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Categorifying reduced rings

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Abstract. Given a domain of characteristic zero R, we functorially construct a rigid symmetric monoidal dg-category whose K_0 is R, solving a problem of Khovanov. We also functorially construct, for any reduced commutative ring R, a rigid braided monoidal dg-category whose K_0 is R.

One prevalent insight in mathematics is that many classical invariants admit categorifications. Namely, instead of assigning a number or polynomial to an object X, one assigns an object of some stable category C (throughout this paper, we use category to mean ∞-category in the sense of Joyal and Lurie (see [15], [16], or [17]). If the reader prefers, they may read the phrase 'stable category' as 'dg-category'. In our language, a dg-category over k is a k-linear stable category [7]. All stable categories of interest in this paper are k-linear for some commutative ring k), which encodes richer information about X. The original invariant can then be recovered via applying an Euler characteristic, i.e., a homomorphism out of $K_0(C)$ ($K_0(C)$, i.e., the Grothendieck group of C, is the abelian group generated by $[c]$ for each $c \in C$ with the relation $[c] = [c'] + [c'']$ whenever there is a cofibre sequence $c' \to c \to c''$), to the K_0 class of the categorified invariant.

A prototypical example of this is in algebraic topology, where the Euler characteristic of a space is categorified by its homology. Another example is in algebraic geometry, where the Hilbert polynomial is categorified by coherent cohomology. Finally, in low-dimensional topology, classical knot invariants, such as the Jones polynomial and Alexander polynomial, are categorified by Khovanov homology [11] and Knot Floer homology [19], respectively.

A natural question to ask is whether the type of invariant can obstruct categorifications from existing. For example, the Witten–Reshetikhin–Turaev invariants, constructed in [20, 22], are invariants taking values in the complex numbers, and it is not known whether these always admit a categorification.

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One can build stable categories whose K_0 is an arbitrary rational vector space, as is done in [3]. The reason is that the multiplication by n map on K_0 can be implemented by a functor: for example, the functor sending $c \mapsto \oplus_1^n c$. By taking a filtered colimit along such a functor, since K_0 preserves filtered colimits, one obtains a category whose K_0 is that of C, but with n inverted.

The method above does not naturally produce monoidal categories, which is something usually desired for the target categories of categorifications of invariants, to allow for Künneth-type formulas for the invariants one is categorifying. This is also a fundamental feature of categorifications of manifold invariants that come from field theories. Along these lines, Khovanov posed the following problems.

Problem 1 ([12, Problem 2.3]). Construct a stable[1] monoidal category C with

$$K_0 \cong \mathbb{Q}.$$

Problem 2 ([12, Problem 2.4]). Construct a stable monoidal category C with

$$K_0 \cong \mathbb{Z}\left[\frac{1}{n}\right].$$

Khovanov and Tian [13] solved Problem 2 in the case $n = 2$, but the general case was previously open.

We solve both problems below in Theorem 9, even producing categories that are rigid[2] symmetric monoidal. The fundamental input is the ability to produce a category whose K_0 is a characteristic zero field, which we construct as an ultraproduct of the stable module categories of $\mathbb{F}_p[C_p]$. We further refine this in Theorem A below.

Let $\mathrm{Ring}^0_{\mathrm{red}}$ be the category of commutative rings which are reduced such that every minimal prime ideal is characteristic zero[3]. Let $\mathbb{Z}\mathrm{Cat}^{\infty}_{\mathrm{rig}}$ denote the category of rigid symmetric monoidal $\mathbb{Z}$-linear stable categories.

Theorem A. *There is a filtered colimit preserving functor* $C^{\infty}_{(-)} : \mathrm{Ring}^0_{\mathrm{red}} \to \mathbb{Z}\mathrm{Cat}^{\infty}_{\mathrm{rig}}$ *and a natural isomorphism of rings* $K_0(C^{\infty}_R) \cong R$.

To illustrate the purpose of Theorem A, $C^{\infty}_{\mathbb{Q}}$ is a rigid symmetric monoidal $\mathbb{Z}$-linear category with a continuous action of the absolute Galois group of $\mathbb{Q}$ such that $K_0(C^{\infty}_{\bar{\mathbb{Q}}}) \cong \bar{\mathbb{Q}}$ with its standard action.

We next remove the characteristic zero assumption of Theorem A in the setting of braided monoidal categories. The key input here is the category of tilting modules in

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[1] By a monoidal stable category we mean an algebra object in the category of categories that is stable such that the tensor product preserves finite colimits in both variables.

[2] We recall that a monoidal category is rigid if every object is strongly left and right dualizable.

[3] For example, this includes all domains of characteristic zero.

the mixed case for Lusztig's quantum group $\mathfrak{sl}_2$, which we semisimplify and tensor with the stable module category of $\mathbb{F}_p[C_p]$ in order to categorify finite fields. Let $\mathrm{Ring}_{\mathrm{red}}$ be the category of reduced commutative rings and $\mathbb{Z}\mathrm{Cat}^2_{\mathrm{rig}}$ the category of rigid braided monoidal $\mathbb{Z}$-linear stable categories.

Theorem B. *There is a filtered colimit preserving functor* $C^2_{(-)} : \mathrm{Ring}_{\mathrm{red}} \to \mathbb{Z}\mathrm{Cat}^2_{\mathrm{rig}}$ *and a natural isomorphism of rings* $K_0(C^2_R) \cong R$.

There is a natural condition on a category for which our constructions are essentially sharp.

Definition 3. A braided monoidal category C is trace-zero if, for every nilpotent endomorphism $f : c \to c$, the trace of f is zero.

Our constructions of braided monoidal categories as described above give the following result[4].

Theorem C. *For every reduced commutative ring R, there is a $\mathbb{Z}$-linear ribbon braided monoidal trace-zero stable category C_R with an isomorphism of rings*

$$K_0(C_R) \cong R.$$

The following result is an obstruction to producing trace-zero categorifications of rings that are more symmetric than those in Theorem C.

Theorem 4 ([8, Theorem 5.15]). *Let C be a ribbon braided monoidal trace-zero stable*[5] *category with $[\mathbb{1}_C, \mathbb{1}_C]$ a characteristic p ring. If $v \in \mathrm{Aut}(\mathbb{1}_C)$ is the twist automorphism, suppose that v^ℓ is unipotent for some ℓ coprime to p. Let n be the smallest integer such that $p^n - 1$ is divisible by ℓ^2. Then, for any object $V \in C$, we have* $\dim V \in \mathbb{F}_{p^n}$. *In particular, $K_0(C)$ cannot contain any field extension of $\mathbb{F}_{p^n}$.*

In particular, for trace-zero symmetric monoidal categories, the dimension homomorphism $K_0(C) \to \mathrm{End}(\mathbb{1}_C)$ must land inside $\mathbb{F}_p$ (see [8, Lemma 3.13]), so $K_0(C)$ cannot contain any field extension of $\mathbb{F}_p$.

Despite the above obstruction, there are a number of possible directions for improvement to our theorems. Notably, the categories of Theorem A are not idempotent-complete, so one could ask whether it is possible to build idempotent-complete categories. Another possible improvement to Theorem A would be an extension to all

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[4] Note that the dimension gives a ring homomorphism $K_0(C_R) \to \mathrm{End}(\mathbb{1}_{C_R})$ to endomorphism ring of the unit. In particular, $\mathrm{End}(\mathbb{1}_{C_R})$ is characteristic p when $p = 0$ in R.

[5] The cited result is for additive 1-categories rather than stable categories, but the relevant part of the assumptions only depends the homotopy 1-category of C which is additive, so this is okay.

commutative rings, or at least the removal of the assumption about the minimal primes being characteristic zero. The method of proof of Theorem A works without the characteristic zero assumption, given a positive solution to the following conjecture.

Conjecture 5. For each prime p, there exists a rigid symmetric monoidal $\mathbb{Z}$-linear category C with $K_0(C) \cong \overline{\mathbb{F}}_p$.

By Theorem 4, any positive solution to Conjecture 5 must not be trace-zero.

1. Examples

In this section, we exhibit symmetric monoidal categories whose K_0 is an arbitrary domain of characteristic zero. Combining this with categories built from $\mathfrak{sl}_2$-tilting modules in the mixed case, we produce ribbon categories whose K_0 is an arbitrary reduced ring.

We first introduce some categories and operations that we use. Consider the category $\mathrm{StMod}^{\omega}_{C_p}$,[6] the compact objects of the stable module category of the group ring $\mathbb{F}_p[C_p]$ (see, for example, [18, Section 2] for an overview of this category). This is an idempotent-complete rigid symmetric monoidal $\mathbb{Z}$-linear category, and

$$K_0(\mathrm{StMod}^{\omega}_{C_p}) \cong \mathbb{F}_p.$$

The construction in Lemma 6 below uses an ultraproduct of the categories $\mathrm{StMod}^{\omega}_{C_p}$ over the set $\mathbb{P}$ of primes. We briefly recall how ultraproducts work, referring the reader to [2, Section 3] or [8, Section 3.6] for more details. Given a family $C_\alpha, \alpha \in A$ of objects in a category D (D could be the category of categories), we can choose a non-principal ultrafilter U on the set A, which is the data of a maximal proper filter in the poset of subsets of A containing all cofinite subsets. Then, an ultraproduct[7], which we denote by $\Pi'_{\alpha\in A} C_\alpha$, is the object of D defined as the filtered colimit along the subsets of S in U of the product $\Pi_{s\in S} C_s$. An ultrapower of an object C is an ultraproduct where all of the C_α, are the same object, C.

Lemma 6. *$\Pi'_{p\in\mathbb{P}} \mathrm{StMod}^{\omega}_{C_p}$ is an idempotent-complete rigid symmetric monoidal $\mathbb{Z}$-linear category with K_0 a field of characteristic zero. It is possible to choose an ultrafilter so that K_0 contains $\overline{\mathbb{Q}}$.*

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[6] The category $\mathrm{StMod}^{\omega}_{C_p}$ can also be described as the category of perfect modules over the $\mathbb{E}_\infty$-ring $\mathbb{F}_p^{tC_p}$, the ring whose homotopy ring is the Tate cohomology of $\mathbb{F}_p$ with a trivial C_p-action.

[7] The ultraproduct depends on the choice of ultrafilter, but we disregard this in our notation.

Proof. The functor K_0 preserves arbitrary products and filtered colimits, so it preserves ultraproducts. Since $K_0(\mathrm{StMod}^{\omega}_{C_p})$ is $\mathbb{F}_p$, the result follows, as an ultraproduct of fields of characteristic p for different p is one of characteristic 0. It follows from [1, Theorem 5] that one can choose an ultrafilter so that K_0 contains $\overline{\mathbb{Q}}$. ■

Remark 7. We chose a non-principal ultrafilter on an infinite set in the proof of Lemma 6 which is a non-constructive operation, but it is possible to do the proof in a constructive way, with the cost of the category not being idempotent-complete. Namely, one can simply choose a filter on the set of primes such that the corresponding colimit of products of prime fields still contains $\overline{\mathbb{Q}}$ as in the proof of [1, Theorem 5]. By then considering the subcategory of objects whose K_0 class lives in $\overline{\mathbb{Q}}$, we obtain an explicit construction of a category with K_0 isomorphic to $\overline{\mathbb{Q}}$.

Definition 8. Given a rigid braided monoidal category C, and an object $x \in C$, we can form the dual x^*, giving an equivalence $(-)^* : C \cong C^{\mathrm{op}}$. We refer to this as the *dual functor*.

Theorem 9. *For any reduced commutative ring R such that every minimal prime ideal of R is characteristic* 0*, there is a rigid symmetric monoidal $\mathbb{Z}$-linear category C such that $K_0(C) \cong R$, and the dual functor is the identity on K_0.*

Proof. We claim that the set of commutative rings R satisfying the theorem is closed under products, ultraproducts, and subrings. The claim for products and ultraproducts follows since K_0 commutes with products and ultraproducts. For subrings, if

$$K_0(C) \cong R, \quad R' \subset R$$

is a subring and the dual is the identity on $K_0(C)$, then the full subcategory of C consisting of objects whose K_0 class is in R' is a category showing that the result is true for R'.

Any reduced ring embeds into the product of its localizations at all of its minimal primes. Indeed, this is because the kernel of a ring mapping into all of its residue fields is exactly its nilradical, and the kernel of any residue field contains the kernel of some minimal prime ideal. In the case of a minimal prime, the localization at the prime is exactly the residue field associated to that prime.

By assumption, each of these localizations is a field of characteristic zero, so we have thus reduced the theorem to the case R is a characteristic zero field. An arbitrary field of characteristic zero embeds into an algebraically closed field, and ultrapowers of $\overline{\mathbb{Q}}$ give examples of algebraically closed fields of arbitrarily large transcendence degree, so it suffices to produce a rigid symmetric monoidal $\mathbb{Z}$-linear category with the dual acting by the identity such that K_0 contains $\overline{\mathbb{Q}}$. Now, we apply Lemma 6 and conclude by observing that the dual functor induces the identity on $K_0(\mathrm{StMod}^{\omega}_{C_p})$. ■

We next explain an improvement of Theorem 9 in the setting of ribbon braided monoidal categories. The key example of a ribbon braided monoidal category that we use is constructed in the following proposition.

Proposition 10. *For every pair of distinct primes* $p, \ell > 2$*, there is an idempotent-complete semisimple* $\overline{\mathbb{F}}_p$*-linear ribbon category* C *with* $K_0(C) \cong \mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$.

Proof. Let q be a primitive ℓ-th root of unity in $\overline{\mathbb{F}}_p$. We consider the category $\mathrm{Tilt}^{\overline{\mathbb{F}}_p, q}$ of tilting modules of Lusztig's divided power quantum group associated to $\mathfrak{sl}_2$ over the field $\overline{\mathbb{F}}_p$. We refer the reader to [21] for a good reference for this category: it is an idempotent-complete $\overline{\mathbb{F}}_p$-linear ribbon category with indecomposable objects $T(v)$ for $v \geq 0$, where $T(0)$ is the unit.

Let C' be the semisimplification of this category: see [9] for an overview of this construction. C' is then an idempotent-complete $\overline{\mathbb{F}}_p$-linear semisimple ribbon category. We will show that an appropriate subcategory $C \subset C'$ has K_0 isomorphic to $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$. The simple objects in the semisimplification C' correspond to indecomposable objects in $\mathrm{Tilt}^{\overline{\mathbb{F}}_p, q}$ with nonzero categorical dimension, which is exactly $T(i)$ for $0 \leq i \leq \ell - 2$ by [21, Proposition 3.23]. The tensor product by [21, Lemma 4.1] agrees with the truncated Clebsch–Gordon rule for the tensor product in the Verlinde category Ver_p [8, Section 4.2].

We let C be the full subcategory of C' generated by $T(i)$ for i even: this is also an idempotent-complete k-linear semisimple ribbon category since the even objects are closed under tensor product, duals and contain the unit. Then, $K_0(C)$ agrees with $K_0(\mathrm{Ver}_p^+)$, which is indeed $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$ by [4, Theorem 4.5]. ■

Remark 1.1. The key properties of the categories of Proposition 10 that we use are that their mod p reductions can contain arbitrarily large field extensions of $\mathbb{F}_p$. If we were interested in constructing categories that are just monoidal as opposed to braided monoidal, there are other candidates, such as the categories constructed in [14], whose K_0 is an arbitrary cyclotomic extension of the integers.

The next goal is to realize the operation $K_0 \mapsto K_0/(p)$ at the level of categories. To do this, we use the tensor product of $\mathbb{F}_p$-linear stable categories. If C, D are $\mathbb{F}_p$-linear stable categories, then $C \otimes_{\mathbb{F}_p} D$ is also such a category that is generated as a stable category by objects $c \otimes d$, $c \in C$, $d \in D$, and $\mathrm{map}(c \otimes d, c' \otimes d') \cong \mathrm{map}(c, c') \otimes_{\mathbb{F}_p} \mathrm{map}(d, d')$. In particular, $K_0(C \otimes_{\mathbb{F}_p} D)$ has a surjective map from $K_0(C) \otimes K_0(D)$.

Proposition 11. *Let* C *be an* $\mathbb{F}_p$*-linear abelian category. Then, if* $D^b(C)$ *is the bounded derived category of* C*, then*

$$K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{StMod}^{\omega}_{C_p}) \cong K_0(C)/(p).$$

Proof. Let $C' = D^b(C) \otimes_{\mathbb{F}_p} \mathrm{StMod}^{\omega}_{C_p}$. There is a surjective map from

$$K_0(D^b(C)) \otimes K_0(\mathrm{StMod}^{\omega}_{C_p}) \to K_0(C').$$

By the Gillet–Waldhausen theorem, $K_0(D^b(C)) \cong K_0(C)$, and since

$$K_0(\mathrm{StMod}^{\omega}_{C_p}) \cong \mathbb{F}_p,$$

the above map is really a surjective map $K_0(C)/(p) \to K_0(C')$. It suffices to show that this map is injective.

Let $\mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p)$ denote the bounded derived category of finite-dimensional representations of the cyclic group C_p over the field $\mathbb{F}_p$ so that $\mathrm{StMod}^{\omega}_{C_p}$ is the quotient of $\mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p)$ by the full stable subcategory generated by the projective representation, which is equivalent to $\mathrm{Mod}^{\omega}(\mathbb{F}_p[C_p])$.

Tensoring with $D^b(C)$, and applying the connective K-theory functor (see [5]), we get a sequence

$$K(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Mod}^{\omega}(\mathbb{F}_p[C_p])) \to K(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p)) \to K(C').$$

This is a cofibre sequence since connective K-theory sends localization sequences of categories to cofibre sequences if the quotient is surjective on K_0.

Let $f : \mathrm{Mod}(\mathbb{F}_p)^{\omega} \to \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p)$ denote the functor giving an $\mathbb{F}_p$-module the trivial C_p-action. It suffices to show that the natural map

$$K_0(D^b(C)) \xrightarrow{K_0(D^b(C) \otimes f)} K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p))$$

is an equivalence, and that the image of $K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Mod}^{\omega}(\mathbb{F}_p[C_p]))$ under the second map is divisible by p so that the kernel of the map $K_0(C)/(p) \to K_0(C')$ is contained in the ideal (p) and hence is zero.

The first claim follows from Lemma 12 below. For the second, we note that the functor f has a retraction

$$g : \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p) \to \mathrm{Mod}^{\omega}(\mathbb{F}_p)$$

given by taking the underlying $\mathbb{F}_p$-module. $K_0(D^b(C) \otimes g)$ is an equivalence since it is an inverse of $K_0(D^b(C) \otimes f)$, which we have already seen is an isomorphism. Thus, it suffices to show that the composite

$$\begin{aligned} K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Mod}^{\omega}(\mathbb{F}_p[C_p])) &\to K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p)) \\ &\xrightarrow{g} K_0(D^b(C) \otimes_{\mathbb{F}_p} \mathrm{Mod}^{\omega}(\mathbb{F}_p)) \end{aligned}$$

has image in the ideal (p). This is because this functor has a filtration given by the filtration of $\mathbb{F}_p[C_p]$ by the powers of the maximal ideal, whose associated graded is a direct sum of p copies of the functor given by base change along the map of rings $\mathbb{F}_p[C_p] \to \mathbb{F}_p$. ■

Lemma 12. *Let C be an $\mathbb{F}_p$-linear category with bounded t-structure. Then, the natural map $K(C) \xrightarrow{K(C\otimes f)} K(C \otimes_{\mathbb{F}_p} \mathrm{Rep}^{\omega}_{\mathbb{F}_p}(C_p))$ is an equivalence, and K_{-1} of both categories vanish.*

Proof. To prove the lemma, we will show that $C \otimes f$ satisfies the conditions of [6, Theorem 1.3]. The image of f generates the target since each finite-dimensional representation of C_p over $\mathbb{F}_p$ always has a nonzero fixed vector. It thus suffices to show that the functor is fully faithful on the heart. If $a, b \in C^{\heartsuit}$, then $\pi_* \operatorname{map}(C \otimes f(a), C \otimes f(b)) \cong \pi_* \operatorname{map}(a, b) \otimes_{\mathbb{F}_p} H^{-*}(C_p; \mathbb{F}_p)$. Since $\operatorname{map}(a, b)$ is coconnective and $H^{-*}(C_p; \mathbb{F}_p)$ is concentrated in nonpositive degrees with $H^0 \cong \mathbb{F}_p$, it follows that

$$\pi_0 \operatorname{map}(a, b) \cong \pi_0 \operatorname{map}(C \otimes f(a), C \otimes f(b)),$$

i.e., that the functor is fully faithful. ■

Trace-zero categories are closed under the operations we use.

Lemma 13. *If C is a rigid braided monoidal abelian category, then C is trace-zero. Any rigid braided monoidal stable category with bounded t-structure is also trace-zero. Trace-zero stable categories are also closed under idempotent completion, products, and filtered colimits.*

Proof. The fact that C is trace-zero is well known: given a nilpotent endomorphism $f : c \to c$, one can filter c by the kernel of the powers of f to find that the map f is zero on the associated graded. Thus, the trace of f is zero since it is on the associated graded.

To see that a rigid braided monoidal category with bounded t-structure is trace-zero, we use the fact that every map is canonically filtered by the Postnikov tower, with associated graded maps in shifts of the heart. In particular, using additivity of traces, if f is a nilpotent endomorphism of X, it suffices to check for each i that the map

$$\pi_i^{\heartsuit} f : \pi_i^{\heartsuit} X \to \pi_i^{\heartsuit} X$$

has trace zero. Each $\pi_i^{\heartsuit} f$ is nilpotent since f is.

The heart of the t-structure is an abelian category, and so, the proof of the previous case applies to $\pi_i^{\heartsuit} f$.[8]

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[8] Note that this does not directly reduce to the previous case, since the dual of an object in the heart may not be in the heart.

We omit the proof that trace-zero categories are closed under idempotent completion, products, and filtered colimits, as it is straightforward. ■

We are now ready to produce ribbon categorifications of all reduced rings.

Theorem 14. *Given any reduced ring R, there exists a trace-zero ribbon braided monoidal $\mathbb{Z}$-linear stable category C with $K_0(C) \cong R$ such that the dual is the identity on K_0.*

Proof. As in Theorem 9, the set of R satisfying the theorem are closed under subrings and ultraproducts. The collection of reduced rings is generated under subrings and ultraproducts by the collection of finite fields $\mathbb{F}_{p^n}$.

Let C be a tensor product (relative to $\overline{\mathbb{F}}_p$) of categories coming from Proposition 10. Since these categories are semisimple over an algebraically closed field, C is also semisimple, and the K_0 of this tensor product is the tensor product of the K_0 of each factor. By Proposition 11, $D^b(C) \otimes \mathrm{StMod}^{\omega}_{C_p}$ is a ribbon braided monoidal $\mathbb{Z}$-linear stable category with K_0 a tensor product of the mod p reductions of $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$ for various $\ell > 2$. To see it is trace-zero, we first observe that since C is semisimple, the underlying stable category of $D^b(C) \otimes \mathrm{StMod}^{\omega}_{C_p}$ is a product of copies of $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathrm{StMod}^{\omega}_{C_p}$ corresponding to the simple objects of C. Now, the category is trace-zero for the same reason that $\mathrm{StMod}^{\omega}_{C_p}$ is, which we now explain.

Given a nilpotent endomorphism f of an object $x \in D^b(C) \otimes \mathrm{StMod}^{\omega}_{C_p}$, we can choose a lift $\tilde{f}$ of f to an endomorphism of an object $\tilde{x}$ in the heart of $D^b(C) \otimes \mathrm{Rep}_{\mathbb{F}_p}(C_p)$ containing no projective summand. Traces of endomorphisms factor through the semisimplification, which kills the projective representation. By assumption $\tilde{f}^n$, factors through some projective representation for sufficiently large n, so $\tilde{f}$ is nilpotent in the semisimplification, and thus has zero trace.

Since $\mathbb{F}_{p^n}$ is the tensor product of finite fields of the form $\mathbb{F}_{p^{q^n}}$, it suffices to show that, for each p, q, n with p, q prime, there is an $\ell > 2$ such that $\mathbb{F}_{p^{q^n}}$ is contained in the mod p reduction of $\mathbb{Z}[\zeta_\ell]$. Indeed, to see that the claim for $\mathbb{Z}[\zeta_\ell]$ implies the one for $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$, the claim for $\mathbb{Z}[\zeta_\ell]$ is equivalent to the claim that $\mathbb{F}_{p^{q^n}}$ is contained in all the characteristic p residue fields of $\mathbb{Z}[\zeta_\ell]$. Since

$$\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}] \to \mathbb{Z}[\zeta_\ell]$$

is a degree 2 extension, it follows that, in $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$, the residue fields mod p contain a subextension of index at most 2, and so in particular contain $\mathbb{F}_{p^{q^{n-1}}}$.

To see the claim for $\mathbb{Z}[\zeta_\ell]$, we first observe that whenever there is a prime ℓ dividing $|\mathbb{F}_{p^{q^n}}|^\times$ that does not divide $|\mathbb{F}_{p^{q^{n-1}}}|^\times$, then $\mathbb{Z}[\zeta_\ell]/p$ contains $\mathbb{F}_{p^{q^n}}$.

We thus need to show that, for infinitely many values of n,

$$|\mathbb{F}_{p^{q^n}}|^\times = p^{q^n} - 1$$

contains primes not dividing $p^{q^{n-1}}-1$. This will be true if

$$\gcd\left(\frac{p^{q^n}-1}{p^{q^{n-1}}-1}, p^{q^{n-1}}-1\right) = \gcd(q, p^{q^{n-1}}-1)\,^{9}$$

is 1, i.e., if p is not congruent to 1 mod q.

If $q|p-1$, then, for large n, the q-adic valuation of $p^{q^{n-1}}-1$ is larger than 1. Thus, since $\gcd(\frac{p^{q^n}-1}{p^{q^{n-1}}-1}, p^{q^{n-1}}-1) = q$ by the above equality, it follows that for large n, the q-adic valuation of $\frac{p^{q^n}-1}{p^{q^{n-1}}-1}$ is 1. Thus, $\frac{p^{q^n}-1}{p^{q^{n-1}}-1}$ must have a prime factor that is not a prime factor of $p^{q^{n-1}}-1$, allowing us to conclude. ■

The categories constructed in Theorem 14 and Theorem 9 are stable categories that do not arise from additive categories. Moreover, they are not idempotent-complete, and the dual functor is the identity. Therefore, we ask the following questions.

Question 15. Given R a commutative ring, when does there exist an additive rigid symmetric monoidal category C with $K_0(C) \cong R$?

Question 16. Given R a commutative ring, when does there exist an idempotent-complete rigid symmetric monoidal stable category C with $K_0(C) \cong R$?

Question 17. Given R a commutative ring with an involution, when does there exist a rigid symmetric monoidal stable category C such that $K_0(C)$ acted on by the dual functor is equivalent to R?

2. Universal and functorial examples

Next, we show it is possible to produce categories whose K_0 admit maps from a ring R equipped with universal witnesses of relations in K_0. We then combine this with the results of the previous theorem to prove Theorem A and Theorem B. Given a symmetric monoidal stable category C, let $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(C))$ denote the category of $\mathbb{E}_n$-monoidal C-linear categories[10]. For a discrete commutative ring R, we use $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(R)^{\heartsuit})$ to denote the category of (discrete) associative R-algebras for $n = 1$ and commutative R-algebras for $n > 1$. For a category E and object $e \in E$, we use $E_{e/}$ to denote the slice category of objects in E equipped with a map from e.

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[9] This equality follows by writing $\frac{p^{q^n}-1}{p^{q^{n-1}}-1} = 1 + \cdots + p^{q^{n-1}} + \cdots + p^{(q-1)q^{n-1}}$ and using that $p^{q^{n-1}} \equiv 1 \bmod (p^{q^{n-1}}-1)$.

[10] By an $\mathbb{E}_n$-monoidal C-linear category we mean an $\mathbb{E}_n$-algebra object (see [16, Definition 5.1.0.4] and [16, Definition 2.1.3.1]) in the symmetric monoidal category of modules over C in the category of categories such that the action map preserves finite colimits in both variables.

Proposition 18. *Let* $1 \leq n \leq \infty$*, and let* C *be a symmetric monoidal stable category and* C' *a* $\mathbb{E}_n$*-monoidal* C*-linear category. There is a functor*

$$D_{(-)} : \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(K_0(C))^{\heartsuit})_{K_0(C')/} \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(C))_{C'/}$$

and a natural transformation

$$\eta_{(R)} : R \to K_0(D_R)$$

such that η_R *makes* D_R *a versal*[11] *object in* $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(C))_{C'/}$ *equipped with a map* $R \to K_0(D_R)$ *in* $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(K_0(C))^{\heartsuit})_{K_0(C')/}$*. The functor* $D_{(-)}$ *functorially depends on* C, C'*, and a choice of representative of each* K_0*-class of* $K_0(C')$*, and preserves filtered colimits with respect to all parameters. We can moreover require that* $D_{(-)}$ *take values and have a versal property instead in the subcategory of rigid categories.*

Proof. We will prove the proposition simultaneously with and without the rigid assumption, using (rigid) to indicate which operations must be done as rigid categories with the rigid assumption.

For each class in $x \in K_0(C')$, let us fix a choice of representative of the K_0-class of x, which we call O_x. Let us fix $R \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(K_0(C))^{\heartsuit})_{K_0(C')/}$. We choose a presentation of R as an associative $K_0(C)$-algebra under C' as follows: let each $r \in R$ itself be the set of generators. To write the set of relations, consider all formal expression of the form $\sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij}$, where x_{ij} is an element of the disjoint union $R \amalg K_0(C')$. For each such data, we get a relation if the relation $\sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij} = 0 \in R$ holds where, for elements of $K_0(C')$, we consider their image in R.

Now, we let D'_R be the free (rigid) $\mathbb{E}_n$-monoidal C-linear category under C' equipped with objects X_r for each generator $r \in K_0(C')$. These objects give a natural map $\eta'_R : K_0(C')\{R\} \to K_0(D'_R)$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(K_0(C))^{\heartsuit})_{K_0(C')/}$.

Next, for each relation $\sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij}$, consider the object

$$O = \bigoplus_{i=1}^{n} \left(\bigotimes_{j=1}^{m_i} P_{ij} \right),$$

where P_{ij} is $O_{x_{ij}}$ whenever $x_{ij} \in K_0(C')$ and is $X_{x_{ij}}$ whenever $x_{ij} \in R$. Note that we consider the operations $\oplus$ and $\otimes$ as only giving monoidal structures (as opposed to symmetric monoidal and $\mathbb{E}_n$-monoidal) so that O is an object defined up to a contractible space of choices from the data defining the relation.

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[11] or weakly initial

For each relation, we freely adjoin to D'_R as a (rigid) $\mathbb{E}_n$-monoidal C-linear category objects Y, Z, maps

$$Y \xrightarrow{f} O \oplus Z, \quad Y \xrightarrow{g} Z,$$

and an isomorphism $\mathrm{cof}(f) \cong \mathrm{cof}(g)$. Let D_R be the object in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(C))_{C'/}$ constructed via these operations.

The fact that $\mathrm{cof}(f) \cong \mathrm{cof}(g)$ implies that $[O] + [Z] - [Y] = [Z] - [Y]$, i.e., $[O] = 0$. Thus, the composite $K_0(C')\{R\} \xrightarrow{\eta'_R} K_0(D'_R) \to K_0(D_R)$ factors through R, so we obtain the natural transformation η_R as this factorization.

It remains to show the versal property of D_R. To do this, let E be a (rigid) $\mathbb{E}_n$-monoidal C-linear category under C' with a factorization $K_0(C') \to R \to K_0(E)$. We must then produce a map $D_R \to E$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}(C))_{C'/}$ compatible with this factorization.

First, observe that, by choosing representatives of the K_0-classes of each element of R, we obtain a map $D'_R \to E$ by sending the object $X_r, r \in R$ to this object. It suffices to show then that this functor $F : D'_R \to E$ admits a factorization through D_R. To do this, we use the following lemma.

Lemma 19 (Heller's criterion [10, Lemma 2.12]). *Let C be a stable category. For any relation $[A] = [B]$ in $K_0(C)$, there exist cofibre sequences of the form*

$$Y \xrightarrow{f} A \oplus Z \to J,$$
$$Y \xrightarrow{g} B \oplus Z \to J.$$

For a given relation $\sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij}$, by applying Lemma 19 to $[F(O)] = [0]$ in E, we obtain the data as in the lemma. For each relation, by sending the Y, Z, f, g used to form D_R to these objects and maps, and by choosing an isomorphism between the Js appearing in the cofibre sequences, we obtain the desired factorization through D_R. ∎

Remark 20. There is a version of Proposition 18 for additive categories: one needs merely replace the application of Lemma 19 with the more basic fact that, in an additive category, the relation $[A] = [B]$ holds iff there is some Y such that

$$A \oplus Y \cong B \oplus Y.$$

Now, we combine Theorem 9, Theorem 14, and Proposition 18 to prove Theorem A and Theorem B, whose statements we recall for convenience.

Theorem A. *There is a filtered colimit preserving functor $C^{\infty}_{(-)} : \mathrm{Ring}^0_{\mathrm{red}} \to \mathbb{Z}\mathrm{Cat}^{\infty}_{\mathrm{rig}}$ and a natural isomorphism of rings $K_0(C^{\infty}_R) \cong R$.*

Theorem B. *There is a filtered colimit preserving functor* $C^2_{(-)} : \mathrm{Ring}_{\mathrm{red}} \to \mathbb{Z}\mathrm{Cat}^2_{\mathrm{rig}}$ *and a natural isomorphism of rings* $K_0(C^2_R) \cong R$.

Proof of Theorem A and Theorem B. We first prove Theorem A, and then indicate the necessary changes to prove Theorem B. Letting $C = C' = \mathrm{Mod}(\mathbb{Z})^\omega$, $n = \infty$, choosing any representatives of $K_0(C')$, and applying Proposition 18, we obtain a functor $D_{(-)}$, taking a discrete commutative ring R to D_R, a $\mathbb{Z}$-linear rigid symmetric monoidal category with a natural map $R \to K_0(D_R)$. The category D_R comes functorially equipped with objects $X_r, r \in R$ (see the proof of Proposition 18) whose K_0-class is the image of r.

We consider the ideal I_R of $K_0(D_R)$ generated by $r - r^*$, where r^* is the class of the dual of r. We apply Proposition 18 again with $C = \mathrm{Mod}(\mathbb{Z})^\omega$, $C' = D_R$, $n = \infty$ and X_r as our choice of representatives to obtain a category D'_R versally equipped with a map $K_0(D_R)/I_R \to K_0(D'_R)$. Note that the image of the composite

$$R \to K_0(D_R)/I_R \to K_0(D'_R)$$

by construction has the property that it is fixed by the dual functor, so the subcategory of D'_R consisting of objects whose K_0-class is in this image is a rigid symmetric monoidal subcategory, which we define to be C^∞_R.

It remains to show that the natural surjection $R \to K_0(C^\infty_R)$ is an isomorphism, which is equivalent to the claim that $R \to K_0(D'_R)$ is injective. We now assume that each minimal prime of R is characteristic 0. Let C be a $\mathbb{Z}$-linear rigid symmetric monoidal category as in Theorem 9 with the property that $K_0(C) \cong R$ and the action of the dual functor on K_0 is trivial. The fact that

$$K_0(C) \cong R$$

by versality gives the existence of a symmetric monoidal $\mathbb{Z}$-linear functor $D_R \to C$. Because the action of the dual functor on C is trivial, versality of D'_R further allows us to extend this to a symmetric monoidal functor $D'_R \to C$. It then follows that the composite

$$R \to K_0(D'_R) \to K_0(C)$$

is an isomorphism, so $R \to K_0(D'_R)$ is injective, as desired.

To prove Theorem B, we run the same proof, except changing n from ∞ to 2 and replacing the use of Theorem 9 with Theorem 14. ■

Remark 21. One can use Proposition 18 to make an ‘obstruction theory’ for constructing *idempotent-complete* categories with a specified K_0. We briefly indicate how this works for a symmetric monoidal category.

Given C_0 an idempotent-complete symmetric monoidal rigid stable category and a map $K_0(C_0) \to R$, we can by Proposition 18 construct a versal idempotent-complete rigid symmetric monoidal functor $C_0 \to C_1$ with a factorization $K_0(C_0) \to R \to K_0(C_1)$. The first obstruction to constructing an idempotent-complete C_0-linear category with K_0 isomorphic to R is whether the map $R \to K_0(C_1)$ admits a retraction.

If a retraction exists, we can choose a retraction and form C_2 as a versal idempotent-complete C_1-linear category with a factorization $K_0(C_1) \to R \to K_0(C_2)$. The second obstruction is whether the map $R \to K_0(C_2)$ admits a retraction.

One can keep going, and if all obstructions vanish, one can produce for each i a category C_i, and the filtered colimit of C_i is an idempotent-complete C_0-linear category with the desired K_0. On the other hand, versality shows that if an idempotent-complete C_0-linear category exists with the desired K_0, then choices of retractions can be made so that all obstructions vanish.

Remark 22. Using Proposition 18 as in Theorem A, one can reduce the problem of categorifying arbitrary commutative rings to the case of finite commutative rings. This is because the versal category C_R with a surjective map $R \to K_0(C_R)$ preserves filtered colimits in the variable R, so to show the natural map $R \to K_0(C_R)$ is an isomorphism, one can assume that R is finitely presented. In this case, R is a Noetherian Jacobson ring, thus embedding into a product of finite rings (namely, the quotients by powers of all maximal ideals), so it suffices to solve the problem for those. It is not clear if there is a ‘small’ collection of finite rings that generate the rest under subrings, products, and ultraproducts, but rings such as $\mathbb{Z}/p^n$ and $\mathbb{F}_q[x_1, \dots, x_n]/(x_i^{m_i})$ can be categorified via methods similar to those presented in this paper and seem to generate a lot of finite rings.

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