

Derivations and the first Hochschild cohomology group of the quantum grassmannian

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Abstract. We calculate the derivations and the first Hochschild cohomology group of the quantum grassmannian over a field of characteristic zero in the generic case when the deformation parameter is not a root of unity. Using graded techniques and two special homogeneous normal elements of the quantum grassmannian, we reduce the problem to computing derivations of the quantum grassmannian that act trivially on these two normal elements. We then use the dehomogenisation equality which shows that a localisation of the quantum grassmannian is equal to a skew Laurent extension of quantum matrices. This equality is used to connect derivations of the quantum grassmannian with those of quantum matrices. More precisely, again using graded techniques, we show that derivations of the quantum grassmannian that act trivially on our two normal elements restrict to homogeneous derivations of quantum matrices. The derivations of quantum matrices are known in the square case, and technical details needed to deal with the general case are given in the appendix. This allows us to explicitly describe the first Hochschild cohomology group of the quantum grassmannian.

1. Introduction

Let K denote a field of characteristic zero and $q \in K$ be a nonzero element which is not a root of unity. The quantum grassmannian $\mathcal{O}_q(G(k, n))$ is a noncommutative algebra that is a deformation of the homogeneous coordinate ring of the classical grassmannian of k -planes in n -space. In this paper, we calculate the derivations and the first Hochschild cohomology group of the quantum grassmannian.

One of the motivations for this paper is [7] where a connection between the totally nonnegative grassmannian – or rather its cell decomposition into the union of so-called positroid cells – and Hochschild (co)homology of $\mathcal{O}_q(G(k, n))$ was established. More precisely, a link between the volume form of positroids and Hochschild (co)homology of the quantum grassmannian $\mathcal{O}_q(G(k, n))$ is established, allowing the contribution of a positroid cell to the scattering amplitude (in the $N = 4$ supersymmetric Yang–Mills theory) to be q -deformed. The present paper is the first one in a project to compute the Hochschild (co)homology of quantum positroid varieties. More precisely, the quantum grassmannian can be viewed as a “quantum positroid variety” (corresponding to the totally positive

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grassmannian) thanks to [5], and we compute the first Hochschild cohomology group of the quantum grassmannian. We will come back to the general case in future work.

Assume $k \leq n$. Recall that $\mathcal{O}_q(G(k, n))$ is the subalgebra of the quantum matrix algebra $\mathcal{O}_q(M(k, n))$ that is generated by the $k \times k$ quantum minors of $\mathcal{O}_q(M(k, n))$. We call these $k \times k$ quantum minors quantum Plücker coordinates of $\mathcal{O}_q(G(k, n))$. They are indexed by k -subsets of $\{1, \dots, n\}$ and denoted by $[I]$ for each k -subset I . The algebra $\mathcal{O}_q(G(k, n))$ is graded with all quantum Plücker coordinates homogeneous of degree one.

Quantum Plücker coordinates formed on consecutive columns play a special role: they are normal, and each one generates a completely prime ideal of $\mathcal{O}_q(G(k, n))$. They are referred to as prime quantum Plücker coordinates. Among the prime quantum Plücker coordinates, two are special: $[u] = [1, \dots, k]$ and $[w] = [n - k + 1, \dots, n]$. The reason for this is that $[u]$ (respectively, $[w]$) q -commutes with every quantum Plücker coordinate $[I]$ with a nonnegative (respectively, nonpositive) power of q , that is,

$$[u][I] = q^{\geq 0}[I][u] \quad \text{and} \quad [w][I] = q^{\leq 0}[I][w].$$

We exploit this observation and other graded techniques in order to reduce the problem of computing derivations of $\mathcal{O}_q(G(k, n))$ to computing derivations D of $\mathcal{O}_q(G(k, n))$ that act trivially on both $[u]$ and $[w]$, that is, derivations D such that $D([u]) = D([w]) = 0$.

We then employ the dehomogenisation equality which shows that a localisation of the quantum grassmannian is equal to a skew Laurent extension of quantum matrices. This equality is used to connect the set of derivations of the quantum grassmannian acting trivially on $[u]$ and $[w]$ with that of quantum matrices. More precisely, again using graded techniques, we show that derivations of the quantum grassmannian that act trivially on $[u]$ and $[w]$ restrict to homogeneous derivations of quantum matrices. To conclude, we use our knowledge of derivations of quantum matrices: the set of derivations is known in the case of square quantum matrices; see [3]. Some technical details that are needed to deal with the fact that the non-square case has not yet been covered are given in the appendix.

This allows us to show that the first Hochschild cohomology group of $\mathcal{O}_q(G(k, n))$ is a K -vector space of dimension n , with basis given by the cosets of the derivations $D_1, \dots, D_n$, where D_i is the derivation of $\mathcal{O}_q(G(k, n))$ defined by

$$D_i([I]) = \begin{cases} [I] & \text{if } i \in I; \\ 0 & \text{otherwise.} \end{cases}$$

2. Basic definitions

Throughout the paper, K denotes a field of characteristic zero and $q \in K$ is a nonzero element which is not a root of unity.

The algebra of $m \times n$ quantum matrices over K , denoted by $\mathcal{O}_q(M(m, n))$, is the algebra generated over K by mn indeterminates x_{ij} , with $1 \leq i \leq m$ and $1 \leq j \leq n$,

which commute with the elements of K and are subject to the relations

$$\begin{aligned} x_{ij}x_{il} &= qx_{il}x_{ij} && \text{for } 1 \leq i \leq m \text{ and } 1 \leq j < l \leq n; \\ x_{ij}x_{kj} &= qx_{kj}x_{ij} && \text{for } 1 \leq i < k \leq m \text{ and } 1 \leq j \leq n; \\ x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for } 1 \leq k < i \leq m \text{ and } 1 \leq j < l \leq n; \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj} && \text{for } 1 \leq i < k \leq m \text{ and } 1 \leq j < l \leq n. \end{aligned}$$

It is well known that $\mathcal{O}_q(M(m, n))$ is an iterated Ore extension over K with the x_{ij} added in lexicographic order. An immediate consequence is that $\mathcal{O}_q(M(m, n))$ is a noetherian domain.

When $m = n$, the *quantum determinant* D_q is defined by

$$D_q := \sum (-q)^{l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where the sum is over all permutations σ of $\{1, \dots, n\}$ and where $l(\sigma)$ denotes the length of the permutation σ .

The quantum determinant is a central element in the algebra of quantum matrices $\mathcal{O}_q(M(n, n))$.

If I and J are t -element subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively, then the *quantum minor* $[I \mid J]$ is defined to be the quantum determinant of the $t \times t$ quantum matrix subalgebra generated by the variables x_{ij} with $i \in I$ and $j \in J$.

The quantum matrix algebra $\mathcal{O}_q(M(m, n))$ is a connected $\mathbb{N}$ -graded algebra with each generator x_{ij} given degree one. Note that each $t \times t$ quantum minor has degree t .

Definition 2.1. Let $k \leq n$. The *homogeneous coordinate ring of the $k \times n$ quantum grassmannian*, $\mathcal{O}_q(G(k, n))$ (known informally as the *quantum grassmannian*), is the subalgebra of the quantum matrix algebra $\mathcal{O}_q(M(k, n))$ that is generated by the $k \times k$ quantum minors of $\mathcal{O}_q(M(k, n))$; see, for example, [2].

A $k \times k$ quantum minor of $\mathcal{O}_q(M(k, n))$ must use all of the k rows, and so we can specify the quantum minor by specifying the columns that define it. With this in mind, we will write $[J]$ for the quantum minor $[1, \dots, k \mid J]$ for any k -element subset J of $\{1, \dots, n\}$. Quantum minors of this type are called *quantum Plücker coordinates*. The quantum grassmannian $\mathcal{O}_q(G(k, n))$ is a connected $\mathbb{N}$ -graded algebra with each quantum Plücker coordinate given degree one. The set of quantum Plücker coordinates in $\mathcal{O}_q(G(k, n))$ is denoted by Π . There is a natural partial order on Π defined in the following way: if $I = [i_1 < \dots < i_k]$ and $J = [j_1 < \dots < j_k]$, then $[I] < [J]$ if and only if $i_l \leq j_l$ for each $l = 1, \dots, k$. A *standard monomial* in the quantum Plücker coordinates is an expression of the form $[I_1][I_2] \cdots [I_t]$, where $I_1 \leq I_2 \leq \dots \leq I_t$ in this partial order. The set of all standard monomials forms a vector space basis of $\mathcal{O}_q(G(k, n))$ over K ; see, for example, [2, Corollary 2.1]. We will refer to the process of rewriting a monomial in terms of standard monomials as *straightening*.

The quantum grassmannian $\mathcal{O}_q(G(1, n))$ is a quantum affine space, and, as such, its derivations are known; see [1, Corollaire 1.3.3]; so we will assume throughout this paper that $k > 1$. As $\mathcal{O}_q(G(n-1, n)) \cong \mathcal{O}_q(G(1, n))$, by [4, Proposition 3.4], we exclude the case $k = n-1$ as well. Thus, we will assume throughout the paper that $2 \leq k \leq n-2$, and so $n \geq 4$.

Also, it is shown in [4, Proposition 3.4] that $\mathcal{O}_q(G(k, n)) \cong \mathcal{O}_q(G(n-k, n))$, when $2k \leq n$. In Section 7, we need to assume that $2k \leq n$, and the general result is obtained from this case in Subsection 8.2 by using the isomorphism just mentioned.

3. Derivations for $\mathcal{O}_q(G(k, n))$ inherited from $\mathcal{O}_q(M(k, n))$

We use the following notation for the delta truth function: if P is a proposition, then $\delta(P) = 1$ if P is true, and $\delta(P) = 0$ if P is false. As is traditional, we write δ_{ij} for $\delta(i = j)$.

Recall from [3] that for each column of $\mathcal{O}_q(M(k, n))$ there is a derivation that is the identity on any quantum minor that contains that column and is zero on the other quantum minors. These derivations restrict to the subalgebra $\mathcal{O}_q(G(k, n))$ to give us n derivations which we denote by D_i for $i = 1, \dots, n$. We refer to these derivations as the *column derivations* of $\mathcal{O}_q(G(k, n))$.

Lemma 3.1. *For each $i = 1, \dots, n$, there is a derivation D_i whose action on quantum Plücker coordinates is given by $D_i([I]) = \delta(i \in I)[I]$.*

Proof. This is immediate from the definition of D_i as the restriction to $\mathcal{O}_q(G(k, n))$ of a column derivation of $\mathcal{O}_q(M(k, n))$. ■

Corollary 3.2. *We have*

$$\frac{1}{k} \left(\sum_{i=1}^n D_i \right) ([I]) = [I]$$

for each quantum Plücker coordinate $[I]$ in $\mathcal{O}_q(G(k, n))$.

Proof. This follows immediately from the previous lemma, as there are k occurrences of D_i for which $D_i([I]) = [I]$, and otherwise $D_i([I]) = 0$. ■

Our main aim in this paper is to prove the following conjecture.

Conjecture 3.3. *Let K be a field of characteristic zero, and let q be a nonzero element of K that is not a root of unity. Then every derivation of $\mathcal{O}_q(G(k, n))$ can be written as a linear combination of inner derivations and column derivations. Furthermore, the column derivations are linearly independent modulo the space generated by the inner derivations.*

As a consequence of the truth of this conjecture, we identify the first Hochschild cohomology group of the quantum grassmannian.

4. The dehomogenisation equality for $\mathcal{O}_q(G(k, n))$

In this section, we recall results concerning the dehomogenisation equality that occur in [4, Section 3]. We refer the reader to that paper for detailed definitions and proofs.

Set $u = \{1, \dots, k\}$. Then $[u]$ commutes with all other quantum Plücker coordinates up to a power of q , by [4, Lemma 3.1]. As $\mathcal{O}_q(G(k, n))$ is generated by the quantum Plücker coordinates, it follows that the element $[u]$ is a normal element, and so we may invert $[u]$ to obtain the overring $T := \mathcal{O}_q(G(k, n))[u]^{-1}$.

For $1 \leq i \leq k$ and $1 \leq j \leq n - k$, set

$$x_{ij} := [1, \dots, \widehat{k+1-i}, \dots, k, j+k][u]^{-1} \in T.$$

Set $p := n - k$. The elements x_{ij} generate a subalgebra of T that is a quantum matrix algebra $\mathcal{O}_q(M(k, p))$. In fact, T is generated over $\mathcal{O}_q(M(k, p))$ by $[u]^{\pm 1}$, so

$$T = \mathcal{O}_q(G(k, n))[u]^{-1} = \mathcal{O}_q(M(k, p))[u]^{\pm 1}; \sigma,$$

where σ is the automorphism of $\mathcal{O}_q(M(k, p))$ defined by $\sigma(x_{ij}) := qx_{ij}$ (since $[u]x_{ij} = qx_{ij}[u]$ for all i, j).

When we operate on the right-hand side of this equality, we will write y for $[u]$. Thus, $yx_{ij} = \sigma(x_{ij})y$ and

$$T = \mathcal{O}_q(G(k, n))[u]^{-1} = \mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma]. \quad (4.1)$$

We refer to this equality as the *dehomogenisation equality*. We will make extensive use of the dehomogenisation equality to transfer derivations from $\mathcal{O}_q(M(k, p))$ to $\mathcal{O}_q(G(k, n))$ and vice versa.

Note that [4, Lemma 3.2] gives the formula for general quantum minors of the algebra $\mathcal{O}_q(M(k, p))$ when viewed as elements in $\mathcal{O}_q(G(k, n))[u]^{-1}$, and each quantum Plücker coordinate, multiplied by $[u]^{-1}$, occurs in the formula. The formula is

$$[I | J] = [\{1, \dots, k\} \setminus (k+1-I) \sqcup (k+J)][u]^{-1}.$$

In the reverse direction, for a quantum Plücker coordinate $[L]$ of $\mathcal{O}_q(G(k, n))$, we have

$$[L] = [L_{\leq k} \sqcup L_{> k}] = [I | J][u],$$

where $I = \{(k+1) - (\{1, \dots, k\} \setminus L_{\leq k})\}$ and $J = L_{> k} - k$ with $L_{\leq k} := L \cap \{1, \dots, k\}$ and $L_{> k} := L \cap \{k+1, \dots, n\}$.

5. Derivations arising via dehomogenisation

The n column derivations D_i of $\mathcal{O}_q(G(k, n))$ that we have defined in Section 3 satisfy $D_i([u]) = [u]$ for $1 \leq i \leq k$ and $D_i([u]) = 0$ for $k < i \leq n$, so they extend to n derivations $\widetilde{D}_i$ of $T = R[y, y^{-1}; \sigma] = \mathcal{O}_q(G(k, n))[u]^{-1}$ with $\widetilde{D}_i(y) = y$ for $1 \leq i \leq k$ and $\widetilde{D}_i(y) = 0$ for $k < i \leq n$.

We will show that these derivations $\widetilde{D}_i$ of T coincide with extensions of known derivations of $\mathcal{O}_q(M(k, p))$ which we now recall from [3].

For $1 \leq i \leq k$, there are derivations D_{i*} of $\mathcal{O}_q(M(k, p))$ defined by $D_{i*}(x_{rs}) := \delta_{ir}x_{rs}$, and for $1 \leq j \leq p$, there are derivations D_{*j} of $\mathcal{O}_q(M(k, p))$ defined by $D_{*j}(x_{rs}) := \delta_{js}x_{rs}$. In other words, D_{i*} fixes row i and kills all the other rows of $\mathcal{O}_q(M(k, p))$, while D_{*j} fixes column j and kills all other columns of $\mathcal{O}_q(M(k, p))$. For these observations, see the comment after [3, Corollary 2.8].

Note that the Leibniz formula for derivations shows that $D_{i*}([I | J]) = \delta(i \in I)[I | J]$ and that $D_{*j}([I | J]) = \delta(j \in J)[I | J]$ for any quantum minor $[I | J]$ of $\mathcal{O}_q(M(k, p))$. The derivations that we have just defined are not linearly independent: it is easy to check that $\sum_{i=1}^k D_{i*} = \sum_{j=1}^p D_{*j}$.

We will extend these derivations of $\mathcal{O}_q(M(k, p))$ to derivations $\widetilde{D}_{i*}$ and $\widetilde{D}_{*j}$ of $T = \mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma] = \mathcal{O}_q(G(k, n))[u]^{-1}$ as follows. For each $i = 1, \dots, k$, set $\widetilde{D}_{i*}(y) = y$ and note that by applying the Leibniz formula for derivatives to $1 = yy^{-1}$ we obtain $\widetilde{D}_{i*}(y^{-1}) = -y^{-1}$. For each $j = 1, \dots, p$, set $\widetilde{D}_{*j}(y) = \widetilde{D}_{*j}(y^{-1}) = 0$. These choices for the action on y are necessary to obtain Corollary 5.2.

In the next proposition, we calculate the effect of the derivations $\widetilde{D}_{i*}$ and $\widetilde{D}_{*j}$ on quantum Plücker coordinates in $\mathcal{O}_q(G(k, n))$.

Proposition 5.1. *Let $[I]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$, and let $1 \leq i \leq k$ and $1 \leq j \leq n - k$. Then*

- (i) $\widetilde{D}_{i*}([I]) = -\delta(k + 1 - i \in I)[I]$,
- (ii) $\widetilde{D}_{*j}([I]) = \delta(k + j \in I)[I]$.

Proof. Write $I = I_{\leq k} \sqcup I_{> k}$. Then, with $A = (k + 1) - (\{1, \dots, k\} \setminus I_{\leq k})$ and $B = I_{> k} - k$, we see that $[I] = [A | B]y$.

In case (i),

$$\begin{aligned} \widetilde{D}_{i*}([I]) &= \widetilde{D}_{i*}([A | B]y) \\ &= D_{i*}([A | B])y + [A | B]D_{i*}(y) \\ &= \delta(i \in A)[A | B]y - [A | B]y \\ &= (\delta(i \in A) - 1)[I]. \end{aligned}$$

Notice that $A = (k + 1) - (\{1, \dots, k\} \setminus I_{\leq k}) = (k + 1) - (\{1, \dots, k\} \setminus I)$. Hence, $i \in A$ if and only if $k + 1 - i \notin I$. It follows that $\delta(i \in A) + \delta(k + 1 - i \in I) = 1$. Hence, $\delta(i \in A) - 1 = -\delta(k + 1 - i \in I)$, and the result follows.

In case (ii),

$$\begin{aligned} \widetilde{D}_{*j}([I]) &= \widetilde{D}_{*j}([A | B]y) \\ &= \widetilde{D}_{*j}([A | B])y + [A | B]\widetilde{D}_{*j}(y) \\ &= D_{*j}([A | B])y + 0 \\ &= \delta(j \in B)[I]. \end{aligned}$$

Note that $B = I_{>k} - k$. Hence, $j \in B = I_{>k} - k$ if and only if $k + j \in I$, and the result follows. ■

Corollary 5.2. *We have*

- (i) $\widetilde{D}_{i*} = -\widetilde{D_{k+1-i}}$ for $1 \leq i \leq k$,
- (ii) $\widetilde{D}_{*j} = \widetilde{D_{k+j}}$ for $1 \leq j \leq n - k$.

Example 5.3. For $\mathcal{O}_q(G(2, 4))$, we have $\widetilde{D}_{1*} = -\widetilde{D}_2$, $\widetilde{D}_{2*} = -\widetilde{D}_1$ and $\widetilde{D}_{*1} = \widetilde{D}_3$, $\widetilde{D}_{*2} = \widetilde{D}_4$.

Remark 5.4. Note, for later use, that

$$\begin{aligned} & (\widetilde{D}_1 + \cdots + \widetilde{D}_k + \widetilde{D_{k+1}} + \cdots + \widetilde{D_{k+p}})|_{\mathcal{O}_q(M(k,p))} \\ & = D_{k+1} + \cdots + D_{k+p} - (D_k + \cdots + D_1) = 0, \end{aligned}$$

while $(\widetilde{D}_1 + \cdots + \widetilde{D}_k + \widetilde{D_{k+1}} + \cdots + \widetilde{D_{k+p}})(y) = ky$.

6. Adjusting derivations

In cohomology calculations, we can adjust our original derivation by adding or subtracting derivations that arise as inner derivations. In this section, we consider what can happen when we do this or when adjusting by column derivations.

Let $[u] = [1, \dots, k]$ and $[w] = [n - k + 1, \dots, n]$ denote the leftmost and rightmost quantum Plücker coordinates of $\mathcal{O}_q(G(k, n))$, respectively. For any quantum Plücker coordinate $[I]$, set $d(I) := \#(I \setminus (I \cap u))$ and $e(I) := \#(I \setminus (I \cap w))$. This notation is fixed for the rest of the paper. We will use the commutation relations for $[u]$ and $[w]$ with other quantum Plücker coordinates that are described in the following lemma without comment throughout the paper.

Lemma 6.1. *We have*

- (i) $[u][I] = q^{d(I)}[I][u]$,
- (ii) $[w][I] = q^{-e(I)}[I][w]$.

Proof. (i) is proved in [4, Lemma 3.1] and (ii) is proved in [2, Lemma 1.5]. ■

If $S = [I_1] \cdots [I_m]$ is a standard monomial, then set $d(S) := \sum d(I_i)$ and note that each $d(I_i) \geq 0$ with $d(I_i) = 0$ if and only if $[I_i] = [u]$. Then, $S[u] = q^{d(S)}[u]S$, and so $Su = uS$ if and only if $d(S) = 0$ (in which case $S = [u]^m$).

In any case, note that $[u]S$ is a standard monomial, as $[u]$ is the unique minimal quantum Plücker coordinate.

Recall from Section 2 that $\mathcal{O}_q(G(k, n))$ is a graded algebra with each quantum Plücker coordinate having degree one.

Lemma 6.2. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$. Suppose that $D([I]) = b_0 + \cdots + b_t$ is the homogeneous decomposition of $D([I])$. Then $b_0 = 0$.*

Proof. Suppose that $[I] \neq [u]$, and suppose that $D([u]) = a_0 + \cdots + a_s$ is the homogeneous decomposition of $D([u])$. Let $d = d([I])$ and note that $d > 0$ so that $q^d \neq 1$. By applying D to the equation $[u][I] = q^d [I][u]$, we obtain

$$D([u])[I] + [u]D([I]) = q^d D([I])[u] + q^d [I]D([u]).$$

Examination of the terms in degree one in this equation reveals that

$$a_0[I] + b_0[u] = q^d b_0[u] + q^d a_0[I],$$

from which it follows that $a_0 = q^d a_0$ and $b_0 = q^d b_0$. Then $a_0 = b_0 = 0$, as $q^d \neq 1$. ■

The following lemma is deduced from results in [5]. In the proof of the lemma, we use the notation developed in that paper without further explanation.

Lemma 6.3. *Let $v = \{1, \dots, k-1, k+1\}$, and let $[\alpha] > [v]$ in the standard order on quantum Plücker coordinates. Set $t := |\alpha \setminus (v \cap \alpha)|$ and note that $t > 0$. Then $[v][\alpha] = q^t [\alpha][v]$ modulo $\langle u \rangle$.*

Proof. There is an isomorphism

$$\Psi : \frac{\mathcal{O}_q(G(k, n))}{\langle u \rangle} [[v]^{-1}] \longrightarrow \mathcal{O}_{q^{-1}}(Y_\lambda)[Y^{\pm 1}; \sigma],$$

where $\sigma(x_{ij}) = qx_{ij}$ for the generators x_{ij} of the partition subalgebra Y_λ ; see [5, Theorem 9.17]. If $[I | J]_{q^{-1}}$ is a $t \times t$ pseudo-quantum minor in Y_λ , then we have the following commutation relation: $Y[I | J]_{q^{-1}} = q^t [I | J]_{q^{-1}} Y$.

Let $\alpha \in \Pi \setminus \{u, v\}$, and set $t := |\alpha \setminus v|$. Then

$$\Psi(\bar{\alpha} \bar{v}^{-1}) = \beta [I | J]_{q^{-1}}$$

for some $\beta \in K$, and $t \times t$ pseudo-quantum minor $[I | J]_{q^{-1}}$, by [5, Theorem 9.17].

Hence,

$$\begin{aligned} \Psi(\bar{v} \bar{\alpha}) &= \Psi(\bar{v}) \Psi(\bar{\alpha} \bar{v}^{-1}) \Psi(\bar{v}) \\ &= Y \beta [I | J]_{q^{-1}} Y \\ &= q^t \beta [I | J]_{q^{-1}} Y^2 \\ &= q^t \Psi(\bar{\alpha} \bar{v}^{-1}) \Psi(\bar{v}^2) \\ &= q^t \Psi(\bar{\alpha} \bar{v}), \end{aligned}$$

and so $\bar{v} \bar{\alpha} = q^t \bar{\alpha} \bar{v}$, as required. ■

Definition 6.4. Suppose that $a = \sum_{i=1}^s a_i S_i$ is an expression for an element $a \in \mathcal{O}_q(G(k, n))$ in terms of standard monomials S_i and that each a_i is nonzero. Then the support of a , written $\text{Supp}(a)$, is defined to be $\{S_i \mid i = 1, \dots, s\}$.

Lemma 6.5. *Let D be a derivation on $\mathcal{O}_q(G(k, n))$. Then*

- (i) *there are no standard monomials of the form $[I_1]^{a_1}[I_2]^{a_2} \cdots [I_m]^{a_m}$, with $I_1 \neq u$ and $\sum a_i > 0$, occurring in $\text{Supp}(D[u])$,*
- (ii) *there are no standard monomials of the form $[I_1]^{a_1}[I_2]^{a_2} \cdots [I_m]^{a_m}$, with $I_m \neq w$ and $\sum a_i > 0$, occurring in $\text{Supp}(D[w])$.*

Proof. (i) The proof is by contradiction, so suppose that such a standard monomial exists. Without loss of generality, we may assume that $I_1 = v = \{1, \dots, k-1, k+1\}$ as we are allowing the possibility that some $a_i = 0$. As u and v differ only in one value, we know that $[u][v] = q[v][u]$.

Apply D to the equation $[u][v] = q[v][u]$ to obtain

$$D([u])[v] + [u]D([v]) = qD([v])[u] + q[v]D([u]).$$

Suppose that $D([u]) = \sum_i \alpha_i S_i + \sum_i \alpha'_i S'_i$ where $\alpha_i \in K$ and the S_i, S'_i are standard monomials such that $[u]$ does not occur in the S_i , but does occur in the S'_i . Also, suppose that $D([v]) = \sum_j \beta_j T_j$ where $\beta_j \in K$ and the T_j are standard monomials. Then

$$\begin{aligned} \sum_i \alpha_i S_i[v] + \sum_i \alpha'_i S'_i[v] + \sum_j \beta_j [u]T_j \\ = \sum_j q\beta_j T_j[u] + \sum_i q\alpha_i [v]S_i + \sum_i q\alpha'_i [v]S'_i. \end{aligned}$$

Now, $S_i[v] = q^{-t_i}[v]S_i$ modulo $\langle u \rangle$ for some $t_i \geq 0$, by Lemma 6.3 (we allow zero as it might be that S_i is a power of $[v]$).

The image of the above equation in $\mathcal{O}_q(G(k, n))/\langle [u] \rangle$ gives

$$\sum_i q^{-t_i} \alpha_i [v] \overline{S_i} = \sum_i q\alpha_i [v] \overline{S_i}$$

or

$$\sum_i (q^{-t_i} - q)\alpha_i [v] \overline{S_i} = 0.$$

Now, $\mathcal{O}_q(G(k, n))/\langle [u] \rangle$ is the quantum Schubert variety determined by the quantum Plücker coordinate $[v]$ (see [6] for the definition of quantum Schubert varieties of the quantum grassmannian). As such, $\mathcal{O}_q(G(k, n))/\langle [u] \rangle$ has a basis consisting of the images of the standard monomials in $\mathcal{O}_q(G(k, n))$ that do not involve $[u]$; see [6, Example 2.1.3]. Consequently, each $(q^{-t_i} - q)\alpha_i$ in the equation above is equal to zero. As at least one α_i is nonzero, this gives $q^{-t_i} - q = 0$, for that i , which is a contradiction as q is not a root of unity and $t_i \geq 0$.

(ii) Follows in a similar fashion. ■

The next corollary follows immediately from the previous lemma.

Corollary 6.6. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$. Suppose that the homogeneous decomposition of $D([u])$ is given by $D([u]) = a_1 + \cdots + a_s$ and that $D([w]) = b_1 + \cdots + b_t$ is the homogeneous decomposition of $D([w])$. Then a_1 is a scalar multiple of $[u]$ and b_1 is a scalar multiple of $[w]$.*

In order to prove Conjecture 3.3, we will start by showing that we can adjust an arbitrary derivation D by adding or subtracting derivations coming from the derivations mentioned in the conjecture (column derivations and inner derivations), so that the adjusted derivation, which we will continue to denote by D , satisfies $D([u]) = D([w]) = 0$. This will enable us to transfer the study of D into a quantum matrix problem by using the dehomogenisation equality. We show that we can make this adjustment in a number of steps that demonstrate how to remove standard monomials that occur in $D([u])$ (and $D([w])$).

As a result of Lemma 6.5, any standard monomial that occurs in the support of $D([u])$ must start with $[u]$. Similarly, any standard monomial that occurs in the support of $D([w])$ must finish with $[w]$.

The next lemma shows that, by adjusting D by suitable inner derivations, we can remove terms of $D([u])$ that are not of the form $\alpha[u]^a$ for values of $a \geq 1$.

Lemma 6.7. *Suppose that $S = [u]^a[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}$ is a standard monomial with $[v] \leq [I_1]$ and $\sum_i b_i > 0$, while $a \geq 1$. Let D be a derivation of $\mathcal{O}_q(G(k, n))$, and suppose that S occurs in the support of $D([u])$ with nonzero scalar coefficient α . Set $d := d(S) = b_1 d(I_1) + \cdots + b_m d(I_m)$, and set $z := \frac{\alpha}{q^{-d}-1}[u]^{a-1}[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}$. Also, set $D' := D - \text{ad}_z$. Then*

$$\text{Supp}(D'([u])) = \text{Supp}(D([u])) \setminus \{S\}.$$

Proof. Note that $d > 0$, so $q^{-d} - 1 \neq 0$. We calculate

$$\begin{aligned} \text{ad}_z([u]) &= zu - uz \\ &= \frac{\alpha}{q^{-d}-1}([u]^{a-1}[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}[u] - [u]^a[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}) \\ &= \frac{\alpha}{q^{-d}-1}(q^{-d}[u]^a[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m} - [u]^a[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}) \\ &= \frac{\alpha}{q^{-d}-1}((q^{-d}-1)[u]^a[I_1]^{b_1}[I_2]^{b_2} \cdots [I_m]^{b_m}) \\ &= \alpha S. \end{aligned}$$

It follows that $\text{Supp}(D'([u])) = \text{Supp}(D([u])) \setminus \{S\}$, as required. ■

Corollary 6.8. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$. Then there is a derivation D' such that $D' - D$ is a sum of inner derivations and such that the homogeneous terms of $D'([u])$ are of the form $\lambda_a[u]^a$ for $a \geq 1$ and $\lambda_a \in K$.*

Proof. Lemma 6.5 shows that $\text{Supp}(D([u]))$ has no terms whose standard monomials do not begin with $[u]$. By using Lemma 6.7 an appropriate number of times, we can remove terms in $\text{Supp}(D([u]))$ that involve $[u]$ and at least one other quantum Plücker coordinate by adjusting with suitable inner derivations. What remains is a derivation D' whose support only involves terms of the form $[u]^a$. ■

Our next task is to show that we can adjust further, if necessary, to see that we can reduce to $a = 1$ being the only possibility.

Lemma 6.9. *Suppose that D is a derivation of $\mathcal{O}_q(G(k, n))$ such that $[u]^{a-1}[w]$ occurs in $\text{Supp}(D([w]))$ with $a > 1$. Then there is a derivation D' of $\mathcal{O}_q(G(k, n))$ such that $D' - D$ is inner and $D'([u]) = D([u])$ while $\text{Supp}(D'([w])) = \text{Supp}(D([w])) \setminus [u]^{a-1}[w]$.*

Proof. Suppose that $[u]^{a-1}[w]$ occurs in $\text{Supp}(D([w]))$, say with nonzero coefficient β . Now, $\text{ad}_{[u]^{a-1}}([w]) = [u]^{a-1}[w] - [w][u]^{a-1} = (1 - q^{-d(w)(a-1)})[u]^{a-1}[w]$, and note that $1 - q^{-d(w)(a-1)} \neq 0$, as both $d(w)$ and $a - 1$ are nonzero and q is not a root of unity. Set $D' := D - \beta(1 - q^{-d(w)(a-1)})^{-1} \text{ad}_{[u]^{a-1}}$. Then $D'([u]) = D([u])$, as $\text{ad}_{[u]^{a-1}}([u]) = 0$. Also, $D'([w]) = D([w]) - \beta(1 - q^{-d(w)(a-1)})^{-1} \text{ad}_{[u]^{a-1}}([w]) = D([w]) - \beta[u]^{a-1}[w]$, so that $\text{Supp}(D'([w])) = \text{Supp}(D([w])) \setminus [u]^{a-1}[w]$, as required. ■

The following corollary now follows by applying the previous lemma an appropriate number of times.

Corollary 6.10. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$. Then there is a derivation D' of $\mathcal{O}_q(G(k, n))$ such that $D' - D$ is a sum of inner derivations, $D'([u]) = D([u])$ and such that there are no terms of the form $[u]^{a-1}[w]$ with $a > 1$ occurring in $\text{Supp}(D([w]))$.*

Lemma 6.11. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$ such that $D([u]) = \sum \lambda_i [u]^i$ for some $\lambda_i \in K$, and suppose that no terms of the form $[u]^{a-1}[w]$ with $a > 1$ occur in $\text{Supp}(D([w]))$. Then $D([u]) = \lambda_1 [u]$.*

Proof. Suppose that $D([w]) = \sum \beta_i S_i$ for some standard monomials S_i and $0 \neq \beta_i \in K$, and note that there is no S_i such that $S_i = [u]^{a-1}[w]$ for any $a > 1$, by assumption. Apply D to the equation $[u][w] = q^{d(w)}[w][u]$ to obtain

$$D([u])[w] + [u]D([w]) = q^{d(w)}D([w])[u] + q^{d(w)}[w]D([u]).$$

Hence,

$$\sum \lambda_i [u]^i [w] + \sum \beta_i [u] S_i = \sum q^{d(w)} \beta_i S_i [u] + \sum q^{d(w)} \lambda_i [w] [u]^i$$

and so

$$\sum \lambda_i [u]^i [w] + \sum \beta_i [u] S_i = \sum q^{d(w)-d(S_i)} \beta_i [u] S_i + \sum q^{d(w)-i \cdot d(w)} \lambda_i [u]^i [w].$$

The terms in this equation are all scalar multiples of standard monomials. Consider the occurrences of the standard monomial $[u]^a[w]$ for a given $a > 1$. If $[u]S_i = [u]^a[w]$, then $S_i = [u]^{a-1}[w]$, which does not occur, by assumption. Hence, the second term on the left side of this equation and the first term on the right side do not contain $[u]^a[w]$. It follows that $\lambda_a[u]^a[w] = q^{(1-a)d(w)}\lambda_a[u]^a[w]$, and this forces $\lambda_a(1 - q^{(1-a)d(w)}) = 0$. Now, $(1 - a)d(w) \neq 0$, as $a \neq 1$ so $(1 - q^{(1-a)d(w)}) \neq 0$. Hence, $\lambda_a = 0$. As this is true for all $a > 1$, we obtain the required result. ■

Recall from Lemma 3.1 that the column derivation D_1 is acting on a quantum Plücker coordinate $[I]$ by $D_1([I]) = \delta(1 \in I)[I]$.

Corollary 6.12. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$ such that $D([u]) = \lambda[u]$ for some $\lambda \in K$, and suppose that no terms of the form $[u]^{a-1}[w]$ with $a > 1$ occur in $\text{Supp}(D([w]))$. Then there is a derivation D' of D such that the following hold:*

- (i) $D'([u]) = 0$.
- (ii) $D' - D = \lambda D_1$.
- (iii) *There are no terms of the form $[u]^{a-1}[w]$ with $a > 1$ that occur in $\text{Supp}(D'([w]))$.*

Proof. We start by observing that $D_1([u]) = D_1([1, \dots, k]) = [u]$ and $D_1([w]) = D_1([n - k + 1, \dots, n]) = 0$, as $1 < n - k + 1$. Set $D' = D - \lambda D_1$, so that $D'([u]) = D([u]) - \lambda D_1([u]) = \lambda[u] - \lambda[u] = 0$. Also, $D'([w]) = D([w]) - \lambda D_1([w]) = D([w])$, so no terms of the form $[u]^{a-1}[w]$ with $a > 1$ occur in $\text{Supp}(D([w]))$. ■

Lemma 6.13. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$ with the following properties:*

- (i) $D([u]) = 0$.
- (ii) $\text{Supp}(D([w]))$ contains no term of the form $[u]^{a-1}[w]$ for $a > 1$.

Then $D([w]) = \alpha[w]$ for some $\alpha \in K$.

Proof. Recall, from Lemma 6.5, that $[w]$ must occur in any standard monomial contained in $\text{Supp}(D([w]))$. Suppose that $S = [u]^a[I_1]^{a_1} \dots [I_m]^{a_m}[w]^b \in \text{Supp}(D([w]))$ with $u < I_1$ and $I_m < w$, while $a, a_i \geq 0, b > 0$. Suppose that S occurs in $D([w])$ with nonzero coefficient $\alpha \in K$. Note that $d(I_i) > 0$ for each i . Apply D to the equation $[u][w] = q^{d(w)}[w][u]$, remembering that $D([u]) = 0$, to obtain

$$[u]D([w]) = q^{d(w)}D([w])[u].$$

Examination of the standard monomials of the form $[u]S$ in this equation reveals that

$$\begin{aligned} & \alpha[u]^{a+1}[I_1]^{a_1} \dots [I_m]^{a_m}[w]^b \\ &= \alpha q^{d(w)}[u]^a[I_1]^{a_1} \dots [I_m]^{a_m}[w]^b[u] \\ &= \alpha q^{d(w) - (d(w)b + a_1 d(I_1) + \dots + a_m d(I_m))}[u]^{a+1}[I_1]^{a_1} \dots [I_m]^{a_m}[w]^b. \end{aligned}$$

As q is not a root of unity, the only possibility is that the power of q on the right-hand side is q^0 , and this is only possible for $b = 1$ and $a_1 = \cdots = a_m = 0$. Thus, the only possible terms in $\text{Supp}(D[w])$ are of the form $[u]^a[w]$. Taking into account condition (ii) in the statement of the lemma, we see that $a > 0$ is not allowed, so $D[w] = \alpha[w]$, as required. ■

Recall from Lemma 3.1 that there are column derivations D_i for $i = 1, \dots, n$ such that $D_i([I]) = \delta(i \in I)[I]$ for each quantum Plücker coordinate $[I]$. The results of this section are summarised in the following proposition.

Proposition 6.14. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$. Then there is a derivation D' of $\mathcal{O}_q(G(k, n))$ with $D'([u]) = D'([w]) = 0$ and such that $(D' - D)$ is a linear combination of derivations of the form ad_z , with $z \in \mathcal{O}_q(G(k, n))$, and column derivations D_i for $i = 1, \dots, n$.*

Proof. We know the following facts hold for any derivation D of $\mathcal{O}_q(G(k, n))$ and so will hold for the any derivation that occurs when we adjust a given derivation by adding or subtracting inner derivations and scalar multiples of the D_i : (i) the degree zero parts of $D([u])$ and $D([w])$ are both zero (see Lemma 6.2); (ii) the degree one part of $D([u])$ is a scalar multiple of $[u]$ and, similarly, the degree one part of $D([w])$ is a scalar multiple of $[w]$ (see Corollary 6.6); (iii) any standard monomial occurring in the support of $D([u])$ must start with at least one occurrence of $[u]$ and, similarly, any standard monomial occurring in the support of $D([w])$ must end with at least one occurrence of $[w]$ (see Lemma 6.5).

Let D be an arbitrary derivation of $\mathcal{O}_q(G(k, n))$. By Corollary 6.8, there is a derivation $D^{(1)}$ of $\mathcal{O}_q(G(k, n))$ such that $D^{(1)} - D$ is a sum of inner derivations and such that the homogeneous terms of $D^{(1)}([u])$ are of the form $\lambda_a[u]^a$ for $a \geq 1$ and $\alpha_a \in K$.

By Corollary 6.10, there is a derivation $D^{(2)}$ of $\mathcal{O}_q(G(k, n))$ such that $D^{(2)} - D^{(1)}$ is a sum of inner derivations; the homogeneous terms of $D^{(2)}([u]) = D^{(1)}([u])$ are of the form $\lambda_a[u]^a$ for $a \geq 1$ and $\alpha_a \in K$, and there are no terms of the form $[u]^{a-1}[w]$ with $a > 1$ occurring in $\text{Supp}(D^{(2)}([w]))$. By Lemma 6.11, $D^{(2)}([u]) = \lambda[u]$ for some $\lambda \in K$.

Set $D^{(3)} := D^{(2)} - \lambda D_1$. Then $D^{(3)}([u]) = 0$, while

$$D^{(3)}([w]) = D^{(2)}([w]) - \lambda D_1([w]) = D^{(2)}([w]),$$

as $D_1([w]) = 0$. Hence, there are no terms of the form $[u]^{a-1}[w]$ with $a > 1$ in $D^{(3)}([w])$. It follows from Lemma 6.13 that $D^{(3)}([w]) = \alpha[w]$ for some $\alpha \in K$.

Finally, set $D' := D^{(3)} - \alpha D_n$. Then $D'([u]) = D^{(3)}([u]) = 0$, as $D_n([u]) = 0$ and $D'([w]) = D^{(3)}([w]) - \alpha D_n([w]) = 0$. The passage from D to D' via $D^{(1)}, D^{(2)}, D^{(3)}$ only involves adjustments by adding or subtracting derivations of the form ad_z , with $z \in \mathcal{O}_q(G(k, n))$, and D_i for $i = 1, \dots, n$ at each stage, so the required result follows. ■

7. Transferring derivations of $\mathcal{O}_q(G(k, n))$ to $\mathcal{O}_q(M(k, p))$

Throughout this section, we assume that $2k \leq n$.

Recall the dehomogenisation equality from Section 4

$$T = \mathcal{O}_q(G(k, n))[u]^{-1} = \mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma].$$

Given a derivation D of $\mathcal{O}_q(G(k, n))$ with $D([u]) = D([w]) = 0$, we may extend D to T by setting $D([u]^{-1}) = 0$ and then transfer to $\mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma]$ via the dehomogenisation equality. We then know that $D(y) = D([u]) = 0$. We retain the notation D for this extension to T .

Recall that in Section 4 we set $R := \mathcal{O}_q(M(k, p))$ where $p = n - k$. The quantum matrix generators x_{ij} of R are defined in Section 4

$$x_{ij} := [1, \dots, \widehat{k+1-i}, \dots, k, j+k][u]^{-1} \in T.$$

Our aim in this section is to show that $D(R) \subseteq R$ for such a derivation D . We will use a pair of gradings of T that were developed in [4, Section 6] to discuss a similar result for certain automorphisms of $\mathcal{O}_q(G(k, n))$.

As $2k \leq n$, we know that $k \leq n - k = p$, and so R has at least k columns and the quantum minor $[I \mid J] := [1 \cdots k \mid p+1-k, \dots, p] \in R$ is defined (we are using all the rows of $R = \mathcal{O}_q(M(k, p))$ and the last k columns). As noted in [4, Lemma 6.1], $x_{ij}[I \mid J] = q[I \mid J]x_{ij}$ when $j < p+1-k$ while $x_{ij}[I \mid J] = [I \mid J]x_{ij}$ when $j \geq p+1-k$. As a consequence, $[I \mid J]$ is a normal element in R and also in T .

Lemma 7.1. *Let D be a derivation of $\mathcal{O}_q(G(k, n))$, where $2k \leq n$, with $D([u]) = D([w]) = 0$. Let $[I \mid J]$ be defined as in the previous paragraph. Then $D([I \mid J]) = 0$.*

Proof. The discussion at the end of Section 4 shows that

$$[I \mid J] = [p+1 \cdots n][1 \cdots k]^{-1} = [w][u]^{-1},$$

and so $D([I \mid J]) = D([w][u]^{-1}) = 0$. ■

Also, we can calculate how $[I \mid J]$ commutes with $[u] = [1 \cdots k]$. Note that $k < n - k + 1 = p + 1$, as $2k \leq n$. Thus, the index sets $\{1, \dots, k\}$ and $\{p+1, \dots, n\}$ do not overlap, and

$$[u][I \mid J] = [u][w][u]^{-1} = q^k[w][u][u]^{-1} = q^k[w][u]^{-1}[u] = q^k[I \mid J][u],$$

where the second equality comes from Lemma 6.1.

The two gradings that were used in [4, Section 6] are defined by considering how elements of T commute with $y = [u]$ and with $[I \mid J]$.

We set $T_i := \{a \in T \mid y a y^{-1} = q^i a\}$ and $T^{(i)} := \{a \in T \mid [I \mid J] a [I \mid J]^{-1} = q^{-i} a\}$.

Lemma 7.2. *We have*

- (i) $T = \bigoplus_{i=1}^{\infty} T_i$,
- (ii) $T = \bigoplus_{i \in \mathbb{Z}} T^{(i)}$,
- (iii) $(T^{(0)} \cup T^{(1)}) \cap T_1 \subseteq \mathcal{O}_q(M(k, p)) = R$.

Proof. These results are established in [4, Lemmas 6.2 (i), 6.3 (i) and 6.4]. ■

Theorem 7.3. *Suppose that $2k \leq n$ and that D is a derivation of $\mathcal{O}_q(G(k, n))$ such that $D([u]) = D([w]) = 0$. Then $D(R) \subseteq R$.*

Proof. It is enough to show that $D(x_{ij}) \in R$ for each generator x_{ij} of R .

Note that $x_{ij} \in (T^{(0)} \cup T^{(1)}) \cap T_1$. This claim follows from the commutation rules given in [4, Lemma 6.1] and the fact that $yx_{ij} = qx_{ij}y$.

Let $a \in T_1$. Then $ya = qay$. Apply D to this equation, noting that $D(y) = 0$, to obtain $yD(a) = qD(a)y$, so that $D(T_1) \subseteq T_1$. Similar calculations, using the fact that $D([I \mid J]) = 0$ show that $D(T^{(0)}) \subseteq T^{(0)}$ and $D(T^{(1)}) \subseteq T^{(1)}$. It follows that $D(x_{ij}) \in D((T^{(0)} \cup T^{(1)}) \cap T_1) \subseteq (T^{(0)} \cup T^{(1)}) \cap T_1 \subseteq R$.

As D takes each generator x_{ij} of R into R , we see that $D(R) \subseteq R$, as required. ■

8. The main theorem

Recall that we are assuming that K is a field of characteristic zero and that $q \in K$ is a nonzero element that is not a root of unity. We are also assuming that $k \neq 1$, as in this case the quantum grassmannian is a quantum affine space, where the results are known; see [1]. As $\mathcal{O}_q(G(n-1, n)) \cong \mathcal{O}_q(G(1, n))$, we exclude this case as well. Thus, we are assuming that $2 \leq k \leq n-2$.

In this section, we prove Conjecture 3.3. The proof proceeds by first analysing the case where $2k \leq n$. The general case is then obtained by using the isomorphism $\mathcal{O}_q(G(k, n)) \cong \mathcal{O}_q(G(n-k, n))$. In order to avoid breaking the flow of the main result, we relegate to the appendix a discussion concerning homogeneous derivations on non-square quantum matrices that we use in obtaining the truth of the conjecture in the case where $2k < n$.

In this section, in order to make reading easier, we will use δ to denote an arbitrary derivative.

8.1. The case where $2k \leq n$

In this subsection, we consider $\mathcal{O}_q(G(k, n))$ in the case that $2k \leq n$. In this case, the dehomogenisation equality is

$$\mathcal{O}_q(G(k, n))[[u]^{-1}] = T = \mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma],$$

where $p = n - k$. We set $R := \mathcal{O}_q(M(k, p))$.

When $2k = n$, so that $R = \mathcal{O}_q(M(k, k))$, it is well known that the centre of R is $K[D_q]$, where D_q is the quantum determinant of $\mathcal{O}_q(M(k, k))$. It is also well known that when $2k < n$, so that R is non-square, the centre of R is K . We also recall that the centre of $\mathcal{O}_q(G(k, n))$ is always reduced to scalars. This follows easily from the basis of standard monomials by first observing that an element is central in $\mathcal{O}_q(G(k, n))$ if and only if all the standard monomials in its support are central, and next by noting that there are no non-trivial central standard monomials since the only standard monomial commuting with both $[u]$ and $[w]$ are scalars by Lemma 6.1.

Proposition 8.1. *Assume that $2k \leq n$. Then any derivation δ of $\mathcal{O}_q(G(k, n))$ is equal, modulo inner derivations, to a linear combination of $D_1, \dots, D_n$. Furthermore, these n derivations are linearly independent modulo the inner derivations.*

Proof. We use the same notation δ for the extension to T . After possibly adjusting δ by inner derivations and linear combinations of $D_1, \dots, D_n$, we may assume that $\delta(y) = \delta([u]) = \delta([w]) = 0$, by Proposition 6.14, and then $\delta(R) \subseteq R$, by Theorem 7.3. Apply δ to the equation $yx_{ij} = qx_{ij}y$ to obtain

$$y\delta(x_{ij}) = q\delta(x_{ij})y.$$

Given this equation and the fact that $\delta(R) \subseteq R$, we conclude that $\delta(x_{ij})$ is homogeneous of degree one.

In order to prove the first claim, we consider the two cases (i) $2k = n$ and (ii) $2k < n$ separately in order to show the *subclaim* that $\delta|_R$ can be written as a linear combination of the row and column derivations D_{i*}, D_{*j} of the quantum matrix algebra R introduced in Section 5.

Subclaim: Case (i). First, suppose that $2k = n$, so that $k = n - k$. In this case, $R = \mathcal{O}_q(M(k, k))$, and so R is a square quantum matrix algebra and the centre of R is $K[D_q]$, where D_q is the quantum determinant of $\mathcal{O}_q(M(k, k))$.

By [3, Theorem 2.9], there are polynomials $P_1, \dots, P_k, Q_1, \dots, Q_k \in K[D_q]$ and an element $z \in R$ such that

$$\delta|_R = \text{ad}_z + \sum_{i=1}^k P_i D_{i*} + \sum_{j=1}^{n-k} Q_j D_{*j},$$

where D_{i*}, D_{*j} are the row and column derivations of R introduced in Section 5.

Let a_i be the constant term in P_i and b_j be the constant term in Q_j . Then

$$\begin{aligned} \delta(x_{rs}) &= \left(\sum_{i=1}^k a_i D_{i*}(x_{rs}) + \sum_{j=1}^{n-k} b_j D_{*j}(x_{rs}) \right) \\ &= zx_{rs} - x_{rs}z + \sum_{i=1}^k (P_i - a_i) D_{i*}(x_{rs}) + \sum_{j=1}^{n-k} (Q_j - b_j) D_{*j}(x_{rs}) \end{aligned}$$

for all r, s . The terms on the left-hand side of this equation all have degree one, whereas the terms on the right-hand side have degree greater than one, because $zx_{rs} - x_{rs}z$ has no degree zero or degree one terms.

It follows that both sides are zero, and so $\delta|_R = \sum_{i=1}^k a_i D_{i*} + \sum_{j=1}^{n-k} b_j D_{*j}$, which establishes the subclaim in the case that $2k = n$.

Subclaim: Case (ii). Next, suppose that $2k < n$. In this case, $R = \mathcal{O}_q(M(k, p))$, where $p = n - k > k$ and so R is a non-square quantum matrix algebra with more columns than rows, and the centre of R is K . The proof of this case is substantially more complicated than that of case (i) due to the fact that [3, Theorem 2.9] only covers derivations for square quantum matrices. To avoid disturbing the flow of the proof of this proposition, the proof of this subclaim is treated in the appendix and finally established in Proposition A.6.

Having established the subclaim, we revert to the condition that $2k \leq n$.

By using Corollary 5.2, we see that

$$\delta|_R = \sum_{i=1}^k a_i D_{i*} + \sum_{j=1}^{n-k} b_j D_{*j} = \left(-\sum_{i=1}^k a_i \widetilde{D}_i + \sum_{j=1}^{n-k} b_j \widetilde{D}_{k+j} \right)|_R.$$

We define $\widetilde{\delta} = -\sum_{i=1}^k a_i \widetilde{D}_i + \sum_{j=1}^{n-k} b_j \widetilde{D}_{k+j}$, so that $\delta|_R = \widetilde{\delta}|_R$. Note that $\widetilde{\delta}(y) = -(\sum_{i=1}^k a_i)y$, as $\widetilde{D}_i(y) = y$ for $i = 1, \dots, k$, while $\widetilde{D}_{k+j}(y) = 0$ for $j = 1, \dots, n-k$. Recall from Remark 5.4 that $\sum_{i=1}^n \widetilde{D}_i$ acts trivially on R , while $\sum_{i=1}^n \widetilde{D}_i(y) = ky$.

Set $\widehat{\delta} := (\frac{1}{k} \sum_{i=1}^k a_i)(\sum_{i=1}^n \widetilde{D}_i)$. Then, for all r and s , we have $(\widetilde{\delta} + \widehat{\delta})(x_{rs}) = \widetilde{\delta}(x_{rs}) + \widehat{\delta}(x_{rs}) = \widetilde{\delta}(x_{rs}) + 0 = \delta(x_{rs})$, while $(\widetilde{\delta} + \widehat{\delta})(y) = \widetilde{\delta}(y) + \widehat{\delta}(y) = -(\sum a_i)y + (\frac{1}{k} \sum a_i)(ky) = 0 = \delta(y)$. As δ and $\widetilde{\delta} + \widehat{\delta}$ agree on the generating set x_{ij}, y , they are equal as derivatives.

For the proof of the second part, suppose that

$$\text{ad}_z + \sum_{i=1}^n a_i D_i = 0$$

for some $z \in \mathcal{O}_q(G(k, n))$ and $a_i \in K$. Thus, $\text{ad}_z([I]) + \sum_{i=1}^n a_i \delta(i \in I)[I] = 0$ for each quantum Plücker coordinate $[I] \in \mathcal{O}_q(G(k, n))$. The first term has no components in degree one and the other terms are all in degree one, so we deduce that $\text{ad}_z([I]) = 0$ for each quantum Plücker coordinate $[I] \in \mathcal{O}_q(G(k, n))$, so that $\text{ad}_z = 0$. Thus, we obtain $\sum_{i=1}^n a_i D_i = 0$.

For $r = 1, \dots, n + 1 - k$, set $[I_r] := [1, \dots, k - 1, k - 1 + r]$, and observe that

$$0 = \sum_{i=1}^n a_i D_i[I_r] = ((a_1 + \dots + a_{k-1}) + a_{k-1+r})[I_r].$$

Thus, $(a_1 + \dots + a_{k-1}) + a_{k-1+r} = 0$ for each of these values of r . It follows that $a_k = a_{k+1} = \dots = a_n$. In a similar manner, set $[J_r] := [n - k + 1 - r, n - k + 2, \dots, n]$

for $r = 0, \dots, n - k$ to observe that $a_{n-k+1-r} + (a_{n-k+2} + \dots + a_n) = 0$ for these values of r . It follows that $a_1 = \dots = a_{n-k+1}$. These two ranges of values must overlap, or else $n - k + 1 < k$, so that $n + 1 < 2k \leq n$, a contradiction. Thus, $a_1 = a_2 = \dots = a_n$, and from this and Corollary 3.2, it follows that each $a_i = 0$. ■

8.2. The general case

We have now proved our conjecture for $\mathcal{O}_q(G(k, n))$ in the case where $2k \leq n$. In order to remove this restriction, we use the fact that $\mathcal{O}_q(G(k, n)) \cong \mathcal{O}_q(G(n - k, n))$; see, for example, [4, Proposition 3.1]. We will use $\overline{D}$ to distinguish derivations of $\mathcal{O}_q(G(n - k, n))$ from derivations of $\mathcal{O}_q(G(k, n))$. Let $\psi : \mathcal{O}_q(G(k, n)) \rightarrow \mathcal{O}_q(G(n - k, n))$ be the automorphism of [4, Proposition 3.1], so that $\psi([I]) = [w_0(\hat{I})]$ for each quantum Plücker coordinate $[I]$ of $\mathcal{O}_q(G(k, n))$, where $\hat{I} := \{1, \dots, n\} \setminus I$ and w_0 is the longest element of the symmetric group S_n . Let D be a derivation of $\mathcal{O}_q(G(k, n))$. It is easy to check that $\psi D \psi^{-1}$ is a derivation of $\mathcal{O}_q(G(n - k, n))$. Similarly, if $\overline{D}$ is a derivation of $\mathcal{O}_q(G(n - k, n))$, then $\psi^{-1} \overline{D} \psi$ is a derivation of $\mathcal{O}_q(G(k, n))$.

Recall that we have the derivations D_i of $\mathcal{O}_q(G(k, n))$ for $i = 1, \dots, n$, with $D_i([I]) = \delta(i \in I)[I]$, and similarly we have derivations $\overline{D}_j$ of $\mathcal{O}_q(G(n - k, n))$ for $j = 1, \dots, n$.

For each i , we need to see how $\psi D_i \psi^{-1}$ acts as a derivation on $\mathcal{O}_q(G(n - k, n))$ in terms of the $\overline{D}_j$.

Note from Corollary 3.2 that $\frac{1}{n-k} (\sum_{j=1}^n \overline{D}_j)([J]) = [J]$ for each quantum Plücker coordinate $[J] \in \mathcal{O}_q(G(n - k, n))$.

Lemma 8.2. *We have*

$$\psi D_i \psi^{-1} = \frac{1}{n-k} \left(\sum_{j=1}^n \overline{D}_j \right) - \overline{D}_{w_0(i)}.$$

Proof. Let $[J]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(n - k, n))$ and suppose that $[J] = \psi([I]) = [w_0(\hat{I})]$ for a quantum Plücker coordinate $[I]$ of $\mathcal{O}_q(G(k, n))$.

Before we do the calculation of $\psi D_i \psi^{-1}$, note the following evaluation of a truth function:

$$\delta(i \in I) = 1 - \delta(i \in \hat{I}) = 1 - \delta(w_0(i) \in w_0(\hat{I})) = 1 - \delta(w_0(i) \in J).$$

We obtain

$$\begin{aligned} \psi D_i \psi^{-1}([J]) &= \psi D_i([I]) = \psi(\delta(i \in I)[I]) = \delta(i \in I)[J] \\ &= (1 - \delta(w_0(i) \in J))[J] = [J] - \overline{D}_{w_0(i)}([J]) \\ &= \left\{ \frac{1}{n-k} \left(\sum_{j=1}^n \overline{D}_j \right) - \overline{D}_{w_0(i)} \right\}([J]), \end{aligned}$$

as required. ■

We can now obtain our main theorem without any restriction other than $1 < k < n - 1$. Given that we have proved the conjecture in the case that $2k \leq n$, it is enough to prove the result for $\mathcal{O}_q(G(n - k, n))$ when $2k \leq n$.

Proposition 8.3. *Assume that $2k \leq n$. In $\mathcal{O}_q(G(n - k, n))$ any derivation is equal, modulo inner derivations, to a linear combination of $\overline{D}_1, \dots, \overline{D}_n$. Furthermore, these n derivations are linearly independent modulo the inner derivations.*

Proof. Let $\overline{D}$ be a derivation on $\mathcal{O}_q(G(n - k, n))$. Then $\psi^{-1}\overline{D}\psi$ is a derivation of the algebra $\mathcal{O}_q(G(k, n))$. Hence,

$$\psi^{-1}\overline{D}\psi = \text{ad}_z + \sum_{i=1}^n a_i D_i$$

for some $z \in \mathcal{O}_q(G(k, n))$ and $a_i \in K$, by Proposition 8.1.

Therefore,

$$\overline{D} = \text{ad}_{\psi(z)} + \sum_{i=1}^n a_i \psi D_i \psi^{-1} = \text{ad}_{\psi(z)} + \sum_{i=1}^n a_i \left\{ \frac{1}{n-k} \left(\sum_{j=1}^n \overline{D}_j \right) - \overline{D}_{w_0(i)} \right\},$$

and the first claim follows.

The proof of the second part follows in the same way as for the second part of Proposition 8.1. ■

We are now ready to state and prove our main result.

Theorem 8.4. *Let $2 \leq k \leq n - 2$. Then any derivation of $\mathcal{O}_q(G(k, n))$ is equal, modulo inner derivations, to a linear combination of $D_1, \dots, D_n$. Furthermore, these n derivations are linearly independent modulo the inner derivations.*

Proof. The result in the case that $2k \leq n$ has been established in Proposition 8.1, so suppose that $2k > n$. Set $k' := n - k$, then $2k' < n$ and $n - k' = k$. As $2k' < n$, the result holds for $\mathcal{O}_q(G(k', n))$. It then follows that the result holds for $\mathcal{O}_q(G(k, n)) = \mathcal{O}_q(G(n - k', n))$, by using Proposition 8.3. ■

Recall that the Hochschild cohomology group in degree one of a ring R , denoted by $\text{HH}^1(R)$, is defined by

$$\text{HH}^1(R) := \text{Der}(R)/\text{InnDer}(R),$$

where $\text{InnDer}(R) := \{\text{ad}_z \mid z \in R\}$ is the Lie algebra of inner derivations of R . It is well known that $\text{HH}^1(R)$ is a module over $\text{HH}^0(R) := Z(R)$.

The following corollary is immediate from the above theorem.

Corollary 8.5. *Let $2 \leq k \leq n - 2$. The first Hochschild cohomology group of the quantum grassmannian, $\text{HH}^1(\mathcal{O}_q(G(k, n)))$, is an n -dimensional vector space over K with basis (the cosets of) $D_1, \dots, D_n$.*

A. Derivations on non-square quantum matrices

In this appendix, we prove case (ii) of the subclaim in the proof of Proposition 8.1. To be more specific, in case (ii) of the subclaim, we are dealing with a derivation of $\mathcal{O}_q(M(k, p))$, where $k < p$, that arises, via the dehomogenisation equality, from a derivation D of $\mathcal{O}_q(G(k, n))$ that has the following properties: (i) $D([u]) = D([w]) = 0$; (ii) D is a homogeneous derivative of $\mathcal{O}_q(M(k, p))$. As a consequence of Lemma 7.1, the first condition implies that the derivative D acts trivially on the rightmost quantum minor of $\mathcal{O}_q(M(k, p))$.

Hence, with a change of notation, throughout this appendix, we assume that we are considering derivations on $\mathcal{O}_q(M(m, n))$ where $m < n$ and that we have a derivation D acting on $\mathcal{O}_q(M(m, n))$ with the following properties:

- (i) The derivation D is homogeneous; that is, all terms appearing non-trivially in $D(x_{rs})$ have degree one.
- (ii) $D([1, \dots, m \mid n+1-m, \dots, n]) = 0$.

The aim in this appendix is to show that such a derivation can be written as a linear combination of the row and column derivations D_{i*} and D_{*j} that were introduced in Section 5. With this in mind, we fix the following notation.

Notation A.1. Throughout the appendix, B denotes the quantum matrix subalgebra of $\mathcal{O}_q(M(m, n))$ generated by x_{rs} in the first $n-m$ columns of $\mathcal{O}_q(M(m, n))$ and C denotes the square quantum matrix subalgebra of $\mathcal{O}_q(M(m, n))$ generated by the x_{rs} in the final m columns. The quantum determinant of C is $[I \mid J] := [1, \dots, m \mid n-m+1, \dots, n]$ and $[I \mid J]$ is in the centre of C .

A.1. Action of derivations on first m columns of $\mathcal{O}_q(M(m, n))$

Lemma A.2. Use Notation A.1. Let D be a derivation on $\mathcal{O}_q(M(m, n))$ such that $D(x_{ij})$ is homogeneous of degree one for all x_{ij} , and suppose that $D([I \mid J]) = 0$. Then $D(B) \subseteq B$.

Proof. It is enough to show that $D(x_{ij}) \in B$ for $j \leq m$. For such an x_{ij} , suppose that

$$D(x_{ij}) = \sum_{\substack{r \leq m \\ s \leq n}} a_{rs} x_{rs}$$

with $a_{rs} \in K$. Now, $x_{ij}[I \mid J] = q[I \mid J]x_{ij}$, by [4, Lemma 6.1 (ii)]. Apply D to this equation, noting that $D([I \mid J]) = 0$, to obtain

$$\sum_{\substack{1 \leq r \leq m \\ 1 \leq s \leq n}} a_{rs} x_{rs} [I \mid J] = q[I \mid J] \sum_{\substack{1 \leq r \leq m \\ 1 \leq s \leq n}} a_{rs} x_{rs}.$$

As $x_{rs}[I | J] = q[I | J]x_{rs}$ when $s \leq m$, and $x_{rs}[I | J] = [I | J]x_{rs}$ for $s > m$, this gives

$$\sum_{\substack{r \leq m \\ s > m}} (1 - q) a_{rs} x_{rs} [I | J] = 0,$$

and it follows that $a_{rs} = 0$ when $s > m$, so that $D(x_{ij}) \in B$, as required. ■

A.2. We can adjust D , so that D is trivial on C

Lemma A.3. *Use Notation A.1. Let D be a derivation on $\mathcal{O}_q(M(m, n))$ such that $D(x_{ik})$ is homogeneous of degree one for all x_{ik} , and suppose that $D([I | J]) = 0$. Then $D(C) \subseteq C$.*

Proof. This is proved in a similar manner to the proof of Lemma A.2, using the fact that x_{ik} commutes with $[I | J]$ for x_{ik} in C . ■

We now show that we can adjust D by row and column derivations, so that the adjusted derivation acts trivially on C .

Lemma A.4. *Use Notation A.1. Let D be a derivation on $\mathcal{O}_q(M(m, n))$ such that $D(x_{rs})$ is homogeneous of degree one for all x_{rs} , and suppose that $D([I | J]) = 0$. There is a homogeneous derivation D' of $\mathcal{O}_q(M(m, n))$ with $D'|_C = 0$ and $D'(B) \subseteq B$ such that $D' - D$ is a linear combination of D_{i*} , for $i = 1, \dots, m$, and D_{*j} , for $n - m + 1 \leq j \leq n$, the row and column derivations of $\mathcal{O}_q(M(m, n))$ introduced in Section 5.*

Proof. Note that $D(C) \subseteq C$, by Lemma A.3. Set $Y := [I | J]$, the quantum determinant of C . By [3, Theorem 2.9], there are polynomials $P_1, \dots, P_m, Q_{n-m+1}, \dots, Q_n \in K[Y]$ and an element $z \in C$ such that

$$D|_C = \text{ad}_z + \sum_{i=1}^m P_i D_{i*}|_C + \sum_{j=n-m+1}^n Q_j D_{*j}|_C.$$

Let $s \geq n - m + 1$, so that $x_{rs} \in C$. Then

$$D|_C(x_{rs}) = \text{ad}_z(x_{rs}) + \sum_{i=1}^m P_i D_{i*}|_C(x_{rs}) + \sum_{j=n-m+1}^n Q_j D_{*j}|_C(x_{rs}).$$

Let a_i be the constant term in P_i and b_j be the constant term in Q_j , so that

$$\begin{aligned} D|_C(x_{rs}) &= \sum_{i=1}^m a_i D_{i*}|_C(x_{rs}) + \sum_{j=n-m+1}^n b_j D_{*j}|_C(x_{rs}) \\ &= z x_{rs} - x_{rs} z + \sum_{i=1}^m (P_i - a_i) D_{i*}|_C(x_{rs}) + \sum_{j=n-m+1}^n (Q_j - b_j) D_{*j}|_C(x_{rs}). \end{aligned}$$

As $D(x_{rs})$ is homogeneous of degree one, the terms on the left-hand side of the equation all have degree one, whereas the nonzero terms on the right-hand side all have degree greater than one. It follows that both sides are zero, and so

$$D|_C(x_{rs}) = \sum_{i=1}^m a_i D_{i*}|_C(x_{rs}) + \sum_{i=n-m+1}^n b_j D_{*j}|_C(x_{rs}).$$

Set

$$D' := D - \sum_{i=1}^m a_i D_{i*} + \sum_{i=n-m+1}^n b_j D_{*j}.$$

Then $D'|_C = 0$. Note that D' is a homogeneous derivation as D , D_{i*} and D_{*j} are all homogeneous. Lemma A.2 shows that $D'(B) \subseteq B$. ■

A.3. Derivatives column by column

Let $\varepsilon_1, \dots, \varepsilon_m$ and $\varepsilon'_1, \dots, \varepsilon'_n$ be the standard bases for $\mathbb{Z}^m$ and $\mathbb{Z}^n$, respectively. The quantum matrix algebra $\mathcal{O}_q(M(m, n))$ has a natural $\mathbb{Z}^m \times \mathbb{Z}^n$ bigrading defined by giving each x_{rs} bidegree, also called *bicontent*, $(\varepsilon_r, \varepsilon'_s)$. Observe that any quantum minor $[U | V]$ is homogeneous of bidegree (χ_U, χ'_V) , where χ_S (respectively, χ'_S) stands for the characteristic function of a subset S of $\{1, \dots, m\}$ (respectively, of $\{1, \dots, n\}$). If the bidegree of a homogeneous element is (χ_r, χ_c) , we refer to χ_r as the *row content* of the element and χ_c as the *column content* of the element. The main use of the notion of content will be that in any equation of the form $P = Q$, the terms of the same bicontent on each side of the equation must be equal.

We consider $\mathcal{O}_q(M(m, n))$ with $m < n$ and assume that we have a homogeneous derivation D that acts trivially on C and that $D(B) \subseteq B$.

Lemma A.5. *Use Notation A.1. Let D be a derivation on $\mathcal{O}_q(M(m, n))$ such that $D(x_{ik})$ is homogeneous of degree one for all x_{ik} and that $D(C) = 0$. Then D is a linear combination of D_{*k} for $k = 1, \dots, n - m$.*

Proof. Let $x_{ik} \in B$, so that $1 \leq i \leq m$ and $1 \leq k \leq n - m$. As $D(B) \subseteq B$, by Lemma A.2, we can write

$$D(x_{ik}) = \sum_{\substack{1 \leq r \leq m \\ 1 \leq s \leq n-m}} a_{rs}^{(ik)} x_{rs}$$

for some $a_{rs}^{(ik)} \in K$. Our first aim is to show that $a_{rs}^{(ik)} = 0$ whenever $r \neq i$, so that

$$D(x_{ik}) = \sum_{1 \leq s \leq n-m} a_{is}^{(ik)} x_{is}.$$

Set $l := n - m + 1$. Choose any $j < i$. Note that $x_{ik}x_{jl} = x_{jl}x_{ik}$. As $x_{jl} \in C$, we know that $D(x_{jl}) = 0$, so that when we apply the derivative D to this equation, we obtain

$$\sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(ik)} x_{rs} x_{jl} = \sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(ik)} x_{jl} x_{rs}$$

or

$$\sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(ik)} (x_{rs} x_{jl} - x_{jl} x_{rs}) = 0.$$

Consider terms in this equation with bicontent $(2\varepsilon_j; \varepsilon_s + \varepsilon_l)$ (which from now on we also denote by $(j, j; s, l)$ to ease notation) to see that

$$\sum_{1 \leq s < l} a_{js}^{(ik)} (x_{js} x_{jl} - x_{jl} x_{js}) = 0.$$

As $s < l$, this gives

$$\sum_{1 \leq s < l} a_{js}^{(ik)} (q - 1) x_{jl} x_{js} = 0.$$

From which it follows that $a_{js}^{(ik)} = 0$ for all s when $j < i$.

Next, choose any $j > i$. As $D(x_{jk}) \in B$, write

$$D(x_{jk}) = \sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(jk)} x_{rs}.$$

Apply D to the equation $x_{ik} x_{jl} - x_{jl} x_{ik} = \hat{q} x_{il} x_{jk}$ (where $\hat{q} := q - q^{-1}$), noting that $D(x_{*l}) = 0$ as $x_{*l} \in C$, to obtain

$$\sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(ik)} x_{rs} x_{jl} - \sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(ik)} x_{jl} x_{rs} = \hat{q} \left(\sum_{\substack{1 \leq r \leq m \\ 1 \leq s < l}} a_{rs}^{(jk)} x_{il} x_{rs} \right). \quad (\text{A.1})$$

Look at terms with bicontent $(j, j; s, l)$ in equation (A.1). There are no such terms on the right-hand side, as i occurs as a row index and $i \neq j$, while on the left-hand side, we get such terms when $r = j$. Hence, $a_{js}^{(ik)} x_{js} x_{jl} - a_{js}^{(ik)} x_{jl} x_{js} = 0$ for each s . As $s < l$, this gives the equality $(q - 1) a_{js}^{(ik)} x_{js} x_{jl} = 0$ and so $a_{js}^{(ik)} = 0$ for all s and each $j > i$. As we already know that $a_{js}^{(ik)} = 0$ for all s when $j < i$, this gives

$$D(x_{ik}) = \sum_{1 \leq s < l} a_{is}^{(ik)} x_{is}$$

for $1 \leq i \leq m$ as required.

Fix k , with $1 \leq k < l$ and $j > 1$. Apply the derivative D to the equation $x_{1k} x_{jl} - x_{jl} x_{1k} = \hat{q} x_{1l} x_{jk}$, using the expressions we have just derived, to obtain

$$\sum_{1 \leq s < l} a_{1s}^{(1k)} x_{1s} x_{jl} - \sum_{1 \leq s < l} a_{1s}^{(1k)} x_{jl} x_{1s} = \hat{q} \sum_{1 \leq s < l} a_{js}^{(jk)} x_{1l} x_{js}. \quad (\text{A.2})$$

Look at terms with content $(1, j; s, l)$ in equation (A.2) for fixed s . On the left-hand side, we have $a_{1s}^{(1k)} x_{1s} x_{jl} - a_{1s}^{(1k)} x_{jl} x_{1s}$, and this is equal to $\hat{q} a_{1s}^{(1k)} x_{1l} x_{js}$. On the right-hand

side, we have $\hat{q} a_{js}^{(jk)} x_{1l} x_{js}$. It follows that $a_{js}^{(jk)} = a_{1s}^{(1k)}$ for each s with $1 \leq s < l$, so that

$$D(x_{jk}) = \sum_{1 \leq s < l} a_{1s}^{(1k)} x_{js}$$

for each j with $1 \leq j \leq m$.

Hence,

$$D \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix} = a_{11}^{(1k)} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \end{pmatrix} + a_{12}^{(1k)} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \end{pmatrix} + \cdots + a_{1,l-1}^{(1k)} \begin{pmatrix} x_{1,l-1} \\ x_{2,l-1} \\ \vdots \\ x_{m,l-1} \end{pmatrix}. \quad (\text{A.3})$$

For each $k = 1, \dots, l-1$, we want to show that $a_{1r}^{(1k)} = 0$ when $r \neq k$, so that

$$D \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix} = a_{1k}^{(1k)} \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix} = a_{1k}^{(1k)} D_{*k} \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix}.$$

Fix k with $1 \leq k \leq l-1$. First, we show that $a_{1r}^{(1k)} = 0$ for each r such that $k < r \leq l-1$. If $k = l-1$, there is no such r to consider, so assume that $k < l-1$, in which case $k+1 < l$ and $x_{1,k+1} \in B$. Apply D to the equation $x_{1k} x_{1,k+1} = q x_{1,k+1} x_{1k}$ for $k+1 < l$ to obtain

$$D(x_{1k}) x_{1,k+1} + x_{1k} D(x_{1,k+1}) = q D(x_{1,k+1}) x_{1k} + q x_{1,k+1} D(x_{1k}),$$

which gives

$$\begin{aligned} \sum_{1 \leq s < l} a_{1s}^{(1k)} x_{1s} x_{1,k+1} + \sum_{1 \leq s < l} a_{1s}^{(1,k+1)} x_{1k} x_{1s} \\ = \sum_{1 \leq s < l} q a_{1s}^{(1,k+1)} x_{1s} x_{1k} + \sum_{1 \leq s < l} q a_{1s}^{(1k)} x_{1,k+1} x_{1s}. \end{aligned} \quad (\text{A.4})$$

Consider terms in this equation with content $(1, 1; k+1, r)$ for $r > k$. These occur in the first and fourth sums when $s = r$ and do not occur in the second and third sums because k is in the column content of the terms in these sums. Hence, $a_{1r}^{(1k)} x_{1r} x_{1,k+1} = q a_{1r}^{(1k)} x_{1,k+1} x_{1r}$. If $r = k+1$, then we get $a_{1,k+1}^{(1k)} x_{1,k+1} x_{1,k+1} = q a_{1,k+1}^{(1k)} x_{1,k+1} x_{1,k+1}$ and so $a_{1,k+1}^{(1k)} = 0$. If $r > k+1$, then $x_{1r} x_{1,k+1} = q^{-1} x_{1,k+1} x_{1r}$. So we see that $q^{-1} a_{1r}^{(1k)} x_{1,k+1} x_{1r} = q a_{1r}^{(1k)} x_{1,k+1} x_{1r}$ and so $a_{1r}^{(1k)} = 0$. Hence, $a_{1r}^{(1,k)} = 0$ for all $r > k$.

Next, we show $a_{1r}^{(1k)} = 0$ when $r < k$. If $k = 1$, then there is no such r to consider, so assume that $k > 1$, in which case $k-1 \geq 1$, so that $x_{1,k-1}$ exists and is in B . Apply D to the equation $x_{1,k-1} x_{1,k} = q x_{1,k} x_{1,k-1}$ to obtain

$$D(x_{1,k-1}) x_{1,k} + x_{1,k-1} D(x_{1,k}) = q D(x_{1,k}) x_{1,k-1} + q x_{1,k} D(x_{1,k-1}),$$

which gives

$$\begin{aligned} & \sum_{1 \leq s < l} a_{1s}^{(1,k-1)} x_{1s} x_{1,k} + \sum_{1 \leq s < l} a_{1s}^{(1,k)} x_{1,k-1} x_{1s} \\ &= \sum_{1 \leq s < l} q a_{1s}^{(1k)} x_{1s} x_{1,k-1} + \sum_{1 \leq s < l} q a_{1s}^{(1,k-1)} x_{1k} x_{1s}. \end{aligned} \quad (\text{A.5})$$

Consider terms in this equation with content $(1, 1; r, k-1)$ with $r < k$. There are no such terms in the first and fourth terms, as k is in the column content of the terms in these sums, and such terms occur in the second and third terms when $s = r$. Hence, $a_{1r}^{(1,k)} x_{1,k-1} x_{1r} = q a_{1r}^{(1,k)} x_{1r} x_{1,k-1}$.

When $r = k-1$, we see that $a_{1,k-1}^{(1,k)} x_{1,k-1} x_{1,k-1} = q a_{1,k-1}^{(1,k)} x_{1,k-1} x_{1,k-1}$, so that $a_{1,k-1}^{(1,k)} = 0$.

When $r < k-1$, then $x_{1,k-1} x_{1r} = q^{-1} x_{1r} x_{1,k-1}$, so we see that $q^{-1} a_{1r}^{(1,k)} x_{1r} x_{1,k-1} = q a_{1r}^{(1,k)} x_{1r} x_{1,k-1}$ and so $a_{1r}^{(1,k)} = 0$.

Hence, $a_{1,r}^{(1,k)} = 0$ for all $r < k$. Thus, $a_{1r}^{(1k)} = 0$ whenever $r \neq k$, so that $D(x_{ik}) = a_{ik}^{(1k)} x_{ik}$ for each $1 \leq i \leq m$, as required to show that

$$D \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix} = a_{1k}^{(1k)} \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{mk} \end{pmatrix}$$

for each $1 \leq k < l$. As we already know that D acts trivially on columns $n+1-m$ up to n , this gives

$$D = a_{11}^{(1,1)} D_{*1} + a_{12}^{(1,2)} D_{*2} + \cdots + a_{1,l-1}^{(1,l-1)} D_{*,l-1},$$

as required. ■

A.4. Conclusion of appendix

The preceding analysis proves the following proposition.

Proposition A.6. *Suppose that $m < n$ and that we have a derivation D of $\mathcal{O}_q(M(m, n))$ such that*

- (i) *the derivation D is homogeneous; that is, all terms appearing non-trivially in $D(x_{ij})$ have degree one,*
- (ii) *$D([1, \dots, m \mid n+1-m, \dots, n]) = 0$.*

Then

$$D = \sum_{i=1}^m a_i D_{i*} + \sum_{k=1}^n b_k D_{*k}$$

for some $a_i, b_k \in K$.

Proof. By Lemma A.4, there is a homogeneous derivation $D^{(1)}$ such that $D^{(1)} - D$ is a linear combination of the D_{i*} , with $1 \leq i \leq m$, and D_{*j} , with $n - m + 1 \leq j \leq n$, and such that $D^{(1)}|_C = 0$. By Lemma A.5, $D^{(1)}$ is a linear combination of the D_{*k} with $1 \leq k \leq n - m$. Writing $D = D^{(1)} - (D^{(1)} - D)$ gives the required result. ■

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