Doc. Math. (Online first)
DOI 10.4171/DM/1078

Equivalences of the form $\Sigma^V X \simeq \Sigma^W X$ in equivariant stable homotopy theory

William Balderrama

Abstract. We study equivalences of the form $\Sigma^V X \simeq \Sigma^W X$, where G is a compact Lie group, X is a G-spectrum, and V and W are G-representations. These equivalences encode a periodicity phenomenon in G-equivariant homotopy theory which generalizes the classical James periodicity for $G = C_2$.

In the case where $X = C(a_\lambda)$ is the cofiber of an Euler class, we construct an $RO(G)$-graded J-homomorphism $J\colon \pi_\lambda KO_G \to \pi_\star^G C(a_\lambda)^\times$ which gives control over these periodicities. It also produces infinite periodic families in the G-equivariant stable stems. We illustrate this with several explicit examples.

More generally, our work gives information about $RO(G)$-graded units in equivariant stable cohomotopy rings. We apply this to construct universal periodicities and differentials in the G-homotopy fixed point spectral sequence, and other equivariant Atiyah–Hirzebruch spectral sequences.

1. Introduction

1.1. Background

In [7,8], Bredon introduced the C_2-equivariant stable stems. In modern notation, these are the groups $\pi_\star S_{C_2}$ comprising the $RO(C_2)$-graded coefficient ring of the C_2-equivariant sphere spectrum. At the same time, he introduced groups that, in modern notation, may be identified as the $RO(C_2)$-graded homotopy groups $\pi_\star^{C_2} C(a_\sigma^m)$ of the cofiber of the Euler class $a_\sigma^m\colon S_{C_2}^{-m\sigma} \to S_{C_2}^0$.

In addition to fitting these homotopy groups into the evident long exact sequences, Bredon observed that they satisfy a certain periodicity:

$$\pi^{C_2}_{\star+2^{\gamma(m)}} C(a_\sigma^{m+1}) \cong \pi^{C_2}_{\star+2^{\gamma(m)}\sigma} C(a_\sigma^{m+1}), \tag{1.1}$$

where $\gamma(m) = \#\{0 < k \leq m : k \equiv 0,1,2,4 \pmod 8\}$.

This periodicity was further studied by Araki [2]. The C_2-spectrum $C(a_\sigma^{m+1})$ is dual to the m-sphere $\underline{S}^m$ with its free antipodal action. The theory of Clifford algebras provides a trivialization of the C_2-equivariant vector bundle $2^{\gamma(m)}\sigma \times \underline{S}^m \to \underline{S}^m$, and Araki used this to produce and study invertible elements

$$\omega_m \in \pi^{C_2}_{2^{\gamma(m)}(1-\sigma)} C(a_\sigma^{m+1})^\times. \tag{1.2}$$

Mathematics Subject Classification 2020: 55Q91 (primary); 19L20, 19L47, 55P42 (secondary).
Keywords: James periodicity, equivariant homotopy, J-homomorphism.

Additional properties of these were established by Araki and Iriye in [3], where they were applied to the computation of C_2-equivariant stable stems.

As already observed by Bredon, the groups $\pi_\star^{C_2} C(a_\sigma^{m+1})$ can be identified as nonequivariant stable homotopy groups of real stunted projective spaces. From this perspective, (1.1) is a consequence of *James periodicity* [16], as also noted by Landweber [19]. In [6], Behrens and Shah used the Adams isomorphism and James periodicity directly to produce invertible elements of the form (1.2). Their work emphasizes the interaction with the C_2-equivariant Adams spectral sequence, constructing these elements to lift powers of the orientation class $u_\sigma \in \pi_{1-\sigma} H\mathbb{F}_2^{C_2}$.

Our goal in this paper is to put these results in the context of a more general periodicity phenomenon in equivariant stable homotopy theory, and give applications.

1.2. Summary

The starting point of our investigation is a reinterpretation and generalization of these classical results in terms of the *J-homomorphism*. Classically, the stable J-homomorphism for a space Z is a homomorphism

$$KO^{-1}(Z) \to \pi_0 D(\Sigma^\infty_+ Z)^\times \tag{1.3}$$

from the K-theory of real vector bundles over the suspension of Z to the group of units in the stable cohomotopy ring of Z. The J-homomorphism was originally introduced by G. W. Whitehead [38], and the construction works just as well equivariantly, as has been studied by several people going back to Segal [31].

We introduce the following refinement of the equivariant J-homomorphism. Fix a compact Lie group G and compact G-space Z, with *unreduced* suspension SZ.

Theorem A (2.2.1). *The equivariant J-homomorphism refines to a homomorphism*

$$J\colon \widetilde{KO}^0_G(SZ) \to \pi_\star^G D(\Sigma^\infty_+ Z)^\times,$$

defined up to signs, interacting with degrees as follows: if $\xi \in \widetilde{KO}^0_G(SZ)$ restricts along the inclusion $S^0 \to SZ$ to $\alpha \in \widetilde{KO}^0_G(S^0) \cong RO(G)$, then $J(\xi) \in \pi_\alpha^G D(\Sigma^\infty_+ Z)$.

This refines the classical equivariant J-homomorphism as in (1.3) in the sense that the latter is obtained by restricting along the canonical map $SZ \to \Sigma(Z_+)$. By "defined up to signs" we mean that J takes values in orbits for the action by the subgroup of orthogonal units in $(\pi_0 S_G)^\times$, i.e., those units which are obtained by compactifying an automorphism of a G-representation. See Remark 2.1.1 for further discussion.

Example 1.2.1. Let $Z = S(\lambda)$ be the unit sphere in a G-representation λ. In this case the cofiber sequence $S(\lambda)_+ \to S^0 \to S^\lambda$ yields equivalences

$$SZ \simeq S^\lambda, \quad D(\Sigma^\infty_+ Z) \simeq C(a_\lambda),$$

where $a_\lambda \in \pi_{-\lambda} S_G$ is the Euler class of λ, so our J-homomorphism takes the form

$$J\colon \pi_\lambda KO_G \to \pi_\star^G C(a_\lambda)^\times,$$

where if $\xi \in \pi_\lambda KO_G$ then

$$\left|J(\xi)\right| = \alpha \quad \text{when } a_\lambda \xi = \alpha \in \pi_0 KO_G = RO(G).$$

Example 1.2.2. It follows from the structure of $\pi_0 KO_{C_2} \otimes C(a_\sigma^{m+1}) \cong KO^0(\mathbb{R}P^m)$, computed by Adams [1], that

$$\operatorname{Im}\left(a_\sigma^{m+1}\colon \pi_{(m+1)\sigma} KO_{C_2} \to RO(C_2)\right) = \mathbb{Z}\left\{2^{\gamma(m)}(1-\sigma)\right\},$$

so Theorem A encodes the classical James periodicity of Section 1.1.

Remark 1.2.3. In general, an invertible element $u \in \pi_\alpha^G C(a_\lambda)$ gives rise to a secondary *periodicity operator*

$$P_u\colon \pi_\star^G(-) \rightharpoonup \pi_{\star+\alpha}^G(-),$$

defined on the kernel of a_λ and defined modulo the projection $\partial(u) \in \pi_{\alpha+\lambda-1} S_G$ of u onto the top cell of $C(a_\lambda)$. These operators are periodic in the sense that if $a_\lambda x = 0$ then $x \in P_{u^{-1}}(P_u(x))$. Thus Example 1.2.1 implies that the equivariant K-theory of representation spheres parametrizes certain nontrivial periodicities defined on Euler-torsion in G-equivariant stable homotopy theory.

As Remark 1.2.3 suggests, Theorem A can be used to produce *infinite periodic families* in the $RO(G)$-graded equivariant stable stems.

Example 1.2.4 (§6.6). Let $\mathbb{H}$ denote the tautological representation of the quaternion group $Q_8 \subset \mathcal{S}\mathrm{p}(1)$ of order 8. If we define

$$p(n) = \begin{cases} 2n, & n \text{ even}, \\ 2n+1, & n \text{ odd}, \end{cases}$$

then $2^{p(n)}(4-\mathbb{H}) = a_{\mathbb{H}}^{n+1} \cdot b_{n+1}$ for some $b_{n+1} \in \pi_{(n+1)\mathbb{H}} KO_{Q_8}$ which lifts a generator of $\pi_{4(n+1)} KO$ when n is even and 4 times a generator when n is odd. From this one obtains invertible elements

$$u_{2^{p(n)}\mathbb{H}} = J(b_{n+1}) \in \pi^{Q_8}_{2^{p(n)}(4-\mathbb{H})} C(a_{\mathbb{H}}^{n+1}).$$

By taking powers and projecting onto the top cell of $C(a_{\mathbb{H}}^{n+1})$, this gives for each n the infinite periodic family

$$\partial(u^k_{2^{p(n)}\mathbb{H}}) \in \pi_{2^{p(n)}k(4-\mathbb{H})+(n+1)\mathbb{H}-1} S_{Q_8},$$

all nonzero for $k \neq 0$, satisfying

$$\operatorname{res}^{Q_8}_e\left(\partial(u^k_{2^{p(n)}})\right) = \begin{cases} k \cdot j_{4n+3}, & n \text{ even}, \\ k \cdot 4 j_{4n+3}, & n \text{ odd}, \end{cases}$$

where $j_n \in \pi_n S$ is a generator of the image of J in this degree.

In Section 3, we apply Theorem A to construct universal periodicities and differentials in equivariant Atiyah–Hirzebruch spectral sequences. We refer the reader there for the general situation and just state here a special case.

Given a finite group G and G-ring spectrum R, there is a *homotopy fixed point spectral sequence*

$$E_2(R) = H^*(G;\pi_\star^e R) \implies \pi_{\star-*}^G R_h^\wedge$$

computing the $RO(G)$-graded homotopy of the Borel completion $R_h^\wedge = F(EG_+, R)$. This spectral sequence exhibits periodic behavior: there are invertible elements $u_\alpha \in \pi_{|\alpha|-\alpha}^e R \cong \pi_0^e R$ for each $\alpha \in RO(G)$, canonical up to a sign, allowing one to identify

$$E_2(R) = H^*\big(G;\pi_*^e R[u_\lambda^{\pm 1} : \lambda \text{ irreducible } G\text{-representation}]\big).$$

Here, $g \in G$ acts on u_λ by ± 1 according to whether the action of g on λ is oriented. Theorem A allows us to construct universal differentials on these classes. If $\mathbb{Z} \leftarrow \mathbb{Z}[G] \leftarrow \mathbb{Z}[G]\{I_1\} \leftarrow \mathbb{Z}[G]\{I_2\} \leftarrow \cdots$ is the free $\mathbb{Z}[G]$-module resolution associated to a cell structure on EG, then an E_1-page for the homotopy fixed point spectral sequence is given by

$$E_1^{\alpha,f}(R) = \pi_{|\alpha|+f}^e R^{\times I_f} \implies \pi_\alpha^G R_h^\wedge.$$

In the case $R = KO_G$ and in integer degrees, this is the classical Atiyah–Hirzebruch spectral sequence for $KO^* BG$, and the classical J-homomorphism defines a map

$$\widetilde{J}\colon E_1^{0,f}(KO_G) = \pi_f KO^{\times I_f} \to \pi_{f-1} S^{\times I_f} = E_1^{-1,f}(S_G).$$

Given a 0-dimensional virtual G-representation α, set $t_\alpha = u_{-\alpha} \in E_1^{\alpha,0}(R)$.

Theorem B (3.2.1). *Suppose that $\alpha \in RO(G)$ is detected in the Atiyah–Hirzebruch spectral sequence for $KO^0(BG)$ in positive filtration f by $b \in E_1^{0,f}(KO)$. Then $t_\alpha \in E_1^{\alpha,0}(R)$ survives to the E_f-page, whereupon*

$$d_f(t_\alpha) = \pm \widetilde{J}(b) t_\alpha$$

in $E_f^{\alpha-1,f}(R)$.

The sign in Theorem B is present due to choices involved in constructing $\widetilde{J}\colon \pi_f KO \to \pi_{f-1} S$; we make no attempt to pin this down further.

Example 1.2.5. When $G = C_2$, the universal space $EC_2 = S(\infty\sigma)$ admits a cell structure with one free cell in each degree, giving

$$E_1^{s,f}(KO_{C_2}) = \pi_{s+f} KO \implies KO^{-s}(BC_2),$$
$$E_1^{\star,*}(S_{C_2}) = \pi_* S[u_\sigma^{\pm 1}, a_\sigma] \implies \pi_\star(S_{C_2})_h^\wedge.$$

Here, a_σ is the generator of $H^1(C_2;\pi_{1-\sigma}^e S_{C_2}) \cong \mathbb{Z}/(2)$. Let $\rho(n)$ denote the nth Hurwitz–Radon number, i.e., if $\nu_2(n) = 4a + b$ with $0 \le b \le 3$ then $\rho(n) = 8a + 2^b$. Then $k(1-\sigma)$

is detected in the Atiyah–Hirzebruch spectral sequence for $KO^0(BC_2)$ by a generator of $\pi_{\rho(k)}KO$, and it follows that

$$d_{\rho(k)}(u_\sigma^k) = j_{\rho(k)-1} \cdot a_\sigma^{\rho(k)} u_\sigma^{-\rho(k)} \cdot u_\sigma^k,$$

where $j_n \in \pi_n S$ is a generator of the image of J in degree, with the understanding that $j_0 = \pm 2$. These can also be interpreted as primary differentials in the Atiyah–Hirzebruch spectral sequence for real projective space. This essentially goes back to Adams [1], compare also [3, Theorem 3.5] and [9, Proposition 2.17].

Example 1.2.6. Theorem B only produces universal differentials, which can vanish in a given G-ring spectrum. For example, the homotopy fixed point spectral sequence for Real bordism $M\mathbb{R}$ takes the form

$$E_1^{\star,*}(M\mathbb{R}) = \pi_* MU[u_\sigma^{\pm 1}, a_\sigma] \implies \pi_\star^{C_2} M\mathbb{R}.$$

Theorem B implies $d_1(u_\sigma) = \pm 2a_\sigma$ and $d_2(u_\sigma^2) = \eta a_\sigma^2$. As η vanishes in MU, this implies that u_σ^2 must survive to (at least) the E_3-page for $M\mathbb{R}$. In fact, one has

$$d_3(u_\sigma^2) = a_\sigma^3 \bar{v}_1,$$

where $\bar{v}_1 \in \pi_{1+\sigma} M\mathbb{R}$ is detected by $u_\sigma^{-1} v_1$ [15]. Given this, one can instead interpret Theorem B as saying that $a_\sigma \bar{v}_1 \in \pi_1 M\mathbb{R}$ is the Hurewicz image of a class in $\pi_1 S_{C_2}$ which is killed by a_σ^2 and lifts η. The only such class is $\eta_{\mathrm{cl}} + a_\sigma^2 \nu_{C_2}$, where $\eta_{\mathrm{cl}} \in \pi_1 S_{C_2}$ and $\nu_{C_2} \in \pi_{1+2\sigma} S_{C_2}$ are the nonequivariant and equivariant complex and quaternionic Hopf maps respectively. As ν_{C_2} has trivial Hurewicz image in $M\mathbb{R}$, this encodes the identity $a_\sigma \bar{v}_1 = \eta_{\mathrm{cl}}$. Similar considerations apply to higher differentials and other G-spectra.

Example 1.2.7. When $G = Q_8$, as $\mathbb{H}$ is oriented the homotopy fixed point spectral sequence restricted to degrees $* + *\mathbb{H}$ takes the form

$$E_2 = H^*(Q_8; \pi_*^e R)[u_{\mathbb{H}}^{\pm 1}] \implies \pi_{*+*\mathbb{H}}^{Q_8} R_h^\wedge,$$

where $|u_{\mathbb{H}}| = 4 - \mathbb{H}$. Using the fact that the unit sphere $S(n\mathbb{H})$ is a $(4n-1)$-skeleton of EQ_8, it follows from Theorem B and Example 1.2.4 that there are differentials

$$d_{4(n+1)}(u_{\mathbb{H}}^{2^{p(n)}}) = \begin{cases} j_{4n+3} \cdot a_{\mathbb{H}}^{n+1} u_{\mathbb{H}}^{-(n+1)} \cdot u_{\mathbb{H}}^{2^{p(n)}}, & n \text{ even}, \\ 4 j_{4n+3} \cdot a_{\mathbb{H}}^{n+1} u_{\mathbb{H}}^{-(n+1)} \cdot u_{\mathbb{H}}^{2^{p(n)}}, & n \text{ odd} \end{cases}$$

for $n \geq 0$, where $a_{\mathbb{H}}$ is detected by the generator of $H^4(Q_8; \pi_{4-\mathbb{H}}^e S_{Q_8}) \cong \mathbb{Z}/(8)$.

In Section 4 we investigate an additional property of the invertible elements produced by Theorem A that makes precise a certain analogy between these equivariant "James-type periodicities" and v_n-periodicity in classical stable homotopy theory. This analogy was first highlighted by Behrens and Shah [6] in their construction of C_2-equivariant u_σ-self maps. Given a 0-dimensional virtual complex G-representation α, there is a canonical invertible *Thom class*

$$t_\alpha \in \pi_\alpha MU_G,$$

where MU_G is tom Dieck's homotopical G-equivariant complex cobordism spectrum.

Definition 1.2.8 (4.1.2). A t_α*-element* of order $n > 0$ in a G-ring spectrum R is an invertible element $t \in \pi_{n\alpha}^G R$ lifting t_α^n under the Hurewicz map $R \to R \otimes MU_G$.

When $\alpha = |V| - V$ one might call these u_V*-elements*. Note that multiplication by a t_α-element induces a t_α*-self map*: a self-map inducing multiplication by a power of t_α in MU_G-theory. The observation here is now the following.

Proposition C (4.2.3). *The complex J-homomorphism*

$$J\colon \widetilde{KU}{}_G^0(SZ) \to \pi_\star^G D(\Sigma_+^\infty Z)^\times$$

takes values in t_α-elements for various α: if $|J(\xi)| = \alpha$ then $J(\xi)$ is a t_α-element.

The analogous proposition holds with *KU* and *MU* replaced by *KO* and *MO*, or by *KSp* and *MSp*.

Example 1.2.9. As $H\mathbb{F}_2^{C_2} \otimes C(a_\sigma^{m+1}) \simeq F(EC_{2+}, H\mathbb{F}_2) \otimes C(a_\sigma^{m+1})$ is MO_{C_2}-oriented, it follows from the real analogue of Proposition C that the invertible elements $u_{2^{\gamma(m)}\sigma} \in \pi^{C_2}_{2^{\gamma(m)}(1-\sigma)} C(a_\sigma^{m+1})$ guaranteed by Example 1.2.2 have Hurewicz image

$$u_\sigma^{2^{\gamma(m)}} \in (H\mathbb{F}_2^{C_2})_{2^{\gamma(m)}(1-\sigma)} C(a_\sigma^{m+1}),$$

where $u_\sigma \in \pi_{1-\sigma} H\mathbb{F}_2^{C_2}$ is the orientation class. In other words, we recover the u_σ-elements produced by Behrens–Shah in [6, Theorem 7.7].

Using equivariant nilpotence techniques we are able to bootstrap Proposition C into a general existence criterion for t_α-elements.

Theorem D (4.5.4). *Let G be a finite group and R be a G-ring spectrum, and suppose*

$$\Phi^C R \neq 0 \implies \operatorname{res}^G_C \alpha = 0$$

for all cyclic subgroups $C \subset G$. Then R admits a t_α-element of order dividing a power of $|G|$.

Example 1.2.10. Let $A \to R$ be a faithful G-Galois extension of $\mathbb{E}_\infty$-rings. Then R together with its G-action defines a G-ring spectrum satisfying $\Phi^H R \simeq 0$ for all nontrivial subgroups $H \subset G$ [22, Example 4.10], and which therefore admits a t_α-element of order dividing a power of $|G|$ for any zero-dimensional virtual representation α.

There are more $RO(G)$-graded periodicities than are accounted for by t_α-self maps. For example, there are stable equivalences $S^V \simeq S^W$ that do not come from an isomorphism $V \cong W$ of G-representations. Section 5 contains some observations about this situation, mostly adapting work of tom Dieck, Hauschild, Petrie, and Tornehave [35–37] on homotopy equivalent representation spheres and the equivariant Adams conjecture. For example, we prove the following.

Theorem E (5.2.3). *Let G be a finite group and X be a compact G-spectrum. Given $\alpha \in RO(G)$, there exists an equivalence $\Sigma^{n\alpha} X \simeq X$ for some $n \geq 1$ if and only if*

$$\Phi^C X \neq 0 \implies |\alpha^C| = 0$$

for all cyclic subgroups $C \subset G$.

Here, the assumption that X is compact (i.e., finite) guarantees that $\Phi^C X = 0$ if and only if $\Phi^C \operatorname{End}(X) = 0$; we deduce Theorem E as a special case of a more general statement about the existence of units in G-ring spectra, see Theorem 5.2.2.

Example 1.2.11. Given a finite group G, G-representation λ, and $\alpha \in RO(G)$, there exists an equivalence $\Sigma^{n\alpha} C(a_\lambda) \simeq C(a_\lambda)$ for some $n \geq 1$ if and only if

$$|\lambda^C| \neq 0 \implies |\alpha^C| = 0$$

for all cyclic subgroups $C \subseteq G$. Thus it is a purely representation-theoretic condition that determines when a_λ-torsion is α-periodic (of some possibly large period).

As a general statement this is quite satisfactory, but as a practical matter it gives no control over the equivalences $\Sigma^{n\alpha} X \simeq X$. More can be said after inverting some primes not dividing the order of G, and we make some comments about this situation in Section 5.1. A clean general statement is available when G is a p-group, where we prove the following.

Fix a prime p and finite p-group G. In this case, Bousfield localization $L_{KU_G/p}$ behaves similarly in G-equivariant homotopy theory to how $K(1)$-localization behaves in nonequivariant homotopy theory. In particular, if ℓ generates a dense subgroup of $\mathbb{Z}_p^\times/\{\pm 1\}$ and we set

$$J_G = \operatorname{Fib}\left(\psi^\ell - \psi^1 \colon (KO_G)_p^\wedge \to (KO_G)_p^\wedge\right),$$

then $J_G \simeq L_{KU_G/p} S_G$, and more generally

$$L_{KU_G/p} D(\Sigma^\infty_+ Z) \simeq F(\Sigma^\infty_+ Z, J_G)$$

for any compact G-space Z [5, Corollary A.4.13]. Write

$$j^Z_{K(1)} \colon RO(G) \to KO^0_G(Z) \to J^1_G(Z)$$

for the resulting boundary map.

Theorem F (5.3.5). *Let G be a finite p-group and Z be a compact G-space. Given $\alpha \in RO(G)$, there exists an invertible element in $\pi_\alpha D(\Sigma^\infty_+ Z)_{(p)}$ if and only if $j^Z_{K(1)}(\alpha) = 0$.*

Thus the location of units in $\pi_\star^G D(\Sigma_+^\infty Z)_{(p)}$ is completely determined by $K(1)$-local information.

In Section 6 we work out several explicit examples. In particular, we compute $\pi_{*\lambda}KO_G$ for a variety of finite groups G and G-representations λ, producing explicit periodicities on $C(a_\lambda^n)$ and infinite periodic families in $\pi_\star S_G$.

2. An $RO(G)$-graded J-homomorphism

In this section we construct the equivariant J-homomorphism promised in the introduction. Throughout this section G is a compact Lie group, and for simplicity we shall restrict our attention to compact G-spaces.

2.1. Preliminaries

We begin by fixing a bit of notation. Fix F to be one of the real division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. All vector spaces, vector bundles, and so forth are understood to be with respect to F. Write

$$KF_G = \begin{cases} KO_G, & F = \mathbb{R}, \\ KU_G, & F = \mathbb{C}, \\ KSp_G, & F = \mathbb{H} \end{cases}$$

for the G-spectrum representing the K-theory of G-equivariant vector bundles, and

$$RF(G) = \pi_0 KF_G$$

for the corresponding Grothendieck group of G-representations.

If Z is a compact G-space, then $KF_G^0(Z)$ can be identified explicitly as

$$KF_G^0(Z) = \{G\text{-equivariant vector bundles over } Z\}/(\sim),$$

where

$$\xi \sim \zeta \quad \text{when } \xi \oplus V \cong \zeta \oplus V \text{ for some representation } V.$$

The reduced K-group $\widetilde{KF}_G^0(Z)$, generally only defined when Z is pointed, may be similarly identified as

$$\widetilde{KF}_G^0(Z) = \{G\text{-equivariant vector bundles over } Z\}/(\approx),$$

where

$$\xi \approx \zeta \quad \text{when } \xi \oplus V \cong \zeta \oplus W \text{ for some representations } V \text{ and } W.$$

Write $SZ = S^0 * Z$ for the unreduced suspension of Z, and $a_Z\colon S^0 \to SZ$ for the inclusion of cone points. Then the restriction

$$a_Z\colon \widetilde{KF}_G^0(SZ) \to \pi_0 KF_G \cong RF(G)$$

sends a vector bundle ξ over SZ to the difference $V - W$, where V and W are the restrictions of ξ to the cone points -1 and 1 respectively.

We separate out the following remark for easy reference.

Remark 2.1.1. If V is a G-representation, then the representation sphere S^V is the pointed G-space defined as the one-point compactification of V. The suspension spectrum of S^V is invertible as a G-spectrum, allowing one to construct virtual representation spheres $S^\alpha \in \mathcal{S}\mathrm{p}^G$ for any virtual representation $\alpha \in RO(G)$. However, whereas the assignment $V \mapsto S^V$ is natural in V, the G-spectrum S^α is generally defined only up to *noncanonical* isomorphism.

This issue arises from the fact that if V is a G-representation, then $\mathrm{Aut}(V)$ may act nontrivially on S^V in the homotopy category of G-spectra. If V is defined over $F = \mathbb{C}$ or $\mathbb{H}$ and we consider only F-linear automorphisms, then in fact $\mathrm{Aut}(V)$ acts trivially on S^V up to homotopy and no such issues arise. If $F = \mathbb{R}$ then we will sidestep this issue by taking the convention that, for our purposes, S^α and $\pi_\alpha(-)$ will only be *defined* up to signs, where "signs" refers to the orthogonal units in $A(G)$, i.e., the image of tom Dieck's homomorphism

$$j\colon RO(G) \to (\pi_0 S_G)^\times$$

sending a G-representation V to the class obtained by compactifying $-1\colon V \to V$. Unfortunately we see no way to avoid this while maintaining that the source of the J-homomorphism of Theorem 2.2.1 is $\widetilde{KO}{}^0_G(SZ)$, and not some more rigid but less uniformly computable object (such as $\mathrm{Map}^G(Z, L(V,W))$ for fixed V and W).

We warn the reader that, having taken this convention, we will generally not take the care needed to pin down precise orientations and signs in intermediate arguments as they will not affect the final results.

2.2. Constructing the J-homomorphism

We now proceed to the construction. Fix a G-space Z and vector bundle ξ over SZ, and write V and W for the restrictions of ξ to the cone points -1 and 1. Let $L(V,W)$ denote the G-space of linear isometries $V \to W$, and write

$$j\colon L(V,W) \to \mathrm{Map}_*(S^V, S^W)$$

for the one-point compactification map. As SZ is obtained by gluing two contractible spaces along a copy of Z, the vector bundle ξ is classified by a clutching function

$$\varphi_\xi\colon Z \to L(V,W),$$

and the composite $j \circ \varphi_\xi\colon Z \to \mathrm{Map}_*(S^V, S^W)$ is adjoint to a map

$$J(\xi)\colon \Sigma^V(Z_+) \to S^W.$$

After stabilization, this defines a class

$$J(\xi) \in \pi^G_{V-W} D(\Sigma^\infty_+ Z)$$

that we will denote by the same name.

Theorem 2.2.1. *The above construction extends to a natural homomorphism*

$$J\colon \widetilde{KF}{}^0_G(SZ) \to \pi_\star^G D(\Sigma^\infty_+ Z)^\times,$$

defined up to signs when $F = \mathbb{R}$*, with the property that*

$$\bigl|J(\xi)\bigr| = \alpha \quad \textit{when } a_Z\xi = \alpha \in RF(G).$$

Proof. If Z_V is the trivial bundle $V \times Z \to Z$, then by construction $J(Z_V)\colon \Sigma^V(Z_+) \to S^V$ is the collapse map, adjoint to the unit $1 \in \pi_0^G D(\Sigma^\infty_+ Z)$. Recall that if ξ is any vector bundle over SZ, then there exists another vector bundle ξ' for which $\xi \oplus \xi' \cong Z_V$ for some G-representation V [30, Proposition 2.4]. Therefore we may reduce to just verifying that $J(\xi \oplus \xi') = J(\xi) \cdot J(\xi')$ for any two vector bundles ξ and ξ' over SZ. Write

$$\varphi_\xi\colon Z \to L(V, W), \quad \varphi_{\xi'}\colon Z \to L(V', W')$$

for clutching functions for ξ and ξ'. Then a clutching function for $\xi \oplus \xi'$ is given by

$$\varphi_{\xi\oplus\xi'}\colon Z \to L(V \oplus V', W \oplus W'), \quad \varphi_{\xi\oplus\xi'}(z)(v, v') = \bigl(\varphi_\xi(z)(v), \varphi_{\xi'}(z)(v')\bigr).$$

It follows that $J(\xi \oplus \xi')$ is the composite

$$\Sigma^{V\oplus V'}(Z_+) \xrightarrow{\Delta} \Sigma^{V\oplus V'}\bigl((Z \times Z)_+\bigr) \xrightarrow{\cong} \Sigma^V(Z_+) \wedge \Sigma^{V'}(Z_+)$$
$$\xrightarrow{J(\xi)\wedge J(\xi')} S^W \wedge S^{W'} \xrightarrow{\cong} S^{W\oplus W'},$$

which exactly corresponds to $J(\xi) \cdot J(\xi')$. ∎

Example 2.2.2. There is a cofiber sequence $Z_+ \to S^0 \to SZ$, and precomposing J with restriction along the boundary map $SZ \to \Sigma(Z_+)$ gives an equivariant J-homomorphism

$$KF_G^{-1}(Z) \to \pi_0 D(\Sigma^\infty_+ Z)^\times,$$

as considered, for example, in [31].

Example 2.2.3. Write

$$\widetilde{J} = \partial \circ J\colon \widetilde{KF}{}^0_G(SZ) \to \pi^G_{\star-1} D(\Sigma^\infty SZ).$$

If λ is a G-representation and $Z = S^{\lambda+1}$, then $a_Z = 0$ and $\widetilde{J}$ recovers the equivariant J-homomorphism

$$\pi_{\lambda+2} KF_G \to \pi_{\lambda+1} S_G,$$

as considered, for example, in [10, 20, 26].

Remark 2.2.4. If Z is G-pointed, then $SZ \simeq \Sigma Z$ and $a_Z = 0$, so the J-homomorphism is of the form

$$\widetilde{KF}{}^{-1}_G(Z) \to \pi_0^G D(\Sigma^\infty_+ Z).$$

The basepoint of Z induces a splitting $D(\Sigma_+^\infty Z) \simeq S_G \oplus D(\Sigma^\infty Z)$, and so we may project onto the latter summand to obtain

$$\widetilde{J} = \partial \circ J : \widetilde{KF}_G^{-1}(Z) \to \pi_0^G D(\Sigma^\infty Z).$$

Note that this need not be a group homomorphism, though it is if Z is G-connected. On the other hand, one can also consider the (perhaps more familiar) J-homomorphism

$$J' : \widetilde{KF}_G^{-1}(Z) \to \pi_0^G D(\Sigma^\infty Z)$$

sending a *pointed* map $\phi: Z \to L(V, V)$ to the adjoint $\Sigma^V Z \to S^V$ of the composite $j \circ \phi: Z \to \mathrm{Map}_*(S^V, S^V)$. If Z is G-connected then $J' = \widetilde{J}$, but they can differ in general, as the next example demonstrates.

Example 2.2.5. Take $Z = S^0$. The homomorphism $j : RO(G) \to A(G)^\times$ indicated in Remark 2.1.1 factors through the surjection $\eta: RO(G) = \pi_0 KO_G \to \pi_1 KO_G$, and we have

$$J'(\eta \cdot \alpha) = j(\alpha), \quad \widetilde{J}(\eta \cdot \alpha) = \pm\big(1 - j(\alpha)\big).$$

For example $\widetilde{J}(\eta) = \pm 2$, and if σ is a 1-dimensional representation with index 2 kernel $K \subset G$ then $\widetilde{J}(\eta \cdot \sigma) = \pm\,\mathrm{tr}_K^G(1)$.

3. The equivariant Atiyah–Hirzebruch spectral sequence

If Z is a G-complex and R is a G-spectrum, then there is an *Atiyah–Hirzebruch spectral sequence*

$$E_2^{\alpha,f}(Z; R) = H_G^f(Z; \underline{\pi}_{\alpha+f} R) \implies R_G^{-\alpha}(Z),$$

with E_2-page the Bredon cohomology of Z with coefficients in the Mackey functor $\underline{\pi}_\star R$.

Example 3.0.1. For $Z = EG$ one obtains the *homotopy fixed point spectral sequence*

$$E_2^{\alpha,f} = H^f(G; \pi_{\alpha+f}^e R) \implies \pi_\alpha^G F(EG_+, R) \cong \pi_0\big(F(S^\alpha, R)^{\mathrm{h}G}\big),$$

with E_2-page the group cohomology of G with coefficients in the underlying homotopy groups $\pi_\star^e R$.

In this section we apply the J-homomorphism constructed in Section 2 to obtain universal periodicities and differentials in these spectral sequences.

3.1. A technical lemma

We will need a technical lemma. Let $p: Z \to T$ be a map of G-spaces with a homotopy retraction $i: T \to Z$, so that $p \circ i \simeq \mathrm{id}_T$. In particular, there is an equivalence

$$T/Z \simeq S(Z/T).$$

Write

$$q\colon Z \to Z/T, \quad \partial\colon S(Z/T) \simeq T/Z \to SZ$$

for the canonical maps, and note that

$$Sq \circ \partial\colon S(Z/T) \to SZ \to S(Z/T)$$

is an equivalence. The following diagram of coCartesian squares may help illustrate the situation:

$$\begin{array}{ccccccccc}
T & \longrightarrow & Z & \longrightarrow & T & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
* & \longrightarrow & Z/T & \longrightarrow & * & \longrightarrow & ST & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & S(Z/T) \simeq T/Z & \longrightarrow & SZ & \longrightarrow & SZ/ST \simeq S(Z/T).
\end{array}$$

The retraction $p \circ i \simeq \mathrm{id}_T$ induces a stable splitting

$$D(\Sigma^\infty_+ Z) \simeq D(\Sigma^\infty_+ T) \oplus D(\Sigma^\infty Z/T).$$

Here, $D(\Sigma^\infty Z/T)$ is a module over $D(\Sigma^\infty_+ Z)$, and thus over $D(\Sigma^\infty_+ T)$ by restriction along p. We now have the following.

Lemma 3.1.1. *If $\xi \in \widetilde{KO}{}^0_G(SZ)$, then under the above splitting we have*

$$J(\xi) = \big(J(Si^*\xi), \widetilde{J}\big(\partial^*(\xi)\big) \cdot J(Si^*\xi)\big),$$

where $\widetilde{J}$ is as in Remark 2.2.4.

Proof. The relevant diagram is

$$\begin{array}{ccccc}
\widetilde{KO}{}^0_G\big(S(Z/T)\big) & \xrightarrow{J} & \pi_0 D(\Sigma^\infty_+ Z/T) & \xrightarrow{\simeq} & \pi_0\big(S_G \oplus D(\Sigma^\infty Z/T)\big) \\
{\scriptstyle \partial^*}\uparrow \downarrow{\scriptstyle Sq^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle \iota \oplus \mathrm{id}} \\
\widetilde{KO}{}^0_G(SZ) & \xrightarrow{J} & \pi_\star D(\Sigma^\infty_+ Z) & \xrightarrow{\simeq} & \pi_\star\big(D(\Sigma^\infty_+ T) \oplus D(\Sigma^\infty Z/T)\big). \\
{\scriptstyle Sp^*}\uparrow \downarrow{\scriptstyle Si^*} & & {\scriptstyle p^*}\uparrow \downarrow{\scriptstyle i^*} & \overset{\mathrm{in}_1}{\nearrow} \underset{\mathrm{pr}_1}{\swarrow} & \\
\widetilde{KO}{}^0_G(ST) & \xrightarrow{J} & \pi_\star D(\Sigma^\infty_+ T) & &
\end{array}$$

First suppose $\xi = Sp^*\zeta$ for some $\zeta \in \widetilde{KO}{}^0_G(ST)$. In this case we have

$$J(\xi) = J(Sp^*\zeta) = p^* J(\zeta) = \big(J(\zeta), 0\big) = \big(J(Si^*\xi), 0\big)$$

as claimed. Next suppose $Si^*\xi = 0$. In this case $\xi = Sq^*\partial^*\xi$ and thus

$$J(\xi) = J(Sq^*\partial^*\xi) = q^* J(\partial^*\xi).$$

Under the splitting

$$D(\Sigma^\infty_+ Z/T) \simeq S_G \oplus D(\Sigma^\infty Z/T)$$

we have by definition

$$J(\partial^*\xi) = \big(1, \widetilde{J}(\partial^*\xi)\big),$$

and thus

$$q^* J(\partial^*\xi) = q^*\big(1, \widetilde{J}(\partial^*\xi)\big) = \big(1, \widetilde{J}(\partial^*\xi)\big).$$

Finally, for general ξ we combine the preceding cases to compute

$$\begin{aligned} J(\xi) &= J(\xi - Sp^* Si^* \xi) \cdot J(Sp^* Si^* \xi) \\ &= \big(1, \widetilde{J}\big(\partial^*(\xi - Sp^* Si^* \xi)\big)\big) \cdot \big(J(Si^* Sp^* Si^* \xi), 0\big) \\ &= \big(1, \widetilde{J}(\partial^*\xi)\big) \cdot \big(J(Si^*\xi), 0\big) = \big(J(Si^*\xi), \widetilde{J}(\partial^*\xi) \cdot J(Si^*\xi)\big) \end{aligned}$$

as claimed. ∎

3.2. Periodicities and differentials in the equivariant AHSS

A *G-complex* is a filtered G-space

$$Z = \operatorname*{colim}_{n\to\infty} Z^{\le n},$$

where $Z^{\le n}$ is obtained from $Z^{<n}$ by attaching n-cells. For our purposes, this will mean that we have specified G-sets I_n for each $n \ge 0$, together with pushout squares

$$\begin{array}{ccc} S^{n-1} \times I_n & \longrightarrow & Z^{<n} \\ \downarrow & & \downarrow \\ D^n \times I_n & \longrightarrow & Z^{\le n} \end{array}$$

for $n \ge 0$. Note in particular

$$Z^{\le n}/Z^{<n} \simeq (D^n \times I_n)/(S^{n-1} \times I_n) \simeq \Sigma^n(I_{n+}).$$

For a G-set I and G-spectrum R, write

$$(\underline{\pi}_\star R)(I) = \pi^G_\star F(\Sigma^\infty_+ I, R)$$

for the evaluation of the Mackey functor $\underline{\pi}_\star R$ on I. Then the *Atiyah–Hirzebruch spectral sequence*

$$E_1^{\alpha,f}(Z; R) = (\underline{\pi}_{\alpha+f} R)(I_f) \implies R_G^{-\alpha}(Z)$$

is the spectral sequence associated to the filtration

$$F(\Sigma^\infty_+ Z, R) \simeq \lim_{f\to\infty} F(\Sigma^\infty_+ Z^{<f}, E).$$

The E_2-page is given by the Bredon cohomology groups

$$E_2(Z;R) = H^*_G(Z;\underline{\pi}_\star R).$$

If $f \geq 1$ then the J-homomorphism defines a map

$$\widetilde{J}\colon E_1^{0,f}(Z;KO_G) = (\underline{\pi}_f KO_G)(I_f) \to (\underline{\pi}_{f-1} S_G)(I_f) = E_1^{-1,f}(Z;S_G).$$

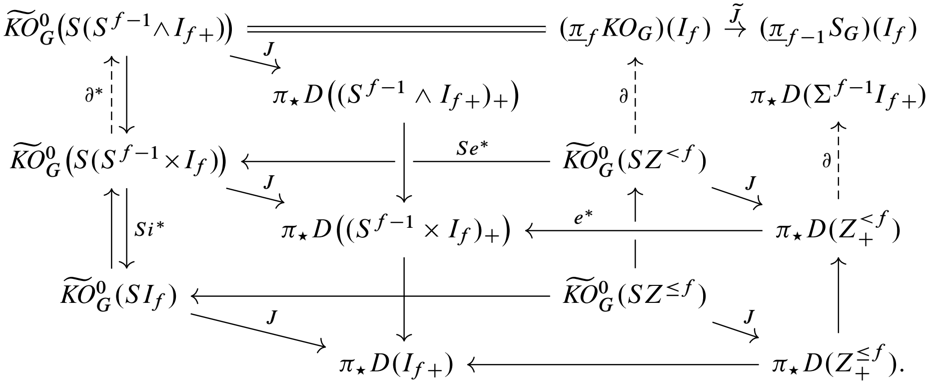
In general $E_r(Z;R)$ is a module over $E_r(Z;S_G)$, and we have the following.

Theorem 3.2.1. *Fix $\alpha \in RO(G)$, and suppose that the image of α in $KO_G^0(Z)$ is detected by $b \in E_1^{0,f}(Z;KO_G)$ with $f \geq 1$. Then there exists an invertible element $t_\alpha \in E_1^{\alpha,0}(Z;R)$ which survives to the E_f-page, whereupon*

$$d_f(t_\alpha) = \pm \widetilde{J}(b) t_\alpha$$

in $E_f^{\alpha-1,f}(Z;R)$. Here, the $\pm$ refers to signs in the sense of Remark 2.1.1, and is present just as $\widetilde{J}(b)$ has only been defined up to signs.

Proof. We may as well suppose $R = S_G$. The relevant diagram is the following:



Here we have abbreviated $D(X) = D(\Sigma^\infty X)$ for a pointed G-space X.

As α is detected by $b \in E_1^{0,f}(Z;KO_G)$, there exists $\tilde{b} \in \widetilde{KO}^0_G(SZ^{<f})$ satisfying $\partial(\tilde{b}) = b$ and restricting to α along $S^0 \to SZ^{<f}$. Applying the J-homomorphism we obtain an invertible element

$$J(\tilde{b}) \in \pi_\alpha^G D(\Sigma^\infty_+ Z^{<f}).$$

We take t_α to be the image of $J(\tilde{b})$ under restriction along $I_0 = Z^{\leq 0} \to Z^{<f}$; we will also abuse notation by writing the same for the image of $J(\tilde{b})$ under restriction along the map $i_f\colon I_f \to Z^{<f}$. Then to say that $d_f(t_\alpha) = \widetilde{J}(b)\cdot t_\alpha$ is to say that $\partial(J(\tilde{b})) = \widetilde{J}(b)\cdot t_\alpha$.

Under the splitting

$$D\big((S^{f-1}\times I_{f+})_+\big) \simeq D(I_{f+}) \oplus D(\Sigma^{f-1} I_{f+})$$

we may identify

$$e^*(x) = \big(i_f^*(x), \partial(x)\big)$$

for any $x \in \pi_\star D(Z_+^{<f})$. Using Lemma 3.1.1, we compute

$$e^* J(\tilde{b}) = J(Se^*\tilde{b}) = \big(J\big(Si^*(Se)^*\tilde{b}\big),\ \widetilde{J}(\partial^* Se^*\tilde{b}) \cdot J(Si^* Se^*\tilde{b})\big).$$

Here, by construction we have $\partial^*(Se)^*\tilde{b} = b$ and $J(Si^*(Se)^*\tilde{b}) = t_\alpha$, implying that

$$e^* J(\tilde{b}) = \big(t_\alpha, \widetilde{J}(b) \cdot t_\alpha\big),$$

and so $\partial(J(\tilde{b})) = \widetilde{J}(b)t_\alpha$ as claimed. ∎

Remark 3.2.2. In Theorem 3.2.1, the choice of $t_\alpha \in E_1^{\alpha,0}(Z; R)$ depends on the choice of a lift of $\alpha \in RO(G)$ to $\widetilde{\alpha} \in \widetilde{KO}_G^0(SI_0)$, and this choice puts constraints on the detecting class b. However, once $\widetilde{\alpha}$ is chosen, the proof shows that $\widetilde{J}(b) \in E_1^{-1,f}(Z; S_G)$ descends to a class in $E_r^{-1,f}(Z; S_G)$ which is independent of the choice of detecting class chosen from

$$\begin{aligned} b &\in \operatorname{Im}\big(\{\widetilde{\alpha}\} \times_{\widetilde{KO}_G^0(SI_0)} \widetilde{KO}_G^0(SZ^{<f}) \to \underline{\pi}_f KO_G(I_f)\big) \\ &\subset \operatorname{Im}\big(\{\alpha\} \times_{RO(G)} \widetilde{KO}_G^0(SZ^{<f}) \to \underline{\pi}_f KO_G(I_f)\big). \end{aligned}$$

It would be desirable to have more control over all these choices, perhaps via a suitable "synthetic J-homomorphism"; we shall refrain from any precise speculation.

4. Equivariant t_α-self maps and t_α-elements

This section contains our study of t_α-elements and t_α-self maps, connecting the periodicities produced by the J-homomorphism of Section 2 to periodicities in equivariant cobordism.

4.1. Equivariant cobordism and t_α-elements

We begin by recalling the definitions needed to make sense of t_α-elements. Let F denote one of the real division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and write

$$MF_G = \begin{cases} MO_G, & F = \mathbb{R}, \\ MU_G, & F = \mathbb{C}, \\ MSp_G, & F = \mathbb{H} \end{cases}$$

for the corresponding G-equivariant homotopical cobordism spectrum. This was originally constructed by tom Dieck [34] when $F = \mathbb{C}$, and the construction works just as well for $F = \mathbb{R}$ and $F = \mathbb{H}$. To be precise tom Dieck constructs MF_G as a G-equivariant cohomology theory; a good construction of MF_G as a G-spectrum suitable for our purposes may be found in [32]. The G-spectra MF_G are compatible as G varies, and assemble together to form a highly structured globally equivariant ring spectrum. We refer the reader to [29, Chapter 6] for a careful construction of MF_G which incorporates this additional structure.

A vector bundle γ over a compact G-space Z has associated sphere bundle $S(\gamma)$ and disk bundle $D(\gamma)$, and the *Thom space* of γ is given by

$$\mathrm{Th}(\gamma) = D(\gamma)/S(\gamma).$$

This is a pointed G-space, and we will generally not distinguish between it and its suspension spectrum. Associated to γ is a *Thom class*

$$t_\gamma \in \widetilde{MF}_G^{-|\gamma|}\,\mathrm{Th}(\gamma),$$

cupping with which induces the *Thom isomorphism*

$$\widetilde{MF}_G^\star\,\mathrm{Th}(\gamma) \cong MF_G^{\star+|\gamma|}(Z).$$

Moreover, MF_G is the universal G-spectrum with such Thom isomorphisms [28].

We only need the simplest example of a Thom class. Given a G-representation V, we write

$$t_V \in \pi_{V-|V|}MF_G$$

for the Thom class of V considered as a vector bundle over a point, noting that $\mathrm{Th}(V) = S^V$. This satisfies

$$t_{V\oplus W} = t_V t_W, \quad t_{F^n} = 1,$$

and has inverse the orientation class $u_V \in \pi_{|V|-V}MF_G$. Moreover, t_V depends only on the isomorphism class of V, allowing for the following definition.

Definition 4.1.1. The *Thom class* $t_\alpha \in \pi_{\alpha-|\alpha|}MF_G$ of a virtual G-representation $\alpha = V - W$ is defined as $t_\alpha = t_V t_W^{-1}$.

From now on, we shall assume that α has virtual dimension zero. This loses no real generality for our purposes, and is convenient as it ensures $|t_\alpha| = \alpha$.

Definition 4.1.2. A *t_α-element* of order $n > 0$ in a G-ring spectrum R is an invertible element $t \in \pi_{n\alpha}^G R$ lifting t_α^n under the Hurewicz map $R \to R \otimes MU_G$.

The condition that a t_α-element is invertible is automatic if, for example, R is MF_G-local. We include it as our examples satisfy it.

Definition 4.1.3. A *t_α-self map* of order $n \geq 1$ on a G-spectrum X is an equivalence

$$f\colon \Sigma^{n\alpha}X \xrightarrow{\sim} X$$

which induces multiplication by t_α^n after smashing with MF_G, that is for which $MF_G \otimes f = t_\alpha^n \otimes X$ as self maps of $MF_G \otimes X$.

We record the following for easy reference.

Lemma 4.1.4. *If $t \in \pi_{n\alpha} R$ is a t_α-element and M is an R-module, then multiplication by t defines a t_α-self map $\Sigma^{n\alpha} M \to M$. Conversely, every R-linear t_α-self map $\Sigma^{n\alpha} R \to R$ is given by multiplication with a t_α-element.*

Proof. Immediate from the definitions. ∎

4.2. Vector bundles and K-theory

Let Z be a compact G-space. We now relate the study of t_α-elements in $D(\Sigma^\infty_+ Z)$ to the study of vector bundles on Z.

Given a G-representation V, write Z_V for the vector bundle $V \times Z \to Z$. This has Thom spectrum

$$\mathrm{Th}(Z_V) \simeq \Sigma^V \Sigma^\infty_+ Z.$$

In particular, a stable equivalence $Z_V \cong Z_W$ of vector bundles over Z induces an equivalence $\Sigma^V \Sigma^\infty_+ Z \simeq \Sigma^W \Sigma^\infty_+ Z$. The basic observation is that this is determined by a t_{V-W}-element in $D(\Sigma^\infty_+ Z)$, and that such elements are parametrized by the J-homomorphism of Section 2, as we now explain.

If ξ is a vector bundle over Z, then the Thom diagonal $\mathrm{Th}(\xi) \to \mathrm{Th}(\xi) \otimes \Sigma^\infty_+ Z$ transposes to make $\mathrm{Th}(\xi)$ into a module over the Spanier–Whitehead dual $D(\Sigma^\infty_+ Z)$. When $\xi = Z_V$, one obtains the usual module structure on $\mathrm{Th}(Z_V) = \Sigma^V \Sigma^\infty_+ Z$. In particular, a map $Z_V \to Z_W$ of vector bundles induces a $D(\Sigma^\infty_+ Z)$-linear map $\Sigma^V \Sigma^\infty_+ Z \to \Sigma^W \Sigma^\infty_+ Z$. By duality, such a map is given by capping with an element of $\pi^G_{V-W} D(\Sigma^\infty_+ Z)$.

Remark 4.2.1. Explicitly, given a map $f\colon \Sigma^V \Sigma^\infty_+ Z \to \Sigma^W \Sigma^\infty_+ Z$, one defines the class $t \in \pi^G_{V-W} D(\Sigma^\infty_+ Z)$ to be represented by the composite

$$\Sigma^V \Sigma^\infty_+ Z \xrightarrow{f} \Sigma^W \Sigma^\infty_+ Z \longrightarrow \Sigma^W \Sigma^\infty_+ (*) \xrightarrow{\simeq} S^W,$$

and when f is $D(\Sigma^\infty_+ Z)$-linear it may be recovered from t as the composite

$$\Sigma^V \Sigma^\infty_+ Z \xrightarrow{\Sigma^V \Delta} \Sigma^V \Sigma^\infty_+ Z \otimes \Sigma^\infty_+ Z \xrightarrow{t \otimes \Sigma^\infty_+ Z} \Sigma^W \Sigma^\infty_+ Z.$$

We can now give the following.

Lemma 4.2.2. *Fix an equivalence $Z_V \xrightarrow{\sim} Z_W$ of vector bundles over Z with associated equivalence $f\colon \Sigma^V \Sigma^\infty_+ Z \xrightarrow{\sim} \Sigma^W \Sigma^\infty_+ Z$ of Thom spectra. Then f is a t_{V-W}-self map and the associated class $t \in \pi^G_{V-W} D(\Sigma^\infty_+ Z)$ is a t_{V-W}-element.*

Proof. By the above discussion and Lemma 4.1.4, it suffices to show that t is a t_{V-W}-element. By definition $t = f^*(1)$, where $f^*\colon MF^\star_G(\Sigma^W \Sigma^\infty_+ Z) \to MF^\star_G(\Sigma^V \Sigma^\infty_+ Z)$. By $MF^\star_G$-linearity of f^* and naturality of Thom classes, we can identify

$$t = f^*(1) = f^*(t_W^{-1} t_W) = t_W^{-1} f^*(t_W) = t_W^{-1} t_{f^* W} = t_W^{-1} t_V = t_{V-W}$$

in $MF^{W-V}_G(Z)$, and so t is a t_{V-W}-element as claimed. ∎

Proposition 4.2.3. *Consider the J-homomorphism*

$$J\colon \widetilde{KF}^0_G(SZ) \to \pi_\star^G D(\Sigma^\infty_+ Z)^\times.$$

If $\xi \in \widetilde{KF}^0_G(SZ)$ satisfies $a_Z\xi = \alpha \in RF(G)$, then $J(\xi) \in \pi_\alpha^G D(\Sigma^\infty_+ Z)$ is a t_α-element.

Proof. Write $\alpha = V - W$ for two G-representations V and W. By construction, $J(\xi)$ is obtained from a stable equivalence $Z_V \simeq Z_W$ after passing to Thom spectra, i.e., from an equivalence of vector bundles $Z_{V\oplus U} \simeq Z_{W\oplus U}$ for some G-representation U. It follows from Lemma 4.2.2 that $J(\xi)$ is a $t_{(V\oplus U)-(W\oplus U)} = t_\alpha$-element as claimed. ∎

Corollary 4.2.4. *There exists a t_α-element in $\pi_\alpha D(\Sigma^\infty_+ Z)$ provided α is in the kernel of $RF(G) \to KF^0_G(Z)$.*

Proof. By the cofiber sequence $Z_+ \to S^0 \to SZ$, if $\alpha \in \operatorname{Ker}(RF(G) \to KF^0_G(Z))$ then $\alpha = a_Z b$ for some $b \in \widetilde{KF}^0_G(SZ)$, and by Proposition 4.2.3,

$$J(b) \in \pi_\alpha^G D(\Sigma^\infty_+ Z)$$

is our desired t_α-element. ∎

4.3. Character theory

Suppose that G is finite. In this case, we can use Corollary 4.2.4 to give criteria for $D(\Sigma^\infty_+ Z)$ to admit a t_α-element of some order based only on the isotropy groups of Z. We will extend this to arbitrary G-ring spectra in Section 4.5 below.

We require some general equivariant localization theory.

Lemma 4.3.1. *Write $\mathcal{M}\mathrm{ack}_G$ for the category of G-Mackey functors, and for $H \subset G$ write $W_G H = N_G H/H$ for the Weyl group. Then there are functors*

$$\tau_H\colon \mathcal{M}\mathrm{ack}_G \to \mathcal{M}\mathrm{od}_{\mathbb{Z}[W_G H]}, \quad \tau_H \underline{M} = \operatorname{Coker}\left(\mathrm{tr}\colon \bigoplus_{K\subsetneq H} \underline{M}(G/K) \to \underline{M}(G/H)\right)$$

for which the following hold:

(1) *For any G-Mackey functor $\underline{M}$, there is a natural splitting*

$$\mathbb{Z}[\tfrac{1}{|G|}] \otimes \underline{M}(G) \cong \mathbb{Z}[\tfrac{1}{|G|}] \otimes \bigoplus_{(H)} (\tau_H \underline{M})^{W_G H},$$

this sum being over the conjugacy classes of subgroups of G.

(2) *If E is a G-spectrum, then the geometric fixed point maps $\phi_H\colon \pi_0^G E \to \pi_0 \Phi^H E$ factor through $\tau_H(\underline{\pi}_0 E)$, and induce isomorphisms*

$$\mathbb{Z}[\tfrac{1}{|G|}] \otimes \tau_H(\underline{\pi}_0 E) \cong \mathbb{Z}[\tfrac{1}{|G|}] \otimes \pi_0 \Phi^H E.$$

Proof. See for example [29, Section 3.4]. ∎

Lemma 4.3.2. *Let E be a G-spectrum. Fix $u \in E^0_G(Z)$, and for $H \subset G$ abbreviate $u_H = \mathrm{res}^G_H u$. Suppose that*

$$Z^H \neq \emptyset \implies u_H = 0$$

for all $H \subset G$. Then u has finite order dividing a power of $|G|$.

Proof. As Z is compact we have

$$\Phi^H \mathcal{S}\mathrm{p}^G(\Sigma^\infty_+ Z, E) \cong \mathcal{S}\mathrm{p}^G\big(\Sigma^\infty_+(Z^H), \Phi^H E\big),$$

and therefore Lemma 4.3.1 implies

$$E^0_G(Z)\Big[\tfrac{1}{|G|}\Big] \subset \prod_{H \subset G} (\Phi^H E)^0(Z^H)\Big[\tfrac{1}{|G|}\Big].$$

It thus suffices to show that $u_H = 0$ in $(\Phi^H E)^0(Z^H)$ for all $H \subset G$. If $Z^H = \emptyset$ then this group vanishes. Otherwise $u_H = 0$ by assumption. In either case this shows that $u = 0$ in $E^0_G(Z)[\frac{1}{|G|}]$, and so u has finite order dividing a power of $|G|$. ■

Combining these two lemmas leads to the following.

Proposition 4.3.3. *A class $\alpha \in KF^0_G(Z)$ has finite order if and only if $Z^C \neq \emptyset \Rightarrow \alpha_C = 0$ for all cyclic $C \subset G$, in which case its order divides a power of $|G|$.*

Proof. First suppose that $Z^C \neq \emptyset \Rightarrow \alpha_C = 0$ for all cyclic $C \subset G$. If $Z^H \neq \emptyset$, then $Z^C \neq \emptyset$ for all cyclic $C \subset H$. It follows that $\alpha_C = 0$ for all cyclic $C \subset H$, and as a representation is determined by its restriction to cyclic subgroups we deduce $\alpha_H = 0$. Thus Z satisfies $Z^H \neq \emptyset \Rightarrow \alpha_H = 0$ for all $H \subset G$. By Lemma 4.3.2, this implies that $\alpha \in KF^0_G(Z)$ has finite order dividing a power of $|G|$.

Conversely, if $Z^C \neq \emptyset$, then there is some equivariant map $p\colon G/C \to Z$. This must satisfy $p^*(\alpha) = \alpha_C \in KF^0_G(G/C) \cong RF(C)$. It follows that if α has finite order, then so does $\alpha_C \in RF(C)$, implying that $\alpha_C = 0$ as $RF(C)$ is torsion-free. ■

Corollary 4.3.4. *If $Z^C \neq \emptyset \Rightarrow \alpha_C = 0$ for all cyclic subgroups $C \subset G$, then $D(\Sigma^\infty_+ Z)$ admits a t_α-element of order dividing a power of $|G|$.*

Proof. Combine Proposition 4.3.3 and Corollary 4.2.4. ■

4.4. $\mathcal{F}$-nilpotence

Our next goal is to upgrade Corollary 4.3.4 to arbitrary G-ring spectra. To do this we will make use of the nilpotence machinery developed by Mathew–Nauman–Noel in [23, 24]. In this subsection we extract the parts of this theory that we will need.

Given a subgroup $H \subset G$, abbreviate

$$G/H_+ = \Sigma^\infty_+ G/H.$$

Let $\mathcal{F}$ be a family of subgroups of G, i.e., a collection of subgroups closed under subconjugacy.

Definition 4.4.1 ([23, Definition 6.36]). A G-spectrum X is *$\mathscr{F}$-nilpotent* if it lies in the thick $\otimes$-ideal generated by G/H_+ for $H \in \mathscr{F}$.

Up to G-homotopy equivalence, there is a unique G-space $E\mathscr{F}$ characterized by

$$E\mathscr{F}^H \simeq \begin{cases} \emptyset, & H \notin \mathscr{F}, \\ *, & H \in \mathscr{F}. \end{cases}$$

A convenient model for this space is given as follows. Write

$$G/\mathscr{F} = \coprod_{H \in \mathscr{F}} G/H.$$

Then $E\mathscr{F}$ is equivalent to an infinite join of copies of $G/\mathscr{F}$:

$$E\mathscr{F} = \operatorname{colim}(G/\mathscr{F} \to G/\mathscr{F} * G/\mathscr{F} \to G/\mathscr{F} * G/\mathscr{F} * G/\mathscr{F} \to \cdots).$$

Write $E\mathscr{F}^{<m} = (G/\mathscr{F})^{*m-1}$ for the resulting $(m-1)$-skeleton of $E\mathscr{F}$. This satisfies

$$(E\mathscr{F}^{<m})^H \simeq \left((G/\mathscr{F})^H\right)^{*m-1} \simeq \begin{cases} \emptyset, & H \notin \mathscr{F}, \\ \bigvee S^{m-1}, & H \in \mathscr{F}. \end{cases}$$

We also abbreviate $E\mathscr{F}_+^{<m} = \Sigma_+^\infty E\mathscr{F}^{<m}$.

Lemma 4.4.2. *Let X be a G-spectrum, and consider the following statements.*

(1) *X is $\mathscr{F}$-nilpotent.*

(2) *$E\mathscr{F}_+^{<m} \otimes X \to X$ admits a section for some m.*

(3) *$X \to \mathcal{S}\mathrm{p}^G(E\mathscr{F}_+^{<m}, X) \simeq D(E\mathscr{F}_+^{<m}) \otimes X$ admits a retraction for some m.*

(4) *$\Phi^H X \neq 0 \Rightarrow H \in \mathscr{F}$.*

Always (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Rightarrow$(4). *If X is finite or a G-ring spectrum, then* (4)$\Rightarrow$(2).

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3): See [24, Theorem 2.25, Remark 2.27].

(2)$\Rightarrow$(4): If $H \notin \mathscr{F}$ then

$$\Phi^H(E\mathscr{F}_+^{<m} \otimes X) \simeq \Phi^H(E\mathscr{F}_+^{<m}) \otimes \Phi^H X \simeq 0 \otimes \Phi^H X \simeq 0.$$

Thus if X is a retract of $E\mathscr{F}_+^{<m} \otimes X$ then $\Phi^H X \simeq 0$ for $H \notin \mathscr{F}$.

(4)$\Rightarrow$(2) when X is finite: If $\Phi^H X \neq 0 \Rightarrow H \in \mathscr{F}$, then the map $E\mathscr{F}_+ \otimes X \to X$ is an equivalence. As X is finite, the inverse

$$X \to E\mathscr{F}_+ \otimes X \simeq \operatorname*{colim}_{n\to\infty} E\mathscr{F}_+^{<m} \otimes X$$

factors through some $E\mathscr{F}_+^{<m} \otimes X$.

(4)$\Rightarrow$(2) when X is a ring: See [23, Theorem 6.41]. ∎

This allows for the following explicit quantification of $\mathscr{F}$-nilpotence, see [23, Definition 6.36] and [24, Proposition 2.26].

Definition 4.4.3. The $\mathscr{F}$*-exponent* $\exp_{\mathscr{F}}(X)$ of an $\mathscr{F}$-nilpotent G-spectrum X is the minimal m for which $X \to D(E\mathscr{F}_+^{<m}) \otimes X$ admits a retraction.

If R is a $\mathscr{F}$-nilpotent G-ring spectrum with $\exp_{\mathscr{F}}(R) \le m$, then precomposing the guaranteed retraction $D(E\mathscr{F}_+^{<m}) \otimes R \to R$ with the unit $D(E\mathscr{F}_+^{<m}) \to D(E\mathscr{F}_+^{<m}) \otimes R$ provides a map

$$r\colon D(E\mathscr{F}_+^{<m}) \to R$$

satisfying $r(1) = 1$. This map is *not* guaranteed to be multiplicative. It is possible to show by a formal argument that the composite $D(E\mathscr{F}_+^{<2m}) \to D(E\mathscr{F}_+^{<m}) \to R$ is multiplicative, and that although the composite $D(E\mathscr{F}_+^{<m+1}) \to D(E\mathscr{F}_+^{<m}) \to R$ may fail to be multiplicative the induced map on homotopy groups preserves invertible elements. However, in our particular context it turns out we can do just a little better.

Lemma 4.4.4. *Let R be a G-ring spectrum. Fix a map $r\colon D(E\mathscr{F}_+^{<m}) \to R$ satisfying $r(1) = 1$, and fix $t \in \pi_\alpha^G D(E\mathscr{F}_+^{<m})$.*

(1) *If t lifts $t_\alpha \in \pi_\alpha^G(MF_G \otimes D(E\mathscr{F}_+^{<m}))$, then $r(t)$ lifts $t_\alpha \in \pi_\alpha^G(MF_G \otimes R)$.*

(2) *If t is invertible and $m \ge 2$, then $r(t) \in \pi_\alpha^G R$ is invertible.*

In particular, if $m \ge 2$ then r preserves t_α-elements.

Proof. (1) This holds as the maps $D(E\mathscr{F}_+^{<m}) \to MF_G \otimes D(E\mathscr{F}_+^{<m})$ and $R \to MF_G \otimes R$ are defined using only the unit maps of $D(E\mathscr{F}_+^{<m})$ and R, and r is compatible with these.

(2) It suffices to show that $\Phi^H r(t)$ is invertible for all $H \subset G$. Set $W = (E\mathscr{F}^{<m})^H$. As $t \in \pi_\alpha^G D(E\mathscr{F}_+^{<m})$ is invertible, necessarily $\Phi^H t \in \pi_{\alpha^H} D(\Sigma_+^\infty W)$ is invertible, implying also $|\alpha^H| = 0$. It now suffices to prove the nonequivariant assertion that if $T = \Phi^H R$ is a ring spectrum,

$$f = \Phi^H r\colon D(\Sigma_+^\infty W) \to T$$

satisfies $f(1) = 1$, and $s = \Phi^H t \in \pi_0 D(\Sigma_+^\infty W)$ is invertible, then $f(s) \in \pi_0 T$ is invertible.

If $H \notin \mathscr{F}$, then $W = \emptyset$ and there is nothing to check. If $H \in \mathscr{F}$, then there is an equivalence $W \simeq \bigvee S^{m-1}$. Choosing such an equivalence, the unit $S \to D(\Sigma_+^\infty W)$ splits off, giving

$$D(\Sigma_+^\infty W) \simeq S \oplus \bigvee S^{-(m-1)}.$$

The unit s appears in this splitting as $s = (\pm 1, \varepsilon)$ for some $\varepsilon \in \pi_0 \bigvee S^{-(m-1)} \cong \bigoplus \pi_{m-1} S$. As $f(1) = 1$, it follows that $f(s) = \pm 1 + g(\varepsilon)$ for some $g\colon \bigvee S^{-(m-1)} \to T$. As $m \ge 2$, Nishida nilpotence [27] implies that ε is $\otimes$-nilpotent, and thus $g(\varepsilon)$ is nilpotent. This implies that $f(s)$ is invertible as claimed. ∎

Remark 4.4.5. The condition $m \ge 2$ in Lemma 4.4.4 is necessary. For example, let $G = C_2$ and write $\underline{S}^1 = S(2\sigma)$ for the 1-sphere with its free antipodal action. Then $R = D(\Sigma_+^\infty \underline{S}^1)[\frac{1}{2}]$ satisfies $\exp_{\{e\}}(R) = 1$, but $\pi_{1-\sigma}^{C_2} R = 0$ and so every map $s\colon D(C_{2+}) \to R$ sends the invertible element $u_\sigma \in \pi_{1-\sigma}^{C_2} D(C_{2+}) \cong \pi_0 S \cong \mathbb{Z}$ to zero. On the other hand, if $m = 1$ then the ε appearing in the proof of Lemma 4.4.4 must be divisible by 2, and so the proof goes through provided each $\Phi^H R$ is 2-complete.

4.5. Existence of t_α-elements

Fix $\alpha \in RF(G)$ of virtual dimension zero. We can now formulate and prove our general existence theorems on t_α-elements and t_α-self maps. We first introduce a convenient definition.

Definition 4.5.1. The *$\mathcal{F}$-order* $\mathrm{ord}_{\mathcal{F}}(\alpha)$ of α is the minimal integer m for which $\alpha \neq 0$ in $KF^0_G(E\mathcal{F}^{<m+1})$, with $\mathrm{ord}_{\mathcal{F}}(\alpha) = \infty$ if no such m exists.

In other words,

$$m \leq \mathrm{ord}_{\mathcal{F}}(\alpha) \iff \alpha = 0 \text{ in } KF^0_G(E\mathcal{F}^{<m}),$$

and $\mathrm{ord}_{\mathcal{F}}(\alpha) = m$ precisely when α is detected on the m-line of the equivariant Atiyah–Hirzebruch spectral sequence

$$H^*_G(E\mathcal{F}; \underline{\pi}_* KF_G) \implies KF^*_G(E\mathcal{F}).$$

The most familiar case is when $\mathcal{F} = \{e\}$, where $E\mathcal{F} = EG$ and this is equivalent to the non-equivariant Atiyah–Hirzebruch spectral sequence for $KF^* BG$. Observe that $\mathrm{ord}_{\mathcal{F}}(\alpha) \geq 0$, with equality if and only if $\alpha_H \neq 0$ for some $H \in \mathcal{F}$. Thus if we define the family

$$\mathcal{F}(\alpha) = \{H \subset G : \alpha_H = 0\},$$

then $\mathrm{ord}_{\mathcal{F}}(\alpha) > 0$ if and only if $\mathcal{F} \subset \mathcal{F}(\alpha)$. This refines to the following.

Lemma 4.5.2. *If $\mathcal{F} \subset \mathcal{F}(\alpha)$, then the sequence $\{\mathrm{ord}_{\mathcal{F}}(|G|^k \alpha) : k \geq 0\}$ is unbounded.*

Proof. The claim is that for all $m \geq 0$ there exists some $k \geq 0$ for which $m \leq \mathrm{ord}_{\mathcal{F}}(|G|^k \alpha)$, that is for which $|G|^k \alpha = 0$ in $KF^0_G(E\mathcal{F}^{<m})$. As $\mathcal{F} \subset \mathcal{F}(\alpha)$ we have $(E\mathcal{F}^{<m})^H \neq \emptyset \Rightarrow \alpha_H = 0$, and so this follows from Proposition 4.3.3. ■

Lemma 4.5.3. *Suppose that X is finite or a G-ring spectrum. Then X is $\mathcal{F}(\alpha)$-nilpotent if and only if $\Phi^C X \neq 0 \Rightarrow \alpha_C = 0$ for all cyclic subgroups $C \subset G$.*

Proof. Suppose that $\Phi^C X \neq 0 \Rightarrow \alpha_C = 0$ for all cyclic subgroups $C \subset G$. As X is finite or a G-ring spectrum, the collection $\{H \subset G : \Phi^H X \neq 0\}$ forms a family of subgroups of G closed under subconjugacy. It follows that $\Phi^H X \neq 0 \Rightarrow (C \subset H \text{ cyclic} \Rightarrow \alpha_C = 0)$. As a representation is determined by its restriction to cyclic subgroups we deduce $\Phi^H X \neq 0 \Rightarrow \alpha_H = 0$ for all $H \subset G$. Thus X is $\mathcal{F}(\alpha)$-nilpotent by Lemma 4.4.2, which also gives the converse. ■

We now give the main theorem of this section. Fix a family $\mathcal{F} \subset \mathcal{F}(\alpha)$.

Theorem 4.5.4. *Let R be a $\mathcal{F}$-nilpotent G-ring spectrum, and suppose that* $\mathrm{ord}_{\mathcal{F}}(n\alpha) \geq \max(2, \exp_{\mathcal{F}}(R))$. *Then R admits a t_α-element of order n. In particular, if $\Phi^C R \neq 0 \Rightarrow \alpha_C = 0$ for all cyclic subgroups $C \subset G$, then R admits a t_α-element of order dividing a power of $|G|$.*

Proof. Set $m = \max(2, \exp_{\mathcal{F}}(R))$. As $m \le \mathrm{ord}_{\mathcal{F}}(n\alpha)$, we have $n\alpha = 0$ in $KF_G^0(E\mathcal{F}^{<m})$. By Corollary 4.2.4, there is a t_α-element $t \in \pi_{n\alpha}^G D(E\mathcal{F}_+^{<m})$. As $m \ge \exp_{\mathcal{F}}(R)$, there is a map $r\colon D(E\mathcal{F}_+^{<m}) \to R$ satisfying $r(1) = 1$. As $m \ge 2$, this preserves t_α-elements by Lemma 4.4.4. Thus $r(t) \in \pi_{n\alpha}^G R$ is a t_α-element of order n. The final statement follows from Lemma 4.5.3, which ensures that R is $\mathcal{F}(\alpha)$-nilpotent, and Lemma 4.5.2, which ensures that $\mathrm{ord}_{\mathcal{F}(\alpha)}(n\alpha) \ge \max(2, \exp_{\mathcal{F}(\alpha)}(R))$ for some n dividing a power of $|G|$. ■

Corollary 4.5.5. *Let X be an $\mathcal{F}$-nilpotent G-spectrum, and suppose that*

$$\mathrm{ord}_{\mathcal{F}}(n\alpha) \ge \max\left(2, \exp_{\mathcal{F}}(R)\right).$$

Then X admits a t_α-self map of order n. In particular, if X is $\mathcal{F}(\alpha)$-nilpotent then X admits a t_α-self map of order dividing a power of $|G|$; and if X is compact, then it suffices to verify just that $\Phi^C X \neq 0 \Rightarrow \alpha_C = 0$ for all cyclic subgroups $C \subset G$.

Proof. By [23, Corollary 4.15], we have $\exp_{\mathcal{F}}(X) = \exp_{\mathcal{F}}(\mathrm{End}(X))$. As X is an $\mathrm{End}(X)$-module, a t_α-element in $\mathrm{End}(X)$ induces a t_α-self map on X, so apply Theorem 4.5.4 to $\mathrm{End}(X)$. ■

5. General and local equivalences $\Sigma^V X \simeq \Sigma^W X$

In this section we make some observations about equivalences $\Sigma^V X \simeq \Sigma^W X$, possibly after localization, which need not be t_{V-W}-self maps.

Given a virtual G-representation α and subgroup $H \subset G$, we will write $|\alpha^H|$ for the virtual dimension of the H-fixed points of α, i.e., if $\alpha = V - W$ for two G-representations V and W then $|\alpha^H| = \dim(V^H) - \dim(W^H)$.

5.1. Stable equivalences of representation spheres

We begin by summarizing work of tom Dieck, Petrie, and Tornehave on stable equivalences between representation spheres. Given a compact Lie group G, the representation rings $RU(G)$ and $RO(G)$ come equipped with *Adams operations* ψ^k. These can be computed by their action on characters, given by

$$\chi_{\psi^k V}(g) = \chi_V(g^k).$$

If G is finite then these operations satisfy $\psi^k = \psi^l$ when $k \equiv l \pmod{|G|}$, and thus the operations ψ^k for k coprime to G induce an action of

$$\Gamma = \left(\mathbb{Z}/|G|\right)^\times$$

on $RU(G)$ and $RO(G)$. See [4, II §3] for a good discussion of this action (as well as for earlier work on stably equivalent representations). Write $I(\Gamma)$ for the augmentation ideal of $\mathbb{Z}[\Gamma]$, so that if M is a Γ-module then $M/I(\Gamma)M$ is identified with the orbits M_Γ.

Theorem 5.1.1. *If G is finite and k is coprime to $|G|$, then there exists a stable map*

$$f\colon S^V \to S^{\psi^k V}$$

which is an equivalence after inverting k. Moreover, if V is complex then one can choose f to satisfy

$$\Phi^H f = k^{\lfloor |V^H|/2 \rfloor}$$

for all $H \subset G$.

Proof. The first statement follows from the proof of [35, Theorem 10.12]. The refinement is due to Tornehave and appears in [36, Theorem 4]. See also [37, Theorem 4.1] for the more delicate case where V is not assumed to be complex. ∎

Example 5.1.2. Let $T \subset \mathbb{C}^\times$ denote the circle group and $L = S(\mathbb{C})$ be the tautological complex character of T. Then the kth power map

$$\psi_k\colon S(L^n) \to S(L^{nk})$$

on unit spheres is T-equivariant. Passing to unreduced suspensions this yields a map

$$\psi_k\colon S^{L^n} \to S^{L^{nk}}$$

with the property that

$$\Phi^{C_d}\psi_k = \begin{cases} k, & d \mid n, \\ 0, & d \mid nk \text{ but } d \nmid n, \\ 1, & d \nmid n. \end{cases}$$

In particular, if m is coprime to k then ψ_k is an equivalence after restricting to $C_m \subset T$ and inverting k.

The following theorem now summarizes information about stably equivalent representation spheres.

Write $\operatorname{Pic}(A(G))$ for the Picard group of $A(G) = \pi_0 S_G$. For a spectrum or abelian group M and integer n, write $M_{(n)} = M[p^{-1} : \gcd(p,n) = 1]$.

Theorem 5.1.3. *Let G be a compact Lie group and $\alpha \in RO(G)$. The following are equivalent:*

(1) $|\alpha^H| = 0$ *for all $H \subset G$.*

(2) $\pi_\alpha S_G \in \operatorname{Pic}(A(G))$, *with*

$$\pi_{\alpha+\star} X \cong \pi_\alpha S_G \otimes_{A(G)} \pi_\star X$$

for any G-spectrum X.

(3) *There exists an equivalence $S^{n\alpha} \simeq S^0$ for some $n \geq 1$.*

If G is finite, then these are moreover equivalent to the following:

(4) $|\alpha^C| = 0$ *for all cyclic* $C \subset G$.

(5) $\alpha \in I(\Gamma) \cdot RO(G)$.

(6) *There exists an equivalence* $S^{\alpha}_{(|G|)} \simeq S_{(|G|)}$.

In addition,

(7) *If* $\alpha \in I(\Gamma)^2 \cdot RO(G)$ *then* $S^{\alpha} \simeq S^0$, *and*

(8) *The converse holds if G is a p-group.*

Proof. (1)⇒(2): This is [36, Theorem 1].

(2)⇒(3): The picard group $\mathrm{Pic}(A(G))$ has finite exponent [36, Equation 32], and thus

$$A(G) \cong \pi_0 S_G \cong (\pi_\alpha S_G)^{\otimes_\alpha n} \cong \pi_{n\alpha} S_G$$

for some $n \geq 1$. The image of 1 under such an isomorphism is an invertible element in $\pi_{n\alpha} S_G$, giving an equivalence $S^{n\alpha} \simeq S^0$.

(3)⇒(1): If $S^{n\alpha} \simeq S^0$ for some $n \geq 1$, then applying Φ^H implies $S^{n|\alpha^H|} \simeq S^0$, and thus $|\alpha^H| = 0$ for all $H \subset G$.

(1)⇔(4)⇔(5): This is [35, Proposition 9.2.6].

(5)⇒(6): This follows from Theorem 5.1.1.

(6)⇒(1): Same proof as (3)⇒(1).

(7,8): These are [35, Theorems 9.1.4, 9.1.5]. ■

We end this subsection with some comments on how these results can be applied in practice. Say that two virtual representations α and α' are *locally J-equivalent* if $\alpha - \alpha' \in I(\Gamma) \cdot RO(G)$. Theorem 5.1.3 says that this is equivalent to the existence of a unit $p_{\alpha'-\alpha} \in \pi_{\alpha'-\alpha}(S_G)_{(|G|)}$. If one finds an algebraic witness to $\alpha' - \alpha \in I(\Gamma) \cdot RO(G)$, for example if $\alpha' = \psi^k \alpha$, then Theorem 5.1.1 gives some control over the behavior of (a choice for) $p_{\alpha'-\alpha}$. Moreover, in many basic cases Example 5.1.2 and variations are already sufficient and give completely explicit choices of $p_{\alpha'-\alpha}$.

Example 5.1.4. If Z is a compact G-space and α vanishes in $KO^0_G(Z)$, then α lifts to $b \in \widetilde{KO}^0_G(SZ)$ and Theorem 2.2.1 produces a unit $t_\alpha = J(b) \in \pi_\alpha D(\Sigma^\infty_+ Z)$ with good properties: for example, $\mathrm{res}^G_e(t_\alpha) = J(\mathrm{res}^G_e(b)) \in \pi^e_0 D(\Sigma^\infty_+ Z)$ is in the image of the classical J-homomorphism. Thus if α' is locally J-equivalent to α then one obtains a unit

$$p_{\alpha'-\alpha} \cdot t_\alpha \in \pi^G_{\alpha'} D(\Sigma^\infty_+ Z)_{(|G|)}$$

with similarly good properties: for example, $\mathrm{res}^G_e(p_{\alpha'-\alpha} \cdot J(b)) = k \cdot J(\mathrm{res}^G_e(b))$ for some integer k coprime to $|G|$.

Example 5.1.5. In the situation of Theorem B, if α' is locally J-equivalent to α then one also has

$$d_f(t_{\alpha'}) = \pm \widetilde{J}(b) t_{\alpha'}$$

in the homotopy fixed point spectral sequence. Indeed, $p_{\alpha'-\alpha} \in \pi^G_{\alpha'-\alpha}(S_G)_{(|G|)}$ is detected by $k \cdot t_{\alpha'} t_\alpha^{-1}$ for some integer k coprime to $|G|$, which must then be a permanent cycle. As the spectral sequence is $|G|$-torsion in positive filtration, it follows that $t_{\alpha'} t_\alpha^{-1}$ is a permanent cycle. Thus

$$d_f(t_{\alpha'}) = d_f(t_{\alpha'} t_\alpha^{-1} t_\alpha) = t_{\alpha'} t_\alpha^{-1} \cdot d_f(t_\alpha) = t_{\alpha'} t_\alpha^{-1} \cdot \pm \widetilde{J}(b) t_\alpha = \pm \widetilde{J}(b) t_{\alpha'}$$

as claimed. Similarly considerations hold for the general equivariant Atiyah–Hirzebruch spectral sequences handled in Section 3.

One might ask to what extent these techniques account for everything, and to that end we leave the following question.

Question 5.1.6. Let G be a finite group and Z be a compact G-space. Is every unit in $\pi^G_\star D(\Sigma^\infty_+ Z)_{(|G|)}$ of the form $c \cdot J(b)$, where $b \in \widetilde{KO}^0_G(SZ)$ and c lifts to $\pi_\star(S_G)^\times_{(|G|)}$?

5.2. General periodicities

Throughout this subsection G is a finite group. We now explain how Theorem 5.1.3 implies a general theorem about the eventual existence of equivalences $\Sigma^{n\alpha} X \simeq X$.

Lemma 5.2.1. *Let Z be a 1-dimensional G-complex. If $Z^H \neq \emptyset \Rightarrow |\alpha^H| = 0$ for all subgroups $H \subset G$, then there is an invertible element $t \in \pi^G_{n\alpha} D(\Sigma^\infty_+ Z)$ for some $n \geq 1$.*

Proof. As Z is 1-dimensional, it can be built as a homotopy coequalizer of the form

$$\textstyle\coprod_{i\in I} G/H_i \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{j\in J} G/H_j \longrightarrow Z.$$

It follows that $D(\Sigma^\infty_+ Z)$ is an equalizer of the form

$$D(\Sigma^\infty_+ Z) \longrightarrow \textstyle\prod_{j\in J} D(G/H_{j+}) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \prod_{i\in I} D(G/H_{i+}).$$

Note that in general $\pi^G_\alpha D(G/H_+) \cong \pi_{\alpha_H} S_H$. Applying Theorem 5.1.3 ((1)⇒(3)) to the restrictions α_{H_j}, as G is finite we can find some $n \geq 1$ for which there exist equivalences $u_j\colon S^{n\alpha}_{H_j} \simeq S^0_{H_j}$. These determine an invertible element $u \in \pi^G_{n\alpha} \prod_{j\in J} D(G/H_{j+})$. Our conventions from Remark 2.1.1 are such that $\pi^G_{n\alpha}$ is only defined up to a sign, but this issue goes away after passing to $u^2 \in \pi^G_{\mathbb{C}\otimes n\alpha} \prod_{j\in J} D(G/H_{j+})$. Moreover, this square is guaranteed to satisfy $f^*(u^2) = g^*(u^2)$, so u^2 lifts to an invertible element $t \in \pi^G_{2n\alpha} D(\Sigma^\infty_+ Z)$. ■

Theorem 5.2.2. *Let G be a finite group and R be a G-ring spectrum, and suppose that $\Phi^C R \neq 0 \Rightarrow |\alpha^C| = 0$ for all cyclic subgroups $C \subset G$. Then there exists an invertible element $t \in \pi^G_{n\alpha} R$ for some $n \geq 1$. The converse holds if each $\Phi^C R$ is bounded below.*

Proof. Define the family

$$\mathcal{F}[\alpha] = \{H \subset G : C \subset H \text{ cyclic} \implies |\alpha^C| = 0\},$$

and consider the Atiyah–Hirzebruch spectral sequence

$$E_2 = H_G^*\big(E\mathcal{F}[\alpha]; \underline{\pi}_\star R\big) \implies R_G^{*-\star}\big(E\mathcal{F}[\alpha]\big).$$

The argument in Lemma 4.5.3 adapts, in conjunction with Theorem 5.1.3 ((1)⇔(4)), to show that R is $\mathcal{F}[\alpha]$-nilpotent, so this converges to $\pi_\star^G R$ with a horizontal vanishing line at a finite page. Lemma 5.2.1 ensures that there exists an invertible class $u \in H_G^0(\mathcal{F}; \underline{\pi}_{k\alpha} R)$ for some $k \geq 1$. A theorem of Dress shows that $H_G^{>0}(E\mathcal{F}; \underline{\pi}_\star R)$ is killed by $|G|$ [14, Proposition 21.3]. Thus the Leibniz rule implies that if u^i survives to the E_r-page then $u^{i|G|}$ survives to the E_{r+1}-page. By the horizontal vanishing line it follows that some power $u^{|G|^m}$ is a permanent cycle, and we can take t to be any class detected by this power.

Conversely, if $t \in \pi_{n\alpha}^G R$ is invertible, then $\Phi^C t \in \pi_{n|\alpha^C|} \Phi^C R$ is invertible. As $\Phi^C R$ is bounded below, this is only possible if either $\Phi^C R = 0$ or $|\alpha^C| = 0$ as claimed. ∎

We deduce Theorem E as a corollary.

Theorem 5.2.3. *Let G be a finite group and X be a compact G-spectrum. Then there is an equivalence $\Sigma^{n\alpha} X \simeq X$ for some $n \geq 1$ if and only if $\Phi^C X \neq 0 \Rightarrow |\alpha^C| = 0$ for all cyclic subgroups $C \subset G$.*

Proof. Apply Theorem 5.2.2 to the G-ring spectrum $\mathrm{End}(X)$. ∎

5.3. The equivariant Adams conjecture

Throughout this subsection G is a finite group and Z is a compact G-space. We now use work of tom Dieck and Hauschild [35, Chapter 11] on the equivariant Adams conjecture to obtain more information about units in $\pi_\star^G D(\Sigma_+^\infty Z)_{(|G|)}$. See also work of McClure [25]. The starting point is the following. Say that a stable map $f\colon S(\xi) \to S(\zeta)$ of sphere bundles over Z has *fiberwise degree dividing a power of k* if for all $H \subset G$ and $z \in Z^H$, the induced map $f_z^H\colon S(\xi^H)_z \to S(\zeta^H)_z$ on fibers has degree dividing a power of k.

Theorem 5.3.1 ([35, Theorem 11.3.8, Proposition 11.4.4]). *Let ξ be a vector bundle over Z and k be an odd positive integer coprime to $|G|$. Then there exist stable maps $f\colon S(\xi) \to S(\psi^k \xi)$ and $g\colon S(\psi^k \xi) \to S(\xi)$ of fiberwise degree dividing a power of k.* ∎

The Thom spectrum $\mathrm{Th}(\xi)$ of a vector bundle ξ depends only on its associated sphere bundle $S(\xi)$, and is functorial in stable maps. The condition that a map $f\colon S(\xi) \to S(\zeta)$ has fiberwise degree dividing a power of k ensures that it induces an equivalence $\mathrm{Th}(\xi)[\frac{1}{k}] \to \mathrm{Th}(\zeta)[\frac{1}{k}]$. Hence if we define

$$J_G^{\mathrm{alg}}(Z) = KO_G^0(Z)/\big(x - \psi^k x : x \in KO_G^0(Z),\ 2 \nmid k,\ \gcd\big(k, |G|\big) = 1\big),$$

and write $j\colon KO_G^0(Z) \to J_G^{\mathrm{alg}}(Z)$ for the quotient map, then we have the following.

Proposition 5.3.2. *If $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)$, then there is a unit in $\pi_\alpha^G D(\Sigma_+^\infty Z)_{(|G|)}$.*

Proof. This follows from Theorem 5.3.1, using the same considerations as in Section 4.2 to translate between stable maps $S(Z_V) \to S(Z_W)$ and elements in $\pi^G_{V-W} D(\Sigma^\infty_+ Z)$. ∎

Our main goal in the rest of this section is to prove a converse to Proposition 5.3.2 when G is a p-group.

Lemma 5.3.3. *Write $\alpha = V - W$ as a difference of representations. If there is an invertible class in $\pi^G_\alpha D(\Sigma^\infty_+ Z)[\frac{1}{k}]$, then there is a stable map $f\colon S(Z_V) \to S(Z_W)$ of fiberwise degree dividing a power of k.*

Proof. We may suppose ourselves given a map $\phi\colon \Sigma^V \Sigma^\infty_+ Z \to S^W$ associated to an element $u \in \pi^G_\alpha D(\Sigma^\infty_+ Z)$ which becomes invertible after inverting k. As Z is compact, after possibly enlarging V and W we may write ϕ as the stabilization of a map

$$f\colon \Sigma^V(Z_+) \to S^W.$$

The assumption that u is invertible after inverting k ensures that for all $H \subset G$ and $z \in Z^H$, the induced map

$$f_z^H\colon S^{V^H} \to S^{W^H}, \quad f_z^H(v) = f(v \wedge z)$$

has degree dividing a power of k. As $S^V \cong S(V+1)$, it follows that

$$\tilde{f}\colon S^V \times Z \to S^W \times Z, \quad \tilde{f}(v,z) = \big(f(v \wedge z), z\big)$$

defines a map $S(Z_{V+1}) \to S(Z_{W+1})$, hence a stable map $S(Z_V) \to S(Z_W)$, of fiberwise degree dividing a power of k. ∎

Lemma 5.3.4. *If $j(\alpha) = 0$ in $J^{\mathrm{alg}}_G(Z)_{(|G|)}$, then $j(\alpha) = 0$ in $J^{\mathrm{alg}}_G(Z)$.*

Proof. Deferred to the next subsection, where it appears as Corollary 5.4.3. ∎

Now fix a prime p, and suppose that G is a p-group. As discussed in the introduction, the G-spectrum KU_G/p plays a similar role in G-equivariant homotopy theory as KU/p does in nonequivariant homotopy theory. For example, if ℓ generates a dense subgroup of $\mathbb{Z}_p^\times/\{\pm 1\}$—and we may as well assume ℓ is odd here—and we define

$$J_G = \operatorname{Fib}\big(\psi^\ell - \psi^1\colon (KO_G)^\wedge_p \to (KO_G)^\wedge_p\big),$$

then for any compact G-space Z there is an equivalence

$$L_{KU_G/p} D(\Sigma^\infty_+ Z) \simeq F(\Sigma^\infty_+ Z, J_G),$$

where $L_{KU_G/p}$ denotes Bousfield localization with respect to KU_G/p. Write

$$j^Z_{K(1)}\colon RO(G) \to KO^0_G(Z) \to J^1_G(Z)$$

for the resulting boundary map. We can now give the following.

Theorem 5.3.5. *Let Z be a compact G-space and $\alpha \in RO(G)$. Then there exists an invertible element in $\pi_\alpha^G D(\Sigma_+^\infty Z)_{(p)}$ if and only if $j_{K(1)}^Z(\alpha) = 0$.*

Proof. If there exists an invertible element in $\pi_\alpha^G D(\Sigma_+^\infty Z)_{(p)}$, then after writing $\alpha = V - W$ as a difference of representations, Lemma 5.3.3 provides a stable map $S(Z_V) \to S(Z_W)$ of fiberwise degree coprime to p. Now [35, Theorem 11.4.1, Proposition 11.4.2] implies that $j_{K(1)}^Z(\alpha) = 0$ in $J_G^1(Z)$.

Conversely, if $j_{K(1)}^Z(\alpha) = 0$, then as p-completion is faithful for finitely generated $\mathbb{Z}_{(p)}$-modules we can decomplete to say that α is sent to zero in

$$\mathrm{Coker}\left(\psi^\ell - \psi^1 \colon KO_G^0(Z)_{(p)} \to KO_G^0(Z)_{(p)}\right),$$

and thus also in $J_G^{\mathrm{alg}}(Z)_{(p)}$. By Lemma 5.3.4 we deduce $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)$ and thus there is an invertible element in $\pi_\alpha^G D(\Sigma_+^\infty Z)_{(p)}$ by Proposition 5.3.2. ∎

5.4. Localization arguments

We now make good on Lemma 5.3.4. Given odd positive integers $\vec{r} = (r_1, \ldots, r_t)$ coprime to $|G|$, set $r = r_1 \cdots r_t$ and define

$$J_{G,\vec{r}}^{\mathrm{alg}}(Z) = KO_G^0(Z)[\tfrac{1}{r}] / \left(x - \psi^{r_i} x : x \in KO_G^0(Z),\ 1 \le i \le r\right).$$

As $\psi^{r_i} \colon KO_G^0(Z)[\frac{1}{r}] \to KO_G^0(Z)[\frac{1}{r}]$ is a stable operation, it commutes with restrictions and transfers, and so $J_{G,\vec{r}}^{\mathrm{alg}}(Z)$ is a quotient Mackey functor of $KO_G^0(Z)[\frac{1}{r}]$. It relates to $J_G^{\mathrm{alg}}(Z)$ via a commutative diagram

$$\begin{array}{ccc} KO_G^0(Z) & \xrightarrow{\ j\ } & J_G^{\mathrm{alg}}(Z) \\ \downarrow{\scriptstyle j_{\vec{r}}} & & \downarrow \\ J_{G,\vec{r}}^{\mathrm{alg}}(Z) & \twoheadrightarrow & J_G^{\mathrm{alg}}(Z)[\frac{1}{r}]. \end{array}$$

If $k \equiv l \pmod{|G|}$ then $\psi^k = \psi^l$ on $RO(G)$, and it follows that the bottom surjection is an isomorphism for $Z = *$ provided that $\vec{r}$ generates $(\mathbb{Z}/|G|)^\times$.

Lemma 5.4.1. *Suppose that $\vec{r}$ generates $(\mathbb{Z}/|G|)^\times$ and $Z^C \neq \emptyset \Rightarrow |\alpha^C| = 0$ for all cyclic $C \subset G$. Then $j(\alpha) \in J_{G,\vec{r}}^{\mathrm{alg}}(Z)$ is $|G|$-power torsion.*

Proof. The claim is that $j(\alpha) = 0$ in $J_{G,\vec{r}}^{\mathrm{alg}}(Z)[\frac{1}{|G|}]$. As $J_{-,\vec{r}}^{\mathrm{alg}}(Z)$ is a Mackey functor, we may use Lemma 4.3.1 to reduce to showing that if $Z^H \neq \emptyset$ then $j_{H,\vec{r}}(\alpha) = 0$.

So suppose $Z^H \neq \emptyset$. It follows that $Z^C \neq \emptyset$ for all cyclic subgroups $C \subset H$. Thus $|\alpha^C| = 0$ for all cyclic subgroups $C \subset H$ by assumption, and so $j_H(\alpha) = 0$ in $J_H^{\mathrm{alg}}(*)$ by Theorem 5.1.3 ((1)⇔(4)). As $\vec{r}$ generates $(\mathbb{Z}/|G|)^\times$ we have $J_H^{\mathrm{alg}}(*)[\frac{1}{r}] \cong J_{H,\vec{r}}^{\mathrm{alg}}(*)$, implying $j_{H,\vec{r}}(\alpha) = 0$ in $J_{H,\vec{r}}^{\mathrm{alg}}(*)$. Pulling back along $Z \to *$ it follows that $j_{H,\vec{r}}(\alpha) = 0$ in $J_{H,\vec{r}}^{\mathrm{alg}}(Z)$. ∎

Proposition 5.4.2. *The class $j(\alpha) \in J_G^{\mathrm{alg}}(Z)$ has finite order if and only if $Z^C \neq \emptyset \Rightarrow |\alpha^C| = 0$ for all cyclic subgroups $C \subset G$. In this case, its order divides a power of $|G|$.*

Proof. First suppose $Z^C \neq \emptyset \Rightarrow |\alpha^C| = 0$ for all cyclic subgroups $C \subset G$. By Lemma 5.4.1 and the comparison map $J_{G,\vec{r}}^{\mathrm{alg}}(Z) \to J_G^{\mathrm{alg}}(Z)[\frac{1}{r}]$, we find that if $\vec{r}$ generates $(\mathbb{Z}/|G|)^\times$ then $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)[\frac{1}{r|G|}]$. So fix $\vec{r}$ generating $(\mathbb{Z}/|G|)^\times$ and $\vec{s}$ generating $(\mathbb{Z}/r|G|)^\times$. Then $\vec{s}$ also generates $(\mathbb{Z}/|G|)^\times$, so $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)[\frac{1}{r|G|}]$ and $J_G^{\mathrm{alg}}(Z)[\frac{1}{s|G|}]$, and $\gcd(r, s) = 1$ then implies $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)[\frac{1}{|G|}]$. Thus $j(\alpha)$ has finite order dividing a power of $|G|$.

Conversely, if $Z^C \neq \emptyset$, then there is some equivariant map $p\colon G/C \to Z$. This must satisfy $p^*(j(\alpha)) = j_C(\alpha) \in J_G^{\mathrm{alg}}(G/C) \cong J_C^{\mathrm{alg}}(*)$. It follows that if $j(\alpha)$ is torsion, then so is $j_C(\alpha)$, implying that $j_C(\alpha) = 0$ as $J_C^{\mathrm{alg}}(*)$ is torsion-free. Thus $|\alpha^C| = 0$ by Theorem 5.1.3. ∎

Corollary 5.4.3. *If $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)_{(|G|)}$, then $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)$.*

Proof. If $j(\alpha) = 0$ in $J_G^{\mathrm{alg}}(Z)_{(|G|)}$, then

$$j(\alpha) \in J_G^{\mathrm{alg}}(Z)$$

has finite order coprime to $|G|$. Proposition 5.4.2 then implies that $j(\alpha)$ also has order dividing a power of $|G|$, so $j(\alpha) = 0$. ∎

At this point, one could also carry out a J-theoretic analogue of Section 4.5, giving information about invertible elements in $|G|$-local G-ring spectra. We leave the details to the interested reader.

6. Examples

In this section we give examples of the material of the previous sections, focusing especially on the J-homomorphism

$$\pi_\lambda KO_G \to \pi_\star C(a_\lambda)^\times$$

derived from Theorem 2.2.1 and the equivalence $D(\Sigma_+^\infty S(\lambda)) \simeq C(a_\lambda)$ guaranteed by the cofiber sequence $S(\lambda)_+ \to S^0 \to S^\lambda$. Our goal is to demonstrate that this is in fact quite computable, and that it produces explicit computational information about G-equivariant homotopy theory and G-equivariant stable stems.

The bulk of our work in this section lies in computing the groups $\pi_\star KO_G$, particularly information about $a_\lambda\colon \pi_\lambda KO_G \to RO(G)$. We discuss this in general Section 6.1, and the examples we give have been chosen to illustrate such computations. We focus on the case where G is finite.

In our examples we write

$$j_n \in \pi_n S$$

for the J-image of a generator of $\pi_{n+1} KO$, defined up to a sign, with the understanding that $j_0 = \pm 2$.

6.1. Computing with equivariant K-theory

Let G be a compact Lie group and λ be a G-representation. To apply our machinery to produce periodicities on a_λ-torsion, one needs to be able to understand the groups $\pi_{*\lambda}KF_G$:

$$RF(G) = \pi_0 KF_G \xleftarrow{a_\lambda} \pi_\lambda KF_G \xleftarrow{a_\lambda} \pi_{2\lambda} KF_G \xleftarrow{a_\lambda} \cdots .$$

When $F = \mathbb{C}$ and λ is a complex representation, equivariant Bott periodicity implies

$$\pi_\lambda KU_G \cong RU(G)\{\beta_\lambda\}, \quad a_\lambda \beta_\lambda = e_\lambda,$$

where if V is a complex G-representation then we write

$$e_V = \sum_i (-1)^i \Lambda^i V \in RU(G)$$

for the K-theory Euler class of V, see for example [4, Section IV.1]. This Euler class can be computed using character information, for example combining the character identity $\chi_{\psi^k V}(g) = \chi_V(g^k)$ with Newton's identity $k\Lambda^k V = \sum_{i=1}^k (-1)^{i-1} \psi^i V \cdot \Lambda^{k-i} V$. It can also be computed using representation information: if $p_V(g,t)$ is the characteristic polynomial of $g\colon V \to V$, then $\chi_{e_V}(g) = p_V(g,1)$.

In general we do not have a complete recipe for $\pi_{*\lambda}KU_G$ when λ does not admit a complex structure. As one always has Bott periodicity isomorphisms $\pi_{(*+2)\lambda}KU_G \cong \pi_{*\lambda}KU_G\{\beta_{\mathbb{C}\otimes\lambda}\}$, we are left with the following problem.

Problem 6.1.1. For a real G-representation λ, describe the sequence

$$RU(G) \cong \pi_0 KU_G \xleftarrow{a_\lambda} \pi_\lambda KU_G \xleftarrow{a_\lambda} \pi_{2\lambda} KU_G \cong RU(G)\{\beta_{\mathbb{C}\otimes\lambda}\},$$

the composite of which is multiplication by $e_{\mathbb{C}\otimes\lambda}$.

Remark 6.1.2. If G is finite, then for any $\alpha \in RO(G)$, there is a natural character isomorphism

$$\mathbb{C} \otimes \pi_\alpha KU_G \cong \prod_{\langle g\rangle} \widetilde{H}^0\big(S^{\alpha^g}/C(g), \mathbb{C}[\beta^{\pm 1}]\big).$$

Here, the product is over the conjugacy classes of elements $g \in G$, and $C(g)$ is the centralizer of g acting on the fixed points S^{α^g}. In particular, the sequence of Problem 6.1.1 is easily understood after complexification.

Remark 6.1.3. Karoubi [17] has shown that $\pi_\alpha KU_G$ is a free abelian group for any $\alpha \in RO(G)$. The ranks of these free abelian groups are determined by Remark 6.1.2, as described in [17, Theorem 1.8], so this completely describes $\pi_\star KU_G$ additively.

Once enough is known about $\pi_\star KU_G$, one can descend to $\pi_\star KO_G$ using the homotopy fixed point spectral sequence (HFPSS)

$$E_2 = H^*(C_2; \pi_\star KU_G) \implies \pi_{\star-*} KO_G,$$

where $C_2 = \{\psi^{\pm 1}\}$ acts on KU_G by complex conjugation. This also describes $\pi_\star KSp_G$ as $KSp_G \simeq \Sigma^4 KO_G$. We make some observations about this spectral sequence.

Remark 6.1.4. The structure of KO^G was determined by Segal [30], and this in turn determines the HFPSS for $\pi_* KO_G$ in integer degrees. See [23, Section 9] for a detailed discussion. In particular, KU^G is a free KU-module with basis indexed by the irreducible complex G-representations, and likewise KO^G splits into a sum of KO-modules indexed by the irreducible real G-representations, where these summands are of the form KO, KU, or KSp, corresponding to the orthogonal, complex, and symplectic irreducibles.

Remark 6.1.5. The symplectic orientation of KO implies that if V is a quaternionic G-representation, then the associated KU-Thom class in $\pi_{V-|V|} KU_G$ descends to KO_G. See [12, Section 5] for further discussion. In this case, $F(S^V, KO_G) \simeq \Sigma^{|V|} KO_G$ has fixed points determined by the representation theory of G as in Remark 6.1.4. If for example V is a complex G-representation (such as $\mathbb{C} \otimes_{\mathbb{R}} U$ for a real representation U), then $V + \psi^{-1} V = \mathbb{H} \otimes_{\mathbb{C}} V$ is a quaternionic representation, so this reduces the computation of $\pi_\star KO_G$ to essentially finitely many degrees.

Now suppose that G is finite. We can further cut down the amount of work needed to understand $\pi_\star KF_G$ as follows. For $F = \mathbb{R}$ or $\mathbb{C}$, write

$$\varepsilon \colon KF^G \to KF, \quad \varepsilon \colon RF(G) \to \mathbb{Z}, \quad \varepsilon(V) = \dim_F(V/G)$$

for the projection onto the summand corresponding to the trivial representation.

Lemma 6.1.6. *For $F = \mathbb{R}$ or $\mathbb{C}$, the composite*

$$\langle -, - \rangle \colon \; KF^G \otimes_{KF} KF^G \xrightarrow{\mu} KF^G \xrightarrow{\varepsilon} KF$$

is adjoint to an equivalence $KF^G \simeq \mathcal{M}\mathrm{od}_{KF}(KF^G, KF)$.

Proof. As base change $\mathcal{M}\mathrm{od}_{KO} \to \mathcal{M}\mathrm{od}_{KU}$ is conservative, satisfies $KU \otimes_{KO} (KO^G) \simeq KU^G$, and is compatible with ε, it suffices to consider the case $F = \mathbb{C}$. As KU^G is free over KU, it suffices to show that $\langle -, - \rangle$ induces a perfect pairing $RU(G) \otimes_{\mathbb{Z}} RU(G) \to \mathbb{Z}$ on π_0. This pairing acts on irreducible G-representations V and W by

$$\langle V, W \rangle = \dim_{\mathbb{C}} \big((V \otimes W)/G \big) = \dim_{\mathbb{C}} \big(\mathrm{Hom}(V^\vee, W)^G \big) = \begin{cases} 1, & W \cong \psi^{-1} V, \\ 0, & \text{otherwise.} \end{cases}$$

This is evidently a perfect pairing, though note that it is not quite the usual inner product on $RU(G)$. ∎

This is an integral version of the height 1 case of the $K(n)$-local duality considered by Strickland in [33]. It has the following consequence.

Proposition 6.1.7. *Let $F = \mathbb{R}$ or $\mathbb{C}$ and let M be a KF_G-module. Then there is an equivalence*

$$\begin{gathered} \mathcal{M}\mathrm{od}_{KF_G}(M, KF_G) \xrightarrow{\sim} \mathcal{M}\mathrm{od}_{KF}(M^G, KF), \\ (f \colon M \to KF_G) \mapsto (\varepsilon \circ f^G \colon M^G \to KF^G \to KF) \end{gathered}$$

of KF-modules.

Proof. This is a natural transformation

$$\mathcal{M}\mathrm{od}_{KF_G}(-, KF_G) \to \mathcal{M}\mathrm{od}_{KF}\big((-)^G, KF\big)$$

between limit-preserving functors $\mathcal{M}\mathrm{od}_{KF_G}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_{KF}$. It therefore suffices to check that it is an equivalence when evaluated on $G/H_+ \otimes KF_G$ for $H \subset G$. But in this case the map is exactly the equivalence

$$KF^H \simeq \mathcal{M}\mathrm{od}_{KF}(KF^H, KF)$$

guaranteed by Lemma 6.1.6. ∎

Corollary 6.1.8. *For any $\alpha \in RO(G)$, the map*

$$\langle -, - \rangle \colon \pi_\alpha KU_G \otimes_{\mathbb{Z}} \pi_{-\alpha} KU_G \to \mathbb{Z}, \quad \langle x, y \rangle = \varepsilon(xy)$$

is adjoint to an isomorphism $\pi_\alpha KU_G \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_{-\alpha} KU_G, \mathbb{Z})$. Moreover, if $\alpha = \lambda$ is a G-representation then $a_\lambda \colon \pi_0 KU_G \to \pi_{-\lambda} KU_G$ is dual to $a_\lambda \colon \pi_\lambda KU_G \to \pi_0 KU_G$.

Proof. The adjoint $\pi_\alpha KU_G \to \mathrm{Hom}_{\mathbb{Z}}(\pi_{-\alpha} KU_G, \mathbb{Z})$ may be written as

$$\begin{aligned} \pi_\alpha KU_G &\cong \pi_0\big(\mathcal{M}\mathrm{od}_{KU_G}(KU_G \otimes S^\alpha, KU_G)\big) \\ &\xrightarrow{\sim} \pi_0 \mathcal{M}\mathrm{od}_{KU}\big((KU_G \otimes S^\alpha)^G, KU\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\pi_{-\alpha} KU_G, \mathbb{Z}). \end{aligned}$$

Here, the second equivalence is obtained from Proposition 6.1.7, and the third holds as $(KU_G \otimes S^\alpha)^G$ is a free KU-module by Karoubi [17]. The final statement holds just as $a_\lambda \colon S^0 \to S^\lambda$ is dual to $a_\lambda \colon S^{-\lambda} \to S^0$. ∎

Remark 6.1.9. By the Leibniz rule we have

$$0 = d_r\big(\langle x, y\rangle\big) = \big\langle d_r(x), y\big\rangle + \big\langle x, d_r(y)\big\rangle.$$

In particular, the HFPSS for $\pi_{*+\alpha} KO_G$ is determined by the HFPSS for $\pi_{*-\alpha} KO_G$, cutting the work needed to compute $\pi_\star KO_G$ in half.

6.2. Example: Cyclic groups

Character theory ensures that much of the structure of G-equivariant K-theory is controlled by the case where G is a cyclic group. So before giving more exotic examples, we begin by summarizing the structure of cyclic-equivariant K-theory.

6.2.1. Complex circle-equivariant K-theory. Let $T \subset \mathbb{C}^\times$ be the circle group. We begin by describing KU_T. Let L be the tautological complex character of T, so that

$$\pi_0 KU_T \cong RU(T) \cong \mathbb{Z}[L^{\pm 1}].$$

If $\alpha \in RO(T)$, then either α or $\alpha + 1$ lifts to $RU(T)$, but this lift is not canonical. There are several constructions that depend on the choice of a complex structure, and for this reason it can be convenient to think of T-equivariant homotopy groups as graded not over $RO(T)$, but over "$RU(T)$ adjoined $\mathbb{R} = \frac{1}{2}\mathbb{C}$", i.e., the group

$$\big(\mathbb{Z}\{L^n : n \in \mathbb{Z}\} \oplus \mathbb{Z}\{1\}\big)/(L^0 - 2).$$

Associated to every virtual complex representation $\alpha = \sum_i n_i L^i$ is the invertible Bott class

$$\beta_\alpha = \prod_i \beta_{L^i}^{n_i} \in \pi_\alpha KU_T \cong RU(T)\{\beta_\alpha\}.$$

Together with $\pi_1 KU_T = 0$, this completely determines $\pi_\star KU_T$.

The Adams operation ψ^{-1} acts on $\pi_\star KU_T$ by multiplicative automorphisms, satisfying

$$\psi^{-1}(L) = L^{-1}, \quad \psi^{-1}(\beta_{L^i}) = -L^{-i} \cdot \beta_{L^i}.$$

The Adams operation ψ^k for $k > 0$ acts on $\pi_V KU_T \cong \widetilde{KU}_0^T(S^V)$ for V an actual representation by ring endomorphisms, where it is determined by

$$\psi^k(L) = L^k, \quad \psi^k(\beta_{L^i}) = (1 + L^i + L^{2i} + \cdots + L^{(k-1)i}) \cdot \beta_{L^i}.$$

See for example [4]. Finally, if $V = \sum_i n_i L^i$ with $n_i \geq 0$, then the Euler class $e_V \in RU(T)$ is given by

$$e_V = \prod_i (1 - L^i)^{n_i}.$$

6.2.2. Real circle-equivariant *K*-theory. A virtual T-representation $\alpha = \sum_i n_i L^i$ admits a Spin structure if and only if its second Stiefel–Whitney class

$$w_2(\alpha) \equiv \sum_i n_i \cdot i \pmod 2$$

vanishes. In this case α is KO-orientable, in the sense that there is an equivalence

$$F(S^\alpha, KO_T) \simeq F(S^{|\alpha|}, KO_T).$$

If $|\alpha|$ is a multiple of 8, then this is realized by an invertible Bott class

$$\beta_\alpha^{\mathrm{Spin}} \in \pi_\alpha KO_T.$$

However, if $\psi^{-1}(\alpha) \neq \alpha$, then the image of $\beta_\alpha^{\mathrm{Spin}}$ under the complexification $c\colon KO_T \to KU_T$ is *not* guaranteed to be the complex Bott class $\beta_\alpha \in \pi_\alpha KU_T$. Instead, the value of $c(\beta_\alpha^{\mathrm{Spin}})$ can be determined as follows. Abbreviate $n = \sum_i n_i \cdot i$. Under the assumption that $|\alpha|$ is a multiple of 8, we then have

$$\psi^{-1}(\beta_\alpha) = L^{-n} \cdot \beta_\alpha.$$

It follows that $\pm L^{-n/2}\beta_\alpha$ are the only units in $\pi_\alpha KU_T$ fixed by ψ^{-1}. As the Bott class is compatible with restriction, we must have $\mathrm{res}_e^T(c(\beta_\alpha^{\mathrm{Spin}})) = \beta^{|\alpha|/2}$, and the only possibility is that

$$c(\beta_\alpha^{\mathrm{Spin}}) = L^{-n/2}\beta_\alpha.$$

We can now describe $\pi_\star KO_T$ in general. For any $\alpha = \sum_i n_i L^i$, either α or $\alpha + L$ is Spin, so the above discussion provides a KO_T-linear equivalence between $\Sigma^{-\alpha} KO_T$ and an integer suspension of one of KO_T or $\Sigma^{-L} KO_T$. Hence it suffices to describe the fixed points of these.

In the former case we have $\pi_0 KO_T = RO(T)$ and in general

$$KO^T \cong KO\{1\} \oplus \bigoplus_{i>0} KU,$$

where

$$\mathrm{Im}(\pi_{2n} KO_T \to \pi_{2n} KU_T) = \begin{cases} \big(\mathbb{Z}\{1\} \oplus \mathbb{Z}\{L^i + L^{-i} : i \geq 1\}\big)\beta^n, & n \equiv 0 \pmod 4, \\ \big(\mathbb{Z}\{2\} \oplus \mathbb{Z}\{L^i + L^{-i} : i \geq 1\}\big)\beta^n, & n \equiv 2 \pmod 4, \\ \mathbb{Z}\{L^i - L^{-i} : i \geq 1\}\beta^n, & n \equiv 1,3 \pmod 4. \end{cases}$$

In the latter case, $H^{>0}(C_2; \pi_{*+L} KU_T) = 0$, and it follows

$$(\Sigma^{-L} KO_T)^T \simeq \bigoplus_{i \geq 0} KU,$$

where

$$\mathrm{Im}(\pi_{2n+L} KO_T \to \pi_{2n+L} KU_T) = \begin{cases} \mathbb{Z}\{L^i - L^{-i-1}\}\beta^n \beta_L, & n \equiv 0 \pmod 2, \\ \mathbb{Z}\{L^i + L^{-i-1}\}\beta^n \beta_L, & n \equiv 1 \pmod 2. \end{cases}$$

This completely determines $\pi_\star KO_T$.

6.2.3. Finite cyclic groups. Let $C_n \subset T$ denote the finite subgroup of order n, so

$$\pi_0 KU_{C_n} \cong RU(C_n) \cong \mathbb{Z}[L]/(L^n - 1).$$

If $\alpha = \sum_i n_i L^i$, then by restricting the above KO_T-linear equivalences we obtain KO_{C_n}-linear equivalences between $\Sigma^\alpha KO_{C_n}$ and an integer suspension of KO_{C_n} or $\Sigma^{-L} KO_{C_n}$. For n even, write σ for the real sign representation of C_n, satisfying $\mathbb{C} \otimes \sigma = L^{n/2}$. Then we can identify

$$KO^{C_n} \simeq \begin{cases} KO\{1, \sigma\} \oplus KU^{n/2-1}, & n \equiv 0 \pmod 2, \\ KO\{1\} \oplus KU^{(n-1)/1}, & n \equiv 1 \pmod 2, \end{cases}$$

and

$$(\Sigma^{2-L} KO_{C_n})^{C_n} \simeq \begin{cases} KU^{n/2}, & n \equiv 0 \pmod 2, \\ KO\{L^{(n-1)/2}\beta_L \beta^{-1}\} \oplus KU^{(n-1)/2}, & n \equiv 1 \pmod 2, \end{cases}$$

with the images of $\pi_* KO_{C_n}$ and $\pi_{*+L} KO_{C_n}$ in $\pi_\star KU_{C_n}$ easily determined as in the T-equivariant case. This completely describes $\pi_\star KO_{C_n}$ if n is odd, but if n is even then one must also account for degrees involving σ. For this reason it can be convenient to grade C_n-equivariant computations over "$RU(C_n)$ adjoined $\mathbb{R} = \frac{1}{2}\mathbb{C}$ and $\sigma = \frac{1}{2}L^{n/2}$", i.e.,

$$\big(\mathbb{Z}\{L^0, \dots, L^{n-1}\} \oplus \mathbb{Z}\{1,\sigma\}\big)/(L^0 - 2,\ L^{n/2} - \sigma),$$

in order to incorporate a choice of complex structure into the grading. If α is an element therein, then the above discussion gives a KO_{C_n}-linear equivalence between $\Sigma^\alpha KO_{C_n}$ and an integer suspension of one of

$$KO_{C_n}, \quad \Sigma^{-L}KO_{C_n}, \quad \Sigma^{-\sigma}KO_{C_n}, \quad \Sigma^{-L-\sigma}KO_{C_n}.$$

The first two cases were described above, and the latter two can be computed using the cofiber sequence

$$S^{-\sigma} \to S^0 \to D(C_n/C_{n/2+}),$$

ultimately allowing one to identify

$$(\Sigma^{-\sigma}KO_{C_n})^{C_n} \simeq \begin{cases} KO \oplus \Sigma^{-1}KO \oplus KU^{n/4}, & n \equiv 0 \pmod 4, \\ KO\{1\} \oplus KU^{(n-2)/4}, & n \equiv 2 \pmod 4, \end{cases}$$

as an augmentation ideal of KO^{C_n}, and similarly

$$(\Sigma^{2-L-\sigma}KO_{C_n})^{C_n} \simeq \begin{cases} KU^{n/4}, & n \equiv 0 \pmod 4, \\ KU^{(n-2)/4} \oplus \Sigma^{-2}KO, & n \equiv 2 \pmod 4. \end{cases}$$

Example 6.2.3.1. Let us give details for the identification of $(\Sigma^{2-L-\sigma}KO_{C_n})^{C_n}$ for $n \equiv 2 \pmod 4$, as the rest are similar or easier. The fiber sequence

$$(\Sigma^{2-L-\sigma}KO_{C_n})^{C_n} \to (\Sigma^{2-L}KO_{C_n})^{C_n} \xrightarrow{\text{res}} (\Sigma^{2-L}KO_{C_{n/2}})^{C_{n/2}},$$

may be identified as

$$\begin{aligned}(\Sigma^{2-L-\sigma}KO_{C_n})^{C_n} &\to KU\big\{(L^i - L^{-i-1})\beta_L\beta^{-1} : 0 \le i < \tfrac{n}{2}\big\} \\ &\xrightarrow{\text{res}} KO\{L^{(n-2)/4}\beta^{-1}\beta_L\} \oplus KU\big\{(L^i - L^{-i-1})\beta_L\beta^{-1} : 0 \le i < \tfrac{n-2}{4}\big\},\end{aligned}$$

where we name summands for how generators in π_0 are named after base change to KU. We always have

$$\text{res}^{C_n}_{C_{n/2}}\big((L^i + L^{-i-1})\beta_L\beta^{-1}\big) = \text{res}^{C_n}_{C_{n/2}}\big((L^{i+n/2} + L^{-i-n/2-1})\beta_L\beta^{-1}\big).$$

If $0 \le i < (n-2)/4$ then these terms are distinct before restriction, allowing us to split off a copy of

$$KU\big\{\big((L^i + L^{-i-1}) - (L^{i+n/2} + L^{-i-n/2-1})\big)\beta_L\beta^{-1} : 0 \le i < \tfrac{n-2}{4}\big\}$$

in the fiber. Thus $(\Sigma^{2-L-\sigma} KO_{C_n})^{C_n} \simeq KU^{(n-2)/4} \oplus F$ where

$$F = \mathrm{Fib}\big(r\colon KU\big\{(L^{(n-2)/4} + L^{(-n-2)/4})\beta^{-1}\beta_L\big\} \to KO\{L^{(n-2)/4}\beta^{-1}\beta_L\}\big),$$

which we are claiming is equivalent to $\Sigma^{-2}KO$. Observe that r sends the generator of $\pi_4 KU$ to that of $\pi_4 KO$. The Wood cofiber sequence $\Sigma KO \xrightarrow{\eta} KO \to KU$ yields

$$\mathcal{M}\mathrm{od}_{KO}(KU, KO) \simeq KU,$$

so this in fact characterizes r. Thus r is equivalent to the twofold desuspension of the boundary map in the Wood cofiber sequence, implying $F \simeq \Sigma^{-2}KO$ as claimed.

6.2.4. Examples. There is already an extensive literature on the K-theory and J-theory of the lens spaces $S(kL)/C_n$, more than we can hope to summarize here. We just give some small examples illustrating general phenomena relevant to our study of periodicities.

Example 6.2.4.1. Take $n = 2$. The behavior of the units in $\pi_{2\gamma(m)(1-\sigma)} C(a_\sigma^{m+1})$ discussed in the introduction was analyzed in detail by Araki and Iriye in [3, Section 3]. We give an example to highlight the nature of compatibility between these elements as m varies.

As $4\sigma = \mathbb{H} \otimes_{\mathbb{R}} \sigma$ is 4-dimensional quaternionic, we have

$$\pi_{4\sigma} KO_{C_2} \cong RO(C_2) \cdot 2\beta_{4\sigma}, \quad \pi_{8\sigma} KO_{C_2} \cong RO(C_2) \cdot \beta_{8\sigma},$$

where

$$a_\sigma^4 \cdot \beta_{8\sigma} = (1-\sigma) \cdot 2\beta_{4\sigma}, \quad a_\sigma^4 \cdot 2\beta_{4\sigma} = 4(1-\sigma), \quad a_\sigma^8 \cdot \beta_{8\sigma} = 16(1-\sigma).$$

Hence there exist u_σ-elements

$$u_{4\sigma} = J(2\beta_{4\sigma}) \in \pi_{4(1-\sigma)} C(a_\sigma^4), \quad u_{8\sigma} = J(\beta_{8\sigma}) \in \pi_{8(1-\sigma)} C(a_\sigma^8),$$

and up to choices of orientations these satisfy

$$\mathrm{res}_e^{C_2}\big(\partial(u_{4\sigma})\big) = \nu \in \pi_3 S, \quad \mathrm{res}_e^{C_2}\big(\partial(u_{8\sigma})\big) = \sigma \in \pi_7 S.$$

As $\mathrm{res}_e^{C_2}(\partial(u_{4\sigma}^2)) = 2\nu \neq 0$, it follows $u_{4\sigma}^2$ cannot lift to $C(a_\sigma^5)$, so the map $q\colon C(a_\sigma^8) \to C(a_\sigma^4)$ must satisfy $q(u_{8\sigma}) \neq u_{4\sigma}^2$. The difference $\varepsilon = u_{4\sigma}^2 \cdot q(u_{8\sigma})^{-1}$ is measured by

$$\begin{aligned}\varepsilon = u_{4\sigma}^2 \cdot q(u_{8\sigma})^{-1} &= J(2\beta_{4\sigma})^2 \cdot J(a_\sigma^4 \beta_{8\sigma})^{-1} \\ &= J(2 \cdot 2\beta_{4\sigma} - a_\sigma^4 \beta_{8\sigma}) = J\big((1+\sigma) \cdot 2\beta_{4\sigma}\big).\end{aligned}$$

Thus $\varepsilon \in \pi_0 C(a_\sigma^4)^\times$ is a class satisfying $\mathrm{res}_e^{C_2}(\partial(\varepsilon)) = 2\nu$. In fact,

$$J\big((1+\sigma) \cdot 2\beta_{4\sigma}\big) = J\big(\mathrm{tr}_e^{C_2}(2\beta^2)\big) = N_e^{C_2}\big(J(2\beta^2)\big),$$

where $N_e^{C_2}\colon \pi_0^e C(a_\sigma^4) \to \pi_0^{C_2} C(a_\sigma^4)$ is the norm, and where if we write $\pi_0^e C(a_\sigma^4) \cong \pi_0 D(S_+^3) \cong \mathbb{Z}[\overline{\nu}]/(\overline{\nu}^2, 24\overline{\nu})$ then $J(2\beta^2) = 1 + \overline{\nu}$.

Example 6.2.4.2. We give an example of the remarks at the end of Section 5.1. We were guided to existence of an example like this by [18]. Take $G = C_8$ and let

$$\rho = L + L^3 + L^5 + L^7.$$

This representation is quaternionic of real dimension 8, and thus the Bott class $\beta_\rho \in \pi_\rho KU_{C_8}$ descends to KO_{C_8}, yielding

$$\pi_\rho KO_{C_8} \cong RO_{C_8}\{\beta_\rho\}.$$

Set $RO_{C_8} = \mathbb{Z}\{1, \sigma, \lambda, \mu, \mu'\}$ where

$$\mathbb{C} \otimes \sigma = L^4, \quad \mathbb{C} \otimes \lambda = L^2 + L^6, \quad \mathbb{C} \otimes \mu = L + L^7, \quad \mathbb{C} \otimes \mu' = L^3 + L^5.$$

Then $\psi^3(\mu) = \mu'$ implies $S^\mu[\frac{1}{3}] \simeq S^{\mu'}[\frac{1}{3}]$, and $(\psi^3 - \psi^1)^2(\mu) = 2(\mu - \mu')$ implies $S^{2\mu} \simeq S^{2\mu'}$. If we identify μ and μ' as the underlying real representations of L^7 and L^5 respectively, then a particular choice of invertible element $\psi_3 \in \pi_{\mu'-\mu} S_{C_8}[\frac{1}{3}]$ is constructed in Example 5.1.2.

The Euler class of ρ is given by

$$e_\rho = (1 - L)(1 - L^3)(1 - L^5)(1 - L^7) = 4 + 2\sigma + 2\lambda - 2\mu - 2\mu'.$$

A calculation reveals that $2 - \mu$ has order 16 in $RO(C_8)/(e_\rho)$, with

$$16(2 - \mu) = e_\rho \cdot (11 - 5\sigma - \lambda - 4\mu + 4\mu');$$

so the best that the J-homomorphism gives is an invertible element

$$u_{16\mu} = J\big((11 - 5\sigma - \lambda - 4\mu + 4\mu')\beta_\rho\big) \in \pi^{C_8}_{16(2-\mu)} C(a_\rho).$$

As $\mathrm{res}^{C_8}_e((11 - 5\sigma - \lambda - 4\mu + 4\mu')\beta_\rho) = 4\beta^4 \in \pi_8 KO$, this invertible element satisfies $\mathrm{res}^{C_8}_e(\partial(u_{16\mu})) = \pm 4\sigma$, and in the context of Theorem B this yields a differential

$$d_8(u_\mu^{16}) = \pm 4\sigma \cdot a_\rho u_\rho^{-1} \cdot u_\mu^{16}$$

in the C_8-homotopy fixed point spectral sequence. Here, $a_\rho u_\rho^{-1}$ generates $H^8(C_8; \pi_0 S)$.

On the other hand, 8μ is locally J-equivalent to $3\mu + 5\mu'$, and another calculation shows

$$16 - (3\mu + 5\mu') = e_\rho \cdot (5 - 3\sigma - \lambda - \mu'),$$

thus giving $u_{3\mu+5\mu'} \in \pi^{C_8}_{16-3\mu-5\mu'} C(a_\rho)$. Hence we obtain an invertible element

$$\psi_3^5 \cdot u_{3\mu+5\mu'} \in \pi^{C_8}_{8(2-\mu)} C(a_\rho)[\tfrac{1}{3}]$$

playing the role of "$u_{8\mu}$". In the context of Theorem B, by Example 5.1.5 this yields a differential

$$d_8(u_\mu^8) = \pm 2\sigma \cdot a_\rho u_\rho^{-1} \cdot u_\mu^8$$

in the C_8-homotopy fixed point spectral sequence, refining the above differential on u_μ^{16}.

Example 6.2.4.3. Let $G = C_6$ and $\rho = L + L^5$. Then $\alpha = (1 - L^2 - L^3 + L^5)$ satisfies

$$e_\rho \cdot \alpha = \alpha,$$

so there exist compatible t_α-elements in $\pi_\alpha^{C_6} C(a_\rho^n)$ for all $n \geq 1$. As $S(\infty\rho)$ is a model for EC_6, by taking $n \to \infty$ this produces a t_α-element in the completion $F(EC_{6+}, S_{C_6})$. This phenomenon is generic for composite order groups, corresponding to the kernel of the completion map $RU(G) \to KU^0 BG$.

6.3. Example: The symmetric group on 3 letters

We give an example with an orthogonal irreducible. Let $G = \Sigma_3$ be the symmetric group on 3 letters. Write σ for its real sign representation and λ for its reduced real canonical permutation representation, so that

$$RO(\Sigma_3) \cong \mathbb{Z}\{1, \sigma, \lambda\}, \quad \sigma^2 = 1, \quad \sigma\lambda = \lambda, \quad \lambda^2 = 1 + \sigma + \lambda.$$

Complexification $RO(\Sigma_3) \to RU(\Sigma_3)$ is an isomorphism; write $RU(\Sigma_3) = \mathbb{Z}\{1, \sigma_\mathbb{C}, \lambda_\mathbb{C}\}$. Consider the sequence

$$RU(\Sigma_3) \cong \pi_0 KU_{\Sigma_3} \xleftarrow{a_\lambda} \pi_\lambda KU_{\Sigma_3} \xleftarrow{a_\lambda} \pi_{2\lambda} KU_{\Sigma_3} \cong RU(\Sigma_3)\{\beta_{\lambda_\mathbb{C}}\}.$$

The composite is determined by

$$a_\lambda^2 \beta_{\lambda_\mathbb{C}} = a_{\lambda_\mathbb{C}} \beta_{\lambda_\mathbb{C}} = e_{\lambda_\mathbb{C}} = 1 - \lambda_\mathbb{C} + \sigma_\mathbb{C},$$

and has image the rank 1 subspace $\mathbb{Z}\{1 - \lambda_\mathbb{C} + \sigma_\mathbb{C}\} \subset RU(\Sigma_3)$. As $\pi_\lambda KU_{\Sigma_3}$ is a free abelian group of rank 1 [17], the only possibility is that $\pi_\lambda KU_{\Sigma_3} \cong \mathbb{Z}\{a_\lambda \beta_{\lambda_\mathbb{C}}\}$ with $\sigma_\mathbb{C} a_\lambda = a_\lambda$ and $\lambda_\mathbb{C} a_\lambda = -a_\lambda$. Thus $\pi_{*\lambda} KU_{\Sigma_3}$ has the 2-periodic pattern

$$\mathbb{Z}\{1, \sigma_\mathbb{C}, \lambda_\mathbb{C}\} \xleftarrow{a_\lambda} \mathbb{Z}\{a_\lambda \beta_{\lambda_\mathbb{C}}\} \xleftarrow{a_\lambda} \mathbb{Z}\{1, \sigma_\mathbb{C}, \lambda_\mathbb{C}\}\beta_{\lambda_\mathbb{C}} \xleftarrow{a_\lambda} .$$

Complex conjugation acts trivially on $RU(\Sigma_3)$, and using $\mathrm{res}^{\Sigma_3}_{C_2}(\lambda) \cong 1 + \sigma$ and $\mathrm{res}^{\Sigma_3}_{C_3}(\lambda_\mathbb{C}) \cong L + L^{-1}$ we find

$$\psi^{-1}(\beta_{\lambda_\mathbb{C}}) = \sigma_\mathbb{C} \beta_{\lambda_\mathbb{C}}.$$

Thus $H^0(C_2; \pi_{*\lambda} KU_{\Sigma_3})$ is 4-periodic with

$$H^0(C_2; \pi_{m\lambda} KU_{\Sigma_3}) = \begin{cases} \mathbb{Z}\{1, \sigma_\mathbb{C}, \lambda_\mathbb{C}\}, & m = 0, \\ \mathbb{Z}\{a_\lambda \beta_{\lambda_\mathbb{C}}\}, & m = 1, \\ \mathbb{Z}\{1 + \sigma_\mathbb{C}, \lambda_\mathbb{C}\}\beta_{\lambda_\mathbb{C}}, & m = 2, \\ \mathbb{Z}\{a_\lambda \beta_{\lambda_\mathbb{C}}^2\}, & m = 3. \end{cases}$$

We claim that $H^0(C_2; \pi_{*\lambda} KU_{\Sigma_3})$ consists of permanent cycles. As $4\lambda = \lambda \otimes_\mathbb{R} \mathbb{H}$ is 8-dimensional quaternionic, we reduce to considering $H^0(C_2; \pi_{m\lambda} KU_{\Sigma_3})$ for $m \in \{0, 1, 2, 3\}$. As $\eta \cdot (1 + \sigma_\mathbb{C})\beta_{\lambda_\mathbb{C}} = 0$, necessarily $(1 + \sigma_\mathbb{C})\beta_{\lambda_\mathbb{C}}$ is a permanent cycle.

As a_λ is a permanent cycle and

$$\langle a_\lambda, a_\lambda\beta_{\lambda_\mathbb{C}}\rangle = 1, \quad \langle a_\lambda^2, \lambda_\mathbb{C}\beta_{\lambda_\mathbb{C}}\rangle = -1, \quad \langle a_\lambda^3, a_\lambda\beta_{\lambda_\mathbb{C}}^2\rangle = 3,$$

we deduce following Remark 6.1.9 that the remaining classes are permanent cycles. Thus $\pi_{*\lambda}KO_{\Sigma_3}$ has the 4-periodic pattern

$$\begin{aligned} &\mathbb{Z}\{1,\sigma,\lambda\} \xleftarrow{a_\lambda} \mathbb{Z}\{a_\lambda\beta_{\lambda_\mathbb{C}}\} \xleftarrow{a_\lambda} \mathbb{Z}\big\{(1+\sigma),\lambda\big\}\beta_{\lambda_\mathbb{C}} \\ &\xleftarrow{a_\lambda} \mathbb{Z}\{a_\lambda\beta_{\lambda_\mathbb{C}}^2\} \xleftarrow{a_\lambda} \mathbb{Z}\{1,\sigma,\lambda\}\beta_{\lambda_\mathbb{C}}^2 \xleftarrow{a_\lambda} , \end{aligned}$$

where we now write σ and λ for classes that complexify to $\sigma_\mathbb{C}$ and $\lambda_\mathbb{C}$. In particular, the identities

$$3^{2m}(1+\sigma-\lambda) = a_\lambda^{4m+2}\cdot -\lambda\beta_{\lambda_\mathbb{C}}^{2m+1}, \quad 3^{2m-1}(1+\sigma-\lambda) = a_\lambda^{4m}\cdot\beta_{\lambda\mathbb{C}}^{2m}$$

imply that $C(a_\lambda^{m+1})$ admits a real $t_{1+\sigma-\lambda}$-element of order $3^{\lfloor m/2\rfloor}$. The real $t_{1+\sigma-\lambda}$-elements

$$t_{3^{n-1}(1+\sigma-\lambda)} = \begin{cases} J(\beta_{\lambda_\mathbb{C}}^n), & k \text{ even,} \\ J(-\lambda\beta_{\lambda_\mathbb{C}}^n), & k \text{ odd,}\end{cases} \in \pi^{\Sigma_3}_{3^{n-1}(1+\sigma-\lambda)}C(a_\lambda^{2m})$$

give rise to infinite periodic families

$$\partial(t^k_{3^{n-1}(1+\sigma-\lambda)}) \in \pi_{3^{n-1}k(1+\sigma-\lambda)+2n\lambda-1}S_{\Sigma_3}$$

with the property that

$$\mathrm{res}^{\Sigma_3}_e\left(\partial(t^k_{3^{n-1}(1+\sigma-\lambda)})\right) = k\cdot j_{4n-1}\in\pi_{4n-1}S.$$

6.4. The dihedral group of order 8

We give a more delicate example with an orthogonal irreducible. Let D_8 be the dihedral group of order 8, generated by a rotation r and reflection f. We then have

$$RO(D_8) = \mathbb{Z}\{1,\sigma_r,\sigma_f,\sigma_{rf},\rho\},$$

where σ_g is the character with kernel $\langle r^2, g\rangle$ and ρ is the tautological 2-dimensional representation. These satisfy the usual identity between characters together with

$$\sigma_g\cdot\rho = \rho, \quad \rho^2 = 1+\sigma_r+\sigma_f+\sigma_{rf}.$$

We begin by describing $\pi_{*\rho}KU_{D_8}$. Complexification $RO(D_8)\to RU(D_8)$ is an isomorphism. We will use this to write $RU(D_8) = \mathbb{Z}\{1,\sigma_r,\sigma_f,\sigma_{rf},\rho\}$, only with the understanding that in the context of representation-grading these symbols refer only to their real counterparts. Write $\beta_{2\rho}\in\pi_{2\rho}KU_{D_8} = \pi_{\mathbb{C}\otimes\rho}KU_{D_8}$ for the Bott class of the complex representation $\mathbb{C} \otimes \rho$ of real dimension 4, and let $e_{\rho_{\mathbb{C}}} = a_\rho^2 \beta_{2\rho} \in RU(D_8)$. As r acts by orientation-preserving automorphisms of ρ but f and rf do not, we find $\Lambda^2\rho = \sigma_r$, so that

$$e_{\rho_{\mathbb{C}}} = 1 + \sigma_r - \rho.$$

This determines $\pi_{2*\rho} KU_{D_8}$. We next compute $\pi_\rho KU_{D_8}$. A computation shows

$$\operatorname{Ker}(e_{\rho_{\mathbb{C}}}) = \mathbb{Z}\{1 - \sigma_r, \sigma_f - \sigma_{rf}, 1 + \sigma_f + \rho\},$$
$$(e_{\rho_{\mathbb{C}}}) = \mathbb{Z}\{1 + \sigma_r - \rho, \sigma_f + \sigma_{rf} - \rho\}.$$

In the terminology of [17], only the conjugacy classes of e, r, and r^2 are oriented and even with respect to ρ, and thus $\pi_\rho KU_{D_8} = \mathbb{Z}^3$. As $\operatorname{Ker}(a_\rho\colon \pi_{2\rho} KU_{D_8} \to \pi_\rho KU_{D_8})$ has rank 3 and $\operatorname{Im}(a_\rho^2\colon \pi_{2\rho} KU_{D_8} \to \pi_0 KU_{D_8}) = (e_{\rho_{\mathbb{C}}}) \subset RU(D_8)$ is a split summand, the only possibility is that

$$\pi_\rho KU_{D_8} = \mathbb{Z}\{a_\rho \beta_{2\rho}, a_\rho \sigma_f \beta_{2\rho}, b\}$$

for some b generating the kernel of a_ρ. This determines $\pi_{*\rho} KU_{D_8}$.

We next descend to $\pi_{*\rho} KO_{D_8}$. As $4\rho = \mathbb{H} \otimes_{\mathbb{R}} \rho$ is 8-dimensional quaternionic, we reduce to considering $0 \le * \le 3$. As

$$\operatorname{res}^{D_8}_{\langle r\rangle}(\rho) = L + L^{-1}, \quad \operatorname{res}^{D_8}_{\langle f\rangle}(\rho) = 1 + \sigma = \operatorname{res}^{D_8}_{\langle rf\rangle}(\rho),$$

we must have

$$\psi^{-1}(\beta_{2\rho}) = \sigma_r \beta_{2\rho}.$$

As b must lift a multiple of β and generates the kernel of a_ρ we also have $\psi^{-1}(b) = -b$. Thus

$$H^0(C_2; \pi_{m\rho} KU_{D_8}) = \begin{cases} \mathbb{Z}\{1, \sigma_r, \sigma_f, \sigma_{rf}, \rho\}, & m = 0, \\ \mathbb{Z}\{a_\rho, a_\rho \sigma_f\}\beta_{2\rho}, & m = 1, \\ \mathbb{Z}\big\{(1 + \sigma_r), (\sigma_f + \sigma_{rf}), \rho\big\}\beta_{2\rho}, & m = 2, \\ \mathbb{Z}\{a_\rho, a_\rho \sigma_f\}\beta_{2\rho}^2, & m = 3. \end{cases}$$

We claim that all of these are permanent cycles. For $m = 0$ this holds as $RO(D_8) \cong RU(D_8)$, and using a_ρ this implies it for $m = 1$. Necessarily $(1 + \sigma_r)\beta_{2\rho}$ and $(\sigma_f + \sigma_{rf})\beta_{2\rho}$ are permanent cycles as they annihilate η, and $a_\rho \beta_{2\rho}^2$ and $a_\rho \sigma_f \beta_{2\rho}^2$ are permanent cycles because the same is true of each of a_ρ, $\beta_{2\rho}^2$, and σ_f. Finally, as $a_\rho \cdot \rho\beta_{2\rho} = a_\rho(1 + \sigma_f)\beta_{2\rho}$, we find that $a_\rho\colon H^3(C_2; \pi_{2\rho+2} KU_{D_8}) \to H^3(C_2; \pi_{\rho+2} KU_{D_8})$ is an injection, forcing $\rho\beta_{2\rho}$ to be a permanent cycle. In the end we find that $\pi_{*\rho} KO_{D_8}$ (modulo possible torsion classes) has the 4-periodic pattern

$$\mathbb{Z}\{1, \sigma_r, \sigma_f, \sigma_{rf}, \rho\} \xleftarrow{a_\rho} \mathbb{Z}\{a_\rho, a_\rho \sigma_f\}\beta_{2\rho} \xleftarrow{a_\rho} \mathbb{Z}\big\{(1 + \sigma_r), (\sigma_f + \sigma_{rf}), \rho\big\}\beta_{2\rho}$$
$$\xleftarrow{a_\rho} \mathbb{Z}\{a_\rho, a_\rho \sigma_f\}\beta_{2\rho}^2 \xleftarrow{a_\rho} \mathbb{Z}\{1, \sigma_r, \sigma_f, \sigma_{rf}, \rho\}\beta_{2\rho}^2.$$

Let $\alpha = 1 + \sigma_r - \rho$. Then the identities

$$2^{4m-1}\alpha = a_\rho^{4m}\big((1 + 2^{2m-1}) + (1 - 2^{2m-1})\sigma_f\big)\beta_{2\rho}^{2m},$$
$$2^{4m+1}\alpha = a_\rho^{4m+2}\big(2^{2m-1}(1 + \sigma_r) - 2^{2m-1}(\sigma_f + \sigma_{rf}) - \rho\big)\beta_{2\rho}^{2m+1}$$

for $m > 0$ produce t_α-elements

$$t_{2^{2n-1}\alpha} \in \pi^{D_8}_{2^{2n-1}\alpha} C(a_\rho^{2n})$$

satisfying

$$\operatorname{res}^{D_8}_e\left(\partial(t_{2^{2n-1}\alpha})\right) = \begin{cases} 2j_{4n-1}, & n \equiv 0 \pmod 2, \\ j_{4n-1}, & n \equiv 1 \pmod 2). \end{cases}$$

Likewise, because

$$\operatorname{res}^{D_8}_{\langle r^2\rangle}(a_\rho\beta_{2\rho}) = (a_\sigma\beta_{2\sigma})^2 = \eta_{C_2}^2,$$

the element

$$t_\alpha = J(a_\rho\beta_{2\rho}) \in \pi^{D_8}_\alpha C(a_\rho)$$

satisfies

$$\operatorname{res}^{D_8}_{\langle r^2\rangle}(t_\alpha) = u_{2\sigma} \in \pi^{C_2}_{2(1-\sigma)} C(a_\sigma^2).$$

6.5. Example: The nonabelian group of order 21

We give an example with a complex irreducible. Let $G = C_7 \rtimes C_3 = \langle x,\ y : x^7 = e = y^3, xy = yx^2\rangle$ be the nonabelian group of order 21. Consulting [11] we see that this group has the five conjugacy classes

$$C(e) = \{e\},\quad C(x) = \{x, x^2, x^4\},\quad C(x^3) = \{x^3, x^5, x^6\},$$
$$C(y) = \{y, yx, \ldots, yx^6\},\quad C(y^2) = \{y^2, y^2x, \ldots, y^2x^6\},$$

and character table

	$C(e)$	$C(x)$	$C(x^3)$	$C(y)$	$C(y^2)$
1	1	1	1	1	1
ω	1	1	1	ζ_3^2	ζ_3
$\overline{\omega}$	1	1	1	ζ_3	ζ_3^2
ρ	3	$\zeta_7^3 + \zeta_7^6 + \zeta_7^5$	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	0	0
$\overline{\rho}$	3	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	$\zeta_7^3 + \zeta_7^6 + \zeta_7^5$	0	0

.

Write ω_0 and ρ_0 for the underlying real representations of the complex representations ω and ρ. We describe $\pi_{*\rho_0} KO_{C_7\rtimes C_3}$. Observe

$$\omega\rho = \rho = \overline{\omega}\rho,\quad \rho^2 = \rho + 2\overline{\rho},\quad \rho\overline{\rho} = 1 + \omega + \overline{\omega} + \rho + \overline{\rho}.$$

Using the general character identity

$$\chi_{\Lambda^2 U}(g) = \frac{\chi_U(g)^2 - \chi_U(g^2)}{2},$$

we compute $\Lambda^2\rho = \overline{\rho}$ and thus $e_\rho = \overline{\rho} - \rho$ This determines $\pi_{*\rho} KU_{C_7 \rtimes C_3}$. Restriction gives an injection $\pi_\rho KU_{C_7 \rtimes C_3} \to \pi_{L^3+L^5+L^6} KU_{C_7} \times \pi_{L^0+L^1+L^2} KU_{C_3}$. Using this we compute $\psi^{-1}(\beta_\rho) = -\beta_\rho$. The same calculations works swapping ρ and $\overline{\rho}$, and so we find

$$H^0(C_2; \pi_{m_1\rho+m_2\overline{\rho}} KU_{C_7 \rtimes C_3}) = \begin{cases} \mathbb{Z}\{1, \omega+\overline{\omega}, \rho+\overline{\rho}\}\beta_\rho^{m_1}\beta_{\overline{\rho}}^{m_2}, & m_1+m_2 \text{ even}, \\ \mathbb{Z}\{\omega-\overline{\omega}, \rho-\overline{\rho}\}\beta_\rho^{m_1}\beta_{\overline{\rho}}^{m_2}, & m_1+m_2 \text{ odd}. \end{cases}$$

As $\mathbb{H} \otimes_{\mathbb{C}} \rho = \rho + \overline{\rho}$ is 12-dimensional quaternionic, $\beta_\rho^2\beta_{\overline{\rho}}^2$ is a permanent cycle but $\beta_\rho\beta_{\overline{\rho}}$ is not. Thus we can compute $\pi_{*\rho_0} KO_{C_7 \rtimes C_3}$, with a convenient choice of complex structure on multiples of ρ_0, as having the form

$$\begin{aligned} \mathbb{Z}\{1, \omega+\overline{\omega}, \rho+\overline{\rho}\} \xleftarrow{a_\rho} \mathbb{Z}\{\omega-\overline{\omega}, \rho-\overline{\rho}\}\beta_\rho \xleftarrow{a_{\overline{\rho}}} \mathbb{Z}\{2, \omega+\overline{\omega}, \rho+\overline{\rho}\}\beta_\rho\beta_{\overline{\rho}} \\ \xleftarrow{a_\rho} \mathbb{Z}\{\omega-\overline{\omega}, \rho-\overline{\rho}\}\beta_\rho^2\beta_{\overline{\rho}} \xleftarrow{a_{\overline{\rho}}} \mathbb{Z}\{1, \omega+\overline{\omega}, \rho+\overline{\rho}\}\beta_\rho^2\beta_{\overline{\rho}}^2 \xleftarrow{a_\rho} . \end{aligned}$$

In writing $RO(C_7 \rtimes C_3) = \mathbb{Z}\{1, \omega+\overline{\omega}, \rho+\overline{\rho}\} \subset RU(C_7 \rtimes C_3)$, the symbols $\omega+\overline{\omega}$ and $\rho+\overline{\rho}$ represent the real representations ω_0 and ρ_0 underlying ω and ρ. For example, the identity $(\overline{\rho}-\rho)\cdot\overline{\rho} = \rho - (1+\omega+\overline{\omega})$ in $RU(C_7 \rtimes C_3)$ gives a complex $t_{\rho-(\mathbb{C}+\omega+\overline{\omega})}$-element $J(\overline{\rho}\beta_\rho) \in \pi^{C_7 \rtimes C_3}_{\rho-(\mathbb{C}+\omega+\overline{\omega})} C(a_\rho)$, with underlying real $t_{\rho_0-(2+2\omega_0)}$-element living in $\pi^{C_7 \rtimes C_3}_{\rho_0-(2+2\omega_0)} C(a_{\rho_0})$. The identity $a_\rho a_{\overline{\rho}}(\rho+\overline{\rho})\beta_\rho\beta_{\overline{\rho}} = (\rho+\overline{\rho}) - (2+2(\omega+\overline{\omega}))$ then implies that this in lifts to $t \in \pi^{C_7 \rtimes C_3}_{\rho_0-(2+2\omega_0)} C(a^2_{\rho_0})$ satisfying $\mathrm{res}^{C_7 \rtimes C_3}_e(\partial(t)) = 3j_{11}$.

6.6. Example: The quaternion group of order 8

We give an example with a symplectic irreducible. See also [13, 21]. Let $G = Q_8$ be the quaternion group of order 8. This sits in a short exact sequence

$$1 \to C_2 \to Q_8 \to C_2 \times C_2 \to 1,$$

and we have

$$RU(Q_8) = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3, H\},$$

with $\mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3\} \cong RO(C_2 \times C_2)$ consisting of those representations lifted from $C_2 \times C_2$ and H the tautological representation of $Q_8 \subset Sp(1) \cong SU(2)$. Write $\mathbb{H}$ for the underlying 4-dimensional real representation of H. As Q_8 acts freely on $S(\mathbb{H})$, the Q_8-spectra $C(a^m_{\mathbb{H}})$ admit all possible periodicities. Observe

$$\sigma_i H = H, \quad H^2 = 1 + \sigma_1 + \sigma_2 + \sigma_3, \quad e_H = 2 - H.$$

This determines $\pi_{*\mathbb{H}} KU_{Q_8}$. Complex conjugation acts trivially on $\pi_{*\mathbb{H}} KU_{Q_8}$, so the E_2-page of the HFPSS for $\pi_{*+*\mathbb{H}} KO_{Q_8}$ is given by

$$H^*(C_2; \pi_{*+*\mathbb{H}} KU_{Q_8}) \cong \pi_{*H} KU_{Q_8} \otimes \mathbb{Z}[\beta^{\pm 2}, \eta]/(2\eta).$$

As H is symplectic and the σ_i are orthogonal we have

$$RO(Q_8) = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3, 2H\} \subset RU(Q_8),$$

where "$2H$" represents the real representation $\mathbb{H}$. This manifests in the HFPSS as a nontrivial differential $d_3(H) = H\beta^{-2}\eta^3$. As $\mathbb{H}$ is quaternionic of real dimension 4, the class $\beta_H^2 \in \pi_{2\mathbb{H}} KU_{Q_8}$ is a permanent cycle but $\beta_H \in \pi_{\mathbb{H}} KU_{Q_8}$ is not, and so $\pi_{*\mathbb{H}} KO_{Q_8}$ has the 2-periodic pattern

$$\mathbb{Z}\{1,\sigma_1,\sigma_2,\sigma_3,2H\} \xleftarrow{a_{\mathbb{H}}} \mathbb{Z}\{2,2\sigma_1,2\sigma_2,2\sigma_3,H\}\beta_H \xleftarrow{a_{\mathbb{H}}} \mathbb{Z}\{1,\sigma_1,\sigma_2,\sigma_3,2H\}\beta_H^2 \xleftarrow{a_{\mathbb{H}}} .$$

A calculation reveals that

$$\begin{aligned} 2^{4m-1}(4-\mathbb{H}) &= a_{\mathbb{H}}^{2m}\big(2^{2m+1} - 2^{2m-1}(1+\sigma_1+\sigma_2+\sigma_3) - 2H\big)\beta_H^{2m}, \\ 2^{4m}(4-\mathbb{H}) &= a_{\mathbb{H}}^{2m+1}\big(2^{2m+1} - 2^{2m-1}(1+\sigma_1+\sigma_2+\sigma_3) - H\big)\beta_H^{2m+1} \end{aligned}$$

for $m > 0$: for example, the identities

$$\begin{aligned} (2-H)\cdot(1-\sigma_i) &= 2(1-\sigma_i), \\ (2-H)\cdot(1+\sigma_1+\sigma_2+\sigma_3-2H) &= 8(1+\sigma_1+\sigma_2+\sigma_3-2H), \\ (2-H)\cdot(-H) &= 1+\sigma_1+\sigma_2+\sigma_3-2H \end{aligned}$$

together imply

$$\begin{aligned} a_{\mathbb{H}}^{2m}\cdot\big(2^{2m+1} - 2^{2m-1}(1+\sigma_1+\sigma_2+\sigma_3)\big)\beta_H^{2m} &= 2^{4m-1}\big(4-(1+\sigma_1+\sigma_2+\sigma_3)\big), \\ a_{\mathbb{H}}^{2m}\cdot(-2H)\beta_H^{2m} &= 2^{4m-1}(1+\sigma_1+\sigma_2+\sigma_3-\mathbb{H}), \end{aligned}$$

and adding these yields the first; the second is similar. Hence if we define

$$p(n) = \begin{cases} 2n, & n \text{ even}, \\ 2n+1, & n \text{ odd}, \end{cases}$$

then there are $t_{4-\mathbb{H}}$-elements

$$u_{2^{p(n)}\mathbb{H}} \in \pi^{Q_8}_{2^{p(n)}(4-\mathbb{H})} C(a_{\mathbb{H}}^{n+1})$$

for $n \geq 0$, satisfying

$$\operatorname{res}^{Q_8}_e\big(\partial(u^k_{2^{p(n)}})\big) = \begin{cases} k\cdot j_{4n+3}, & n \text{ even}, \\ k\cdot 4j_{4n+3}, & n \text{ odd} \end{cases}$$

for $k \in \mathbb{Z}$. By identifying $S(n\mathbb{H})$ as a $(4n-1)$-skeleton of EQ_8, this implies that if R is a Q_8-ring spectrum then in the homotopy fixed point spectral sequence

$$E_2 = H^*\big(Q_8; \pi_*^e R[u_{\mathbb{H}}^{\pm 1}, u_{\sigma_1}^{\pm 1}, u_{\sigma_2}^{\pm 1}, u_{\sigma_3}^{\pm 1}]\big) \implies \pi_\star R_h^\wedge$$

there are differentials

$$d_{4(n+1)}(u_{\mathbb{H}}^{2^{p(n)}}) = \begin{cases} j_{4n+3}\cdot a_{\mathbb{H}}^{n+1} u_{\mathbb{H}}^{-(n+1)}\cdot u_{\mathbb{H}}^{2^{p(n)}}, & n \text{ even}, \\ 4j_{4n+3}\cdot a_{\mathbb{H}}^{n+1} u_{\mathbb{H}}^{-(n+1)}\cdot u_{\mathbb{H}}^{2^{p(n)}}, & n \text{ odd} \end{cases}$$

for $n \geq 0$, where $a_{\mathbb{H}}$ is detected by the generator of $H^4(Q_8; \pi^e_{4-\mathbb{H}} S_{Q_8}) \cong \mathbb{Z}/(8)$. In other words,

$$d_4(u_{\mathbb{H}}) = \nu a_{\mathbb{H}}, \quad d_4(u_{\mathbb{H}}^2) = 2\nu a_{\mathbb{H}} u_{\mathbb{H}}, \quad d_4(u_{\mathbb{H}}^4) = 4\nu a_{\mathbb{H}} u_{\mathbb{H}}^3,$$
$$d_8(u_{\mathbb{H}}^8) = 4\sigma a_{\mathbb{H}}^2 u_{\mathbb{H}}^6,$$
$$d_{12}(u_{\mathbb{H}}^{16}) = j_{11} a_{\mathbb{H}}^3 u_{\mathbb{H}}^{13}, \quad d_{12}(u_{\mathbb{H}}^{32}) = 2 j_{11} a_{\mathbb{H}}^3 u_{\mathbb{H}}^{29}, \quad d_{12}(u_{\mathbb{H}}^{64}) = 4 j_{11} a_{\mathbb{H}}^3 u_{\mathbb{H}}^{61},$$
$$d_{16}(u_{\mathbb{H}}^{128}) = 4 j_{15} a_{\mathbb{H}}^4 u_{\mathbb{H}}^{124},$$

and so forth, up to orientation of ν.

6.7. Example: The binary octahedral group

We give a larger symplectic example. Let $2O \subset Sp(1)$ denote the binary octahedral group, of order 48. This group is of interest to chromatic homotopy theorists as the maximal subgroup $G_{48} \subset \mathbb{G}_2$ of the extended Morava stabilizer group associated to the Honda formal group law at the prime 2 and height 2. Consulting [11] we find that $2O$ has character table

	1	4A	3	4B	2	8A	6	8B
1	1	1	1	1	1	1	1	1
ρ_2	1	−1	1	1	1	−1	1	−1
ρ_3	2	0	−1	2	2	0	−1	0
ρ_4	2	0	−1	0	−2	$-\sqrt{2}$	1	$\sqrt{2}$
ρ_5	2	0	−1	0	−2	$\sqrt{2}$	1	$-\sqrt{2}$
ρ_6	3	1	0	−1	3	−1	0	−1
ρ_7	3	−1	0	−1	3	1	0	1
ρ_8	4	0	1	0	−4	0	−1	0

,

where ρ_4, ρ_5, ρ_8 are symplectic and the rest are orthogonal.

The tautological representation $\mathbb{H}$ of $2O \subset Sp(1) \cong SU(2)$ can be identified with ρ_4, with Euler class $e_{\rho_4} = 2 - \rho_4$. The element $\alpha = 1 + \rho_2 - \rho_3 + \rho_4 + \rho_5 - \rho_8 \in RU(2O)$ satisfies $e_{\rho_4} \cdot \alpha = \alpha$, and as $E2O = \operatorname{colim}_{n \to \infty} S(n\mathbb{H})$ this produces a complex t_α-element in $\pi_\alpha F(E2O_+, S_{2O})$.

Because $\mathbb{H}$ is real 4-dimensional quaternionic, as with $G = Q_8$ we can compute

$$\pi_{2m\mathbb{H}} KO_{2O} = \mathbb{Z}\{1, \rho_2, \rho_3, 2\rho_4, 2\rho_5, \rho_6, \rho_7, 2\rho_8\}\beta_{\mathbb{H}}^{2m},$$
$$\pi_{(2m+1)\mathbb{H}} KO_{2O} = \mathbb{Z}\{2, 2\rho_2, 2\rho_3, \rho_4, \rho_5, 2\rho_6, 2\rho_7, \rho_8\}\beta_{\mathbb{H}}^{2m+1},$$

with the action of $a_{\mathbb{H}}$ determined by $a_{\mathbb{H}}\beta_{\mathbb{H}} = e_{\rho_4} = 2 - \rho_4$. Hence for example

$$48(4 - \mathbb{H}) = a_{\mathbb{H}}^2 \cdot (64 - 8\rho_2 - 8\rho_3 + 17 \cdot 2\rho_4 - 7 \cdot 2\rho_5 - 16\rho_6 + 8\rho_7 - 6 \cdot 2\rho_8)\beta_{\mathbb{H}}^2,$$

yielding an invertible class $t \in \pi^{2O}_{48(4-\mathbb{H})} C(a_{\mathbb{H}}^2)$ satisfying $\operatorname{res}^{2O}_e(\partial(t)) = \pm 8\sigma \in \pi_7 S$. If for example $u_{8\mathbb{H}} \in \pi^{Q_8}_{8(4-\mathbb{H})} C(a_{\mathbb{H}}^2)$ is as in Section 6.6, then $\operatorname{res}^{2O}_{Q_8}(t) \neq u_{8\mathbb{H}}^6$; instead

$$\varepsilon = \operatorname{res}^{2O}_{Q_8}(t) \cdot u_{8\mathbb{H}}^{-6} \in \pi_0^{Q_8} C(a_{\mathbb{H}}^2)^\times$$

is a unit satisfying $\operatorname{res}^{Q_8}_e(\partial(\varepsilon)) = \pm 24\sigma$.

Funding. This work was supported by NSF RTG grant DMS-1839968.

References

[1] J. F. Adams, Vector fields on spheres. *Ann. of Math. (2)* **75** (1962), 603–632 Zbl 0112.38102 MR 0139178

[2] S. Araki, Forgetful spectral sequences. *Osaka Math. J.* **16** (1979), no. 1, 173–199 Zbl 0411.55017 MR 0527025

[3] S. Araki and K. Iriye, Equivariant stable homotopy groups of spheres with involutions. I. *Osaka Math. J.* **19** (1982), no. 1, 1–55 Zbl 0488.55012 MR 0656233

[4] M. F. Atiyah and D. O. Tall, Group representations, λ-rings and the J-homomorphism. *Topology* **8** (1969), 253–297 Zbl 0159.53301 MR 0244387

[5] W. Balderrama, Total power operations in spectral sequences. *Trans. Amer. Math. Soc.* **377** (2024), no. 7, 4779–4823 Zbl 1554.19001 MR 4778062

[6] M. Behrens and J. Shah, C_2-equivariant stable homotopy from real motivic stable homotopy. *Ann. K-Theory* **5** (2020), no. 3, 411–464 Zbl 1475.55011 MR 4132743

[7] G. E. Bredon, Equivariant stable stems. *Bull. Amer. Math. Soc.* **73** (1967), 269–273 Zbl 0152.21803 MR 0206947

[8] G. E. Bredon, Equivariant homotopy. In *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*, pp. 281–292, Springer, New York, 1968 Zbl 0175.20502 MR 0250303

[9] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*. Lecture Notes in Math. 1176, Springer, Berlin, 1986 Zbl 0585.55016 MR 0836132

[10] M. C. Crabb, *$\mathbf{Z}/2$-homotopy theory*. London Math. Soc. Lecture Note Ser. 44, Cambridge University Press, Cambridge, 1980 Zbl 0443.55001 MR 0591680

[11] T. Dokchitser, GroupNames. https://people.maths.bris.ac.uk/~matyd/GroupNames/ visited on 3 June 2026

[12] C. French, The equivariant J-homomorphism. *Homology Homotopy Appl.* **5** (2003), no. 1, 161–212 Zbl 1032.55016 MR 1989617

[13] K. Fujii, On the KO-ring of S^{4n+3}/H_m. *Hiroshima Math. J.* **4** (1974), 459–475 Zbl 0289.55016 MR 0365608

[14] J. P. C. Greenlees and J. P. May, Generalized Tate cohomology. *Mem. Amer. Math. Soc.* **113** (1995), no. 543, viii+178 Zbl 0876.55003 MR 1230773

[15] P. Hu and I. Kriz, Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology* **40** (2001), no. 2, 317–399 Zbl 0967.55010 MR 1808224

[16] I. M. James, Cross-sections of Stiefel manifolds. *Proc. London Math. Soc. (3)* **8** (1958), 536–547 Zbl 0089.39303 MR 0100840

[17] M. Karoubi, Equivariant K-theory of real vector spaces and real projective spaces. *Topology Appl.* **122** (2002), no. 3, 531–546 Zbl 1011.19005 MR 1911698

[18] T. Kobayashi and M. Sugawara, On stable homotopy types of stunted lens spaces. II. *Hiroshima Math. J.* **7** (1977), no. 3, 689–705 Zbl 0398.55007 MR 0464220

[19] P. S. Landweber, Conjugations on complex manifolds and equivariant homotopy of MU. *Bull. Amer. Math. Soc.* **74** (1968), 271–274 Zbl 0181.26801 MR 0222890

[20] P. Löffler, Equivariant framability of involutions on homotopy spheres. *Manuscripta Math.* **23** (1977/78), no. 2, 161–171 Zbl 0367.57009 MR 0461532

[21] N. Mahammed, *K-theorie des formes spheriques*. Ph.D. thesis, Université des sciences et technologies de Lille, 1975, https://pepite-depot.univ-lille.fr/LIBRE/Th_Num/1975/50376-1975-117.pdf visited on 3 June 2026

[22] A. Mathew, Examples of descent up to nilpotence. In *Geometric and topological aspects of the representation theory of finite groups*, pp. 269–311, Springer Proc. Math. Stat. 242, Springer, Cham, 2018 Zbl 1544.18019 MR 3901164

[23] A. Mathew, N. Naumann, and J. Noel, Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.* **305** (2017), 994–1084 Zbl 1420.55024 MR 3570153

[24] A. Mathew, N. Naumann, and J. Noel, Derived induction and restriction theory. *Geom. Topol.* **23** (2019), no. 2, 541–636 Zbl 1422.19001 MR 3939042

[25] J. E. McClure, On the groups $JO_G X$. I. *Math. Z.* **183** (1983), no. 2, 229–253 Zbl 0521.55011 MR 0704106

[26] H. Minami, On equivariant J-homomorphism for involutions. *Osaka Math. J.* **20** (1983), no. 1, 109–122 Zbl 0518.55012 MR 0695620

[27] G. Nishida, The nilpotency of elements of the stable homotopy groups of spheres. *J. Math. Soc. Japan* **25** (1973), 707–732 Zbl 0261.55013 MR 0341485

[28] C. Okonek, Der Conner-Floyd-Isomorphismus für Abelsche Gruppen. *Math. Z.* **179** (1982), no. 2, 201–212 Zbl 0485.55007 MR 0645496

[29] S. Schwede, *Global homotopy theory*. New Math. Monogr. 34, Cambridge University Press, Cambridge, 2018 Zbl 1451.55001 MR 3838307

[30] G. Segal, Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.* (1968), no. 34, 129–151 Zbl 0199.26202 MR 0234452

[31] G. B. Segal, Equivariant stable homotopy theory. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pp. 59–63, Gauthier-Villars Éditeur, Paris, 1971 Zbl 0225.55014 MR 0423340

[32] D. P. Sinha, Computations of complex equivariant bordism rings. *Amer. J. Math.* **123** (2001), no. 4, 577–605 Zbl 0997.55008 MR 1844571

[33] N. P. Strickland, $K(N)$-local duality for finite groups and groupoids. *Topology* **39** (2000), no. 4, 733–772 Zbl 0953.55005 MR 1760427

[34] T. tom Dieck, Bordism of G-manifolds and integrality theorems. *Topology* **9** (1970), 345–358 Zbl 0209.27504 MR 0266241

[35] T. tom Dieck, *Transformation groups and representation theory*. Lecture Notes in Math. 766, Springer, Berlin, 1979 Zbl 0445.57023 MR 0551743

[36] T. tom Dieck and T. Petrie, Geometric modules over the Burnside ring. *Invent. Math.* **47** (1978), no. 3, 273–287 Zbl 0389.57008 MR 0501372

[37] J. Tornehave, Equivariant maps of spheres with conjugate orthogonal actions. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, pp. 275–301, CMS Conf. Proc. 2, American Mathematical Society, Providence, RI, 1982 Zbl 0546.55019 MR 0686150

[38] G. W. Whitehead, On the homotopy groups of spheres and rotation groups. *Ann. of Math. (2)* **43** (1942), 634–640 Zbl 0060.41105 MR 0007107

Communicated by Vesna Stojanoska

Received 8 July 2024; revised 17 February 2026.

William Balderrama
Mathematical Institute, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany;
williamb@math.uni-bonn.de
Author IDs: zbMATH balderrama.william MR 1534659 ORCID 0009-0000-1813-0602