Doc. Math. (Online first)
DOI 10.4171/DM/1079

Isocrystals and de Rham–Witt connections

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Abstract. We introduce the notion of integrable connections for a sheaf of differential graded algebras on a topological space. We then describe them in the finite locally projective setting, when the sheaf is either the de Rham complex of a formal or a weakly formal scheme, or for the convergent or the overconvergent de Rham–Witt complex on a smooth scheme over a perfect field of positive characteristic. This enables us to give a new description of convergent and overconvergent isocrystals with a Frobenius structure.

Introduction

The search for a p-adic Weil cohomology theory for schemes over a field of positive characteristic has a long and rich history. The construction of Monsky–Washnitzer cohomology [24], which was the earliest attempt to find such a theory, was inspired from the fact that on a smooth manifold, de Rham cohomology computes singular cohomology. This allowed them to find a formula à la Lefschetz computing the zeta function of an affine scheme, under some assumptions.

In positive characteristic, the de Rham complex can have infinite-dimensional cohomology groups. In order to get a computable trace formula, one wants to work with finite-dimensional vector spaces. The idea of Monsky and Washnitzer was to lift the de Rham complex to another de Rham complex in characteristic 0 having finite dimensional cohomology. Moreover, to get an action on these cohomology groups they also lift the Frobenius endomorphism.

This action is paramount to the theory. However, it is not possible in general to lift the absolute Frobenius of a scheme in positive characteristic to characteristic 0; one can only do this locally.

In order to somehow glue these Frobenius lift globally, the first approach was crystalline cohomology [3]. The idea here is to consider a Grothendieck topology on the scheme, giving rise to the crystalline site. The category of crystals are the coefficients for this cohomology, whose good properties hold for proper schemes.

A more general theory is rigid cohomology [4]. It retrieves both Monsky–Washnitzer and crystalline cohomology. Berthelot's original construction did not use a site, but rather

Mathematics Subject Classification 2020: 14F30 (primary); 13F35, 14F40, 14G17 (secondary).
Keywords: F-isocrystals, de Rham–Witt complex, overconvergence.

relied on constructions using rigid geometry. It is now known that there actually is an overconvergent site computing rigid cohomology [21].

There have been other strategies to understand the global Frobenius action. One can cite, for instance, the motivic approach which enables one to remove all the choices involved [31].

In this article, we are interested in another viewpoint. Crystalline and rigid cohomology can be computed without introducing any Grothendieck topology. By using p-adic analytic methods, one can instead consider sheaves of differential graded algebras on the underlying topological space of the scheme.

This approach has been started with the introduction of the de Rham–Witt complex [16]. This complex yields a quasi-isomorphism with crystalline cohomology. Moreover, there exists a canonical and global lift of the Frobenius on the de Rham–Witt complex. Also, by restricting to locally finite free crystals, one can see that they are also coefficients for this cohomology theory [6, 14].

There have been two different proposals for the computation of rigid cohomology. The first one, the celebrated theory of arithmetic $\mathcal{D}$-modules, yields a category stable under the 6-functor formalism. Nevertheless, there are no global Frobenius lifts in that setting, so one has to resort to glueing to get the action. Another viewpoint is the overconvergent de Rham–Witt complex [11], which is known to compute rigid cohomology, but which had no theory of coefficients yet. Still, as in the crystalline setting, this complex is endowed with a global Frobenius lift.

In rigid cohomology, two categories play an important role: convergent and overconvergent F-isocrystals. Despite not having a 6-functor formalism, they are a powerful tool; for instance, they are used to describe arithmetic $\mathcal{D}$-modules [10].

Inspired by the crystalline setting, a preprint of Ertl suggests that locally free overconvergent F-isocrystals can also be interpreted as coefficients for overconvergent de Rham–Witt cohomology [13].

In this paper, the third and last of a series, we back up her claim: there is an equivalence of categories between overconvergent F-isocrystals, and a category of locally projective overconvergent de Rham–Witt connections. Our methods also give an equivalence of categories between convergent F-isocrystals and a category of locally projective de Rham–Witt connections. This allows one to consider both these categories using only Zariski topology, and a global Frobenius lift.

1. Connections

Throughout this article, we let p be a prime number, and k be a perfect field of characteristic p.

In this section, X shall denote a topological space, R shall denote a commutative ring, and $\mathcal{F} = (\bigoplus_{n\in\mathbb{N}} \mathcal{F}_n, d)$ a sheaf of strictly commutative R-dgas on X. We first extend [1, Définition 2.2.1] to the context of sheaves.

Definition 1.1. An $\mathcal{F}$*-connection* is an $\mathcal{F}_0$-module M endowed with a morphism of sheaves of groups $\nabla: M \to M \otimes_{\mathcal{F}_0} \mathcal{F}_1$ over X. We furthermore require that there exists an open cover $\mathcal{C}$ of X such that for every open subset U of an open in $\mathcal{C}$, the morphism $\nabla(U)$ factors through a group morphism $\widetilde{\nabla}: M(U) \to M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{F}_1(U)$ satisfying the Leibniz rule:

$$\forall s \in \mathcal{F}_0(U),\ \forall m \in M(U), \quad \widetilde{\nabla}(sm) = s\widetilde{\nabla}(m) + m \otimes d(s).$$

The morphisms are defined as usual.

Definition 1.2. Let (M, ∇_M) and (N, ∇_N) two $\mathcal{F}$-connections. A *horizontal morphism* is a morphism $\varphi: M \to N$ of $\mathcal{F}_0$-modules such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad\varphi\quad} & N \\ \big\downarrow{\scriptstyle \nabla_M} & & \big\downarrow{\scriptstyle \nabla_N} \\ M \otimes_{\mathcal{F}_0} \mathcal{F}_1 & \xrightarrow{\varphi \otimes_{\mathcal{F}_0} \mathrm{Id}_{\mathcal{F}_1}} & N \otimes_{\mathcal{F}_0} \mathcal{F}_1. \end{array}$$

We shall denote by $\mathcal{F}$-MC the category of $\mathcal{F}$-connections and horizontal morphisms.

We endow the category of graded abelian groups with the usual translation functor $\bigoplus_{i \in \mathbb{N}} G_i \mapsto \bigoplus_{i \in \mathbb{N}} G_{i+1}$. We will denote by dGAb the category of differential objects in graded abelian groups.

We will construct a functor $\mathcal{F}$-MC $\to$ Sh(X, dGAb) from the category of $\mathcal{F}$-connections to the category of sheaves on X of differential objects in graded abelian groups. To define it, on any cover $\mathcal{C}$ satisfying the assumptions of the definition of an $\mathcal{F}$-connection, and on any open U contained in an open of $\mathcal{C}$, for each $k \in \mathbb{N}$ put:

$$\widetilde{\nabla}_k: \quad \begin{aligned} M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{F}_k(U) &\to M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{F}_{k+1}(U) \\ m \otimes s &\mapsto \widetilde{\nabla}(m) \wedge s + m \otimes d(s). \end{aligned}$$

These maps are well defined, by applying the same computations as in [29, 07I0]. Therefore, for every open $V \in \mathcal{C}$ they yield glueable morphisms of sheaves of groups $\nabla_k|_V: (M \otimes_{\mathcal{F}_0} \mathcal{F}_k)|_V \to (M \otimes_{\mathcal{F}_0} \mathcal{F}_{k+1})|_V$, which give rise to the sheaf $(M \otimes_{\mathcal{F}_0} \mathcal{F}, \nabla_\bullet)$ of Sh(X, dGAb). Of course, $\nabla_0 = \nabla$.

Definition 1.3. An $\mathcal{F}$-connection ∇ is said to be *integrable* if $\nabla_1 \circ \nabla_0 = 0$. We shall denote by $\mathcal{F}$-MIC the full subcategory of $\mathcal{F}$-MC whose objects are integrable connections.

The computations in the proofs of [3, Propositions II 3.2.3 and II 3.2.5] still hold in this new context. In particular, this means that an $\mathcal{F}$-connection ∇ is integrable if and only if $(M \otimes_{\mathcal{F}_0} \mathcal{F}, \nabla_\bullet)$ is a sheaf of complexes in graded abelian groups, thus if and only if it is a right differential graded $\mathcal{F}$-module.

Proposition 1.4. *The categories* $\mathcal{F}$-MC *and* $\mathcal{F}$-MIC *are additive.*

Proof. It is straightforward to check that the addition of two horizontal morphisms is still horizontal, so the categories are preadditive. Moreover, the product of a finite family of

$\mathscr{F}_0$-modules endowed with (integrable) $\mathscr{F}$-connections is also endowed with a canonical (integrable) $\mathscr{F}$-connection since the tensor products and direct sums commute. It is easily checked that the projections are horizontal, and that this new connection is the finite product in these categories. ■

Definition 1.5. Let ∇ be an $\mathscr{F}$-connection on a free $\mathscr{F}_0$-module M. Let $\{m_i\}_{i\in I}$ be an $\mathscr{F}_0(X)$-basis of $M(X)$, where I is a set.

The unique matrix $(n_{i,j})_{i,j\in I} \in \mathrm{Mat}_{\#I}(\mathscr{F}_1(X))$ such that $\nabla(m_i) = \sum_{j\in I} m_j \otimes n_{j,i}$ for every $i \in I$ is called the *representative matrix* of ∇ for the basis $\{m_i\}_{i\in I}$.

Similarly, the unique matrix $(n_{i,j})_{i,j\in I} \in \mathrm{Mat}_{\#I}(\mathscr{F}_2(X))$ such that for every $i \in I$ we have $\nabla_1 \circ \nabla_0(m_i) = \sum_{j\in I} m_j \otimes n_{j,i}$ is called the *curvature matrix* of ∇ for the basis $\{m_i\}_{i\in I}$.

A representative matrix uniquely determines an $\mathscr{F}$-connection on a free $\mathscr{F}_0$-module for a given basis, and conversely using the following result.

Proposition 1.6. *Let $\nabla\colon M \to M \otimes_{\mathscr{F}_0} \mathscr{F}_1$ be an $\mathscr{F}$-connection on a free $\mathscr{F}_0$-module. Let $\{m_i\}_{i\in I}$ be an $\mathscr{F}_0(X)$-basis of $M(X)$, where I is a set. Let $U = \{u_i\}_{i\in I} \in \mathscr{F}_0(X)^I$ be a vertical vector, and let $N = (n_{i,j})_{i,j\in I} \in \mathrm{Mat}_{\#I}(\mathscr{F}_1(X))$ be the representative matrix of ∇ for $\{m_i\}_{i\in I}$.*

Then $\nabla(\sum_{i\in I} u_i m_i) = \sum_{i\in I} m_i \otimes v_i$, where:

$$\{v_i\}_{i\in I} = NU + d(U).$$

Proof. Clear from the Leibniz rule. ■

Proposition 1.7. *Let $\nabla\colon M \to M \otimes_{\mathscr{F}_0} \mathscr{F}_1$ be an $\mathscr{F}$-connection on a free $\mathscr{F}_0$-module. Let $\{m_i\}_{i\in I}$ be an $\mathscr{F}_0(X)$-basis of $M(X)$, where I denotes a set. Consider an invertible matrix $U = \{u_{i,j}\}_{i,j\in I} \in \mathrm{GL}_{\#I}(\mathscr{F}_0(X))$, and let $N = (n_{i,j})_{i,j\in I} \in \mathrm{Mat}_{\#I}(\mathscr{F}_1(X))$ be the representative matrix of ∇ for $\{m_i\}_{i\in I}$.*

The representative matrix of ∇ for the basis $\{\sum_{j\in I} u_{j,i} m_j\}_{i\in I}$ is then:

$$U^{-1}NU + U^{-1}d(U).$$

Proof. This is a simple consequence of Proposition 1.6. ■

Proposition 1.8. *Let ∇ be an $\mathscr{F}$-connection. Then, for each $k \in \mathbb{N}$ the morphism $\nabla_{k+1} \circ \nabla_k$ is $\mathscr{F}_0$-linear.*

Proof. Just follow the end of the proof of [3, Proposition II 3.2.3], and acknowledge that it still holds in our slightly different setting. ■

Proposition 1.9. *Let $\nabla\colon M \to M \otimes_{\mathscr{F}_0} \mathscr{F}_1$ be an $\mathscr{F}$-connection on a free $\mathscr{F}_0$-module. Let $\{m_i\}_{i\in I}$ be an $\mathscr{F}_0(X)$-basis of $M(X)$, where I is a set, and let N be a representative matrix of ∇ for this basis. Then, the curvature matrix of ∇ for $\{m_i\}_{i\in I}$ is:*

$$N^2 + d(N).$$

In particular, ∇ is integrable if and only if $d(N) = -N^2$.

Proof. Write $\{n_{i,j}\}_{i,j\in I} := N$. By definition, $\nabla(m_i) = \sum_{j\in I} m_j \otimes n_{j,i}$ for every $i \in I$. Hence, we find:

$$\begin{aligned}\nabla_1 \circ \nabla_0(m_i) &= \nabla_1\Big(\sum_{j\in I} m_j \otimes n_{j,i}\Big) = \sum_{j\in I}\big(\nabla(m_j)n_{j,i} + m_j \otimes d(n_{j,i})\big)\\ &= \sum_{j\in I}\Big(\sum_{k\in I} m_k \otimes n_{k,j}n_{j,i} + m_j \otimes d(n_{j,i})\Big),\\ \nabla_1 \circ \nabla_0(m_i) &= \sum_{j\in I} m_j \otimes \Big(\sum_{k\in I} n_{j,k}n_{k,i} + d(n_{j,i})\Big).\end{aligned}$$

The second part of the statement is an application of Proposition 1.8. ∎

Proposition 1.10. *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of strictly commutative R-dgas on X. Let $\nabla\colon M \to M \otimes_{\mathcal{F}_0} \mathcal{F}_1$ be an $\mathcal{F}$-connection. For every open U of X, let:*

$$\varphi_M\colon \quad \begin{aligned} M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{F}_1(U) &\to \big(M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{G}_0(U)\big) \otimes_{\mathcal{G}_0(U)} \mathcal{G}_1(U)\\ m \otimes f &\mapsto (m \otimes 1) \otimes \varphi(f).\end{aligned}$$

Then, assuming that ∇ factors through $\tilde{\nabla}\colon M(U) \to M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{F}_1(U)$ satisfying the Leibniz rule on U, then the data of the following maps for varying U covering X yields a $\mathcal{G}$-connection $\varphi^*(\nabla)$ on $M \otimes_{\mathcal{F}_0} \mathcal{G}_0$:

$$\widetilde{\varphi^*(\nabla)}(U)\colon \quad \begin{aligned} M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{G}_0(U) &\to \big(M(U) \otimes_{\mathcal{F}_0(U)} \mathcal{G}_0(U)\big) \otimes_{\mathcal{G}_0(U)} \mathcal{G}_1(U)\\ m \otimes g &\mapsto g\varphi_M\big(\tilde{\nabla}(m)\big) + (m \otimes 1) \otimes d(g).\end{aligned}$$

More precisely, it defines a functor:

$$\varphi^*\colon \mathcal{F}\text{-MC} \to \mathcal{G}\text{-MC}.$$

Proof. The map is well defined. Indeed, it is biadditive and for every open U of X, every $f \in \mathcal{F}_0(U)$, every $m \in M(U)$ and every $g \in \mathcal{G}_0(U)$ one has:

$$\begin{aligned}\widetilde{\varphi^*(\nabla)}(U)(fm \otimes g) &= g\varphi_M\big(\tilde{\nabla}(fm)\big) + (fm \otimes 1) \otimes d(g)\\ &= g\varphi_M\big(f\tilde{\nabla}(m) + m \otimes d(f)\big) + (fm \otimes 1) \otimes d(g)\\ &= \varphi(f)g\varphi_M\big(\tilde{\nabla}(m)\big) + (m \otimes 1) \otimes \big(g\varphi\big(d(f)\big) + \varphi(f)d(g)\big),\\ \widetilde{\varphi^*(\nabla)}(U)(fm \otimes g) &= \widetilde{\varphi^*(\nabla)}(U)(m \otimes \varphi(f)g).\end{aligned}$$

Moreover, we get a $\mathcal{G}$-connection because for all $m \in M$ and all $g, g' \in \mathcal{G}_0(U)$ one finds:

$$\begin{aligned}\widetilde{\varphi^*(\nabla)}(U)\big(g'(m \otimes g)\big) &= \widetilde{\varphi^*(\nabla)}(U)(m \otimes g'g)\\ &= g'g\varphi_M\big(\tilde{\nabla}(m)\big) + (m \otimes 1) \otimes d(g'g)\\ &= g'g\varphi_M\big(\tilde{\nabla}(m)\big) + (m \otimes 1) \otimes g'd(g) + (m \otimes 1) \otimes gd(g'),\end{aligned}$$

$$\widetilde{\varphi^*(\nabla)}(U)\big(g'(m \otimes g)\big) = g'\widetilde{\varphi^*(\nabla)}(U)(m \otimes g) + (m \otimes g) \otimes d(g').$$

The functoriality is clear. ∎

Proposition 1.11. *Let us keep the notations of Proposition 1.10. Then, the functor φ^* preserves integrability; that is, by restriction we have a functor:*

$$\varphi^* \colon \mathcal{F}\text{-MIC} \to \mathcal{G}\text{-MIC}.$$

Proof. By construction, the functor $\mathcal{F}\text{-MC} \to \mathsf{Sh}(X, \mathsf{dGAb})$ has the following property: when the $\mathcal{F}$-connection $\nabla \colon M \to M \otimes_{\mathcal{F}_0} \mathcal{F}_1$ is integrable, its image is the unique couple $(M \otimes_{\mathcal{F}_0} \mathcal{F}, \nabla_\bullet)$ turning $M \otimes_{\mathcal{F}_0} \mathcal{F}$ into a right differential graded $\mathcal{F}$-module satisfying $\nabla_0 = \nabla$. Moreover, recall that $(M \otimes_{\mathcal{F}_0} \mathcal{F}, \nabla_\bullet)$ is a right differential graded $\mathcal{F}$-module if and only if ∇ is integrable.

Also, we have a tensor product on the category $\mathsf{RDiffGr}\text{-}\mathcal{F}\text{-}\mathsf{Mod}$ of right differential graded $\mathcal{F}$-modules [29, 09LL and 0FR2] yielding a base change functor. As can be seen from its definition, it coincides up to an isomorphism of graded $\mathcal{F}$-modules (truncated in degree greater than 1) to the base change φ^* we have defined. Thus, the following diagram is essentially commutative (that is, commutative up to isomorphism of functors):

$$\begin{array}{ccccc}
\mathcal{F}\text{-MIC} & & \xrightarrow{\quad\varphi^*\quad} & & \mathcal{G}\text{-MC} \\
\Big\downarrow{\scriptstyle \nabla \mapsto \nabla_\bullet} & & & & \Big\downarrow{\scriptstyle \nabla \mapsto \nabla_\bullet} \\
\mathsf{RDiffGr}\text{-}\mathcal{F}\text{-}\mathsf{Mod} & \xrightarrow{\bullet \otimes_{\mathcal{F}} \mathcal{G}} & \mathsf{RDiffGr}\text{-}\mathcal{G}\text{-}\mathsf{Mod} & \xrightarrow{\text{Forgetful}} & \mathsf{Sh}(X, \mathsf{dGAb}).
\end{array}$$

∎

2. Weak formal schemes

We assume the reader is familiar with Meredith's theory of weak formal schemes, which he called weak formal preschemes in [23, 4. Definition 1]. In what follows, the weak completion, in the sense of Monsky–Washnitzer [24, Definition 1.1], of a commutative $W(k)$-algebra R shall be denoted by $R^\dagger$, and its ring of Witt vectors by $W(R)$. Its associated affine weak formal scheme shall be denoted by $\mathrm{Spff}(R^\dagger)$.

In all the article, for every morphism of commutative rings $R \to S$ we shall denote:

$$\boldsymbol{\Omega}_{S/R} := \Omega_{S/R} / \bigcap_{i \in \mathbb{N}} p^i \Omega_{S/R}.$$

It is the universal p-adically separated de Rham complex associated to $R \to S$. These similar notations shall lead to no confusion, as we shall always work with the p-adically separated complex, and never with the usual one.

Proposition 2.1. *Let R be a commutative $W(k)$-algebra such that $\boldsymbol{\Omega}^1_{R/W(k)}$ is a finite R-module. Then, we have a canonical isomorphism of $R^\dagger$-modules:*

$$\boldsymbol{\Omega}^1_{R/W(k)} \otimes_R R^\dagger \cong \boldsymbol{\Omega}^1_{R^\dagger/W(k)}.$$

Proof. Let $n \in \mathbb{N}$ and $(P_j)_{j\in\mathbb{N}} \in (W(k)[X_1,\dots,X_n])^{\mathbb{N}}$ such that there exists $c \in \mathbb{N}$ satisfying $\deg(P_j) \leqslant c(j+1)$ for every $j \in \mathbb{N}$. If $\underline{r} \in R^n$, then by definition $\sum_{j\in\mathbb{N}} p^j P_j(\underline{r})$ is in $R^\dagger$, and every element of this ring can be written as such. If $d\colon R \to \boldsymbol{\Omega}^1_{R/W(k)}$ is the canonical derivation, then for every $j \in \mathbb{N}$ we have:

$$d\big(P_j(\underline{r})\big) = \sum_{i=1}^{n} P_{j,i}(\underline{r}) d(r_i)$$

where $P_{j,i} \in W(k)[X_1,\dots,X_n]$ satisfies $\deg(P_{j,i}) \leqslant c(j+1)-1$ for every $i \in [\![1,n]\!]$. We let:

$$d'\colon \begin{array}{c} R^\dagger \to \boldsymbol{\Omega}^1_{R/W(k)} \otimes_R R^\dagger \\ \sum_{j\in\mathbb{N}} p^j P_j(\underline{r}) \mapsto \sum_{i=1}^{n} d(r_i) \otimes \sum_{j\in\mathbb{N}} p^j P_{j,i}(\underline{r}). \end{array}$$

This map does not depend on neither n, nor $(P_j)_{j\in\mathbb{N}}$ nor $\underline{r}$.

Indeed, $\boldsymbol{\Omega}^1_{R/W(k)} \otimes_R R^\dagger$ is a finite $R^\dagger$-module. So by [24, Theorem 1.6] and [7, Chapitre III, §3, Proposition 6], it is also p-adically separated. Hence, if we were to choose other $n' \in \mathbb{N}$, $(P'_j)_{j\in\mathbb{N}} \in (W(k)[X_1,\dots,X_n])^{\mathbb{N}}$ and $\underline{r}' \in R^n$ satisfying the same conditions, and such that $\sum_{j\in\mathbb{N}} p^j P_j(\underline{r}) = \sum_{j\in\mathbb{N}} p^j P'_j(\underline{r}')$, then for every $k \in \mathbb{N}$ we get:

$$\begin{aligned} &\sum_{i=1}^{n} d(r_i) \otimes \sum_{j\in\mathbb{N}} p^j P_{j,i}(\underline{r}) - \sum_{i=1}^{n'} d(r'_i) \otimes \sum_{j\in\mathbb{N}} p^j P'_{j,i}(\underline{r}') \\ &\quad = \sum_{j=0}^{k-1} \left(\sum_{i=1}^{n} p^j P_{j,i}(\underline{r}) d(r_i) - \sum_{i=1}^{n'} p^j P'_{j,i}(\underline{r}') d(r'_i) \right) \otimes 1 \\ &\quad\quad + p^k \sum_{j=k}^{+\infty} \left(\sum_{i=1}^{n} d(r_i) \otimes p^{j-k} P_{j,i}(\underline{r}) - \sum_{i=1}^{n'} d(r'_i) \otimes p^{j-k} P'_{j,i}(\underline{r}') \right). \end{aligned}$$

Hence, the difference is divisible by p^k for every $k \in \mathbb{N}$, so the map is well defined.

We see that d' is a $W(k)$-derivation because for every $j, j' \in \mathbb{N}$ we have:

$$\begin{aligned} d\big(P_j(\underline{r}) P'_{j'}(\underline{r}')\big) &= P_j(\underline{r}) d\big(P'_{j'}(\underline{r}')\big) + P'_{j'}(\underline{r}') d\big(P_j(\underline{r})\big) \\ &= \sum_{i=1}^{n'} P_j(\underline{r}) P'_{j',i}(\underline{r}') d(r'_i) + \sum_{i=1}^{n} P'_{j'}(\underline{r}') P_{j,i}(\underline{r}) d(r_i). \end{aligned}$$

If we let $d_\dagger\colon R^\dagger \to \boldsymbol{\Omega}^1_{R^\dagger/W(k)}$ be the universal derivation, we then know there exists a unique morphism of $R^\dagger$-modules $\varphi\colon \boldsymbol{\Omega}^1_{R^\dagger/W(k)} \to \boldsymbol{\Omega}^1_{R/W(k)} \otimes_R R^\dagger$ such that $\varphi \circ d_\dagger = d'$. Let $\iota\colon \boldsymbol{\Omega}^1_{R/W(k)} \to \boldsymbol{\Omega}^1_{R^\dagger/W(k)}$ be the canonical morphism of R-modules. We let:

$$\varphi^{-1}\colon \begin{array}{c} \boldsymbol{\Omega}^1_{R/W(k)} \otimes_R R^\dagger \to \boldsymbol{\Omega}^1_{R^\dagger/W(k)} \\ \omega \otimes r \mapsto \iota(\omega) r. \end{array}$$

For every $r \in R$ we have $d'(r) = d(r) \otimes 1$, which yields $\varphi \circ \varphi^{-1} \circ d'(r) = d'(r)$ and $\varphi^{-1} \circ \varphi \circ d_\dagger(r) = d_\dagger(r)$. By using p-adically overconvergent series, one gets similar equalities when $r \in R^\dagger$. Since $d'(R^\dagger)$ is an $R^\dagger$-generating set of $\mathbf{\Omega}^1_{R/W(k)} \otimes_R R^\dagger$ and $d_\dagger(R^\dagger)$ is an $R^\dagger$-generating set of $\mathbf{\Omega}^1_{R^\dagger/W(k)}$, one can conclude. ■

Proposition 2.2. *Let k be a perfect field of characteristic p. Assume that R is a commutative $W(k)$-algebra such that $\mathbf{\Omega}^1_{R/W(k)}$ is a finite R-module. Then, there is a canonical isomorphism of $W(k)$-dgas:*

$$\mathbf{\Omega}_{R/W(k)} \otimes_R R^\dagger \cong \mathbf{\Omega}_{R^\dagger/W(k)}.$$

Proof. This is Proposition 2.1 combined with [9, III.83, Proposition 8]. ■

The convergent version of this statement, that is $\mathbf{\Omega}_{R/W(k)} \otimes_R \widehat{R} \cong \mathbf{\Omega}_{\widehat{R}/W(k)}$ with the above assumptions, is well known by [29, 00RV and 0315] and [7, Chapitre III, §3, Proposition 6].

We recall the following definition of Caro [10, Définition 1.2.1].

Definition 2.3. A weak formal scheme $\mathfrak{X}^\dagger$ over $W(k)$ is said to be *smooth* if for every $i \in \mathbb{N}$, the scheme $(\mathfrak{X}^\dagger, \mathcal{O}_{\mathfrak{X}^\dagger}/p^i\mathcal{O}_{\mathfrak{X}^\dagger})$ is smooth over $W(k)/p^iW(k)$.

If $(X, \mathcal{O}_X)$ is a ringed space, for every $\mathcal{O}_X(X)$-module M we denote by $\widetilde{M}$ the sheaf of $\mathcal{O}_X$-modules associated to M. We refer to [29, 01BH] for its properties. This sheaf depends on the choice of the ringed space, but the notation should lead to no confusion in context.

If $(\mathfrak{X}^\dagger, \mathcal{O}_{\mathfrak{X}^\dagger})$ is an affine weak formal scheme, then by [23, 3. Theorem 3] we know that the functor $\widetilde{\bullet}$ induces an equivalence of categories between $\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)$-modules of finite type and coherent $\mathcal{O}_{\mathfrak{X}^\dagger}$-modules.

The following proposition explains us that, in the context of weak affine schemes, one can extend this equivalence of categories to see that our notion of connection coincides with the usual algebraic one; so we will identify both points of view in the paper.

Proposition 2.4. *Let k be a perfect field of characteristic p. Let $(\mathfrak{X}^\dagger, \mathcal{O}_{\mathfrak{X}^\dagger})$ be a smooth affine weak formal scheme over $W(k)$. Then, there exists a unique quasi-coherent sheaf of strictly commutative $W(k)$-dgas $\mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}$ on $\mathfrak{X}^\dagger$ such that for every $\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)$-module M and every affine open set $U \subset \mathfrak{X}^\dagger$ we have:*

$$\widetilde{M} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}} \mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}(U) \cong M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)}.$$

Proof. Let U be an affine open set of $\mathfrak{X}^\dagger$. Let M be an $\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)$-module. Let $\mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}$ be the sheafification of $\mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(\bullet)/W(k)}$.

There is a canonical morphism $M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)} \to \widetilde{M} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}} \mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}(U)$ of $\mathcal{O}_{\mathfrak{X}^\dagger}(U)$-modules which yields by adjunction [29, 01BH] a morphism of sheaves of $\mathcal{O}_{\mathfrak{X}^\dagger}|_U$-modules $\widetilde{M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)}} \to \widetilde{M} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}} \mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}|_U$. For every $x \in U$, this morphism induces a morphism of $\mathcal{O}_{\mathfrak{X}^\dagger,x}$-modules:

$$M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)} \operatorname{colim}_{x \in V \subset U} \mathcal{O}_{\mathfrak{X}^\dagger}(V) \to M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \operatorname{colim}_{x \in V \subset U} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(V)/W(k)}. \quad (2.1)$$

One can restrict the above open sets V to standard opens $V = D(\bar{f})$, where $f \in \mathcal{O}_{\mathfrak{X}^\dagger}(U)$ satisfies $x \in D(\bar{f})$. By [29, 00RT] and [9, III.83, Proposition 8], there is an isomorphism $\Omega_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)} \mathcal{O}_{\mathfrak{X}^\dagger}(U)_f \cong \Omega_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)_f/W(k)}$ of $W(k)$-dgas. By applying Proposition 2.2, one finds an isomorphism of $W(k)$-dgas:

$$\begin{aligned}\mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)} \mathcal{O}_{\mathfrak{X}^\dagger}(V) &\cong \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)} \mathcal{O}_{\mathfrak{X}^\dagger}(U)_f{}^\dagger \\ &\cong \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(V)/W(k)}.\end{aligned}$$

Moreover, as tensor products commute with direct limits we see that for every $x \in U$, the morphism (2.1) is an isomorphism. In particular, we have an isomorphism of sheaves of $\mathcal{O}_{\mathfrak{X}^\dagger}|_U$-modules

$$\widetilde{M \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}(\mathfrak{X}^\dagger)} \mathbf{\Omega}_{\mathcal{O}_{\mathfrak{X}^\dagger}(U)/W(k)}} \to \tilde{M} \otimes_{\mathcal{O}_{\mathfrak{X}^\dagger}} \mathbf{\Omega}_{\mathfrak{X}^\dagger/W(k)}|_U$$

and we conclude. ∎

Of course, there is a similar statement for formal schemes with the usual p-adic completion. We only stated this proposition for affine weak formal schemes since quasi-coherent sheaves on weak formal schemes are in general not as well behaved as in the case of schemes.

3. The overconvergent de Rham–Witt complex

We let $\bar{R}$ be a commutative k-algebra of finite type. In this paper, we shall denote by $W\Omega_{\bar{R}/k}$ the de Rham–Witt complex of $\bar{R}$. We refer to [16, 20] for the definition and basic properties.

Let $n \in \mathbb{N}$. In this article, we shall denote $k[\underline{X}] := k[X_1, \dots, X_n]$, and similarly for $W(k)[\underline{X}]$. The truncated de Rham–Witt complex of $\bar{R}$ will be denoted by $W_n\Omega_{\bar{R}/k}$.

Recall that when $\bar{R}$ is smooth, a theorem of Elkik tells us that we can find a smooth commutative $W(k)$-algebra R lifting $\bar{R}$ [29, 07M8]. Moreover, its weak completion $R^\dagger$ does not depend, up to an isomorphism of $W(k)$-algebras, on the choice of R, and there is a morphism of rings $F\colon R^\dagger \to R^\dagger$ lifting the Frobenius and compatible with $F\colon W(k) \to W(k)$ [30, Theorem 2.4.4]. This yields in turn a morphism of δ-rings $t_F\colon R^\dagger \to W(\bar{R})$ by [17, Théorème 4], which extends by the universal property of the p-adically separated de Rham complex [24, Theorem 4.2] to a morphism of $W(k)$-dgas:

$$t_F\colon \mathbf{\Omega}_{R^\dagger/W(k)} \to W\Omega_{\bar{R}/k}.$$

Let us now recall some definitions needed for the description of the de Rham–Witt complex.

Definition 3.1. A *weight function* is a map $a\colon [\![1,n]\!] \to \mathbb{N}[\frac{1}{p}]$; for any $i \in [\![1,n]\!]$, its values will be written a_i. We define:

$$|a| := \sum_{i=1}^{n} a_i.$$

Let a be a weight function. Let $J \subset [\![1,n]\!]$. We will denote by $a|_J$ the weight function which for any $i \in [\![1,n]\!]$ satisfies:

$$a|_J(i) = \begin{cases} a_i & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.2. The *support* of a weight function a is the set:

$$\mathrm{Supp}(a) := \{ i \in [\![1,n]\!] \mid a_i \neq 0 \}.$$

Definition 3.3. A *partition* of a weight function a is a subset of $\mathrm{Supp}(a)$. We denote by $\mathcal{P}$ the set of all (a, I), where a is a weight function and I is a partition of a. Throughout this paper, the p-adic valuation will be denoted by v_p. We fix the total order $\preceq$ on $\mathrm{Supp}(a)$:

$$\forall i, i' \in \mathrm{Supp}(a),\ i \preceq i' \iff \Big(\big(\mathrm{v}_p(a_i) \leqslant \mathrm{v}_p(a_{i'}) \big) \wedge \Big(\big(\mathrm{v}_p(a_i) = \mathrm{v}_p(a_{i'}) \big) \implies (i \leqslant i') \Big) \Big).$$

This order depends on the weight function a, but in practice no confusion will arise. We denote by $\prec$ the associated strict total order, and we will denote by $\min(a) \in \mathrm{Supp}(a)$ the only element such that $\min(a) \preceq i$ for any $i \in \mathrm{Supp}(a)$.

Let $I := \{i_j\}_{j \in [\![1,m]\!]}$ be a partition of a. By convention, we will always assume that $i_j \prec i_{j'}$ for all $j, j' \in [\![1,m]\!]$ such that $j < j'$. We will also say that $i_0 \preceq i$ and $i \prec i_{m+1}$ for any $i \in \mathrm{Supp}(a)$. For any $l \in [\![0,m]\!]$, we define the following subsets of $\mathrm{Supp}(a)$:

$$I_l := \{ i \in \mathrm{Supp}(a) \mid i_l \preceq i \prec i_{l+1} \}.$$

We set:

$$\mathrm{v}_p(a) := \min \{ \mathrm{v}_p(a_i) \mid i \in [\![1,n]\!] \},$$
$$u(a) := \max \{ 0, -\mathrm{v}_p(a) \}.$$

And if F denotes de Frobenius endomorphism, V the Verschiebung, and $[\bullet]$ the multiplicative representative, we put:

$$g(a) := F^{u(a)+\mathrm{v}_p(a)}\Big(d\Big(V^{u(a)}\big([\underline{X}^{p^{-\mathrm{v}_p(a)} a}] \big) \Big) \Big).$$

Furthermore, if I is a partition of a, and for any $\eta \in W(k)$, we set:

$$e(\eta, a, I) := \begin{cases} V^{u(a)}\big(\eta [\underline{X}^{p^{u(a)} a|_{I_0}}] \big) \times \prod_{l=1}^{\#I} g(a|_{I_l}) & \text{if } I_0 \neq \emptyset \text{ or } u(a) = 0, \\ d\Big(V^{u(a)}\big(\eta [\underline{X}^{p^{u(a)} a|_{I_1}}] \big) \Big) \times \prod_{l=2}^{\#I} g(a|_{I_l}) & \text{otherwise.} \end{cases}$$

Any element $w \in W\Omega_{k[\underline{X}]/k}$ in the de Rham–Witt complex has a unique description as a convergent series $\sum_{(a,I) \in \mathcal{P}} e(\eta_{a,I}, a, I)$ by [20, Theorem 2.8], where all $\eta_{a,I} \in W(k)$.

We use this description to define for every $\varepsilon > 0$ a map:

$$\zeta_\varepsilon\colon \begin{aligned} &W\Omega_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\} \\ &w \mapsto \begin{cases} \inf_{(a,I)\in\mathcal{P}} \left\{2n\, \mathrm{v}_V(\eta_{a,I}) + \#I\, u(a) - \varepsilon|a|\right\} & \text{if } I_0 = \emptyset, \\ \inf_{(a,I)\in\mathcal{P}} \left\{2n\, \mathrm{v}_V(\eta_{a,I}) + (\#I + 1)u(a) - \varepsilon|a|\right\} & \text{if } I_0 \neq \emptyset. \end{cases} \end{aligned}$$

Let $\varphi\colon k[\underline{X}] \to \overline{R}$ be a surjective map of k-algebras from a polynomial ring. This yields a morphism of $W(k)$-dgas

$$\varphi^*\colon W\Omega_{k[\underline{X}]/k} \to W\Omega_{\overline{R}/k}.$$

We put:

$$\zeta_{\varepsilon,\varphi}\colon \begin{aligned} &W\Omega_{\overline{R}/k} \to \mathbb{R} \cup \{+\infty, -\infty\}, \\ &w \mapsto \sup\left\{\zeta_\varepsilon(x) \mid \varphi^*(x) = w\right\}. \end{aligned}$$

First of all, by [26, Proposition 5.15] and the subsequent discussion, we can make the following definition which coincides with [11, Definition 1.1] in the case where k is a perfect field, and which does not depend on the choice of φ.

Definition 3.4. The *overconvergent de Rham–Witt complex of* $\overline{R}$ is:

$$W^\dagger\Omega_{\overline{R}/k} := \left\{w \in W\Omega_{\overline{R}/k} \mid \exists \varepsilon > 0,\ \zeta_{\varepsilon,\varphi}(w) \neq -\infty\right\}.$$

Recall that when $\overline{R} = k[\underline{X}]$ we have a decomposition of the overconvergent de Rham–Witt complex as three graded sub-$W(k)$-modules [25, (8)]:

$$W^\dagger\Omega_{k[\underline{X}]/k} = W^\dagger\Omega^{\mathrm{int}}_{k[\underline{X}]/k} \oplus W^\dagger\Omega^{\mathrm{frp}}_{k[\underline{X}]/k} \oplus d\left(W^\dagger\Omega^{\mathrm{frp}}_{k[\underline{X}]/k}\right). \tag{3.1}$$

By [2, 3.3.2 Théorème], there exists a lift of the Frobenius on $W(k)[\underline{X}]^\dagger$, which we shall also denote by F, such that we can find a δ-ring morphism $W(k)[\underline{X}]^\dagger \to R^\dagger$ lifting φ. More details can be found in [26, Lemma 7.1]. Moreover, there is a canonical isomorphism of $W(k)$-dgas $\boldsymbol{\Omega}_{W(k)[\underline{X}]^\dagger/W(k)} \cong W^\dagger\Omega^{\mathrm{int}}_{k[\underline{X}]/k}$ [11, pp. 231–233] (see also [26, Proposition 7.11] for a self-contained statement), so that we shall identify them in what follows.

We introduce the following sets for $t \in \mathbb{N}$:

$$G(t) = \left\{ e\left(1, \frac{a + p^u\chi_J}{p^u}, I \cup J\right) \;\middle|\; \begin{array}{l} u \in \mathbb{N},\ (a, I) \in \mathcal{P}, \\ J \subset [\![1, n]\!] \setminus \mathrm{Supp}(a), \\ \mathrm{v}_p(a) = 0,\ \forall i \in [\![1, n]\!],\ a_i < p^u, \\ I_0 \neq \emptyset,\ \#I + \#J = t \end{array} \right\},$$

$$H(t) = \left\{ e(1, \chi_I, I) \;\middle|\; \begin{array}{l} (\chi_I, I) \in \mathcal{P}, \\ \#I = t \end{array} \right\}.$$

We also have the convention:

$$G(-1) := \emptyset.$$

Finally, we introduce the following $W(k)[\underline{X}]^\dagger$-modules:

$$W^\dagger\Omega^{\underline{X}-\mathrm{int},t}_{k[\underline{X}]/k} := \left\{ \sum_{e\in H(t)} t_F(s_e e) \;\middle|\; \begin{array}{l} \exists C\in\mathbb{R},\ \exists\varepsilon>0,\ \forall e\in H(t), \\ \left(s_e\in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}\right)\wedge\left(\zeta_\varepsilon(s_e)+\zeta_\varepsilon(e)\geqslant C\right) \end{array}\right\},$$

$$W^\dagger\Omega^{\underline{X}-\mathrm{frp},t}_{k[\underline{X}]/k} := \left\{ \sum_{e\in G(t)} t_F(s_e)e \;\middle|\; \begin{array}{l} \exists C\in\mathbb{R},\ \exists\varepsilon>0,\ \forall e\in G(t), \\ \left(s_e\in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}\right)\wedge\left(\zeta_\varepsilon(s_e)+\zeta_\varepsilon(e)\geqslant C\right) \end{array}\right\}.$$

And we put:

$$W^\dagger\Omega^{\underline{X}}_{k[\underline{X}]/k} := W^\dagger\Omega^{\underline{X}-\mathrm{int}}_{k[\underline{X}]/k} + W^\dagger\Omega^{\underline{X}-\mathrm{frp}}_{k[\underline{X}]/k} + d\left(W^\dagger\Omega^{\underline{X}-\mathrm{frp}}_{k[\underline{X}]/k}\right).$$

For details on these constructions, see [26, 9]. Note that in that paper, there are hypothesis on the morphism φ, but we can safely assume here that they are satisfied.

By [26, Theorem 9.8], the previous decomposition as graded sub-$W(k)$-modules actually extends for any $\overline{R}$ finite étale over $\mathbb{A}^n_k$:

$$W^\dagger\Omega_{\overline{R}/k} = W^\dagger\Omega^{\mathrm{int}}_{\overline{R}/k} \oplus W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k} \oplus d\left(W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}\right).$$

More precisely, this decomposition depends on φ as

$$W^\dagger\Omega^{\mathrm{int}}_{\overline{R}/k} = \varphi^*\left(W^\dagger\Omega^{\mathrm{int}}_{k[\underline{X}]/k}\right) \quad\text{and}\quad W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k} = \varphi^*\left(W^\dagger\Omega^{\mathrm{frp}}_{k[\underline{X}]/k}\right).$$

In practice, there shall be no confusion so we omit it in our notations. This enables us to define the following map:

$$\check{\zeta}_{\varepsilon,\varphi}\colon \begin{array}{l} W\Omega_{\overline{R}/k} \to \mathbb{R}\cup\{+\infty,-\infty\} \\ \qquad w \mapsto \sup\left\{\zeta_\varepsilon(x)\mid x\in{\varphi^*}^{-1}(w)\cap W^\dagger\Omega^{\underline{X}}_{k[\underline{X}]/k}\right\}. \end{array}$$

For $x\in W^\dagger\Omega_{\overline{R}/k}$, we will denote by $x|_{\mathrm{int}}$ its projection to $W^\dagger\Omega^{\mathrm{int}}_{\overline{R}/k}$, by $x|_{\mathrm{frp}}$ its projection to $W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}$, by $x|_{\mathrm{d(frp)}}$ its projection to $d(W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k})$, and by $x|_{\mathrm{frac}}$ its projection to:

$$W^\dagger\Omega^{\mathrm{frac}}_{\overline{R}/k} := W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k} + d\left(W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}\right).$$

A similar decomposition exists for the usual de Rham–Witt complex [26, Theorem 4.5]. The construction work the same way as here, except that we use the p-adic completion instead of the weak completion. We omit the details. We shall keep the same notations in that context.

Applying [26, Proposition 7.14 and Theorem 9.8], we get $\delta>0$ such that for every $\varepsilon\in\,]0,\delta]$ we have:

$$\forall x\in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k},\quad \zeta_\varepsilon\left(t_F(x)-x\right)\geqslant\zeta_\varepsilon\left(t_F(x)\right)+\frac{3}{4}, \tag{3.2}$$

$$\forall x\in W^\dagger\Omega_{\overline{R}/k},\qquad \check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{int}})\geqslant\zeta_{\varepsilon,\varphi}(x)-\frac{1}{2}, \tag{3.3}$$

$$\forall x\in W^\dagger\Omega_{\overline{R}/k},\qquad \check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{frp}})\geqslant\zeta_{\varepsilon,\varphi}(x)-\frac{1}{2}, \tag{3.4}$$

$$\forall x \in W^\dagger\Omega_{\overline{R}/k}, \qquad \check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{d(frp)}}) \geqslant \zeta_{\varepsilon,\varphi}(x) - \frac{1}{2}, \tag{3.5}$$

$$\forall x \in W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}, \qquad \check{\zeta}_{\varepsilon,\varphi}(x) = \check{\zeta}_{\varepsilon,\varphi}\big(d(x)\big), \tag{3.6}$$

$$\forall x \in W^\dagger\Omega_{\overline{R}/k}, \qquad \zeta_{\varepsilon,\varphi}(x) \geqslant \check{\zeta}_{\varepsilon,\varphi}(x). \tag{3.7}$$

Also, by [25, (9)], we have:

$$\forall \varepsilon > 0,\ \forall x \in W^\dagger\Omega_{k[\underline{X}]/k}, \quad \zeta_\varepsilon\big(d(x)\big) \geqslant \zeta_\varepsilon(x). \tag{3.8}$$

Proposition 3.5. *Let $t \in \mathbb{N}$. For every $\varepsilon \in \,]0,\delta]$ and every $x \in W^\dagger\Omega^{\underline{X}-\mathrm{frp},t}_{k[\underline{X}]/k}$ we have for the decomposition (3.1):*

$$\zeta_\varepsilon(x|_{\mathrm{int}}) \geqslant \zeta_\varepsilon(x) + \frac{3}{4},$$
$$\zeta_\varepsilon(x|_{\mathrm{frp}}) \geqslant \zeta_\varepsilon(x),$$
$$\zeta_\varepsilon(x|_{\mathrm{d(frp)}}) \geqslant \zeta_\varepsilon(x) + \frac{3}{4}.$$

Proof. For every $e \in G(t)$, we let $s_e \in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}$ such that:

$$\sum_{e\in G(t)} t_F(s_e)e = x,$$
$$\zeta_\varepsilon(s_e) + \zeta_\varepsilon(e) \geqslant \zeta_\varepsilon(x).$$

Now, write $x = \sum_{e\in G(t)}(t_F(s_e) - s_e)e + \sum_{e\in G(t)} s_e e$. Following [26, Remark 9.2], the second series is in $W^\dagger\Omega^{\mathrm{frp},t}_{k[\underline{X}]/k}$. So now we can conclude thanks to (3.2) and to the fact that ζ_ε is a pseudovaluation [25, Theorem 3.17]. ∎

Given $t, s \in \mathbb{N}$, we write $[\![t,s]\!] := [t,s] \cap \mathbb{N}$. We recall that a rng is defined like a ring, without the need of an identity for the multiplication.

Lemma 3.6. *Let R be a rng. Let $t \in \mathbb{N}$. Let $(x_i)_{i\in[\![0,t]\!]}, (y_i)_{i\in[\![0,t]\!]} \in R^{t+1}$. Let us denote by $\mathcal{F}_{((x_i,y_i))_{i\in[\![0,t]\!]}}$ the set of functions $[\![0,t]\!] \to R$ such that for every $i \in [\![0,t]\!]$ we have $f_i \in \{x_i + y_i, -y_i\}$, and $f_i = -y_i$ for at least one such i. Then:*

$$\prod_{i=0}^{t}(x_i + y_i) = \prod_{i=0}^{t} x_i - \sum_{f\in\mathcal{F}_{((x_i,y_i))_{i\in[\![0,t]\!]}}} \prod_{i=0}^{t} f_i.$$

Proof. This is done by induction on $t \in \mathbb{N}$, the case $t = 0$ being easy. Assuming the result for some $t \in \mathbb{N}$, we get:

$$\prod_{i=0}^{t} x_i(x_{t+1} + y_{t+1}) = \prod_{i=0}^{t+1} x_i - \Bigg(\prod_{i=0}^{t}(x_i + y_i) + \sum_{f\in\mathcal{F}_{((x_i,y_i))_{i\in[\![0,t]\!]}}} \prod_{i=0}^{t} f_i\Bigg)(-y_{t+1}). \quad ∎$$

Proposition 3.7. *Let $t \in \mathbb{N}$. Let $x \in W^\dagger\Omega^{\mathrm{int},t}_{k[\underline{X}]/k}$. For the decomposition (3.1), for every $\varepsilon \in]0, \delta]$ we have:*

$$\zeta_\varepsilon\big(t_F(x)|_{\mathrm{int}}\big) \geqslant \zeta_\varepsilon\big(t_F(x)\big),$$
$$\zeta_\varepsilon\big(t_F(x)|_{\mathrm{frp}}\big) \geqslant \zeta_\varepsilon\big(t_F(x)\big) + \frac{3}{4},$$
$$\zeta_\varepsilon\big(t_F(x)|_{\mathrm{d(frp)}}\big) \geqslant \zeta_\varepsilon\big(t_F(x)\big) + \frac{3}{4}.$$

Proof. We derive from [26, Proposition 9.1] that for every $e \in H(t)$ there exists a unique $s_e \in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}$ such that:

$$\zeta_\varepsilon(s_e) + \zeta_\varepsilon(e) \geqslant \zeta_\varepsilon\big(t_F(x)\big),$$
$$\sum_{e\in H(t)} t_F(s_e e) = t_F(x).$$

By definition, each $e \in H(t)$ has the form $\prod_{i=1}^t d(y_{e,i})$ where $y_{e,i} \in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}$ for each $i \in [\![1, n]\!]$, and $H(t)$ is a finite set.

Now, fix $e \in H(t)$. Let $x_0 := s_e$, $x_i := d(y_{e,i})$ for $i \in [\![1, t]\!]$, $y_0 := t_F(s_e) - s_e$ and $y_i := d(t_F(y_{e,i}) - y_{e,i})$ for $i \in [\![1, t]\!]$. Using Lemma 3.6 and its notations we get:

$$t_F(s_e e) = s_e e - \sum_{f\in\mathcal{F}_{((x_i,y_i))_{i\in[\![0,t]\!]}}} \prod_{i=0}^{t} f_i.$$

For each f in the above sum, we have $\zeta_\varepsilon(\prod_{i=0}^t f_i) \geqslant \zeta_\varepsilon(s_e) + \zeta_\varepsilon(e) + \frac{3}{4}$ by formulas (3.2) and (3.8), and since ζ_ε is a pseudovaluation [25, Theorem 3.17]. This enables us to conclude with the same argument because $s_e e$ is in the $W(k)$-dga $W^\dagger\Omega^{\mathrm{int},t}_{k[\underline{X}]/k}$, and because ζ_ε is compatible with this decomposition. ■

Proposition 3.8. *Let $t \in \mathbb{N}^*$ and $s \in \mathbb{N}$. Let $x \in W^\dagger\Omega^{\mathrm{int},t}_{\overline{R}/k}$ and $y \in W^\dagger\Omega^{\mathrm{frp},s}_{\overline{R}/k}$. For every $\varepsilon \in]0, \delta]$ we have:*

$$\check{\zeta}_{\varepsilon,\varphi}(xy) \geqslant \check{\zeta}_{\varepsilon,\varphi}(x) + \check{\zeta}_{\varepsilon,\varphi}(y) + \frac{1}{4}.$$

Proof. Let $\mu > 0$. Let $\tilde{x} \in W^\dagger\Omega^{\underline{X}-\mathrm{int},t}_{k[\underline{X}]/k}$ and $\tilde{y} \in W^\dagger\Omega^{\underline{X}-\mathrm{frp},s}_{k[\underline{X}]/k}$ be respective preimages of x and y by φ^* such that

$$\zeta_\varepsilon(\tilde{x}) \geqslant \check{\zeta}_{\varepsilon,\varphi}(x) - \mu \quad \text{and} \quad \zeta_\varepsilon(\tilde{y}) \geqslant \check{\zeta}_{\varepsilon,\varphi}(y) - \mu.$$

If we prove the inequality $\zeta_\varepsilon(\tilde{x}\tilde{y}) \geqslant \check{\zeta}_{\varepsilon,\varphi}(x) + \check{\zeta}_{\varepsilon,\varphi}(y) + \frac{3}{4} - 2\mu$, we can conclude by applying (3.3), (3.4) and (3.5).

Decompose $\tilde{x}$ and $\tilde{y}$ in (3.1). Applying Propositions 3.5 and 3.7, we are almost done using the pseudovaluation properties of ζ_ε [25, Theorem 3.17], except for the term $\tilde{x}|_{\mathrm{int}}\tilde{y}|_{\mathrm{frp}}$. For that last one, we need [25, Proposition 3.8] to conclude. ■

Lemma 3.9. *Let $x \in W^\dagger\Omega_{\overline{R}/k}$. After shrinking δ if needed, one can assume that for every $\varepsilon \in]0,\delta]$:*

$$\check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{int}}) \geqslant -\frac{1}{4},$$
$$\check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{frp}}) \geqslant \frac{1}{2},$$
$$\check{\zeta}_{\varepsilon,\varphi}(x|_{\mathrm{d(frp)}}) \geqslant \frac{3}{4}.$$

Proof. Let $\mu > 0$. Let us choose $\tilde{x} \in W^\dagger\Omega^{X}_{k[\underline{X}]/k}$ a preimage of x by φ^* which satisfies $\zeta_\varepsilon(\tilde{x}) \geqslant \check{\zeta}_{\varepsilon,\varphi}(x) - \mu$.

Recall that $n \in \mathbb{N}$ is the number of indeterminates in $k[\underline{X}]$. When $n = 0$, we are working in the ring of Witt vectors $W(k)$ so that we only have an integral part, and in that case $\zeta_\varepsilon = 0$.

When $n > 0$, then by [26, Proposition 5.9] we get by induction on $m \in \mathbb{N}$ the inequality $\zeta_{\frac{\varepsilon}{(2n)^m}}(\tilde{x}) \geqslant \frac{\zeta_\varepsilon(\tilde{x})}{(2n)^m}$, from which we get $\check{\zeta}_{\varepsilon,\varphi}(x) \geqslant \zeta_\varepsilon(x) \geqslant -\frac{1}{4}$ when ε is small enough. So we only have to focus on the fractional part.

As ζ_ε is a pseudovaluation [25, Theorem 3.17], and applying (3.6), we can assume without loss of generality that $\tilde{x} \in W^\dagger\Omega^{X-\mathrm{frp},t}_{k[\underline{X}]/k}$, and prove that $\check{\zeta}_{\varepsilon,\varphi}(\tilde{x}) \geqslant \frac{3}{4}$.

By definition, for every $e \in G(t)$ there is $s_e \in W^\dagger\Omega^{\mathrm{int},0}_{k[\underline{X}]/k}$ such that:

$$\sum_{e\in G(t)} t_F(s_e)e = \tilde{x},$$
$$\zeta_\varepsilon(s_e) + \zeta_\varepsilon(e) \geqslant -\frac{1}{4}.$$

The definition of the elements of $G(t)$ and of ζ_ε immediately yield $\zeta_\varepsilon(e) \geqslant 1 - \varepsilon$ for every $e \in G(t)$. So for every such e, we have $\zeta_\varepsilon(s_e) \geqslant \varepsilon - \frac{5}{4}$. We can use the same trick as above to see that, by dividing ε as much as needed, we have $\zeta_\varepsilon(s_e) \geqslant \varepsilon - \frac{1}{4}$ independently on e, and we are done. ∎

4. Finite projective modules

In all this section, $(X, \mathcal{O}_X)$ denotes a ringed space. We will study finite projective modules, their relationship with F-isocrystals, and how one can roll back to the locally free setting.

Definition 4.1. An $\mathcal{O}_X$-module M is *globally finite projective* if there exists finite sets J and L and an exact sequence of $\mathcal{O}_X$-modules:

$$\bigoplus_{j\in J}\mathcal{O}_X \longrightarrow \bigoplus_{l\in L}\mathcal{O}_X \xrightarrow{\psi} M \longrightarrow 0$$

such that ψ has a section.

An $\mathcal{O}_X$-module M is *locally finite projective* if there exists a cover $\{U_i\}_{i\in I}$ of X by opens, where I is any set, such that for each $i \in I$ the $\mathcal{O}_X|_{U_i}$-module $M|_{U_i}$ is globally finite projective.

In other terms, M is an $\mathcal{O}_X$-module of finite presentation, for which we can globally (resp. locally) choose a finite presentation which splits.

In many settings, the category of $\mathcal{O}_X$-modules has no projective objects except for the zero module. What justifies the terminology chosen here is rather the following fact.

Proposition 4.2. *Let M be a globally finite projective module. Then, the $\mathcal{O}_X(X)$-module $M(X)$ is finite and projective. Moreover, we have $M \cong \widetilde{M(X)}$ as $\mathcal{O}_X$-modules.*

Proof. This is really just an immediate consequence of the definition. Since the map r has a section for all I, we get that $M(X)$ is a direct summand of $\bigoplus_{l\in L} \mathcal{O}_X(X)$ in the category of $\mathcal{O}_X(X)$-modules.

The last part of the statement is given by the description of $\widetilde{M(X)}$ in [29, 01BH]. ■

Take care, however, not to confuse these sheaves with finite locally free ones. On general ringed spaces, these notions are not equivalent. For instance, if X only has a single point and $\mathcal{O}_X(X) = \mathbb{Z}/6\mathbb{Z}$, then $\widetilde{\mathbb{Z}/2\mathbb{Z}}$ yields a counter-example. Even in more geometric contexts, it is unclear whether these notions are equivalent; see for instance Question 5.7 below.

For a commutative ring R, and a topological space X endowed with a sheaf $\mathcal{F}$ of strictly commutative R-dgas, we shall denote by $\mathcal{F}$-$\mathsf{LocFProjMIC}$ the full subcategory of $\mathcal{F}$-MIC whose objects are integrable $\mathcal{F}$-connections on a locally finite projective $\mathcal{F}_0$-module. Similarly, $\mathcal{F}$-$\mathsf{GlobFProjMIC}$ shall denote the full subcategory of $\mathcal{F}$-MIC whose objects are integrable $\mathcal{F}$-connections on a globally finite projective $\mathcal{F}_0$-module.

Notice that the functors defined in Proposition 1.11 restrict in the above categories.

A globally finite projective $\mathcal{O}_X$-module is a direct summand of a finite free $\mathcal{O}_X$-module. As we are about to see, this is also the case for $\mathcal{F}$-connections on a globally finite projective $\mathcal{O}_X$-module, but not necessarily in the category $\mathcal{F}$-$\mathsf{GlobFProjMIC}$.

In the remainder of this paper, except where otherwise stated for any function $f\colon E \to F$ we shall denote also by $f\colon \mathrm{Mat}(E) \to \mathrm{Mat}(F)$ the function it induces on matrices, by applying f to each coefficients. We shall also do so for functions using unusual notations, such as $\bullet|_{\mathrm{int}}$.

Lemma 4.3. *Let ∇ be an $\mathcal{F}$-connection on a globally finite projective $\mathcal{O}_X$-module M. Let $\psi\colon \bigoplus_{i=1}^r \mathcal{F}_0 \to M$ be a surjective morphism of $\mathcal{F}_0$-modules with a section s, where $r \in \mathbb{N}^*$. Fix an $\mathcal{F}_0(X)$-basis $\mathcal{B}$ of $\bigoplus_{i=1}^r \mathcal{F}_0(X)$, and let P be the representative matrix of $s \circ \psi$ for $\mathcal{B}$.*

Then, there exists an $\mathcal{F}$-connection $\widetilde{\nabla}$ on $\bigoplus_{i=1}^r \mathcal{F}_0$ such that ψ and s promote to horizontal morphisms between ∇ and $\widetilde{\nabla}$. Moreover, if ∇ is integrable and N denotes the representative matrix of $\widetilde{\nabla}$ for $\mathcal{B}$, then one can choose $\widetilde{\nabla}$ so that its curvature matrix for $\mathcal{B}$ is:

$$d(P)Pd(P).$$

Proof. Put $(b_j)_{j\in[\![1,r]\!]} := \mathcal{B}$. Let $A = (a_{i,j})_{i,j\in[\![1,r]\!]} \in \mathrm{Mat}_r(\mathcal{F}_1)$ be the unique matrix such that for every $i \in [\![1,r]\!]$ we have:

$$(s \otimes \mathrm{Id}_{\mathcal{F}_1}) \circ \nabla\big(\psi(b_i)\big) = \sum_{j=1}^{r} b_j \otimes a_{j,i}.$$

Notice that $P^2 = P$ and $A = PA$. Moreover, $AP + Pd(P) = A$ by the Leibniz rule. Now, put $N := A - d(P)P$. Thus:

$$PN = PA - Pd(P)P = A - Pd(P) + PPd(P) = A.$$

Also:

$$NP + d(P) = AP - d(P)P + d(P) = AP + Pd(P) = A.$$

If we denote by $\tilde{\nabla}$ the $\mathcal{F}$-connection on $\bigoplus_{i=1}^{r} \mathcal{F}_0$ whose representative matrix is N for $\mathcal{B}$, these two equalities imply that ψ promotes to a morphism of $\mathcal{F}$-connections $\tilde{\nabla} \to \nabla$, and so do its section s by Proposition 1.6.

For now on, we assume that ∇ is integrable. If we denote by C the curvature matrix of $\tilde{\nabla}$ for $\mathcal{B}$, this implies that $PC = CP = 0$. Also, $C = N^2 + d(N)$ by Proposition 1.9. Therefore:

$$C = (1-P)\big(N^2 + d(N)\big)(1-P).$$

Now, the above relations give us $(1-P)N = N - A = -d(P)P$, and we also derive $N(1-P) = N - A + d(P) = Pd(P)$. In particular, $(1-P)N(1-P) = 0$.

Thus:

$$\begin{aligned}(1-P)&d(N)(1-P)\\ &= d\big((1-P)N(1-P)\big) - d(1-P)N(1-P) + (1-P)Nd(1-P)\\ &= 0 + d(P)Pd(P) + d(P)Pd(P).\end{aligned}$$

Hence, $C = 2d(P)Pd(P) + (1-P)N^2(1-P) = d(P)Pd(P)$. ∎

Fix a smooth commutative k-algebra $\overline{R}$ and a smooth commutative $W(k)$-algebra R lifting $\overline{R}$. On its weak completion $R^\dagger$, fix a morphism of rings $F\colon R^\dagger \to R^\dagger$ lifting the Frobenius and compatible with $F\colon W(k) \to W(k)$ as we did previously. This yields a morphism of weak formal schemes $F\colon \mathrm{Spff}(R^\dagger) \to \mathrm{Spff}(R^\dagger)$ by [10, 1.1.2]. It extends, by the universal property of the p-adically complete de Rham complex, to a morphism of sheaves of $W(k)$-dgas $F\colon \boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}] \to \boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$ on $\mathrm{Spec}(\overline{R})$. It also yields a morphism of sheaves of $W(k)$-dgas on $\mathrm{Spec}(\overline{R})$:

$$t_F\colon \boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}\left[\frac{1}{p}\right] \to W^\dagger\Omega_{\mathrm{Spec}(\overline{R})/k}\left[\frac{1}{p}\right].$$

Let $\phi\colon W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}] \to W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$ defined on the gradation by the relation

$$\phi|_{W^\dagger\Omega^i_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]} := p^i F|_{W^\dagger\Omega^i_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]} \quad \text{for all } i \in \mathbb{N}.$$

Notice that we have $t_F \circ F = \phi \circ t_F$. This is a morphism of sheaves of $W(k)$-dgas by [20, (1.19)].

Of course, all these constructions have a convergent analogue for which we shall use the same notations. We state the following proposition only for overconvergent de Rham–Witt connections, the case in the classical convergent case being totally similar.

Proposition 4.4. *Let ∇ be an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection on a globally finite projective $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-module M. Assume that there is an isomorphism $\nabla \cong \phi^*(\nabla)$ of $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connections, and that $W^\dagger\Omega_{\bar{R}/k}$ has no p-torsion.*

Then, there exists an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection on a globally finite projective $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-module K such that $M \oplus K$ is a finite free $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-module.

Proof. Let $r \in \mathbb{N}^*$, and let $\psi\colon \bigoplus_{i=1}^r W^\dagger(\bar{R})[\frac{1}{p}] \to M(\mathrm{Spec}(\bar{R}))$ be a surjective morphism of $W^\dagger(\bar{R})[\frac{1}{p}]$-modules. By Proposition 4.2, $M(\mathrm{Spec}(\bar{R}))$ is projective, so ψ has a section s.

Since we assumed that $\nabla \cong \phi^*(\nabla)$ as $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connections, we also have an isomorphism $\varphi\colon M(\mathrm{Spec}(\bar{R})) \to M(\mathrm{Spec}(\bar{R}))_F$ of $W^\dagger(\bar{R})[\frac{1}{p}]$-modules, where we have denoted by $M(\mathrm{Spec}(\bar{R}))_F$ the extension of scalars of $M(\mathrm{Spec}(\bar{R}))$ by the Frobenius morphism $F\colon W^\dagger(\bar{R})[\frac{1}{p}] \to W^\dagger(\bar{R})[\frac{1}{p}]$. Similarly, in what follows we shall also denote by $\psi_F\colon \bigoplus_{i=1}^r W^\dagger(\bar{R})[\frac{1}{p}] \to M(\mathrm{Spec}(\bar{R}))_F$ the extension of scalars of ψ by F.

Next, we derive from ψ another surjective morphism of $W^\dagger(\bar{R})[\frac{1}{p}]$-modules

$$\tilde{\psi}\colon \bigoplus_{i=1}^{2r} W^\dagger(\bar{R})\left[\frac{1}{p}\right] \to M\big(\mathrm{Spec}(\bar{R})\big)$$

by sending every new direct summand to zero. It has a section $\tilde{s}$. By Schanuel's lemma, we have the following isomorphisms of $W^\dagger(\bar{R})[\frac{1}{p}]$-modules:

$$\ker(\tilde{\psi}) \cong \bigoplus_{i=1}^{r} W^\dagger(\bar{R})\left[\frac{1}{p}\right] \oplus \ker(\psi) \cong \bigoplus_{i=1}^{r} W^\dagger(\bar{R})\left[\frac{1}{p}\right] \oplus \ker(\psi_F) \cong \ker(\tilde{\psi})_F.$$

Let $K := \ker(\tilde{\psi})$. Denote by $\iota\colon K \to K_F$ the isomorphism we get by composing the three above ones. We thus have an isomorphism $\varphi \oplus \iota$ of finite free $W^\dagger(\bar{R})[\frac{1}{p}]$-modules.

Let us consider a $W^\dagger(\bar{R})[\frac{1}{p}]$-basis $\mathcal{B} = (b_i)_{i\in[\![1,2r]\!]}$ of $M(\mathrm{Spec}(\bar{R})) \oplus K$. Denote by $P = (p_{i,j})_{i,j\in[\![1,2r]\!]}$ the representative matrix of $\tilde{s} \circ \tilde{\psi}$ for $\mathcal{B}$. By [22, Proposition V 5.1], the representative matrix of $\tilde{s}_F \circ \tilde{\psi}_F$ for $(b_i \otimes 1)_{i\in[\![1,2r]\!]}$ is $F(P)$, that is, the matrix whose coefficients are the images through F of the ones of P.

Notice that we have the following commutative diagram, in which every horizontal line is an isomorphism:

$$\begin{array}{ccc} M\big(\mathrm{Spec}(\bar{R})\big) \oplus K & \xrightarrow{\varphi \oplus \iota} & M\big(\mathrm{Spec}(\bar{R})\big)_F \oplus K_F \\ \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \tilde{\psi}_F} \\ M\big(\mathrm{Spec}(\bar{R})\big) & \xrightarrow{\varphi} & M\big(\mathrm{Spec}(\bar{R})\big)_F \\ \downarrow{\scriptstyle \tilde{s}} & & \downarrow{\scriptstyle \tilde{s}_F} \\ M\big(\mathrm{Spec}(\bar{R})\big) \oplus K & \xrightarrow{\varphi \oplus \iota} & M\big(\mathrm{Spec}(\bar{R})\big)_F \oplus K_F. \end{array}$$

For every $i \in [\![1, 2r]\!]$ we have:

$$\begin{aligned} \sum_{j=1}^{2r} p_{j,i}(b_j \otimes 1) &= (\varphi \oplus \iota)^{-1}\Big(\sum_{j=1}^{2r} p_{j,i}(\varphi \oplus \iota)(b_j \otimes 1)\Big) \\ &= \big((\varphi \oplus \iota)^{-1} \circ \tilde{s} \circ \tilde{\psi} \circ (\varphi \oplus \iota)\big)(b_j \otimes 1), \\ \sum_{j=1}^{2r} p_{j,i}(b_j \otimes 1) &= (\tilde{s}_F \circ \tilde{\psi}_F)(b_j \otimes 1). \end{aligned}$$

This means that $P = F(P)$. In particular, for every integer $n \in \mathbb{N}$, we have $d(P) = p^n F^n(d(P))$ by [20, (1.19)]. This implies that the coefficients of the matrix $d(P)$ are in $\bigcap_{m \in \mathbb{N}} p^m W^\dagger \Omega_{\bar{R}/k}$ because we assumed that $W^\dagger\Omega_{\bar{R}/k}$ had no p-torsion. Thus, $d(P) = 0$.

Therefore, by Lemma 4.3, there exists an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection $\widetilde{\nabla}$ on $\bigoplus_{i=1}^{2r} W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$ such that $\tilde{\psi}$ and $\tilde{s}$, seen here as morphisms of $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-modules, promote to horizontal morphisms between $\widetilde{\nabla}$ and ∇. To conclude, notice that $\widetilde{\nabla}$ restricts to an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection on $\widetilde{K}$. ∎

5. The functor

We shall denote here by X a scheme smooth over k, and by $\mathcal{O}_{X/W(k)}$ the structure sheaf of the crystalline site $\mathrm{Cris}(X/W(k))$. The goal of this section is to study Etesse's functor $\acute{E}\colon \mathrm{Cris}(X/W(k)) \to W\Omega_{X/k}\text{-MIC}$ from crystals over $X/W(k)$ to integrable $W\Omega_{X/k}$-connections [14, Définition 1.2.5]. We shall see here how it yields a functor on convergent isocrystals on X.

First, we describe it locally.

Proposition 5.1. *Assume that $X = \mathrm{Spec}(\bar{R})$ is finite étale over $\mathbb{A}_k^n$ for some $n \in \mathbb{N}$. Let R be a smooth commutative $W(k)$-algebra lifting $\bar{R}$. Then, the restriction of $\acute{E}$ to crystals in finite $\mathcal{O}_{X,W(k)}$-modules factors in the following essentially commutative diagram, in which ι is fully faithful:*

$$\begin{array}{ccc} \mathrm{Cris}_{fin}\big(X/W(k)\big) & \xrightarrow{\acute{E}} & W\Omega_{X/k}\text{-MIC} \\ \downarrow{\scriptstyle \iota} & \nearrow{\scriptstyle t_F{}^*} & \\ \mathbf{\Omega}_{\mathrm{Spf}(\hat{R})/W(k)}\text{-MIC} & & \end{array}$$

Proof. The functor ι is given by [5, Proposition 1.3.3]. For each $n \in \mathbb{N}^*$ both $R/p^n R$ and $W_n(\bar{R})$ yield divided power thickenings of $\bar{R}$. Hence, given a crystal A over $X/W(k)$, the morphism t_F induces an isomorphism $A(R/p^n R) \otimes_{R/p^n R} W_n(\bar{R}) \cong A(W_n(\bar{R}))$ of $W(\bar{R})$-modules.

In fact, [5, Corollaire 1.3.2] and [14, Proposition 1.1.6] explain how we get an integrable $\mathbf{\Omega}_{\mathrm{Spec}(R/p^n R)/W(k)}$-connection $\nabla_{R,n}$ on the $\mathcal{O}_{\mathrm{Spec}(R/p^n R)}$-module $\widetilde{A(R/p^n R)}$, and an integrable $W_n\Omega_{X/k}$-connection $\nabla_{W,n}$ on the $\mathcal{O}_{\mathrm{Spec}(W_n(\bar{R}))}$-module $\widetilde{A(W_n(\bar{R}))}$. From the construction of the above connections, the aforementioned isomorphism actually yields an isomorphism $t_F{}^*(\nabla_{R,n}) \cong \nabla_{W,n}$ of $W_n\Omega_{X/k}$-connections. Moreover, by definition we have $\iota(A) = \lim_n \nabla_{R,n}$ and $\acute{E}(A) = \lim_n \nabla_{W,n}$.

Let K_n fit in the following exact sequence of $W(\bar{R})$-modules:

$$K_n \to A(R/p^n R) \otimes_R W\Omega_{\bar{R}/k}/p^n W\Omega_{\bar{R}/k} \to A(R/p^n R) \otimes_R W_n\Omega_{\bar{R}/k} \to 0.$$

Using [26, Theorem 4.5], we see that the limit of the K_n must be naught, so taking the limit of these sequences tells us that $\acute{E}(A)$ is p-adically complete.

Assume now that A is a crystal in finite $\mathcal{O}_{X,W(k)}$-modules. As $\widehat{R}$ is a coherent ring, we find that t_F is a flat morphism because $W(\bar{R})$ is a product of copies of $\widehat{R}$ by [26, Theorem 4.5]. By base change, so are all its reductions modulo p^n. Thus, by [29, 0912] we find that $t_F{}^*(\iota(A)) \cong \acute{E}(A)$ as $W\Omega_{X/k}$-connections. ■

Recall that given a lax monoidal functor $F\colon \mathsf{C} \to \mathsf{D}$, we have a change of enriching category functor $F_*\colon \mathsf{C}\text{-}\mathsf{Cat} \to \mathsf{D}\text{-}\mathsf{Cat}$ from the big category of categories enriched over C to the big category of categories enriched over D. Given a category K enriched over C, this functor is given by defining a category enriched over D with the same objects as K, and defining $\mathrm{Hom}_{F_*(\mathsf{K})}(A, B) := F(\mathrm{Hom}_{\mathsf{K}}(A, B))$ for each objects A and B of K.

The extension of scalars $\bullet \otimes_{\mathbb{Z}} \mathbb{Q}\colon \mathbb{Z}\text{-}\mathsf{Mod} \to \mathbb{Q}\text{-}\mathsf{Mod}$ is a lax monoidal functor which we shall denote by i. We use this functor and the above formalism to rephrase the notion of an isogeny category, using the fact that preadditive categories can be seen as categories enriched over $\mathbb{Z}\text{-}\mathsf{Mod}$.

Definition 5.2. Let C be a preadditive category. The *isogeny category of* C is the category $\mathsf{i}_*(\mathsf{C})$ enriched over $\mathbb{Q}\text{-}\mathsf{Mod}$.

Functorially on C, we have a functor $\mathsf{C} \to \mathsf{i}_*(\mathsf{C})$ given by the tensorisation of $\mathbb{Z} \to \mathbb{Q}$. This functor is an isomorphism of categories when all the hom-objects of C are $\mathbb{Q}$-vector spaces.

Definition 5.3. Let C be a category. Let $F\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor. A *Frobenius structure* on an object A is an isomorphism $A \cong F(A)$ in C.

A morphism between Frobenius structures i_A and i_B respectively on objects A and B of C is a morphism $\phi \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ such that $F(\phi) \circ i_A = i_B \circ \phi$.

We will denote by F-C the category of Frobenius structures on objects of C with these morphisms.

In this article, we shall prove various equivalences of categories between such categories. Let us give a first one to begin with.

Example 5.4. The absolute Frobenius endomorphism induces an endomorphism of topoi $F\colon \mathsf{Cris}_{\mathsf{fin}}(X/W(k)) \to \mathsf{Cris}_{\mathsf{fin}}(X/W(k))$, whose pullback yields an endofunctor F^* on the category of crystals in finite $\mathcal{O}_{X/W(k)}$-modules.

By applying [28, Theorem 0.7.2] and [4, Théorème 2.4.2], we know that the category $F\text{-}\mathsf{Isoc}(X/W(k))$ of F-isocrystals on X is equivalent to $\mathrm{i}_*(F^*)\text{-}\mathrm{i}_*(\mathsf{Cris}_{\mathsf{fin}}(X/W(k)))$, the category of Frobenius structures on objects of the isogeny category of crystals in finite $\mathcal{O}_{X,W(k)}$-modules. Indeed, by virtue of [4, (2.3.4)], Berthelot's definition coincides with the one made by Ogus in [27, Definition 2.7]. See [19, Theorem 2.2] for this modern formulation of the equivalence of categories.

Proposition 5.5. *Let us fix a smooth commutative k-algebra $\overline{R}$, and a smooth commutative $W(k)$-algebra R lifting $\overline{R}$. Recall that we have previously constructed a morphism of sheaves of $W(k)$-dgas $F\colon \mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)} \to \mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}$ which is compatible with the Frobenius endomorphisms.*

Then, the category $F\text{-}\mathsf{Isoc}^\dagger(\mathrm{Spec}(\overline{R})/W(k))$ of overconvergent F-isocrystals on the affine scheme $\mathrm{Spec}(\overline{R})$ is equivalent to the category $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$ of Frobenius structures on integrable $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connections on a locally finite projective $\mathcal{O}_{\mathrm{Spff}(R^\dagger)}[\frac{1}{p}]$-module, and such a module is actually globally finite projective.

In other words, the category $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$ is actually equal to $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]\text{-}\mathsf{GlobFProjMIC}$, the category of Frobenius structures on integrable $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connections on a globally finite projective $\mathcal{O}_{\mathrm{Spff}(R^\dagger)}[\frac{1}{p}]$-module.

Proof. By [4, Corollaire 2.5.8], the category $F\text{-}\mathsf{Isoc}^\dagger(\mathrm{Spec}(\overline{R})/W(k)[\frac{1}{p}])$ is equivalent to the category of finite projective $R^\dagger[\frac{1}{p}]$-modules with an integrable connection, in the usual algebraic meaning, and a Frobenius structure.

The functor $\widetilde{\bullet}$ allows one to define an integrable $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connection with a Frobenius structure from this data. Indeed, it commutes with direct sums, so that the module of global sections is the finite projective $R^\dagger[\frac{1}{p}]$-module one started with, and same with the tensor product. So we only need to show that this functor is essentially surjective and fully faithful.

Given a Frobenius structure on an integrable $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connection on a locally finite projective $\mathcal{O}_{\mathrm{Spff}(R^\dagger)}[\frac{1}{p}]$-module, one can cover $\mathrm{Spff}(R^\dagger)$ with affine opens $\mathrm{Spff}(R_i)$ for a finite set $i \in I$, on which the associated module has a finite presentation which splits. Using Proposition 4.2, we find using [4, Corollaire 2.5.6] that we actually have a compatible family of overconvergent F-isocrystals on $\mathrm{Spec}(\overline{R_i})$, which we can glue by [4, Remarques 2.3.3] to an essential preimage of our Frobenius structure. The full faithfulness works similarly. ■

As in the overconvergent case, one has a endomorphism of sheaves of $W(k)$-dgas $F\colon \mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}] \to \mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]$ on $\mathrm{Spec}(\overline{R})$.

Proposition 5.6. *With the above notations, the category* $F\text{-}\mathsf{Isoc}(\mathrm{Spec}(\overline{R})/W(k))$ *of convergent* F*-isocrystals on* $\mathrm{Spec}(\overline{R})$ *is equivalent to the category of Frobenius structures on integrable* $\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]$*-connections on a locally finite projective* $\mathcal{O}_{\mathrm{Spf}(\widehat{R})}[\frac{1}{p}]$*-module* $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$.

Moreover, the category $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$ *is actually equal to the category* $F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-}\mathsf{GlobFProjMIC}$.

Proof. This is the same proof as before, only using a result of Etesse [15, Corollaire 1.2.3] instead of the one by Berthelot, which he closely follows. ∎

Take care that, in the previous propositions, it is not clear that locally finite projective is equivalent to finite locally free, as it would be for instance in the classical case of a sheaf of quasi-coherent modules on a scheme. Actually, after discussing with many experts, which the author thanks here, it seems that the following question is still open.

Question 5.7. With the above notations, is the category of convergent (resp. overconvergent) F-isocrystals on $\mathrm{Spec}(\overline{R})$ equivalent to the category of Frobenius structures on integrable $\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]$-connections (resp. integrable $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connections) on a locally free $\mathcal{O}_{\mathrm{Spf}(\widehat{R})}[\frac{1}{p}]$-module (resp. locally free $\mathcal{O}_{\mathrm{Spff}(R^\dagger)}[\frac{1}{p}]$-module)?

Would the answer to this question be positive, the proofs in the next sections could be simplified. Without knowing it, we will therefore need to work with projective modules, and not just free ones. But as we have seen, Proposition 4.4 allows us to reduce to the case of free modules.

Let E be an F-isocrystal on a scheme X smooth over k. The previous discussion explains how applying $\acute{E}$ yields a Frobenius structure on an integrable $W\Omega_{X/k}[\frac{1}{p}]$-connection on a locally finite projective $W(\mathcal{O}_X)[\frac{1}{p}]$-module, for $\phi\colon W\Omega_{X/k}[\frac{1}{p}] \to W\Omega_{X/k}[\frac{1}{p}]$ the morphism of $W(k)$-dgas that we introduced previously. Let us describe this locally to explain what we mean.

Proposition 5.8. *Assume that* $X = \mathrm{Spec}(\overline{R})$ *is finite étale over* $\mathbb{A}^n_k$ *for some* $n \in \mathbb{N}$*. Let* R *be a smooth commutative* $W(k)$*-algebra lifting* $\overline{R}$*. Then, the isogeny functor* $\acute{E}$ *from the category of isocrystals on* X *factors in the following essentially commutative diagram, in which* ι *is an equivalence of categories:*

$$\begin{array}{ccc} F\text{-}\mathsf{Isoc}(X/W(k)) & \xrightarrow{\ \acute{E}\ } & \phi^*\text{-}W\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{GlobFProjMIC} \\ \downarrow{\scriptstyle \iota} & \nearrow{\scriptstyle t_F{}^*} & \\ F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-}\mathsf{GlobFProjMIC} & & \end{array}$$

Proof. To construct this diagram, we shall first consider the composition of the two functors $\mathsf{Cris}_{\mathsf{fin}}(X/W(k)) \to W\Omega_{X/k}\text{-}\mathsf{MIC} \to W\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{MIC}$, where the first arrow is given by Proposition 5.1, and the second arrow is given by the localisation. The target category is actually enriched over $\mathbb{Q}\text{-}\mathsf{Mod}$, so this yields a functor $i_*(\mathsf{Cris}_{\mathsf{fin}}(X/W(k))) \to W\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{MIC}$ whose source is the isogeny category of $\mathsf{Cris}_{\mathsf{fin}}(X/W(k))$.

Now, the same can be done with ι, but by Example 5.4, we see that this coincides with the functor described in [15, Corollaire 1.2.3]. Conclude with Proposition 5.6. ∎

Our next goal is to show that $\acute{E}$ is actually an equivalence of categories, and to describe the essential image of overconvergent F-isocrystals on X.

6. Working with de Rham–Witt matrices

In this section, $\overline{R}$ denotes a commutative k-algebra such that $\mathrm{Spec}(\overline{R})$ is finite étale over $\mathbb{A}_k^n$ for some $n \in \mathbb{N}$.

Let $r \in \mathbb{N}^*$. We extend the definition of the p-adic pseudovaluation v_p to matrices in the following manner:

$$\mathrm{v}_p\colon \begin{aligned} \mathrm{Mat}_r(W^\dagger\Omega_{\overline{R}/k}) &\to \mathbb{R} \cup \{+\infty, -\infty\} \\ (x_{i,j})_{i,j\in[\![1,r]\!]} &\mapsto \min\left\{ \mathrm{v}_p(x_{i,j}) \mid i,j \in [\![1,r]\!] \right\}. \end{aligned}$$

For simplicity, we only state the overconvergent version of the results here. However, all results stated here hold for the usual de Rham–Witt complex. The difficulty of the proofs are in the control of the overconvergence, and depend on the results of deep theorems given in the author's previous articles. Needless to say, the reader can easily translate all these results to the convergent case, by removing the daggers and replacing weak completion by p-adic completion. In the proofs, the translating reader only needs to ignore all arguments about overconvergence, and see that all structure theorems here used also have a classical de Rham–Witt flavour.

Proposition 6.1. *Let $x \in W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}$ and $y \in W^\dagger\Omega_{\overline{R}/k}$. Then, there exists $z \in W^\dagger\Omega^{\mathrm{frp}}_{\overline{R}/k}$ such that $xy \equiv z \pmod{p}$.*

Put differently, $\mathrm{v}_p((xy)|_{\mathrm{int}}) > 0$ and $\mathrm{v}_p((xy)|_{\mathrm{d(frp)}}) > 0$.

Proof. This result holds when $\overline{R} = k[\underline{X}]$ and one takes the canonical Frobenius lift by [25, Lemma 2.7 and Proposition 2.8]. Because for each $t \in \mathbb{N}$, we have $G(t) \subset W^\dagger\Omega^{\mathrm{frp}}_{k[\underline{X}]/k}$ from the definition, the result holds for any Frobenius lift by [26, (7.8)] and [8, IX. §1 Proposition 3]. Using [26, Lemma 9.6], one can conclude for a general $\overline{R}$. ∎

The next lemma is a constructive version of [6, Lemma 5.2].

Lemma 6.2. *Let $r, s \in \mathbb{N}^*$ and let $N \in \mathrm{Mat}_r(W^\dagger\Omega^1_{\overline{R}/k})$ be a matrix such that $N^2 + d(N) = 0$ and that $\mathrm{v}_p(N|_{\mathrm{frac}}) \geqslant s$. Let*

$$U \in \mathrm{Mat}_r\left(W^\dagger\Omega^{\mathrm{frp},0}_{\overline{R}/k}\right)$$

be the unique matrix which satisfies $d(p^sU) = -N|_{\mathrm{d(frp)}}$. Then:

$$\mathrm{v}_p\left(\left((1+p^sU)^{-1}N(1+p^sU) + (1+p^sU)^{-1}d(1+p^sU)\right)|_{\mathrm{frac}}\right) \geqslant s+1.$$

Proof. Let us first write $N = E + p^s H$, where the matrices $E \in \mathrm{Mat}_r(t_F(\boldsymbol{\Omega}^1_{R^\dagger/W(k)}))$ and $H \in \mathrm{Mat}_r(W^\dagger\Omega^{\mathrm{frac},1}_{\overline{R}/k})$. We get:

$$\begin{aligned} E^2 + d(E) &= E^2 + d(N) - p^s d(H) \\ &= E^2 - (E + p^s H)^2 - p^s d(H), \\ E^2 + d(E) &= -p^s(EH + HE + d(H)) - p^{2s} H^2. \end{aligned}$$

We thus get that $d(H) \equiv -(EH + HE)|_{\mathrm{d(frp)}} \pmod p$. Using Proposition 6.1, we find that $EH|_{\mathrm{frp}} + H|_{\mathrm{frp}}E$ is congruent to a matrix with coefficients in $W^\dagger\Omega^{\mathrm{frp},2}_{\overline{R}/k}$ modulo p, hence:

$$d(H) \equiv -(EH|_{\mathrm{d(frp)}} + H|_{\mathrm{d(frp)}}E)|_{\mathrm{d(frp)}} \pmod p. \tag{6.1}$$

Moreover, $V := \sum_{i\in\mathbb{N}^*} p^{-s}(-p^s U)^i$ satisfies $(1 + p^s U)^{-1} = 1 + p^s V$, and its coefficients are in $W^\dagger(\overline{R})$ according to [26, Lemma 6.4] and [25, Theorem 3.17]. Also:

$$\begin{aligned} &(1 + p^s V)(E + p^s H)(1 + p^s U) + (1 + p^s V)d(1 + p^s U) \\ &\quad = E + p^s\big(VE + H + EU + d(U)\big) + p^{2s}\big(VH + VU + HU + Vd(U)\big) \\ &\quad\quad + p^{3s} VHU. \end{aligned} \tag{6.2}$$

By Proposition 6.1, $VE + EU$ is congruent to a matrix with coefficients in $W^\dagger\Omega^{\mathrm{frp},1}_{\overline{R}/k}$ modulo p. We also have:

$$d(VE + EU) = d(V)E + Vd(E) + d(E)U - Ed(U).$$

By construction, $U \equiv -V \pmod p$ and $d(U) = -H|_{\mathrm{d(frp)}}$, thus:

$$d(V)E - Ed(U) \equiv H|_{\mathrm{d(frp)}}E + EH|_{\mathrm{d(frp)}} \pmod p.$$

The same argument as above tells us that $Vd(E) + d(E)U$ is congruent to a matrix with coefficients in $W^\dagger\Omega^{\mathrm{frp},2}_{\overline{R}/k}$ modulo p, so:

$$d(VE + EU) \equiv (H|_{\mathrm{d(frp)}}E + EH|_{\mathrm{d(frp)}})|_{\mathrm{d(frp)}} \pmod p.$$

Hence, we derive from (6.1) that $d(VE + H + EU) \equiv 0 \pmod p$. Using Proposition 6.1 again, we get $VE + H|_{\mathrm{frp}} + EU \equiv 0 \pmod p$. So in the end $VE + H + EU + d(U) \equiv 0 \pmod p$, and we conclude with (6.2). ∎

Our assumption on $\overline{R}$ implies that $W^\dagger\Omega_{\mathrm{Spec}(\overline{R})/k}$ is p-torsion free. It thus injects into $W^\dagger\Omega_{\mathrm{Spec}(\overline{R})/k}[\frac{1}{p}]$, and the p-adic pseudovaluation v_p extends naturally in the localisation.

Proposition 6.3. *Let (M, ∇_M) and (N, ∇_N) be two $W^\dagger\Omega_{\mathrm{Spec}(\overline{R})/k}[\frac{1}{p}]$-connections on free $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\overline{R})})[\frac{1}{p}]$-modules, and let $\varphi\colon (M, \nabla_M) \to (N, \nabla_N)$ be a horizontal morphism. Let I and J denote sets, and let $\{m_i\}_{i\in I}$ and $\{n_j\}_{j\in J}$ be respective $W^\dagger(\overline{R})[\frac{1}{p}]$-bases of $M(\mathrm{Spec}(\overline{R}))$ and $N(\mathrm{Spec}(\overline{R}))$. Let E, F and G be the respective representative matrices* *of ∇_M, ∇_N and φ in these bases. Let $t \in \mathbb{N}$ be such that $\mathrm{v}_p(E) \geqslant t$, $\mathrm{v}_p(F) \geqslant t$ and $\mathrm{v}_p(G) \geqslant -t$.*

Then, if for $s \in \mathbb{N}$ we have $\mathrm{v}_p(E|_{\mathrm{frac}}) \geqslant s+t$ and $\mathrm{v}_p(F|_{\mathrm{frac}}) \geqslant s+t$, we also have $\mathrm{v}_p(G|_{\mathrm{frac}}) \geqslant s$.

Proof. Write $(e_{l,i})_{l,i\in I} := E$, $(f_{l,j})_{l,j\in J} := F$ and $(g_{j,i})_{i\in I}^{j\in J} := G$ for the coefficients of the representative matrices. That is, we have:

$$\begin{aligned}
\forall i \in I, \quad \nabla_M(m_i) &= \sum_{l\in I} m_l \otimes e_{l,i}, \\
\forall i \in J, \quad \nabla_N(n_i) &= \sum_{j\in J} n_j \otimes f_{j,i}, \\
\forall i \in I, \quad \varphi(m_i) &= \sum_{j\in J} g_{j,i} n_j.
\end{aligned}$$

By definition of a horizontal morphism, for each $i \in I$, we find:

$$\sum_{l\in I} \varphi(m_l) \otimes e_{l,i} = \sum_{j\in J} \big(g_{j,i} \nabla_N(n_j) + n_j \otimes d(g_{j,i})\big).$$

In other words:

$$\forall i \in I, \sum_{j\in J} n_j \otimes \Big(\sum_{l\in I} g_{j,l} e_{l,i}\Big) = \sum_{j\in J} n_j \otimes \Big(d(g_{j,i}) + \sum_{l\in J} g_{l,i} f_{j,l}\Big).$$

Therefore:

$$\forall i \in I, \ \forall j \in J, \quad d(g_{j,i}) = \sum_{l\in I} g_{j,l} e_{l,i} - \sum_{l\in J} g_{l,i} f_{j,l}.$$

Let $u := \min\{\mathrm{v}_p((g_{j,i})|_{\mathrm{frac}}) \mid i \in I,\ j \in J\}$. We want to prove that $u \geqslant s$.

If $u = +\infty$, there is nothing to prove. Otherwise, let $i \in I$ and $j \in J$ be such that $\mathrm{v}_p((g_{j,i})|_{\mathrm{frac}}) = u$. We have:

$$\begin{aligned}
d(g_{j,i}) = &\sum_{l\in I} (g_{j,i}|_{\mathrm{int}} e_{l,i}|_{\mathrm{int}} + g_{j,i}|_{\mathrm{frp}} e_{l,i}|_{\mathrm{int}} + g_{j,i} e_{l,i}|_{\mathrm{frac}}) \\
&- \sum_{l\in J} (g_{l,i}|_{\mathrm{int}} f_{j,l}|_{\mathrm{int}} + g_{l,i}|_{\mathrm{frp}} f_{j,l}|_{\mathrm{int}} + g_{l,i} f_{j,l}|_{\mathrm{frac}}).
\end{aligned}$$

By Proposition 6.1, for every $l \in I$ we have

$$\mathrm{v}_p\left((g_{j,i}|_{\mathrm{frp}} e_{l,i}|_{\mathrm{int}})|_{\mathrm{d(frp)}}\right) \geqslant u+t+1 \quad \text{and} \quad \mathrm{v}_p\left((g_{j,i} e_{l,i}|_{\mathrm{frac}})|_{\mathrm{d(frp)}}\right) \geqslant s,$$

and similarly for every $l \in J$ we derive the two inequalities

$$\mathrm{v}_p\left((g_{l,i}|_{\mathrm{frp}} f_{j,l}|_{\mathrm{int}})|_{\mathrm{d(frp)}}\right) \geqslant u+t+1 \quad \text{and} \quad \mathrm{v}_p\left((g_{l,i} f_{j,l}|_{\mathrm{frac}})|_{\mathrm{d(frp)}}\right) \geqslant s,$$

so we conclude. ∎

7. The equivalences of categories

As in the previous section, $\overline{R}$ denotes a commutative k-algebra such that $\operatorname{Spec}(\overline{R})$ is finite étale over $\mathbb{A}^n_k$ for some $n \in \mathbb{N}$. We let $\varphi\colon k[\underline{X},\underline{Y}] \to \overline{R}$ be a surjective morphism of $k[\underline{X}]$-algebras.

For every $\varepsilon > 0$, we extend the definition of $\check{\zeta}_{\varepsilon,\varphi}$ to matrices:

$$\check{\zeta}_{\varepsilon,\varphi}\colon \begin{array}{rcl} \operatorname{Mat}_r(W^\dagger\Omega_{\overline{R}/k}) & \to & \mathbb{R}\cup\{+\infty,-\infty\} \\ (x_{i,j})_{i,j\in[\![1,r]\!]} & \mapsto & \min\left\{\check{\zeta}_{\varepsilon,\varphi}(x_{i,j}) \mid i,j\in[\![1,r]\!]\right\}. \end{array}$$

Let $r \in \mathbb{N}^*$ and $N \in \operatorname{Mat}_r(W^\dagger\Omega_{\overline{R}/k})$. By Lemma 3.9, it makes sense to consider this condition for some fixed $\varepsilon > 0$:

$$\begin{cases} \check{\zeta}_{\varepsilon,\varphi}(N|_{\mathrm{int}}) \geqslant -\dfrac{1}{4}, \\ \check{\zeta}_{\varepsilon,\varphi}(N|_{\mathrm{frp}}) \geqslant \dfrac{1}{2}, \\ \check{\zeta}_{\varepsilon,\varphi}(N|_{\mathrm{d(frp)}}) \geqslant \dfrac{3}{4}. \end{cases} \tag{7.1}$$

Lemma 7.1. *Let $\varepsilon > 0$. Let $r \in \mathbb{N}^*$. Let*

$$U \in \operatorname{Mat}_r\left(V\left(W^\dagger(\overline{R})\right)\right) \quad \textit{and} \quad N \in \operatorname{Mat}_r(W^\dagger\Omega_{\overline{R}/k})$$

be two matrices such that $\check{\zeta}_{\varepsilon,\varphi}(U) \geqslant \frac{3}{4}$ and that N satisfies condition (7.1).

Then, the matrices $(1+U)N$ and $(1+U)^{-1}N(1+U)$ also satisfy condition (7.1).

Proof. We have $NU = (N|_{\mathrm{frp}} + N|_{\mathrm{d(frp)}})(U|_{\mathrm{int}} + U|_{\mathrm{frp}}) + N|_{\mathrm{int}}(U|_{\mathrm{int}} + U|_{\mathrm{frp}})$. Once the first product is expended, all the products satisfy condition (7.1) by applying (3.7), (3.3), (3.4), (3.5) and the pseudovaluation properties of ζ_ε [25, Theorem 3.17].

Now, $N|_{\mathrm{int}}U|_{\mathrm{int}}$ clearly satisfies condition (7.1) by [25, Theorem 3.17]. For $N|_{\mathrm{int}}U|_{\mathrm{frp}}$, we need the stronger Proposition 3.8. We have thus shown that NU satisfies (7.1). Permuting N and U yields the same result.

Moreover, we have $(1+U)^{-1} = \sum_{i\in\mathbb{N}}(-U)^i$. In particular, we find using the same arguments as above that:

$$\check{\zeta}_{\varepsilon,\varphi}\left((1+U)^{-1}-1\right) \geqslant \frac{3}{4}.$$

This implies that we can use the same reasoning to conclude. ∎

Lemma 7.2. *Let $\varepsilon > 0$. Let $r \in \mathbb{N}^*$. Let $U \in \operatorname{Mat}_r(V(W^\dagger(\overline{R})))$ such that $\check{\zeta}_{\varepsilon,\varphi}(U) \geqslant \frac{3}{4}$. Then:*

$$\check{\zeta}_{\varepsilon,\varphi}\left((1+U)^{-1}d(1+U)\right) \geqslant \frac{3}{4}.$$

Proof. Of course, $d(1) = 0$. Recall that $\check{\zeta}_{\varepsilon,\varphi}((1+U)^{-1}-1) \geqslant \frac{3}{4}$. So one concludes with (3.8) and the above arguments. ∎

We have seen previously that one can always choose a lift of the Frobenius endomorphism $F\colon \mathrm{Spff}(R^\dagger) \to \mathrm{Spff}(R^\dagger)$ from which we derive a morphism of sheaves of $W(k)$-dgas $t_F\colon \boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}] \to W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$ on $\mathrm{Spec}(\bar{R})$. Recall that we also have a morphism of sheaves of $W(k)$-dgas

$$\phi\colon W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}\left[\frac{1}{p}\right] \to W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}\left[\frac{1}{p}\right]$$

which we use for Frobenius structures on $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connections.

Proposition 7.3. *The following functor, given by Proposition 1.11, going from the category of Frobenius structures on integrable $\boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connections on a globally finite projective $\mathcal{O}_{\mathrm{Spff}(R^\dagger)}[\frac{1}{p}]$-module to the category of Frobenius structures on integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connections on a globally finite projective $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-module, is an equivalence of categories:*

$$t_F{}^*\colon F^*\text{-}\boldsymbol{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}\left[\frac{1}{p}\right]\text{-}\mathsf{GlobFProjMIC} \to \phi^*\text{-}W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}\left[\frac{1}{p}\right]\text{-}\mathsf{GlobFProjMIC}.$$

Proof. Start with a Frobenius structure ∇ on an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection on a globally finite projective $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-module M.

By Proposition 4.4, there exist an integer $r \in \mathbb{N}$, a morphism of $W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$-modules

$$\psi\colon \bigoplus_{i=1}^{r} W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})\left[\frac{1}{p}\right] \to M,$$

which is surjective with a section s, as well as an integrable $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection $\tilde{\nabla}$ on $\bigoplus_{i=1}^r W^\dagger(\mathcal{O}_{\mathrm{Spec}(\bar{R})})[\frac{1}{p}]$, which promotes ψ and s to horizontal morphisms between $\tilde{\nabla}$ and ∇.

Denote by $\mathcal{B}_0 = (b_i)_{i\in[\![1,r]\!]}$ the canonical $W^\dagger(\bar{R})[\frac{1}{p}]$-basis of $\bigoplus_{i=1}^r W^\dagger(\bar{R})[\frac{1}{p}]$. Let N_0 be the representative matrix of $\tilde{\nabla}$ for that basis. Recall that $\phi^*(\tilde{\nabla})$ is a $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connection on $\bigoplus_{i=1}^r W^\dagger(\mathcal{O}_X)[\frac{1}{p}]$, and its representative matrix for $\mathcal{B}_0$ is $\phi(N_0) = pF(N_0)$. In particular, after applying ϕ^* enough times, we end up with a representative matrix with coefficients in $pW^\dagger\Omega_{\bar{R}/k}$.

Since we have a Frobenius structure, then $\nabla \cong \phi^*(\nabla)$ as $W^\dagger\Omega_{\mathrm{Spec}(\bar{R})/k}[\frac{1}{p}]$-connections. This means that we could have assumed that N_0 had coefficients in $pW^\dagger\Omega_{\bar{R}/k}$ to start with.

By Lemma 3.9, we can furthermore assume that N_0 satisfies condition (7.1) for a fixed $\varepsilon > 0$. We have the same condition on $\mathcal{B}_0$, seen as a matrix.

We shall now build by induction on $l \in \mathbb{N}$ families $\mathcal{B}_l \in W^\dagger(\bar{R})[\frac{1}{p}]^r$ as well as matrices $N_l \in \mathrm{Mat}_r(W^\dagger\Omega^1_{\bar{R}/k}[\frac{1}{p}])$ such that:

$$\mathcal{B}_l \text{ is a } W^\dagger(\bar{R})\left[\frac{1}{p}\right]\text{-basis of } \bigoplus_{i=1}^{r} W^\dagger(\bar{R})\left[\frac{1}{p}\right], \tag{7.2}$$

$$N_l \text{ is the representative matrix of } \tilde{\nabla} \text{ for } \mathcal{B}_l, \tag{7.3}$$

$$\mathrm{v}_p(\mathcal{B}_l) \geqslant 0, \tag{7.4}$$

$$\mathrm{v}_p(N_l) \geqslant 0, \tag{7.5}$$

$$\mathcal{B}_l \text{ satisfies (7.1) for } \varepsilon, \tag{7.6}$$

$$N_l \text{ satisfies (7.1) for } \varepsilon, \tag{7.7}$$

$$\mathrm{v}_p(N_l|_{\mathrm{frac}}) > l, \tag{7.8}$$

$$\mathcal{B}_l \equiv \mathcal{B}_{l+1} \pmod{p^{l+1}}, \tag{7.9}$$

$$N_l \equiv N_{l+1} \pmod{p^{l+1}}. \tag{7.10}$$

We have such data for $l = 0$.

Assume that we have it for a given $l \in \mathbb{N}$. Put $U := 1 - d^{-1}(N_l|_{\mathrm{d(frp)}})$. Let $\mathcal{B}_{l+1} := U\mathcal{B}_l$. As U is overconvergent and invertible, we get condition (7.2).

Let $N_{l+1} := U^{-1}N_lU + U^{-1}d(U)$. By Proposition 1.7, $\mathcal{B}_{l+1}$ and N_{l+1} also satisfy condition (7.3).

The induction hypothesis (7.4) and (7.5) imply conditions (7.4) and (7.5).

By (3.6) and Lemma 7.1, we get condition (7.6).

Similarly, and using also Lemma 7.2 we get condition (7.7).

Proposition 1.9 and the induction hypothesis (7.8) allow us to apply Lemma 6.2 and get condition (7.8).

Induction hypothesis (7.8) implies conditions (7.9) and (7.10).

Now that we have these families, by p-adic overconvergence given by (3.7) we can consider the limit $\mathcal{B}$ of these $W^\dagger(\overline{R})$-bases, which is also a $W^\dagger(\overline{R})$-basis of $\bigoplus_{i=1}^r W^\dagger(\overline{R})$. Then, the limit N of the N_l overconverges to the representative matrix of $\widetilde{\nabla}$ for $\mathcal{B}$.

But most importantly, N has coefficients in $t_F(\mathbf{\Omega}_{R^\dagger/W(k)})$. We can thus apply Proposition 6.3 to the representative matrix P of $s \circ \psi$ for $\mathcal{B}$ and see that it also has coefficients in $t_F(\mathbf{\Omega}_{R^\dagger/W(k)})$. This means that $\widetilde{\nabla}$ is in the essential image of an $\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]$-connection through the functor $t_F{}^*$, and so does ∇.

Moreover, we can use Proposition 6.3 again, to get the full faithfulness. Indeed, if E and F are two representative matrices as in the statement of that proposition, if their associated connection is in the essential image of $t_F{}^*$, then we have

$$\mathrm{v}_p(E|_{\mathrm{frac}}) = \mathrm{v}_p(F|_{\mathrm{frac}}) = +\infty.$$

Thus, from the proposition we see that we also must have $\mathrm{v}_p(G|_{\mathrm{frac}}) = +\infty$, where G is a representative matrix of a horizontal morphism between these connections. ■

As usual, the previous proposition has a convergent analogue, with the same proof except that we do not need to take care of overconvergence. It is thus natural to ask whether the canonical functor from the category of Frobenius structures on integrable $W^\dagger\Omega_{X/k}$-connections on a locally projective $W^\dagger(\mathcal{O}_X)$-module to the category of Frobenius structures on integrable $W\Omega_{X/k}$-connections on a locally projective $W(\mathcal{O}_X)$-module is fully faithful, where X is a scheme smooth over k. Indeed, an integral version of this result has been demonstrated by Ertl in the locally free setting [12].

Proposition 7.4. *Consider the inclusion*

$$\iota\colon W^{\dagger}\Omega_{X/k}\left[\frac{1}{p}\right] \to W\Omega_{X/k}\left[\frac{1}{p}\right].$$

The following functor is fully faithful:

$$\iota^{*}\colon \phi^{*}\text{-}W^{\dagger}\Omega_{X/k}\left[\frac{1}{p}\right]\text{-LocFProjMIC} \to \phi^{*}\text{-}W\Omega_{X/k}\left[\frac{1}{p}\right]\text{-LocFProjMIC}.$$

Proof. The question is local on X, so we can assume that $X = \mathrm{Spec}(\overline{R})$ satisfies the conditions introduced at the beginning of this section. By Propositions 5.5 and 5.6, we can also consider the categories where the connections are on globally finite projective modules.

Let

$$i\colon \mathbf{\Omega}_{\mathrm{Spff}(R^{\dagger})/W(k)}\left[\frac{1}{p}\right] \to \mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}\left[\frac{1}{p}\right]$$

be the inclusion. We then have the following essentially commutative diagram:

$$\begin{array}{ccc} F^{*}\text{-}\mathbf{\Omega}_{\mathrm{Spff}(R^{\dagger})/W(k)}\left[\frac{1}{p}\right]\text{-GlobFProjMIC} & \xrightarrow{i^{*}} & \widehat{F}^{*}\text{-}\mathbf{\Omega}_{\mathrm{Spff}(\widehat{R})/W(k)}\left[\frac{1}{p}\right]\text{-GlobFProjMIC} \\ \downarrow t_{F}{}^{*} & & \downarrow t_{\widehat{F}}{}^{*} \\ \phi^{*}\text{-}W^{\dagger}\Omega_{\mathrm{Spec}(\overline{R})/k}\left[\frac{1}{p}\right]\text{-GlobFProjMIC} & \xrightarrow{\iota^{*}} & \phi^{*}\text{-}W\Omega_{\mathrm{Spec}(\overline{R})/k}\left[\frac{1}{p}\right]\text{-GlobFProjMIC}. \end{array}$$

By Proposition 7.3, which as we mentioned also applies in the convergent case, both vertical arrows are equivalences of categories. But by [15, Corollaire 1.2.3], the functor i^{*} is fully faithful because it is equivalent to the functor from overconvergent F-isocrystals to convergent F-isocrystals. ■

Notice that the proof here relies on the fact that the result is equivalent to the full faithfulness of the functor from overconvergent F-isocrystals to convergent F-isocrystals. Should we get a proof of this result using only the theory of de Rham–Witt connections as in [12], we would deduce a new proof of that theorem.

Theorem 7.5. *Let X be a scheme smooth on k. The functor $\acute{E}$ yields and equivalence of categories between the category F-$\mathrm{Isoc}(X/W(k))$ of convergent F-isocrystals on X and the category ϕ^{*}-$W\Omega_{X/k}[\frac{1}{p}]$-LocFProjMIC of Frobenius structures on integrable $W\Omega_{X/k}[\frac{1}{p}]$-connections on a locally finite projective $W(\mathcal{O}_X)[\frac{1}{p}]$-module.*

Proof. The functor $\acute{E}$ is of local nature, so we can work locally on X. We can assume that X is of the form $\mathrm{Spec}(\overline{R})$ needed to apply Proposition 5.8 by [18, Theorem 2]. In that case, this follows from Proposition 5.6 and the convergent version of Proposition 7.3.

To see that these local constructions glue, fix a cover of X of affines of the above form. Let us consider two of them, $\overline{R}$ and $\overline{R'}$, with respective Frobenius lifts F and F' to the respective p-adically complete lifts $\widehat{R}$ and $\widehat{R}'$ in characteristic zero. Let us denote by $Y := \mathrm{Spec}(\overline{R}) \cap \mathrm{Spec}(\overline{R'})$. We get from Proposition 5.8 the following essentially commutative diagram:

$$\begin{tikzcd}
F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[rd, "t_F{}^*"] & \\
F\text{-Isoc}(\mathrm{Spec}(\overline{R})/W(k)) \arrow[u] \arrow[r, "\acute{E}"] \arrow[d, "|_Y"] & \phi^*\text{-}W\Omega_{\mathrm{Spec}(\overline{R})/k}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[d, "|_Y"] \\
F\text{-Isoc}(Y/W(k)) \arrow[r, "\acute{E}"] & \phi^*\text{-}W\Omega_{Y/k}[\frac{1}{p}]\text{-GlobFProjMIC} \\
F\text{-Isoc}(\mathrm{Spec}(\overline{R'})/W(k)) \arrow[u, "|_Y"] \arrow[r, "\acute{E}"] \arrow[d] & \phi^*\text{-}W\Omega_{\mathrm{Spec}(\overline{R'})/k}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[u, "|_Y"'] \\
F^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R'})/W(k)}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[ru, "t_{F'}{}^*"] &
\end{tikzcd}$$

Proposition 7.3 tells us that all arrows, except for the restriction ones and the middle horizontal one, are equivalences of categories. Starting with an object in the category ϕ^*-$W\Omega_{X/k}[\frac{1}{p}]$-LocFProjMIC, we can restrict them to both Spec($\overline{R}$) and Spec($\overline{R'}$), which from the above equivalences yield two F-isocristals $\mathcal{I}$ and $\mathcal{I}'$ on Y.

Then, one can cover Y again with affines of the above form. We can endow each affine with two compatible Frobenius lifts to characteristic zero, which we also respectively denote by F and F' [26, Lemma 7.15]. Applying Proposition 5.8 on these affines, we see that $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic on them. The same kind of reasoning tell us that these isomorphisms satisfy the cocycle condition, so that $\mathcal{I}$ and $\mathcal{I}'$ are actually isomorphic on Y.

Again, this reasoning tells us that such isomorphisms on the chosen cover of X will satisfy the cocyle condition, so that we can glue the preimages to get the essential surjectivity of $\acute{E}$.

The same thing can be done for essential surjectivity. ■

Now, if $X = \mathrm{Spec}(\overline{R})$, we can extend the diagram in Proposition 5.8 in the following way:

$$\begin{tikzcd}
F\text{-Isoc}(X/W(k)) \arrow[rr, "\acute{E}"] \arrow[rd] & & \phi^*\text{-}W\Omega_{X/k}[\frac{1}{p}]\text{-GlobFProjMIC} \\
& \widehat{F}^*\text{-}\mathbf{\Omega}_{\mathrm{Spf}(\widehat{R})/W(k)}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[ru, "t_{\widehat{F}}{}^*"] & \\
& F^*\text{-}\mathbf{\Omega}_{\mathrm{Spff}(R^\dagger)/W(k)}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[u, "i^*"] \arrow[rd, "t_F{}^*"] & \\
F\text{-Isoc}^\dagger(X/W(k)) \arrow[uuu] \arrow[ru] \arrow[rr] & & \phi^*\text{-}W^\dagger\Omega_{X/k}[\frac{1}{p}]\text{-GlobFProjMIC} \arrow[uuu, "\iota^*"']
\end{tikzcd}$$

By Proposition 7.4 and its proof, all the vertical arrows are fully faithful functors. The upper triangle is essentially commutative by Proposition 5.8, and so is the quadrilateral on the left by [15, Corollaire 1.2.3], and the one on the right. This implies that the whole diagram is essentially commutative.

In particular, by glueing we can only assume that X is a scheme smooth over k, and get a functor $\acute{E}^\dagger\colon F\text{-}\mathsf{Isoc}^\dagger(X/W(k)) \to \phi^*\text{-}W^\dagger\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$ from the category of overconvergent isocrystals on X to the category of Frobenius structures on integrable $W^\dagger\Omega_{X/k}[\frac{1}{p}]$-connections on a locally finite projective $W^\dagger(\mathcal{O}_X)[\frac{1}{p}]$-module.

Theorem 7.6. *The functor $\acute{E}^\dagger$ is an equivalence of categories.*

Proof. Same proof as Theorem 7.5, but with the overconvergent versions of our previous results. ∎

These theorems enable us to reformulate Question 5.7 in the following way.

Question 7.7. Let $\nabla \to \phi^*(\nabla)$ be an object in $\phi^*\text{-}W\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$ (respectively $\phi^*\text{-}W^\dagger\Omega_{X/k}[\frac{1}{p}]\text{-}\mathsf{LocFProjMIC}$). Is the $W(\mathcal{O}_X)[\frac{1}{p}]$-module (resp. the $W^\dagger(\mathcal{O}_X)[\frac{1}{p}]$-module) on which ∇ is defined locally free?

Acknowledgements. This work generalises the main result of my PhD thesis, so it is my pleasure to thank my advisor Daniel Caro. I am also very grateful to all the members of my jury Andreas Langer, Tobias Schmidt, Andrea Pulita, Jérôme Poineau, and especially Christine Huyghe for her long term support.

I warmly thank Veronika Ertl for her time and explanations on her work which was the main motivation for this article, as well as Bernard Le Stum for all the discussions we had on these topics, and his help throughout the years.

I also thank the anonymous referee for their comments which helped improving this article.

Funding. The author was under "Contrat EDGAR-CNRS no 277952 UMR 6623, financé par la région Bourgogne-Franche-Comté" at Université Marie et Louis Pasteur, CNRS, LmB (UMR 6623), 25000 Besançon, France during the writing of this article. This work has been financed by ANR-21-CE39-0009-BARRACUDA. It was also written in part during my position at Université Paris-Saclay, UVSQ, CNRS, Laboratoire de mathématiques de Versailles, 78000, Versailles, France. This work was supported by the HYPERFORM consortium, funded by France through Bpifrance.

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Communicated by Takeshi Saito

Received 5 February 2025; revised 17 March 2026.

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