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Limits over orbit categories of locally finite groups

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Abstract. We correct an error in the paper [Geom. Topol. 11 (2007), 315–427], and take the opportunity to examine in more detail the derived functors of inverse limits over orbit categories of (infinite) locally finite groups. The main results show how to reduce this in many cases to limits over orbit categories of finite groups, but we also look at generalizations of the Lyndon–Hochschild–Serre spectral sequence for higher limits over orbit categories for an extension of locally finite groups.

Introduction

The orbit category of a group G is the category $\mathcal{O}(G)$ whose objects are the subgroups of G, and where a morphism from U to V is a G-map $G/U \to G/V$ between the corresponding orbits. (A different, but equivalent definition of the morphism sets is given in Definition 2.1) When p is a prime, $\mathcal{O}_p^f(G) \subseteq \mathcal{O}_p(G) \subseteq \mathcal{O}(G)$ denote the full subcategories whose objects are the finite p-subgroups, and all p-subgroups, respectively, of G. Here, by a p-group, we mean a group each of whose elements has (finite) order a power of p.

When G is a group and M is a $\mathbb{Z}G$-module, let $F_M: \mathcal{O}_p(G)^{\mathrm{op}} \to \mathsf{Ab}$ be the functor defined by setting $F_M(P) = 0$ if $P \neq 1$, and $F_M(1) = M$ with the given action of $\mathrm{Aut}_{\mathcal{O}_p(G)}(1) \cong G$. Graded groups $\Lambda^*(G; M) \overset{\mathrm{def}}{=} \lim^*_{\mathcal{O}_p(G)}(F_M)$ were defined by Jackowski, McClure, and Oliver in [8] (at least for finite G), and they play an important role (as described in Proposition 3.2) when describing right derived functors of inverse limits of arbitrary functors on $\mathcal{O}_p(G)$. We refer to these higher derived functors as "higher limits" for short.

These functors appeared again in work by Broto, Levi, and Oliver, mostly for finite groups, but also in [3] when G is a (possibly infinite) locally finite group. Unfortunately, there was an error in the proof of [3, Lemma 5.12] (see the discussion before Theorem 3.7 below), and the original purpose of this paper was to correct that proof. Lemma 5.12 in [3] was needed to prove Theorems 8.7 and 8.10 in the same paper, which say among other things that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ for every torsion group G that is linear in characteristic different from p. We correct the proof of that lemma in Theorem 3.7 in this paper (see also Theorem D). The result in [3] for which this is needed is also restated here as Theorem 3.8.

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While fixing the error in [3], we also take the opportunity to develop more of the theory of higher limits over orbit categories, and in particular the graded groups $\Lambda^*(G;M)$, for infinite groups G, especially when G is locally finite (i.e., every finitely generated subgroup of G is finite). For example, one of our main results is the following, where $\mathfrak{Fin}(G)$ is the poset of finite subgroups of a group G.

Theorem A. *Fix a prime p, and let G be a locally finite group. Then for each $\mathbb{Z}G$-module M, there is a first quarter spectral sequence*

$$E_2^{ij} \cong \lim_{\mathfrak{Fin}(G)}{}^i \big(\Lambda^j(-;M)\big) \implies \Lambda^{i+j}(G;M).$$

Theorem A follows from the spectral sequence of Theorem 3.3, together with Theorem B. Theorem 3.3 is a special case of Proposition 2.4, which involves higher limits of functors on orbit categories with more general sets of objects. Proposition 2.4 is in turn a special case of the still more general Proposition 1.7.

The next theorem is more technical, but is a key tool for many of the results in the paper. It is proven as Theorem 3.6. Roughly, it says that Λ^* is the same, whether we take limits with respect to all p-subgroups of G or only the finite p-subgroups.

Theorem B. *Fix a prime p, and let G be a group all of whose p-subgroups are locally finite. Then for each $\mathbb{Z}G$-module M,*

$$\Lambda^*(G;M) \cong \Lambda^*_f(G;M) \overset{\text{def}}{=} \lim_{\mathcal{O}_p^f(G)}{}^* \,(F_M).$$

We also construct a spectral sequence that describes how the functors $\Lambda^*(-;-)$ behave under an extension of locally finite groups.

Theorem C. *Fix a prime p, a locally finite group G, a normal subgroup $H \trianglelefteq G$, and a $\mathbb{Z}G$-module M. Let $\chi\colon G \to G/H$ be the natural map. Then there is a first quarter spectral sequence*

$$E_2^{ij} = \lim_{\mathcal{O}_p(G/H)}{}^i \big(\Lambda^j\big(\chi^{-1}(-);M\big)\big) \implies \Lambda^{i+j}(G;M).$$

Theorem C is proven below as Theorem 3.5. It is a special case of Theorem 2.5: a more general spectral sequence involving higher limits over orbit spaces that also includes the Lyndon–Hochschild–Serre spectral sequence (see [11, Theorem XI.10.1] or [15, Theorem 6.8.2]) and Theorem B as other special cases.

Finally, we fill in the gap in [3] by proving the following result, shown below as Theorem 3.7.

Theorem D. *Fix a prime p. Let G be a locally finite group, and let M be an $\mathbb{Z}_{(p)}G$-module such that $C_G(M)$ contains an element of order p. Then $\Lambda^*(G;M) = 0$.*

Section 1 contains some general results on categories and limits. In Section 2, we specialize to the case of orbit categories of groups, and construct among other things spectral sequences of which those in Theorems A and C are special cases. The functors

$\Lambda^*(-;-)$ are then defined in Section 3, where we study their basic properties and prove most of our main results, including Theorems A, B, C, and D. Then, in Section 4, we give some examples to show how the functors $\Lambda^*(G;M)$ for locally finite groups G can behave quite differently when G is infinite than in the finite case. We end with an appendix, where conditions are given on a pair of p-groups $Q < P$ that imply that $Q < N_P(Q)$, and where we also construct an example to show that this is not always true, not even when the p-groups are locally finite.

Notation. Composition is always from right to left. By a *p-group* (for a prime p), we always mean a group each of whose elements has p-power order. When G is a group, we write

- $\mathfrak{Fin}(G)$ for the poset of finite subgroups of G; and
- ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for $g \in G$ and $H \leq G$.

Also, when M is a $\mathbb{Z}G$-module and $H \leq G$, we let $\mathrm{Fix}(H, M)$ denote the fixed set of the action of H on M.

We will frequently regard a poset $(X, \leq)$ as a category, where X is the set of objects, and where there is a unique morphism $x \to y$ whenever $x \leq y$.

1. Background on categories and limits

We begin by fixing some terminology. When $\mathcal{C}$ is a small category, a *$\mathcal{C}$-module* is a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$. We always work with contravariant functors, since those are the ones that appear most naturally in our work, but replacing them by covariant functors would mean only minor rephrasing of the definitions and results.

More generally, when R is a commutative ring, an *$R\mathcal{C}$-module* is a functor $\mathcal{C}^{\mathrm{op}} \to R\text{-}\mathsf{mod}$. Let $\mathcal{C}\text{-}\mathsf{mod}$ and $R\mathcal{C}\text{-}\mathsf{mod}$ denote the categories of $\mathcal{C}$-modules and $R\mathcal{C}$-modules, respectively, where morphisms are the natural transformations of functors. (In earlier papers such as [12], these categories are often denoted $\mathsf{Ab}^{\mathcal{C}^{\mathrm{op}}}$ and $R\text{-}\mathsf{mod}^{\mathcal{C}^{\mathrm{op}}}$, respectively, but we find the "-mod" notation natural and typographically simpler.)

If $\mathcal{C}$ is a small category and R is a commutative ring, then the category $R\mathcal{C}\text{-}\mathsf{mod}$ has enough injectives by Proposition 1.2 (d) below. So the right derived functors of inverse limits of an $R\mathcal{C}$-module Φ are defined, and we denote them $\lim^*_{\mathcal{C}}(\Phi)$. Thus if $(0 \to \Phi \to I_0 \to I_1 \to \cdots)$ is a resolution of Φ by injective $R\mathcal{C}$-modules, then

$$\lim_{\mathcal{C}}{}^*(\Phi) = H^*\bigl(0 \longrightarrow \lim_{\mathcal{C}}(I_0) \longrightarrow \lim_{\mathcal{C}}(I_1) \longrightarrow \lim_{\mathcal{C}}(I_2) \longrightarrow \cdots\bigr).$$

We usually refer to these derived functors of limits as the "higher limits" of Φ.

The following notation for certain injective or acyclic $\mathcal{C}$-modules will often be useful.

Definition 1.1. Let $\mathcal{C}$ be a small category, and let R be a commutative ring. For each c in $\mathcal{C}$ and each R-module M, let $\mathfrak{I}^M_{\mathcal{C},c} = \mathfrak{I}^M_c$ be the $R\mathcal{C}$-module that sends an object d to

$$\mathfrak{I}^M_c(d) = \mathrm{map}\bigl(\mathrm{Mor}_{\mathcal{C}}(c,d), M\bigr) \cong \prod_{\mathrm{Mor}_{\mathcal{C}}(c,d)} M.$$

A morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(d, d')$ induces a map of sets $\mathrm{Mor}_{\mathcal{C}}(c, d) \to \mathrm{Mor}_{\mathcal{C}}(c, d')$ via composition, and through that induces a homomorphism $\mathfrak{I}_c^M(\varphi)\colon \mathfrak{I}_c^M(d') \to \mathfrak{I}_c^M(d)$.

The following result is well known and elementary, but we include a proof here since we have been unable to find a precise reference. The key ideas all go back at least to Mitchell [12, p. 27], but he states his result under somewhat different assumptions.

Proposition 1.2. *Let $\mathcal{C}$ be a small category, and let R be a commutative ring.*

(a) *For each c in $\mathcal{C}$, each R-module M, and each $R\mathcal{C}$-module Φ, $\mathrm{Mor}_{R\mathcal{C}\text{-mod}}(\Phi, \mathfrak{I}_c^M) \cong \mathrm{Hom}_R(\Phi(c), M)$. Hence $\mathfrak{I}_c^M$ is an injective $R\mathcal{C}$-module if M is an injective R-module.*

(b) *Every injective $R\mathcal{C}$-module is a direct factor in a product of $R\mathcal{C}$-modules $\mathfrak{I}_c^M$, for objects c in $\mathcal{C}$ and injective R-modules M.*

(c) *For each c in $\mathcal{C}$ and each R-module M (injective or not), $\mathfrak{I}_c^M$ is acyclic as an $R\mathcal{C}$-module, in the sense that $\lim^i_{\mathcal{C}}(\mathfrak{I}_c^M) = 0$ for all $i > 0$.*

(d) *The category $R\mathcal{C}$-mod has enough injectives.*

Proof. (a) Define homomorphisms

$$\mathrm{Mor}_{R\mathcal{C}\text{-mod}}(\Phi, \mathfrak{I}_c^M) \underset{\Omega}{\overset{\Psi}{\rightleftarrows}} \mathrm{Hom}_R\big(\Phi(c), M\big)$$

as follows. Each α in $\mathrm{Mor}_{R\mathcal{C}\text{-mod}}(\Phi, \mathfrak{I}_c^M)$ induces an R-linear homomorphism $\alpha(c)$ from $\Phi(c)$ to $\mathfrak{I}_c^M(c)$ by evaluation at c, and we let $\Psi(\alpha) = \mathrm{ev}_{\mathrm{Id}_c} \circ \alpha(c)$ where $\mathrm{ev}_{\mathrm{Id}_c}$ from $\mathfrak{I}_c^M(c)$ to M is evaluation at Id_c. Conversely, given $\beta \in \mathrm{Hom}_R(\Phi(c), M)$, define $\Omega(\beta) \in \mathrm{Hom}_{R\mathcal{C}\text{-mod}}(\Phi, \mathfrak{I}_c^M)$ by setting

$$\Omega(\beta)(d)(c \xrightarrow{\ \rho\ } d) = \beta \circ \Phi(\rho)\colon \Phi(d) \longrightarrow M.$$

That $\Psi\Omega(\beta) = \beta$ is immediate from these definitions, and the relation $\Omega\Psi(\alpha) = \alpha$ follows from the naturality properties of α.

If M is injective, and $\Phi_1 \to \Phi_2$ is an injective morphism of $R\mathcal{C}$-modules, then each morphism $\Phi_1 \to \mathfrak{I}_c^M$ extends to Φ_2 since each homomorphism $\Phi_1(c) \to M$ extends to $\Phi_2(c)$. So $\mathfrak{I}_c^M$ is injective as an $R\mathcal{C}$-module.

(b) Fix $\Phi\colon \mathcal{C}^{\mathrm{op}} \to R\text{-mod}$. For each $c \in \mathrm{Ob}(\mathcal{C})$, choose an injective R-module $I(c)$ and an injective homomorphism $\chi(c)\colon \Phi(c) \to I(c)$. By (a), there is an injective homomorphism of $R\mathcal{C}$-modules from Φ into $\prod_{c \in \mathrm{Ob}(\mathcal{C})} \mathfrak{I}_c^{I(c)}$. If Φ is injective, then this injection splits, and Φ is a direct factor in this product.

(c) Let $0 \to M \to I_0 \to I_1 \to \cdots$ be a resolution of M by injective R-modules. Then by (a), $0 \to \mathfrak{I}_c^M \to \mathfrak{I}_c^{I_0} \to \mathfrak{I}_c^{I_1} \to \cdots$ is a resolution of $\mathfrak{I}_c^M$ by injective $R\mathcal{C}$-modules. For each $n \geq 0$,

$$\lim_{\mathcal{C}}(\mathfrak{I}_c^{I_n}) \cong \mathrm{Hom}_{R\mathcal{C}\text{-mod}}(\underline{R}, \mathfrak{I}_c^{I_n}) \cong \mathrm{Hom}_R(R, I_n) \cong I_n,$$

where $\underline{R}$ is the constant functor that sends all objects in $\mathcal{C}$ to R, and where the second isomorphism holds by (a). So

$$\lim_{\mathcal{C}}{}^i(\mathfrak{I}_c^M) \cong H^i(0 \to I_0 \to I_1 \to \cdots) = \begin{cases} M & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

(d) Let Φ be an $R\mathcal{C}$-module. For each c in $\mathcal{C}$, choose an injective R-linear homomorphism $\psi_c\colon \Phi(c) \to M_c$ where M_c is injective in R-mod. (The category R-mod has enough injectives by [11, Theorem III.7.4].) Set $\mathfrak{I} = \prod_{c\in\mathrm{Ob}(\mathcal{C})} \mathfrak{I}_c^{M_c}$: this is a product of injective $R\mathcal{C}$-modules by (a) and hence is itself injective. By (a) again, there is an injective homomorphism $\Phi \to \mathfrak{I}$, and thus $R\mathcal{C}$-mod has enough injectives. ■

The "bar resolution" for higher limits over small categories will often be useful.

Proposition 1.3. *Let $\mathcal{C}$ be a small category, and let Φ be a $\mathcal{C}$-module. For each $n \geq 0$, set*

$$C^n(\mathcal{C};\Phi) = \prod_{c_0\to\cdots\to c_n} \Phi(c_0),$$

where the product is taken over all composable sequences of n morphisms in $\mathcal{C}$ (over all objects in $\mathcal{C}$ if $n = 0$). Define $d^n\colon C^n(\mathcal{C};\Phi) \to C^{n+1}(\mathcal{C};\Phi)$ by setting, for $\xi \in C^n(\mathcal{C};\Phi)$,

$$(d^n\xi)(c_0 \xrightarrow{\chi} c_1 \to \cdots \to c_{n+1}) \\ = \chi^*\bigl(\xi(c_1 \to \cdots \to c_{n+1})\bigr) + \sum_{i=1}^{n+1} (-1)^i \xi(c_0 \to \cdots \widehat{c_i} \cdots \to c_{n+1}).$$

Here, "$\widehat{c_i}$" means that the term c_i is removed from the sequence. Then there is a natural isomorphism

$$\lim_{\mathcal{C}}{}^*(\Phi) \cong H^*\bigl(0 \longrightarrow C^0(\mathcal{C};\Phi) \xrightarrow{d^0} C^1(\mathcal{C};\Phi) \xrightarrow{d^1} C^2(\mathcal{C};\Phi) \xrightarrow{d^2} \cdots\bigr).$$

Proof. This is shown in [5, Appendix II, Proposition 3.3]. A different proof, based on taking a projective resolution of the constant functor $\underline{\mathbb{Z}}$, is given in [13, Lemma 2] and [1, Proposition III.5.3]. It can also be shown by constructing a resolution of Φ

$$0 \longrightarrow \Phi \longrightarrow \prod_{c_0} \mathfrak{I}_{c_0}^{\Phi(c_0)} \longrightarrow \prod_{c_0\to c_1} \mathfrak{I}_{c_1}^{\Phi(c_0)} \longrightarrow \prod_{c_0\to c_1\to c_2} \mathfrak{I}_{c_2}^{\Phi(c_0)} \longrightarrow \cdots$$

by acyclic $\mathcal{C}$-modules (Proposition 1.2 (c)), and then applying Lemma 1.4 below to show that its homology after taking limits is isomorphic to $\lim_{\mathcal{C}}^*(\Phi)$. ■

The next lemma gives us a tool in many cases for computing the homology of the limit of a chain complex of $\mathcal{C}$-modules. It is particularly useful when $\mathcal{C}$ is the category of a directed poset.

Lemma 1.4. *Let $\mathcal{C}$ be a small category, let R be a commutative ring, and let*

$$0 \longrightarrow \Phi^0 \xrightarrow{d^0} \Phi^1 \xrightarrow{d^1} \Phi^2 \xrightarrow{d^2} \Phi^3 \xrightarrow{d^3} \cdots$$

be a chain complex of $R\mathcal{C}$-modules. Assume that each Φ^j is acyclic; i.e., that $\lim^i_{\mathcal{C}}(\Phi^j) = 0$ for all $i \geq 1$ and all $j \geq 0$. Set $\mathbf{H}^j = H^j(\Phi^*, d^*) = \operatorname{Ker}(d^j)/\operatorname{Im}(d^{j-1})$ as an $R\mathcal{C}$-module. Then there is a spectral sequence

$$E_2^{ij} = \lim_{\mathcal{C}}{}^i\big(\mathbf{H}^j\big) \implies H^{i+j}\big(\lim_{\mathcal{C}}(\Phi^*), \lim_{\mathcal{C}}(d^*)\big).$$

Proof. Set $Z^j = \operatorname{Ker}(d^j)$ and $B^{j+1} = \operatorname{Im}(d^j)$ for each $j \geq 0$ (and set $B^0 = 0$), all regarded as $R\mathcal{C}$-modules. Thus there are short exact sequences of $R\mathcal{C}$-modules

$$\begin{aligned} &0 \longrightarrow B^j \longrightarrow Z^j \longrightarrow \mathbf{H}^j \longrightarrow 0, \\ &0 \longrightarrow Z^j \longrightarrow \Phi^j \longrightarrow B^{j+1} \longrightarrow 0 \end{aligned} \tag{1.1}$$

for all $j \geq 0$. For each $j \geq 0$, choose injective resolutions

$$\begin{aligned} &0 \longrightarrow B^j \longrightarrow I^{0j}_{(B)} \longrightarrow I^{1j}_{(B)} \longrightarrow I^{2j}_{(B)} \longrightarrow \cdots, \\ &0 \longrightarrow \mathbf{H}^j \longrightarrow I^{0j}_{(H)} \longrightarrow I^{1j}_{(H)} \longrightarrow I^{2j}_{(H)} \longrightarrow \cdots. \end{aligned}$$

By the horseshoe lemma (see [14, Proposition 6.5], or [4, Proposition I.3.5] for the dual version), there are injective resolutions

$$\begin{aligned} &0 \longrightarrow Z^j \longrightarrow I^{0j}_{(Z)} \longrightarrow I^{1j}_{(Z)} \longrightarrow I^{2j}_{(Z)} \longrightarrow \cdots, \\ &0 \longrightarrow \Phi^j \longrightarrow I^{0j}_{(\Phi)} \longrightarrow I^{1j}_{(\Phi)} \longrightarrow I^{2j}_{(\Phi)} \longrightarrow \cdots \end{aligned}$$

which fit into short exact sequences of resolutions

$$\begin{aligned} &0 \longrightarrow I^{*j}_{(B)} \xrightarrow{\alpha^{*j}} I^{*j}_{(Z)} \xrightarrow{\beta^{*j}} I^{*j}_{(H)} \longrightarrow 0, \\ &0 \longrightarrow I^{*j}_{(Z)} \xrightarrow{\gamma^{*j}} I^{*j}_{(\Phi)} \xrightarrow{\delta^{*j}} I^{*,j+1}_{(B)} \longrightarrow 0 \end{aligned} \tag{1.2}$$

of the sequences in (1.1).

Now consider the sequence of injective resolutions

$$0 \longrightarrow I^{*0}_{(\Phi)} \longrightarrow I^{*1}_{(\Phi)} \longrightarrow I^{*2}_{(\Phi)} \longrightarrow \cdots$$

where each morphism of resolutions is the composite

$$I^{*j}_{(\Phi)} \xrightarrow{\delta^{*j}} I^{*,j+1}_{(B)} \xrightarrow{\alpha^{*,j+1}} I^{*,j+1}_{(Z)} \xrightarrow{\gamma^{*,j+1}} I^{*,j+1}_{(\Phi)}.$$

We regard this as a double complex $\{I^{ij}_{(\Phi)}\}_{i,j\geq 0}$ of injective objects in $\mathcal{C}$-mod, where the j-th row is the injective resolution $I^{*j}_{(\Phi)}$ of Φ^j. Set $X^{ij} = \lim_{\mathcal{C}}(I^{ij}_{(\Phi)})$.

Consider the two spectral sequences induced by the double complex X^{ij}: let $\widehat{E}$ be that obtained by taking homology first of the rows and then of the columns, and let E be the other one. Since the j-th row $I^{*j}_{(\Phi)}$ is an injective resolution of Φ^j, we have $\widehat{E}_1^{ij} \cong \lim^i_{\mathcal{C}}(\Phi^j)$. Thus for all $j \geq 0$, we have $\widehat{E}_1^{0j} \cong \lim_{\mathcal{C}}(\Phi^j)$, while $\widehat{E}_1^{ij} = 0$ for $i > 0$ by assumption. So $\widehat{E}$ collapses, and E and $\widehat{E}$ both converge to $H^*(\lim_{\mathcal{C}}(\Phi^*), \lim_{\mathcal{C}}(d^*))$.

The short exact sequences of injectives (1.2) are still exact after taking limits over $\mathcal{C}$. So the homology of the i-th column X^{i*} is $E_1^{i*} \cong \lim_{\mathcal{C}}(I^{i*}_{(H)})$. Thus the j-th row in the E_1-term is $E_1^{*j} \cong \lim_{\mathcal{C}}(I^{*j}_{(H)})$, where $\{I^{*j}_{(H)}\}_{j\geq 0}$ is an injective resolution of $\mathbf{H}^j$. It follows that $E_2^{ij} \cong \lim^i_{\mathcal{C}}(\mathbf{H}^j)$. ■

The following definition of a directed category is that used by Bass [2, p. 44].

Definition 1.5. A category $\mathcal{C}$ is *directed* if it satisfies the following conditions:

(a) For each pair of objects $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$, there is a third object d in $\mathcal{C}$, together with morphisms $c_1 \to d \leftarrow c_2$.

(b) For each pair of morphisms $\varphi, \psi \in \mathrm{Mor}_{\mathcal{C}}(c_1, c_2)$ between the same two objects c_1 and c_2 in $\mathcal{C}$, there is an object d in $\mathcal{C}$ and $\chi \in \mathrm{Mor}_{\mathcal{C}}(c_2, d)$ such that $\chi\varphi = \chi\psi$.

Note that a poset is directed if and only if its category is directed. (Condition (b) always holds for the category of a poset, since there is at most one morphism between any pair of objects.)

We recall here the definition of over- and undercategories, as well as Kan extensions. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. For each object d in $\mathcal{D}$, let $d{\downarrow}F$ (the undercategory) and $F{\downarrow}d$ (the overcategory) be the categories with objects

$$\mathrm{Ob}(d{\downarrow}F) = \big\{(c, \varphi) \mid c \in \mathrm{Ob}(\mathcal{C}),\ \varphi \in \mathrm{Mor}_{\mathcal{D}}\big(d, F(c)\big)\big\}$$
$$\mathrm{Ob}(F{\downarrow}d) = \big\{(c, \varphi) \mid c \in \mathrm{Ob}(\mathcal{C}),\ \varphi \in \mathrm{Mor}_{\mathcal{D}}\big(F(c), d\big)\big\}.$$

A morphism in $d{\downarrow}F$ from (c_1, φ_1) to (c_2, φ_2) is a morphism $\rho \in \mathrm{Mor}_{\mathcal{C}}(c_1, c_2)$ such that $\varphi_2 = F(\rho) \circ \varphi_1$, and similarly for $F{\downarrow}d$.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories and $\Phi\colon \mathcal{C}^{\mathrm{op}} \to R\text{-mod}$ is an $R\mathcal{C}$-module, then the left and right Kan extensions along F are functors $F_*^L, F_*^R\colon R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ that are left and right adjoint, respectively, to the functor sending Ψ to $\Psi \circ F$. They are described explicitly by the formulas

$$(F_*^L \Psi)(d) = \operatorname*{colim}_{d\downarrow F}\Big((d{\downarrow}F)^{\mathrm{op}} \xrightarrow{\mu^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\Psi} R\text{-mod}\Big), \tag{1.3}$$

$$(F_*^R \Psi)(d) = \lim_{F\downarrow d}\Big((F{\downarrow}d)^{\mathrm{op}} \xrightarrow{\mu^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\Psi} R\text{-mod}\Big), \tag{1.4}$$

where in both cases, μ is the forgetful functor sending an object (c, φ) to c. We refer to [10, §X.3] for the description of $F_*^R\Phi$ (but note that we assume that Φ is contravariant). The formula for $F_*^L\Phi$ can then be obtained by replacing each category by its opposite.

We next note some conditions under which the restriction of a functor to a subcategory has the same higher limits.

Lemma 1.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Assume, for each $d \in \mathrm{Ob}(\mathcal{D})$, that $(d\downarrow F)^{\mathrm{op}}$ is nonempty and directed. Then for each commutative ring R and each functor $\Phi\colon \mathcal{D}^{\mathrm{op}} \to R\text{-}\mathsf{mod}$, we have $\lim^q_{\mathcal{D}}(\Phi) \cong \lim^q_{\mathcal{C}}(\Phi \circ F)$ for all $q \geq 0$.*

Proof. For each $d \in \mathrm{Ob}(\mathcal{D})$, the undercategory $d\downarrow F$ is nonempty by assumption, and is connected since it is directed (Definition 1.5 (a)). So by [10, §IX.3, Theorem 1], $\lim_{\mathcal{D}}(\Phi) \cong \lim_{\mathcal{C}}(\Phi \circ F)$ for each $R\mathcal{D}$-module Φ.

Fix an $R\mathcal{D}$-module Φ, and let $0 \to \Phi \to I_0 \to I_1 \to \cdots$ be a resolution of Φ by injective $R\mathcal{D}$-modules. The sequence

$$0 \longrightarrow \Phi \circ F \longrightarrow I_0 \circ F \longrightarrow I_1 \circ F \longrightarrow I_2 \circ F \longrightarrow \cdots$$

is an exact sequence of $R\mathcal{C}$-modules, and we just saw that $\lim_{\mathcal{C}}(I_n \circ F) \cong \lim_{\mathcal{D}}(I_n)$ for each n. So if $I_n \circ F$ is injective for each n, then $\lim^*_{\mathcal{C}}(\Phi \circ F) \cong \lim^*_{\mathcal{D}}(\Phi)$, which is what we want to show.

It thus remains to prove that composition with F sends injective $R\mathcal{D}$-modules to injective $R\mathcal{C}$-modules. To see this, it suffices to show that left Kan extension along F sends injective maps of $R\mathcal{C}$-modules to injective maps of $R\mathcal{D}$-modules (since it is left adjoint to composition with F). By (1.3), it suffices to show that colimits of functors on $(d\downarrow F)^{\mathrm{op}}$ are left exact, and this follows from [15, Theorem 2.6.15], applied with $(d\downarrow F)^{\mathrm{op}}$ in the role of I. (What Weibel calls a filtering category is what we call here a nonempty directed category.) ∎

The next proposition is an application of Lemma 1.4, and is useful in certain cases when comparing higher limits over a category to those over a family of subcategories.

Proposition 1.7. *Fix a commutative ring R. Let $\mathcal{C}$ be a nonempty small category, and let $\mathbb{D}$ be a set of subcategories of $\mathcal{C}$, regarded as a poset via inclusion. Assume that*

- (i) *$\mathcal{C}$ is the union of the subcategories in $\mathbb{D}$; and*
- (ii) *$\mathbb{D}$ is a directed poset.*

Then for each $R\mathcal{C}$-module Φ, there is a first quarter spectral sequence of R-modules

$$E_2^{ij} \cong \lim_{\mathbb{D}}{}^i\big(\mathcal{D} \mapsto \lim_{\mathcal{D}}{}^j(\Phi|_{\mathcal{D}})\big) \implies \lim_{\mathcal{C}}{}^{i+j}(\Phi).$$

Proof. For each $n \geq 0$ and each $\mathcal{D} \subseteq \mathcal{C}$, let $\mathfrak{ch}_n(\mathcal{D})$ be the set of all sequences $c_0 \to \cdots \to c_n$ of objects and morphisms in $\mathcal{D}$. For each $\sigma = (c_0 \to \cdots \to c_n) \in \mathfrak{ch}_n(\mathcal{C})$, set $\mathrm{init}(\sigma) = c_0$. By (i) and (ii), $\mathfrak{ch}_n(\mathcal{C})$ is the union of the $\mathfrak{ch}_n(\mathcal{D})$ for all $\mathcal{D} \in \mathbb{D}$. So in the notation of Proposition 1.3, for each $n \geq 0$,

$$C^n(\mathcal{C};\Phi) = \prod_{\sigma\in\mathfrak{ch}_n(\mathcal{C})} \Phi\big(\mathrm{init}(\sigma)\big) \cong \lim_{\mathcal{D}\in\mathbb{D}}\Big(\prod_{\sigma\in\mathfrak{ch}_n(\mathcal{D})} \Phi\big(\mathrm{init}(\sigma)\big)\Big) \cong \lim_{\mathcal{D}\in\mathbb{D}}\big(C^n(\mathcal{D};\Phi|_{\mathcal{D}})\big).$$

Hence the proposition follows from Lemma 1.4 once we show that each of the functors $(\mathcal{D} \mapsto C^n(\mathcal{D}; \Phi|_{\mathcal{D}}))$ is acyclic as an $R\mathbb{D}$-module.

For each $\sigma \in \mathfrak{ch}_n(\mathcal{C})$, let $X(\sigma)$ be the set of all members of $\mathbb{D}$ that contain σ (which we just saw is nonempty). For each $\sigma \in \mathfrak{ch}_n(\mathcal{C})$ and each abelian group A, let $\Omega^A_{X(\sigma)}$ be the $\mathbb{D}$-module that sends each member of $X(\sigma)$ to A and each inclusion between them to Id_A, and sends all members of $\mathbb{D} \smallsetminus X(\sigma)$ to 0. Then as $\mathbb{D}$-modules,

$$\big(\mathcal{D} \mapsto C^n(\mathcal{D}; \Phi|_{\mathcal{D}})\big) \cong \prod_{\sigma \in \mathfrak{ch}_n(\mathcal{C})} \Omega^{\Phi(\mathrm{init}(\sigma))}_{X(\sigma)}. \tag{1.5}$$

Also, for each σ and A, we have $\lim^*_{\mathbb{D}}(\Omega^A_{X(\sigma)}) \cong \lim^*_{X(\sigma)}(\underline{A})$, where $\underline{A}$ denotes the constant functor, by Proposition 1.3 and since $C^*(\mathbb{D}; \Omega^A_{X(\sigma)}) \cong C^*(X(\sigma); \underline{A})$. So it remains to show, for each $\sigma \in \mathfrak{ch}_n(\mathcal{C})$, that every constant functor $X(\sigma)^{\mathrm{op}} \to R\text{-}\mathsf{mod}$ is acyclic. The bar resolution for a constant functor is the same up to isomorphism whether we regard it as a functor on $X(\sigma)^{\mathrm{op}}$ or on $X(\sigma)$, so we will be done upon showing that every constant functor $X(\sigma) \to R\text{-}\mathsf{mod}$ is acyclic.

Let $*$ be the category with one object o and the identity morphism, and let $F\colon X(\sigma) \to *$ be the (unique) functor. Fix an R-module M, and let $\Phi_M\colon * \to R\text{-}\mathsf{mod}$ be the functor that sends o to M. Then $o{\downarrow}F \cong X(\sigma)$, we already showed that it is nonempty, and it is directed by (ii). So by Lemma 1.6, applied with $F^{\mathrm{op}}\colon X(\sigma)^{\mathrm{op}} \to *$ in the role of $F\colon \mathcal{C} \to \mathcal{D}$, the constant functor $\underline{M} = \Phi_M \circ F$ is acyclic. ■

2. Limits over orbit categories

In this section, we first give the definition and some of the basic properties of orbit categories of groups, and then construct two spectral sequences that involve higher limits over orbit categories. We are mostly interested in orbit categories of locally finite groups, but many of these results hold just as easily for arbitrary groups.

Definition 2.1. Let G be a group.

(a) Let $\mathcal{O}(G)$ be the *orbit category* of G: the category whose objects are the subgroups of G, and where for each $H, K \le G$,

$$\mathrm{Mor}_{\mathcal{O}(G)}(H,K) = K\backslash T_G(H,K) = \{Kg \mid g \in G,\ {}^gH \le K\}.$$

We identify $\mathrm{Mor}_{\mathcal{O}(G)}(H,K) \cong \mathrm{map}_G(G/H, G/K)$, where a morphism Kg sends $xH \in G/H$ to $xg^{-1}K \in G/K$.

(b) When X is a nonempty set of subgroups of G invariant under conjugation, we let $\mathcal{O}_X(G) \subseteq \mathcal{O}(G)$ denote the full subcategory whose objects are the members of X.

(c) As a special case of (b), when p is a prime, $\mathcal{O}^f_p(G) \subseteq \mathcal{O}_p(G) \subseteq \mathcal{O}(G)$ denote the full subcategories whose objects are the finite p-subgroups and arbitrary p-subgroups of G, respectively.

When it does not cause confusion, we write $[g] = Kg$ to denote a morphism in $\mathcal{O}(G)$ induced by $g \in G$.

A morphism $f : a \to b$ in a category $\mathcal{C}$ is an *epimorphism* (in the categorical sense) if $gf = hf$ implies $g = h$ for each pair of morphisms $g, h : b \to c$. It will be useful to know that morphisms in an orbit category are epimorphisms.

Lemma 2.2. *For every group G, all morphisms in $\mathcal{O}(G)$, and hence all morphisms in every subcategory of $\mathcal{O}(G)$, are epimorphisms in the categorical sense.*

Proof. Let $Kg \in \mathrm{Mor}_{\mathcal{O}_X(G)}(H,K)$ and $Lx, Ly \in \mathrm{Mor}_{\mathcal{O}_X(G)}(K,L)$ be such that $Lx \circ Kg = Ly \circ Kg$ (for some $H, K, L \le G$). Then $Lxg = Lyg$, so $Lx = Ly$ as cosets and hence as morphisms in $\mathcal{O}_X(G)$. Thus Kg is an epimorphism. ■

We next show a version of Lemma 1.6 specialized to orbit categories.

Lemma 2.3. *Let G be a group, and let $X_0 \subseteq X$ be nonempty sets of subgroups of G, both invariant under conjugation. Assume that*

(i) *each member of X is contained in a member of X_0, and*

(ii) *X_0 is closed under finite intersections.*

Then for each $\mathcal{O}_X(G)$-module Φ,

$$\lim_{\mathcal{O}_X(G)}{}^{*}(\Phi) \cong \lim_{\mathcal{O}_{X_0}(G)}{}^{*}(\Phi|_{\mathcal{O}_{X_0}(G)}). \tag{2.1}$$

Proof. We apply Lemma 1.6, with the inclusion functor $\mathcal{I} : \mathcal{O}_{X_0}(G) \to \mathcal{O}_X(G)$ in the role of F. For each $H \in X$, the undercategory $H{\downarrow}\mathcal{I}$ is nonempty by (i), and it satisfies condition (a) in the definition of a directed category by (ii). Since all morphisms in $\mathcal{O}_X(G)$ are epimorphisms by Lemma 2.2, there is at most one morphism between any given pair of objects in $H{\downarrow}\mathcal{I}$, and so condition (b) in Definition 1.5 also holds. Thus $H{\downarrow}\mathcal{I}$ is directed, and (2.1) follows from Lemma 1.6. ■

We now turn to some spectral sequences involving orbit categories.

As usual, we say that a group G is *locally finite* if every finitely generated subgroup of G is finite. In general, orbit spaces of locally finite groups are easier to work with than those of arbitrary infinite groups. For example, in a locally finite group G, the poset $\mathfrak{Fin}(G)$ of its finite subgroups is a directed poset, and this is important when we want to describe limits over $\mathcal{O}_p(G)$ in terms of finite subgroups of G.

Proposition 2.4. *Let G be a locally finite group, and let X be a nonempty set of finite subgroups of G invariant under conjugation. For each $K \le G$, let $X \cap K$ be the set of members of X that are contained in K. Then for each commutative ring R and each $R\mathcal{O}_X(G)$-module Φ, there is a first quarter spectral sequence*

$$E_2^{ij} \cong \lim_{K \in \mathfrak{Fin}(G)}{}^{i}\Big(\lim_{\mathcal{O}_{X\cap K}(K)}{}^{j}(\Phi|_{\mathcal{O}_{X\cap K}(K)})\Big) \Longrightarrow \lim_{\mathcal{O}_X(G)}{}^{i+j}(\Phi) \tag{2.2}$$

of R-modules. In particular,

$$\lim_{\mathcal{O}_X(G)}(\Phi) \cong \lim_{K\in\mathfrak{Fin}(G)}\Big(\lim_{\mathcal{O}_{X\cap K}(K)}(\Phi|_{\mathcal{O}_{X\cap K}(K)})\Big),$$

and if G is countable, then for each $n \geq 1$ there is a short exact sequence

$$0 \longrightarrow \lim_{K\in\mathfrak{Fin}(G)}{}^{1}\Big(\lim_{\mathcal{O}_{X\cap K}(K)}{}^{n-1}(\Phi|_{\mathcal{O}_{X\cap K}(K)})\Big) \longrightarrow \lim_{\mathcal{O}_X(G)}{}^{n}(\Phi) \\ \longrightarrow \lim_{K\in\mathfrak{Fin}(G)}\Big(\lim_{\mathcal{O}_{X\cap K}(K)}{}^{n}(\Phi|_{\mathcal{O}_{X\cap K}(K)})\Big) \longrightarrow 0.$$

Proof. Let $\mathbb{D}$ be the poset of all subcategories $\mathcal{O}_{X\cap K}(K) \subseteq \mathcal{O}_X(G)$ for $K \in \mathfrak{Fin}(G)$, and let $F\colon \mathfrak{Fin}(G) \to \mathbb{D}$ be the surjective functor that sends K to $\mathcal{O}_{X\cap K}(K)$. Then $\mathcal{O}_X(G)$ is the union of the members of $\mathbb{D}$ since all members of X are finite and G is locally finite, and $\mathfrak{Fin}(G)$ and $\mathbb{D}$ are directed posets since G is locally finite. So by Proposition 1.7, for each $R\mathcal{O}_X(G)$-module Φ, there is a first quarter spectral sequence

$$E_2^{ij} \cong \lim_{\mathbb{D}}{}^{i}\big(\mathcal{D} \mapsto \lim_{\mathcal{D}}{}^{j}(\Phi|_{\mathcal{D}})\big) \implies \lim_{\mathcal{O}_X(G)}{}^{i+j}(\Phi). \tag{2.3}$$

For each $\mathcal{D}\in\mathbb{D}$, the undercategory $\mathcal{D}\downarrow F$ is isomorphic to the poset of all $H\in\mathfrak{Fin}(G)$ such that $\mathcal{O}_{X\cap H}(H)\geq\mathcal{D}$, and hence is nonempty and directed. So by Lemma 1.6, for each $R\mathbb{D}$-module Ψ,

$$\lim_{\mathfrak{Fin}(G)}{}^{*}(\Psi\circ F) \cong \lim_{\mathbb{D}}{}^{*}(\Psi).$$

In particular, (2.2) follows from (2.3).

If G is countable, then $\mathfrak{Fin}(G)$ is a countable directed poset, so higher limits over $\mathfrak{Fin}(G)$ vanish in degrees greater than 1, and the E_2-term of the spectral sequence vanishes except for the first two columns. ∎

More generally, if G has cardinality $\aleph_m$ for some finite $m \geq 0$, then $\mathfrak{Fin}(G)$ also has cardinality $\aleph_m$. So by a theorem of Goblot [6, Proposition 2] (see also [9, Théorème 3.1]), in the spectral sequence of Proposition 2.4, the terms E_2^{ij} are always zero for $i \geq m+2$.

We next construct a spectral sequence that describes higher limits of functors on orbit categories for a group extension. We refer to (1.4) and the discussion before that for the definition and description of right Kan extensions.

Theorem 2.5. *Let R be a commutative ring, let G be a (discrete) group, let $H \trianglelefteq G$ be a normal subgroup, and let $\chi\colon G \to G/H$ be the natural homomorphism. Let X and Y be nonempty sets of subgroups of G and G/H, respectively, both invariant under conjugation, and such that $K \in X$ implies $\chi(K) \in Y$. Let $\widehat{\chi}\colon \mathcal{O}_X(G) \to \mathcal{O}_Y(G/H)$ be the functor that sends $K \in X$ to $\chi(K)$ and sends a morphism $[g]$ to $[\chi(g)]$, and let*

$$\widehat{\chi}^*\colon R\mathcal{O}_Y(G/H)\text{-mod} \longrightarrow R\mathcal{O}_X(G)\text{-mod}$$

be the functor that sends Φ to $\Phi\circ\widehat{\chi}$. Let $\widehat{\chi}_*^R\colon R\mathcal{O}_X(G)\text{-mod} \to R\mathcal{O}_Y(G/H)\text{-mod}$ be the right Kan extension along $\widehat{\chi}$.

(a) *For each $R\mathcal{O}_X(G)$-module Φ, there is a first quarter spectral sequence*

$$E_2^{ij} = \lim_{\mathcal{O}_Y(G/H)}{}^{i}\big((R^j\hat{\chi}_*^R)(\Phi)\big) \implies \lim_{\mathcal{O}_X(G)}{}^{i+j}(\Phi),$$

where $R^j\hat{\chi}_*^R\colon R\mathcal{O}_X(G)\text{-mod} \to R\mathcal{O}_Y(G/H)\text{-mod}$ is the j-th right derived functor of $\hat{\chi}_*^R$.

(b) *For each $R\mathcal{O}_X(G)$-module Φ, each subgroup $K/H \in Y$, and each $j \ge 0$,*

$$(R^j\hat{\chi}_*^R)(\Phi)(K/H) \cong \lim_{\mathcal{O}_{X\cap K}(K)}{}^{j}(\Phi|_{\mathcal{O}_{X\cap K}(K)}),$$

where $X \cap K$ is the set of members of X contained in K. Under this identification, a morphism $[g]\colon K_1/H \to K_2/H$ in $\mathcal{O}_Y(G/H)$ (where $g \in G$ and ${}^gK_1 \le K_2$) is sent to the homomorphism

$$\lim_{\mathcal{O}_{X\cap K_2}(K_2)}{}^{j}(\Phi|_{\mathcal{O}_{X\cap K_2}(K_2)}) \longrightarrow \lim_{\mathcal{O}_{X\cap K_1}(K_1)}{}^{j}(\Phi|_{\mathcal{O}_{X\cap K_1}(K_1)})$$

induced by the functor $c_g^{K_1}\colon \mathcal{O}_{X\cap K_1}(K_1) \to \mathcal{O}_{X\cap K_2}(K_2)$ that sends $L \in X \cap K_1$ to ${}^gL \in X \cap K_2$ and sends a morphism $[x]$ to $[{}^gx]$, together with the natural transformation of functors

$$\Phi|_{\mathcal{O}_{X\cap K_2}(K_2)} \circ c_g^{K_1} \longrightarrow \Phi|_{\mathcal{O}_{X\cap K_1}(K_1)}$$

that sends an object L in $\mathcal{O}_{X\cap K_2}(K_2)$ to $\Phi([g]) \in \mathrm{Hom}_R(\Phi({}^gL), \Phi(L))$.

Proof. Set $\bar{G} = G/H$ for short. For each $K \le G$, set $\bar{K} = \chi(K) = KH/H$.

By (1.4), for each $R\mathcal{O}_X(G)$-module Φ and each $\bar{K} \in Y$,

$$\hat{\chi}_*^R(\Phi)(\bar{K}) = \lim_{\hat{\chi}\downarrow\bar{K}}\Big((\hat{\chi}\downarrow\bar{K})^{\mathrm{op}} \xrightarrow{\mu^{\mathrm{op}}} \mathcal{O}_X(G)^{\mathrm{op}} \xrightarrow{\Phi} R\text{-mod}\Big),$$

where μ is the forgetful functor sending $(L, \bar{L} \to \bar{K})$ to L. Since inverse limits are left exact, so is $\hat{\chi}_*^R$.

(a) This is a special case of the Grothendieck spectral sequence, in the form described in [15, Theorem 5.8.3], applied to the triangle

$$\begin{array}{ccc} \mathcal{O}_X(G)\text{-mod} & \xrightarrow{\hat{\chi}_*^R} & \mathcal{O}_Y(\bar{G})\text{-mod} \\ & {\scriptstyle\lim}\searrow \quad \swarrow{\scriptstyle\lim} & \\ & R\text{-mod} & \end{array}$$

of categories and functors. We already saw that $\hat{\chi}_*^R$ is left exact, and limits are always left exact. So to be able to apply the spectral sequence (with $\hat{\chi}_*^R$ and lim in the roles of G and F), it remains to check that the triangle commutes up to natural isomorphism, and that $\widehat{\chi}^R_*$ sends injectives to injectives. This last condition (sending injectives to injectives) holds by [15, Proposition 2.3.10] and since $\widehat{\chi}^R_*$ is right adjoint to the exact functor $\widehat{\chi}^*$.

Let $\underline{R}$ denote the constant functor on $\mathcal{O}_Y(\bar{G})$ that sends all objects to R and all morphisms to Id_R. For each $R\mathcal{O}_X(G)$-module Φ, there are natural isomorphisms

$$\lim_{\mathcal{O}_Y(\bar{G})}\big(\widehat{\chi}^R_*(\Phi)\big) \cong \mathrm{Mor}_{R\mathcal{O}_Y(\bar{G})\text{-mod}}\big(\underline{R}, \widehat{\chi}^R_*(\Phi)\big) \cong \mathrm{Mor}_{R\mathcal{O}_X(G)\text{-mod}}\big(\widehat{\chi}^*(\underline{R}), \Phi\big) \cong \lim_{\mathcal{O}_X(G)}(\Phi),$$

where the second holds since $\widehat{\chi}^R_*$ is right adjoint to $\widehat{\chi}^*$. So the triangle commutes up to natural isomorphism.

(b) Let $0 \to \Phi \to I_0 \to I_1 \to I_2 \to \cdots$ be a resolution of Φ by injective $\mathcal{O}_X(G)$-modules. Thus $(R^*\widehat{\chi}^R_*)(\Phi)$ is the homology of the complex of $R\mathcal{O}_Y(\bar{G})$-modules

$$0 \longrightarrow \widehat{\chi}^R_*(I_0) \longrightarrow \widehat{\chi}^R_*(I_1) \longrightarrow \widehat{\chi}^R_*(I_2) \longrightarrow \cdots . \tag{2.4}$$

We first claim that for each subgroup $K \le G$,

$$\Phi \text{ injective in } \mathcal{O}_X(G)\text{-mod} \implies \Phi|_{\mathcal{O}_{X\cap K}(K)} \text{ injective in } \mathcal{O}_{X\cap K}(K)\text{-mod}, \tag{2.5}$$

where $X \cap K$ is the set of members of X contained in K. Recall (Proposition 1.2) that each injective object in $\mathcal{O}_X(G)$-mod is a direct factor in a product of injectives of the form $\mathfrak{I}^A_L$ for $L \in X$ and A an injective abelian group. So it suffices to prove that the restriction of each such $\mathfrak{I}^A_L$ is injective. Fix a set $W \subseteq G$ of right coset representatives for K in G (thus $G = KW$). For given L and A, we have

$$\mathfrak{I}^A_{\mathcal{O}_X(G),L}|_{\mathcal{O}_{X\cap K}(K)} \cong \prod_{\substack{w\in W\\ {}^wL\le K}} \mathfrak{I}^A_{\mathcal{O}_{X\cap K}(K),{}^wL}$$

(and vanishes if no conjugate of L is contained in K), since for each $U \le K$ and $L \le G$, there is a bijection

$$\begin{aligned}\mathrm{Mor}_{\mathcal{O}(G)}(L,U) &= \{Ug \mid g\in G,\ {}^gL \le U\}\\ &= \coprod_{w\in W}\{Uhw \mid h\in K,\ {}^{hw}L\le U\} \cong \coprod_{w\in W}\mathrm{Mor}_{\mathcal{O}(K)}({}^wL,U)\end{aligned}$$

that sends $[g] \in \mathrm{Mor}_{\mathcal{O}(G)}(L,U)$, for $g \in Kw$, to $[gw^{-1}] \in \mathrm{Mor}_{\mathcal{O}(G)}({}^wL,U)$. This proves (2.5).

Fix some $\bar{K} = K/H$ in Y. Consider the functor

$$\lambda\colon \mathcal{O}_{X\cap K}(K) \longrightarrow \widehat{\chi}\downarrow\bar{K}$$

that sends an object L in $\mathcal{O}_{X\cap K}(K)$ to the object $(L, \bar{L} \xrightarrow{[1]} \bar{K})$ and similarly for morphisms. Then λ is injective (split by the forgetful functor). Its image is a full subcategory, since for each morphism

$$\big(L_1, \bar{L}_1 \xrightarrow{[1]} \bar{K}\big) \xrightarrow{[g]} \big(L_2, \bar{L}_2 \xrightarrow{[1]} \bar{K}\big)$$

in $\widehat{\chi}{\downarrow}\overline{K}$ between objects of $\mathrm{Im}(\lambda)$, we have $[1]\circ[\chi(g)]=[1]$ in $\mathrm{Mor}_{\mathcal{O}(\overline{G})}(\overline{L}_1,\overline{K})$, and hence $g\in K$ and $[g]\in\mathrm{Mor}_{\mathcal{O}(K)}(L_1,L_2)$. Each object in $\widehat{\chi}{\downarrow}\overline{K}$ has the form $(L,\overline{L}\xrightarrow{[\chi(g)]}\overline{K})$ for some $L\in X$ and $g\in G$ such that ${}^gL\le K$, and this pair is isomorphic to the object $({}^gL,\overline{{}^gL}\xrightarrow{[1]}\overline{K})=\lambda({}^gL)$. So λ is an equivalence of categories.

Thus for each $n\ge 0$,

$$\begin{aligned}\widehat{\chi}^R_*(I_n)(\overline{K}) &= \lim_{\widehat{\chi}\downarrow\overline{K}}\big((\widehat{\chi}{\downarrow}\overline{K})^{\mathrm{op}}\xrightarrow{\mu^{\mathrm{op}}}\mathcal{O}_X(G)^{\mathrm{op}}\xrightarrow{I_n}R\text{-mod}\big)\\ &\cong \lim_{\mathcal{O}_{X\cap K}(K)}\big(\mathcal{O}_{X\cap K}(K)^{\mathrm{op}}\xrightarrow{\lambda^{\mathrm{op}}}(\widehat{\chi}{\downarrow}\overline{K})^{\mathrm{op}}\xrightarrow{\mu^{\mathrm{op}}}\mathcal{O}_X(G)^{\mathrm{op}}\xrightarrow{I_n}R\text{-mod}\big)\\ &= \lim_{\mathcal{O}_{X\cap K}(K)}(I_n|_{\mathcal{O}_{X\cap K}(K)}),\end{aligned}$$

where the isomorphism holds by Lemma 1.6 and since each undercategory $x{\downarrow}\lambda$ has an initial object (hence $(x{\downarrow}\lambda)^{\mathrm{op}}$ is directed). So

$$\begin{aligned}(R^*\widehat{\chi}^R_*)(\Phi)(\overline{K}) &\cong H^*\big(0\longrightarrow\widehat{\chi}^R_*(I_0)(\overline{K})\longrightarrow\widehat{\chi}^R_*(I_1)(\overline{K})\longrightarrow\cdots\big)\\ &\cong H^*\big(0\longrightarrow\lim(I_0|_{\mathcal{O}_{X\cap K}(K)})\longrightarrow\lim(I_1|_{\mathcal{O}_{X\cap K}(K)})\longrightarrow\cdots\big)\\ &\cong \lim_{\mathcal{O}_{X\cap K}(K)}{}^{*}(\Phi|_{\mathcal{O}_{X\cap K}(K)}),\end{aligned}$$

where the last isomorphism holds since each $I_n|_{\mathcal{O}_{X\cap K}(K)}$ is injective.

Now assume that $K_1/H,K_2/H\in Y$, and $g\in G$ is such that ${}^gK_1\le K_2$. We must identify, for each $n\ge 0$, the homomorphism

$$[g]^*\colon \lim_{\mathcal{O}_{X\cap K_2}(K_2)}(I_n|_{\mathcal{O}_{X\cap K_2}(K_2)})\longrightarrow\lim_{\mathcal{O}_{X\cap K_1}(K_1)}(I_n|_{\mathcal{O}_{X\cap K_1}(K_1)})$$

induced by $[g]\in\mathrm{Mor}_{\mathcal{O}_Y(G/H)}(K_1/H,K_2/H)$. Fix an element $y=(y_L)_{L\in X\cap K_2}$ in the first inverse limit; thus $y_L\in I_n(L)$ for each L. This extends to an element

$$\widehat{y}=(\widehat{y}_{(L,\overline{L}\xrightarrow{[h]}\overline{K}_2)})\in\lim_{\widehat{\chi}\downarrow\overline{K}_2}(I_n\circ\mu^{\mathrm{op}})\quad\text{where }\widehat{y}_{(L,\overline{L}\xrightarrow{[h]}\overline{K}_2)}=I_n\big([h]\big)(y_{{}^xL}),$$

and $[g]$ sends $\widehat{y}$ to

$$\widehat{z}=(\widehat{z}_{(L,\overline{L}\xrightarrow{[h]}\overline{K}_1)})\quad\text{where }\widehat{z}_{(L,\overline{L}\xrightarrow{[h]}\overline{K}_1)}=\widehat{y}_{(L,\overline{L}\xrightarrow{[gh]}\overline{K}_2)}.$$

Then $[g]^*(y_L)_{L\in X\cap K_2}=(z_L)_{L\in X\cap K_1}$, where for each $L\in X\cap K_1$,

$$z_L=\widehat{z}_{(L,\overline{L}\xrightarrow{[1]}\overline{K}_1)}=\widehat{y}_{(L,\overline{L}\xrightarrow{[g]}\overline{K}_2)}=I_n\big([g]\big)(y_{({}^gL)}).$$

Thus $[g]^*$ is induced by $c_g^{K_1}\colon\mathcal{O}_{X\cap K_1}(K_1)\to\mathcal{O}_{X\cap K_2}(K_2)$, together with the natural transformation of functors induced by $[g]\in\mathrm{Mor}_{\mathcal{O}_X(G)}(L,{}^gL)$ for $L\in X\cap K_1$. ■

Note that the Lyndon–Hochschild–Serre spectral sequence in cohomology (see, e.g., [15, Theorem 6.8.2]) for the group extension $1\to H\to G\to G/H\to 1$ is the special case of the spectral sequence of Theorem 2.5 when X and Y both contain only the trivial subgroup. Other special cases of this spectral sequence will be looked at in the next section.

3. Λ-functors

We now restrict attention to certain functors $\Lambda^*(G;M)$, defined originally in [8, Definition 5.3] when G is a finite group. Most of our main results, including Theorems A–D in the introduction, are proven in this section.

Definition 3.1. Fix a prime p. Let G be a group, and let M be a $\mathbb{Z}G$-module. Define a functor $F_M\colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathsf{Ab}$ by setting

$$F_M(P) = \begin{cases} M & \text{if } P = 1, \\ 0 & \text{if } P \neq 1 \end{cases}$$

for P a p-subgroup of G, where $\mathrm{Aut}_{\mathcal{O}_p(G)}(1) \cong G$ has the given action on M. Set

$$\Lambda^*(G;M) = \lim_{\mathcal{O}_p(G)}{}^{*}\,(F_M) \quad \text{and} \quad \Lambda^*_f(G;M) = \lim_{\mathcal{O}_p^f(G)}{}^{*}\,(F_M|_{\mathcal{O}_p^f(G)}).$$

More generally, if X is a set of p-subgroups of G invariant under conjugation, set

$$\Lambda^*_X(G;M) = \lim_{\mathcal{O}_X(G)}{}^{*}\,(F_M|_{\mathcal{O}_X(G)}).$$

Note that the functors $\Lambda^*(-;-)$ depend on the prime p, even though that has been left out of the notation to keep it simple.

The importance of these graded groups comes from the following proposition, which was shown in the finite case in [8, Lemma 5.4], and in a more general situation in [3, Lemma 5.10]. (In fact, it was formulated in [8, Lemma 5.4] in a way that held more generally for compact Lie groups.)

Proposition 3.2. *Let p be a prime, and let G be a group. Let X be a set of p-subgroups of G invariant under conjugation, fix $Q \in X$, and assume that either*

(i) *each member of X contains an abelian subgroup of finite index; or*

(ii) *each member of X that contains Q contains it with finite index; or*

(iii) *for each $Q < P \in X$, we have $Q < N_P(Q)$.*

Assume also that $Q \le P \in X$ implies $N_P(Q) \in X$. Let Y be the set of all subgroups P/Q for $Q \trianglelefteq P \in X$. Then for each $\Phi\colon \mathcal{O}_X(G)^{\mathrm{op}} \to \mathsf{Ab}$ such that $\Phi(P) = 0$ for all $P \in X$ not G-conjugate to Q,

$$\lim_{\mathcal{O}_X(G)}{}^{*}\,(\Phi) \cong \Lambda^*_Y\big(N_G(Q)/Q;\Phi(Q)\big). \tag{3.1}$$

Proof. Since (i) and (ii) imply (iii) by Lemma A.1, we assume from now on that (iii) holds. Set $\Gamma = N_G(Q)/Q$. We want to apply [3, Proposition 5.3] with $\mathcal{C} = \mathcal{O}_X(G)$. Let $\alpha\colon \mathcal{O}_Y(\Gamma) \to \mathcal{O}_X(G)$ be the functor that sends an object $P/Q \in Y$ to $P \in X$, and sends a morphism $P/Q \xrightarrow{[gQ]} R/Q$ to $P \xrightarrow{[g]} R$ for $g \in N_G(Q)$ such that ${}^gP \le R$. To prove the lemma, we must show that conditions (a)–(d) in [3, Proposition 5.3] all hold.

(a) By definition, α sends $\mathrm{Aut}_{\mathcal{O}_Y(\Gamma)}(1) \cong \Gamma$ isomorphically to $\mathrm{Aut}_{\mathcal{O}_X(G)}(Q)$.

(b) We must show, for all $P \in X$ not isomorphic to Q in $\mathcal{O}_X(G)$, that all isotropy subgroups of the action of $\Gamma = N_G(Q)/Q$ on $\mathrm{Mor}_{\mathcal{O}_X(G)}(Q, P)$ are nontrivial and in Y. To see this, fix $[x] \in \mathrm{Mor}_{\mathcal{O}_X(G)}(Q, P)$. For $g \in N_G(Q)$, the class $[g] \in \Gamma$ is in the isotropy subgroup if and only if $Pxg = Px$, equivalently, $xgx^{-1} \in P$. Thus the isotropy subgroup is $(P^x \cap N_G(Q))/Q = N_{P^x}(Q)/Q$, and this is nontrivial by (iii) (and is in Y by definition of Y).

(c) Since all morphisms in $\mathcal{O}_X(G)$ are epimorphisms by Lemma 2.2, this holds automatically.

(d) We must show, for all $P/Q \in Y$, all $R \in X$, and all $[x] \in \mathrm{Mor}_{\mathcal{O}_X(G)}(Q, R)$ fixed by the action of P/Q on this morphism set, that $[x]$ extends to a morphism defined on P. We just saw in the proof of (b) that $P \leq R^x$ since P/Q fixes $[x]$, so ${}^xP \leq R$, and $[x]$ also defines a morphism from P to R. ∎

For example, the conclusion of Proposition 3.2 holds if X and Y are the sets of all finite p-subgroups, or if X is the set of all p-subgroups of G that contain an abelian subgroup of finite index and Y is the set of all P/Q for $Q \trianglelefteq P \in X$.

The following theorem is a special case of Proposition 2.4, and helps reduce computations of $\Lambda^*_f(G; -)$ to the finite case.

Theorem 3.3. *Fix a prime p. Let G be a locally finite group, and let M be a $\mathbb{Z}G$-module. Then there is a first quarter spectral sequence*

$$E_2^{ij} \cong \lim_{\mathfrak{Fin}(G)}{}^i \left(\Lambda^j(-; M)\right) \Longrightarrow \Lambda_f^{i+j}(G; M).$$

If G is countable, then this reduces to short exact sequences

$$0 \longrightarrow \lim_{\mathfrak{Fin}(G)}{}^1 \left(\Lambda^{i-1}(-; M)\right) \longrightarrow \Lambda^i_f(G; M) \longrightarrow \lim_{\mathfrak{Fin}(G)} \Lambda^i(-; M) \longrightarrow 0$$

for each $i \geq 0$.

Proof. This follows from Proposition 2.4, applied with X the set of finite p-subgroups of G and $\Phi = F_M$. ∎

The following result was shown for finite G and $\mathbb{Z}_{(p)}G$-modules in [8, Proposition 6.1 (ii)].

Proposition 3.4. *Fix a prime p, a group G, and a $\mathbb{Z}G$-module M.*

(1) *If $O_p(G) \neq 1$, then $\Lambda^*(G; M) = 0$.*

(2) *If there is a nontrivial finite normal p-subgroup $1 \neq Q \trianglelefteq G$, then $\Lambda^*_f(G; M) = 0$.*

(3) *If G is a nontrivial locally finite p-group, then $\Lambda^*_f(G; M) = 0$.*

Proof. (a) Set $Q = O_p(G)$ for short, and let $\mathcal{O}_p^0(G) \subseteq \mathcal{O}_p(G)$ be the full subcategory with objects the p-subgroups of G that contain Q.

We apply Lemma 2.3 with X the set of all p-subgroups of G and X_0 the set of those that contain Q. Every p-subgroup of G is contained in one that contains Q since Q is a normal p-subgroup, and intersections of p-subgroups containing Q again contain Q. Thus conditions (i) and (ii) in Lemma 2.3 hold, and hence

$$\lim_{\mathcal{O}_p(G)}^* (\Phi) \cong \lim_{\mathcal{O}_p^0(G)}^* (\Phi|_{\mathcal{O}_p^0(G)})$$

for each $\mathcal{O}_p(G)$-module Φ by Lemma 2.3. So $\Lambda^*(G; M) = 0$ since $F_M|_{\mathcal{O}_p^0(G)} = 0$.

(b) Now apply Lemma 2.3 with X the set of finite p-subgroups of G and X_0 the set of those that contain Q. Since Q is finite, each member of X is contained in a member of X_0. The rest of the proof goes through exactly as in the proof of (a).

(c) Assume $G \neq 1$ is a locally finite p-group. For each nontrivial finite subgroup $1 \neq Q \leq P$, $\Lambda^*(Q; M) = 0$ by (a). Also, $\Lambda^i(1; M) = H^i(1; M) = 0$ for $i > 0$, while $\Lambda^0(1; M) = M$. So $E_2^{ij} = 0$ for all $j > 0$ in the spectral sequence of Theorem 3.3. Hence by that spectral sequence,

$$\Lambda_f^i(G; M) \cong E_2^{i0} \cong \lim_{\mathfrak{Fin}(G)}^i (\Psi_M) \quad \text{for all } i \geq 0,$$

where $\Psi_M \colon \mathfrak{Fin}(G) \to \mathsf{Ab}$ sends the trivial subgroup to M and all others to 0.

Fix a nontrivial finite subgroup $1 \neq R \leq G$, and let $\mathfrak{Fin}^0(G)$ be the set of finite subgroups of G that contain R. Then each $H \in \mathfrak{Fin}(G)$ is contained in $\langle H, R \rangle \in \mathfrak{Fin}^0(G)$ since G is locally finite. Also, if $H \in \mathfrak{Fin}(G)$ is contained in $K_1, K_2 \in \mathfrak{Fin}^0(G)$, then $K_1 \cap K_2 \in \mathfrak{Fin}^0(G)$ also contains H. So the categories $\mathfrak{Fin}^0(G) \subseteq \mathfrak{Fin}(G)$ satisfy the hypotheses of Lemma 1.6, and hence

$$\lim_{\mathfrak{Fin}(G)}^* (\Psi_M) \cong \lim_{\mathfrak{Fin}^0(G)}^* (\Psi_M|_{\mathfrak{Fin}^0(G)}) = 0$$

since $\Psi|_{\mathfrak{Fin}^0(G)} = 0$. ■

When G is locally finite, $\Phi = F_M$ for some $\mathbb{Z}G$-module M, and X and Y are the sets of all p-subgroups of G and G/H, the spectral sequence of Theorem 2.5 takes the following form.

Theorem 3.5. *Fix a prime p, a locally finite group G, a normal subgroup $H \trianglelefteq G$, and a $\mathbb{Z}G$-module M. Then there is a first quarter spectral sequence*

$$E_2^{ij} = \lim_{\mathcal{O}_p(G/H)}^i \big(P/H \mapsto \Lambda^j(P; M)\big) \implies \Lambda^{i+j}(G; M).$$

Another special case of Theorem 2.5 is the following.

Theorem 3.6. *Fix a prime p, and assume G is a group all of whose p-subgroups are locally finite. Then for each $\mathbb{Z}G$-module M, the natural homomorphism $\Lambda^*(G; M) \to \Lambda_f^*(G; M)$ is an isomorphism.*

Proof. We again apply Theorem 2.5, this time with $H = 1$ the trivial subgroup, X the set of finite p-subgroups of G, and Y the set of all p-subgroups. For each $\mathcal{O}_p(G)$-module Φ, the spectral sequence takes the form

$$E_2^{ij} = \lim_{\mathcal{O}_p(G)}{}^i \big(K \mapsto \lim_{\mathcal{O}_p^f(K)}{}^j (\Phi|_{\mathcal{O}_p^f(K)})\big) \implies \lim_{\mathcal{O}_p^f(G)}{}^{i+j} (\Phi).$$

When $\Phi = F_M$, this yields a spectral sequence

$$E_2^{ij} = \lim_{\mathcal{O}_p(G)}{}^i \big(\Lambda_f^j(-;M)\big) \implies \Lambda_f^{i+j}(G;M). \tag{3.2}$$

If $K \neq 1$ is a nontrivial p-subgroup of G, then $\Lambda_f^*(K;M) = 0$ by Proposition 3.4 (c) and since K is locally finite. If $K = 1$, then $\Lambda_f^*(K;M) \cong H^*(K;M)$ is M in degree 0 and zero in higher degrees. Thus $\Lambda_f^0(-;M) \cong F_M|_{\mathcal{O}_p^f(G)}$, while $\Lambda_f^j(-;M) = 0$ (as a $\mathcal{O}_p^f(G)$-module) for $j > 0$. So by (3.2), for each $i \geq 0$,

$$\Lambda_f^i(G;M) \cong E_2^{i0} \cong \lim_{\mathcal{O}_p(G)}{}^i (F_M) = \Lambda^i(G;M). \qquad \blacksquare$$

We are now ready to prove [3, Lemma 5.12], and through that finish the proof of [3, Theorem 8.7]. In Step 1 of the argument given in [3], we claimed that certain injective functors $\mathfrak{I}_k$ are inverse systems of groups and surjections. Unfortunately, the maps in the inverse systems need not be surjective, which means that we can get into trouble with nonvanishing $\lim^1(-)$. This is due to the fact that if Q and $P_0 < P$ are p-subgroups of a locally finite group G, then the natural map $\mathrm{Mor}_{\mathcal{O}_p(G)}(Q, P_0) \to \mathrm{Mor}_{\mathcal{O}_p(G)}(Q, P)$ need not be injective.

Instead, we use the following argument, based on Theorems 3.3 and 3.6.

Theorem 3.7. *Fix a prime p. Let G be a locally finite group, and let M be a $\mathbb{Z}G$-module. Assume, for some finite subgroup $H_0 \leq G$, that $\Lambda^*(H;M) = 0$ for each finite subgroup $H \leq G$ that contains H_0. Then $\Lambda^*(G;M) = 0$. In particular, this holds if M is a $\mathbb{Z}_{(p)}G$-module and $C_G(M)$ contains at least one element of order p.*

Proof. Let $\mathfrak{Fin}_0(G)$ be the set of finite subgroups of G that contain H_0. Thus by assumption, $\Lambda^*(K;M) = 0$ for all $K \in \mathfrak{Fin}_0(G)$. Since G is locally finite, each finite subgroup $K \in \mathfrak{Fin}(G)$ is contained in $\langle K, H_0\rangle \in \mathfrak{Fin}_0(G)$. Also, the intersection of two members of $\mathfrak{Fin}_0(G)$ again lies in the set. Since there is at most one morphism between any pair of objects in the category $\mathfrak{Fin}(G)$, this proves that opposite categories of undercategories for the inclusion $\mathcal{I}\colon \mathfrak{Fin}_0(G) \to \mathfrak{Fin}(G)$ are nonempty and directed, and hence by Lemma 1.6 that

$$\lim_{\mathfrak{Fin}(G)}{}^* \big(\Lambda^*(-;M)\big) \cong \lim_{\mathfrak{Fin}_0(G)}{}^* \big(\Lambda^*(-;M)|_{\mathfrak{Fin}_0(G)}\big) = 0.$$

Hence $\Lambda_f^*(G;M) = 0$ by the spectral sequence of Theorem 3.3, and so $\Lambda^*(G;M) = 0$ by Theorem 3.6.

If M is a $\mathbb{Z}_{(p)}G$-module and there is $g \in C_G(M)$ of order p, then $\Lambda^*(H; M) = 0$ for each $H \in \mathfrak{Fin}(G)$ containing g by [8, Proposition 6.1 (ii)]. So $\Lambda^*(G; M) = 0$ by the first part of the statement, applied with $H_0 = \langle g \rangle$. ■

Lemma 5.12 of [3] was applied only once in that paper: in the proof of Theorem 8.7 when we showed that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ for an LFS(p)-group G. We restate the full theorem here, but it is only the last statement in it ($|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$) that was affected by the error.

Theorem 3.8 ([3, Theorem 8.7]). *Let G be a locally finite group all of whose p-subgroups are discrete p-toral. Assume in addition that for each increasing sequence $A_1 \leq A_2 \leq \cdots$ of finite abelian p-subgroups of G, there is $k \geq 1$ such that $C_G(A_n) = C_G(A_k)$ for all $n \geq k$. Then G has a unique conjugacy class $\mathrm{Syl}_p(G)$ of maximal p-subgroups, and for each $S \in \mathrm{Syl}_p(G)$, the triple $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p-local compact group with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.*

4. Examples

We finish with a few examples that show how higher limits over orbit categories of locally finite groups can differ from those over orbit categories of finite groups. The following groups will be used whenever we need a more concrete example to see this. When $G_1, G_2, \ldots$ is a sequence of groups, we write $\bigoplus_{i=1}^\infty G_i$ to mean the group of elements of finite support in the direct product of the G_i.

Fix a prime p. Let $\mathbb{F}_0$ be a finite field of characteristic p and order at least 3, and let $\mathbb{F} \supseteq \mathbb{F}_0$ be the extension of degree p. Choose $1 \neq U \leq \mathbb{F}^\times$ such that $U \cap \mathbb{F}_0^\times = 1$. Let $S \leq \mathrm{Aut}(\mathbb{F})$ be the subgroup of order p (so $\mathrm{Fix}(S, \mathbb{F}) = \mathbb{F}_0$). Set

$$H = \bigoplus_{i=1}^\infty \mathbb{F}^\times, \quad H_0 = \bigoplus_{i=1}^\infty U \leq H, \quad M = \bigoplus_{i=1}^\infty \mathbb{F}, \qquad \Gamma = H \rtimes S, \quad \Gamma_0 = H_0 \rtimes S \leq \Gamma, \quad \Gamma_* = (U \times H) \rtimes S. \tag{4.1}$$

Let U and H act on M by setting $u(x_1, x_2, \ldots) = (ux_1, ux_2, \ldots)$ and

$$(h_1, h_2, \ldots)(x_1, x_2, \ldots) = (h_1x_1, h_2x_2, \ldots),$$

and let Γ and Γ_* act on M via those actions and the Galois action of S.

We first note the following very general vanishing result.

Lemma 4.1. *Fix a prime p, and let G be a countable locally finite group with finite Sylow p-subgroup $S \leq G$ of order p^n (some $n \geq 0$). Then for each $\mathbb{Z}_{(p)}G$-module M, $\Lambda^i(G; M) = 0$ for all $i \geq n + 2$.*

Proof. For each $K \in \mathfrak{Fin}(G)$, and each $i \geq 1$ such that $\Lambda^i(K; M) \neq 0$, we have $i \leq n$ by [1, Lemma III.5.27]. So for each $i \geq n + 2$, $\Lambda^i(K; M) \cong \Lambda^{i-1}(K; M) = 0$ for all $K \in \mathfrak{Fin}(G)$, and hence $\Lambda^i(G; M) = \Lambda_f^i(G; M) = 0$ by Theorem 3.3. ■

By a result of Jackowski and McClure (see [8, Proposition 5.2]), for any finite group G and any $\mathbb{Z}_{(p)}G$-module M, the functor $P \mapsto \mathrm{Fix}(P, M)$ is acyclic. The following example, an application of the spectral sequence of Proposition 2.4, shows that this does not hold in general for locally finite groups.

Example 4.2. Fix a prime p, let G be a locally finite group, and let M be a (left) $\mathbb{Z}_{(p)}G$-module. Consider the $\mathbb{Z}_{(p)}\mathcal{O}_p^f(G)$-module

$$\Phi_M^G\colon \mathcal{O}_p^f(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-mod}$$

defined by setting $\Phi_M^G(P) = \mathrm{Fix}(P, M)$ (the P-invariant elements of M) for each finite p-subgroup $P \le G$, and by sending a morphism $P \xrightarrow{[g]} Q$ to the composite

$$\mathrm{Fix}(Q, M) \xrightarrow{\text{incl}} \mathrm{Fix}({}^gP, M) \xrightarrow{g^{-1}} \mathrm{Fix}(P, M).$$

Then

$$\lim_{\mathcal{O}_p^f(G)}{}^i\,(\Phi_M^G) \cong \lim_{\mathfrak{Fin}(G)}{}^i\,\big(K \mapsto \mathrm{Fix}(K, M)\big) \tag{4.2}$$

$$\cong \begin{cases} \mathrm{Fix}(G, M) & \text{if } i = 0, \\ \mathrm{Coker}\big[M \to \big(\lim_{\mathfrak{Fin}(G)} \big(M/\mathrm{Fix}(-, M)\big)\big)\big] & \text{if } i = 1, \\ 0 & \text{if } |G| \le \aleph_m \text{ and } i \ge m+2 \\ & \quad\text{(some } m \ge 0). \end{cases} \tag{4.3}$$

Proof. By [8, Proposition 5.2 and Corollary 1.8], for each $K \in \mathfrak{Fin}(G)$, we have that $\lim^0(\Phi_M^G|_{\mathcal{O}_p(K)}) \cong \mathrm{Fix}(K, M)$, while $\lim^i(\Phi_M^G|_{\mathcal{O}_p(K)}) = 0$ for all $i > 0$. So the spectral sequence of Proposition 2.4 collapses, and (4.2) follows directly from that. If $|G| \le \aleph_m$ where $0 \le m < \infty$, then $|\mathfrak{Fin}(G)| \le \aleph_m$, and so $\lim^i_{\mathfrak{Fin}(G)}(\Psi) = 0$ for each functor Ψ and each $i \ge m+2$ by a theorem of Goblot [6, Proposition 2] (see also [9, Théorème 3.1]).

The constant functor on $\mathfrak{Fin}(G)$ with value M is isomorphic to $\mathfrak{I}_1^M$, and hence is acyclic by Proposition 1.2 (c). So the short exact sequence

$$0 \longrightarrow \mathrm{Fix}(-, M) \longrightarrow M \longrightarrow M/\mathrm{Fix}(-, M) \longrightarrow 0$$

of $\mathbb{Z}_{(p)}\mathfrak{Fin}(G)$-modules induces an exact sequence

$$0 \longrightarrow \lim_{\mathfrak{Fin}(G)}{}^0\,\big(\mathrm{Fix}(-, M)\big) \longrightarrow M \longrightarrow \lim_{\mathfrak{Fin}(G)}{}^0\,\big(M/\mathrm{Fix}(-, M)\big) \longrightarrow \lim_{\mathfrak{Fin}(G)}{}^1\,\big(\mathrm{Fix}(-, M)\big) \longrightarrow 0,$$

and $\lim^i_{\mathfrak{Fin}(G)}(\mathrm{Fix}(-, M))$ is as described in (4.3) for $i = 0, 1$. ∎

For example, if we return to the pair (Γ, M) defined in (4.1), we get

$$\lim_{\mathcal{O}_p(\Gamma)}{}^0\,(\Phi_M^\Gamma) = 0 \quad\text{and}\quad \lim_{\mathcal{O}_p(\Gamma)}{}^1\,(\Phi_M^\Gamma) \cong \Big(\prod_{i=1}^\infty \mathbb{F}_0\Big)\Big/\Big(\bigoplus_{i=1}^\infty \mathbb{F}_0\Big) \ne 0,$$

Hence Φ_M^Γ is not acyclic as an $\mathcal{O}_p(\Gamma)$-module. (Note that $\mathcal{O}_p^f(\Gamma) = \mathcal{O}_p(\Gamma)$ in this case.)

In the situation of Example 4.2, if G has no elements of order p, then $\mathcal{O}_p(G) = \mathcal{O}_p^f(G)$ has only one object, so an $\mathcal{O}_p(G)$-module is the same as a $\mathbb{Z}G$-module, and $\Lambda^*(G; M) \cong H^*(G; M)$ for each such module M. This situation has been studied by Holt, who showed in [7, Theorem 1] that for each locally finite group G with no p-torsion satisfying a certain condition $(*)$ (which is always satisfied if G is abelian), if $|G| = \aleph_n$ for some $n \geq 0$, then $H^{n+1}(G; M) \neq 0$ for some $\mathbb{F}_pG$-module M. In particular, in the situation of Example 4.2, this shows that $\lim^n_{\mathcal{O}_p^f(G)}(\Phi_M^G)$ can be nonzero for arbitrarily large n, while in the situation of Proposition 2.4, the terms E_2^{ij} can be nonzero for arbitrarily large i.

The pair (H, M) in (4.1) gives an example in the countable case of the type studied by Holt: Example 4.2 implies that

$$H^1(H; M) \cong \Lambda^1(H; M) \cong \Big(\prod_{i=1}^{\infty} \mathbb{F}\Big) \Big/ \Big(\bigoplus_{i=1}^{\infty} \mathbb{F}\Big) \neq 0. \tag{4.4}$$

Many of the basic properties of the functors $\Lambda^*(G; M)$ for finite G were listed in [8, Propositions 6.1–6.2]. For example, [8, Proposition 6.2 (i)] states that if G is a finite group, M is a $\mathbb{Z}_{(p)}G$-module, and $S \in \mathrm{Syl}_p(G)$ has order p, then

$$\Lambda^1(G; M) \cong \mathrm{Fix}\big(N_G(S), M\big) / \mathrm{Fix}(G, M),$$

and $\Lambda^i(G; M) = 0$ for $i \neq 1$. Neither of these holds in general when G is locally finite and infinite.

Example 4.3. Fix a prime p. Let G be a countable locally finite group that contains a maximal p-subgroup $S \leq G$ of order p, and let $\mathfrak{Fin}_0(G)$ be the poset of finite subgroups of G that contain S. Then for each $\mathbb{Z}_{(p)}G$-module M, we have

$$\Lambda^1(G; M) \cong \lim_{\mathfrak{Fin}_0(G)} \Big(K \mapsto \tfrac{\mathrm{Fix}(N_K(S), M)}{\mathrm{Fix}(K, M)}\Big), \tag{4.5}$$

$$\Lambda^2(G; M) \cong \mathrm{Coker}\Big[\lim_{\mathfrak{Fin}_0(G)} \Big(K \mapsto \tfrac{\mathrm{Fix}(S,M)}{\mathrm{Fix}(K,M)}\Big) \longrightarrow \lim_{\mathfrak{Fin}_0(G)} \Big(K \mapsto \tfrac{\mathrm{Fix}(S,M)}{\mathrm{Fix}(N_K(S),M)}\Big)\Big] \tag{4.6}$$

and $\Lambda^i(G; M) = 0$ for all $i \neq 1, 2$.

Proof. If $P \leq G$ is a finite p-subgroup, then $\langle S, P\rangle$ is finite, and $S \in \mathrm{Syl}_p(\langle S, P\rangle)$ since it is a maximal p-subgroup of G. Thus $|P| \leq p$. In particular, every p-subgroup of G is finite, so $\Lambda^*(G; M) = \Lambda^*_f(G; M)$.

For each $K \in \mathfrak{Fin}_0(G)$, we have $S \in \mathrm{Syl}_p(K)$ since it is a maximal p-subgroup. So by [8, Proposition 6.2 (i)], $\Lambda^1(K; M) \cong \mathrm{Fix}(N_K(S), M)/\mathrm{Fix}(K, M)$, and $\Lambda^i(K; M) = 0$ for all $i \neq 1$. Theorem 3.3 now says that

$$\Lambda^i(G; M) \cong \lim^{i-1}_{K \in \mathfrak{Fin}_0(G)} \big(\mathrm{Fix}\big(N_K(S), M\big) / \mathrm{Fix}(K, M)\big)$$

when $i = 1, 2$, while $\Lambda^i(G; M) = 0$ for $i \neq 1, 2$. This proves (4.5) and the last statement.

Upon applying limits to the short exact sequence

$$0 \longrightarrow \mathrm{Fix}(K,M) \longrightarrow \mathrm{Fix}(N_K(S),M) \longrightarrow \mathrm{Fix}(N_K(S),M)/\,\mathrm{Fix}(K,M) \longrightarrow 0$$

of $\mathfrak{Fin}_0(G)$-modules, we get an exact sequence

$$\lim_{K\in\mathfrak{Fin}_0(G)}{}^1 (\mathrm{Fix}(K,M)) \longrightarrow \lim_{K\in\mathfrak{Fin}_0(G)}{}^1 (\mathrm{Fix}(N_K(S),M)) \longrightarrow \Lambda^2(G;M) \longrightarrow 0 \quad (4.7)$$

(recall that $\mathfrak{Fin}(G)$ is a countable directed poset). For $T=1$ or $T=S$, the extension

$$0 \longrightarrow \mathrm{Fix}\big(N_K(T),M\big) \longrightarrow \mathrm{Fix}(S,M) \longrightarrow \mathrm{Fix}(S,M)/\,\mathrm{Fix}\big(N_K(T),M\big) \longrightarrow 0$$

of $\mathfrak{Fin}_0(G)$-modules induces an isomorphism

$$\lim_{K\in\mathfrak{Fin}_0(G)}{}^1 \big(\mathrm{Fix}\big(N_K(T),M\big)\big) \cong \mathrm{Coker}\Big[\mathrm{Fix}(S,M) \to \lim_{K\in\mathfrak{Fin}_0(G)} \Big(\tfrac{\mathrm{Fix}(S,M)}{\mathrm{Fix}(N_K(T),M)}\Big)\Big] \quad (4.8)$$

and (4.6) follows from (4.7) and (4.8). ■

Upon returning to the groups and modules defined in (4.1), formulas (4.5) and (4.6) imply

$$\Lambda^1(\Gamma_0;M) \cong \prod_{i=1}^{\infty} \mathbb{F}_0 \quad \text{and} \quad \Lambda^2(\Gamma_*;M) \cong \Big(\prod\nolimits_{i=1}^{\infty} \mathbb{F}_0\Big)\Big/\Big(\bigoplus\nolimits_{i=1}^{\infty} \mathbb{F}_0\Big) \neq 0. \quad (4.9)$$

In contrast, $\mathrm{Fix}(N_{\Gamma_0}(S),M)/\,\mathrm{Fix}(\Gamma_0,M) \cong \bigoplus_{i=1}^{\infty} \mathbb{F}_0$. So when Γ is locally finite with Sylow p-subgroups of order p, then $\Lambda^1(\Gamma;M)$ and $\Lambda^2(\Gamma;M)$ can both be larger than would be predicted by the formulas for finite Γ.

When G is a product of two locally finite groups, Theorem 3.5 takes the following form.

Proposition 4.4. *Let G_1 and G_2 be locally finite groups, and let M be a $\mathbb{Z}[G_1{\times}G_2]$-module. Then there is a first quarter spectral sequence*

$$E_2^{ij} = \Lambda^i\big(G_1;\Lambda^j(G_2;M)\big) \implies \Lambda^{i+j}(G_1\times G_2;M).$$

Here, G_1 acts on $\Lambda^*(G_2;M)$ via its action on M.

Proof. By Theorem 3.5, there is a spectral sequence converging to $\Lambda^*(G_1\times G_2;M)$ with $E_2^{ij} \cong \lim^i_{\mathcal{O}_p(G_1)}(\Psi^j)$, where for a p-subgroup $P \le G_1$, we have $\Psi^j(P) = \Lambda^j(P\times G_2;M)$. When $P\neq 1$, we have $O_p(P\times G_2)\neq 1$, and so $\Lambda^j(P\times G_2;M)=0$ by Proposition 3.4 (a). So $\Psi^j = F_{\Lambda^j(G_2;M)}$, and E_2^{ij} is as described above. ■

By [8, Proposition 6.1 (v)], when H and K are finite groups, M is an $\mathbb{F}_pH$-module, and N is an $\mathbb{F}_pK$-module, then $\Lambda^*(H\times K;M\otimes_{\mathbb{F}_p}N)$ is isomorphic to the tensor product of $\Lambda^*(H;M)$ with $\Lambda^*(K;N)$. (A more general Künneth formula is given there.) This is not in general the case for products of locally finite groups, since tensor products do not in general commute with taking limits (and also since higher limits are involved). For example, if (Γ_*,M) and (H_0,M) are as in (4.1), then by Lemma 4.1, (4.4), and (4.9),

$$\Lambda^3(\Gamma_*\times H;M\otimes_{\mathbb{F}_p}M)=0 \quad \text{while } \Lambda^2(\Gamma_*;M)\otimes_{\mathbb{F}_p}\Lambda^1(H;M)\neq 0,$$
$$\Lambda^4(\Gamma_*\times \Gamma_*;M\otimes_{\mathbb{F}_p}M)=0 \quad \text{while } \Lambda^2(\Gamma_*;M)\otimes_{\mathbb{F}_p}\Lambda^2(\Gamma_*;M)\neq 0.$$

A. Locally finite p-groups

In the proof of Proposition 3.2, we needed to know, for certain pairs $Q < P$ of p-groups, that Q is strictly contained in its normalizer $N_P(Q)$. This motivates the following lemma, which gives conditions under which this holds, and also motivates Example A.2 where we show that it is not always the case, not even for locally finite p-groups.

Lemma A.1. *Let P be a p-group, and let $Q < P$ be a proper subgroup. Assume either*

(i) *Q has finite index in P, or*

(ii) *P contains an abelian subgroup of finite index.*

Then $Q < N_P(Q)$.

Proof. We first recall the well known fact that

$$H \le G \text{ groups and } [G : H] < \infty \implies \exists\, N \trianglelefteq G \text{ with } N \le H \text{ and } [G : N] < \infty. \quad \text{(A.1)}$$

This follows upon setting $N = C_G(G/H)$, where G acts on the finite set G/H via translation of cosets.

Assume first that (i) holds. By (A.1), there is $N \trianglelefteq P$ of finite index and contained in Q. Then P/N is a finite p-group and $Q/N < P/N$ is a proper subgroup, so $Q/N < N_{P/N}(Q/N) = N_P(Q)/N$. Thus $Q < N_P(Q)$.

Now assume (ii) holds, and let $A \le P$ be an abelian subgroup of finite index. By (A.1) again, we can assume that $A \trianglelefteq P$. If $Q \ge A$, then $[P : Q] < \infty$, and so $Q < N_P(Q)$ by case (i). So assume $Q \not\ge A$, choose $a \in A \smallsetminus Q$, and let $\hat{a}$ be its conjugacy class in P: a finite subset of A since P/A is finite and A is abelian. Then $\langle \hat{a} \rangle$ is a finite normal subgroup of P, so Q has finite index in $Q\langle \hat{a} \rangle$, and hence $Q < N_{Q\langle \hat{a} \rangle}(Q) \le N_P(Q)$ by case (i). ■

The following example shows why some conditions are needed to ensure that $N_P(Q) > Q$ for a pair of p-groups $P > Q$.

Example A.2. Define inductively a sequence of p-groups $P_0, P_1, P_2, \ldots$ with subgroups $A_n, Q_n \le P_n$ as follows:

$$\begin{aligned} P_0 &= C_p, & Q_0 &= 1 \le P_0, & A_0 &= P_0, \\ P_{n+1} &= P_n \wr C_p, & Q_{n+1} &= Q_n \wr C_p \le P_{n+1}, & A_{n+1} &= (A_n)^{\times p} \trianglelefteq P_{n+1}. \end{aligned}$$

Here, $(-)^{\times p}$ means the p-fold direct product. Thus for each n, A_n is elementary abelian of rank p^n, P_n is an $(n+1)$-times iterated wreath product $C_p \wr \cdots \wr C_p$, and $A_n Q_n = P_n$ and $A_n \cap Q_n = 1$.

Define injective homomorphisms $\varphi_n \colon P_n \to P_{n+1}$ by setting

$$\varphi_n(g) = \begin{cases} (g, 1, \ldots, 1) \in (P_n)^{\times p} \trianglelefteq P_{n+1} & \text{if } n \text{ is odd} \\ (g, g, \ldots, g) \in (P_n)^{\times p} \trianglelefteq P_{n+1} & \text{if } n \text{ is even.} \end{cases}$$

Set

$$P = \operatorname*{colim}_n (P_n, \varphi_n), \quad Q = \operatorname*{colim}_n (Q_n, \varphi_n|_{Q_n}), \quad \text{and} \quad A = \operatorname*{colim}_n (A_n, \varphi_n|_{A_n}).$$

Then P is a locally finite p-group, and $A, Q \le P$ have the following properties:

(a) $Z(P) = 1$ and P has no finite nontrivial normal subgroups;

(b) $[P, P] = P$ and P has no proper subgroups of finite index;

(c) $A \trianglelefteq P$, $A \cap Q = 1$, and $AQ = P$; and

(d) $N_P(Q) = Q$.

Proof. Let $\psi_n \colon P_n \to P$ be the natural map: an injective homomorphism since the φ_n are all injective. Set $P_n^* = \psi_n(P_n) < P$ for short. Thus $P_n^* \cong P_n$, and P is the union of the increasing sequence $P_0^* < P_1^* < P_2^* < \cdots$ of finite p-groups. In particular, P is a locally finite p-group.

(a) If $z \in Z(P)$, then $z \in P_n^*$ for some n, and so $z \in Z(P_m^*)$ for all $m \ge n$. Since $|Z(P_m)| = p$ for each m, and $Z(P_{m+1}) \cap \varphi_m(Z(P_m)) = 1$ if m is odd, we get that $Z(P) = 1$.

Assume $1 \ne N \trianglelefteq P$ is a finite nontrivial normal subgroup. Then $1 \ne Z(N) \trianglelefteq P$, and $\mathrm{Aut}_P(Z(N))$ is a finite p-group of automorphisms of $Z(N)$. So

$$1 \ne C_{Z(N)}\big(\mathrm{Aut}_P\,(Z(N))\big) \le Z(P),$$

which we just saw is impossible.

(b) When n is even, the subgroup $\varphi_n(P_n) < P_{n+1}$ is generated by elements of the form $(g, g, \dots, g)$ for $g \in P_n$ of order p. For such g, $(1, g, g^2, \dots, g^{p-1})$ is conjugate in P_{n+1} to $(g, g^2, g^3, \dots, g^p)$, so $(g, g, \dots, g) \in [P_{n+1}, P_{n+1}]$, and hence $P_n^* \le [P, P]$. Thus $P = [P, P]$.

If $R < P$ is a proper subgroup of finite index, then it contains a normal subgroup $N \trianglelefteq P$ of finite index by (A.1), which is impossible since that would imply $[P/N, P/N] = [P, P]/N = P/N$ when P/N is a nontrivial finite p-group.

(c) We have $A \cap Q = \bigcup_{n=0}^{\infty} \psi_n(A_n \cap Q_n) = 1$ and $AQ \ge \bigcup_{n=0}^{\infty} \psi_n(A_n Q_n) = P$. For each $g \in P$, let n be such that $g \in P_n^*$: then g normalizes $\psi_m(A_m)$ for each $m \ge n$ and hence g normalizes A.

(d) Set $B = N_P(Q) \cap A$. Then $[Q, B] \le Q \cap A = 1$, and hence $B \le C_P(AQ) = Z(P)$ since $P = AQ$ and $A \ge B$ is abelian. So $B = 1$ by (a).

Each $g \in N_P(Q)$ has the form $g = ah$ for $a \in A$ and $h \in Q$ by (b), and $a \in N_P(Q) \cap A = B$. So $N_P(Q) = QB = Q$. ∎

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