J. Noncommut. Geom. 20 (2026), 1155–1168
DOI 10.4171/JNCG/632

A note on traces for the Heisenberg calculus

Alexander Gorokhovsky and Erik van Erp

Abstract. We gave a local formula for the index of Heisenberg elliptic operators on contact manifolds in our earlier work (Gorokhovsky and van Erp, 2022). We constructed a cocycle in periodic cyclic cohomology which, when paired with the Connes–Chern character of the principal Heisenberg symbol, calculates the index. A crucial ingredient of our index formula was a new trace on the algebra of Heisenberg pseudodifferential operators. The construction of this trace was rather involved. In the present paper, we clarify the nature of this trace and thus simplify the index formula from our previous paper (Gorokhovsky and van Erp, 2022).

1. Introduction

Let $\mathfrak{g} = \mathbb{R}^3$ be the Lie algebra of the Heisenberg group with basis X, Y, Z and

$$[X,Z] = [Y,Z] = 0 \quad [X,Y] = Z.$$

Elements of $\mathfrak{g}$ can be identified with right-invariant vector fields on the Heisenberg group $G = \mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z} \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z} \quad Z = \frac{\partial}{\partial z}.$$

Here (x, y, z) are the standard coordinates on $\mathbb{R}^3$. The Heisenberg calculus is a pseudodifferential calculus that is built on the idea that, for certain purposes, it is natural to treat Z as a differential operator of degree 2. For example, in the classical calculus, the leading term of the second order differential operator

$$P = X^2 + Y^2 + i\gamma Z \quad \gamma \in \mathbb{C}$$

is the sublaplacian $X^2 + Y^2$. The principal symbol of P is not invertible, and P is not elliptic. But in the Heisenberg calculus, the term $i\gamma Z$ has order 2 and is part of the leading term. It turns out that the principal symbol of P is invertible in the Heisenberg calculus iff γ is not an odd integer. In that case, P has a parametrix (an inverse modulo smoothing operators), and, while P is not elliptic, it is still hypoelliptic.

The construction of the Heisenberg pseudodifferential calculus generalizes naturally to contact manifolds (see [1]), which can be locally identified with the Heisenberg group.

Mathematics Subject Classification 2020: 47G30 (primary); 58J40 (secondary).
Keywords: trace, pseudodifferential operators, homogeneous group, Heisenberg calculus.

On a compact contact manifold, an operator with invertible principal symbol in the Heisenberg calculus is a Fredholm operator. In [5], we gave a local formula for the index of Heisenberg elliptic operators on contact manifolds in terms of the principal symbol. Principal symbols of pseudodifferential operators in the Heisenberg calculus form a non-commutative algebra $\mathcal{S}_H$. An invertible symbol determines an element in the K-theory group $K_1(\mathcal{S}_H)$. The Connes–Chern character maps $K_1(\mathcal{S}_H)$ to periodic cyclic homology $HP_1(\mathcal{S}_H)$. In [5], we constructed a cocycle in periodic cyclic cohomology $HP^1(\mathcal{S}_H)$ which, when paired with the Connes–Chern character of the symbol, calculates the index.

A crucial ingredient of our index formula was a trace τ on the symbol algebra $\mathcal{S}_H$. The purpose of the present paper is to clarify the nature of this trace and thus to simplify the index formula of [5]. The construction of τ in [5] was rather involved. We show here how τ can be understood with reference to a natural class of traces that can be defined for a general homogeneous group.

A homogeneous group G is a nilpotent Lie group with underlying manifold $\mathbb{R}^m$ that has a graded Lie algebra. Let $\Psi^0_H(G)$ be the algebra of translation invariant order zero operators in the Heisenberg calculus associated to G (as, e.g., in [2]). An operator in $\Psi^0_H(G)$ is given by convolution (on the left) with a compactly supported distribution $k \in \mathcal{E}'(G)$ that is regular (i.e., k restricts to a smooth function on $G - \{0\}$) and that has an asymptotic expansion

$$k \sim k_0 + k_1 + k_2 + \cdots .$$

The principal term k_0 is homogeneous of degree $-Q$, where Q is the homogeneous dimension of G. In Section 2, we show that evaluation of the principal term k_0 at a central element $z \in G$, $z \neq 0$ defines a trace:

$$\tau_z : \Psi^0_H(G) \to \mathbb{C} \quad \tau_z(k) := k_0(z).$$

In Section 3, we briefly review our construction of the trace τ from [5], which is defined if G is the $2n+1$-dimensional Heisenberg group. The main result of this paper is Theorem 3.3, which expresses τ as a linear combination of traces defined in Section 2:

$$\tau = \frac{(2\pi)^{n+1}}{2(n!)i^{n+1}}(\tau_{(0,+1)} - (-1)^n \tau_{(0,-1)}). \tag{1.1}$$

Section 4 introduces a technical lemma about the Fourier transform of a homogeneous distribution. We use this lemma in Section 5 to prove Theorem 3.3.

Remark 1.1. In [5], we worked with traces on the Weyl algebra, rather than on the convolution algebra of the Heisenberg group. The connection between the two involves a Fourier transform. The constant in equation (1.1) depends on a choice of isomorphism between the convolution algebra and the algebra used in [5]. The relevant conventions are spelled out in Section 3.1.

2. Traces for homogeneous groups

2.1. Homogeneous groups

Let G be a nilpotent Lie group with underlying manifold $\mathbb{R}^n$, with graded Lie algebra $\mathfrak{g}$. This means that $\mathfrak{g}$ decomposes as a direct sum:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^r,$$

where $[\mathfrak{g}^j, \mathfrak{g}^k] \subseteq \mathfrak{g}^{j+k}$ if $j + k \leq r$, and $[\mathfrak{g}^j, \mathfrak{g}^k] = 0$ if $j + k > r$. Dilations $\delta_t : \mathfrak{g} \to \mathfrak{g}$ are defined by

$$\delta_t(X) = t^j X \quad \text{if } X \in \mathfrak{g}^j,\ t > 0.$$

The dilation δ_t is a Lie algebra automorphism. We identify $G = \mathfrak{g}$ via the exponential map. Then δ_t is a group automorphism.

If μ is a Haar measure on G, then $\mu \circ \delta_t = t^Q \mu$, where Q is the homogeneous dimension of G:

$$Q = \sum_{j=1}^{r} j \cdot \dim \mathfrak{g}^j.$$

We shall use the term “homogeneous” always in relation to the dilations δ_t. A function f on G is homogeneous of degree k if $f(\delta_t x) = t^k f(x)$. A distribution $u \in \mathcal{D}'(G)$ is homogeneous of degree k if $\langle u, \varphi \circ \delta_t \rangle = t^{-Q-k} \langle u, \varphi \rangle$.

Example 2.1. The Lie algebra of the Heisenberg group $G = \mathbb{R}^{2n+1}$ is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \quad \mathfrak{g}^1 = \mathbb{R}^{2n}, \quad \mathfrak{g}^2 = \mathbb{R}.$$

The Lie bracket is

$$[(x,t),(y,s)] = (0, \omega(x,y)) \quad x, y \in \mathbb{R}^{2n},\ t, s \in \mathbb{R},$$

where $\omega(x, y) = \sum_{i=1}^{n} x_i y_{n+i} - x_{n+i} y_i$ is the standard symplectic form on $\mathbb{R}^{2n}$. The group operation is

$$(x,t)(y,s) = \left(x + y, t + s + \frac{1}{2}\omega(x,y)\right) \quad x, y \in \mathbb{R}^{2n},\ t, s \in \mathbb{R}.$$

The dilations are $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$. The homogeneous dimension is $Q = 2n + 2$.

2.2. The algebra

For a homogeneous group G as above, let $\Psi^0_H(G)$ denote the algebra of right-invariant pseudodifferential operators of order zero in the generalized Heisenberg calculus on G (as defined, e.g., in [2,6]). An operator in $\Psi^0_H(G)$ is given by convolution (on the left) with a compactly supported distribution $k \in \mathcal{E}'(G)$ that is regular (i.e., smooth on $G - \{0\}$) and that has an asymptotic expansion

$$k \sim k_0 + k_1 + k_2 + \cdots,$$

where the principal term k_0 is homogeneous of degree $-Q$.

The algebra $\mathcal{S}_H$ of principal symbols is the quotient in the short exact sequence

$$0 \to \Psi_H^{-1}(G) \to \Psi_H^0(G) \to \mathcal{S}_H(G) \to 0.$$

Elements of $\mathcal{S}_H(G)$ can be represented by regular distributions on G that are homogeneous of degree $-Q$. Note that the Dirac delta distribution on G is homogeneous of degree $-Q$. The Dirac delta is the unit of the algebra $\mathcal{S}_H(G)$.

Convolution of two homogeneous distributions $f, g \in \mathcal{S}_H(G)$ is defined as follows. Let $\phi \in C_c^\infty(G)$ be such that $\phi = 1$ in a neighborhood of 0. Then $f * g$ is, by definition, the principal part (of homogeneous degree $-Q$) in the asymptotic expansion of $\phi f * \phi g$. The result does not depend on the choice of ϕ.

2.3. Traces

For a central element $z \in G$, evaluation at z defines a trace on the convolution algebra $C_c^\infty(G)$:

$$(f * g)(z) = \int_G f(zx^{-1})g(x)dx = \int_G g(x)f(x^{-1}z)dx = (g * f)(z).$$

This generalizes in various ways.

Lemma 2.2. *If z is a central element of G, and $u \in \mathcal{E}'(G)$, $f \in C^\infty(G)$, then*

$$(u * f)(z) = (f * u)(z).$$

Proof. This is immediate from the definitions:

$$(u * f)(z) := \langle u, \tilde{f}^z \rangle \quad \tilde{f}^z(y) = f(y^{-1}z)$$

and

$$(f * u)(z) := \langle u, \tilde{f}_z \rangle \quad \tilde{f}_z(y) = f(zy^{-1}). \qquad \blacksquare$$

For a regular distribution f, we can evaluate $f(z)$ as long as $z \neq 0$.

Lemma 2.3. *If $z \neq 0$ is a central element in G, and f, g are regular distributions with compact support on G, then*

$$(f * g)(z) = (g * f)(z).$$

Proof. Let U be a neighborhood of $0 \in G$ such that $z \notin U^2$. Let $\phi \in C_c^\infty(G)$ be such that $\phi = 1$ in a neighborhood of 0, and $\operatorname{supp} \phi \subset U$. Then

$$f * g = \phi f * \phi g + (1 - \phi) f * \phi g + \phi f * (1 - \phi) g + (1 - \phi) f * (1 - \phi) g.$$

Since $(1-\phi)f, (1-\phi)g \in C_c^\infty(G)$, we have

$$(1-\phi)f * (1-\phi)g(z) = (1-\phi)g * (1-\phi)f(z).$$

By Lemma 2.2,

$$(1-\phi)f * \phi g(z) = \phi g * (1-\phi)f(z), \quad \phi f * (1-\phi)g(z) = (1-\phi)g * \phi f(z).$$

Finally, since $\operatorname{supp} \phi f * \phi g$ and $\operatorname{supp} \phi g * \phi f$ are contained in U^2, both of them vanish at z. ■

Proposition 2.4. *If $z \neq 0$ is a central element in G, then*

$$\tau_z : \mathcal{S}_H(G) \to \mathbb{R} \quad \tau_z(f) := f(z)$$

is a trace on $\mathcal{S}_H(G)$.

Proof. By definition of the product in $\mathcal{S}_H(G)$,

$$(f * g)(z) = \lim_{t \downarrow 0} t^Q (\phi f * \phi g)(\delta_t z)$$

which, by Lemma 2.3, is equal to

$$(g * f)(z) = \lim_{t \downarrow 0} t^Q (\phi g * \phi f)(\delta_t z).$$ ■

3. The main result

In this section, we specialize to the Heisenberg group. Throughout this section, $G = \mathbb{R}^{2n+1}$ will denote the Heisenberg group, with notations as in Example 2.1.

3.1. Fourier transform

The Fourier transform is defined for Schwartz class functions $f \in \mathcal{S}(\mathbb{R}^{2n+1})$ by

$$\hat{f}(y,s) := \int f(x,t) e^{-i(xy+ts)} dx dt$$

and is then extended to tempered distributions $\mathcal{S}'(\mathbb{R}^{2n+1})$.

Given a compactly supported distribution $k \in \Psi_H^0(G)$, let $\hat{k} \in C^\infty(\mathbb{R}^{2n+1})$ be its Fourier transform. Define two functions $\sigma_\pm \in C^\infty(\mathbb{R}^{2n})$:

$$\sigma_+(x) := \lim_{\lambda \to +\infty} \hat{k}(\lambda x, \lambda^2) \quad \sigma_-(x) := \lim_{\lambda \to -\infty} \hat{k}(\lambda x, -\lambda^2) \quad x \in \mathbb{R}^{2n}.$$

Alternatively, in the asymptotic expansion

$$k \sim k_0 + k_1 + k_2 + \cdots,$$

the principal term k_0 is homogeneous of degree $-Q$ and represents an element in $\mathcal{S}_H(G)$. The Fourier transform $\widehat{k}_0$ is a smooth function on $\mathbb{R}^{2n+1} - \{0\}$ that is homogeneous of degree 0 for the dilations, that is,

$$\widehat{k}_0(\lambda x, \lambda^2 t) = \widehat{k}_0(x,t) \quad x \in \mathbb{R}^{2n},\ t \in \mathbb{R},\ \lambda > 0.$$

Then

$$\sigma_+(x) := \widehat{k}_0(x,+1) \quad \sigma_-(x) := \widehat{k}_0(x,-1).$$

In particular, if $k \in \Psi_H^{-1}(G)$, then $k_0 = 0$ and $\sigma_\pm = 0$.

If we wish to emphasize the dependence of $\sigma_\pm$ on k, we denote it as $\sigma_\pm^k$. The map $k \mapsto \sigma_+^k$ is an algebra homomorphism from the convolution algebra $\Psi^0(G)$ to the Weyl algebra of $\mathbb{R}^{2n}$:

$$\sigma_+^{k*h}(v) = (\sigma_+^k \# \sigma_+^h)(v) = \frac{1}{(2\pi)^{2n}} \iint e^{2i\,\omega(x,y)} \sigma_+^k(v+x)\sigma_+^h(v+y)\,dx\,dy. \tag{3.1}$$

The map $k \to \sigma_-^k$ is an anti-homomorphism, with $\sigma_-^{k*h} = \sigma_-^h \# \sigma_-^k$.

3.2. The old trace

Our construction of the trace in [5] was strongly influenced by the work of Epstein and Melrose in [3,4]. We summarize our construction here. For an in-depth treatment, see [5].

The smooth functions $\sigma_\pm$, defined above, have asymptotic expansions for large $\|x\| \to \infty$ of the form

$$\sigma_+(x) \sim \sum_{l=0}^{\infty} w_{2l}(x) \quad \sigma_-(x) \sim \sum_{l=0}^{\infty} (-1)^l w_{2l}(x), \tag{3.2}$$

where w_{2l} is smooth on $\mathbb{R}^{2n} - 0$, and homogeneous of degree $-2l$.

Definition 3.1. The trace $\tau : \mathcal{S}_H(G) \to \mathbb{C}$ is

$$\tau(k) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathcal{R}(x)\,dx \quad k \in \mathcal{S}_H(G),$$

where $\mathcal{R}$ is the function

$$\mathcal{R} := \sigma_+ - (-1)^n \sigma_- - \sum_{l=0}^{n-1} \varepsilon_l w_{2l} \quad \varepsilon_l = 1 - (-1)^{n+l} = \begin{cases} 0 & l+n \text{ even} \\ 2 & l+n \text{ odd} \end{cases}.$$

Note that $\mathcal{R}$ is an integrable smooth function defined on $\mathbb{R}^{2n} - \{0\}$.

For a proof that τ is a trace, see [5, Theorem 5.8 and Proposition 5.10].

Remark 3.2. If $\sigma_{\pm}$ are Schwartz class functions, then $w_{2l} = 0$ for all $l = 0, 1, 2, \ldots$, and simply

$$\tau(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \sigma_+(x)\,dx - (-1)^n \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \sigma_-(x)\,dx.$$

In this case, the integral

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \sigma_+(x)\,dx$$

is the usual trace of the trace class operator on $L^2(\mathbb{R}^n)$ determined by the distribution k_0 in the Schrödinger representation of the Heisenberg group.

Likewise,

$$(-1)^n \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \sigma_-(x)\,dx$$

is the trace of the trace class operator on $L^2(\mathbb{R}^n)$ determined by k in the dual of the Schrödinger representation. (For details, see [5, Section 5.3].)

The main result of this paper is that the trace τ can be expressed as a linear combination of traces defined in Section 2.3.

Theorem 3.3. *Let $G = \mathbb{R}^{2n+1}$ be the Heisenberg group. Let*

$$\tau_+ := \tau_{(0,+1)} \qquad \tau_- := \tau_{(0,-1)}$$

be the traces defined in Section 2.3, for the central elements $(0, \pm 1) \in G$. Then

$$\tau = \frac{(2\pi)^{n+1}}{2(n!)i^{n+1}} (\tau_+ - (-1)^n \tau_-).$$

4. A lemma about Fourier transforms

For an integrable function $g \in L^1(\mathbb{R}^{2n+1})$, denote the Fourier transform by

$$\mathcal{F}(g)(y,s) := \int g(x,t) e^{-i(xy+ts)} dx dt$$

with inverse Fourier transform

$$\mathcal{F}^*(g)(y,s) := \frac{1}{(2\pi)^{2n+1}} \int g(x,t) e^{i(xy+ts)} dx dt.$$

As usual, the Fourier transform $\mathcal{F}(T)$ of a tempered distribution T is defined by $\langle \mathcal{F}(T), \varphi \rangle = \langle T, \mathcal{F}(\varphi) \rangle$, $\varphi \in \mathcal{S}(\mathbb{R}^{2n+1})$, and similarly for $\mathcal{F}^*(T)$.

In this section, we prove the following lemma, which expresses the Fourier transform of a homogeneous distribution as a limit of the Fourier transforms of integrable functions.

Lemma 4.1. *Let f be a smooth function on $\mathbb{R}^{2n+1}-\{0\}$, homogeneous of degree* 0 *with respect to the dilations, that is,*

$$f(\lambda x,\lambda^2 t)=f(x,t)\quad x\in\mathbb{R}^{2n},\ t\in\mathbb{R},\ \lambda>0.$$

Then

$$\mathcal{F}^*(f)(0,s)=\lim_{\beta\downarrow 0}\lim_{\delta\downarrow 0}\mathcal{F}^*(f(x,t)e^{-\beta\|x\|^2-\delta|t|})(0,s).$$

Note that the function $\mathcal{F}^*(f)$ is smooth away from 0, and so are $\mathcal{F}^*(fe^{-\beta\|x\|^2})$ and $\mathcal{F}^*(fe^{-\beta\|x\|^2-\delta|t|})$ for $\delta,\beta>0$. The proof proceeds in two steps.

Lemma 4.2. *Let f be a smooth function on $\mathbb{R}^{2n+1}-\{0\}$, homogeneous of order* 0 *with respect to the dilations. Let $\beta>0$. For $s\neq 0$,*

$$\lim_{\delta\downarrow 0}\mathcal{F}^*(fe^{-\beta\|x\|^2-\delta|t|})(0,s)=\mathcal{F}^*(fe^{-\beta\|x\|^2})(0,s).$$

Proof. Choose a smooth function $\chi\in C^\infty(\mathbb{R})$ with

$$\chi(t)=\begin{cases}0 & \text{if } |t|\leq 1,\\ 1 & \text{if } |t|\geq 2.\end{cases}$$

Then $(1-\chi(t))f(x,t)e^{-\beta\|x\|^2}$ is in $L^1(\mathbb{R}^{2n+1})$. By the dominated convergence theorem,

$$\lim_{\delta\downarrow 0}\mathcal{F}^*((1-\chi)fe^{-\beta\|x\|^2-\delta|t|})(0,s)=\mathcal{F}^*((1-\chi)fe^{-\beta\|x\|^2})(0,s).\tag{4.1}$$

Assume $\beta>0$, $\delta>0$. Then $\chi(t)f(x,t)e^{-\beta\|x\|^2-\delta|t|}$ is a Schwartz class function. Therefore,

$$\begin{aligned}\mathcal{F}^*(\chi fe^{-\beta\|x\|^2-\delta|t|})&=\frac{1}{(-is)^k}\mathcal{F}^*(\partial_t^k(\chi fe^{-\beta\|x\|^2-\delta|t|}))\\ &=\frac{1}{(-is)^k}\sum_{p=0}^{k}\binom{k}{p}\mathcal{F}^*(\partial_t^p(\chi f)\partial_t^{k-p}e^{-\beta\|x\|^2-\delta|t|}).\end{aligned}\tag{4.2}$$

Note that, while $e^{-\beta\|x\|^2-\delta|t|}$ is not smooth at $t\neq 0$, the above formula makes sense because $\partial_t^p(\chi f)$ is zero for $t\in[-1,1]$.

Since differentiating in t reduces the degree of homogeneity of f, for each $p=0,1,2,\ldots$, there exists C such that for all $(x,t)\in\mathbb{R}^{2n+1}$,

$$|\partial_t^p(\chi(x,t)f(x,t))|\leq C|t|^{-p}.$$

Also,

$$|\partial_t^{k-p}e^{-\beta\|x\|^2-\delta|t|}|\leq\delta^{k-p}e^{-\beta\|x\|^2-\delta|t|},$$

and so

$$|\partial_t^p(\chi f)\partial_t^{k-p}e^{-\beta\|x\|^2-\delta|t|}| \leq C\delta^{k-p}|t|^{-p}e^{-\beta\|x\|^2-\delta|t|}.$$

With $\beta > 0$ fixed, we obtain (with different constant C)

$$\|\partial_t^p(\chi f)\partial_t^{k-p}e^{-\beta\|x\|^2-\delta|t|}\|_{L^1} \leq C\delta^{k-p}\int_{\|t\|>1}|t|^{-p}e^{-\delta|t|}dt.$$

For small $\delta > 0$,

$$\int_{|t|\geq 1}|t|^{-p}e^{-\delta|t|}dt = \begin{cases} \mathcal{O}(\delta^{-1}) & \text{if } p = 0,\\ \mathcal{O}(\log\delta) & \text{if } p = 1,\\ \mathcal{O}(1) & \text{if } p \geq 2,\end{cases}$$

and so

$$\|\partial_t^p(\chi f)\partial_t^{k-p}e^{-\beta\|x\|^2-\delta|t|}\|_{L^1} = \begin{cases} \mathcal{O}(\delta^{k-1}) & \text{if } p = 0,\\ \mathcal{O}(\delta^{k-1}\log\delta) & \text{if } p = 1,\\ \mathcal{O}(\delta^{k-p}) & \text{if } p \geq 2.\end{cases}$$

We now assume that $k \geq 2$. Then all these L^1-norms converge to 0 if $\delta \downarrow 0$, except when $p = k$. Since $\|\mathcal{F}^*(g)\|_\infty \leq \|g\|_{L^1}$, equation (4.2) implies

$$\lim_{\delta\downarrow 0}\mathcal{F}^*(\chi f e^{-\beta\|x\|^2-\delta|t|})(0,s) = \lim_{\delta\downarrow 0}\frac{1}{(-is)^k}\mathcal{F}^*(\partial_t^k(\chi f)\, e^{-\beta\|x\|^2-\delta|t|})(0,s).$$

Since $\partial_t^k(\chi f)e^{-\beta\|x\|^2}$ is in L^1, dominated convergence gives

$$\begin{aligned}\lim_{\delta\downarrow 0}\mathcal{F}^*(\partial_t^k(\chi f)e^{-\beta\|x\|^2-\delta|t|})(0,s) &= \mathcal{F}^*(\partial_t^k(\chi f)e^{-\beta\|x\|^2})(0,s)\\ &= \mathcal{F}^*(\partial_t^k(\chi f e^{-\beta\|x\|^2}))(0,s)\\ &= (-is)^k\mathcal{F}^*(\chi f e^{-\beta\|x\|^2})(0,s).\end{aligned}$$

In other words,

$$\lim_{\delta\downarrow 0}\mathcal{F}^*(\chi f e^{-\beta\|x\|^2-\delta|t|})(0,s) = \mathcal{F}^*(\chi f e^{-\beta\|x\|^2})(0,s).$$

Combined with (4.1), this completes the proof. ■

Lemma 4.3. *Let f be a smooth function on $\mathbb{R}^{2n+1} - \{0\}$, homogeneous of degree 0 with respect to the dilations. For $s \neq 0$,*

$$\lim_{\beta\downarrow 0}\mathcal{F}^*(f e^{-\beta\|x\|^2})(0,s) = \mathcal{F}^*(f)(0,s).$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{2n+1})$ be a compactly supported smooth function such that $\varphi(x,t) = 1$ in a neighborhood of 0. Then φf is integrable, and so

$$\lim_{\beta\downarrow 0}\mathcal{F}(\varphi f e^{-\beta\|x\|^2})(0,s) = \mathcal{F}(\varphi f)(0,s).$$

We also have

$$\mathcal{F}^*((1-\varphi)fe^{-\beta\|x\|^2}) = \frac{1}{(-is)^k}\mathcal{F}^*(\partial_t^k((1-\varphi)f)e^{-\beta\|x\|^2}),$$

where $\partial_t^k((1-\varphi)f)$ is integrable if k is sufficiently large, and hence

$$\begin{aligned}\lim_{\beta\downarrow 0}\mathcal{F}^*(\partial_t^k((1-\varphi)f)e^{-\beta\|x\|^2})(0,s) &= \mathcal{F}^*(\partial_t^k((1-\varphi)f))(0,s)\\ &= (-is)^k\mathcal{F}^*((1-\varphi)f)(0,s)\end{aligned}$$

The statement follows. ■

5. Proof of the main result

Throughout this section, $G = \mathbb{R}^{2n+1}$ is the Heisenberg group, with notations as in Example 2.1.

The trace $\tau_z(k)$ is, by definition, evaluation of the principal part k_0 of k at a central element $z \in G$, $z \neq 0$. The definition of the trace $\tau(k)$ is expressed in terms of the principal Heisenberg symbol (σ_+, σ_-), which involves the Fourier transform of k. In order to relate τ and τ_z, we use Lemma 4.1 to express $\tau_z(k)$ in terms of (σ_+, σ_-).

From there, the proof of Theorem 3.3 consists of a series of calculations. These calculations result in Proposition 5.5, which expresses τ_z as a linear combination of τ and a well-known residue trace (see (5.2)). Theorem 3.3 follows immediately from Proposition 5.5.

For readability, we have formalized steps in these calculations as lemmas.

Lemma 5.1. *Let $k \in \Psi_H^0(G)$, and let $z = (0,s) \in G$ be a central element with $s \neq 0$. Then*

$$\tau_z(k) = \frac{n!}{(2\pi)^{2n+1}}\lim_{\beta\downarrow 0}\int_{\mathbb{R}^{2n}}\left(\frac{\sigma_+(x)}{(\beta\|x\|^2 - is)^{n+1}} + \frac{\sigma_-(x)}{(\beta\|x\|^2 + is)^{n+1}}\right)dx.$$

Proof. Let $\delta > 0$. Denote $f = \widehat{k_0}$. For $t > 0$, we have $f(x,t) = \sigma_+(x/\sqrt{t})$; for $t < 0$, we have $f(x,t) = \sigma_-(x/\sqrt{-t})$. Change of variables gives

$$\begin{aligned}&(2\pi)^{2n+1}\mathcal{F}^*(fe^{-\beta\|x\|^2-\delta|t|})(0,s)\\ &\quad= \int fe^{-\beta\|x\|^2-\delta|t|+ist}\,dt\,dx\\ &\quad= \int_{\mathbb{R}^{2n}}\int_0^\infty \sigma_+(x/\sqrt{t})e^{-\beta\|x\|^2-\delta t+ist}\,dt\,dx\\ &\qquad+ \int_{\mathbb{R}^{2n}}\int_{-\infty}^0 \sigma_-(x/\sqrt{-t})e^{-\beta\|x\|^2+\delta t+ist}\,dt\,dx\end{aligned}$$

$$= \int_{\mathbb{R}^{2n}} \sigma_+(x) \int_0^\infty t^n e^{-(\beta\|x\|^2+\delta-is)t}\,dt\,dx$$
$$+ \int_{\mathbb{R}^{2n}} \sigma_-(x) \int_0^\infty t^n e^{-(\beta\|x\|^2+\delta+is)t}\,dt\,dx.$$

Using the Gamma integral

$$\int_0^\infty t^n e^{-ct}\,dt = \frac{1}{c^{n+1}} \int_0^\infty u^n e^{-u}\,du = \frac{\Gamma(n+1)}{c^{n+1}} = \frac{n!}{c^{n+1}},$$

we obtain

$$\mathcal{F}^*(f e^{-\beta\|x\|^2-\delta|t|})(0,s)$$
$$= \frac{n!}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n}} \left(\frac{\sigma_+(x)}{(\beta\|x\|^2+\delta-is)^{n+1}} + \frac{\sigma_-(x)}{(\beta\|x\|^2+\delta+is)^{n+1}} \right) dx.$$

The statement now follows from Lemma 4.1. ■

With $k \in \Psi_H^0(G)$ as above, and the asymptotic expansions of $\sigma_\pm$ as in (3.2), define

$$\tilde{\sigma}_+ := \begin{cases} \sigma_+ - \sum_{l=0}^{n} w_{2l} & \text{if } \|x\| \geq 1, \\ \sigma_+ - \sum_{l=0}^{n-1} w_{2l} & \text{if } \|x\| < 1, \end{cases}$$

and

$$\tilde{\sigma}_- := \begin{cases} \sigma_- - \sum_{l=0}^{n} (-1)^l w_{2l} & \text{if } \|x\| \geq 1, \\ \sigma_- - \sum_{l=0}^{n-1} (-1)^l w_{2l} & \text{if } \|x\| < 1. \end{cases}$$

Note that $\tilde{\sigma}_\pm$ are integrable functions on $\mathbb{R}^{2n} - \{0\}$.

Lemma 5.2. *With notation as above,*

$$\frac{n!}{(2\pi)^{2n+1}} \lim_{\beta\downarrow 0} \int_{\mathbb{R}^{2n}} \left(\frac{\tilde{\sigma}_+(x)}{(\beta\|x\|^2-is)^{n+1}} + \frac{\tilde{\sigma}_-(x)}{(\beta\|x\|^2+is)^{n+1}} \right) dx$$
$$= \frac{n!}{(2\pi)^{n+1}(-is)^{n+1}} \tau(k).$$

Proof. Since $\tilde{\sigma}_\pm$ are integrable, the left-hand side is simply

$$\frac{n!}{(-is)^{n+1}(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n}} (\tilde{\sigma}_+(x) + (-1)^{n+1}\tilde{\sigma}_-(x))\,dx.$$

This is equal to the right-hand side, since

$$\mathcal{R} = \widetilde{\sigma}_+ + (-1)^{n+1}\widetilde{\sigma}_-,$$

where $\mathcal{R}$ is as in Definition 3.1. ■

We get

$$\tau_z(k) = \frac{n!}{(2\pi)^{n+1}(-is)^{n+1}}\tau(k) + \sum_{l=0}^{n} I_l, \tag{5.1}$$

where, for $l = 0, 1, \ldots, n-1$,

$$I_l = \frac{n!}{(2\pi)^{2n+1}} \lim_{\beta \downarrow 0} \int_{\mathbb{R}^{2n}} \left(\frac{w_{2l}(x)}{(\beta\|x\|^2 - is)^{n+1}} + \frac{(-1)^l w_{2l}(x)}{(\beta\|x\|^2 + is)^{n+1}} \right) dx.$$

The formula for I_n is the same (with $l = n$), except that the domain of integration is $\|x\| \geq 1$.

Lemma 5.3. *For $l = 0, 1, \ldots, n-1$, we have $I_l = 0$.*

Proof. Introduce spherical coordinates

$$r := \|x\| \geq 0 \quad \theta := \frac{x}{r} \in S^{2n-1} \quad x \in \mathbb{R}^{2n},$$

and let

$$w_{2l}(x) = a_{2l}(x) r^{-2l} \quad a_{2l} \in C^\infty(S^{2n-1}).$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \left(\frac{a_{2l}(\theta) r^{-2l}}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l a_{2l}(\theta) r^{-2l}}{(\beta r^2 + is)^{n+1}} \right) dx \\ &\quad = \int_{S^{2n-1}} a_{2l}(\theta) d\theta \int_0^\infty \left(\frac{r^{-2l}}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l r^{-2l}}{(\beta r^2 + is)^{n+1}} \right) r^{2n-1} dr \end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \left(\frac{r^{-2l}}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l r^{-2l}}{(\beta r^2 + is)^{n+1}} \right) r^{2n-1} dr \\ &\quad = \frac{1}{2} \int_0^\infty \left(\frac{z^{-l+n-1}}{(\beta z - is)^{n+1}} + \frac{(-1)^l z^{-l+n-1}}{(\beta z + is)^{n+1}} \right) dz \\ &\quad = \frac{1}{2} \int_{-\infty}^\infty \frac{z^{-l+n-1}}{(\beta z - is)^{n+1}} dz. \end{aligned}$$

We used a change of variables $z \mapsto -z$ for the second summand.

Contour integration over an interval $[-R, R]$ and a semicircle of radius R in the upper half-plane if $s < 0$, or the lower half-plane if $s > 0$, shows that

$$\int_{-\infty}^\infty \frac{z^{-l+n-1}}{(\beta z - is)^{n+1}} dz = 0.$$ ■

With $a_{2l} \in C^\infty(S^{2n-1})$ as above, define

$$\operatorname{Res}(k) = -\frac{1}{2(2\pi)^n} \int_{S^{2n-1}} a_{2n}(\theta) d\theta. \tag{5.2}$$

This is a trace on $\mathcal{S}_H(G)$ (see [5]).

Lemma 5.4. *The value of I_n is*

$$I_n = \operatorname{sign}(s) \frac{n!}{(2\pi)^{n+1}(-is)^{n+1}} \pi i \operatorname{Res}(k).$$

Proof. To calculate I_n, we use spherical coordinates, as above. We have

$$\begin{aligned}&\int_{r\geq 1} \left(\frac{a_{2n}(\theta) r^{-2n}}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^n a_{2n}(\theta) r^{-2n}}{(\beta r^2 + is)^{n+1}} \right) r^{2n-1} dr d\theta \\ &\quad = \int_{S^{2n-1}} a_{2n}(\theta) d\theta \int_1^\infty \left(\frac{1}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l}{(\beta r^2 + is)^{n+1}} \right) r^{-1} dr.\end{aligned}$$

Thus

$$I_n = -\frac{2(n!)}{(2\pi)^{n+1}} \operatorname{Res}(k) \cdot \lim_{\beta\downarrow 0} \int_1^\infty \left(\frac{1}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l}{(\beta r^2 + is)^{n+1}} \right) r^{-1} dr.$$

As before,

$$\begin{aligned}&\int_1^\infty \left(\frac{1}{(\beta r^2 - is)^{n+1}} + \frac{(-1)^l}{(\beta r^2 + is)^{n+1}} \right) r^{-1} dr \\ &\quad = \frac{1}{2} \int_{z\in\mathbb{R}, |z|\geq 1} \frac{z^{-1}}{(\beta z - is)^{n+1}} dz = \frac{1}{2} \int_{z\in\mathbb{R}, |z|\geq \beta} \frac{z^{-1}}{(z - is)^{n+1}} dz.\end{aligned} \tag{5.3}$$

A standard application of the residue theorem shows that

$$\lim_{\beta\downarrow 0} \int_{z\in\mathbb{R}, |z|\geq\beta} \frac{z^{-1}}{(z + is)^{n+1}} dz = \begin{cases} -\dfrac{\pi i}{(-is)^{n+1}} & \text{if } s > 0, \\ \dfrac{\pi i}{(-is)^{n+1}} & \text{if } s < 0. \end{cases}$$

This proves the lemma. ■

Proposition 5.5. *Let $z = (0, s) \in G$ be a central element with $s \neq 0$. Then*

$$\frac{(2\pi)^{n+1}(-is)^{n+1}}{n!} \tau_z = \begin{cases} \tau + \pi i \operatorname{Res} & \text{if } s > 0, \\ \tau - \pi i \operatorname{Res} & \text{if } s < 0. \end{cases}$$

Proof. Combine (5.1) with Lemmas 5.3 and 5.4. ■

Acknowledgments. We thank the anonymous referee for carefully reading the paper and making useful suggestions and corrections. The first author is grateful for the hospitality and support of the Max-Planck Institute for Mathematics in Bonn where part of this work was carried out.

References

[1] R. Beals and P. Greiner, *Calculus on Heisenberg manifolds*. Ann. of Math. Stud. 119, Princeton University Press, Princeton, NJ, 1988 Zbl 0654.58033 MR 0953082

[2] M. Christ, D. Geller, P. Głowacki, and L. Polin, Pseudodifferential operators on groups with dilations. *Duke Math. J.* **68** (1992), no. 1, 31–65 Zbl 0764.35120 MR 1185817

[3] C. Epstein and R. Melrose, Contact degree and the index of Fourier integral operators. *Math. Res. Lett.* **5** (1998), no. 3, 363–381 Zbl 0929.58012 MR 1637844

[4] C. Epstein and R. Melrose, The Heisenberg algebra, index theory and homology. Unpublished notes. 2000, https://math.mit.edu/ rbm/book.html visited on 17 July 2025

[5] A. Gorokhovsky and E. van Erp, The Heisenberg calculus, index theory and cyclic cohomology. *Adv. Math.* **399** (2022), article no. 108229, 62 pp. Zbl 1489.58008 MR 4383012

[6] A. Melin, Parametrix constructions for right invariant differential operators on nilpotent groups. *Ann. Global Anal. Geom.* **1** (1983), no. 1, 79–130 Zbl 0524.58044 MR 0739894

Received 17 October 2024; revised 18 February 2025.

Alexander Gorokhovsky
Department of Mathematics, University of Colorado, Boulder, Campus Box 395, Boulder,
CO 80309, USA; gorokhov@colorado.edu

Erik van Erp
Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA;
erikvanerp@dartmouth.edu, jhamvanerp@gmail.com