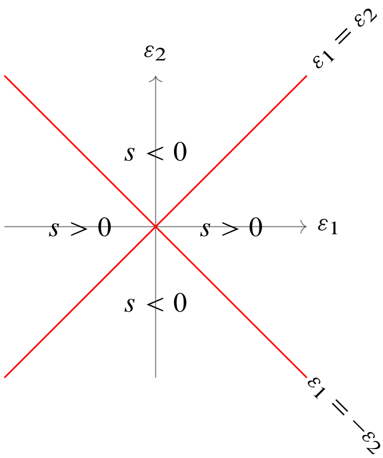


Note that at the bottom, the side of the circle used to pair the half twists together is irrelevant, thanks to the move (PF_3).

We now go back to the case where the last line in the Taylor expansion is $\varepsilon_1^2 - \varepsilon_2^2$

$$f(u_0 + \varepsilon_1, v_0 + \varepsilon_2, w_0) \sim \begin{pmatrix} a_1\varepsilon_1 + b_1\varepsilon_2 \\ d_2\varepsilon_1^2 + e_2\varepsilon_1\varepsilon_2 + g_2\varepsilon_2^2 \\ a_3\varepsilon_1 + b_3\varepsilon_2 \\ \varepsilon_1^2 - \varepsilon_2^2 \end{pmatrix}.$$

Looking at the last line in the above matrix, one sees that $s = 0$ when $\varepsilon_1 = \pm\varepsilon_2$. The s-coordinate is then positive or negative as illustrated below:

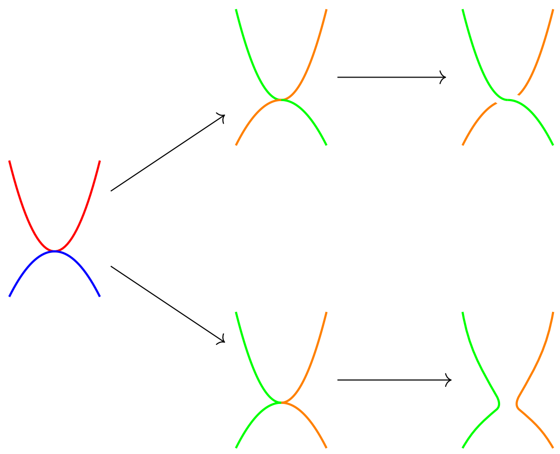


At $s = 0$, then one has

$$f(u_0 + \varepsilon_1, v_0 \pm \varepsilon_1, w_0) \sim \begin{pmatrix} (a_1 \pm b_1)\varepsilon_1 \\ (d_2 \pm e_2 + g_2)\varepsilon_1^2 \\ \cdot \\ 0 \end{pmatrix}.$$

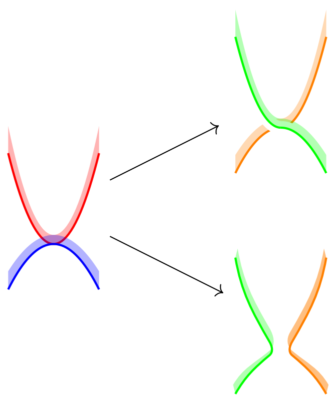
One may assume that generically $a_1 \pm b_1 \neq 0$ and $d_2 \pm e_2 + g_2 \neq 0$. One then sees two parabolas, tangent at one point.

Fixing $s = \mu^2$ with $\mu > 0$, one has that $\varepsilon_1^2 = \varepsilon_2^2 + \mu^2 \geq \mu^2$, thus one expects to see two connected components, depending that $\varepsilon_1 > \mu$ or $\varepsilon_1 < -\mu$. Let us first focus on $\varepsilon_1 > \mu$. The curve one will see is close to the two half-parabolas formed by the $\varepsilon_1 \geq 0$ part at $s = 0$. Similarly, the curve corresponding to $\varepsilon_1 < -\mu$ is close to the two half-lines formed by the $\varepsilon_1 \leq 0$ part at $s = 0$. Depending on the relative signs of $a_1 \pm b_1$, the parts at $s = 0$ assemble either into two smooth curves with one intersection point, or two singular curves meeting at their singular point (that then get smoothed at $\mu \neq 0$). Replacing $s = \mu^2$ by $s = -\mu^2$ passes from one to the other situation.



Notice that if the x-coordinate takes the value 0, then one has $\varepsilon_2 = -\frac{a_1}{b_1}\varepsilon_1$. Replacing ε_2 by this value in the last line implies that if $a_1 > b_1$, then this can only occur for negative value of the s-coordinate, while if $a_1 < b_1$, this will only occur for positive values of s. Since both positive and negative values can be taken in the x-coordinate, this implies the following: if $a_1 > b_1$, then the two connected components living at fixed s-coordinate of positive value live each on one side of the $x = 0$ line (and in particular do not meet); if $a_1 < b_1$, then the two connected components living at fixed s-coordinate of negative value do not meet. This justifies that the green and orange lines at the bottom right corner of the above picture are disjoint.

For codimension reason, the framing can be assumed to be generic at the singular point, and then transported around, giving (notice the framing changes on the crossingless picture)



When one adds the time parameter, the place where the ∂u and ∂v derivatives align will be taken away from the Morse singularity, in one or the other side of the saddle. One gets the movie move (MM_{13}) (and all variants of it).

Let us now go back to framing considerations we had eluded at the beginning of our analysis. The graph of f on $N(F) \times [0,1]$ is 4-dimensional. Requiring that the framing lies in the subspace generated by the tangent plane to F and the vertical direction is a codimension 1 condition. Furthermore, the restriction to F is also a codimension 1 condition. Thus, the set of points where the framing is vertical is of dimension 1 in a generic foam, and of dimension 2 in a time-family of foams.

Assume that

$$(df)_M = \left(\boxed{A} \quad \boxed{B} \quad \boxed{A + B + \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix}} \right).$$

Up to change of coordinates in u and v we can assume that $(\frac{\partial f}{\partial u})_M$ has zero s coordinate. We first assume that $(\frac{\partial f}{\partial u})$ has non-zero coordinates in x or y, which allows us to reduce to the case where its y-coordinate is zero and the x-coordinate is strictly positive. Then, one can change coordinates in u, v, w so that

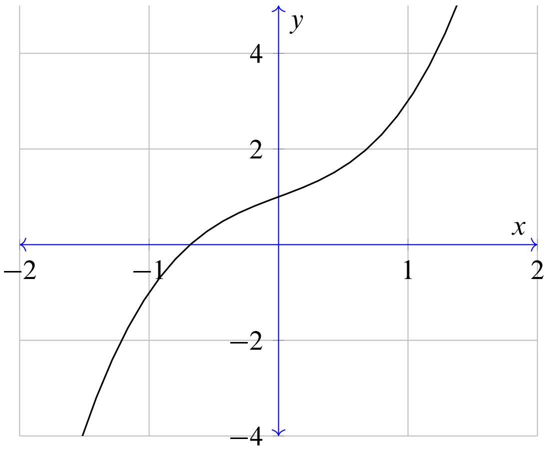
$$(df)_M = \begin{pmatrix} a_1 & 0 & \lambda a_1 \\ 0 & b_2 & \mu b_2 \\ a_3 & b_3 & \mu b_3 + \eta \\ 0 & b_4 & \mu b_4 \end{pmatrix}, \quad a_1 > 0, \eta \neq 0.$$

Let us first assume that $b_4 \neq 0$. In that case, the local shape for the foam is a trivial movie over a single segment and we are basically brought to the case of an isotopy of a web. We will be interested in $\frac{\partial^2 f}{\partial u \partial w}$. Assume that this vector has a non-zero y coordinate f_2. This ensures that the framing change is isolated (the framing has non-vanishing projection through π around M) and we have a trivial movie move.

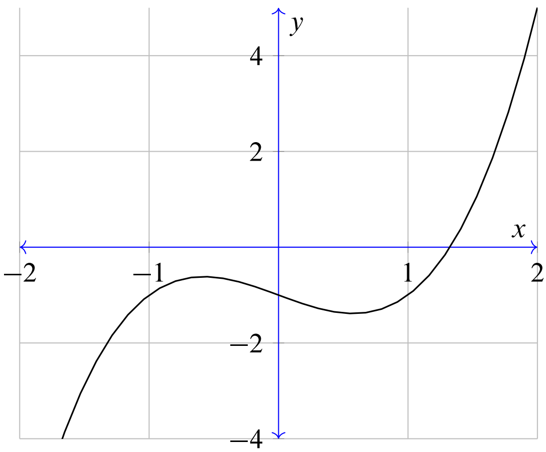
Now, assume that $f_2 = 0$. This is a codimensions 1 condition, so we expect through time a 1-dimensional set of such points. Consider $\beta' = (\frac{\partial^3 f}{\partial u^2 \partial w})_M$. If $\beta' \neq 0$, then we locally are in a situation similar to the one from equation (3.3). If the y-coordinate ν of $\frac{\partial^2 f}{\partial v \partial w}$ is non-zero, then at $t = t_0$ we see the foam from equations (3.4) or (3.5). Along time, we see no change provided $\frac{\partial^2 f}{\partial t \partial w}$ has non-zero y coordinates. Otherwise, one gets the movie move (PF_{10}), which is only a twist version of the framed R_I move.

If $\beta' = 0$, then for codimension reason this only happens at isolated points, and we can assume that the y-coordinate $\beta"$ in $\frac{\partial^4 f}{\partial^3 u \partial w}$ is non-zero as well as ν. The equation for the curve of annihilation of the points is then: $\beta" \varepsilon_1^3 + \nu \varepsilon_2 = 0$. As we add the time parameter, we can assume that κ the y-coordinate in $\frac{\partial^2 f}{\partial w \partial t}$ is non-zero as well ξ the y-coordinate in $\frac{\partial^2 f}{\partial u \partial t}$, and we get an equation: $\beta" \varepsilon_1^3 + \nu \varepsilon_2 + \kappa \delta + \xi \varepsilon_1 \delta = 0$. Below we illustrate the equations: $v = u^3 + u + 1$ and $v = u^3 - u - 1$ (corresponding to

$\beta'' = 1$, $\nu = 1$, $\kappa = 1$, $\xi = 1$, and $\delta = \pm 1$). The point is that in all cases, we pass between 0 and 2 inflection points.



(4.3)



(4.4)

This corresponds to the movie move (PF_2).

We now go back to the case where $b_4 = 0$, that is

$$(df)_M = \begin{pmatrix} a_1 & 0 & \lambda a_1 \\ 0 & b_2 & \mu b_2 \\ a_3 & b_3 & \lambda a_3 + \mu b_3 + \eta \\ 0 & 0 & 0 \end{pmatrix}, \quad a_1 > 0, \eta \neq 0.$$

This is a co-dimension 1 condition, so we expect such points to happen at isolated places (through time). Up to adding multiples of u and v to w, we can simplify the matrix to

$$(df)_M = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ a_3 & b_3 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \quad a_1 > 0,\ \eta \neq 0.$$

One may assume that the 3 by 3 matrix formed by the projections under π of $(\frac{\partial^2 f}{\partial^2 u})_M$, $(\frac{\partial^2 f}{\partial u \partial v})_M$ and $(\frac{\partial^2 f}{\partial^2 v})_M$ is of rank 3. This ensures that the system formed by the first-order expansions of the equation

$$A\left(\frac{\partial f}{\partial u}\right)_{(u_0+\varepsilon_1, v_0+\varepsilon_2, w_0, t_0)} + B\left(\frac{\partial f}{\partial v}\right)_{(u_0+\varepsilon_1, v_0+\varepsilon_2, w_0, t_0)} = \left(\frac{\partial f}{\partial w}\right)_{(u_0+\varepsilon_1, v_0+\varepsilon_2, w_0, t_0)}$$

has a unique solution in A, B for each value of ε_1 and ε_2. Furthermore, we can assume that the right-hand side of the above equation is non-zero, which implies that the solution is non-zero linear combination in ε_1 and ε_2. In other terms, at first order, the locus of framing change draws a line that passes at M.

The same equation will transport through time, except that the line will drift in the direction given by $-\frac{\partial^2 f}{\partial w \partial t}$. This vector can be assumed to have non-zero projection in the x, y plane. One thus gets one of the moves (PF_3) or (PF_4) (with all possibilities for framing changes).

We now look at points where $\vec{z} \in \mathrm{Span}_{\mathbb{R}}(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$. If we recap on codimension, we have: a codimension 1 condition when restricting to F, and a codimension 2 condition coming from the verticality assumption. We thus have isolated points on a foam (this was already analyzed when looking at isotopies of a web) yielding the movies from equation (3.2), and 1-dimensional sets through time.

It can happen that $a_4 = b_4 = 0$: this is the case of Morse extrema which has already been analyzed, yielding movie moves (MM_{12}) and (MM_{13}).

Notice that there cannot be a framing change happening at the same time. Indeed

$$\mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \vec{z}\right) = \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$$

and $\frac{\partial f}{\partial w}$ cannot belong to the above vector space without violating condition (2).

Let us run a process similar to the one that yielded equation (3.2). Up to change of (u, v) basis, one can reduce to

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ b_4 \end{pmatrix}.$$

Up to rotation in the x, y plane, one further reduces to

$$\frac{\partial^2 f}{\partial u^2} = \begin{pmatrix} d_1 \\ 0 \\ d_3 \\ d_4 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial u \partial v} = \begin{pmatrix} e_1 \\ e_2 \\ a_3 \\ e_4 \end{pmatrix}, \quad \frac{\partial f^3}{\partial u^3} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}.$$

If $d_1 \neq 0$, $l_2 \neq 0$ and $e_2 \neq 0$, then we simply have a framed Reidemeister 1 move traveling through time. This will be a trivial movie move provided the line of double points has a tangent at origin with non-zero s-coordinate. If this is not the case, then one gets the classical move (MM_7).

To illustrate this case, consider the following particular situation:

$$f_{t_0}(u_0 + \varepsilon_1, v_0 + \varepsilon_2) = \begin{pmatrix} b_1\varepsilon_2 + d_1\varepsilon_1^2 \\ b_2\varepsilon_2 + e_2\varepsilon_1\varepsilon_2 + l_2\varepsilon_1^3 \\ b_4\varepsilon_2 + d_4\varepsilon_1^2 + e_4\varepsilon_1\varepsilon_2 \end{pmatrix}.$$

Then, the line of double points is governed by equation $\varepsilon_1^2 = -\frac{e_2}{l_2}\varepsilon_2$. The tangent vector at $\varepsilon_2 = 0$ is

$$\begin{pmatrix} b_1 + \frac{d_1 e_2}{l_2} \\ b_2 \\ b_4 + \frac{d_4 e_2}{l_2} \end{pmatrix}.$$

It might indeed happen that the last line vanishes.

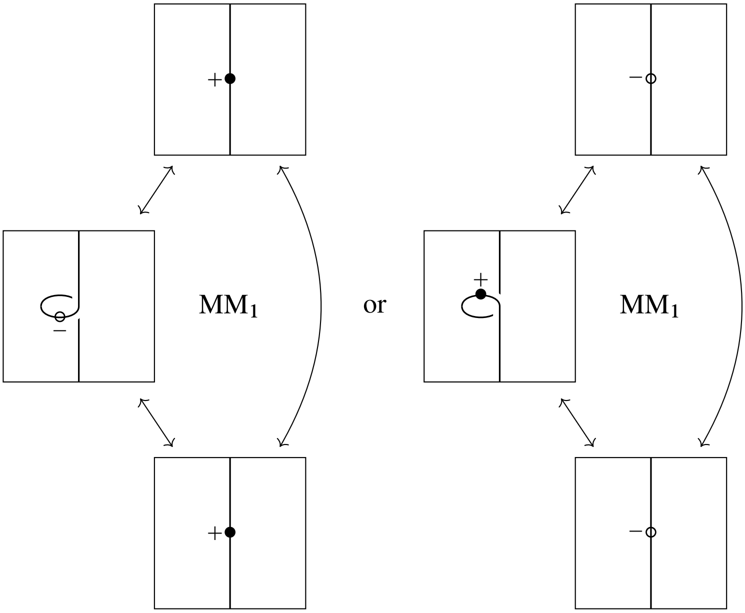
Asking that $d_1 = 0$ corresponds to the simultaneous vanishing of both x and y coordinates in $\frac{\partial^2 f}{\partial u^2}$, so this is of too big codimension and generically does not happen.

Assume that $e_2 = 0$. Then, we will use m_2 the y-coordinate of $\frac{\partial f^3}{\partial u^2 \partial v}$ and n_2 the y-coordinate of $\frac{\partial f^3}{\partial u \partial v^2}$ to write, at fixed ε_2 and ε_1 small enough

$$f_{t_0}(u_0 + \varepsilon_1, v_0 + \varepsilon_2, w_0) \sim \begin{pmatrix} b_1\varepsilon_2 + d_1\varepsilon_1^2 + e_1\varepsilon_1\varepsilon_2 \\ b_2\varepsilon_2 + l_2\varepsilon_1^3 + m_2\varepsilon_1^2\varepsilon_2 + n_2\varepsilon_1\varepsilon_2^2 \\ \cdot \\ b_4\varepsilon_2 + d_4\varepsilon_1^2 + e_4\varepsilon_1\varepsilon_2 \end{pmatrix}.$$

Starting from the cusp seen at $\varepsilon_2 = 0$, one can study the effect of $e_1\varepsilon_1\varepsilon_2$, $m_2\varepsilon_1^2\varepsilon_2$ and $n_2\varepsilon_1\varepsilon_2^2$. One finds that only the latter term plays a role, and this time this role is the same both for ε_2 positive or negative. So, depending on the signs of l_2 and n_2, either a crossing gets created both for $\varepsilon_2 > 0$ and $\varepsilon_2 < 0$, or for none of them.

The first case yields the classical movie move (MM_1), while the second one yields (MM_2). Below we explicitly show two of the four versions, to illustrate how crossings and framing changes are related.



The other two versions can be obtained by exchanging the signs for ∘ and •.

The case where $l_2 = 0$ goes as follows: one replaces $l_2\varepsilon_1^3$ by $p_2\varepsilon_1^4$. One can push the Taylor expansion by adding a term $q_2\varepsilon_1^5$, but then one has to account for terms $r_2\varepsilon_1^3\varepsilon_2$ too (we only keep track of those with odd exponent in ε_1). We find multiple points by looking at the roots of the polynomial $q_2\varepsilon_1^5 + r_2\varepsilon_1^3\varepsilon_2 + e_2\varepsilon_1\varepsilon_2 = 0$, $\varepsilon_1 = 0$ gives one root, and then depending on the values of the coefficients, one can get 0, 2 or 4 real roots. Finding these roots goes by considering the degree two polynomial in the variable X^2 given by $q_2X^4 + (r_2\varepsilon_2)X^2 + (e_2\varepsilon_2)$. Changing ε_2 to $-\varepsilon_2$ does change the sign of the discriminant. So, on one of the two sides in the v direction, one sees no multiple points. On the other side, the discriminant is equivalent to $-4q_2e_2\varepsilon_2$, and one finds that one of the two roots of the degree two polynomial has to be negative. This only creates two real roots, yielding a single double point, as above.

4.6. Neighborhood of a point on the 1-dimensional locus

Recall from the beginning of the section that we have to consider the following degeneracies:

- $\frac{\partial f}{\partial v} = \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$;
- $(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ contains the direction $\vec{z}$;
- $\frac{\partial f}{\partial w}$ is contained in the space spanned by $\vec{z}$, $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$.

Recall that the graph of f is 3-dimensional (+1 through time). Regarding codimension, restricting to a point on F^1 is a codimension 2 condition. Asking that

$$\frac{\partial f}{\partial v} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

is a codimension 1 condition, so this happens at isolated points with no other degeneracy in a foam, and such points form 1-dimensional sets through time.

To obtain a local foam model, consider a point $M = (u_0, v_0, w_0) \in F^1$, and assume that $\frac{\partial f}{\partial v} = \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$. Recall that we use the local chart that makes the v direction parallel to the seam. Up to rotation in the x, y plane, one may assume that

$$\frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{pmatrix}, \quad b_1 > 0.$$

The assumption that $b_1 > 0$ is generic, even through time, as its failure expresses the codimension 2 condition that both x and y are zero. Consider

$$\frac{\partial^2 f}{\partial v^2} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}.$$

One may assume that $g_4 \neq 0$. This is a codimension 1 condition, so through time we will have to consider failure of it.

One may write, following the seam,

$$f(u_0, v_0 + \varepsilon_2, w_0) \sim f(u_0, v_0, w_0) + \begin{pmatrix} b_1\varepsilon_2 \\ g_2\varepsilon_2^2 \\ b_3\varepsilon_2 \\ g_4\varepsilon_2^2 \end{pmatrix}.$$

Depending on the sign of g_4, this corresponds to a Morse minimum ($g_4 > 0$) or maximum ($g_4 < 0$) on the seam. We run our analysis in the case with $g_4 > 0$, the other case being symmetric. Then, denoting by

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix},$$

one may assume that $a_4 \neq 0$ (this will also have to be considered through time). Up to adding to u a scalar multiple of v, one can assume that

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Furthermore, up to adding to w scalar multiples of u and v, one can assume that

$$\frac{\partial f}{\partial w} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \\ 0 \end{pmatrix}.$$

Thus, one may write

$$f(u_0 + \varepsilon_1, v_0 + \varepsilon_2, w_0) \sim \begin{pmatrix} b_1\varepsilon_2 + d_1\varepsilon_1^2 \\ a_2\varepsilon_1 + g_2\varepsilon_2^2 + e_2\varepsilon_1\varepsilon_2 \\ * \\ g_4\varepsilon_2^2 + a_4\varepsilon_1 \end{pmatrix}. \tag{4.5}$$

At fixed level $s = r$, one thus has

$$\varepsilon_1 = \frac{r - g_4\varepsilon_2^2}{a_4},$$

yielding a parametric curve equivalent to

$$\left(d_1\frac{r^2}{a_4^2} + b_1\varepsilon_2, \frac{a_2 r}{a_4} + e_2\frac{r}{a_4}\varepsilon_2\right)$$

provided $e_2 \neq 0$, or

$$\left(d_1\frac{r^2}{a_4^2} + b_1\varepsilon_2, \frac{a_2 r}{a_4} + (g_2 - \frac{a_2 g_4}{a_4})\varepsilon_2^2\right)$$

if $e_2 = 0$ (provided $g_2 - \frac{a_2 g_4}{a_4} \neq 0$, which can be assumed generically if e_2 has already been set to zero).

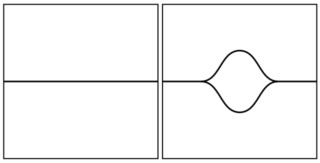
In the first case, one just sees a line, while in the second case one gets a parabola.

Let us now identify the pieces that correspond to $\varepsilon_1 \geq 0$ (this is the part that will be replaced by the two legs of the web). $\varepsilon_1 \geq 0$ corresponds to

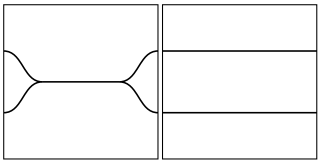
$$\varepsilon_2^2 \leq \frac{r}{g_4} \quad \text{if } a_4 > 0, \quad \varepsilon_2^2 \geq \frac{r}{g_4} \quad \text{if } a_4 < 0.$$

Now, to pass from the line to the web, one will replace the portions corresponding to $\varepsilon_1 \geq 0$ by a two curves, obtained by pushing positively or negatively in the framing direction. Notice that at (u_0, v_0, w_0), the tangent to the line has slope $\frac{e_2 r}{a_4 b_1} \neq \infty$ (since b_1 and a_4 both are non-zero) if $e_2 \neq 0$, and 0 if $e_2 = 0$. Since the framing at (u_0, v_0, w_0) has (x, y) coordinates $(0, c_2)$, in both cases the process of creating the legs creates no crossing nor special framing features (provided $c_2 \neq 0$).

Thus, depending on whether $a_4 > 0$ or $a_4 < 0$, one gets one of the following foam generators:



or



Through time, we have to consider the following special cases (which are mutually exclusive):

(1) $g_4 = 0$;

(2) $a_4 = 0$;

(3) $c_2 = 0$ (which also corresponds to a framing change).

In the case where $g_4 = 0$, one has to go one order further and appeal to

$$\frac{\partial^3 f}{\partial v^3} = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \end{pmatrix}.$$

One may assume that $j_4 \neq 0$ and thus obtain

$$f(u_0, v_0 + \varepsilon_2, w_0) \sim f(u_0, v_0, w_0) + \begin{pmatrix} b_1 \varepsilon_2 \\ d_2 \varepsilon_2^2 \\ b_3 \varepsilon_2 \\ j_4 \varepsilon_2^3 \end{pmatrix}.$$

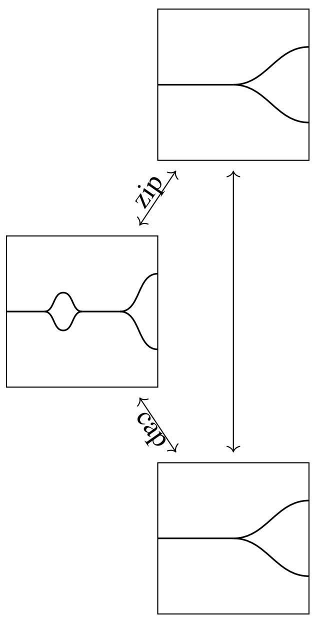
We care about the last line, that draws a polynomial with a triple root at 0.
Let us consider also

$$\frac{\partial f}{\partial t} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial v \partial t} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}.$$

The term l_4 can be assumed to be non-zero. The Taylor expansion thus becomes (we still only focus on the seam)

$$f_{t_0+\delta}(u_0, v_0 + \varepsilon_2, w_0) \sim f_{t_0}(u_0, v_0, w_0) + \begin{pmatrix} b_1\varepsilon_2 + k_1\delta + l_1\varepsilon_2\delta \\ d_2\varepsilon_2^2 + k_2\delta + l_2\varepsilon_2\delta \\ k_3\varepsilon_2 + l_3\delta + g_3\varepsilon_2\delta \\ j_4\varepsilon_2^3 + k_4\delta + l_4\varepsilon_2\delta \end{pmatrix}.$$

Through time and focusing on the x and s coordinates, one is brought back to a situation similar to the one investigated in the figures drawn at (4.3) and (4.4). Both signs for a_4 yield mirror versions of the move (CL_1) (we have indicated the elementary moves to apply on the arrows: going from the top to the bottom, first zip together the two legs of the trivalent vertex, then cap off the digon):



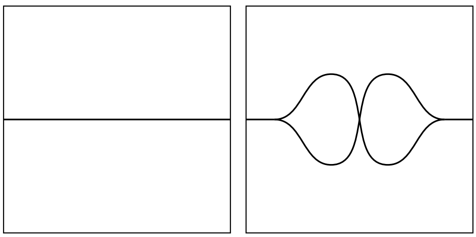
Let us now suppose that $a_4 = 0$. One should consider the second derivative in the u direction, the s-coordinate of which we denote d_4. One may assume that $d_4 \neq 0$. This means that on both sides of the seam, the image of the $w = 0$ sheet goes similarly up or down. Since following both split facets corresponds to $\varepsilon_3 \ll \varepsilon_1$, we have the same behavior for the two thinnest facets. Up to mirror, one can assume that $g_4 > 0$, and we are left with two situations: either $d_4 > 0$ or $d_4 < 0$. In the case where $d_4 > 0$ one reads the move (CL_2).

When $d_4 < 0$, one gets (CL_3).

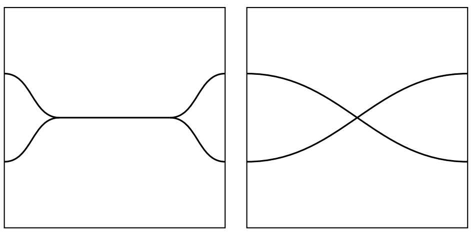
Let us now consider the case where a_4 and g_4 are both non-zero, but $c_2 = 0$. One can start the same analysis as in equation (4.5), but splitting the piece with $\varepsilon_1 \geq 0$ will be more involved (since the framing crosses the line). It is best seen by pushing the Taylor expansion also in the w direction, writing

$$(f(u, v_0 + \varepsilon_2, w))_x \sim b_1\varepsilon_2 + e_1\varepsilon_1\varepsilon_2 + f_1\varepsilon_1\varepsilon_3 + h_1\varepsilon_2\varepsilon_3,$$
$$(f(u, v_0 + \varepsilon_2, w))_y \sim a_2\varepsilon_1 + e_2\varepsilon_1\varepsilon_2 + g_2\varepsilon_2^2 + f_1\varepsilon_1\varepsilon_3 + h_2\varepsilon_2\varepsilon_3 + i_2\varepsilon_3^2.$$

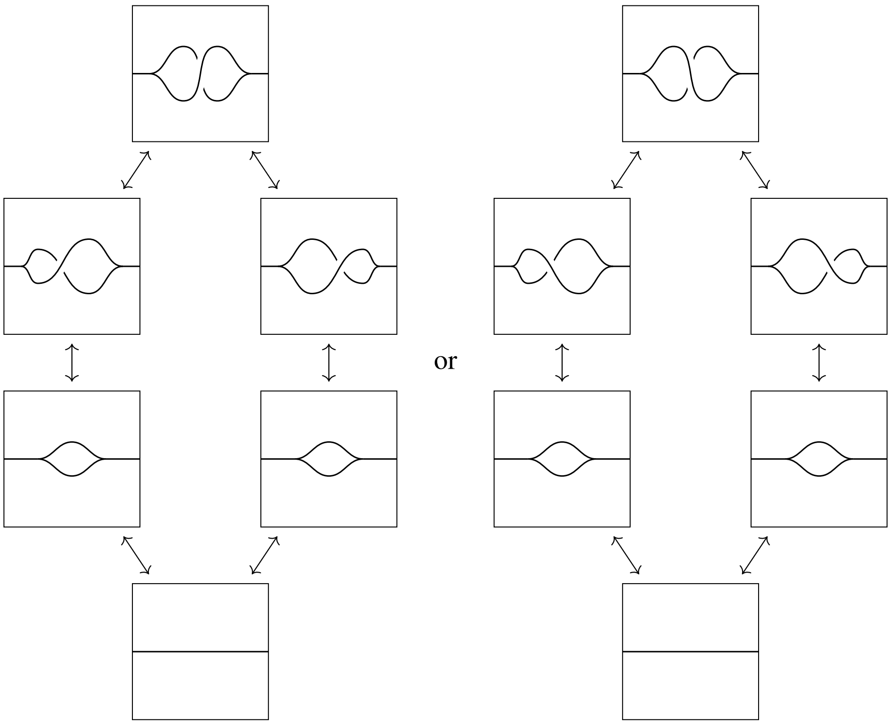
Again, one sees a line at first order in ε_2, but the change when ε_3 becomes $-\varepsilon_3$ is now controlled by $h_2\varepsilon_2\varepsilon_3$. This creates a crossing, and at $t = t_0$ we get a singular version of the zip/unzip or cap/cup foam generators with a crossing being created at the Morse singularity (the crossings come with an over/under information):



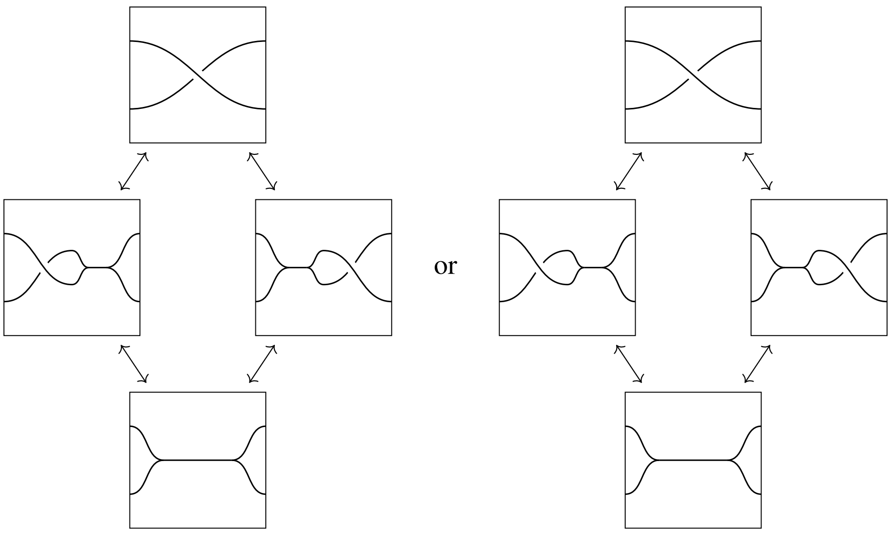
or



Furthermore, the line of multiple points ends at $(u_0, v_0, w_0) \in F^1$, our Morse extremum. Bringing in the time parameter leaves the end of this line to the same point, while the Morse minimum moves in a direction determined by the s coordinate of $\frac{\partial^2 f}{\partial v \partial t}$. One thus reads one of the following moves (we have not yet studied the framing data).



(4.6)



(4.7)

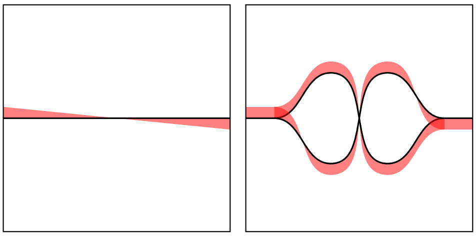
Recall that we have

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial w} = \begin{pmatrix} 0 \\ 0 \\ c_3 \\ 0 \end{pmatrix}.$$

Thus, the framing is purely in the $\vec{z}$ direction, which produces a framing change. Following the seam, one reads

$$\frac{\partial f}{\partial w}_{u_0, v_0+\varepsilon_2, w_0} = \begin{pmatrix} h_1\varepsilon_2 \\ h_2\varepsilon_2 \\ c_3 + h_3\varepsilon_2 \\ h_4\varepsilon_2 \end{pmatrix}.$$

The term h_2 may be assumed to be non-zero. Around the singular point we get a picture of the form:



Depending on the signs for h_2 and h_4 one gets (OM_{10}) or (OM_{11}) (with all their analogs).

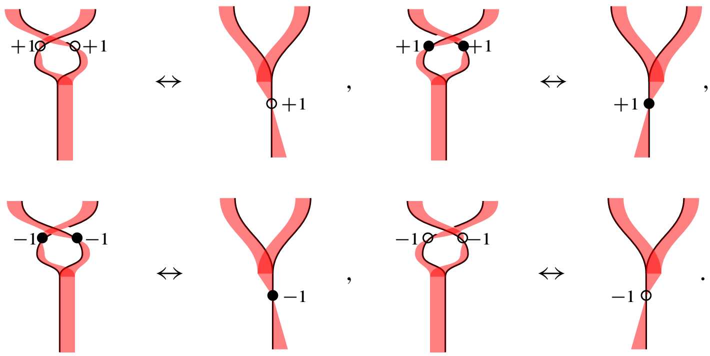
We now consider the second bullet point from the list written at the beginning of this section

$$\vec{z} \in \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right).$$

Recall that we have set the v direction to be the one of the seams. Asking that this is parallel to the $\vec{z}$ direction is a codimension 3 condition, so this is never generic. On the other hand, asking that $\frac{\partial f}{\partial u}$ is a linear combination of the $\vec{z}$ direction and of $\frac{\partial f}{\partial v}$ is a codimension 2 condition. Through time, we expect isolated points. At such a point we can reparametrize u by a scalar multiple of v so that $\frac{\partial f}{\partial u}$ is genuinely vertical.

Interestingly, the resulting move can be read entirely from the ones in equation (3.2). Indeed, forgetting one of the two legs of the web does bring us back to the isotopy investigated in Section 3. Now, when the time moves the vertical vector on the side of the web that only has one strand, one gets the same isotopy as in equation (3.2). When it moves on the other side, focusing on each leg one at a time, we again read the same isotopy. We pretend that the way the two legs entangle can also be read from it: indeed, one leg is the image of the other one by a push along the framing: one just has to take the boundary of the red ribbon in (3.2). The resulting move is equivalent to (OM_{13}) (or to similar ones with other configurations for crossings and twists). Here, the right-hand side of the move just consists in undoing a curl, while on the left-hand side the curl is first passed through the trivalent vertex, creating two curls and extra entanglement of the legs, which get undone progressively.

For the third bullet point on Section 4.6, requiring that $\frac{\partial f}{\partial w} \in \mathrm{Span}_{\mathbb{R}}(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \vec{z})$ at $M \in F^1$ is a codimension 3 condition (2 from restricting to F^1, 1 from the framing assumption). This creates foam generators that we have already investigated in Theorem 3.7, namely,



To get the above generators, one has to assume that we are away from the first two bullet points from the list on Section 4.6, and also that $\frac{\partial^2 f}{\partial u \partial w}$ and $\frac{\partial^2 f}{\partial v \partial w}$ have non-zero y coordinates. To give a bit more detail, one can reduce to the preferred situation where

$$\frac{\partial f}{\partial u} = \begin{pmatrix} a_1 \\ 0 \\ a_3 \\ 0 \end{pmatrix}, \quad a_1 > 0, \quad \frac{\partial f}{\partial v} = \begin{pmatrix} 0 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad b_4 \neq 0, \quad \frac{\partial f}{\partial w} = \begin{pmatrix} 0 \\ 0 \\ c_3 \\ 0 \end{pmatrix}.$$

Consider our usual notations that coordinates in ∂u^2 are denoted by d_i, in $\partial u \partial v$, e_i, in $\partial u \partial w$, f_i, in ∂v^2, g_i, in $\partial v \partial w$, h_i. Then, on a typical slice, the framing grows parallel to a line generated by $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$. The framing vector will be on one or the other side of the tangent plane, the precise side being determined by the following determinant (at $(u_0 + \varepsilon_1, v_0, w_0)$):

$$\begin{vmatrix} f_1\varepsilon_1 & a_1 & e_1\varepsilon_1 \\ f_2\varepsilon_1 & d_2\varepsilon_1 & b_2 \\ f_4\varepsilon_1 & d_4\varepsilon_1 & b_4 \end{vmatrix}.$$

One gets a polynomial in ε_1 of the form $\varepsilon_1(f_4a_1b_2 - f_2a_1b_4 + B\varepsilon_1 + C\varepsilon_1^2)$. Provided $f_4a_1b_2 - f_2a_1b_4 \neq 0$, the sign changes when ε_1 passes zero.

Similarly, following the seam, one gets at $(u_0, v_0 + \varepsilon_2, w_0)$:

$$\begin{vmatrix} h_1\varepsilon_2 & a_1 & g_1\varepsilon_2 \\ h_2\varepsilon_2 & e_2\varepsilon_2 & b_2 \\ h_4\varepsilon_2 & e_4\varepsilon_2 & b_4 \end{vmatrix}.$$

We get that the sign will change if $h_4a_1b_2 - h_2a_1b_4 \neq 0$. This ensures that the line of framing change is generic, in the sense that it goes from the bottom on one side of the seam to the top on the other side.

Through time, there could be more degeneration from the list on Section 4.6, or the two conditions listed above could fail. In each case we get codimensions 1 conditions, so we only have to consider one case at a time. Notice that the second bullet point is not compatible with the third one, as we have assumed that the 3 derivatives form a rank 3 matrix.

In case one has

$$f_4a_1b_2 - f_2a_1b_4 = 0,$$

then the framing line stays on one side of the seam. We get as movie move the invertibility of a half twist passing through a vertex: see (I_3) and (I_4).

In case of a framing change happening at a Morse extremum on the 1-skeleton, corresponding to

$$h_4a_1b_2 - h_2a_1b_4 = 0,$$

one gets the move (OM_{12}).

Finally, if $\frac{\partial f}{\partial v}$ has zero s-coordinate, one gets back the moves (OM_{10}) to (OM_{11}).

4.7. Neighborhood of a point on the 0-dimensional locus

We now consider the neighborhood of a point

$$M = (u_0, v_0, w_0) \in F^0.$$

Restricting to F^0 is already a codimension 3 condition, so at the level of foams no other degeneration will generically occur. Assume as in picture (4.1) that the local chart in u, v, w is such that the tangent vector along both seams at M is parallel to the v direction.

Locally on the vertical seam, one has

$$f(u_0, v_0 + \varepsilon_2, w_0) \sim f(u_0, v_0, w_0) + \begin{pmatrix} b_1\varepsilon_2 \\ b_2\varepsilon_2 \\ b_3\varepsilon_2 \\ b_4\varepsilon_2 \end{pmatrix} + o(\varepsilon_2).$$

Assuming $b_4 \neq 0$, the image of the seam will have a non-trivial component in the s-direction. In order to have the expected generator, one wishes that

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \notin \mathrm{Span}_{\mathbb{R}}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right).$$

Assuming the converse is a codimension 2 condition, so this will never happen, even through time.

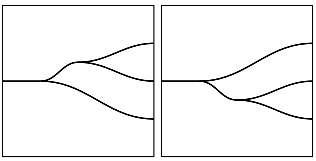
Up to change of coordinates u, v, w and rotation in the x, y plane, one can thus assume that

$$\frac{\partial f}{\partial u} = \begin{pmatrix}a_1\\0\\a_3\\0\end{pmatrix}, \quad a_1 > 0,$$

$$\frac{\partial f}{\partial v} = \begin{pmatrix}0\\b_2\\b_3\\b_4\end{pmatrix},$$

$$\frac{\partial f}{\partial w} = \begin{pmatrix}0\\c_2\\c_3\\0\end{pmatrix}.$$

Finally, one can assume that $c_2 \neq 0$ (which also ensures that the framing is locally constant and non-vertical). This gives the classical foam generator:

.

Through time, we have to analyze the codimension 1 conditions that $b_4 = 0$ or $c_2 = 0$. For codimension reasons, they generically will not happen together.

Let us start with $b_4 = 0$. Then, following the vertical seam, one reads

$$f(u_0, v_0 + \varepsilon_2, w_0) = f(u_0, v_0, w_0) + \begin{pmatrix}\varepsilon_2 b_1\\ \varepsilon_2 b_2\\ \varepsilon_2 b_3\\ \varepsilon_2^2 g_4\end{pmatrix}.$$

For codimension reason, one may assume that $g_4 \neq 0$. One thus sees a bending seam. At $t_0 + \delta$ though, one picks up a contribution of the s-coordinate of $\frac{\partial^2 f}{\partial v \partial t}$, that can be assumed to be non-zero. Depending on the sign of δ, the Morse extremum and the 6-valent point will split in one or the other direction. One gets the movie moves (CL_4) and (CL_5) or their mirror images.

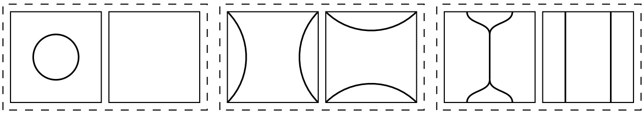
Let us now consider the case where $b_4 \neq 0$ but $c_2 = 0$. Around (u_0, v_0, w_0), the y-coordinate is controlled by

$$b_2\varepsilon_2 + d_2\varepsilon_1^2 + f_2\varepsilon_1\varepsilon_3 + h_2\varepsilon_2\varepsilon_3 + i_2\varepsilon_3^2.$$

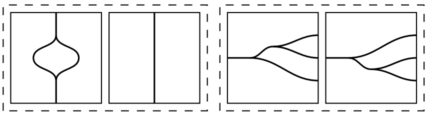
The situation is similar to the one that leads to (3.6), we have either contradictory or parallel effects of $h_2\varepsilon_2\varepsilon_3$ and $f_2\varepsilon_1\varepsilon_3$ depending on the sign of ε_2. One thus gets the movie move (OM_{14}) (and analogs of it with other twist/crossing versions).

4.8. Main statement with half twists

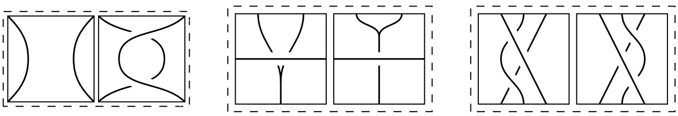
Theorem 4.1. *Framed foams between oriented framed tangled webs admit the following movie generators, in addition to identity movies over tangled webs (each picture is a representative of a family, obtained by crossing changes, changing half twists types and signs, orientations, or taking planar symmetries):*



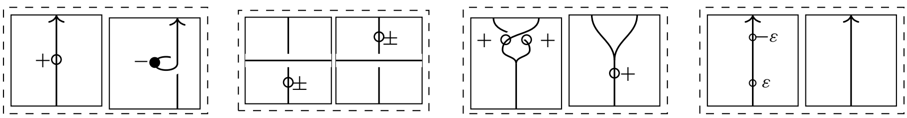
(4.8)



(4.9)



(4.10)

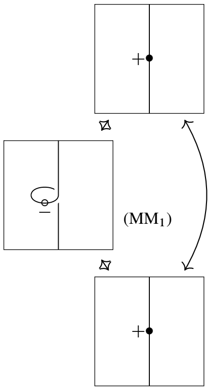


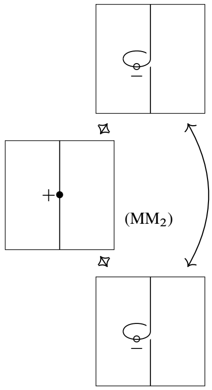
(4.11)

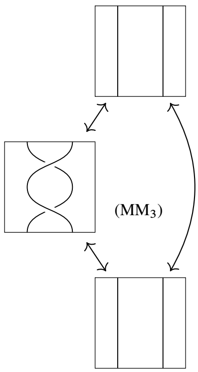
Isotopies of framed foams induce sequences of movie moves from the list given in Section 4.9 (and their analogs under change of crossings, framing, and mirror image).

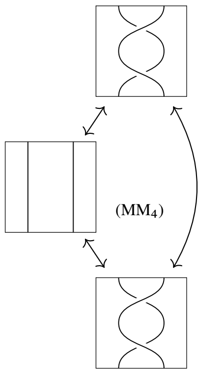
4.9. List of movie moves

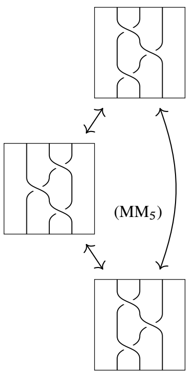
Framed version of classical movie moves.

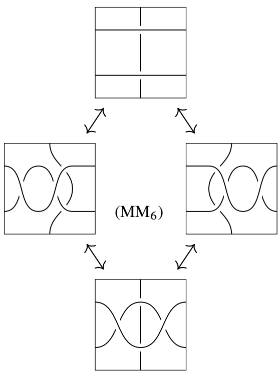


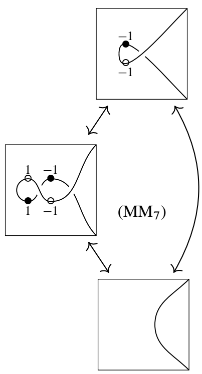


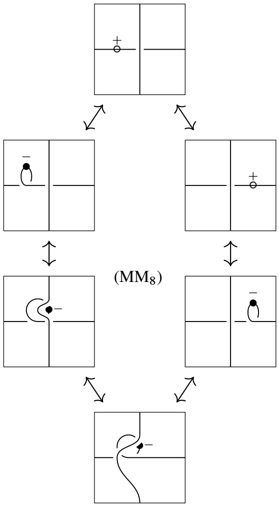




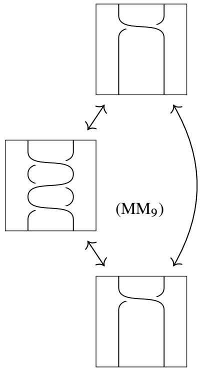




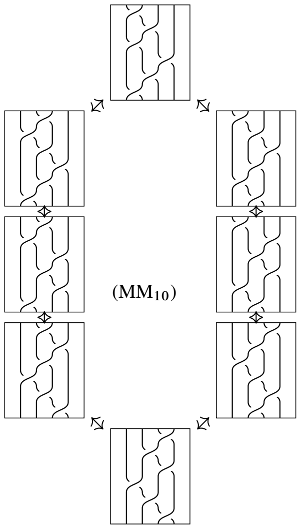




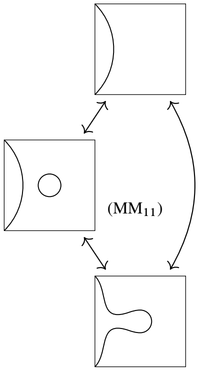
(MM$_8$)

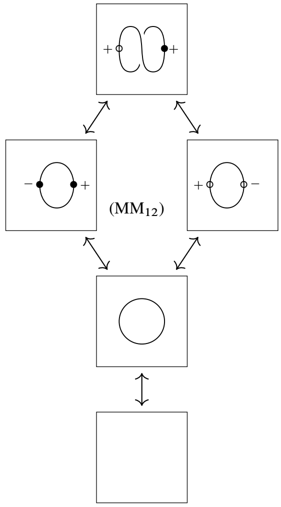


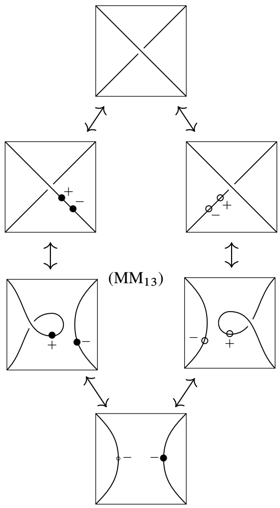
(MM$_9$)

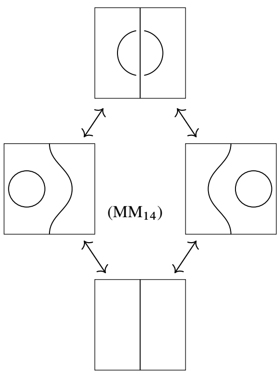


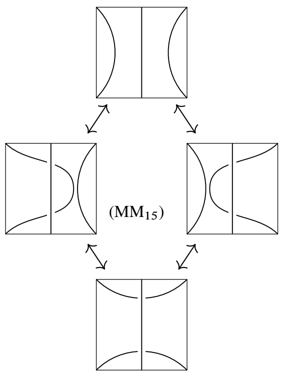
(MM$_{10}$)


(MM$_{11}$)

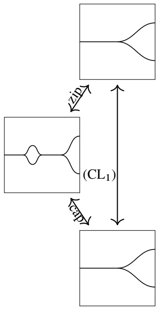

(MM$_{12}$)

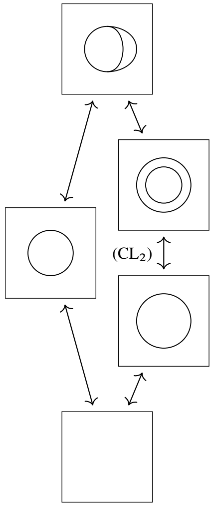

(MM$_{13}$)

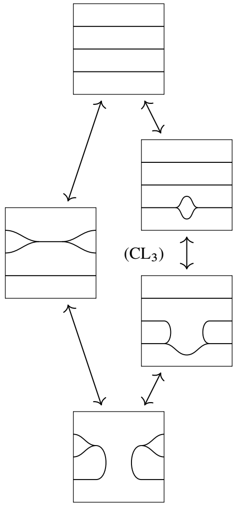

(MM$_{14}$)

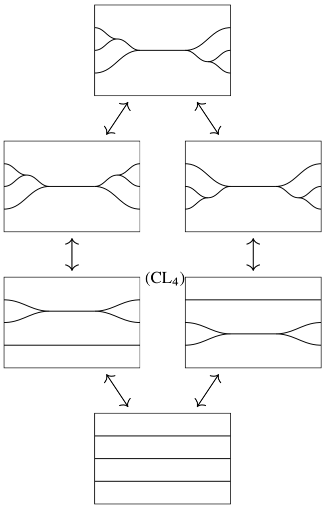

(MM$_{15}$)

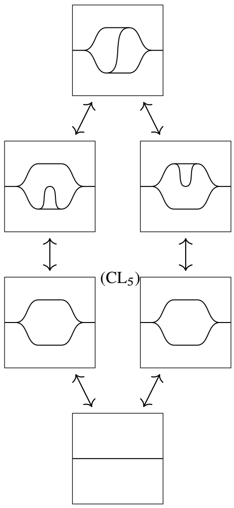
Crossingless moves.



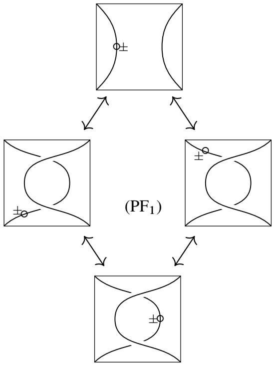


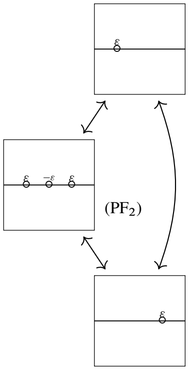


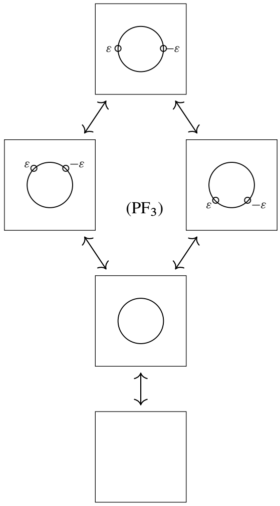


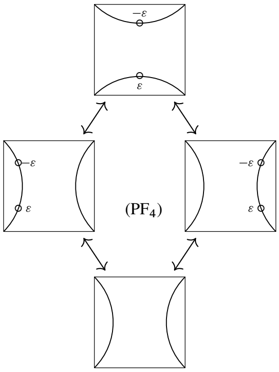


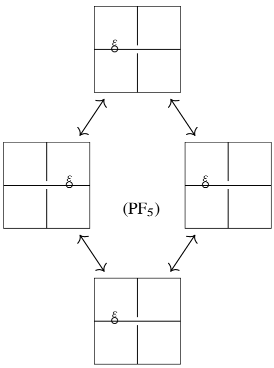
Pure framing moves.

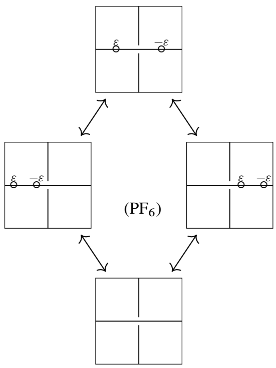


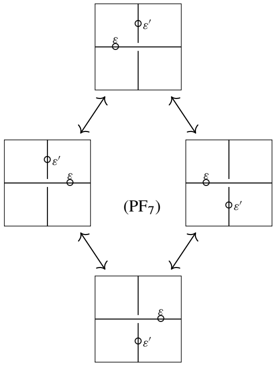


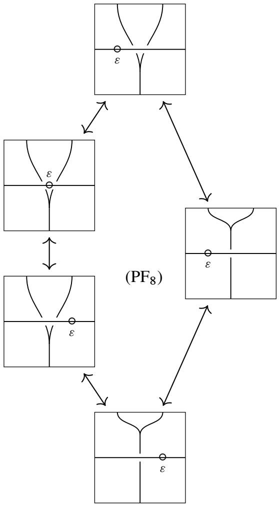


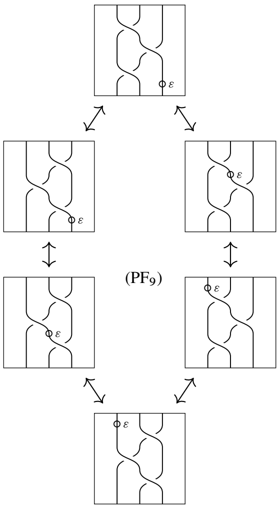


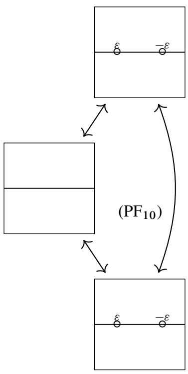




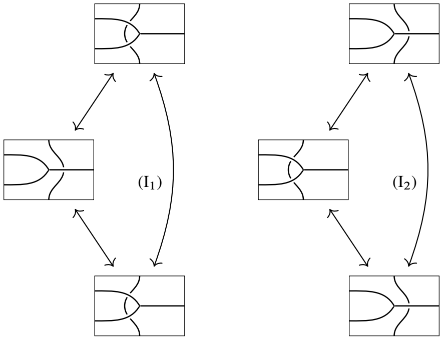

(PF$_7$)

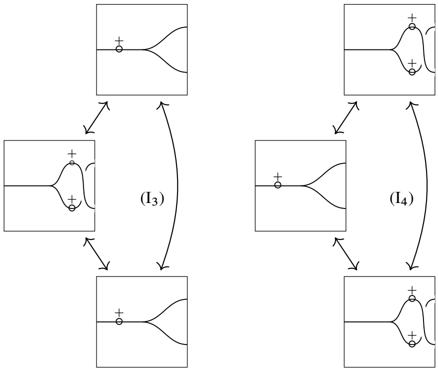

(PF$_8$)


(PF$_9$)

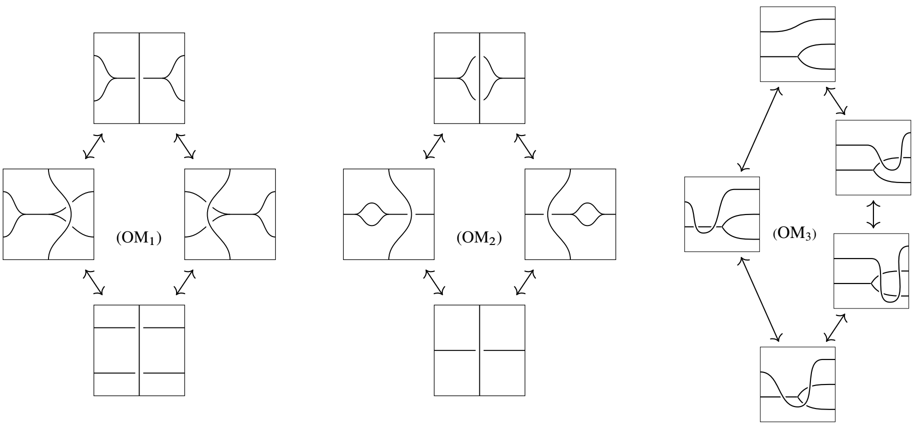

(PF$_{10}$)

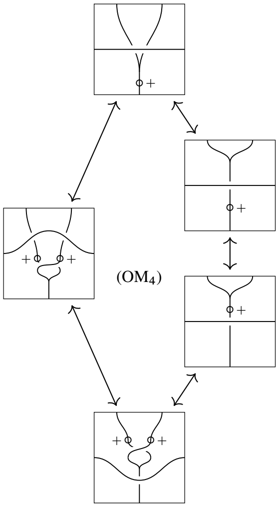
Invertibility of new generators.

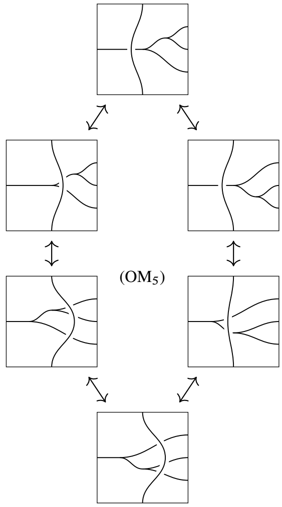


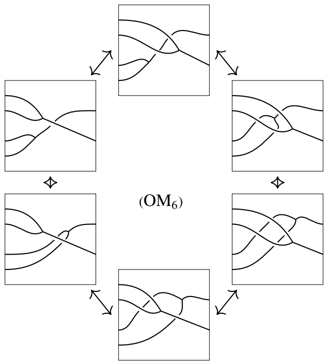


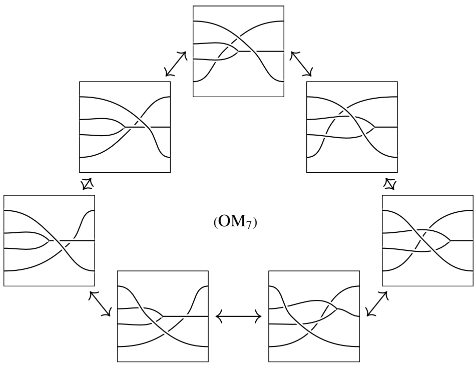
Other moves.

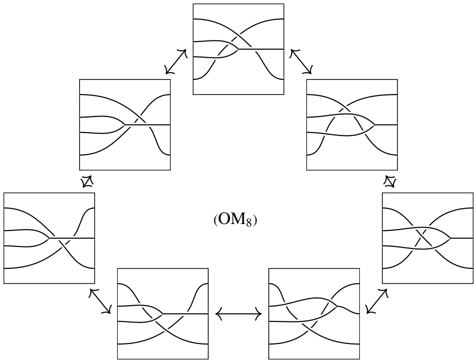


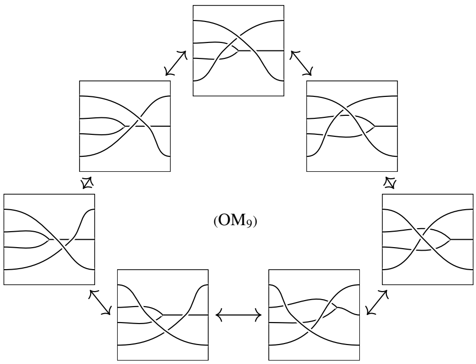

+
+ +
(OM$_4$)
+
+
+ +

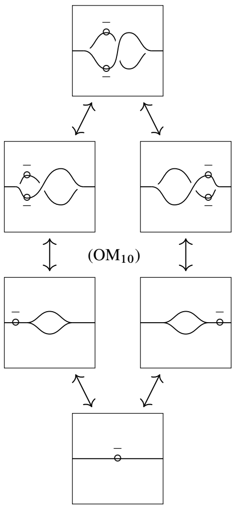

(OM$_5$)

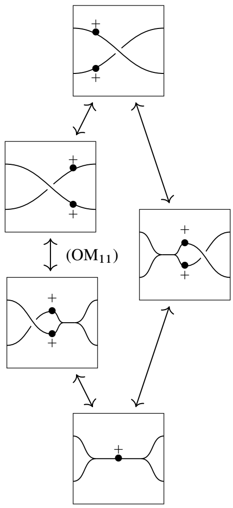

(OM$_6$)

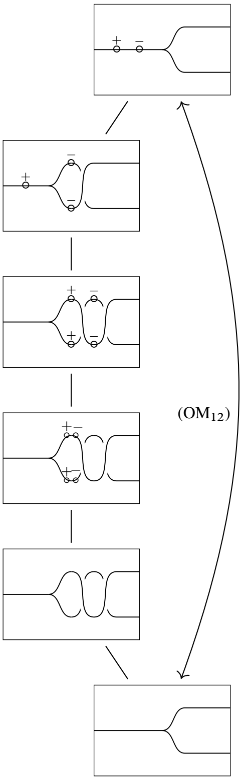

(OM$_7$)


(OM8)

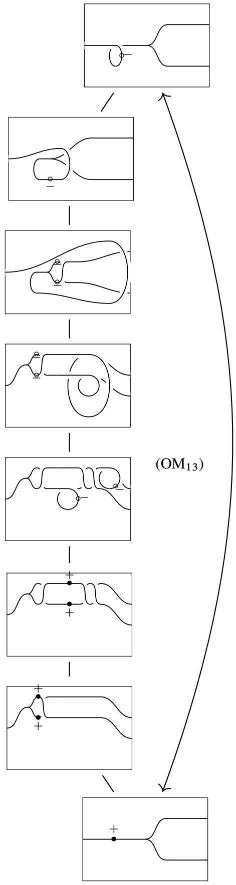

(OM9)


(OM10)
−

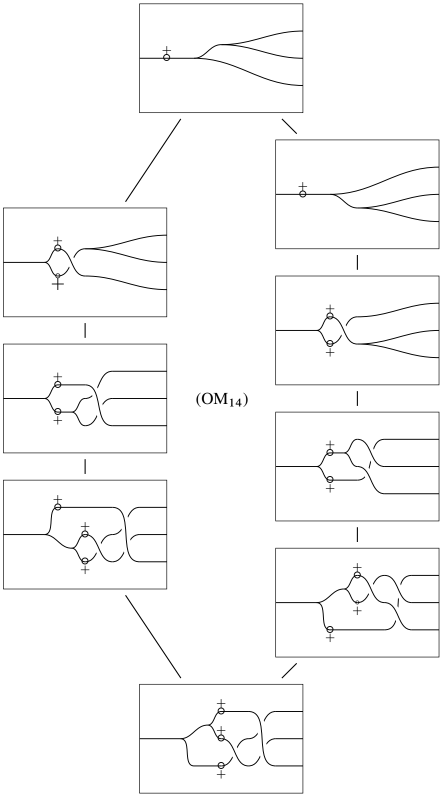

(OM11)
+



(OM$_{12}$)



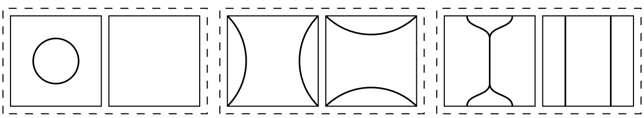
(OM$_{13}$)



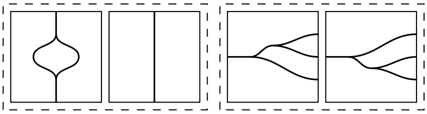
4.10. Main statement with full twists

By restricting our attention to foams that generically bound webs with full twists as in the statement of Theorem 3.8, one can deduce the following statement from the previous one. Indeed, one simply has to pre/post-compose generators by twisting the framing into the standard position, collapsing $\bullet$ and $\circ$ into ◐, where the framing takes a full turn.

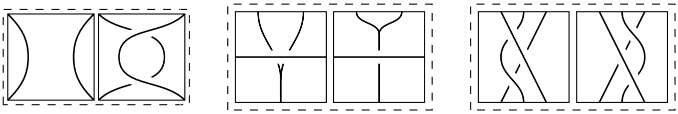
Theorem 4.2. *Framed foams between oriented framed tangled webs with preferred diagrams admit the following movie generators, in addition to identity movies over tangled webs (each picture is a representative of a family, obtained by crossing changes, changing half twists types and signs, orientations, or taking planar symmetries):*



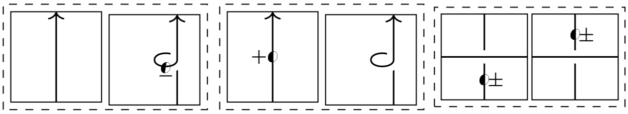
(4.12)



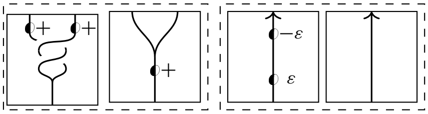
(4.13)



(4.14)



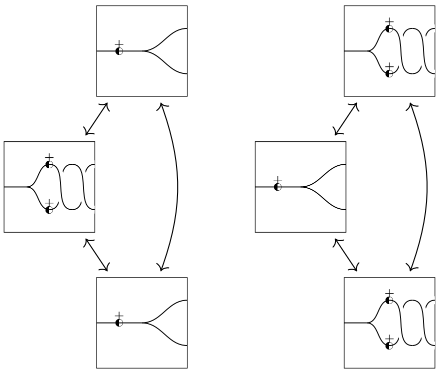
(4.15)



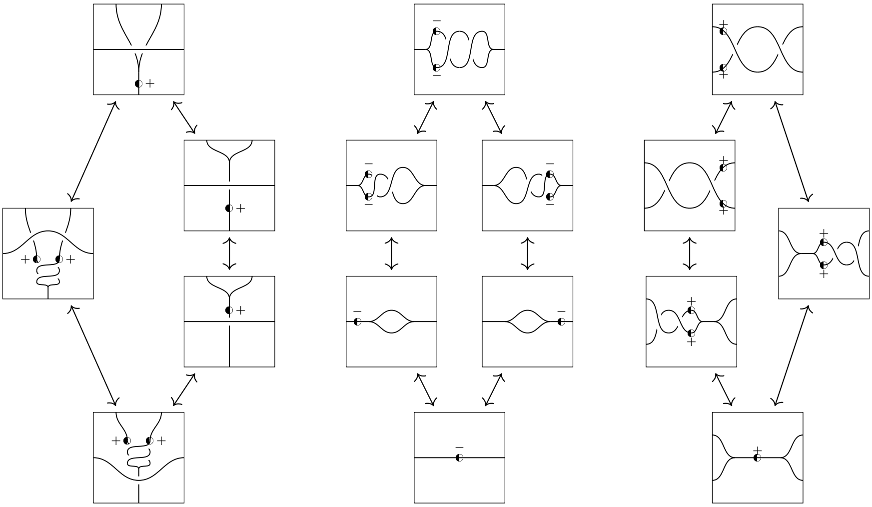
(4.16)

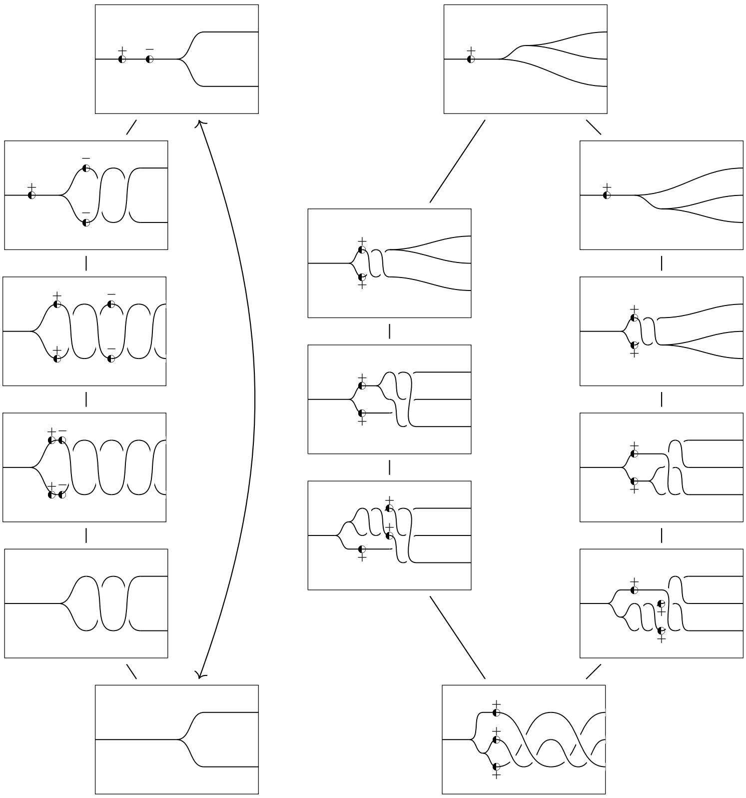
Isotopies of framed foams induce sequences of movie moves that can be deduced from the list in Section 4.9 by merging pairs ∘ and • into ◐ whenever an R_I *move is involved, otherwise replacing half twists by full twists, with the following special cases.*

Invertibility of new generators.



Other moves.





Acknowledgments. We would like to warmly thank Paul Wedrich, who was part of the research project that led to this paper and contributed a lot to the ideas that we present here. H.Q. also would like to thank Scott Carter for generously sharing partial results about movie moves for foams (more than 10 years ago!). Finally, we thank the two reviewers for their very careful reading of our paper and their many helpful suggestions.

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