J. Noncommut. Geom. 20 (2026), 1095–1116
DOI 10.4171/JNCG/655

Limit multiplicities and von Neumann dimensions

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Abstract. Given a connected semisimple Lie group G and an arithmetic subgroup Γ, it is well known that each irreducible representation π of G occurs in the discrete spectrum $L^2_{\text{disc}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$ with at most a finite multiplicity $m_\Gamma(\pi)$. While $m_\Gamma(\pi)$ is unknown in general, we are interested in its limit as Γ is taken to be in a tower of lattices $\Gamma_1 \supset \Gamma_2 \supset \cdots$. For a bounded measurable subset X of the unitary dual $\hat{G}$, we let $m_{\Gamma_k}(X)$ be the integration of the multiplicity $m_{\Gamma_k}(\pi)$ over all π in X, which can be proved finite. Let H_X be the direct integral of the irreducible representations in X with respect to the Plancherel measure of $\hat{G}$, which is also a module over the group von Neumann algebra $\mathcal{L}(\Gamma_k)$. Based on the work of Sauvageot and Finis–Lapid–Müller, we prove

$$\lim_{k\to\infty} \frac{m_{\Gamma_k}(X)}{\dim_{\mathcal{L}(\Gamma_k)} H_X} = 1$$

for any bounded subset X of $\hat{G}$ when (i) $\{\Gamma_k\}_{k\geq 1}$ are cocompact or (ii) $G = \mathrm{SL}(n,\mathbb{R})$ and $\{\Gamma_k\}$ are principal congruence subgroups.

1. Introduction: The motivation from SL(2, ℝ)

We start by recalling a result proved by Langlands and discuss an example on the arithmetic subgroup $\mathrm{SL}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{R})$. This provides our initial motivation. Let G be a connected semisimple real Lie group and Γ be a lattice in G, which is a discrete subgroup of G with finite covolume. One question is the decomposition of the quasi-regular representation $R_\Gamma\colon G \to U(L^2(\Gamma\backslash G))$ of G given by

$$(R_\Gamma(g)f)(x) = f(xg),$$

where $g \in G$, $f \in L^2(\Gamma\backslash G)$, and $x \in \Gamma\backslash G$.

Assuming Γ is cocompact ($\Gamma\backslash G$ is compact), it is well known that R_Γ decomposes completely into a direct sum of irreducible representations with finite multiplicities:

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}} m_\Gamma(\pi)\pi \quad \text{with } m_\Gamma(\pi) < \infty \text{ for each } \pi \in \hat{G},$$

where $\hat{G}$ denotes the unitary dual of G. Recall that an irreducible representation (π, H) of G is called *integrable* (respectively *square integrable*) if the function $g \mapsto \langle \pi(g)v, v\rangle$ is

Mathematics Subject Classification 2020: 46L10 (primary); 11F70, 11S37 (secondary).
Keywords: automorphic representations, arithmetic groups, von Neumann algebras.

in $L^1(G)$ (respectively $L^2(G)$) for some (hence for all) nonzero vector $v \in H$. Note that integrable representations are square integrable. For an irreducible square-integrable representation (π, H) of a group G, we denote its formal dimension by $d(\pi)$[1]. The following result gives an explicit multiplicity for a few special cases.

Theorem 1.1 (Langlands [25]). *If π is an integrable irreducible representation of a semisimple real Lie group G and Γ is a torsion-free cocompact lattice in G, we have*

$$m_\Gamma(\pi) = \mathrm{vol}(\Gamma \backslash G) d(\pi). \tag{1.1}$$

On the other hand, if (π, H_π) is a square-integrable representation of a more general locally compact group G, H_π is a module over the *group von Neumann algebra* $\mathcal{L}(\Gamma)$ of the discrete subgroup Γ (see Section 4.1 for the definition). Viewing H_π as an $\mathcal{L}(\Gamma)$-module yields its *von Neumann dimension* $\dim_{\mathcal{L}(\Gamma)} H_\pi$, which takes values in $[0, \infty]$. Indeed, if a discrete group Γ is an infinite conjugacy class group, or an ICC group for short (see Section 4.1 for the definition), this dimension totally determines the equivalence class of $\mathcal{L}(\Gamma)$-modules, that is, $\dim_{\mathcal{L}(\Gamma)} H_1 = \dim_{\mathcal{L}(\Gamma)} H_2$ if and only if $H_1 \cong H_2$ as $\mathcal{L}(\Gamma)$-modules. In the study of discrete series representations by Atiyah and Schmid, they proved

$$\dim_{\mathcal{L}(\Gamma)} H = \mathrm{vol}(\Gamma \backslash G) d(\pi) \tag{1.2}$$

if π is a square-integrable representation of G (see [6] and also [22, Chapter 3]). By applying the formulas (1.1) and (1.2), we obtain

$$m_\Gamma(\pi) = \dim_{\mathcal{L}(\Gamma)} H_\pi \tag{1.3}$$

if Γ is cocompact and torsion-free and π is integrable. That is to say, the von Neumann dimension is exactly the multiplicity under some assumptions.

However, if Γ is not cocompact or torsion-free, the multiplicity formula (1.3) is false in general. A typical example is the arithmetic subgroups in $G = \mathrm{SL}(2, \mathbb{R})$ such as $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. In these cases, we are interested in the limit behavior of the multiplicities. For example, we let $\Gamma(N)$ be the principal congruence subgroup of level N defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid a, d \equiv 1 \ (\mathrm{mod}\ N), b, c \equiv 0 \ (\mathrm{mod}\ N) \right\}.$$

[1] The formal dimension of an irreducible square-integrable representation (π, H) of a group G is a constant $d(\pi) \geq 0$ such that

$$\int_G \langle \pi(g)u, v \rangle \cdot \overline{\langle \pi(g)u', v' \rangle} d\dot{g} = \frac{1}{d(\pi)} \langle u, u' \rangle \cdot \overline{\langle v, v' \rangle}$$

for $u, v, u', u' \in H$ (see [31, Chapter 16]).

It is well known that (see [24]) the quasi-regular representation $R_{\Gamma(N)}$ is reducible and can be decomposed as

$$L^2(\Gamma(N)\backslash G) = L^2_{\text{cusp}}(\Gamma(N)\backslash G) \oplus L^2_{\text{cont}}(\Gamma(N)\backslash G) \oplus \mathbb{C}.$$

Here $L^2_{\text{cusp}}(\Gamma(N)\backslash G)$ is the cuspidal part, which is a direct sum of irreducible representations with finite multiplicities, that is,

$$L^2_{\text{cusp}}(\Gamma(N)\backslash G) = \bigoplus_{\pi\in\widehat{G}} m_{\Gamma(N)}(\pi)\cdot\pi \quad \text{with } m_{\Gamma(N)}(\pi) < \infty \text{ for each } \pi \in \widehat{G},$$

and $L^2_{\text{cont}}(\Gamma(N)\backslash G)$ is a direct integral of irreducible representations given by the Eisenstein series. Unfortunately, the multiplicities $m_{\Gamma(N)}(\pi)$ are still unknown in general, except for some special families of irreducible representations, including the square-integrable (discrete series) representations of $\mathrm{SL}(2,\mathbb{R})$.

Let $S_k(\Gamma)$ be the space of cusp forms of weight k for a Fuchsian group Γ. Let $\{\pi_k\}_{k\geq 2}$ be the family of holomorphic discrete series representations of $\mathrm{SL}(2,\mathbb{R})$. We have the following result (see [21, Theorem 2.10]).

Lemma 1.2. *For the discrete series π_k, we have $m_{\Gamma(N)}(\pi_k) = \dim S_k(\Gamma(N))$.*

By applying the dimension formulas of cusp forms (see [14, Chapter 3.9]), we obtain

$$m_{\Gamma(N)}(\pi_k) = \left(\frac{k-1}{24} - \frac{1}{4N}\right) N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right) \tag{1.4}$$

for all $N > 2$. On the other hand, by applying the Atiyah–Schmid formula (1.2), we obtain

$$\dim_{\mathcal{L}(\Gamma(N))} H_k = \frac{k-1}{24} N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right). \tag{1.5}$$

Hence the equations (1.4) and (1.5) imply the following corollary.

Corollary 1.3. *For a discrete series (π_k, H_k) of* $\mathrm{SL}(2,\mathbb{R})$*, we have*

$$\lim_{N\to\infty} \frac{m_{\Gamma(N)}(\pi_k)}{\dim_{\mathcal{L}(\Gamma(N))} H_k} = 1.$$

While the explicit multiplicities of most irreducible representations of a semisimple real Lie group are still unknown, the limit multiplicities have been studied since the 1970s. In the case of towers of cocompact lattices, DeGeorge and Wallach got the first results for discrete series of Lie groups [11] and later for bounded sets of irreducible representations in rank one groups [12]. Delorme [13] finally solved the problem for bounded sets of irreducible representations in all Lie groups. See also [1] for a recent approach.

For the non-uniform lattices (or arithmetic subgroups), Savin [34] first obtained the results on discrete series in his thesis, which is based on the work by Rohlfs and Speh [32].

Then Deitmar and Hoffmann proved the results on certain towers of arithmetic subgroups in rank one groups. Recently, Finis, Lapid, and Müller solved the case of congruence subgroups in $\mathrm{SL}(n,\mathbb{R})$ [17, 19], which are based on their study of the spectral side of Arthur's trace formulas [16, 18].

In order to get the multiplicity of representations of Lie groups, we apply the Arthur–Selberg trace formula defined for adelic groups. When a reductive $\mathbb{Q}$-group G satisfies the strong approximation property with respect to $\mathbb{R}$, for example, $G = \mathrm{SL}(n)$, we have a canonical identification

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K \simeq \Gamma_K\backslash G(\mathbb{R}), \tag{1.6}$$

where K is an open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$ and $\Gamma_K = K \cap G(\mathbb{Q})$. Moreover, the identification (1.6) is $G(\mathbb{R})$-equivariant and gives an isomorphism of $G(\mathbb{R})$-modules

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K) \cong L^2(\Gamma_K\backslash G(\mathbb{R})). \tag{1.7}$$

The trace formula is usually used for the left side of (1.7) to describe the representations of $G(\mathbb{R})$ on the right side. In the remainder of this paper, Corollary 1.3 is generalized in the following aspects:

(1) from a single discrete series representation to any bounded subset of the unitary dual $\widehat{G}$ of G,

(2) from $\mathrm{SL}(2,\mathbb{R})$ to the towers of uniform lattices in an arbitrary semisimple Lie group,

(3) from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(n,\mathbb{R})$ with its principal congruence subgroups.

Finally, we prove the following theorem.

Theorem 1.4 (The main theorem). *Let G be a semisimple simply connected Lie group. Let X be a bounded subset of the unitary dual of G and H_X be the direct integral of the irreducible representations of G in X with respect to the Plancherel measure. We have*

$$\lim_{n\to\infty} \frac{m_{\Gamma_k}(X)}{\dim_{\mathcal{L}(\Gamma_k)} H_X} = 1$$

when (i) *$\{\Gamma_k\}$ are cocompact or* (ii) *$G = \mathrm{SL}(n,\mathbb{R})$ and $\{\Gamma_k\}_{k\geq 1}$ are principal congruence subgroups.*

2. The trace formulas and dominant terms

We review the Arthur–Selberg trace formulas and give the dominant terms in them. We mainly follow [3, 5, 18]. Let G be a reductive algebraic group defined over $\mathbb{Q}$. The group $G(\mathbb{A})$ acts naturally on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by

$$R(g)\phi(x) = \phi(xg)$$

for $\phi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and $g \in G(\mathbb{A})$. Let $C_c^\infty(G(\mathbb{A}))$ be the complex algebra of smooth, compactly supported functions on $G(\mathbb{A})$ (see [7, Section 1.9]). Given $f \in C_c^\infty(G(\mathbb{A}))$, we may define the action of $C_c^\infty(G(\mathbb{A}))$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$:

$$(R(f)\phi)(x) = \int_{G(\mathbb{A})} f(g)R(g)\phi(x)dg = \int_{G(\mathbb{A})} f(g)\phi(xg)dg.$$

If we define the *kernel*

$$K(x, y) = K_f(x, y) := \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y),$$

we have $(R(f)\phi)(x) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} K(x, y)\phi(y)dy$.

2.1. The Selberg trace formula

We first assume G is anisotropic, and hence the quotient space $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact. Let $\mathcal{O}$ be the set of conjugacy classes in $G(\mathbb{Q})$ and $o \in \mathcal{O}$ be a conjugacy class. We define

$$K_o(x, y) = \sum_{\gamma \in o} f(x^{-1}\gamma y)$$

and obtain $K(x, y) = \sum_{o \in \mathcal{O}} K_o(x, y)$. On the other hand, if we let $\mathcal{X}$ denote the equivalence classes of irreducible representations of $G(\mathbb{A})$, the representation R decomposes into a direct sum of irreducible representations with finite multiplicities, that is, $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\chi \in \mathcal{X}} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_\chi$. Here $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_\chi = m(\chi) \cdot \chi$, which is $m(\chi)$ copies of the irreducible representation χ. Assume $\mathcal{B}_\chi$ is an orthonormal basis of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_\chi$. Then we let

$$K_\chi(x, y) = K_{f,\chi}(x, y) := \sum_{\phi \in \mathcal{B}_\chi} (R(f))\phi(x) \cdot \overline{\phi(y)},$$

which can be proved to be convergent. Now we let

(1) $k_\chi(x, f) = K_\chi(x, x)$ and $J_\chi(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} k_\chi(x, f)dx$,

(2) $k_o(x, f) = K_o(x, x)$ and $J_o(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} k_o(x, f)dx$.

If we let γ be a representative of $o \in \mathcal{O}$ and $H_\gamma = \{h \in H \mid h\gamma h^{-1} = \gamma\}$ for a group H containing γ, we get

$$J_o(f) = \mathrm{vol}(G(\mathbb{Q})_\gamma\backslash G(\mathbb{A})_\gamma) \int_{G(\mathbb{A})_\gamma\backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx.$$

Theorem 2.1. *Assuming $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact, we have*

$$\mathrm{tr}\, R(f) = \sum_{o \in \mathcal{O}} J_o(f) = \sum_{\chi \in \mathcal{X}} J_\chi(f) \tag{2.1}$$

for any $f \in C_c^\infty(G(\mathbb{A}))$.

For the classical setting, we start with a real Lie group G and an arithmetic lattice $\Gamma \subset G$. Consider the representation R_Γ of G on $L^2(\Gamma\backslash G)$ given by $(R_\Gamma(g)\phi)(x) = \phi(xg)$ for $x, g \in G$. Let $C_c^\infty(G)$ be the space of smooth functions on G with compact support. For $f \in C_c^\infty(G)$, a representation (π, H) of G and $v \in H$, we let

$$\pi(f)v = \int_G f(g)\pi(g)v dg.$$

If π is irreducible, $\pi(f)$ is a trace class operator, and we let $\theta_\pi(f) = \operatorname{tr}\pi(f)$. For the representation R_Γ, we have $(R_\Gamma(f)\phi)(x) = \int_G f(g)R_\Gamma(g)\phi(x)dg = \int_G f(g)\phi(xg)dg$.

It is known that $\Gamma\backslash G$ is compact if and only if the reductive part of G is anisotropic (see [30, Theorem 4.12]). In this case, $L^2(\Gamma\backslash G)$ can be decomposed into a direct sum of irreducible representations of G with each of finite multiplicity, that is,

$$L^2(\Gamma\backslash G) = \bigoplus m_\Gamma(\pi)\cdot\pi$$

with $m_\Gamma(\pi) = \dim \operatorname{Hom}_G(\pi, L^2(\Gamma\backslash G)) < \infty$ for each π.

By taking the test function in Theorem 2.1 to be $f \otimes 1_K$ for a certain compact subgroup K of $G(\mathbb{A}^{\mathrm{fin}})$ with $f \in C_c^\infty(G)$ (see Section 3.1), we get the following result for the lattice Γ in the real Lie group G.

Corollary 2.2 (The Selberg trace formula). *If $\Gamma\backslash G$ is compact, $R_\Gamma(f)$ is of trace class and*

$$\operatorname{tr} R_\Gamma(f) = \sum_{\pi\in\widehat{G}} m_\Gamma(\pi)\theta_\pi(f) = \sum_{\gamma\in[\Gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx, \tag{2.2}$$

where $[\Gamma]$ denotes the set of conjugacy classes in Γ.

2.2. The Arthur trace formula

In this part, we review the trace formula in the cases without the assumption that G is anisotropic or $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact. We will follow [3, 5] for the notations. It is an analogue of the trace formula (2.1) $\sum_{o\in\mathcal{O}} J_o(f) = \sum_{\chi\in\mathcal{X}} J_\chi(f)$ for the compact quotient. There are summations over a geometric index $\mathcal{O}$ and over a spectral index $\mathcal{X}$, which are defined as follows.

(1) The geometric index $\mathcal{O}$ is given as follows:

- For $\gamma \in G(\mathbb{Q})$, let $\gamma = \gamma_s\gamma_u$ be the decomposition such that γ_s is semisimple and γ_u is unipotent.
- Let $\mathcal{O}$ be the set of $G(\mathbb{Q})$-semisimple conjugacy classes of $G(\mathbb{Q})$: $\gamma \sim \beta$ if γ_s and β_s are $G(\mathbb{Q})$-conjugate.
- We denote a semisimple conjugacy class in $G(\mathbb{Q})$ by o, which is an element in $\mathcal{O}$.

(2) The spectral index $\mathcal{X}$ is given as follows:

- Fix a minimal parabolic subgroup P_0 and a Levi component $M_0 = M_{P_0}$ of P_0, which are both defined over $\mathbb{Q}$.
- $P =$ a standard parabolic subgroup defined over $\mathbb{Q}$ ($P_0 \subset P$).
- $N_P = R_u(P)$, the unipotent radical of P.
- $M_P =$ the unique Levi component of P such that $M_0 \subset M_P$.
- $A_P =$ the split component of the center of M_P.
- $\Delta_P =$ the set of simple roots of (P, A_P).
- $\mathfrak{a}_P = \mathrm{Hom}(X(M_P)_{\mathbb{Q}}, \mathbb{R})$, where $X(M_P)_{\mathbb{Q}}$ is the group of $\mathbb{Q}$-characters of M_P.
- $\Omega =$ the restricted Weyl group of (G, A_0).
- $\Omega(\mathfrak{a}_1, \mathfrak{a}_2) =$ the set of distinct isomorphisms from $\mathfrak{a}_1$ to $\mathfrak{a}_2$ obtained by restricting Ω to $\mathfrak{a}$, where $\mathfrak{a}_1 = \mathfrak{a}_{P_1}, \mathfrak{a}_2 = \mathfrak{a}_{P_2}$ are given by two standard parabolic subgroups P_1, P_2.
- $M(\mathbb{A})^1 = \bigcap_{\chi \in \mathrm{Hom}(M(\mathbb{A}), F^\times)} \{\ker|\chi| \colon M(\mathbb{A}) \to \mathbb{R}_+\}$.
- For two Levi subgroups M_1, M_2 of G and $\rho_1 \in \widehat{M_1(\mathbb{A})^1}, \rho_2 \in \widehat{M_2(\mathbb{A})^1}$, we say (M_1, ρ_1) and (M_2, ρ_2) are equivalent, denoted by $(M_1, \rho_1) \sim (M_2, \rho_2)$, if there is an $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ such that the representation $(s\rho)(m') = \rho(w_s^{-1} m w_s)$ is unitarily equivalent to ρ_2.
- $\mathcal{X} = \{(M, \rho)\}/\sim$, the set of equivalence classes of pairs (M, ρ), where M is a Levi subgroup of G and $\rho \in \widehat{M(\mathbb{A})^1}$.

Let $\mathcal{X}(G) = \{(M, \rho) \in \mathcal{X} \mid M = G\}$. There are truncated distributions $J_o^T(f), J_\chi^T(f)$ defined for a given $f \in C_c^\infty(G(\mathbb{A})^1)$, which also depend on the variable $T \in \mathfrak{a}_0^+$ (see [5, Section 5]). The following trace formula was first given in [3, Section 5].

Theorem 2.3 (The Arthur trace formula). *For any $f \in C_c^\infty(G(\mathbb{A})^1)$ and any suitably regular $T \in \mathfrak{a}_0^+$, we have*

$$\sum_{o \in \mathcal{O}} J_o^T(f) = \sum_{\chi \in \mathcal{X}} J_\chi^T(f). \tag{2.3}$$

Moreover, the trace formula of $R(f)$ is given by

$$\mathrm{tr}\, R_{\mathrm{cusp}}(f) = \sum_{o \in \mathcal{O}} J_o^T(f) - \sum_{\chi \in \mathcal{X} \setminus \mathcal{X}(G)} J_\chi^T(f).$$

2.3. The dominant term on the geometric side

We first consider a number field F and denote the set of places, archimedean places, and non-archimedean places of F by V, V_∞, and V_f, respectively. Let $\mathbb{A}$ be adele ring of F and $\mathbb{A}_{\mathrm{fin}} \subset \mathbb{A}$ be restricted product over the finite places. Suppose $S \subset V$ is a finite set containing V_∞. Let $F_S = \prod_{v\in S} F_s$ and $\mathbb{A}^S = \prod'_{v\in V\setminus S} F_s$ so that $\mathbb{A} = F_S \times \mathbb{A}^S$. We define

(1) $G(F_S)^1 = \bigcap_{\chi\in \mathrm{Hom}(G(F_S),F^\times)}\{\ker|\chi|\colon G(F_S) \to \mathbb{R}_+\}$,

(2) $G(\mathbb{A})^1 = \bigcap_{\chi\in \mathrm{Hom}(G(\mathbb{A}),F^\times)}\{\ker|\chi|\colon G(\mathbb{A}) \to \mathbb{R}_+\}$,

where $|\cdot|$ is the product of valuations on F_S and $\mathbb{A}$, respectively.

We will consider the representation of $G(F_S)$ on $L^2(G(F)\backslash G(\mathbb{A})^1/K)$ for an open compact subgroup K of $G(\mathbb{A}^S)$. If we assume the strong approximation property, it reduces to the representation of $G(F_\infty)$ on $L^2(\Gamma_K\backslash F_\infty)$ if we take $S = \{\infty\}$ and $\Gamma_K = G(F) \cap K$.

Let $J(f)$ be the (truncated) distribution, given by equation (2.1) or (2.3) in Section 2 for $f \in C_c^\infty(G(F_S))$, where the truncation is needed if $G(F)\backslash G(\mathbb{A})^1$ is not compact. The goal of this subsection is to prove

$$\lim_{n\to\infty} \frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1) f(1)}{J(f \otimes 1_{K_n})} = 1 \tag{2.4}$$

for certain towers of open compact subgroups $\{K_n\}_{n\geq 1}$ of $G(\mathbb{A}^S)$.

Assume that Γ is a uniform lattice. We let R_Γ denote the representation of G on $L^2(\Gamma\backslash G)$ and $J_\Gamma(f)$ denote trace of $R_\Gamma(f)$ for $f \in C_c^\infty(G)$. By Theorem 2.1, we know

$$J_\Gamma(f) = \mathrm{tr}\, R_\Gamma(f) = \sum_{\pi\in\widehat{G}} J_{\pi,\Gamma}(f) = \sum_{o\in\mathcal{O}} J_{o,\Gamma}(f),$$

where Γ is added as a subscript to indicate the underlying space $L^2(\Gamma\backslash G)$. Let $J_{\{1\},\Gamma}(f) = \mathrm{vol}(\Gamma\backslash G) f(1)$, the contribution of the identity to the geometric side of the trace formula. We take a tower of uniform lattices $\{\Gamma_k\}_{k\geq 1}$ such that $\Gamma_k \trianglelefteq \Gamma_1$, $[\Gamma_1 : \Gamma_k] < \infty$, and $\bigcap_{n\geq 1} \Gamma_k = \{1\}$.

Proposition 2.4. *With the assumption of uniform lattice $\{\Gamma_k\}$ above, we have*

$$\lim_{n\to\infty} \frac{J_{\{1\},\Gamma_k}(f)}{J_{\Gamma_k}(f)} = 1.$$

Proof. Following [12, Corollary 4.6], we obtain

$$\mathrm{tr}\, R_{\Gamma_k}(\phi) = J_{\{1\},\Gamma_k}(\phi) + \sum_{\gamma\neq 1} s_k(\gamma)\,\mathrm{vol}(\Gamma_k\backslash G)\,\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{\Gamma_\gamma\backslash G} \phi(x^{-1}\gamma x)dx,$$

where $0 \leq s_k(\gamma) \leq \mathrm{vol}(\Gamma\backslash G)^{-1}$. As $\bigcap_{k\geq 1} \Gamma_k = \{1\}$, $\lim_{k\to\infty} s_k(\gamma) = 0$ for all $\gamma \neq 1$. Thus, we have $\mathrm{vol}(\Gamma_k\backslash G)^{-1} \cdot \lim_{k\to\infty} \mathrm{tr}\, R_{\Gamma_k}(\phi) = \phi(1)$ by [12, Theorem 5.2] (see also [10, Theorem 2]). Hence we have

$$\lim_{n\to\infty} \frac{J_{\{1\},\Gamma_k}(\phi)}{J_{\Gamma_k}(\phi)} = \lim_{n\to\infty} \frac{J_{\{1\},\Gamma_k}(\phi)}{\mathrm{tr}\, R_{\Gamma_k}(\phi)} = 1. \qquad \blacksquare$$

Now we let G be a reductive group over a number field F. Let $K = K_\infty K_{\text{fin}}$ be a maximal compact subgroup of $G(\mathbb{A}) = G(\mathbb{A}_F)$. By fixing a faithful F-rational representation $\rho\colon G(F) \to \mathrm{GL}(m, F)$ for some $m > 0$, we let $\Lambda \subset F^m$ be an $\mathcal{O}_F$-lattice such that the stabilizer of $\widehat{\Lambda} = \widehat{\mathcal{O}_F} \otimes_{\mathcal{O}_F} \Lambda$[2] in $G(\mathbb{A}_{\text{fin}})$ is K_{fin}.

For a non-trivial ideal I of $\mathcal{O}_F$ (the ring of integers of F), we let

$$K(I) = \{g \in G(\mathbb{A}_{\text{fin}}) \mid \rho(g)v \equiv v \ (\mathrm{mod}\ I \cdot \widehat{\Lambda}),\ v \in \widehat{\Lambda}\}$$

be the *principal congruence subgroup* of level I. We let $N(I) = [\mathcal{O}_F : I]$, the ideal norm of I. Consider a descending tower of ideals

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$$

such that each I_k is prime to (the prime ideals in) S. We obtain the corresponding tower of principal congruence subgroups:

$$K_1 \supsetneq K_2 \supsetneq K_3 \supsetneq \cdots,$$

where $K_n = K(I_n)$.

Lemma 2.5. *We have* $\bigcap_{n\geq 1} I_n = \{0\}$ *and* $\bigcap_{n\geq 1} K_n = \{1\}$.

Proof. By factoring into prime ideals, the family $\{I_n\}_{n\geq 1}$ satisfies either one of the following properties:

(1) there exists a prime ideal $\mathfrak{p}$ such that for each $k \geq 1$, $\mathfrak{p}^k$ is eventually contained in the tower, that is, for any $k \geq 1$, there is $N_k > 0$ such that $\mathfrak{p}^k \subset I_n$ for all $n \geq N_k$ or

(2) there exist infinitely many prime ideals $\{\mathfrak{p}_k\}_{k\geq 1}$ such that for each k, there exists $M_k > 0$ such that $\mathfrak{p}_k \subset I_n$ for all $n \geq M_k$.

Thus, if $x \in \bigcap_{n\geq 1} I_n$, we must have $p^k \mid x$ for all $k \geq 0$, or there are infinitely many primes p_k such that $p_k \mid x$. Thus, $x = 0$. ∎

Recall that the equivalence class of unipotent elements in $G(F)$ consists of the elements $\gamma = \gamma_s\gamma_u$ with the semisimple component $\gamma_s = 1$ (see [4, p. 1240]). For $f \in C_c^\infty(G(\mathbb{A})^1)$, let $J^T_{\text{unip}}(f)$ be the contribution of this equivalence class on the geometric side of the trace formula (2.3). We will consider the function of the form $f = h_S \otimes 1_{K_n}$ with $h_S \in C_c^\infty(G(F_S)^1)$.

Lemma 2.6. *For* $h_S \in C_c^\infty(G(F_S)^1)$, $\lim_{n\to\infty} J(h_S \otimes 1_{K_n}) = \lim_{n\to\infty} J_{\text{unip}}(h_S \otimes 1_{K_n})$.

[2] Here we let $\widehat{\mathcal{O}_F}$ denote the image of the canonical embedding of $\mathcal{O}_F$ into $\mathbb{A}_{\text{fin}}$.

Proof. Suppose $D_{h_S} = \text{supp}(h_S) \subset G(F_S)^1$ is the compact support of h_S. Then $\text{supp}(h_S \otimes 1_{K_n}) = D_{h_S} K_n$ is compact, and hence it intersects finitely many semisimple conjugate classes $o \in \mathcal{O}$.

Consider the trace formula (2.3); only the classes o's (and their conjugacy class) which intersect infinitely many $D_h K_n$ contribute a non-trivial $J_o(h_S \otimes 1_{K_n})$ to the limit $\lim_{n\to\infty} J(h_S \otimes 1_{K_n})$.

Suppose the $G(\mathbb{A})$ conjugacy classes of elements in o intersects $D_{h_S} K_n$ for infinitely many n, that is, $\{g\gamma g^{-1} \mid g \in G(\mathbb{A}),\ \gamma \in o\} \cap D_h K_n \neq \emptyset$ for infinitely many n. Take some $\gamma \in o$. By fixing a faithful F-representation $\rho\colon G(F) \to \text{GL}(m)$, we let $p(x) \in F[x]$ be the characteristic polynomial of $\rho(\gamma) - 1$ (an m-by-m matrix over F). Suppose $p(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with all $a_i \in F$. By Lemma 2.5, we know a_i belongs to infinitely many I_n, or, equivalently, $a_i = 0$. Hence $p(x) = x^m$, and γ is unipotent. ■

The unipotent contribution $J_{\text{unip}}(h_S \otimes 1_{K_n})$ can be further reduced to the one from the identity as follows. We let I_S be a product of prime ideals at the places of S and $K_{S-S_\infty}(I_S)$ be the $S - S_\infty$ component of the compact group $K(I_S)$. We also let $C_\Omega^\infty(G(F_S)^1)$ be the set of smooth functions with compact support contained in a compact subset Ω of $G(F_S)^1$. For each $k \geq 0$, we let $\mathcal{B}_k$ be the k-th component of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$, where $\mathfrak{g}_\mathbb{C}$ is the complexified Lie algebra of the Lie group $G(F_\infty)$. We set $\|h\|_k = \sum_{X\in\mathcal{B}_k} \|X \circ h\|_{L^1(G(\mathbb{A})^1)}$ for $h \in C_\Omega^\infty(G(F_S)^1)$.

The following result is a special case of [19, Proposition 3.1], whose proof is mainly based on [4, Theorems 3.1 and 4.2].

Proposition 2.7 (Finis–Lapid–Müller). *There exists an integer $k \geq 0$ such that for any compact subset Ω of $G(F_S)^1$, we have a constant $C_\Omega > 0$ and*

$$|J_{\text{unip}}(h_S \otimes 1_{K_n}) - \text{vol}(G(F)\backslash G(\mathbb{A})^1)h_S(1)| \leq C_\Omega \frac{(1 + \log N(I_S I))^{d_0}}{N(I)} \|h_S\|_k$$

for any bi-$K_{S-S_\infty}(I_S)$-invariant function $h_S \in C_\Omega^\infty(G(F_S)^1)$.

Then, together with Lemma 2.6, we can obtain the following result.

Corollary 2.8. *For $h_S \in C_c^\infty(G(F_S)^1)$, we have*

$$\lim_{n\to\infty} \frac{\text{vol}(G(F)\backslash G(\mathbb{A})^1)h_S(1)}{J(h_S \otimes 1_{K^S(n)})} = 1.$$

Thus, we prove the formula (2.4) for both uniform and non-uniform cases.

3. The multiplicities problem

This section is devoted to the multiplicity of bounded subsets of the unitary dual of a group instead of a single irreducible representation.

3.1. The multiplicities in $L^2(\Gamma\backslash G)$

We denote the connected component of the real group containing the group identity by G, that is, $G = G(\mathbb{R})^0$. Let $\widehat{G}$ be the unitary dual of G and $\widehat{G}_{\text{temp}} \subset \widehat{G}$ be the tempered dual. By fixing a faithful $\mathbb{Q}$-embedding $\rho : G(\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q})$, we have an arithmetic group Γ commensurable with $G \cap \mathrm{GL}_n(\mathbb{Z})$. Let us consider the following two cases.

(1) $\Gamma\backslash G$ *is compact*. As introduced in Section 2.1, $L^2(\Gamma\backslash G)$ can be decomposed into a direct sum of irreducible representations of G with each of finite multiplicity, that is,

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} m_\Gamma(\pi)\cdot\pi$$

with $m_\Gamma(\pi) := \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) < \infty$ for each $\pi \in \widehat{G}$.

(2) $\Gamma\backslash G$ *is not compact*. If G is semisimple, we have $\Gamma\backslash G$ is of finite volume. The regular representation has both discrete and continuous spectra:

$$L^2(\Gamma\backslash G) = L^2_{\text{disc}}(\Gamma\backslash G) \bigoplus L^2_{\text{cont}}(\Gamma\backslash G).$$

The discrete spectrum can be written as the direct sum of cuspidal and residue subspaces: $L^2_{\text{disc}}(\Gamma\backslash G) = L^2_{\text{cusp}}(\Gamma\backslash G) \bigoplus L^2_{\text{res}}(\Gamma\backslash G)$, which can be decomposed further into a direct sum of irreducible representations with finite multiplicities, that is,

$$L^2_{\text{disc}}(\Gamma\backslash G) = \bigoplus m_\Gamma(\pi)\cdot\pi,$$

where $m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, L^2_{\text{disc}}(\Gamma\backslash G)) = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ is finite for each $\pi \in \widehat{G}$ (note that only the discrete spectrum contributes to $\mathrm{Hom}_G(\pi, -)$).

For a subset $X \subset \widehat{G}$, we call it *bounded* if it is relatively compact under the Fell topology.

Definition 3.1 (The multiplicity for $X \subset \widehat{G}$). For a bounded $X \subset \widehat{G}$, we define the *multiplicity of* X to be the sum of the multiplicities of the irreducible representations in X, that is,

$$m_\Gamma(X) := \int_{\pi\in X} m_\Gamma(\pi).$$

Borel and Garland proved the following finiteness result on $m_\Gamma(X)$ by considering the spectrum of a certain Laplacian (see [9, Theorem 4.6]).

Theorem 3.2 (Borel–Garland). *Let $G = \mathbf{G}(\mathbb{R})^0$ for a connected semisimple group $\mathbf{G}$ over $\mathbb{Q}$ and $X \subset \widehat{G}$ be bounded. We have $m_\Gamma(X) < \infty$.*

Similarly, for a subset $X \subset \widehat{G(F_S)^1}$, we call it *bounded* if it is relatively compact under the Fell topology (see [33]).

Definition 3.3 (The multiplicity for $\widehat{G(F_S)^1}$). Suppose K is a compact open subgroup of $G(\mathbb{A}^S)$. Let σ be an irreducible representation of $G(F_S)^1$ and $X \subset \widehat{G(F_S)^1}$ be a bounded subset.

(1) The *multiplicity of σ with respect to K* is defined as

$$m_K(\sigma) := \dim \operatorname{Hom}_{G(F_S)^1}(\sigma, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K)).$$

(2) The *multiplicity of X with respect to K* is defined as

$$m_K(X) := \int_{\sigma \in X} m_K(\sigma).$$

It is also known that $m_K(X)$ is finite and hence well defined. We will show that the two types of multiplicities mentioned above coincide when $S = V_\infty$ (see Lemma 3.4).

For an irreducible representation π of $G(\mathbb{A})^1$, we write $\pi = \pi_S \otimes \pi^S$, where π_S and π^S denote the components of the representations of $G(F_S)^1$ and $G(\mathbb{A}^S)$, respectively. We treat $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K)$ as the subspace of K-right invariant functions in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. Noting that $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ has both discrete part and continuous part in general, we let $\operatorname{Hom}_{G(\mathbb{A})^1}(\pi, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1))$ be the multiplicity of π in the discrete part, which is known to be finite. We have

$$m_K(\sigma) = \sum_{\pi \in \widehat{G(\mathbb{A})^1}, \pi_S = \sigma} \dim \operatorname{Hom}_{G(\mathbb{A})^1}(\pi, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)) \dim(\pi^S)^K.$$

If we take $S = V_\infty$ and $\mathbf{G}$ is semisimple, simply connected, and without any F-simple factors H such that $H(F_\infty)$ is compact and K is an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$, we know $\Gamma_K = G(F) \cap K$ is a lattice in the semisimple Lie group $G(F_\infty)$.

Lemma 3.4. *With the assumption above, we have $m_{\Gamma_K}(\pi) = m_K(\pi)$ for any $\pi \in \widehat{G(F_\infty)^1}$ and $m_{\Gamma_K}(X) = m_K(X)$ for any bounded $X \subset \widehat{G(F_\infty)^1}$.*

Proof. It follows from the fact that $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ can be identified with $\Gamma_K\backslash G(F_\infty)$, which leads to a $G(F_\infty)$-isomorphism $L^2(\Gamma_K\backslash G(F_\infty)) \cong L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K)$ (see [30, Section 7.4]). ∎

For a finite set S and a suitable function ϕ on $\widehat{G(F_S)^1}$, we define

$$m_K(\phi) := \int_{\widehat{G(F_S)^1}} \phi(\pi) d m_K(\pi)$$

as an integral with respect to the measure given by multiplicities above. If 1_X is the characteristic function of X, we know $m_K(1_X) = m_K(X)$.

For $f \in C_c^\infty(G(F_S)^1)$, we let $\widehat{f}(\pi) = \operatorname{tr} \pi(f)$, the distribution character of π. Let R_{disc} denote the action of $G(\mathbb{A})$ on the discrete subspace $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$.

Proposition 3.5. *For $f \in C_c^\infty(G(F_S)^1)$, we have*

$$\operatorname{tr} R_{\mathrm{disc}}\left(f \otimes \frac{1_K}{\operatorname{vol}(K)}\right) = m_K(\widehat{f}).$$

Proof. Observe that for the component π^S of representation of $G(\mathbb{A}^S)$, we have

$$\begin{aligned}\operatorname{tr} \pi^S(1_K) &= \int_{G(\mathbb{A}^S)} 1_K(x)\pi^S(x^{-1})d\mu^S(x) \\ &= \int_K \pi^S(x^{-1})d\mu^S(x) = \operatorname{vol}(K)\dim(\pi^S)^K,\end{aligned}$$

where we apply the fact that $\int_K \sigma(x)d\mu^S(x) = 0$ for any non-trivial irreducible representation σ of K. Hence we obtain

$$\begin{aligned}\operatorname{tr} R_{\mathrm{disc}}\left(f \otimes \frac{1_K}{\operatorname{vol}(K)}\right) &= \frac{1}{\operatorname{vol}(K)} \sum_{\pi \in \widehat{G(\mathbb{A})^1}} m(\pi) \operatorname{tr} \pi(f \otimes 1_K) \\ &= \frac{1}{\operatorname{vol}(K)} \sum_{\pi \in \widehat{G(\mathbb{A})^1}} m(\pi) \operatorname{tr} \pi_S(f) \operatorname{tr} \pi^S(1_K) \\ &= \frac{1}{\operatorname{vol}(K)} \sum_{\pi \in \widehat{G(\mathbb{A})^1}} m(\pi) \operatorname{tr} \pi_S(f) \operatorname{vol}(K)\dim(\pi^S)^K \\ &= \sum_{\sigma \in \widehat{G(F_S)^1}} m_K(\sigma) \operatorname{tr} \sigma(f) = m_K(\widehat{f}).\end{aligned}$$

∎

The following result connects the trace formulas for adelic groups and Lie groups.

Corollary 3.6. *Let $\Gamma_K = G(F) \cap K$ with an open compact subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$. We have*

$$\operatorname{tr} R_{\mathrm{disc}}\left(f \otimes \frac{1_K}{\operatorname{vol}(K)}\right) = \operatorname{tr} R_{\Gamma_K}(f)$$

for all $f \in C_c^\infty(G(F_\infty)^1)$.

Proof. It follows from the fact that $m_K(\widehat{f}) = m_{\Gamma_K}(\widehat{f})$ in Lemma 3.4, $m_{\Gamma_K}(\widehat{f}) = \operatorname{tr} R_{\Gamma_K}(f)$ and Proposition 3.5. ∎

3.2. Sauvageot's density theorems

We have a brief review of the results in [33] and introduce a variation for towers of lattices. See also [35] for an alternative approach and corrections.

For an open compact subgroup K of $G(\mathbb{A}^S)$, we define a measure on $\widehat{G(F_S)^1}$ by

$$\nu_K(X) := \frac{\operatorname{vol}(K)}{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)} m_K(X)$$

for any bounded subset X of $\widehat{G(F_S)^1}$, and m_K is the multiplicity defined in Section 3.1. Let $K_1 \supsetneq K_2 \supsetneq \cdots$ be a sequence of open compact subgroups of $G(\mathbb{A}^S)$. Given a bounded subset X of $\widehat{G(F_S)^1}$ and $C \geq 0$, we write

$$\lim_{n\to\infty} \nu_{K_n}(X) = C$$

if for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that $|\nu_{K_n}(X) - C| < \varepsilon$ for all $n \geq N$.

Let $\mathcal{H}(G(F_S)^1)$ be the complex algebra of smooth, compactly supported, bi-K_S-finite functions on $G(F_S)^1$. For the measure ν_K above, we let

$$\nu_K(\widehat{\phi}) := \int_{\widehat{G(F_S)^1}} \phi(\pi) d\nu_K(\pi),$$

where $\phi \in \mathcal{H}(G(F_S)^1)$. Let ν denote the Plancherel measure on the unitary dual.

Lemma 3.7 ([33, Corollaire 6.2]). *For $\varepsilon > 0$ and any bounded $X \subset \widehat{G(F_S)^1} \backslash \widehat{G(F_S)^1}_{\text{temp}}$, there is $\Psi \in \mathcal{H}(G(F_S)^1)$ such that*

$$\widehat{\Psi}|_{\widehat{G(F_S)^1}} \geq 0, \quad \nu(\widehat{\Psi}) < \varepsilon, \quad \textit{and} \quad \widehat{\Psi}|_X \geq 1.$$

Given a function f defined on $\widehat{G(F_S)^1}_{\text{temp}}$, we also denote by f the function on $\widehat{G(F_S)^1}$, which is extended by zero on the untempered part.

Lemma 3.8 ([33, Theoreme 7.3(b)]). *For $\varepsilon > 0$ and any ν-integrable function f on $\widehat{G(F_S)^1}_{\text{temp}}$, there exist $\phi, \psi \in \mathcal{H}(G(F_S)^1)$ such that*

$$|f(\pi) - \widehat{\phi}(\pi)| \leq \widehat{\psi}(\pi) \quad \textit{and} \quad \nu(\widehat{\psi}) < \varepsilon.$$

Here we obtain one of the main results in [33]. A short proof is provided here for completeness.

Theorem 3.9 (Sauvageot). *Suppose $\lim_{n\to\infty} \nu_{K_n}(\widehat{\phi}) = \phi(1)$ for all $\phi \in \mathcal{H}(G(F_S)^1)$. We have*

$$\lim_{n\to\infty} \nu_{K_n}(X) = \nu(X)$$

for all bounded subsets X of $\widehat{G(F_S)^1}$.

Proof. First, we show that the contribution from the untempered part is negligible in the limit. For a bounded subset X_0 of $\widehat{G(F_S)^1} \backslash \widehat{G(F_S)^1}_{\text{temp}}$ and $\varepsilon > 0$, we let $\Psi \in \mathcal{H}(G(F_S)^1)$ satisfy Lemma 3.7 with respect to X. We have $\nu_{K_n}(X) \leq \nu_{K_n}(\widehat{\Psi}) \leq |\nu_{K_n}(\widehat{\Psi}) - \psi(1)| + \psi(1) < 2\varepsilon$ for all $n \geq N_1$ with some $N_1 \geq 0$.

For the tempered part, we fix a bounded subset X_1 of $\widehat{G(F_S)^1}_{\text{temp}}$ with the same ε above. Let $\phi, \psi \in \mathcal{H}(G(F_S)^1)$ satisfy Lemma 3.8 with respect to the function $f = 1_{X_1}$ on

$\widehat{G(F_S)^1}_{\text{temp}}$ and ε. By assumption, we have $|\nu_{K_n}(\widehat{\phi}) - \phi(1)| < \varepsilon$ and $|\nu_{K_n}(\widehat{\psi}) - \psi(1)| < \varepsilon$ for all $n \geq N_2$ with some $N_2 \geq 0$. Hence, for $n \geq N_2$, we obtain

$$\begin{aligned}|\nu_{K_n}(X_1) - \nu(X_1)| &\leq |\nu_{K_n}(X_1) - \nu_{K_n}(\widehat{\phi})| + |\nu_{K_n}(\widehat{\phi}) - \phi(1)| + |\phi(1) - \nu(X)| \\ &\leq |\nu_{K_n}(\widehat{\phi}) - \phi(1)| + \nu_{K_n}(\widehat{\psi}) + \psi(1) \\ &\leq |\nu_{K_n}(\widehat{\phi}) - \phi(1)| + |\nu_{K_n}(\widehat{\psi}) - \psi(1)| + 2\psi(1) < 4\varepsilon.\end{aligned}$$

Hence, for the bounded set X of $\widehat{G(F_S)^1}$, let $X = X_0 \sqcup X_1$ be the decomposition into its untempered and tempered parts. We have

$$\begin{aligned}|\nu_{K_n}(X) - \nu(X)| &= |\nu_{K_n}(X) - \nu(X_1)| = |\nu_{K_n}(X_1) - \nu(X)| + \nu_{K_n}(X_0) \\ &\leq 4\varepsilon + 2\varepsilon = 6\varepsilon\end{aligned}$$

for all $N \geq \max\{N_1, N_2\}$. ■

We now give a description of Sauvageot's results in the view of lattices. For a lattice Γ of $G(F_S)$, we define a measure on $\widehat{G(F_S)^1}$ by

$$\nu_\Gamma(X) := \frac{1}{\mathrm{vol}(\Gamma \backslash G(F_S)^1)} m_\Gamma(X)$$

for any bounded subset X of $\widehat{G(F_S)^1}$, and $m_\Gamma(X)$ is integral of the multiplicities of $\pi \in X$ in the discrete part of $L^2(\Gamma \backslash G(F_S)^1)$. For the measure ν_Γ above, we can also define

$$\nu_\Gamma(\widehat{\phi}) := \int_{\widehat{G(F_S)^1}} \phi(\pi) d\nu_\Gamma(\pi),$$

where $\phi \in \mathscr{H}(G(F_S)^1)$. Thus, given a sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ of lattices of $G(F_S)$ and a bounded subset X of $\widehat{G(F_S)^1}$, we can say $\lim_{n\to\infty} \nu_{\Gamma_n}(X) = C$ if for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that $|\nu_{\Gamma_n}(X) - C| < \varepsilon$ for all $n \geq N$. Please note that all these definitions coincide with the ones above for compact open subgroups $\{K\}_{n\geq 1}$ of $G(\mathbb{A}_S)$ when $G(F)$ satisfies the strong approximation property with respect to S and $\Gamma_n = G(F) \cap K_n$. The following result can be directly obtained from Theorem 3.9.

Corollary 3.10. *Suppose* $\lim_{n\to\infty} \nu_{\Gamma_n}(\widehat{\phi}) = \phi(1)$ *for all* $\phi \in \mathscr{H}(G(F_S)^1)$. *We have*

$$\lim_{n\to\infty} \nu_{\Gamma_n}(X) = \nu(X)$$

for all bounded subsets X *of* $\widehat{G(F_S)^1}$.

4. The von Neumann dimensions of direct integrals

4.1. The group von Neumann algebra and the trace

Let Γ be a countable group with the counting measure. Let $\{\delta_\gamma\}_{\gamma\in\Gamma}$ be the usual orthonormal basis of $l^2(\Gamma)$. We also let λ and ρ be the left and right regular representations of Γ on

$l^2(\Gamma)$, respectively. For all $\gamma, \gamma' \in \Gamma$, we have $\lambda(\gamma')\delta_\gamma = \delta_{\gamma'\gamma}$ and $\rho(\gamma')\delta_\gamma = \delta_{\gamma\gamma'^{-1}}$. Let $\mathcal{L}(\Gamma)$ be the strong operator closure of the complex linear span of $\lambda(\gamma)$'s (or equivalently, $\rho(\gamma)$'s). This is the *group von Neumann algebra of* Γ. There is a canonical faithful normal tracial state τ_Γ, or simply τ, on $\mathcal{L}(\Gamma)$, which is given by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle_{l^2(\Gamma)}, \quad x \in \mathcal{L}(\Gamma).$$

Hence $\mathcal{L}(\Gamma)$ is a finite von Neumann algebra (which must be of type I or II_1).

More generally, for a tracial von Neumann algebra M with the trace τ, we consider the GNS representation of M on the Hilbert space constructed from the completion of M with respect to the inner product $\langle x, y\rangle_\tau = \tau(xy^*)$. The underlying space will be denoted by $L^2(M, \tau)$, or simply $L^2(M)$.

Consider a normal unital representation $\pi\colon M \to B(H)$ with both M and H separable. There exists an isometry $u\colon H \to L^2(M) \otimes l^2(\mathbb{N})$, which commutes with the actions of M:

$$u \circ \pi(x) = (\lambda(x) \otimes \mathrm{id}_{l^2(\mathbb{N})}) \circ u \quad \forall x \in M,$$

where $\lambda\colon M \mapsto L^2(M)$ denotes the left action. Then $p = uu^*$ is a projection in $B(L^2(M) \otimes l^2(\mathbb{N}))$ such that $H \cong p(L^2(M) \otimes l^2(\mathbb{N}))$. We have the following result (see [2, Proposition 8.2.3]).

Proposition 4.1. *The correspondence $H \mapsto p$ above defines a bijection between the set of equivalence classes of left M-modules and the set of equivalence classes of projections in $(M' \cap B(L^2(M))) \otimes B(l^2(\mathbb{N}))$.*

The *von Neumann dimension* of the M-module H is defined to be $(\tau \otimes \mathrm{Tr})(p)$ and denoted by $\dim_M(H)$, which takes its value in $[0, \infty]$. We have:

(1) $\dim_M\left(\bigoplus_i H_i\right) = \sum_i \dim_M(H_i)$.

(2) $\dim_M(L^2(M)) = 1$.

Note that $\dim_M(H)$ depends on the trace τ. If M is a finite factor, that is, $Z(M) \cong \mathbb{C}$, there is a unique normal tracial state (see [23, 27]), and we further have:

(3) $\dim_M(H) = \dim_M(H')$ if and only if H and H' are isomorphic as M-modules (provided M is a factor).

When M is not a factor, there is a $Z(M)$-valued trace which determines the isomorphism class of an M-module (see [8]).

In the following sections, we will consider the group von Neumann algebra $\mathcal{L}(\Gamma)$ with the canonical tracial state $\mathrm{tr}(x) = \langle x\delta_e, \delta_e\rangle$. Hence the von Neumann dimension of $\mathcal{L}(\Gamma)$ is the one uniquely determined by this trace. Recall that a discrete group Γ is called an infinite conjugacy class (ICC) group if every non-trivial conjugacy class $C_\gamma = \{g\gamma g^{-1} \mid g \in \Gamma\}$, $\gamma \neq e$, is infinite. It is well known that $\mathcal{L}(\Gamma)$ is a II_1 factor if and only if Γ is a non-trivial ICC group.

We assume that Γ is a discrete subgroup of a locally compact unimodular type I group G. Let μ be a Haar measure of G. A measurable set $D \subset G$ is called a *fundamental* *domain* for Γ if D satisfies $\mu\big(G\backslash \bigcup_{\gamma\in\Gamma}\gamma D\big)=0$ and $\mu(\gamma_1 D\cap\gamma_2 D)=0$ if $\gamma_1\neq\gamma_2$ in Γ. In this section, we always assume Γ is a lattice, that is, $\mu(D)<\infty$. The measure $\mu(D)$ is called *covolume* of Γ and will be denoted by $\mathrm{covol}(\Gamma)$. Note that the covolume depends on the Haar measure μ (see Remark 4.3).

There is a natural isomorphism $L^2(G)\cong l^2(\Gamma)\otimes L^2(D,\mu)$ given by

$$\phi\mapsto\sum_{\gamma\in\Gamma}\delta_\gamma\otimes\phi_\gamma\quad\text{with } \phi_\gamma(z)=\phi(\gamma\cdot z),$$

where $z\in D$ and $\gamma\in\Gamma$. The restriction representation $\lambda_G|_\Gamma$ of Γ is the tensor product of λ_Γ on $l^2(\Gamma)$ and the identity operator id on $L^2(D,\mu)$. Hence we obtain the von Neumann algebra $\lambda_G(\Gamma)''\cong\mathcal{L}(\Gamma)\otimes\mathbb{C}=\mathcal{L}(\Gamma)$, which will be denoted by M throughout this section. Please note $L^2(M)\cong L^2(\mathcal{L}(\Gamma))\cong l^2(\Gamma)$ as modules over Γ and $\mathcal{L}(\Gamma)$.

4.2. A theorem of von Neumann dimensions

Suppose X is a measurable subset of $\widehat{G}$ with the Plancherel measure[3] $\nu(X)<\infty$. Define

$$H_X=\int_X^{\oplus}H_\pi\,d\nu(\pi),$$

which is the direct integral of the spaces H_π over all $\pi\in X$ with respect to the Plancherel measure on $\widehat{G}$. It is a module over G, its lattice Γ, and also the group von Neumann algebra $\mathcal{L}(\Gamma)$.

We state a result on the von Neumann dimension of direct integrals.

Theorem 4.2. *Let G be a locally compact unimodular type I group with Haar measure μ. Let ν be the Plancherel measure on the unitary dual $\widehat{G}$ of G. Suppose Γ is a lattice in G and $\mathcal{L}(\Gamma)$ is the group von Neumann algebra of Γ. Let $X\subset\widehat{G}$ such that $\nu(X)<\infty$ and $H_X=\int_X^{\oplus}H_\pi\,d\nu(\pi)$. We have*

$$\dim_{\mathcal{L}(\Gamma)}(H_X)=\mathrm{covol}(\Gamma)\cdot\nu(X).$$

The proof is mainly based on the G-equivariant embedding $H_X\hookrightarrow L^2(G)$, which generalizes the basic case of the embedding of a single discrete series representation into $L^2(G)$. One can refer to [36, Section 4] for the proof.

[3]Recall that the Plancherel measure of the unitary dual $\widehat{G}$ is a measure ν on $\widehat{G}$ such that there is a G-G isomorphism

$$L^2(G)\cong\int_{\widehat{G}}^{\oplus}H_\pi\otimes H_{\pi^*}\,d\nu(\pi)$$

which is given by the Fourier transform $f\mapsto\hat{f}(\pi)=\int_G f(g)\pi(g^{-1})dg$ for $f\in L^1(G)\cap L^2(G)$ (see [20, Section 7.5]).

Remark 4.3. This dimension is well defined and is related to the Betti number.

(1) If $\mu' = k \cdot \mu$ is another Haar measure on G for some $k > 0$, the covolumes are related by $\mathrm{covol}'(\Gamma) = \mu'(G/\Gamma) = k' \cdot \mu(G/\Gamma) = k \cdot \mathrm{covol}(\Gamma)$. But the induced Plancherel measure $\nu' = k^{-1} \cdot \nu$ and the dependencies cancel out in the formula above.

(2) There is a relevant approach by H. Petersen and A. Valette [29]. They study the von Neumann dimension over locally compact groups. The group von Neumann algebra is equipped with a semifinite tracial weight instead of a tracial state for a discrete group. It is motivated by the study of L^2-Betti number of locally compact groups [28].

If π is an atom in $\widehat{G}$, that is, $\nu(\{\pi\}) > 0$, the irreducible representation π is a discrete series, and $\nu(\{\pi\})$ is just the formal dimension of π [15, 31]. Under this assumption, if G is a real Lie group that has a discrete series and Γ is an ICC group, the theorem reduces to formula (1.2) of a single discrete series representation (see [22, Theorem 3.3.2]).

4.3. The proof of the main theorem

We will prove the main theorem. We first give the proof for a tower of uniform lattices.

Theorem 4.4 (A tower of uniform lattices). *Let $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ be a normal tower of cocompact lattices in a semisimple real Lie group G such that $\bigcap_{n \geq 1} \Gamma_n = \{1\}$. For any bounded subset X of $\widehat{G}$, we have*

$$\lim_{n \to \infty} \frac{m_{\Gamma_n}(X)}{\dim_{\mathcal{L}(\Gamma_n)} H_X} = 1.$$

Proof. Recall that $m_{\Gamma_n}(X) = \mathrm{vol}(\Gamma_n \backslash G(F_\infty)) \cdot \nu_{\Gamma_n}(X)$ (see the definitions above Corollary 3.10). Moreover, Theorem 4.2 gives the formula

$$\dim_{\mathcal{L}(\Gamma_k)} H_X = \mathrm{vol}(\Gamma_k \backslash G(F_\infty)) \nu(X).$$

We need to show $\lim_{n\to\infty} \nu_{\Gamma_n}(X) = \nu(X)$, which reduces to

$$\lim_{n \to \infty} \nu_{\Gamma_n}(\widehat{\phi}) = \phi(1)$$

for all $\phi \in C_c^\infty(G(F_\infty)^1)$ by Corollary 3.10.

If the real rank of G is 1, it follows [12, Theorem 5.2]. We assume that the real rank of G is at least 2. Without loss of generality, we can take the lattices $\{\Gamma_n\}$ to be arithmetic by Margulis' arithmetic theorem (see [37, Theorem 6.1.2]). Thus, we assume $\Gamma_n = G(F) \cap K_n$ for some open compact subgroup K_n of $G(\mathbb{A}^\infty)$. By Proposition 3.5, for such an open compact subgroup K, we know

$$\mathrm{tr}\, R_{\mathrm{disc}}\left(\phi \otimes \frac{1_K}{\mathrm{vol}(K)}\right) = m_K(\widehat{\phi}) = \mathrm{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K) \cdot \nu_K(\widehat{\phi}),$$

which is to say $\operatorname{tr} R_{\mathrm{disc}}(\phi \otimes 1_K) = \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \cdot \nu_K(\widehat{\phi})$. By Proposition 2.4, we have $\lim_{n\to\infty} \operatorname{tr} R_{\mathrm{disc}}(\phi \otimes 1_{K_n}) = \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \cdot \phi(1)$. Hence $\lim_{n\to\infty} \nu_{\Gamma_n}(\widehat{\phi}) = \lim_{n\to\infty} \nu_{K_n}(\widehat{\phi}) = \phi(1)$. ∎

For a non-uniform subgroup Γ, the distribution $J_\Gamma(f)$, or simply $J(f)$ (see equation (2.3)), will no longer be the trace of $R_{\mathrm{disc}}(f)$. Fortunately, Finis–Lapid–Müller proved the following result on the limit of the spectral side of equation (2.3) (see [19, Corollary 7.8]).

Theorem 4.5 (Finis–Lapid–Müller). *Suppose $G = \mathrm{SL}(n)$. Let $\{I_k\}$ be a family of descending integral ideals in $\mathcal{O}_F$ prime to S and $K_k = K(I_k)$ be the compact subgroups of $G(\mathbb{A}^S)$ given by I_k. We have*

$$\lim_{k\to\infty} J(h_S \otimes 1_{K_k}) = \lim_{k\to\infty} \operatorname{tr} R_{\mathrm{disc}}(h_S \otimes 1_{K_k})$$

for any $h_S \in C_c^\infty(G(F_S)^1)$.

Then we are able to prove the following corollary.

Corollary 4.6. *Let $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots$ be a tower of principal congruence subgroups in $G = \mathrm{SL}(n, \mathbb{R})$. For any bounded subset X of $\widehat{G}$, we have*

$$\lim_{k\to\infty} \frac{m_{\Gamma_k}(X)}{\dim_{\mathcal{L}(\Gamma_k)} H_X} = 1.$$

Proof. As shown in the proof of Theorem 4.4, we may apply Theorem 3.9, and it then suffices to prove $\lim_{k\to\infty} \nu_{K_k}(\widehat{\phi}) = \phi(1)$ for all $\phi \in C_c^\infty(G(F_\infty)^1)$.

By Proposition 3.5 and Theorem 4.5, we know

$$\lim_{k\to\infty} \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \cdot \nu_{K_k}(\widehat{\phi}) = \lim_{k\to\infty} \operatorname{tr} R_{\mathrm{disc}}(\phi \otimes 1_{K_k}) = \lim_{k\to\infty} J(\phi \otimes 1_{K_k}).$$

As $\lim_{k\to\infty} \frac{\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)\phi(1)}{J(\phi\otimes 1_{K_k})} = 1$ by Corollary 2.8, we obtain $\lim_{k\to\infty} \nu_{K_k}(\widehat{\phi}) = \phi(1)$. ∎

Remark 4.7. Here we mention some connections with trace formulas and Betti numbers.

(1) The extension to semisimple groups other than $\mathrm{SL}(n)$ is expected with a further study of the refined Arthur's trace formula. Thus, the reductive group cases are related to certain semisimple quotients or subgroups that appeared around the identification (1.6).

(2) The von Neumann dimension is highly related to the term $\operatorname{vol}(\Gamma\backslash G)f(1)$ in the trace formulas. The analysis for the other geometric terms or orbit integrals may lead to a refined description of the spectral side in the view of operator algebras.

(3) The von Neumann dimensions over discrete groups also have a strong connection with the L^2-Betti numbers (see [26]), which have been shown to be related to the multiplicity problem in [1].

Acknowledgments. We thank Arthur Jaffe and Barry Mazur for their encouragement and insightful comments.

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Received 28 August 2024; revised 9 August 2025.

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