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Weighted Poisson polynomial rings in dimension three

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Abstract. We discuss Poisson structures on a weighted polynomial algebra $A := \Bbbk[x, y, z]$ defined by a homogeneous element $\Omega \in A$, called a *potential*. We start with classifying potentials Ω of degree $\deg(x) + \deg(y) + \deg(z)$ with any positive weight $(\deg(x), \deg(y), \deg(z))$ and list all with isolated singularity. Based on the classification, we study the rigidity of A in terms of graded twistings and classify Poisson fraction fields of $A/(\Omega)$ for irreducible potentials. Using Poisson valuations, we characterize the Poisson automorphism group of A when Ω has an isolated singularity extending a nice result of Makar-Limanov–Turusbekova–Umirbaev. Finally, Poisson cohomology groups are computed for new classes of Poisson polynomial algebras.

Contents

1. Introduction

Poisson algebras are used in classical mechanics to describe observable evolution in Hamiltonian systems. They have been widely studied concerning topics such as (twisted) Poincaré duality and modular derivations [32, 34, 53], Poisson Dixmier–Moeglin equivalence [4, 19, 21, 26, 35], Poisson enveloping algebras [1–3, 30, 31], noncommutative discriminant [7, 8, 28, 38] and so on. They have also been utilized to study the representation theory of PI Sklyanin algebras [51, 52]. Additionally, Poisson algebras have

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been investigated in the context of the isomorphism problem, invariant theory and the cancellation problem [16–18, 36].

Let $\Bbbk$ be an algebraically closed base field of characteristic zero throughout. Quadratic Poisson structures with $\deg(x_i) = 1$ for all i on $\Bbbk[x_1, \dots, x_n]$ have been applied in various fields, as discussed in papers [5, 20, 22, 29, 43] and their references. Note that the deformation quantization of such a Poisson structure is the homogeneous coordinate ring of quantum $\mathbb{P}^{n-1}$. It is Calabi–Yau if the corresponding formal Poisson bracket on $\Bbbk[x_1, \dots, x_n]$ is unimodular [10]. The modular derivations can be considered a Poisson analog of the Nakayama automorphisms of skew Calabi–Yau algebras. For more information on skew Calabi–Yau algebras, see [44, 45] and related references. A notable family of quadratic Poisson structures is the elliptic Poisson algebras. These were independently introduced by Feĭgin and Odesskiĭ [14] and Polishchuk [41]. Elliptic Poisson algebras can be viewed as semi-classical limits of the elliptic Sklyanin algebras studied by Feigin and Odesskii [13].

A unimodular Poisson structure on $\Bbbk[x, y, z]$ is determined by a potential $\Omega \in \Bbbk[x, y, z]$. Elliptic Poisson algebras in 3 variables are defined as a particular case by a homogeneous potential Ω of degree 3 with an isolated singularity at the origin. Van den Bergh earlier considered these elliptic Poisson algebras in his work on Hochschild homology of 3-dimensional Sklyanin algebras [50]. Makar-Limanov–Turusbekova–Umirbaev computed Poisson automorphism groups of these algebras [37] when all the generators have degree 1. Recently, it has been proved that every connected graded Poisson polynomial algebra is a twist of a unimodular Poisson polynomial algebra [49].

In associative algebra, Stephenson [46, 47] has classified and studied the weighted version of connected graded Artin–Schelter regular (or skew Calabi–Yau) algebras of global dimension three. However, there is little knowledge about the Poisson analog of these algebras.

1.1. General setup

In this paper, the graded Poisson polynomial algebras A in dimension three are given by a weighted homogeneous potential Ω. We relax two assumptions made in the elliptic ones above: (a) generators being in degree 1 and (b) isolated singularities of potential Ω. We study those A that exhibit similar Poisson cohomological behaviors to elliptic Poisson algebras. Additionally, we are interested in the Poisson automorphism groups of A and some of its quotients $A/(\Omega - \xi)$, where $\xi \in \Bbbk$.

Set $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \dots\}$. An algebra A is said to be *connected graded* if $A = \bigoplus_{i \geq 0} A_i$ is $\mathbb{N}$-graded and $A_0 = \Bbbk$. If so, we use $|f|$ to denote the degree of a homogeneous element $f \in A$. We say A is a *connected w-graded Poisson algebra* (for $w \in \mathbb{Z}$) if $A = \bigoplus_{i \geq 0} A_i$ is a connected graded algebra, where the Poisson bracket of A satisfies $\{A_i, A_j\} \subseteq A_{i+j-w}$ for all $i, j \geq 0$. If $w = 0$, we simply say A is a connected graded Poisson algebra. Below is a general setup for some of the main objects in this paper. We shall make it clear in the context when such a setup or part of it is in place.

Hypothesis 1.1. (1) Suppose that $A := \Bbbk[x, y, z]$ is a weighted polynomial algebra with $\deg(x) = a, \deg(y) = b, \deg(z) = c$ for $a, b, c \in \mathbb{N}_+$.

(2) Let $\Omega \in A$ be a nonzero homogeneous element of degree $n > 0$. We call Ω a potential of A and set $w := n - a - b - c$.

(3) Let A_Ω denote $\Bbbk[x, y, z]$ with a Poisson structure determined by $\Omega \in \Bbbk[x, y, z]$ as follows (Definition 4.1):

$$\{f, g\} = \det\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ \Omega_x & \Omega_y & \Omega_z \end{pmatrix} \quad \text{for all } f, g \in \Bbbk[x, y, z].$$

(4) $w = 0$, equivalently $n = a + b + c$.

Remark 1.2. To save space, the following will be implemented:

(1) Concerning the classification of Ω, it is always assumed that $1 \le a \le b \le c$.

(2) We will use tables, such as Table 1, at the end of the introduction and in Appendix A to present results concisely.

(3) Some computations will not be shown for Theorems 4.6 and 5.2 and Proposition 7.3 but are available from the authors.

(4) When analyzing arguments divided into cases, authors typically provide a detailed analysis for one case and skip details for others if the proofs are similar. All necessary details can be provided upon request.

1.2. Classification

We classify all potentials Ω in $\Bbbk[x, y, z]$ of degree $a + b + c$ (refer to Theorem 3.5 for details). The classification for $(a, b, c) = (1, 1, 1)$ is well known [6, 11, 12, 25, 29]. We characterize all possible weights $(a, b, c) \in \mathbb{N}_+^3$ on $\Bbbk[x, y, z]$ that guarantee the existence of potentials Ω of degree $a + b + c$ with isolated singularities (Lemma 4.10). Together with Theorem 3.5, we identify all three possible parametric families of such potentials.

Theorem 1.3 (Lemma 4.10). *Assume (1), (2) and (4) of Hypothesis 1.1 with $a \le b \le c$ and $\gcd(a, b, c) = 1$. Below is a complete list of potentials with isolated singularities of degree $a + b + c$ (up to graded automorphisms):*

(1) $\Omega_1 = x^3 + y^3 + z^3 + \lambda xyz$ *for* $(-\lambda)^3 \neq 3 \cdot 3 \cdot 3$ *and* $(a, b, c) = (1, 1, 1)$.

(2) $\Omega_2 = x^4 + y^4 + z^2 + \lambda xyz$ *for* $(-\lambda)^4 \neq 4 \cdot 4 \cdot 2^2$ *and* $(a, b, c) = (1, 1, 2)$.

(3) $\Omega_3 = x^6 + y^3 + z^2 + \lambda xyz$ *for* $(-\lambda)^6 \neq 6 \cdot 3^2 \cdot 2^3$ *and* $(a, b, c) = (1, 2, 3)$.

Note that these potentials correspond to the homogeneous coordinate rings of Veronese embeddings of elliptic curves in a weighted projective plane. The embeddings are defined by a divisor $D = kP$ with a marked point $P \in E$ and $k = 3, 2, 1$. Our result will hopefully have independent interests in weighted projective spaces.

Ω-type	Proj. curve $\Omega = 0$	GKdim of A_{sing}	$\text{rgt}(A)$	$\text{Aut}(A)$ graded?	$Q(P_\Omega)$	uPH^2-vacant	K_1-sealed	$h_{\text{PH}^i(A)}$ computed?
ⓘ	Smooth	0	0	Yes	$S_{\xi,\lambda}$	Yes	Yes	Yes
ⓠ	Nodal singularity	1	0	?	K_q	Yes	?	Yes
(bw)	Cusp singularity	1	0	No	K_{Weyl}	Yes	?	Yes
(nw)	Cusp singularity	1	0	No	K_{Weyl}	No	No	No
ⓡ	Reducible	$\{1, 2\}$	≤ -1	Some	Undefined	No	No	Some

Table 1. Potential Ω.

1.3. Rigidities

In [49], the Poisson version of graded twists of graded associative algebras introduced by Zhang [55] was used to define a numerical invariant $\mathrm{rgt}(A)$ (Definition 2.2) for any $\mathbb{Z}$-graded Poisson algebra A. This invariant measures the size of the vector space of graded twists of A. If $\mathrm{rgt}(A) = 0$, then all graded twists of A are isomorphic to A, and we call A rigid. When $A = \Bbbk[x, y, z]$ is a polynomial Poisson algebra generated in degree 1, it was shown in [49, Corollary 6.7] that any connected graded unimodular Poisson structure on A is rigid if and only if the associated potential Ω is irreducible. We generalize this equivalence to the weighted case.

Theorem 1.4 (Theorem 5.2). *Assume Hypothesis 1.1. Then $\mathrm{rgt}(A_\Omega) = 0$ if and only if Ω is irreducible.*

In Subsections 5.2 and 5.3, we will briefly discuss two other types of rigidities.

1.4. Automorphism problem

One of the aims of this paper is to present universal methods for studying Poisson algebra on essential subjects such as Poisson automorphism groups (Theorem 1.5) and Poisson cohomologies (Theorem 1.6).

In [24], Poisson valuations were introduced and used to solve rigidity, automorphism, isomorphism and embedding problems for various Poisson algebras/fields. We use them to determine Poisson automorphism groups for A_Ω and its quotient $A_\Omega/(\Omega - \xi)$ when Ω is a potential of degree $a + b + c$ with isolated singularity. Our approach provides an alternative method for determining the automorphism groups of 3-dimensional elliptic Poisson algebras when the Poisson algebra A_Ω is generated in degree 1, differing from [37].

Theorem 1.5 (Theorem 4.9). *Assume Hypothesis 1.1. Suppose Ω has an isolated singularity. Denote by $P_{\Omega-\xi} = A_\Omega/(\Omega - \xi)$ for $\xi \in \Bbbk$ and write $P_{\Omega-0}$ simply as P_Ω.*

(1) *Every Poisson automorphism of P_Ω is graded, and every Poisson automorphism of $P_{\Omega-\xi}$ is linear when $\xi \neq 0$.*

(2) *Every Poisson automorphism of A_Ω is graded.*

Moreover, the explicit forms of the automorphism groups of $P_{\Omega-\xi}$ and A_Ω are provided in Lemmas 4.11–4.14.

If Ω has no isolated singularity, finding the Poisson automorphism group of A_Ω becomes challenging. According to Theorem 4.6, the Poisson fraction fields $Q = Q(P_\Omega)$ can be divided into three families. For convenience, we call Ω *Weyl type* if Q is isomorphic to the Poisson Weyl field $K_{\mathrm{Weyl}} := \Bbbk(x, y)$ with $\{x, y\} = 1$, and we call Ω *quantum type* if Q is isomorphic to the Poisson quantum field $K_q := \Bbbk(x, y)$ with $\{x, y\} = qxy$ for some $q \in \Bbbk^\times$. When Ω is of Weyl type, we can construct many ungraded Poisson automorphisms of A (see Example 4.15). However, we cannot construct any ungraded

Poisson automorphisms of A if Ω is of quantum type. We are curious if the Poisson automorphisms of their Poisson algebras are similar to those of elliptic Poisson algebras, which are all graded (see Question 4.16).

1.5. Poisson cohomologies

Poisson cohomologies can be notoriously difficult to calculate. We characterize Poisson cohomological groups for various Poisson algebras in dimension three. Inspired by the PH^1-minimality from [49], we introduce the concept of uPH^2-vacancy in Definition 6.8 to control the second Poisson cohomology. The property of being uPH^2-vacant was implicitly used by Pichereau in [40, Remark 3.9] for Ω having an isolated singularity. In the theorem below, we generalize [49, Theorem 0.6] to the weighted case. We call an irreducible potential Ω in $\Bbbk[x, y, z]$ *balanced* if $\Omega_x\Omega_y\Omega_z \neq 0$ for any choice of graded generators (x, y, z); otherwise, we call it *non-balanced*.

Theorem 1.6. *Let $A := \Bbbk[x, y, z]$ be a connected graded Poisson polynomial algebra assuming (1) of Hypothesis 1.1. Denote by Z the Poisson center of A. Then the following statements are equivalent:*

(1) $\mathrm{rgt}(A) = 0$ *and any homogeneous Poisson derivation of A with negative degree is zero.*

(2) *Any graded twist of A is isomorphic to A, and any homogeneous Poisson derivation of A with a negative degree is zero.*

(3) *The Hilbert series of the graded vector space of Poisson derivations of A is*
$$\frac{1}{(1-t^a)(1-t^b)(1-t^c)}.$$

(4) $h_{\mathrm{PH}^1(A)}(t)$ *is* $\frac{1}{1-t^{a+b+c}}$.

(5) $h_{\mathrm{PH}^1(A)}(t)$ *is equal to* $h_Z(t)$.

(6) *Every Poisson derivation ϕ of A has a decomposition $\phi = zE + H_f$, where $z \in Z$ and $f \in A$. Here, z is unique, and f is unique up to a Poisson central element.*

(7) *Every Poisson derivation of A that vanishes on Z is Hamiltonian.*

(8) *A is an unimodular Poisson algebra determined by an irreducible potential Ω that is balanced.*

(9) $h_{\mathrm{PH}^3(A)}(t) - h_{\mathrm{PH}^2(A)}(t) = t^{a+b+c}$.

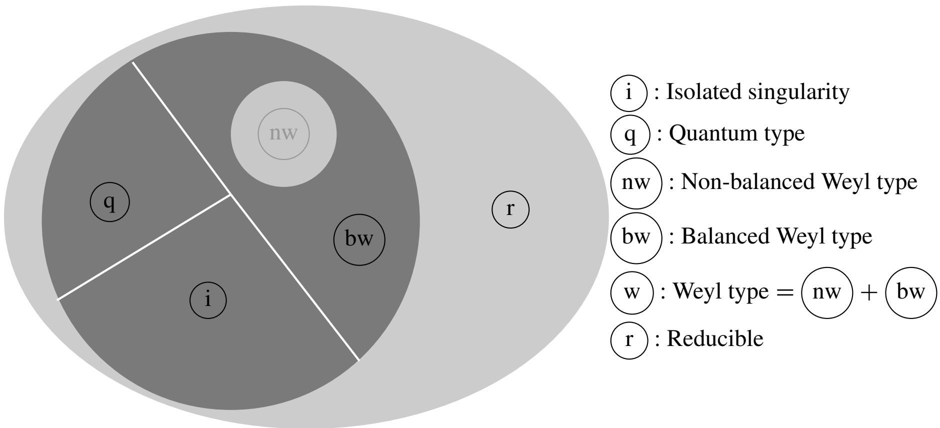
(10) *A is unimodular and* $h_{\mathrm{PH}^2(A)}(t) = \frac{1}{t^{a+b+c}}\Big(\frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{(1-t^{a+b+c})(1-t^a)(1-t^b)(1-t^c)} - 1\Big)$.

(11) *A is unimodular and* $h_{\mathrm{PH}^3(A)}(t) = \frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{t^{a+b+c}(1-t^{a+b+c})(1-t^a)(1-t^b)(1-t^c)}$.

(12) *A is* uPH^2*-vacant.*

It is important to note that van den Bergh [50] already computed the Poisson (co)homology of A for the case where A is generated in degree 1 and Ω is a cubic polynomial with isolated singularity. Moreover, it was later computed by Pichereau [40] for an arbitrary weighted homogeneous Ω with isolated singularity. Additional computations can be found in [39,48]. As stated in Theorem 1.6, the calculation of Poisson cohomology for A_Ω is possible for both quantum and balanced Weyl types of Ω, regardless of whether or not isolated singularity exists.

In the graph below, all potentials of degree $a+b+c$ listed in Theorem 3.5 are divided into irreducible potentials in the black circle and reducible ones in the complement, where irreducible potentials are subdivided into three types: isolated singularity, quantum and (balanced and non-balanced) Weyl.



We conclude the introduction with a table summarizing the main results for each type of Ω. Indeed, the table provides information concerning the smoothness of the projective curve $\Omega = 0$ (see Remark 4.7), the Gelfand–Kirillov dimension (GKdim) of $A_{\text{sing}} := A/(\Omega_x, \Omega_y, \Omega_z)$, the rigidity of A (see Table A.7), the Poisson automorphism group of A (see Theorem 1.5 and Example 4.15), the Poisson fraction field of A/Ω (see Theorem 4.6), the uPH^2-vacancy (see Theorem 1.6) and the K_1-sealedness (see Definition 6.2).

This paper is divided into six sections. Section 2 provides basic notations and results for Poisson algebras and briefly describes Poisson valuations. In Section 3, we classify all homogeneous polynomials Ω in $\Bbbk[x,y,z]$ such that $|\Omega| = |x| + |y| + |z|$ and prove Theorem 1.3. In Section 4, we prove Theorem 1.5; in Section 5, we prove results about several different rigidities, including Theorem 1.4. We study K_1-sealedness and uPH^2-vacancy in Section 6, which will be useful for the following section. In Section 7, we establish the results on Poisson cohomology for Poisson algebras $\Bbbk[x,y,z]$ with irreducible potentials Ω of degree $|x| + |y| + |z|$ as summarized in Theorem 1.6.

2. Preliminaries

2.1. Terminology

Let $A = \Bbbk[x_1, \dots, x_n]$ be a polynomial algebra. We denote by $\mathfrak{X}^\bullet(A) = \bigoplus_{i=0}^{\infty} \mathfrak{X}^i(A)$ the set of skew-symmetric multi-derivations of A. For $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, their *wedge product* $P \wedge Q \in \mathfrak{X}^{p+q}(A)$ is the skew-symmetric $(p+q)$-derivation of A, defined by

$$(P \wedge Q)(a_1, \dots, a_{p+q}) := \sum_{\sigma \in \mathbb{S}_{p,q}} \operatorname{sgn}(\sigma) P(a_{\sigma(1)}, \dots, a_{\sigma(p)}) Q(a_{\sigma(p+1)}, \dots, a_{\sigma(p+q)})$$

for all $a_1, \dots, a_{p+q} \in A$, where $\mathbb{S}_{p,q} \subset \mathbb{S}_{p+q}$ is the set of all (p,q)-shuffles. Note that $(\mathfrak{X}^\bullet(A), \wedge)$ is a graded commutative algebra [27, Proposition 3.1]. Recall that the *Schouten bracket* on $\mathfrak{X}^\bullet(A)$ is given by

$$[\cdot,\cdot]_S : \mathfrak{X}^p(A) \times \mathfrak{X}^q(A) \to \mathfrak{X}^{p+q-1}(A)$$

such that

$$\begin{aligned}
&[P, Q]_S(a_1, \dots, a_{p+q-1}) \\
&\quad = \sum_{\sigma \in \mathbb{S}_{q,p-1}} \operatorname{sgn}(\sigma) P(Q(a_{\sigma(1)}, \dots, a_{\sigma(q)}), a_{\sigma(q+1)}, \dots, a_{\sigma(q+p-1)}) \\
&\qquad - (-1)^{(p-1)(q-1)} \\
&\qquad \cdot \sum_{\sigma \in \mathbb{S}_{p,q-1}} \operatorname{sgn}(\sigma) Q(P(a_{\sigma(1)}, \dots, a_{\sigma(p)}), a_{\sigma(p+1)}, \dots, a_{\sigma(p+q-1)})
\end{aligned}$$

for any $P \in \mathfrak{X}^p(A)$, $Q \in \mathfrak{X}^q(A)$ and $p, q \in \mathbb{N}$. Note that $(\mathfrak{X}^\bullet(A), \wedge, [-,-]_S)$ is a Gerstenhaber algebra [27, Proposition 3.7].

Let $\Omega^1(A)$ be the module of Kähler differentials over A and $\Omega^p(A) = \wedge_A^p \Omega^1(A)$ for $p \geq 2$. The differential $d : A \to \Omega^1(A)$ extends to a well-defined differential of the complex $\Omega^\bullet(A)$, and the complex $(\Omega^\bullet, d)$ is called the algebraic de Rham complex of A.

For every $P \in \mathfrak{X}^p(A)$, the *internal product* with respect to P, denoted by ι_P, is an A-module map

$$\iota_P : \Omega^\bullet(A) \to \Omega^{\bullet - p}(A)$$

which is determined by

$$\begin{aligned}
&\iota_P(dF_1 \wedge dF_2 \wedge \cdots \wedge dF_k) \\
&\quad = \begin{cases} 0, & k < p, \\ \displaystyle\sum_{\sigma \in \mathbb{S}_{p,k-p}} \operatorname{sgn}(\sigma) P(F_{\sigma(1)}, \dots, F_{\sigma(p)}) & \\ \qquad \cdot dF_{\sigma(p+1)} \wedge \cdots \wedge dF_{\sigma(k)} \in \Omega^{k-p}(A), & k \geq p \end{cases}
\end{aligned}$$

for all $dF_1 \wedge dF_2 \wedge \cdots \wedge dF_k \in \Omega^k(A)$. Then the *Lie derivative* with respect to P is defined to be

$$\mathcal{L}_P = [\iota_P, d] : \Omega^\bullet(A) \to \Omega^{\bullet - p + 1}(A)$$

(see [27, equation (3.49)]). Let $\delta \in \mathfrak{X}^1(A)$ be a derivation of A. The *divergence* of δ, denoted by $\mathrm{div}(\delta)$, is an element in A defined by the equation

$$\mathcal{L}_\delta(\nu) = \mathrm{div}(\delta)\nu,$$

where $\nu \in \Omega^n(A)$ is a fixed volume form for A. In particular, if we choose $\nu = dx_1 \wedge \cdots \wedge dx_n$, from [49, Lemma 1.2 (1)], we get

$$\mathrm{div}(\delta) = \sum_{1 \le i \le n} \frac{\partial \delta(x_i)}{\partial x_i}.$$

Let (A, π) be a Poisson algebra with $\pi \in \mathfrak{X}^2(A)$ satisfying $[\pi, \pi]_S = 0$. We usually write the corresponding Poisson bracket on A as $\{-,-\} = \pi(-,-)$. A derivation δ on A is called a *Poisson derivation* if $[\delta, \pi]_S = 0$ or $\delta(\{f, g\}) = \{\delta(f), g\} + \{f, \delta(g)\}$ for all $f, g \in A$. There is a special class of Poisson derivations on A called *Hamiltonian derivations*, which is given by $H_f := \{f, -\}$ for any $f \in A$. The *modular derivation* of A is defined by

$$\mathfrak{m}(f) := -\mathrm{div}(H_f)$$

for all $f \in A$. We call A *unimodular* if $\mathfrak{m} = 0$.

For each $q \ge 0$, the *qth Poisson cohomology* of A is defined to be the qth-cohomology of the cochain complex $(\mathfrak{X}^\bullet(A), d_\pi^\bullet)$ with differential $d_\pi = -[-, \pi]_S$. In particular, for any $f \in \mathfrak{X}^q(A)$, $d_\pi^q(f) \in \mathfrak{X}^{q+1}(A)$ is determined by

$$\begin{aligned} d_\pi^q(f)(a_0, \ldots, a_q) = {} & \sum_{i=0}^{q} (-1)^i \{a_i, f(a_0, \ldots, \widehat{a_i}, \ldots, a_q)\} \\ & + \sum_{0 \le i < j \le q} (-1)^{i+j} f(\{a_i, a_j\}, a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_q) \end{aligned} \tag{2.1}$$

for any $a_0, a_1, \ldots, a_q \in A$. We denote by

$$\mathrm{PH}^q(A) := \ker(d_\pi^q)/\mathrm{im}(d_\pi^{q-1}).$$

Let $\mathrm{Pd}(A)$ be the Lie algebra of all Poisson derivations of A, and let $\mathrm{Hd}(A)$ be the Lie ideal of $\mathrm{Pd}(A)$ consisting of all Hamiltonian derivations. We also denote by $Z_P(A)$ the Poisson center of A. In particular,

$$\mathrm{PH}^0(A) = Z_P(A), \quad \mathrm{PH}^1(A) = \mathrm{Pd}(A)/\mathrm{Hd}(A).$$

For each $q \geq 0$, the *qth Poisson homology* of A is defined to be the qth-homology of the chain complex $(\Omega^\bullet(A), \partial^\pi)$, where the differentials are given by $\partial_q^\pi = \mathcal{L}_\pi = [i_\pi, d] : \Omega^q(A) \to \Omega^{q-1}(A)$. We denote by

$$\mathrm{PH}_q(A) := \ker(\partial_q^\pi)/\mathrm{im}(\partial_{q+1}^\pi).$$

When A has a unimodular Poisson structure π, a duality exists between its Poisson homology and Poisson cohomology [34].

Let us review the concepts of H-ozoneness and PH^1-minimality about a connected graded Poisson algebra.

Definition 2.1 ([49, Definition 7.1]). Let $A = \Bbbk[x_1, \dots, x_n]$ be a connected graded Poisson algebra with its Poisson center denoted by Z.

(1) $\delta \in \mathrm{Pd}(A)$ is called *ozone* if $\delta(Z) = 0$.

(2) Let $\mathrm{Od}(A)$ denote the Lie algebra of all ozone Poisson derivations of A.

(3) We say A is H-ozone if $\mathrm{Od}(A) = \mathrm{Hd}(A)$, namely, any ozone derivation is Hamiltonian.

(4) We say A is PH^1-minimal if $\mathrm{PH}^1(A) \cong ZE$ as graded Z-modules, where E is the Euler derivation (2.2) below.

2.2. Twists of graded Poisson algebras

Let $A = \Bbbk[x_1, \dots, x_n]$ be a graded Poisson polynomial algebra with Poisson bracket $\pi = \{-,-\}$ of degree 0. In [49, Section 2], the notion of graded twists of A was introduced. For any homogeneous element $f \in A$, we use $|f|$ to denote its degree in A. Define the *Euler derivation* E of A by

$$E(f) := |f| f \tag{2.2}$$

for all homogeneous elements $f \in A$. We point out that E is a Poisson derivation and $\mathrm{div}(E) = \sum_{i=1}^n \deg(x_i)$. Recall that a derivation δ on A is said to be a *semi-Poisson derivation* if

$$[E \wedge \delta, \pi]_S = E \wedge [\delta, \pi]_S = 0.$$

The set of all graded semi-Poisson derivations (resp. graded Poisson derivations) of A is denoted by $\mathrm{Gspd}(A)$ (resp. $\mathrm{Gpd}(A)$). When A is a $\mathbb{Z}$-graded Poisson algebra, $\mathrm{Gspd}(A)$ is a $\Bbbk$-vector space. For any $\delta \in \mathrm{Gspd}(A)$, we can define a new Poisson algebra $A^\delta := (A, \pi_{\mathrm{new}})$, called *a graded twist* of A, with

$$\pi_{\mathrm{new}} := \pi + E \wedge \delta \tag{2.3}$$

or namely, $\{f, g\}_{\mathrm{new}} = \{f, g\} + E(f)\delta(g) - \delta(f)E(g)$ for all homogeneous elements $f, g \in A$.

Definition 2.2 ([49, Definition 4.3]). Let $A = \Bbbk[x_1, \dots, x_n]$ be a $\mathbb{Z}$-graded Poisson algebra. The *rigidity* of A is defined to be

$$\operatorname{rgt}(A) := 1 - \dim_{\Bbbk} \operatorname{Gspd}(A).$$

In particular, we say A is *rigid* if $\operatorname{rgt}(A) = 0$.

Let (A, π) be a Poisson algebra with Poisson bracket π. Let ξ be any nonzero scalar. We define a new Poisson bracket $\pi_\xi := \xi\pi$ or $\{-,-\}_\xi := \xi\{-,-\}$ on A. Then it is easy to see that $A' := (A, \pi_\xi)$ is a Poisson algebra. The following lemma shows how Poisson structures and their Poisson cohomologies behave when we replace A (resp. π) by A' (resp. $\pi' := \pi_\xi$).

Lemma 2.3 ([49, Lemma 1.5]). *Retain the notations as above with $\xi \in \Bbbk^\times$. Let d_π^q (resp. $d_{\pi'}^q$) be the differential of $\mathfrak{X}^\bullet(A)$ (resp. $\mathfrak{X}^\bullet(A')$) as defined in (2.1). The following is true:*

(1) $d_{\pi'}^q = \xi d_\pi^q$ *for all* q.

(2) $\ker(d_{\pi'}^q) = \ker(d_\pi^q)$ *for all* q.

(3) $\operatorname{im}(d_{\pi'}^q) = \operatorname{im}(d_\pi^q)$ *for all* q.

(4) $\operatorname{PH}^q(A) = \operatorname{PH}^q(A')$ *for all* q.

(5) $\operatorname{rgt}(A_\Omega) = \operatorname{rgt}(A_{\xi\Omega})$.

(6) A_Ω *is* H*-ozone if and only if* $A_{\xi\Omega}$ *is* H*-ozone.*

(7) A_Ω *is* PH^1*-minimal if and only if* $A_{\xi\Omega}$ *is* PH^1*-minimal.*

2.3. Notations for Poisson (co)homology in dimension three

We consider the polynomial algebra $A = \Bbbk[x, y, z]$ is a polynomial algebra with grading $(\deg(x), \deg(y), \deg(z)) = (a, b, c) \in (\mathbb{N}_+)^3$. Note that a connected w-graded unimodular Poisson structure π on A is determined by a homogeneous polynomial Ω of degree n (not necessarily equal to $a + b + c$). We write (A, π_Ω) for the corresponding Poisson algebra, where the Poisson bracket on A is homogeneous of degree $w := n - a - b - c$. So the cochain complex $(\mathfrak{X}^\bullet(A), d_\pi^\bullet)$ consisting of graded vector spaces of skew-symmetric multi-derivations is given by

$$0 \to \mathfrak{X}^0(A) \xrightarrow{d_\pi^0} \mathfrak{X}^1(A)[w] \xrightarrow{d_\pi^1} \mathfrak{X}^2(A)[2w] \xrightarrow{d_\pi^2} \mathfrak{X}^3(A)[3w] \to 0. \tag{2.4}$$

Here, we choose the natural isomorphisms of graded vector spaces as follows:

$$\begin{cases} \mathfrak{X}^1(A) \xrightarrow{\sim} A[a] \oplus A[b] \oplus A[c], & V \mapsto (V(x), V(y), V(z)), \\ \mathfrak{X}^2(A) \xrightarrow{\sim} A[b+c] \oplus A[a+c] \oplus A[a+b], & V \mapsto (V(y,z), V(z,x), V(x,y)), \\ \mathfrak{X}^3(A) \xrightarrow{\sim} A[a+b+c], & V \mapsto (V(x,y,z)). \end{cases} \tag{2.5}$$

Using these isomorphisms, it becomes convenient to compute the associated Hilbert series. The elements of $A^{\oplus 3}$ are viewed as vector-valued functions on A, and we denote such an element by $\overrightarrow{f} \in A^{\oplus 3}$. Let $\cdot, \times$ denote the usual inner and cross products, respectively, while $\overrightarrow{\nabla}$, $\overrightarrow{\nabla}\times$ and Div denote, respectively, the gradient, the curl, and the divergence operators. Therefore, the cochain complex (2.4) can be identified as

$$0 \to A \xrightarrow{\delta^0_\Omega} \begin{array}{c} A[w+a] \\ \oplus A[w+b] \\ \oplus A[w+c] \end{array} \xrightarrow{\delta^1_\Omega} \begin{array}{c} A[2w+b+c] \\ \oplus A[2w+a+c] \\ \oplus A[2w+a+b] \end{array} \xrightarrow{\delta^2_\Omega} A[3w+a+b+c] \to 0, \quad (2.6)$$

where the differential δ_Ω can be written in a compact form

$$\delta^0_\Omega(f) = \overrightarrow{\nabla} f \times \overrightarrow{\nabla}\Omega, \qquad \text{for } f \in A \xrightarrow{\sim} \mathfrak{X}^0(A), \quad (2.7)$$

$$\delta^1_\Omega(\overrightarrow{f}) = -\overrightarrow{\nabla}(\overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega) + \mathrm{Div}(\overrightarrow{f})\overrightarrow{\nabla}\Omega, \qquad \text{for } \overrightarrow{f} \in A^{\oplus 3} \xrightarrow{\sim} \mathfrak{X}^1(A), \quad (2.8)$$

$$\delta^2_\Omega(\overrightarrow{f}) = -\overrightarrow{\nabla}\Omega \cdot (\overrightarrow{\nabla} \times \overrightarrow{f}) = -\mathrm{Div}(\overrightarrow{f} \times \overrightarrow{\nabla}\Omega), \quad \text{for } \overrightarrow{f} \in A^{\oplus 3} \xrightarrow{\sim} \mathfrak{X}^2(A). \quad (2.9)$$

For any graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ that is locally finite, we use

$$h_M(t) = \sum_{i \in \mathbb{Z}} \dim_{\Bbbk}(M_i) t^i$$

to denote the *Hilbert series* of M. Note that the Hilbert series of A is given by

$$h_A(t) = \frac{1}{(1-t^a)(1-t^b)(1-t^c)}.$$

As a consequence of (2.6), the Poisson cohomology $\mathrm{PH}^\bullet(A)$ (resp. Poisson homology $\mathrm{PH}_\bullet(A)$) are graded vector spaces, and we denote their Hilbert series as $h_{\mathrm{PH}^\bullet(A)}(t)$ (resp. $h_{\mathrm{PH}_\bullet(A)}(t)$). By additivity of the Hilbert series of (2.4)–(2.6), we have

$$\sum_{i=0}^{3} (-t^{-w})^i h_{\mathrm{PH}^i(A)}(t) = -\frac{1}{t^{3w+a+b+c}} \frac{(1-t^{w+a})(1-t^{w+b})(1-t^{w+c})}{(1-t^a)(1-t^b)(1-t^c)}. \quad (2.10)$$

2.4. Poisson valuations and filtrations

In [24], the notion of Poisson valuations was introduced to solve rigidity, automorphism, isomorphism and embedding problems for various classes of Poisson algebras/fields. In this subsection, we recall some basics of Poisson valuations. Let w be any integer.

Definition 2.4 ([24, Definition 1.1]). Let K be a Poisson algebra (or a Poisson field) over $\Bbbk$. A *w-valuation* on K is a map

$$\nu : K \to \mathbb{Z} \cup \{\infty\}$$

which satisfies the following properties: for all $f, g \in K$,

(a) $\nu(f) = \infty$ if and only if $f = 0$,

(b) $\nu(f) = 0$ for all $f \in \Bbbk^\times := \Bbbk \setminus \{0\}$,

(c) $\nu(fg) = \nu(f) + \nu(g)$ (assuming $n + \infty = \infty$ when $n \in \mathbb{Z} \cup \{\infty\}$),

(d) $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, with equality if $\nu(f) \neq \nu(g)$,

(e) $\nu(\{f, g\}) \geq \nu(f) + \nu(g) - w$.

Note that conditions (a)–(d) mean ν is an ordinary valuation on K. Next, we state the definition of w-filtration closely related to the Poisson w-valuation on K.

Definition 2.5 ([24, Definition 2.2]). Let A be a Poisson algebra. Let $\mathbb{F} = \{F_i \mid i \in \mathbb{Z}\}$ be a chain of $\Bbbk$-subspaces of A. We say $\mathbb{F}$ is a *w-filtration* of A if it satisfies

(a) $F_i \supseteq F_{i+1}$ for all $i \in \mathbb{Z}$ and $1 \in F_0 \setminus F_1$,

(b) $F_i F_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$,

(c) $\bigcap_{i \in \mathbb{Z}} F_i = \{0\}$,

(d) $\bigcup_{i \in \mathbb{Z}} F_i = A$,

(e) $\{F_i, F_j\} \subseteq F_{i+j-w}$ for all $i, j \in \mathbb{Z}$.

Let $\mathbb{F} = \{F_i \mid i \in \mathbb{Z}\}$ be a w-filtration of the Poisson algebra A. The *associated graded algebra* of the w-filtration $\mathbb{F}$ of A is defined to be

$$\operatorname{gr}_{\mathbb{F}} A := \bigoplus_{i \in \mathbb{Z}} F_i / F_{i+1}.$$

For any nonzero element $f \in F_i$, we denote $\overline{f}$ the element $f + F_{i+1}$ in the ith degree component $(\operatorname{gr}_{\mathbb{F}} A)_i := F_i / F_{i+1}$. It is clear that $\operatorname{gr}_{\mathbb{F}} A$ is a graded algebra. Moreover, by [24, Lemma 2.3], $\operatorname{gr}_{\mathbb{F}} A$ is a w-graded Poisson algebra with the induced homogeneous Poisson bracket of degree $-w$ such that

$$\{F_i / F_{i+1}, F_j / F_{j+1}\} \subseteq F_{i+j-w} / F_{i+j+1-w},$$

namely, $\{(\operatorname{gr}_{\mathbb{F}} A)_i, (\operatorname{gr}_{\mathbb{F}} A)_j\} \subseteq (\operatorname{gr}_{\mathbb{F}} A)_{i+j-w}$ for all $i, j \in \mathbb{Z}$. We call $\mathbb{F}$ a *good* filtration if $\operatorname{gr}_{\mathbb{F}} A$ is a domain.

Give a good w-filtration $\mathbb{F}$ on A, we define the notion of a *degree* function, denoted by $\deg : A \to \mathbb{Z} \cup \{\infty\}$ via

$$\deg(f) := i \quad \text{if } f \in F_i \setminus F_{i+1} \text{ and } \deg(0) = +\infty.$$

One can see that deg is a valuation on A. Conversely, given a valuation ν on A, we can define a filtration $\mathbb{F}_i^\nu := \{F_i^\nu \mid i \in \mathbb{Z}\}$ of A (associated to ν) by

$$F_i^\nu := \{f \in A \mid \nu(f) \geq i\}.$$

The corresponding associated graded algebra of A is denoted by $\operatorname{gr}_\nu A$.

Proposition 2.6 ([24, Lemma 2.6]). *Let A be a Poisson algebra. There is a one-to-one correspondence between the set of good w-filtrations of A and the set of w-valuations on A via the above constructions.*

In this paper, we will mainly focus on the following special class of Poisson valuations.

Definition 2.7 ([24, Definition 3.1]). Let K be a Poisson field over $\Bbbk$. A w-valuation ν on K is called a *faithful w-valuation* if the following hold:

(a) The image of ν is $\mathbb{Z} \cup \{\infty\}$.

(b) $\mathrm{GKdim}(\mathrm{gr}_\nu K) = \mathrm{GKdim}(K)$.

(c) The w-graded Poisson bracket on $\mathrm{gr}_\nu K$ is nonzero.

A Poisson w-valuation on a Poisson domain A is faithful if its natural extension to the Poisson fractional field $Q(A)$ is faithful.

Note that the above conditions (b) and (c) are different from the original ones [24, Definition 3.1 (1)]. But it is easy to see they are equivalent (see [23, Definitions 0.2 and 2.18]).

3. Classification of potentials Ω in $\Bbbk[x, y, z]$

In this section, we classify all possible homogeneous polynomials Ω satisfying (1), (2) and (4) of Hypothesis 1.1, up to some graded automorphism of A. We will use the following definition.

Definition 3.1. We define the *Jacobian quotient algebra* of A with respect to Ω to be

$$A_{\mathrm{sing}} := \Bbbk[x, y, z]/(\Omega_x, \Omega_y, \Omega_z).$$

It is clear that A_{sing} is independent of the choices of graded generators (x, y, z).

Lemma 3.2. *Assume* (1), (2) *and* (4) *of Hypothesis* 1.1.

(1) *The following are equivalent:*

(1a) *There is an irreducible homogeneous potential Ω in $\Bbbk[x, y]$.*

(1b) $2 < a < b < c = \frac{ab}{\gcd(a,b)} - a - b$.

In this case, $\Bbbk[x, y]_c = 0$, and, up to a graded automorphism of A, $\Omega = x^{\frac{b}{\gcd(a,b)}} + y^{\frac{a}{\gcd(a,b)}}$.

(2) *Let Ω denote a nonzero homogeneous polynomial in $\Bbbk[x, y]$. The following are equivalent:*

(2a) *A has a nonzero graded derivation δ of degree 0 satisfying*

$$\mathrm{div}(\delta) = \delta(\Omega) = 0.$$

(2b) *Ω is reducible.*

Proof. (1) (1b) $\Rightarrow$ (1a): This is clear by taking $\Omega = x^{\frac{b}{\gcd(a,b)}} + y^{\frac{a}{\gcd(a,b)}}$.

(1a) $\Rightarrow$ (1b): Let $\Omega = h(x, y) \in \Bbbk[x, y]$ be an irreducible polynomial of degree $a + b + c$. We have $\deg(h(x, y)) = a + b + c = ka + lb$ for some $k, l \in \mathbb{N}$. As a result, $c = (k - 1)a + (l - 1)b$. If needed, we can divide the degrees of x and y by $\gcd(a, b)$ and thus assume that $\gcd(a, b) = 1$. If no more than one of the $x^{\frac{a+b+c}{a}}$ and $y^{\frac{a+b+c}{b}}$ terms appears in $h(x, y)$, then $h(x, y) = xf(x, y)$ or $h(x, y) = yf(x, y)$ for some nonconstant polynomial $f(x, y) \in \Bbbk[x, y]$. In this case, $\Omega = h(x, y)$ is reducible. Next we consider the case where $h(x, y)$ contains both $x^{\frac{a+b+c}{a}}$ and $y^{\frac{a+b+c}{b}}$ terms. We have $a \mid b + c$ and $b \mid a + c$, which implies that $a \mid l$ and $b \mid k$. Say $l = m_1 a$ and $k = m_2 b$ for some $m_1, m_2 \in \mathbb{N}_+$. Thus, $a + b + c = (m_1 + m_2)ab$. So we have

$$h(x, y) = \lambda_{m_1+m_2} x^{(m_1+m_2)b} + \lambda_{m_1+m_2-1} x^{(m_1+m_2-1)b} y^a + \cdots + \lambda_1 x^b y^{(m_1+m_2-1)a} + \lambda_0 y^{(m_1+m_2)a}$$

for some $\lambda_0, \ldots, \lambda_{m_1+m_2} \in \Bbbk$. We rewrite $p = x^b$ and $q = y^a$. Then we can get

$$h(x, y) = h(p, q) = \lambda_{m_1+m_2} p^{m_1+m_2} + \lambda_{m_1+m_2-1} p^{m_1+m_2-1} q + \cdots + \lambda_0 q^{m_1+m_2}$$

with $\deg(p) = \deg(q) = ab$. If $m_1 + m_2 > 1$, then $h(p, q)$ is always reducible. Since we assume Ω is irreducible, we obtain that $m_1 + m_2 = 1$. Then, after a linear transformation, we can assume that $h(x, y) = x^b + y^a$ (which is irreducible for $\gcd(a, b) = 1$). Since $1 \le a \le b \le c$ and $\gcd(a, b) = 1$, we have $1 \le a < b$. Since $2b < a + b + c = ab$, we have $a > 2$. Note that $c = ab - a - b = (a - 1)b - a$. Thus, we have $c > b$. Thus, we obtain (1b) when $\gcd(a, b) = 1$. Therefore, (1b) holds by lifting to the general case when $\gcd(a, b) > 1$.

One can easily show that the conditions in (1b) imply that $\Bbbk[x, y]_c = 0$.

(2) (2b) $\Rightarrow$ (2a): By assumption, Ω is reducible. By the proof of part (1), $\deg(\Omega) = ka + lb$ for some $k, l \in \mathbb{N}$. If $k, l \ge 1$, we can let $\delta = x^{k-1} y^{l-1} \frac{\partial}{\partial z}$. Otherwise, we may assume $k = 0$ and $\Omega = y^l$ (as Ω is reducible). Then we let $\delta = z\frac{\partial}{\partial z} - x\frac{\partial}{\partial x}$. Then (2a) holds.

(2a) $\Rightarrow$ (2b): Assume to the contrary that Ω is irreducible. Without loss of generality, let $\Omega = x^b + y^a$ with $\gcd(a, b) = 1$ as in part (1). By (1b), we have $m_1 + m_2 = 1$ and $a < b < c = ab - a - b$. This (together with $\Bbbk[x, y]_c = 0$) implies that any graded derivation δ of A of degree 0 must have the form $\delta(x) = \alpha x$, $\delta(y) = \beta y$ and $\delta(z) = \gamma z$ for some $\alpha, \beta, \gamma \in \Bbbk$. So $\operatorname{div}(\delta) = \delta(\Omega) = 0$ yields that $\delta = 0$. This finishes the proof. ∎

Proposition 3.3. *Let $A = \Bbbk[x, y, z]$ be a weighted polynomial algebra with* $\deg x = 1$, $\deg y = 1$, $\deg z = 2$. *Then the nonzero homogeneous degree* 4 *polynomials $\Omega \in A$ can be classified in Table A.2. In particular, Ω has an isolated singularity if and only if $\Omega = z^2 + xy^3 + \lambda x^2 y^2 + x^3 y$ with $\lambda \neq \pm 2$ up to graded isomorphisms of A.*

Proof. Since $\deg(\Omega) = 4$, $\Omega = l_1 z^2 + l_2 z g(x, y) + h(x, y)$, with $\deg g(x, y) = 2$, $\deg h(x, y) = 4$ and $l_1, l_2 \in \Bbbk$. If $l_1 \neq 0$, then after a linear transformation of z, we can assume that $\Omega = z^2 + h(x, y)$. If $l_1 = 0$ and $l_2 \neq 0$, we can assume that $g(x, y) = x^2$ or xy after a further linear transformation of x and y. So, we only need to consider the following cases.

Case 1. $\Omega = z^2 + h(x, y)$.

If $h(x, y) = 0$, then $\Omega = z^2$. If $0 \neq h(x, y)$ has a root of multiplicity 4 in $\mathbb{P}^1$, then without loss of generality, we can assume that $h(x, y) = x^4$. If $h(x, y)$ has a root of multiplicity 3, then we can assume that $h(x, y) = x^3 y$ due to the symmetry between x and y. If $h(x, y)$ has a root of multiplicity 2, then we can assume that $h(x, y) = x^2 y(y + \lambda x)$ for some $\lambda = 0$ or 1. If $h(x, y)$ has no repeated root, then we can assume that $h(x, y) = xy(y + x)(y + kx)$ for some $k \in \Bbbk \setminus \{0, 1\}$. Then $h(x, y) = xy^3 + (k + 1)x^2 y^2 + kx^3 y$. By a suitable re-scaling of x and y, we obtain $h(x, y) = xy^3 + \lambda x^2 y^2 + x^3 y$ for some $\lambda \in \Bbbk$.

Case 2. $\Omega = x^2 z + h(x, y)$.

After a linear transformation of z and re-scaling of x, y and z if necessary, we can assume that $\Omega = x^2 z + \lambda_1 xy^3 + \lambda_2 y^4$ for some $\lambda_1, \lambda_2 \in \{0, 1\}$.

Case 3. $\Omega = xyz + h(x, y)$.

After a linear transformation of z and re-scaling of x, y and z if necessary, we can have $\Omega = xyz + \lambda_1 x^4 + \lambda_2 y^4$ for some $\lambda_1, \lambda_2 \in \{0, 1\}$.

Case 4. $\Omega = h(x, y)$.

If $\Omega = h(x, y)$, then by the same argument of Case 1, we can show that Ω is one of the following forms:

$$x^4, x^3 y, x^2 y^2, x^2 y^2 + x^3 y, xy^3 + \lambda x^2 y^2 + x^3 y \quad \text{for some } \lambda \in \Bbbk.$$

By direct computation, we can verify that $z^2 + xy^3 + \lambda x^2 y^2 + x^3 y$ with $\lambda \neq \pm 2$ has an isolated singularity. ■

Proposition 3.4. *Let $A = \Bbbk[x, y, z]$ be a weighted polynomial algebra with* $\deg x = 1$, $\deg y = 2$, $\deg z = 3$. *Then the nonzero homogeneous degree* 6 *polynomials $\Omega \in A$ can be classified in Table A.5. In particular, Ω has an isolated singularity if and only if $\Omega = z^2 + y^3 + \lambda x^2 y^2 + x^4 y$ with $\lambda \neq \pm 2$ up to graded isomorphisms of A.*

Proof. Since $\deg(\Omega) = 6$, we have $\Omega = l_1 z^2 + l_2 z g(x, y) + h(x, y)$, where $g(x, y) \in \Bbbk[x, y]$ and $h(x, y) \in \Bbbk[x, y]$ have degrees 3 and 6, respectively, and $l_1, l_2 \in \Bbbk$. If $l_1 \neq 0$, then by a linear transformation of z, we can assume that $\Omega = z^2 + h(x, y)$. If $l_1 = 0$ and $l_2 \neq 0$, by a possible linear transformation of x and y, we can have $g(x, y) = x^3$ or $g(x, y) = xy$. We write, in general,

$$h(x, y) = w_1 y^3 + w_2 x^2 y^2 + w_3 x^4 y + w_4 x^6,$$

where $w_i \in \Bbbk$ for $1 \leq i \leq 4$. So, we only need to consider the following cases.

Case 1. $\Omega = z^2 + h(x, y)$.

Subcase 1. If $w_1 \neq 0$, then we can write $h(x, y) = (y + ax^2)(y + bx^2)(y + cx^2)$ for $a, b, c \in \Bbbk$. After a linear transformation of y, we can assume that $h(x, y) = y(y + ax^2)(y + bx^2)$. By a possible re-scaling of x, y and z, we can assume that Ω is one of the following forms:

$$z^2 + y^3, z^2 + y^3 + x^2y^2, z^2 + y^3 + \lambda x^2y^2 + x^4y \quad \text{for some } \lambda \in \Bbbk.$$

Subcase 2. If $w_1 = 0$ and $w_2 \neq 0$, then, similarly, we can assume that $h(x, y) = x^2(y + ax^2)(y + bx^2)$ for some $a, b \in \Bbbk$. A further linear transformation of x, y and z yields $\Omega = z^2 + x^2y^2$ or $\Omega = z^2 + x^2y^2 + x^4y$.

Subcase 3. If $w_1 = w_2 = 0$ and $w_3 \neq 0$, then we can assume that $h(x, y) = x^4(y + ax^2)$ for some $a \in \Bbbk$. A linear transformation of y yields $\Omega = z^2 + x^4y$.

Subcase 4. If $w_1 = w_2 = w_3 = 0$ and $w_4 \neq 0$, then by a re-scaling of x, we get $\Omega = z^2 + x^6$.

Subcase 5. Finally, if $w_1 = w_2 = w_3 = w_4 = 0$, then we have $\Omega = z^2$.

Case 2. $\Omega = x^3z + h(x, y)$.

After a linear transformation of z, we can assume that $\Omega = x^3z + \lambda_1x^2y^2 + \lambda_2y^3$ for some $\lambda_1, \lambda_2 \in \Bbbk$. It is easy to check that Ω is one of the following forms:

$$x^3z, x^3z + y^3, x^3z + x^2y^2, x^3z + x^2y^2 + y^3.$$

Case 3. $\Omega = xyz + h(x, y)$.

Again, via a linear transformation of z, one can assume that

$$\Omega = xyz + \lambda_1x^6 + \lambda_2y^3$$

for some $\lambda_1, \lambda_2 \in \Bbbk$. After re-scaling x and y as needed, we can assume that Ω is of one of the following forms:

$$xyz, xyz + x^6, xyz + y^3, xyz + x^6 + y^3.$$

Case 4. If $\Omega = h(x, y)$, then by the same argument as in Case 1, Ω can be assumed to be one of the following forms:

$$y^3, y^3 + x^2y^2, y^3 + \lambda x^2y^2 + x^4y, x^4y, x^2y^2, x^2y^2 + x^4y, x^6,$$

where $\lambda \in \Bbbk$.

By a direct computation, we can further verify that $\Omega = z^2 + y^3 + \lambda x^2y^2 + x^4y$ has an isolated singularity if and only if $\lambda \neq \pm 2$. ■

Theorem 3.5. *Let* $A = \Bbbk[x, y, z]$ *be a weighted polynomial algebra with* $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$ *for* $1 \leq a \leq b \leq c$. *Let* Ω *be a nonzero homogeneous polynomial of degree* $a + b + c$. *Then, up to a graded automorphism of* A, *we have the following:*

(1) *If* $a = b = c$*, then* Ω *is one of the forms listed in Table* A.1 [6, 11, 12, 25, 29].

(2) *If* $a = b < c$*, then* Ω *is one of the forms listed in Tables* A.2 *and* A.3.

(3) *If* $a < b = c$*, then every* Ω *is reducible and is one of the forms listed in Table* A.4.

(4) *If* $a < b < c$*, then* Ω *is one of the forms listed in Tables* A.5 *and* A.6.

Proof. (1) Since $a = b = c$, we can reduce the classification of Ω to the case where the degrees of x, y and z are all equal to 1. In this case, the classification of Ω is well known. Also, see [49, Corollary 6.7].

(2) Since $a = b < c$, then $\deg(\Omega) = 2a + c < 3c$. So we can write $\Omega = z^2 f(x, y) + zg(x, y) + h(x, y)$, where $\deg(f(x, y)) = a + b - c < a$, $\deg(g(x, y)) = 2a$ and $\deg(h(x, y)) = 2a + c$.

If $a \nmid c$, then in particular, $c \neq a + b$, whence $f(x, y) = 0$. If $h(x, y) \neq 0$, then $a \mid \deg(h(x, y)) = 2a + c$, we get $a \mid c$, yielding a contradiction. So $\Omega = zg(x, y)$ is reducible. By a linear transformation of x, y, we can assume that $\Omega = xyz$ or x^2z.

If $c = ka$ for some integer $k \geq 2$, then $(a, b, c) = (a, a, ka)$. If $k = 2$, then the result is given by Proposition 3.3. Now, assume $k > 2$. Then we have $\Omega = zg(x, y) + h(x, y)$. We can assume $g(x, y) = x^2$ or xy after a linear transformation of x, y. If $\Omega = x^2z + h(x, y)$, after a necessary linear transformation of z, we can write $\Omega = x^2z + \lambda_1 xy^{k+1} + \lambda_2 y^{k+2}$ for $\lambda_1, \lambda_2 \in \{0, 1\}$. If $\Omega = xyz + h(x, y)$, then similarly, we can write $\Omega = xyz + \lambda_1 x^{k+2} + \lambda_2 y^{k+2}$ for some $\lambda_1, \lambda_2 \in \{0, 1\}$.

(3) Since $a < b = c$, we have that $\deg(\Omega) = a + 2b < 3b$. If $a \mid b$, then we have $\Omega = \lambda x^{1+2\frac{b}{a}} + x^{1+\frac{b}{a}} g(y, z) + xf(y, z)$, where $\lambda \in \Bbbk$, $\deg(f(y, z)) = 2b$ and $\deg(g(y, z)) = b$. Thus, Ω is reducible. If $a \nmid b$, then $\Omega = xf(y, z)$ or $\Omega = \lambda x^{1+b} + xf(y, z)$ when $a = 2$, which is again reducible.

If $a \nmid b$ and $a \neq 2$, after a linear transformation of y, z, we can assume that $\Omega = xyz$ or xy^2. If $a \nmid b$ and $a = 2$, we can have $\Omega = x^{1+b} + xyz$ or $\Omega = x^{1+b} + xz^2$ after a linear transformation for x, y and z. If $a \mid b$, we can assume that $\Omega = x\Omega_1$, where $\Omega_1 = \lambda u^2 + ug(y, z) + f(y, z)$ with $u = x^{\frac{b}{a}}$ for some $\lambda \in \Bbbk$. We can rewrite Ω_1 as follows:

$$\Omega_1 = k_1 z^2 + k_2 z h_1(y, u) + h_2(y, u)$$

for some $k_1, k_2 \in \Bbbk$ and $h_1(y, u), h_2(y, u) \in \Bbbk[y, u]$. If $k_1 \neq 0$, then we can assume that Ω is one of the following forms:

$$xz^2, xz^2 + xy^2, xz^2 + x^{1+\frac{2b}{a}}, xz^2 + xy^2 + x^{1+\frac{2b}{a}}, x^{1+\frac{b}{a}} y + xz^2.$$

If $k_1 = k_2 = 0$, then $\Omega = xy^2, x^{1+\frac{2b}{a}}, x^{1+\frac{b}{a}} y, xy^2 + x^{1+\frac{2b}{a}}$. If $k_1 = 0$ and $k_2 \neq 0$, then $\Omega = x^{1+\frac{b}{a}} z, x^{1+\frac{b}{a}} z + xy^2, xyz, xyz + x^{1+\frac{2b}{a}}$.

(4) If $\Omega \in \Bbbk[x, y]$, by Lemma 3.2, the irreducible ones are given by $\Omega = x^{b/d} + y^{a/d}$, where $d = \gcd(a, b)$ and $2 < a < b < c = \frac{ab}{d} - a - b$. Moreover, such irreducible Ω will not occur unless $c = ma + nb, c \neq a + b, a \nmid b$ for some $m, n \in \mathbb{Z}$. Let us assume $\Omega \notin \Bbbk[x, y]$ in the remaining argument. Since $a < b < c$, we have that $\deg(\Omega) =$

$a+b+c<3c$. Thus, we can assume that $\Omega=z^2f(x,y)+zg(x,y)+h(x,y)$, where $\deg(f(x,y))=a+b-c<a$, $\deg(g(x,y))=a+b$ and $\deg(h(x,y))=a+b+c$. We divide the argument into two cases.

Case 1. $c=ma+nb$ for some integers m and n.

Subcase 1. If $c=a+b$, then we have $\deg(\Omega)=2c$ and $f(x,y)\in\Bbbk$. As a result, we can assume that $\Omega=z^2+h(x,y)$ or $zg(x,y)+h(x,y)$. Since $\deg(h(x,y))=2a+2b<4b$, we can write $h(x,y)=h_3(x)y^3+h_2(x)y^2+h_1(x)y+h_0(x)$, where $h_i(x)$ is a monomial in x of degree $2a+(2-i)b$ for $0\le i\le 3$. Let us further assume that $a\nmid b$. After a linear transformation of x,y, we can have $g(x,y)=xy$ if $g(x,y)\neq 0$. If $a\nmid 2b$, then $h(x,y)=\lambda x^2y^2$ for some $\lambda\in\Bbbk$. Hence, by a linear transformation of z, Ω can be one of the following possible forms:

$$z^2,\ z^2+x^2y^2,\ xyz.$$

If $a\mid 2b$, then $h(x,y)$ can be one of the following forms:

$$x^2y^2,\ x^{2+\frac{2b}{a}},\ x^2y^2+x^{2+\frac{2b}{a}}.$$

By a linear transformation of z, Ω can be one of the following possible forms:

$$z^2,\ z^2+x^2y^2,\ z^2+x^{2+\frac{2b}{a}},\ z^2+x^2y^2+x^{2+\frac{2b}{a}},\ xyz,\ xyz+x^{2+\frac{2b}{a}}.$$

Now we assume that $a\mid b$. If $b=2a$, then $c=3a$ and so see Proposition 3.4. We next consider the case $b\neq 2a$. Since $a+b+c=2a+2b<4b$ and $b\neq 2a$, we can write $h(x,y)=\lambda_1x^2y^2+\lambda_2x^{2+\frac{b}{a}}y+\lambda_3x^{2+2\frac{b}{a}}$ for some $\lambda_1,\lambda_2,\lambda_3\in\Bbbk$. If $\Omega=z^2+h(x,y)$, then after a linear transformation of x,y, Ω is one of the following forms:

$$z^2,\ z^2+x^2y^2,\ z^2+x^{2+\frac{b}{a}}y,\ z^2+x^{2+\frac{2b}{a}},\ z^2+x^2y^2+x^{2+\frac{b}{a}}y.$$

If $\Omega=zg(x,y)+h(x,y)$, then after a linear transformation of x,y, we can have $g(x,y)=xy$ or $x^{1+\frac{b}{a}}$. By a linear transformation of z, Ω is one of the following:

$$xyz,\ xyz+x^{2+\frac{2b}{a}},\ x^{1+\frac{b}{a}}z,\ x^{1+\frac{b}{a}}z+x^2y^2.$$

Subcase 2. If $c\neq a+b$, then we can assume that $\Omega=g(x,y)z+h(x,y)$, where $\deg(g(x,y))=a+b$ and $\deg(h(x,y))=(m+1)a+(n+1)b$. Let k,l be the possible integers such that $k=\frac{(n+1)b}{a}$ and $l=\frac{(m+1)a}{b}$. Note that $kl=(m+1)(n+1)$. If $a\nmid b$, then we can assume that $g(x,y)=xy$ if $g(x,y)\neq 0$. If $a\mid b$, then we have $g(x,y)=xy$ or $x^{1+\frac{b}{a}}$ if $g(x,y)\neq 0$. By a linear transformation of z, Ω can be reduced to one of the following forms:

$$xyz,\ xyz+x^{m+1+k},\ xyz+y^{n+1+l},\ xyz+x^{m+1+k}+y^{n+1+l},\ x^{1+\frac{b}{a}}z$$

and

$$x^{1+\frac{b}{a}}z+x^{\frac{b}{a}}y^{n+l},\ x^{1+\frac{b}{a}}z+x^{\frac{b}{a}}y^{n+l}+y^{1+n+l},\ x^{1+\frac{b}{a}}z+y^{1+n+l},$$

where k and l are assumed to be integers in the first place.

Case 2. If $c \neq ma + nb$ for any $m, n \in \mathbb{Z}$, then $f(x, y) = 0$. Suppose $h(x, y) \neq 0$. Then $a + b + c = \deg(h(x, y)) = sa + lb$ for some $s, l \in \mathbb{N}$. Then $c = (s-1)a + (l-1)b$, which is a contradiction. So $h(x, y) = 0$. Then $\Omega = zg(x, y)$ is reducible. After a linear transformation of x, y, we can assume that $\Omega = xyz$ for $a \nmid b$ and $\Omega = xyz$ or $x^{1+\frac{b}{a}}z$ for $a \mid b$.

This completes the proof. ∎

4. Poisson fraction fields and automorphism groups

In this section, we discuss Poisson fraction fields and Poisson automorphism groups related to unimodular Poisson algebras in dimension three. Throughout this section, we will assume Hypothesis 1.1 unless mentioned otherwise.

Definition 4.1. We define a map $\pi : A \to \mathfrak{X}^2(A)$ such that, for each $h \in A$, $\pi_h := \pi(h)$ is defined by

$$\pi_h(f, g) = \det\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix} \cdot \left(= \det\left(\frac{\partial(f, g, h)}{\partial(x, y, z)}\right) = \{f, g\}_h\right) \quad \text{for all } f, g \in A := \Bbbk[x, y, z]. \tag{4.1}$$

Note that the definition of π_h depends on the generating set $\{x, y, z\}$. It is clear that if a new set of generators $\{x, y, z\}$ is used, then the corresponding π_h will be a scalar multiple of the original π_h.

One can check that $[\pi_h, \pi_h]_S = 0$. In particular, π_Ω is a connected graded Poisson bracket on A if Ω satisfies Hypothesis 1.1. To keep things simple, we will introduce the following.

Notation 4.2. Here, $w = n - (a + b + c)$ may not necessarily be zero.

- $A_\Omega := (A, \pi_\Omega)$: the unimodular Poisson algebra with potential Ω.
- $\mathrm{Aut}_{\mathrm{gr}}(A)$: the group of all graded algebra automorphisms of A.
- $\mathrm{Aut}_P(A)$ (resp. $\mathrm{Aut}_{\mathrm{gr}\,P}(A)$): the group of all (resp. graded) Poisson automorphisms of $A = A_\Omega$.
- $P_{\Omega-\xi} := A/(\Omega - \xi)$ ($\xi \in \Bbbk$): the quotient Poisson algebra.
- $\mathrm{Aut}_P(P_\Omega)$ (resp. $\mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega)$): the group of all (resp. graded) Poisson automorphisms of P_Ω.
- $Q(P_{\Omega-\xi})$: the Poisson fraction field of $P_{\Omega-\xi}$ whenever it is an integral domain.

Note that any graded unimodular Poisson structure on A is determined by such a potential Ω. If Ω is homogeneous of degree $|x|+|y|+|z|$, then $P_{\Omega-\xi} \cong P_{\Omega-1}$ whenever $\xi \neq 0$. In this case, we can always assume ξ to be either 0 or 1.

Lemma 4.3. *Let Ω and Ω' be two homogeneous potentials for $A = \Bbbk[x, y, z]$ of degrees $>\max\{|x|, |y|, |z|\}$. Then the two Poisson algebras A_Ω and $A_{\Omega'}$ are isomorphic if and only if there is an algebra automorphism ϕ of A such that $\Omega' = \frac{\phi(\Omega)}{\det(\phi)}$, where $\det(\phi) = \det\left(\frac{\partial(\phi(x),\phi(y),\phi(z))}{\partial(x,y,z)}\right)$. As a consequence,*

$$\mathrm{Aut}_P(A_\Omega) = \{\phi \in \mathrm{Aut}(A) \mid \phi(\Omega) = \det(\phi)\Omega\},$$

and $\mathrm{Aut}_P(A_\Omega) = \mathrm{Aut}_P(A_{\lambda\Omega})$ for any $\lambda \in \Bbbk^\times$.

Proof. Suppose there is a Poisson algebra isomorphism $\phi : A_\Omega \to A_{\Omega'}$. For any polynomials $f, g \in A$, $\phi(f)$ and $\phi(g)$ can be regarded as polynomials of variables $\phi(x), \phi(y)$ and $\phi(z)$ in $\phi(A) = A$. Therefore, according to (4.1), we have

$$\begin{aligned}
\{\phi(f), \phi(g)\}_{\Omega'} &= \phi(\{f, g\}_\Omega) = \phi\Big(\det\Big(\frac{\partial(f, g, \Omega)}{\partial(x, y, z)}\Big)\Big) = \det\Big(\frac{\partial(\phi(f), \phi(g), \phi(\Omega))}{\partial(\phi(x), \phi(y), \phi(z))}\Big) \\
&= \det\Big(\frac{\partial(x, y, z)}{\partial(\phi(x), \phi(y), \phi(z))}\Big) \det\Big(\frac{\partial(\phi(f), \phi(g), \phi(\Omega))}{\partial(x, y, z)}\Big) \\
&= \det(\phi)^{-1}\{\phi(f), \phi(g)\}_{\phi(\Omega)} = \{\phi(f), \phi(g)\}_{\det(\phi)^{-1}\phi(\Omega)}.
\end{aligned}$$

This implies that

$$\{-, -\}_{\Omega'} = \{-, -\}_{\det(\phi)^{-1}\phi(\Omega)}$$

which yields the same Poisson bracket on A. Thus, $\phi(\Omega) = \det(\phi)\Omega' + \lambda$ for some scalar $\lambda \in \Bbbk$. Since Ω and Ω' are homogeneous of degrees $> \max\{|x|, |y|, |z|\}$, their partial derivatives are all homogeneous of positive degree. Therefore, one can check that $A/(\det(\phi)\Omega' + \lambda)$ is regular for $\lambda \neq 0$ while $A/(\Omega)$ is not regular by the Jacobian criterion. As a consequence, the induced algebra isomorphism $\phi : A/(\Omega) \to A/(\phi(\Omega)) = A/(\det(\phi)\Omega' + \lambda)$ implies that $\lambda = 0$. So $\Omega' = \frac{\phi(\Omega)}{\det(\phi)}$.

Conversely, suppose there is an algebra isomorphism ϕ of A such that $\Omega' = \frac{\phi(\Omega)}{\det(\phi)}$. Then we have

$$\begin{aligned}
\phi(\{f, g\}_\Omega) &= \phi\Big(\det\Big(\frac{\partial(f, g, \Omega)}{\partial(x, y, z)}\Big)\Big) = \det\Big(\frac{\partial(\phi(f), \phi(g), \phi(\Omega))}{\partial(\phi(x), \phi(y), \phi(z))}\Big) \\
&= \det(\phi) \det\Big(\frac{\partial(\phi(f), \phi(g), \Omega')}{\partial(\phi(x), \phi(y), \phi(z))}\Big) \\
&= \det(\phi) \det\Big(\frac{\partial(x, y, z)}{\partial(\phi(x), \phi(y), \phi(z))}\Big) \det\Big(\frac{\partial(\phi(f), \phi(g), \Omega')}{\partial(x, y, z)}\Big) \\
&= \det(\phi) \det(\phi)^{-1}\{\phi(f), \phi(g)\}_{\Omega'} = \{\phi(f), \phi(g)\}_{\Omega'}.
\end{aligned}$$

So, ϕ is indeed a Poisson isomorphism. Finally, the consequences follow immediately. ∎

Now, we can classify all connected graded unimodular Poisson algebras in dimension three.

Theorem 4.4. *Any connected graded unimodular Poisson algebra A is isomorphic to some $A_{\lambda\Omega}$, where Ω is listed in Theorem 3.5 and $\lambda \in \Bbbk^\times$.*

Proof. By [42, Theorem 5], any unimodular Poisson structure on $A = \Bbbk[x, y, z]$ is given by π_Ω for some potential Ω. Moreover, π_Ω being graded implies $|\Omega| = |x| + |y| + |z|$. Hence, our classification follows from Definition 4.1, Theorem 3.5 and Lemma 4.3. ■

Example 4.5. We introduce some examples of Poisson fields of transcendental degree 2 over the field $\Bbbk$.

(1) We define $K_{\text{Weyl}} := \Bbbk(x, y)$ to be the Poisson field $\Bbbk(x, y)$ with the Poisson bracket $\{x, y\} = 1$.

(2) We define $K_q := \Bbbk(x, y)$ to be the Poisson field $\Bbbk(x, y)$ with the Poisson bracket $\{x, y\} = qxy$ for some $q \in \Bbbk^\times$. It is shown in [21, Corollary 5.4] that $K_q \cong K_p$ as Poisson fields if and only if $p = \pm q$. Note that K_{Weyl} is not isomorphic to any K_q as Poisson fields by [21, Corollary 5.3].

(3) Consider the irreducible cubic polynomial

$$\Omega_{\zeta,\lambda} := \zeta(x^3 + y^3 + z^3) + \lambda xyz$$

with two parameters $\zeta, \lambda \in \Bbbk$ such that $\zeta \neq 0, \lambda^3 \neq -3^3\zeta^3$. We denote the corresponding graded unimodular Poisson algebra by $A_{\Omega_{\zeta,\lambda}}$, where the Poisson structure on $\Bbbk[x, y, z]$ is defined by

$$\{x, y\} = 3\zeta z^2 + \lambda xy, \{y, z\} = 3\zeta x^2 + \lambda yz, \{z, x\} = 3\zeta y^2 + \lambda xz$$

with $\deg(x) = \deg(y) = \deg(z) = 1$. Let $S_{\zeta,\lambda} := Q(P_{\Omega_{\zeta,\lambda}})$ be the Poisson fraction field. By [24, Corollary 6.4], $S_{\zeta,\lambda}$ is not isomorphic to K_{Weyl} or K_q. However, $S_{\zeta,\lambda} = F_{\zeta,\lambda}(t)$ as fields, where $F_{\zeta,\lambda}$ is the function field of the elliptic curve $\Omega_{\zeta,\lambda} = 0$. Suppose $S_{\zeta,\lambda} \cong S_{\zeta',\lambda'}$ as Poisson fields, whence they are isomorphic as function fields. By [9, Theorem 2], we have $F_{\zeta,\lambda} \cong F_{\zeta',\lambda'}$ as function fields. As a result, $\Omega_{\zeta,\lambda} = 0$ and $\Omega_{\zeta',\lambda'} = 0$ are two birationally equivalent elliptic curves or have the same j-invariant $27\frac{(\lambda/\zeta)^3((\lambda/\zeta)^3+8)^3}{((\lambda/\zeta)^3-1)} = 27\frac{(\lambda'/\zeta')^3((\lambda'/\zeta')^3+8)^3}{((\lambda'/\zeta')^3-1)}$ (for the j-invariant of a Hesse form of a smooth elliptic curve, see [15, Theorem 2.11]). It is unclear if two elliptic curves with equations $\Omega_{\zeta,\lambda} = 0$ and $\Omega_{\zeta',\lambda'} = 0$ being birationally equivalent necessarily implies an isomorphism between their corresponding Poisson fields $S_{\zeta,\lambda}$ and $S_{\zeta',\lambda'}$.

Theorem 4.6. *Let A_Ω be a connected graded unimodular Poisson algebra defined by some homogeneous irreducible potential Ω. Let $Q = Q(P_\Omega)$ be the Poisson fraction field of P_Ω. Then the following hold:*

(1) *If* Ω *does not have an isolated singularity, then* Q *is isomorphic to either* K_{Weyl} *or* K_q *for some* $q \in \Bbbk^{\times}$.

(2) *If* Ω *has an isolated singularity, then* Q *is isomorphic to* $S_{\zeta,\lambda}$ *for some* $\zeta, \lambda \in \Bbbk$ *with* $\zeta \neq 0$ *and* $\lambda^3 \neq -3^3\zeta^3$.

Moreover, we label those Ω *with the corresponding* Q *that are isomorphic to* $S_{\zeta,\lambda}$ *by* ⓘ *and those that are isomorphic to* K_q *by* ⓠ *and those that are isomorphic to* K_{Weyl} *by* ⓦ *in Tables A.1–A.6.*

Proof. We conduct a case-by-case verification for those irreducible potentials Ω listed in Tables A.1–A.6. Then the result follows from Theorem 4.4.

(1) Suppose Ω does not have an isolated singularity. As an illustration, we provide some details when $\Omega = z^2 + y^3 + 2x^2y^2 + x^4y$ ($\deg x = 1, \deg y = 2$ and $\deg z = 3$) in Table A.5 and $\Omega = z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$ with $a \nmid b$ ($\deg x = a > 2, \deg y = b$ and $\deg z = c$) in Table A.6. We also check for $\Omega = z^2 + x^{2+\frac{2b}{a}}$ when $c = a + b$ and $a \nmid b$ in Table A.6. Let $Q = Q(P_{\lambda\Omega})$ for some $\lambda \in \Bbbk^{\times}$.

If $\Omega = z^2 + y^3 + 2x^2y^2 + x^4y$, then we have $z^2 + y(y + x^2)^2 = 0$ in P_Ω. Let $w = \frac{z}{y+x^2}$. Then $y = -w^2$ and $Q = \Bbbk(x, w)$. Consider $\{x, w\} = \left\{x, \frac{z}{y+x^2}\right\} = -\lambda(y + x^2) = \lambda(w^2 - x^2)$. After a linear transformation of x and w, we have $\{x, w\} = pxw$ for some $p \in \Bbbk^{\times}$. In this case, $Q(P_{\lambda\Omega}) \cong K_p$.

If $\Omega = z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$, then $\frac{2b}{a}$ is odd since $a \nmid b$, say $2t + 1$. In Q, we have $x = -\left(\frac{z}{x^{1+t}}\right)^2 - \left(\frac{y}{x^t}\right)^2$. Set $u = \frac{z}{x^{1+t}}$ and $v = \frac{y}{x^t}$. Then $Q = \Bbbk(u, v)$. One can check that $\{u, v\} = \lambda(u^2 + v^2)$. After a linear transformation of u and v, we can obtain $\{u, v\} = puv$ for some $p \in \Bbbk^{\times}$.

If $\Omega = z^2 + x^{2+\frac{2b}{a}}$ with $a \nmid b$ and $(a < b < c) \neq (1, 2, 3)$, then $\frac{2b}{a}$ is odd. Assume that $2 + \frac{2b}{a} = 2l + 1$. Thus, $z^2 + x^{2l+1} = 0$ and $x = -\left(\frac{z}{x^l}\right)^2$ in Q. Set $s = y$ and $t = \frac{z}{x^l}$. We have $Q = \Bbbk(s, t)$ and

$$\begin{aligned}\{s, t\} = \left\{y, \frac{z}{x^l}\right\} &= \frac{1}{x^l}\{y, z\} - \frac{lz}{x^{l+1}}\{y, x\} = \frac{(2l + 1)x^{2l}}{x^l} + \frac{2lz^2}{x^{l+1}} \\ &= \frac{(2l + 1)x^{2l+1} + 2lz^2}{x^{l+1}} = \frac{x^{2l+1}}{x^{l+1}} = x^l = (-t^2)^l = (-1)^l t^{2l}.\end{aligned}$$

Set $u = \frac{s}{(-1)^l t^{2l}}$ and $v = t$. Then we have $Q = \Bbbk(u, v)$ with $\{u, v\} = 1$. Thus, Q is of Weyl type.

(2) Suppose Ω has an isolated singularity. By Theorem 3.5, we need to consider the following three cases. Up to a scalar multiple, we can assume that

(a) $\Omega_1 = z^2x + yx^2 + \lambda xy^2 + y^3$ with $(a, b, c) = (1, 1, 1)$ and $\lambda \neq \pm 2$,

(b) $\Omega_2 = z^2 + x^3y + \lambda x^2y^2 + xy^3$ with $(a, b, c) = (1, 1, 2)$ and $\lambda \neq \pm 2$,

(c) $\Omega_3 = z^2 + y^3 + \lambda x^2y^2 + x^4y$ with $(a, b, c) = (1, 2, 3)$ and $\lambda \neq \pm 2$.

We use an alternative form in (a) following [11, p. 255] instead of the Hessian normal form. We consider $Q_i = Q(P_{\mu\Omega_i})$ for some $\mu \in \Bbbk^{\times}$ for $i = 1, 2, 3$.

Case (a). In Q_1, we have

$$\{x,y\} = 2\mu zx, \quad \{y,z\} = \mu(z^2+2xy+\lambda y^2) \quad \text{and} \quad \{z,x\} = \mu(x^2+2\lambda xy+3y^2).$$

Denote $u = \frac{z}{x}$, $v = \frac{y}{x}$ and $w = x$. We have

$$0 = \frac{\Omega_1}{x^3} = \left(\frac{z}{x}\right)^2 + \frac{y}{x} + \lambda\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 = u^2+v+\lambda v^2+v^3.$$

A direct computation yields that

$$\{u,w\} = \mu(w+2\lambda wv+3wv^2), \quad \{v,w\} = -2\mu wu, \quad \{u,v\} = 0.$$

Case (b). In Q_2, we have

$$\{x,y\} = 2\mu z, \quad \{y,z\} = \mu(3x^2y+2\lambda xy^2+y^3)$$

and

$$\{z,x\} = \mu(x^3+2\lambda x^2y+3xy^2).$$

Denote $u = \frac{z}{x^2}$, $v = \frac{y}{x}$ and $w = x$. We have

$$0 = \frac{\Omega_2}{x^4} = \left(\frac{z}{x^2}\right)^2 + \frac{y}{x} + \lambda\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 = u^2+v+\lambda v^2+v^3.$$

We can easily verify that

$$\{u,w\} = \mu(w+2\lambda wv+3wv^2), \quad \{v,w\} = -2\mu wu, \quad \{u,v\} = 0.$$

Case (c). In Q_3, we have

$$\{x,y\} = 2\mu z, \quad \{y,z\} = \mu(2\lambda xy^2+4x^3y), \quad \{z,x\} = \mu(3y^2+2\lambda x^2y+x^4).$$

Denote $u = \frac{z}{x^3}$, $v = \frac{y}{x^2}$ and $w = x$. Then

$$0 = \frac{\Omega_3}{x^6} = \left(\frac{z}{x^3}\right)^2 + \left(\frac{y}{x^2}\right)^3 + \lambda\left(\frac{y}{x^2}\right)^2 + \left(\frac{y}{x^2}\right) = u^2+v^3+\lambda v^2+v.$$

It is routine to check that

$$\{u,w\} = \mu(w+2\lambda wv+3wv^2), \quad \{v,w\} = -2\mu wu, \quad \{u,v\} = 0.$$

Therefore, $Q_1 \cong Q_2 \cong Q_3$ as Poisson fraction fields. Finally, we reuse the Hesse form for (1) and note that any re-scaling of Ω can be written as $\Omega = \zeta(x^3+y^3+z^3) + \lambda xyz$ for some $\zeta, \lambda \in \Bbbk$ such that $\zeta \neq 0, \lambda^3 \neq -3^3\zeta^3$. The result follows from Example 4.5 (3). ∎

Remark 4.7. Let Ω be irreducible and denote the Poisson fraction field by $Q = Q(P_\Omega)$. Let X be the projective curve in the weighted projective space $\mathbb{P}(a,b,c)$ determined by such Ω. Then the statement of Theorem 4.6 can be refined as: if X is smooth, then $Q \cong S_{\sigma,\lambda}$; if X has nodal singularity, then $Q \cong K_q$ for some $q \in \Bbbk^\times$; and if X has cusp singularity, then $Q \cong K_{\mathrm{Weyl}}$.

Remark 4.8. We note that the isomorphisms between $Q(P_\Omega)$ depend not on grading but on projective curves $\Omega = 0$ when Ω is homogeneous of degree $|x| + |y| + |z|$. The geometry of elliptic curves appears to reflect properties of these Poisson algebras.

We aim to investigate the relationship between the Poisson automorphism group of A_Ω and the type of Ω. We determine the automorphism group of every connected graded unimodular Poisson algebra A_Ω, where the potential Ω has an isolated singularity.

Theorem 4.9. *Let A_Ω be a connected graded unimodular Poisson algebra. If Ω has an isolated singularity, then every Poisson automorphism of A is graded. As a consequence,*

$$\mathrm{Aut}_P(A_\Omega) \cong \mathrm{Aut}_{\mathrm{gr}\,P}(A_\Omega) \cong \mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega) \cong \mathrm{Aut}_P(P_\Omega).$$

We will establish a few lemmas before proving Theorem 4.9. We will use the Hessian normal forms of these elliptic curves in the weighted projective space $\mathbb{P}(a, b, c)$ to prove the lemmas ahead. These forms can be obtained from the Ω listed in part (2) of Theorem 4.6 by a linear transformation, as stated in the following lemma.

Lemma 4.10. *Let $A = \Bbbk[x, y, z]$ be a weighted polynomial algebra with $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$ for some $1 \le a \le b \le c$ and $\gcd(a, b, c) = 1$. If Ω is a homogeneous polynomial of degree $a + b + c$ that has an isolated singularity, then Ω is one of the forms*

(1) $\Omega_1 = x^3 + y^3 + z^3 + \lambda xyz$ *for* $(-\lambda)^3 \neq 27$ *and* $(a, b, c) = (1, 1, 1)$.

(2) $\Omega_2 = x^4 + y^4 + z^2 + \lambda xyz$ *for* $(-\lambda)^4 \neq 64$ *and* $(a, b, c) = (1, 1, 2)$.

(3) $\Omega_3 = x^6 + y^3 + z^2 + \lambda xyz$ *for* $(-\lambda)^6 \neq 432$ *and* $(a, b, c) = (1, 2, 3)$.

Proof. Case (1) follows from a classical result that a smooth cubic projective curve is projectively equivalent to $x^3 + y^3 + z^3 = 3kxyz$ with $k^3 \neq -1$, whose proof can be found in [6, Theorem 2.12]; see also [11, 12, 25, 29]. We will show Case (2) in detail and obtain the third case in a similar fashion. Replacing $z + \frac{1}{2}\lambda xy$ by z, one can rewrite Ω_2 as $z^2 + x^4 + y^4 - \frac{1}{4}\lambda^2 x^2 y^2$, whence it becomes

$$z^2 + (x^2 - uy^2)(x^2 - vy^2) = z^2 + (x + \sqrt{u}y)(x - \sqrt{u}y)(x + \sqrt{v}y)(x - \sqrt{v}y)$$

with $uv = 1$ and $u + v = \frac{1}{4}\lambda^2$. Since $(-\lambda)^4 \neq 64$, it implies $\sqrt{v} \neq \sqrt{u}$. Lastly, after a linear transformation of x and y, we can obtain that $\Omega_2 = z^2 + xy^3 + x^3y + kx^2y^2$ for some $k \in \Bbbk$. Since Ω_2 has isolated singularity, it forces that $k \neq \pm 2$ as desired. ∎

Recall that $P_{\Omega-\xi} = A/(\Omega - \xi)$ for any $\xi \in \Bbbk$. Let $P = P_\Omega$. Since $P = \bigoplus_{i\ge 0} P_i$ is graded, P has two natural 0-filtrations denoted by $\mathbb{F}^{\mathrm{Id}} = \{F_i^{\mathrm{Id}} \mid i \in \mathbb{Z}\}$ and $\mathbb{F}^{-\mathrm{Id}} = \{F_i^{-\mathrm{Id}} \mid i \in \mathbb{Z}\}$, respectively, such that

$$F_i^{\mathrm{Id}} = \bigoplus_{n\ge i} P_n \quad \text{and} \quad F_i^{-\mathrm{Id}} = \bigoplus_{n\le -i} P_n. \tag{4.2}$$

Since $\mathrm{gr}_{\mathbb{F}^{\pm \mathrm{Id}}} P \cong P$ (with grading flipped $i \leftrightarrow -i$ for $\mathbb{F}^{-\mathrm{Id}}$), P has two canonical faithful 0-valuations. We denote them by ν^{Id} and $\nu^{-\mathrm{Id}}$, respectively. In particular, we have

$$\nu^{\mathrm{Id}}(f) = n \quad \text{and} \quad \nu^{-\mathrm{Id}}(f) = -m \tag{4.3}$$

for any $f = \sum_{i=n}^{m} f_i$ with $n \leq m$, $f_i \in P_i$ and $f_n, f_m \neq 0$.

Now consider $P_{\Omega-\xi} = A/(\Omega - \xi)$ for any $\xi \in \Bbbk^\times$. We define a filtration $\mathbb{F}^{-\mathrm{Id}} = \{F_i^{-\mathrm{Id}} \mid i \in \mathbb{Z}\}$ of $P_{\Omega-\xi}$ by

$$F_{-i}^{-\mathrm{Id}} P_{\Omega-\xi} = \left\{ \sum a_j f_j \mid a_j \in \Bbbk,\ f_j \text{ are monomials of degree} \leq i \right\}.$$

One can check that $\mathrm{gr}_{\mathbb{F}^{-\mathrm{Id}}} P_{\Omega-\xi} \cong P$, and the corresponding faithful 0-valuation is given by $\nu^{-\mathrm{Id}}$ via $\nu^{-\mathrm{Id}}(x) = -a$, $\nu^{-\mathrm{Id}}(y) = -b$ and $\nu^{-\mathrm{Id}}(z) = -c$. We say a Poisson algebra automorphism σ of $P_{\Omega-\xi}$ is *linear* if it preserves the specific 0-filtration $\mathbb{F}^{-\mathrm{Id}}$ on $P_{\Omega-\xi}$, or namely $\sigma(F_i^{-\mathrm{Id}}) \subseteq F_i^{-\mathrm{Id}}$ for i.

The Poisson valuations directly apply to Poisson automorphism groups of $P_{\Omega-\xi}$ when the homogeneous potential Ω has an isolated singularity.

Lemma 4.11. *Let A_Ω be a connected graded Poisson algebra. We have the following if the potential Ω has an isolated singularity:*

(1) *Every Poisson automorphism of P_Ω is graded.*

(2) *If $\xi \neq 0$, then every Poisson automorphism of $P_{\Omega-\xi}$ is linear.*

Proof. We only check for Ω_2 (with $(\deg(x), \deg(y), \deg(z)) = (1, 1, 2)$) and the proofs for other cases are similar. For simplicity, we write $\Omega = \Omega_2$, $P = P_{\Omega_2}$ and $P_\xi = P_{\Omega_2 - \xi}$.

(1) By Theorem 4.6 (2), the fraction ring $Q = Q(P)$ is isomorphic to $Q(P_{\Omega_1})$ as Poisson fields. By [24, Theorem 3.8], Q has exactly two faithful 0-valuations, namely $\{\nu^{\pm \mathrm{Id}}\}$ as discussed above. Let ϕ be any Poisson automorphism of P. We extend ϕ to a Poisson field automorphism of Q, which we still denote by ϕ. It is clear that $\nu^{\pm \mathrm{Id}} \circ \phi$ are two distinct faithful 0-valuations of Q. So $\{\nu^{\pm \mathrm{Id}}\} = \{\nu^{\pm \mathrm{Id}} \circ \phi\}$. By (4.3), an element $f \in P$ is homogeneous if and only if $\nu^{\mathrm{Id}}(f) + \nu^{-\mathrm{Id}}(f) = 0$. Let f be any homogeneous element of P. So we have

$$0 = \nu^{\mathrm{Id}}(f) + \nu^{-\mathrm{Id}}(f) = (\nu^{\mathrm{Id}} \circ \phi)(f) + (\nu^{-\mathrm{Id}} \circ \phi)(f) = \nu^{\mathrm{Id}}(\phi(f)) + \nu^{-\mathrm{Id}}(\phi(f)).$$

This implies that $\phi(f)$ is again homogeneous. In particular, $\phi(x), \phi(y), \phi(z)$ are homogeneous and nonzero. By recycling letters a, b, c, respectively, we assume they have degrees a, b, c. Hence, $\phi(\Omega) = \phi(x)^4 + \phi(y)^4 + \phi^2(z) + \lambda \phi(x)\phi(y)\phi(z)$ is homogeneous. So we have $4a = 4b = 2c = a + b + c$. Since P is generated by $\phi(x), \phi(y), \phi(z)$, we must have $a = b = 1$ and $c = 2$. Hence, ϕ is graded.

(2) Let $Q = Q(P_\xi)$ for $\xi \in \Bbbk^\times$. We can use the similar argument of [24, Theorem 3.11] to show that Q has only one faithful 0-valuation, namely $\nu^{-\mathrm{Id}}$ as discussed above. So for any Poisson automorphism ϕ of P_ξ, we have $\nu^{-\mathrm{Id}} = \nu^{-\mathrm{Id}} \circ \phi$. Let $f \in P_\xi$. By definition, we have $f \in F_i^{-\mathrm{Id}}$ if and only if $\nu^{-\mathrm{Id}}(f) \leq -i$ if and only if $(\nu^{-\mathrm{Id}} \circ \phi)(f) = \nu^{-\mathrm{Id}}(\phi(f)) \leq -i$. Hence, ϕ preserves the 0-filtration $\mathbb{F}^{-\mathrm{Id}}$ of P_ξ and so is linear. ■

Next, we explicitly compute the Poisson automorphism groups of $P_{\Omega-\xi}$.

Lemma 4.12. *Let* $\Omega = x^3 + y^3 + z^3 + \lambda xyz$ *for* $(-\lambda)^3 \neq 27$ *with* $\deg(x) = 1$, $\deg(y) = 1$, $\deg(z) = 1$.

(1) *There is a short exact sequence of groups:*

$$1 \to G \to \mathrm{Aut}_P(P_\Omega) \to C_3 \to 1,$$

where $G = \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1^3 = \alpha_2^3 = \alpha_3^3 = \alpha_1\alpha_2\alpha_3\} \subset \Bbbk^\times \times \Bbbk^\times \times \Bbbk^\times$ *and* $\mathrm{Aut}_P(P_\Omega) \cong C_3 \ltimes G$.

(2) *There is a short exact sequence of groups:*

$$1 \to G' \to \mathrm{Aut}_P(P_{\Omega-1}) \to C_3 \to 1,$$

where $G' = \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1^3 = \alpha_2^3 = \alpha_3^3 = \alpha_1\alpha_2\alpha_3 = 1\} \subset \Bbbk^\times \times \Bbbk^\times \times \Bbbk^\times$ *and* $\mathrm{Aut}_P(P_{\Omega-1}) \cong C_3 \ltimes G'$.

Proof. (1) Note that the argument of [37, Theorem 1] works for $\mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega)$ as well. So $\mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega)$ is generated by all possible diagonal actions $G = \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1^3 = \alpha_2^3 = \alpha_3^3 = \alpha_1\alpha_2\alpha_3\}$ and the permutation $\tau(x, y, z) = (y, z, x)$. So our result follows Lemma 4.11.

(2) By Lemma 4.11, every automorphism ϕ of $P_{\Omega-1}$ is linear. Note that for $0 \le i \le 2$, we will identify A_i with $(P_{\Omega-1})_i$ as vector spaces. So we can write

$$\phi(x) = f_1 + f_0, \quad \phi(y) = g_1 + g_0, \quad \phi(z) = h_1 + h_0,$$

where f_i, g_i, h_i are homogeneous polynomials of degree i in $A_i = (P_{\Omega-1})_i$ for all possible $0 \le i \le 1$. For simplicity, we will denote the images of x, y, z in $P_{\Omega-1}$ by x, y, z. We have

$$\{x, y\} = 3z^2 + \lambda xy, \quad \{y, z\} = 3x^2 + \lambda yz, \quad \{z, x\} = 3y^2 + \lambda xz$$

in $P_{\Omega-1}$. We apply ϕ to each of the above three equations. After comparing the constant terms on both sides of the resulting equations, we obtain

$$3f_0^2 + \lambda g_0 h_0 = 3g_0^2 + \lambda f_0 h_0 = 3h_0^2 + \lambda f_0 g_0 = 0.$$

Note that we have $(a + b + c)\Omega = ax\Omega_x + by\Omega_y + cz\Omega_z$ in $A = \Bbbk[x, y, z]$. Since $(a, b, c) = (1, 1, 1)$, it follows that

$$\begin{aligned}\Omega(f_0, g_0, h_0) &= \frac{1}{a+b+c}[af_0\Omega_x(f_0, g_0, h_0) + bg_0\Omega_y(f_0, g_0, h_0) \\ &\qquad + ch_0\Omega_z(f_0, g_0, h_0)] \\ &= \frac{1}{3}[f_0(3g_0^2 + \lambda f_0 h_0) + g_0(3h_0^2 + \lambda f_0 g_0) + h_0(3f_0^2 + \lambda g_0 h_0)] = 0.\end{aligned}$$

So (f_0, g_0, h_0) is a singular point on the surface $\Omega = 0$. Hence, $(f_0, g_0, h_0) = (0, 0, 0)$ since Ω only has an isolated singularity at the origin. So ϕ maps $\Bbbk x + \Bbbk y + \Bbbk z$ to itself. This implies that $\mathrm{gr}(\phi) : \mathrm{gr}(P_{\Omega-1}) \to \mathrm{gr}(P_{\Omega-1})$ is a graded automorphism of $P_\Omega(\cong \mathrm{gr}(P_{\Omega-1}))$ which has been described in part (1). The rest of the proof follows from a direct computation. ■

Lemma 4.13. *Let* $\Omega = x^4 + y^4 + z^2 + \lambda xyz$ *for* $(-\lambda)^4 \neq 64$ *with* $\deg(x) = \deg(y) = 1$ *and* $\deg(z) = 2$.

(1) *There is a short exact sequence of groups:*

$$1 \to G \to \mathrm{Aut}_P(P_\Omega) \to C_2 \to 1,$$

where $G = \{(\alpha_1, \alpha_2) \mid \alpha_1^2 = \alpha_2^2\} \subset \Bbbk^\times \times \Bbbk^\times$ *and* $\mathrm{Aut}_P(P_\Omega) \cong C_2 \ltimes G$.

(2) *There is a short exact sequence of groups:*

$$1 \to G' \to \mathrm{Aut}_P(P_{\Omega-1}) \to C_2 \to 1,$$

where $G' = \{(\alpha_1, \alpha_2) \mid \alpha_1^4 = \alpha_2^4 = 1, \alpha_1^2 = \alpha_2^2\} \subset \Bbbk^\times \times \Bbbk^\times$ *and* $\mathrm{Aut}_P(P_{\Omega-1}) \cong C_2 \ltimes G'$.

Proof. (1) By Lemma 4.11, every Poisson automorphism ϕ of P_Ω is graded. For the convenience of this proof, we write $(x, y, z) = (x_1, x_2, x_3)$. In general, we can assume that ϕ is given by

$$\begin{cases} \phi(x_1) = \alpha_1 x_1 + \alpha_2 x_2, \\ \phi(x_2) = \beta_1 x_1 + \beta_2 x_2, \\ \phi(x_3) = \gamma x_3 + h(x_1, x_2), \end{cases}$$

where γ and $\alpha_1\beta_2 - \alpha_2\beta_1$ are not zero in $\Bbbk$ and $h(x_1, x_2)$ is a quadratic polynomial in $\Bbbk[x_1, x_2]$, as $\det(\phi) = \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) \neq 0$. After a proper re-scaling of variables, we first assume that $\alpha_1\beta_2 - \alpha_2\beta_1 = 1$. Applying ϕ to the equation $\{x_1, x_2\} = 2x_3 + \lambda x_1 x_2$, we get $\gamma = 1$ and $h(x_1, x_2) = \frac{\lambda}{2}(x_1 x_2 - \phi(x_1)\phi(x_2))$. Applying ϕ to the other two equations $\{x_2, x_3\} = 4x_1^3 + \lambda x_2 x_3$ and $\{x_3, x_1\} = 4x_2^3 + \lambda x_1 x_3$, we further obtain the following relations:

$$\begin{cases} 24\alpha_1^2\alpha_2 = \lambda^2(\beta_1 + \alpha_2\beta_1^2 + 2\beta_1\beta_2\alpha_1) = 3\lambda^2\alpha_1\beta_1\beta_2, \\ 24\alpha_1\alpha_2^2 = \lambda^2(-\beta_2 + \alpha_1\beta_2^2 + 2\beta_1\beta_2\alpha_2) = 3\lambda^2\alpha_2\beta_1\beta_2, \\ 24\beta_1^2\beta_2 = \lambda^2(-\alpha_1 + \alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1) = 3\lambda^2\alpha_1\alpha_2\beta_1, \\ 24\beta_1\beta_2^2 = \lambda^2(\alpha_2 + \alpha_2^2\beta_1 + 2\alpha_1\alpha_2\beta_2) = 3\lambda^2\alpha_1\alpha_2\beta_2. \end{cases}$$

Since $\alpha_1\beta_2 - \alpha_2\beta_1 = 1$, α_1 and α_2 cannot be simultaneously equal to zero, and similar situation happens to β_1 and β_2. After simplifying the above equations, we have $8\alpha_1\alpha_2 = \lambda^2\beta_1\beta_2$ and $8\beta_1\beta_2 = \lambda^2\alpha_1\alpha_2$ and so $(64 - \lambda^4)\alpha_1\alpha_2 = (64 - \lambda^4)\beta_1\beta_2 = 0$. Since $\lambda^4 \neq 64$, we get $\alpha_1\alpha_2 = \beta_1\beta_2 = 0$. Thus, we have either $\alpha_2 = \beta_1 = 0$ or

$\alpha_1 = \beta_2 = 0$. Now, by taking care of the re-scaling of variables, we can find a permutation $\sigma \in S_2$ and write ϕ as

$$\phi(x_1) = \alpha_1 x_{\sigma(1)}, \quad \phi(x_2) = \alpha_2 x_{\sigma(2)}, \quad \phi(x_3) = \alpha_3 x_3 + h \tag{4.4}$$

for some nonzero scalars $\alpha_1, \alpha_2, \alpha_3$ and some quadratic polynomial $h(x_1, x_2)$. Then it is routine to check that ϕ is a Poisson automorphism of P_Ω if and only if

$$\alpha_3 = \operatorname{sgn}(\sigma)\alpha_1\alpha_2, \quad \alpha_1^2 = \alpha_2^2, \quad h = \frac{\lambda}{2}(\operatorname{sgn}(\sigma) - 1)\alpha_1\alpha_2 x_1 x_2.$$

Consider the normal subgroup G of $\mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega)$, which is given by $\phi(x) = \alpha_1 x, \phi(y) = \alpha_2 y, \phi(z) = \alpha_1\alpha_2 z$ satisfying $\alpha_1^2 = \alpha_2^2$. Finally, it is clear that $\mathrm{Aut}_{\mathrm{gr}\,P}(P_\Omega)/G \cong S_2$.

(2) By Lemma 4.11, every automorphism ϕ of $P_{\Omega-1}$ is linear. So we can write

$$\phi(x) = f_1 + f_0, \quad \phi(y) = g_1 + g_0, \quad \phi(z) = h_2 + h_1 + h_0,$$

where f_i, g_i, h_i are homogeneous polynomials of degree i in $A_i = (P_{\Omega-1})_i$ for all possible $0 \le i \le 2$. We apply ϕ to the following Poisson brackets:

$$\{x, y\} = 2z + \lambda xy, \quad \{y, z\} = 4x^3 + \lambda yz, \quad \{z, x\} = 4y^3 + \lambda xz.$$

By examining these equations at the degree 0 part, we get

$$2h_0 + \lambda f_0 g_0 = 4f_0^3 + \lambda g_0 h_0 = 4g_0^2 + \lambda f_0 h_0 = 0.$$

This implies that (f_0, g_0, h_0) is a singular point for $\Omega = 0$. So $f_0 = g_0 = h_0 = 0$ since Ω only has an isolated singularity at the origin. Now since $\phi(\Omega) = 1$ in $P_{\Omega-1}$, we get

$$f_1^4 + g_1^4 + (h_2 + h_1)^2 + \lambda f_1 g_1 (h_2 + h_1) = 1.$$

The degree 2 part of the above equation yields that $h_1^2 = 0$ and thus $h_1 = 0$. Hence, ϕ maps $\Bbbk x + \Bbbk y$ (resp. $\Bbbk z$) to itself, and we can write ϕ as in (4.4). Similar to (1), indeed, we have

$$\phi(x_1) = \alpha_1 x_{\sigma(1)}, \quad \phi(x_2) = \alpha_2 x_{\sigma(2)}, \quad \phi(x_3) = \alpha_3 x_3 + \frac{\lambda}{2}(\operatorname{sgn}(\sigma) - 1)\alpha_1\alpha_2 x_1 x_2$$

for some $\sigma \in S_2$ and $\alpha_1^4 = \alpha_2^4 = 1$ and $\alpha_1^2 = \alpha_2^2$ since $\phi(\Omega) = 1$. So the result follows. ∎

Lemma 4.14. *Let* $\Omega = x^6 + y^3 + z^2 + \lambda xyz$ *for* $(-\lambda)^6 \neq 432$ *with* $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$.

(1) *Every Poisson automorphism of* P_Ω *is of the form*

$$x \mapsto \zeta x, \quad y \mapsto \zeta^2 y, \quad z \mapsto \zeta^3 z$$

for some $\zeta \in \Bbbk^\times$.

(2) *Every Poisson automorphism of* $P_{\Omega-1}$ *is of the form*

$$x \mapsto \zeta x, \quad y \mapsto \zeta^2 y, \quad z \mapsto \zeta^3 z,$$

where $\zeta^6 = 1$.

Proof. (1) By Lemma 4.11, every automorphism ϕ of P_Ω is graded. So we can write

$$\phi(x) = \alpha_1 x, \quad \phi(y) = \beta_1 y + \beta_0 x^2, \quad \phi(z) = \gamma_2 z + \gamma_1 xy + \gamma_0 x^3 \tag{4.5}$$

for some $\alpha_1, \beta_1, \gamma_2 \in \Bbbk^\times$ and $\beta_0, \gamma_1, \gamma_0 \in \Bbbk$. Applying ϕ to $\{z, x\} = 3y^2 + \lambda xz$, we obtain that $\beta_0 = \gamma_0 = \gamma_1 = 0$. So ϕ equals a scalar multiple when it is applied to the generators. Finally, from $\{\phi(y), \phi(z)\} = \phi(6x^5 + \lambda yz)$, $\{\phi(x), \phi(y)\} = \phi(2z + \lambda xy)$ and $\{\phi(z), \phi(x)\} = \phi(3y^2 + \lambda xz)$, we obtain $\beta_1\gamma_2 = \alpha_1^5$, $\alpha_1\beta_1 = \gamma_2$ and $\gamma_2\alpha_1 = \beta_1^2$. Let $\zeta = \frac{\gamma_2}{\beta_1}$. We have $\alpha_1 = \zeta, \beta_1 = \zeta^2, \gamma_2 = \zeta^3$. So the result follows.

(2) By Lemma 4.11, every automorphism ϕ of $P_{\Omega-1}$ is linear. Note that $A_i = (P_{\Omega-1})_i$ for $0 \le i \le 5$. So we can write

$$\phi(x) = f_1 + f_0, \quad \phi(y) = g_2 + g_1 + g_0, \quad \phi(z) = h_3 + h_2 + h_1 + h_0,$$

where f_i, g_i, h_i are homogeneous polynomials of degree i in $A_i = (P_{\Omega-1})_i$ for all possible $0 \le i \le 3$. We apply ϕ to the following Poisson brackets:

$$\{x, y\} = 2z + \lambda xy, \quad \{y, z\} = 6x^5 + \lambda yz, \quad \{z, x\} = 3y^2 + \lambda xz$$

and examine the resulting equations at different degrees. First of all, the degree 0 part yields that

$$2h_0 + \lambda f_0 g_0 = 6f_0^5 + \lambda g_0 h_0 = 3g_0^2 + \lambda f_0 h_0 = 0.$$

This is equivalent to the claim that (f_0, g_0, h_0) is a singular point for $\Omega = 0$. So $f_0 = g_0 = h_0 = 0$ since Ω only has an isolated singularity at the origin. Then the degree 1 part further implies that $h_1 = 0$. Now since $\phi(\Omega) = 1$ in $P_{\Omega-1}$, we get

$$f_1^6 + (g_2 + g_1)^3 + (h_3 + h_2)^2 + \lambda f_1 (g_2 + g_1)(h_3 + h_2) = 1.$$

The degree 3 part of the above equation yields that $g_1^3 = 0$ and thus $g_1 = 0$. Furthermore, the degree 4 part implies that $h_2^2 = 0$ and $h_2 = 0$. Hence, we have ϕ as in (4.5). Similar to the proof of part (1), indeed, we have $\phi(x) = \zeta x, \phi(y) = \zeta^2 y, \phi(z) = \zeta^3 z$ for some $\zeta \in \Bbbk^\times$. Finally, $\phi(\Omega) = 1$ implies that $\zeta^6 = 1$. ■

Proof of Theorem 4.9. We will show it case by case which are listed in Lemma 4.10. It suffices to prove the result when Ω is case (2) and case (3), since case (1) was shown in [37]. Though our argument works for all cases, here we only provide the details for Case 2 as an illustration.

Let ϕ be a Poisson algebra automorphism of A_Ω. By the same argument as in Lemma 4.3, the restriction of ϕ on the Poisson center $\Bbbk[\Omega]$ is given by $\phi(\Omega) = \alpha\Omega$ with

$\alpha \in \Bbbk^\times$. So ϕ preserves the principal ideal generated by Ω. Let ϕ' denote the induced Poisson algebra automorphism for P_Ω. By Lemma 4.11, ϕ' is graded. Moreover, since the equations $\{x, y\} = \Omega_z, \{y, z\} = \Omega_x, \{z, x\} = \Omega_y$ are homogeneous of degree $<\deg(\Omega)$, we can lift ϕ' to a unique graded Poisson automorphism of A_Ω, denoted by σ. It is clear that $\sigma' = \phi'$. Let $\varphi = \phi \circ \sigma^{-1}$. Then it satisfies the equation $\varphi' = \mathrm{id}_{P_\Omega}$. It remains to show that $\varphi = \mathrm{id}_{A_\Omega}$. Note that we can write

$$\varphi(x) = x + \Omega f, \quad \varphi(y) = y + \Omega g, \quad \varphi(z) = z + \Omega h$$

for some polynomials $f, g, h \in A_\Omega$. An easy computation yields that $\varphi(\Omega) = \Omega + \Omega\alpha(f, g, h)$, where $\alpha(f, g, h) \in (A_\Omega)_{\geq 1}$. Since φ induces an algebra automorphism of the Poisson center $\Bbbk[\Omega]$ of A_Ω, we must have $\alpha = 0$ and $\varphi(\Omega) = \Omega$.

Now we consider a $\Bbbk$-linear basis $\mathbb{B} := \{1, x, y, z\} \cup \{b_s\}$ of the algebra P_Ω consisting of all possible monomials $\{x^{s_1} y^{s_2} z^{s_3} \mid s_1, s_2 \geq 0, 0 \leq s_3 \leq 1\}$. We also treat $\mathbb{B}$ as a fixed subset of monomials in A_Ω by lifting. We claim that every polynomial f in A_Ω can be written in the form

$$f = 1f^1(\Omega) + xf^x(\Omega) + yf^y(\Omega) + zf^z(\Omega) + \sum_{b_s} b_s f^{b_s}(\Omega), \tag{4.6}$$

where each $f^*(\Omega) \in \Bbbk[\Omega]$. We prove this claim by induction on $\deg(f)$. It is trivial for $\deg(f) = 0$. Suppose our claim holds for $\deg(f) \leq m$. For any polynomial f of degree $m + 1$, we can write

$$f = 1f^1 + xf^x + yf^y + zf^z + \sum_{b_s} b_s f^{b_s} + g\Omega$$

for some scalars $f^* \in \Bbbk$ and $\deg(g) = m - 3$ by looking at the image of f in P_Ω. So, by the induction hypothesis, we can write g in the required form. We get our claim by replacing g above. Therefore, we can write

$$\varphi(x) = 1f^1(\Omega) + xf^x(\Omega) + yf^y(\Omega) + zf^z(\Omega) + \sum_{b_s} b_s f^{b_s}(\Omega) \tag{4.7}$$

for some $f^*(\Omega) \in \Bbbk[\Omega]$.

Now for each scalar $\xi \neq 0$, let $\pi_\xi : A_\Omega \to P_{\Omega-\xi}$ be the quotient map and write φ'_ξ as the induced automorphism of φ since $\varphi(\Omega) = \Omega$. Note that the image of $\mathbb{B}$ via π_ξ is a $\Bbbk$-basis of $P_{\Omega-\xi}$, which we continue to denote by b_s, etc. So we have

$$\varphi'_\xi(x) = 1f^1(\xi) + xf^x(\xi) + yf^y(\xi) + zf^z(\xi) + \sum_{b_s} b_s f^{b_s}(\xi). \tag{4.8}$$

By Lemma 4.11, φ'_ξ is linear. Thus, $f^{b_s}(\xi) = 0$ for all $\xi \neq 0$ and b_s. Hence, $f^{b_s}(\Omega) = 0$, and (4.7) reduces to

$$\varphi(x) = 1f^1(\Omega) + xf^x(\Omega) + yf^y(\Omega) + zf^z(\Omega).$$

Moreover, since $\varphi'(x) = x$ in P_Ω, we have $f^x(\Omega) = 1 + p\Omega \neq 0$ for some $p \in \Bbbk[\Omega]$. If $f^y(\Omega) \neq 0$, we can choose some nonzero ξ_0 such that $f^x(\xi_0), f^y(\xi_0) \neq 0$. But in this case, $\varphi'_{\xi_0}(x)$ in (4.8) contains both terms of x and y. This contradicts to the description of $\mathrm{Aut}_{\mathrm{gr}\,P}(P_{\Omega-\xi_0})$ by Lemma 4.13. So $f^y(\Omega) = 0$, and we get $f^1(\Omega) = f^z(\Omega) = 0$ in the same fashion. This implies that $\varphi(x) = x(1 + p\Omega)$. Similarly, we get $\varphi^{-1}(x) = x(1 + \Omega h)$ for some $h \in \Bbbk[\Omega]$. By using $\varphi(\Omega) = \Omega$, we obtain that

$$x = \varphi(\varphi^{-1}(x)) = \varphi(x(1 + \Omega h)) = x(1 + \Omega p)(1 + \Omega h) = x,$$

which implies that $p = h = 0$. Therefore, $\varphi(x) = x$. We further get $\varphi(y) = y$ and $\varphi(z) = z$ by the same argument. Hence, φ is the identity as required. ∎

Example 4.15. Let $\Omega = z^2 + x^3 y$ in Table A.2 with $\deg x = \deg y = 1$ and $\deg z = 2$. Then the Poisson structure of A_Ω is determined by

$$\{x, y\} = 2z, \quad \{y, z\} = 3x^2 y, \quad \{z, x\} = x^3.$$

Consider the algebra automorphism of the polynomial ring $\Bbbk[x, y, z]$ defined by

$$\phi(x) = x, \quad \phi(y) = y - x^3 - 2z \quad \text{and} \quad \phi(z) = z + x^3.$$

It is straightforward to check that ϕ is an ungraded Poisson automorphism of A_Ω.

We can prove that if Ω belongs to the Weyl type, meaning $Q(P_\Omega) \cong K_{\mathrm{Weyl}}$, then A_Ω possesses ungraded Poisson automorphisms. However, we have not discovered any ungraded automorphisms for other irreducible Ω, that is, those satisfying $Q(P_\Omega) \cong K_q$. Therefore, we pose the following question.

Question 4.16. Let Ω be a homogeneous polynomial of degree $|x| + |y| + |z|$. If $Q(P_\Omega) \cong K_q$ as Poisson fields, is every Poisson automorphism of A_Ω and P_Ω graded?

5. Rigidities

In this section, we discuss several different rigidities of the Poisson algebras related to Ω. We intend to provide general methods for these rigidities and omit some details. Unless mentioned otherwise, we will retain Hypothesis 1.1 for this whole section.

5.1. Rigidity of graded twistings

Note that the *rigidity of graded twistings*, denoted by rgt, was defined in Definition 2.2. The following lemma provides an easy way to compute Poisson derivations and rgt of A_Ω.

Lemma 5.1. *Let A be a connected graded unimodular Poisson algebra A_Ω as given in Hypothesis 1.1. For any derivation δ of A, we have the following:*

(1) *δ is a Poisson derivation if and only if* $\operatorname{div}(\delta)\pi_\Omega = \pi_{\delta(\Omega)}$ *in* $\mathfrak{X}^2(A)$.

(2) *Suppose* $\operatorname{div}(\delta) = 0$. *Then δ is a Poisson derivation if and only if* $\delta(\Omega) = 0$.

(3) *We have*

$$\begin{aligned}\operatorname{rgt}(A) &= 1 - \dim_{\Bbbk} \operatorname{Gspd}(A)\\ &= 1 - \dim_{\Bbbk} \operatorname{Gpd}(A)\\ &= 1 - \dim_{\Bbbk} (\operatorname{PH}^1(A))_0\\ &= -\dim_{\Bbbk}\{\delta \in \operatorname{Gspd}(A) \mid \operatorname{div}(\delta) = 0\}\\ &= -\dim_{\Bbbk}\{\delta \in \operatorname{Gpd}(A) \mid \operatorname{div}(\delta) = 0\}\\ &= -\dim_{\Bbbk}\{\delta \in (\operatorname{Der}(A))_0 \mid \operatorname{div}(\delta) = \delta(\Omega) = 0\}.\end{aligned}$$

Proof. Note that δ is a Poisson derivation if and only if $d_\pi^1(\delta) = 0$. Recall that the cochain complexes (2.4) and (2.6) are isomorphic to each other. So

$$(d_\pi^1(\delta)(y,z), d_\pi^1(\delta)(z,x), d_\pi^1(\delta)(x,y)) = \delta_\Omega^1(\delta(x), \delta(y), \delta(z))$$

as vectors. By (2.8) and (4.1), we conclude that

$$d_\pi^1(\delta) = \operatorname{div}(\delta)\pi_\Omega - \pi_{\delta(\Omega)} \tag{5.1}$$

in $\mathfrak{X}^2(A)$. Thus, (1) follows immediately.

For (2), suppose $\delta(\Omega) = 0$. Then by (5.1), δ is a Poisson derivation. Conversely, suppose δ is a Poisson derivation. By (5.1), $\delta(\Omega) \in \Bbbk = A_0$. We write $\delta = \sum \delta_i$, where each δ_i is a homogeneous derivation of degree i. Since $\deg(\Omega) = n$, we get $\delta_i(\Omega) = 0$ for $i \neq -n$. Moreover, $\delta_{-n}(x)$, if not zero, has degree $a - n < 0$. So $\delta_{-n}(x) = 0$ as A is $\mathbb{N}$-graded. Similarly, we get $\delta_{-n}(y) = \delta_{-n}(z) = 0$. Therefore, $\delta_{-n}(\Omega) = 0$ and so $\delta(\Omega) = 0$, as desired.

For (3), consider the subspace $\operatorname{Gpd}_0(A) = \{\delta \in \operatorname{Gpd}(A) \mid \operatorname{div}(\delta) = 0\}$ of $\operatorname{Gpd}(A)$. Let $\delta \in \operatorname{Gpd}(A)$. By [49, Lemma 1.2 (2)], $\operatorname{div}(\delta) \in \Bbbk$. In particular, $\operatorname{div}(E) = a + b + c \in \Bbbk^\times$ for the Euler derivation E of A. Hence,

$$\delta' = \delta - \frac{\operatorname{div}(\delta)}{\operatorname{div}(E)} E \in \operatorname{Gpd}_0(A). \tag{5.2}$$

So $\operatorname{Gpd}(A) = \operatorname{Gpd}_0(A) \oplus \Bbbk E$. Thus, the formulas of $\operatorname{rgt}(A)$ can be derived from [49, Lemma 4.1], where the last equality follows from (2). ■

When A is generated in degree 1, $\operatorname{rgt}(A_\Omega)$ is computed for each Ω from Table A.1 in [49, Corollary 6.7], and it follows that any graded unimodular structure π_Ω on A is rigid (namely, $\operatorname{rgt}(A_\Omega) = 0$) if and only if Ω is irreducible. Moreover, one can easily obtain that $\operatorname{GK}(A_{\text{sing}}) = 0$ for irreducible Ω with isolated singularity, $\operatorname{GK}(A_{\text{sing}}) = 1$ for irreducible Ω without isolated singularity and $\operatorname{GK}(A_{\text{sing}})$ is 1 or 2 for reducible Ω. Now, we can generalize it to any weighted case.

Theorem 5.2. *Let A_Ω be a connected graded unimodular Poisson algebra given in Hypothesis 1.1. Then A_Ω is rigid if and only if the potential Ω is irreducible. In Table A.7, we will list all* $\operatorname{rgt}(A)$ *and* $\operatorname{GKdim}(A_{\text{sing}})$ *of these Ω from Tables A.2 to A.6.*

Proof. We apply Theorem 4.4 (also see Lemma 2.3) and Lemma 3.2. Indeed, our method is a case-by-case verification. First of all, when $a = b < c$ and $\Omega = h(x, y)$ is reducible, we have $\operatorname{rgt}(A) \neq 0$ according to Lemma 3.2. For the rest of the proof, we only provide the details for the following two cases for illustration: irreducible $\Omega_1 = z^2 + xy^3 + \lambda x^2y^2 + x^3y$ and reducible $\Omega_2 = x^2z + xy^3$ in Table A.2 (with $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$). Let ϕ be any graded Poisson derivation of degree 0. Replacing ϕ by $\phi + cE$ for some suitable scalar $c \in \Bbbk$ if needed, we can always assume $\operatorname{div}(\phi) = 0$ by (5.2). Thus, we can write $\phi(x) = \alpha_1 x + \alpha_2 y$, $\phi(y) = \alpha_3 x + \alpha_4 y$ and $\phi(z) = (-\alpha_1 - \alpha_4)z + \alpha_5 x^2 + \alpha_6 xy + \alpha_7 y^2$ for some $\alpha_i \in \Bbbk$. By Lemma 5.1 (2), we have $\phi(\Omega) = 0$ for $\Omega = \Omega_1$ or Ω_2. From $\phi(\Omega_1) = 0$, one can show that $\alpha_i = 0$ for $i = 1, \ldots, 7$, which implies $\phi = 0$. As a result, $\operatorname{rgt}(A_{\Omega_1}) = 0$. From $\phi(\Omega_2) = 0$, we obtain $\alpha_7 + 3\alpha_3 = 0$ and $\alpha_i = 0$ for any $i \neq 3, 7$. This implies that $\operatorname{Gpd}(A_{\Omega_2}) = \operatorname{span}\{E, \phi\}$, where $\phi(x) = 0$, $\phi(y) = x$, $\phi(z) = -3y^2$. Hence, we get $\operatorname{rgt}(A_{\Omega_2}) = -1$. Finally, a standard Gröbner basis computation yields all possible $\operatorname{GKdim}(A_{\text{sing}})$. We skip the details here. ■

5.2. Rigidity of gradings

In this and the following subsections, we use Poisson valuations to establish results about the rigidity of gradings and filtrations. We believe that these kinds of rigidities deserve more attention.

Theorem 5.3. *Let A_Ω be a connected graded Poisson algebra with Ω having an isolated singularity. Then P_Ω has a unique connected grading such that it is Poisson graded with nonzero degree* 1 *part.*

Proof. We only prove the result for $\Omega := \Omega_1$ in Lemma 4.10. The argument for Ω_2 and Ω_3 is analogous. By [24, Theorem 3.8], $Q(P_\Omega)$ only has two faithful 0-valuations, which are denoted by $\{\nu^{\pm \text{Id}}\}$ as before. They correspond to connected gradings on P_Ω with x, y, z in degree 1 according to (4.2). We assume $\mathbb{G}^{\pm \text{Id}}$ to be the 0-filtrations on $Q(P_\Omega)$ associated with any new grading and denote by $\mu^{\pm \text{Id}}$ the corresponding faithful 0-valuations. It is clear that $\mu^{-\text{Id}}(f) < 0$ for some $f \in P_\Omega$. So we have $\mu^{-\text{Id}} = \nu^{-\text{Id}}$ and $\mu(x) = -1, \mu(y) = -1, \mu(z) = -1$. So we can write $x = x_0 + f$, $y = y_0 + g$ and $z = z_0 + h$, where x_0, y_0, z_0 are homogeneous of new degree 1 and $f, g, h \in \Bbbk$. Since every non-trivial linear combination of x, y, z has $\mu^{-\text{Id}}$-value -1 (as $\mu^{-\text{Id}} = \nu^{-\text{Id}}$), x_0, y_0, z_0 are linearly independent. Since $\Omega = x^3 + y^3 + z^3 + \lambda xyz = 0$ is a sum of homogeneous relations, we get

$$(3f^2 + \lambda gh)x_0 + (3g^2 + \lambda fh)y_0 + (3h^2 + \lambda fg)z_0 = 0.$$

This implies that $3f^2 + \lambda gh = 3g^2 + \lambda fh = 3h^2 + \lambda fg = 0$, or equivalently (f, g, h) is a singular point of $\Omega = 0$. Since Ω has an isolated singularity at the origin, we get $f = g = h = 0$ and $x = x_0$, $y = y_0$, $z = z_0$ are homogeneous of degree 1 in this new grading. Since P_Ω is generated by x, y, z, the new grading agrees with the given grading. ∎

5.3. Rigidity of filtrations

Theorem 5.4. *Assume that Ω has an isolated singularity. Then $P_{\Omega-\xi}$, with $\xi \neq 0$, has a unique filtration $\mathbb{F}$ such that the associated graded ring $\mathrm{gr}_{\mathbb{F}}(P_{\Omega-\xi})$ is a connected graded Poisson domain with nonzero degree 1 part.*

Proof. Again, we only prove the result for $\Omega := \Omega_1$ in Lemma 4.10. The argument for Ω_2 and Ω_3 is analogous. The result follows from [24, Theorem 3.11] that $Q(P_{\Omega-\xi})$ has only one faithful 0-valuation $\nu^{-\mathrm{Id}}$ and Proposition 2.6. ∎

By Theorem 5.2, we have the following:

$$\Omega \text{ being irreducible} \Leftrightarrow \mathrm{rgt}(A_\Omega) = 0. \tag{5.3}$$

Now Theorems 5.3 and 5.4 can be summarized as

$$\begin{aligned} P_\Omega \text{ having a unique grading} \Leftarrow\ & \Omega \text{ having isolated singularity} \\ & \Rightarrow P_{\Omega-1} \text{ having a unique filtration.} \end{aligned} \tag{5.4}$$

There is another diagram for balanced irreducible potentials Ω; see (6.11).

6. K_1-sealedness and uPH^2-vacancy

In this section, we introduce two technical concepts – K_1-sealedness (Definition 6.2 (2)) and uPH^2-vacancy (Definition 6.8 (3)). Together with H-ozoneness (Definition 2.1 (3)), they will play an important role in computing Poisson cohomology in the next section. Note that the uPH^2-vacancy of A_Ω is independent of the choices of the graded generators (x, y, z); however, the K_1-sealedness of Ω may be dependent on the choices of the graded generators (x, y, z). In this section, we assume that n is not necessarily equal to $a + b + c$.

6.1. K_1-sealedness

Let Ω be a homogeneous element of degree $n > 0$ in the weighted polynomial ring $A := \Bbbk[x, y, z]$ with $(\deg(x), \deg(y), \deg(z)) = (a, b, c) \in \mathbb{N}_+^3$. Let us first recall that the *Koszul complex* $K_\bullet(\overrightarrow{\nabla}\Omega)$ given by the sequence $\overrightarrow{\nabla}\Omega := (\Omega_x, \Omega_y, \Omega_z)$ in A is

$$0 \to A[-3n+(a+b+c)] \xrightarrow{\overrightarrow{\nabla}\Omega} \begin{matrix} A[-2n+b+c] \\ \oplus A[-2n+a+c] \\ \oplus A[-2n+a+b] \end{matrix} \xrightarrow{\overrightarrow{\nabla}\Omega\times} \begin{matrix} A[a-n] \\ \oplus A[b-n] \\ \oplus A[c-n] \end{matrix}$$

$$\xrightarrow{\overrightarrow{\nabla}\Omega\cdot} A \to A/(\Omega_x, \Omega_y, \Omega_z) \to 0. \tag{6.1}$$

Note that (6.1) is a complex of graded vector spaces, where the differentials are graded maps of degree 0. A 1-cycle in $\ker(\overrightarrow{\nabla}\Omega\cdot)$ is called *sealed* if $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ when further considered as an element in A_{sing}. Let $s_1(\Omega)$ be the subspace of $\ker(\overrightarrow{\nabla}\Omega\cdot)$ consisting of all sealed 1-cycles in the above complex. The following follows from commutative algebra.

Lemma 6.1. *Retain the above notation.*

(1) *If* $\gcd(\Omega_x, \Omega_y, \Omega_z) = 1$, *then the Koszul complex* (6.1) *is exact everywhere except possibly for the position at* $K_1(\overrightarrow{\nabla}\Omega)$.

(2) *If* GKdim $A_{\text{sing}} \le 1$, *then* $\gcd(\Omega_x, \Omega_y, \Omega_z) = 1$.

(3) *If* Ω *is irreducible (and weighted homogeneous), then* $\gcd(\Omega_x, \Omega_y, \Omega_z) = 1$.

(4) $\operatorname{im}(\overrightarrow{\nabla}\Omega\times) \subseteq s_1(\Omega) \subseteq \ker(\overrightarrow{\nabla}\Omega\cdot)$.

Proof. (1) It follows from [40, Remarks 3.6 and 3.7].

(2) and (3) These are clear.

(4) For any $\overrightarrow{g} \in A^{\oplus 3}$, by a computation, $\overrightarrow{\nabla} \cdot (\overrightarrow{\nabla}\Omega \times \overrightarrow{g}) = 0$ in A_{sing}. The second inclusion follows from the definition. ∎

Definition 6.2. Let $\Omega \in A$ be a potential, and we consider the Koszul complex (6.1).

(1) The *sealed first Koszul homology* of (A, Ω) is defined to be

$$sK_1(A, \Omega) := s_1(\Omega)/\operatorname{im}(\overrightarrow{\nabla}\Omega\times).$$

(2) Ω is said K_1-*sealed* if $sK_1(A, \Omega) = 0$. That is, for any $\overrightarrow{f} \in A^{\oplus 3}$, if $\overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega = 0$ in A and $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ when considered as an element in A_{sing}, then $\overrightarrow{f} = \overrightarrow{\nabla}\Omega \times \overrightarrow{g}$ for some $\overrightarrow{g} \in A^{\oplus 3}$.

The property of being K_1-sealed was implicitly used by Luo in her thesis [33]. It involves computing the Poisson homology using the first homology of the corresponding Koszul complex in certain exceptional cases. It is unclear if the K_1-sealedness of Ω depends on the choice of graded generators (x, y, z). We assume a fixed set of (x, y, z) when discussing K_1-sealedness.

By definition, the K_1-sealedness for Ω can be reflected via the homology of the Koszul complex $K_\bullet(\overrightarrow{\nabla}\Omega)$ in the following way: for any 1-cycle $\overrightarrow{f} \in Z_1(K_\bullet(\overrightarrow{\nabla}\Omega))$, if $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ in A_{sing}, then $\overrightarrow{f}$ belongs to the 1-boundary, namely $\overrightarrow{f} = 0$ in $H_1(K_\bullet(\overrightarrow{\nabla}\Omega))$. If Ω has an isolated singularity at the origin, then $H_1(K_\bullet(\overrightarrow{\nabla}\Omega)) = 0$ [40, Proposition 3.5]. Hence, such an Ω is always K_1-sealed.

In the rest of this subsection, we will show that K_1-sealedness holds for some other families of Ω that do not have isolated singularity.

Lemma 6.3. *Let $\Omega = xyz + g(x, y)$ for some $g(x, y) \in \Bbbk[x, y]$ satisfying the following conditions:*

(1) *$g(x, y)$ is homogeneous with respect to some new grading $\deg_{\text{new}}(x) = a'$ and $\deg_{\text{new}}(y) = b'$ for $a', b' \geq 3$.*

(2) *$g(x, y)$ contains both terms $x^{b'}$ and $y^{a'}$.*

(3) *$xy \mid g_x g_y$.*

Then Ω is K_1-sealed. In this case, by choosing such a new grading together with $\deg_{\text{new}}(z) = a'b' - a' - b' =: c'$, we have $h_{H_1(K_\bullet)}(t) = \frac{t^{c'+a'b'}}{1-t^{c'}}$.

For example, $\Omega = xyz + x^{b'} + y^{a'}$ with $a', b' \geq 3$ is K_1-sealed. In applications/examples in the next section, we have $(a, b, c) = (\frac{a'}{g}, \frac{b'}{g}, \frac{c'}{g})$ for $g = \gcd(a', b', c')$.

Proof of Lemma 6.3. We assign a new grading, denoted by $\deg_{\text{new}}$, on A by choosing $\deg_{\text{new}}(x) := a'$, $\deg_{\text{new}}(y) := b'$ and $\deg_{\text{new}}(z) := a'b' - a' - b'$. It is obvious that Ω becomes a homogeneous potential of degree $n' := a'b'$ under this new grading. It is easy to check that GKdim $A_{\text{sing}} = 1$ and a $\Bbbk$-basis of A_{sing} will be explicitly constructed later on. By Lemma 6.1, the Koszul complex $K_\bullet(\Omega_x, \Omega_y, \Omega_z)$ given in (6.1) is exact everywhere except at K_1. We claim that $H_1(K_\bullet)$ is spanned by the images of the elements

$$\overrightarrow{\varphi_l} := z^l\left(xz, -g_x, \frac{g_x g_y}{xy} - z^2\right)$$

in $(A[a' - n'] \oplus A[b' - n'] \oplus A[c' - n'])_{c'(l+1)+n'}$ for all $l \geq 0$. So $\overrightarrow{\varphi_l} \cdot \overrightarrow{\nabla}\Omega = 0$.

Suppose we have $\overrightarrow{\varphi_l} = \overrightarrow{f} \times \overrightarrow{\nabla}\Omega$ for some

$$\overrightarrow{f} = (f_1, f_2, f_3) \in (A[-a' - n'] \oplus A[-b' - n'] \oplus A[-c' - n'])_{c'(l+1)+n'}.$$

Then the third component of $\overrightarrow{\varphi_l} = \overrightarrow{f} \times \overrightarrow{\nabla}\Omega$ is equal to

$$f_1(xz + g_y) - f_2(yz + g_x) = \frac{g_x g_y}{xy} z^l - z^{l+2}.$$

However, this is impossible since the monomial z^{l+2} cannot appear on the left side. Since each homogeneous element $\overrightarrow{\varphi_l}$ has a different degree for distinct l, their images in $H_1(K_\bullet)$ must be linearly independent. To prove our claim, it suffices to match the Hilbert series of $H_1(K_\bullet)$ with $\frac{t^{c'+n'}}{1-t^{c'}}$, which is the one associated with the $\Bbbk$-subspace spanned by $\{\overrightarrow{\varphi_l} \mid l \geq 0\}$. By our assumption (2), we can apply the diamond lemma to obtain a $\Bbbk$-basis of $A_{\text{sing}} = A/(\Omega_x, \Omega_y, \Omega_y) = \Bbbk[x, y, z]/(xy, xz + g_y, yz + g_x)$ given by

$$\{z^i \mid i \geq 0\} \cup \{x, \dots, x^{b'-1}\} \cup \{y, \dots, y^{a'-1}\}. \tag{6.2}$$

Thus, A_{sing} has a Hilbert series given below

$$h_{A_{\text{sing}}}(t) = \frac{1}{1-t^{c'}} + \frac{t^{a'} - t^{a'b'}}{1-t^{a'}} + \frac{t^{b'} - t^{b'a'}}{1-t^{b'}}.$$

A calculation for the Hilbert series of the terms in (6.1) yields the following:

$$h_{H_1(K_\bullet)}(t) = h_A(t)(t^{a'+b'}-1)(t^{b'+c'}-1)(t^{a'+c'}-1) + h_{A_{\text{sing}}}(t) = \frac{t^{c'+n'}}{1-t^{c'}}.$$

Therefore, we have proved the claim.

Next, we show that Ω is K_1-sealed. Taking an arbitrary homogeneous element $\overrightarrow{f} \in Z_1(K_\bullet(\overrightarrow{\nabla}\Omega))$ satisfying $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ in A_{sing}, up to a boundary, we can take $\overrightarrow{f} = \lambda\overrightarrow{\varphi_l}$ for some $\lambda \in \Bbbk$. One can check that

$$\overrightarrow{\nabla} \cdot \overrightarrow{f} = \lambda(\overrightarrow{\nabla} \cdot \overrightarrow{\varphi_l}) = \lambda\Big(l\frac{g_x g_y}{xy}z^{l-1} - g_{xy}z^l - (l+1)z^{l+1}\Big) = 0 \quad \text{in } A_{\text{sing}}.$$

By our assumption (2), one can check directly $y \nmid g_x$ and $x \nmid g_y$. So our assumption (3) implies $x \mid g_x$ and $y \mid g_y$. Thus, there is a surjection

$$\pi : A \twoheadrightarrow A_{\text{sing}} = A/(\Omega_x, \Omega_y, \Omega_z) \twoheadrightarrow A/(x,y) \cong \Bbbk[t]$$

via $\pi(x) = \pi(y) = 0$ and $\pi(z) = t$. So we have

$$\begin{aligned}\pi(\overrightarrow{\nabla} \cdot \overrightarrow{f}) &= \lambda\left(l\pi\left(\frac{g_x}{x}\right)\pi\left(\frac{g_y}{y}\right)\pi(z)^{l-1} - \pi(g_{xy})\pi(z)^l - (l+1)\pi(z)^{l+1}\right)\\ &= \lambda l\pi\left(\frac{g_x}{x}\right)\pi\left(\frac{g_y}{y}\right)t^{l-1} - \lambda\pi(g_{xy})t^l - \lambda(l+1)t^{l+1}\end{aligned}$$

which cannot be zero in $\Bbbk[t]$ unless $\lambda = 0$. Hence, $\overrightarrow{f} \in B_1(K_\bullet(\overrightarrow{\nabla}\Omega))$, and consequently, $sK_1(A,\Omega) = 0$. Therefore, Ω is K_1-sealed. ∎

Another type of K_1-sealed Ω is given by $\Omega = z^n + g(x,y)$ for some $g(x,y)$ is in $\Bbbk[x,y]$. We need the following definition.

Definition 6.4. A polynomial $g \in \Bbbk[x,y]$ is called *special* if for any polynomial $f \in \Bbbk[x,y]$, $d \mid \left(\frac{g_y}{d}f\right)_x - \left(\frac{g_x}{d}f\right)_y$ implies that $d \mid f$, where $d = \gcd(g_x, g_y)$.

Lemma 6.5. *The following $g \in \Bbbk[x,y]$ are special:*

(1) $g = x^k y^l$ *with* $k, l \geq 1$ *and* $\gcd(k,l) = 1$.

(2) $g = x^k y^l + x^m$ *with* $l \geq 1$ *and* $m \geq k \geq 1$.

(3) $g = x^k y^l + y^m$ *with* $k \geq 1$ *and* $m \geq l \geq 1$.

Proof. (1) We have $d = \gcd(g_x, g_y) = x^{k-1}y^{l-1}$. Suppose there is some $f \in \Bbbk[x,y]$ such that $x^{k-1}y^{l-1} \mid \left(\frac{g_y}{d}f\right)_x - \left(\frac{g_x}{d}f\right)_y = (lxf)_x - (kyf)_y = (l-k)f + lxf_x - kyf_y$.

Without loss of generality, we can take $f = x^m y^n$ to be a monomial. Thus, $x^{k-1}y^{l-1} \mid (l(m+1)-k(n+1))f$. If to the contrary, $x^{k-1}y^{l-1} \nmid f$, then we must have $m < k-1$ or $n < l-1$ and $l(m+1) = k(n+1)$. The latter implies that $l \mid n+1$ and $k \mid m+1$ since $\gcd(k,l) = 1$. This yields a contradiction.

(2) Since $g_x = x^{k-1}(ky^l + mx^{m-k})$ and $g_y = lx^k y^{l-1}$. So $d = \gcd(g_x, g_y) = x^{k-1}$. Suppose there is some $f \in \Bbbk[x,y]$ such that

$$
\begin{aligned}
x^{k-1} \mid \Big(\frac{g_y}{d} f\Big)_x - \Big(\frac{g_x}{d} f\Big)_y &= (lxy^{l-1}f)_x - ((ky^l + mx^{m-k})f)_y \\
&= (1-k)ly^{l-1}f + ly^{l-1}xf_x - (ky^l + mx^{m-k})f_y.
\end{aligned}
$$

We consider f a polynomial in x with coefficients in $\Bbbk[y]$ and denote its lowest term as hx^n for some $0 \neq h \in \Bbbk[y]$. Then the possible lowest term in $\big(\frac{g_y}{d} f\big)_x - \big(\frac{g_x}{d} f\big)_y$ is x^n with its coefficient given by

$$
-l(k-1-n)y^{l-1}h - k(y^l + \delta_{m,k})h_y.
$$

If $n \geq k-1$, then $d \mid f$. Suppose $n < k-1$. Then it implies that the above coefficient is zero, and so we get $\frac{h_y}{h} = -\frac{k-1-n}{k} ly^{l-1}/(y^l + \delta_{m,k})$. Thus, $h = \lambda/(y^l + \delta_{m,k})^{\frac{(k-1-n)}{k}}$ for some nonzero scalar λ, which is not a polynomial. This gives a contradiction.

(3) The proof is similar to (2). ■

Lemma 6.6. *Let $\Omega = z^n + g(x,y)$ where $n \geq 2$ and $g(x,y) \in \Bbbk[x,y]$. Then Ω is K_1-sealed if g is special.*

Proof. Let $\overrightarrow{f} = (f_1, f_2, f_3) \in A^{\oplus 3}$ satisfying $\overrightarrow{\nabla}\Omega \cdot \overrightarrow{f} = 0$ in A and $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ in A_{sing}. Write $f_i = \sum_{j=0}^{n-1} h_{ij} z^j$, where $h_{ij} \in \Bbbk[x,y]$ for $0 \leq j \leq n-2$ and $h_{i(n-1)} \in \Bbbk[x,y,z]$ for $i = 1, 2$. Thus,

$$
\overrightarrow{\nabla}\Omega \cdot \overrightarrow{f} = \sum_{i=0}^{n-2} (h_{1i}g_x + h_{2i}g_y)z^i + (h_{1(n-1)}g_x + h_{2(n-1)}g_y + nf_3)z^{n-1} = 0.
$$

By a direct computation, we get $h_{1i}g_x + h_{2i}g_y = 0$ in $\Bbbk[x,y]$ for $0 \leq i \leq n-2$ and $f_3 = \frac{-(h_{1(n-1)}g_x + h_{2(n-1)}g_y)}{n}$. Set $d = \gcd(g_x, g_y)$, we can further write $h_{1i} = (\frac{g_y}{d})l_i$ and $h_{2i} = -(\frac{g_x}{d})l_i$ with $l_i \in \Bbbk[x,y]$ for all $0 \leq i \leq n-2$. Therefore,

$$
\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{f} &= \overrightarrow{\nabla} \cdot \bigg(\sum_{i=0}^{n-2} l_i \Big(\frac{g_y}{d}, -\frac{g_x}{d}, 0\Big) z^i + h_{1(n-1)} \Big(z^{n-1}, 0, -\frac{g_x}{n}\Big) \\
&\qquad + h_{2(n-1)} \Big(0, z^{n-1}, -\frac{g_y}{n}\Big) \bigg) \\
&= \overrightarrow{\nabla} \cdot \bigg(\sum_{i=0}^{n-2} l_i \Big(\frac{g_y}{d}, -\frac{g_x}{d}, 0\Big) z^i + \overrightarrow{\nabla}\Omega \times \Big(\frac{h_{2(n-1)}}{n}, -\frac{h_{1(n-1)}}{n}, 0\Big) \bigg)
\end{aligned}
$$

$$= \sum_{i=0}^{n-2}\Big(\Big(\frac{g_y}{d}l_i\Big)_x - \Big(\frac{g_x}{d}l_i\Big)_y\Big)z^i - \overrightarrow{\nabla}\Omega \cdot \Big(\overrightarrow{\nabla} \times \Big(\frac{h_{2(n-1)}}{n}, -\frac{h_{1(n-1)}}{n}, 0\Big)\Big)$$
$$= \sum_{i=0}^{n-2}\Big(\Big(\frac{g_y}{d}l_i\Big)_x - \Big(\frac{g_x}{d}l_i\Big)_y\Big)z^i \quad \text{in } A_{\text{sing}}$$
$$= 0 \quad \text{in } A_{\text{sing}}.$$

Note that $A_{\text{sing}} = \Bbbk[x, y, z]/(g_x, g_y, z^{n-1}) = \bigoplus_{i=0}^{n-2}(\Bbbk[x, y]/(g_x, g_y))z^i$. This implies that $\big(\frac{g_y}{d}l_i\big)_x = \big(\frac{g_x}{d}l_i\big)_y$ in $\Bbbk[x, y]/(g_x, g_y)$, and hence $d \mid \big(\frac{g_y}{d}l_i\big)_x - \big(\frac{g_x}{d}l_i\big)_y$ for all $0 \le i \le n-2$. Since g is special, by definition, we can write $l_i = dm_i$ for $m_i \in \Bbbk[x, y]$ for all $0 \le i \le n-2$, and hence

$$\overrightarrow{f} = \sum_{i=0}^{n-2} l_i\Big(\frac{g_y}{d}, -\frac{g_x}{d}, 0\Big)z^i + \overrightarrow{\nabla}\Omega \times \Big(\frac{h_{2(n-1)}}{n}, -\frac{h_{1(n-1)}}{n}, 0\Big)$$
$$= \sum_{i=0}^{n-2}(g_y, -g_x, 0)m_i z^i + \overrightarrow{\nabla}\Omega \times \Big(\frac{h_{2(n-1)}}{n}, -\frac{h_{1(n-1)}}{n}, 0\Big)$$
$$= \overrightarrow{\nabla}\Omega \times \Big(\frac{h_{2(n-1)}}{n}, -\frac{h_{1(n-1)}}{n}, \sum_{i=0}^{n-2} m_i z^i\Big).$$

This proves our result. ∎

6.2. uPH^2-vacancy

In this subsection, we introduce another new concept: uPH^2-vacancy for A_Ω and we shall show that if Ω is K_1-sealed, then its corresponding Poisson algebra A_Ω is uPH^2-vacant. By definition,

$$\ker(d^2_{\pi_\Omega}) = \{\delta \in \mathfrak{X}^2(A) \mid [\delta, \pi_\Omega]_S = 0\},$$

where $d^\bullet_{\pi_\Omega}$ is the differential in the cochain complex $(\mathfrak{X}^\bullet(A_\Omega), d^\bullet_{\pi_\Omega})$ (2.4). Before stating the definitions, we need a lemma.

Definition-Lemma 6.7. *Retain the above notation. Let*

$$M^2(A) := \{f\pi_\Omega + \pi_g \mid f, g \in A\}$$

which is a subspace of $\mathfrak{X}^2(A)$. *Then* $\operatorname{im}(d^1_{\pi_\Omega}) \subseteq M^2(A) \subseteq \ker(d^2_{\pi_\Omega})$.

Proof. The first inclusion follows from (5.1). That is, $d^1_\pi(\delta) = \operatorname{div}(\delta)\pi_\Omega - \pi_{\delta(\Omega)}$ for any $\delta \in \mathfrak{X}^1(A)$. For the second one, it suffices to show that $u := [f\pi_\Omega, \pi_\Omega]_S = 0$ and $v := [\pi_g, \pi_\Omega]_S = 0$ for all $f, g \in A_\Omega$. Note that both u and v are in $\mathfrak{X}^3(A)$. Hence, it remains to show that $u(x, y, z) = 0$ and $v(x, y, z) = 0$. They follow from the definition of the Schouten bracket and (4.1) via a straightforward computation. We omit the details. ∎

Definition 6.8. Let A_Ω be the unimodular Poisson algebra defined in Notation 4.2.

(1) The *upper division of the second Poisson cohomology* of A_Ω is defined to be

$$\mathrm{uPH}^2(A_\Omega) := \ker(d^2_{\pi_\Omega})/M^2(A).$$

(2) The *lower division of the second Poisson cohomology* of A_Ω is defined to be

$$\mathrm{lPH}^2(A_\Omega) := M^2(A)/\mathrm{im}(d^1_{\pi_\Omega}).$$

(3) We say A_Ω is uPH^2-*vacant* if $\mathrm{uPH}^2(A_\Omega) = 0$, or equivalently $\mathrm{lPH}^2(A_\Omega) = \mathrm{PH}^2(A_\Omega)$.

According to Definition 4.1, the definition of $\mathrm{uPH}^2(A_\Omega)$ (and $\mathrm{lPH}^2(A_\Omega)$ as well as uPH^2-vacancy) is invariant under the choice of graded generators (x, y, z). Consequently, uPH^2-vacancy is independent of the choice of graded generators (x, y, z). According to the following lemma, $\mathrm{lPH}^2(A_\Omega)$ is typically nonzero.

Lemma 6.9. *Retain the above notation and assume* (1), (2) *and* (3) *of Hypothesis* 1.1. *If* $h_{\mathrm{PH}^0(A)}(t) = h_{\mathrm{PH}^1(A)}(t) = \frac{1}{1-t^n}$, *then the Hilbert series of* $\mathrm{lPH}^2(A_\Omega)$ *is given by*

$$h_{\mathrm{lPH}^2(A)}(t) = \frac{1}{t^{a+b+c}}\left(\frac{(1-t^{n-a})(1-t^{n-b})(1-t^{n-c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\right).$$

Proof. We claim that the following sequence of graded vector spaces

$$0 \to \Bbbk[\Omega][a+b+c] \xrightarrow{\alpha} A[-n+a+b+c] \oplus A[a+b+c] \xrightarrow{\beta} M^2(A) \to 0 \quad (6.3)$$

is exact, where $\alpha(g) = (\frac{-dg}{d\Omega}, g)$ and $\beta(f, g) = f\pi_\Omega + \pi_g$. It is clear that α, β are graded maps, α is injective and β is surjective, and $\beta\alpha = 0$. Hence, it remains to show that $\ker(\beta) = \mathrm{im}(\alpha)$. Suppose $\beta(f, g) = f\pi_\Omega + \pi_g = 0$. One can check that

$$d^0_\pi(g) = [\pi_\Omega, g]_S = -[\pi_g, \Omega]_S = [f\pi_\Omega, \Omega]_S = 0.$$

By the assumption, we have $Z_P(A) = \Bbbk[\Omega]$. Therefore, we have $g \in Z_P(A) = \Bbbk[\Omega]$ and $f = \frac{-dg}{d\Omega}$. Note that we have the following exact sequences of graded vector spaces:

$$0 \to \mathrm{PH}^0(A) \to \mathfrak{X}^0(A) \to \mathrm{im}(d^0_\pi) \to 0, \quad (6.4)$$

$$0 \to \mathrm{im}(d^0_\pi) \to \mathrm{Pd}(A)[w] \to \mathrm{PH}^1(A)[w] \to 0, \quad (6.5)$$

$$0 \to \mathrm{Pd}(A)[w] \to \mathfrak{X}^1(A)[w] \to M^2(A)[2w] \to \mathrm{lPH}^2(A)[2w] \to 0. \quad (6.6)$$

One can deduce that

$$h_{\mathrm{lPH}^2(A)}(t) = h_{M^2(A)}(t) - h_{\mathfrak{X}^1(A)}(t)t^w + h_{\mathfrak{X}^0(A)}(t)t^{2w} + h_{\mathrm{PH}^1(A)}(t)t^w - h_{\mathrm{PH}^0(A)}(t)t^{2w}. \quad (6.7)$$

The assertion follows from (6.3) via a direct computation. ∎

We also need the *de Rham complex* for A:

$$0 \to \Bbbk \to \Omega_A^0 \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \xrightarrow{d} \Omega_A^3 \to 0. \tag{6.8}$$

We will utilize the following natural isomorphisms of graded vector spaces

$$\begin{cases} \Omega_A^0 \xrightarrow{\sim} A, & \\ \Omega_A^1 \xrightarrow{\sim} A[-a] \oplus A[-b] \oplus A[-c] & f dg \mapsto f \overrightarrow{\nabla} g, \\ \Omega_A^2 \xrightarrow{\sim} A[-b-c] \oplus A[-a-c] \oplus A[-a-b] & f dg \wedge dh \mapsto f \overrightarrow{\nabla} g \times \overrightarrow{\nabla} h, \\ \Omega_A^3 \xrightarrow{\sim} A[-a-b-c] & f dx \wedge dy \wedge dz \mapsto f \end{cases} \tag{6.9}$$

to (6.8) for $f, g, h \in A$. As a result, we have the following complex of graded vector spaces:

$$0 \to \Bbbk \to A \xrightarrow{\overrightarrow{\nabla}} \begin{matrix} A[-a] \\ \oplus A[-b] \\ \oplus A[-c] \end{matrix} \xrightarrow{\overrightarrow{\nabla} \times} \begin{matrix} A[-b-c] \\ \oplus A[-a-c] \\ \oplus A[-a-b] \end{matrix} \xrightarrow{\overrightarrow{\nabla} \cdot} A[-a-b-c] \to 0. \tag{6.10}$$

In the sequel, we will use the well-known fact that the de Rham complex (6.10) for A is always exact.

The following lemma shows that A_Ω is uPH^2-vacant if and only if it is H-ozone. It is important to note that the uPH^2-vacancy of A_Ω plays a significant role in computing the Hilbert series of Poisson cohomology groups. In this lemma, we will give two more equivalent descriptions in terms of a combination of the Koszul complex for the sequence $\overrightarrow{\nabla}\Omega = (\Omega_x, \Omega_y, \Omega_z)$ and the de Rham complex for A.

Lemma 6.10. *Let A be A_Ω satisfying* (1)*,* (2) *and* (3) *of Hypothesis* 1.1*, where Ω is a nonconstant homogeneous potential. Then the following are equivalent:*

(1) *A is H-ozone.*
(2) *For any $\overrightarrow{f} \in A^{\oplus 3}$, $\overrightarrow{\nabla} \cdot \overrightarrow{f} = \overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega = 0$ implies that $\overrightarrow{f} = \overrightarrow{\nabla} g \times \overrightarrow{\nabla}\Omega$ for some $g \in A$.*
(3) *For any $\overrightarrow{f} \in A^{\oplus 3}$, if $(\overrightarrow{\nabla} \times \overrightarrow{f}) \cdot \overrightarrow{\nabla}\Omega = 0$, then $\overrightarrow{f} = \overrightarrow{\nabla} m + g\overrightarrow{\nabla}\Omega$ for some $m, g \in A$.*
(4) *A is* uPH^2*-vacant.*

Proof. (1) $\Leftrightarrow$ (2) Assume (2) holds and suppose δ is an ozone derivation of A, that is, $\delta(\Omega) = 0$. Write $\overrightarrow{\delta} = (\delta(x), \delta(y), \delta(z)) \in A^{\oplus 3}$. So $0 = \delta(\Omega) = \overrightarrow{\delta} \cdot \overrightarrow{\nabla}\Omega$. By (5.1), we have $\overrightarrow{\nabla} \cdot \overrightarrow{\delta} = \operatorname{div}(\delta) = 0$ as $\delta(\Omega) = 0$. So (2) implies that $\overrightarrow{\delta} = \overrightarrow{\nabla} g \times \overrightarrow{\nabla}\Omega$ for some $g \in A$, which is equivalent to $\delta = \{-, g\}$. So (1) holds.

Conversely, suppose there is some $\overrightarrow{f} \in A^{\oplus 3}$ such that $\overrightarrow{\nabla} \cdot \overrightarrow{f} = \overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega = 0$. Consider the derivation δ of A defined by $(\delta(x), \delta(y), \delta(z)) = \overrightarrow{f}$. The assumptions on $\overrightarrow{f}$

yield that $\delta(\Omega) = \mathrm{div}(\delta) = 0$. So Lemma 5.1 (2) implies that δ is a Poisson derivation. Note that in the proof of Lemma 1 of [37], it is shown that the Poisson center $Z_P(A)$ is algebraic over $\Bbbk[\Omega]$. Hence, δ vanishes on $Z_P(A)$, so it is ozone. So by (1), $\delta = \{-, g\}$ for some $g \in A$, that is, $\overrightarrow{f} = \overrightarrow{\nabla} g \times \overrightarrow{\nabla} \Omega$.

(2) $\Leftrightarrow$ (3) Assume (2) holds and suppose that $(\overrightarrow{\nabla} \times \overrightarrow{f}) \cdot \overrightarrow{\nabla}\Omega = 0$ for $\overrightarrow{f} \in A^{\oplus 3}$. Write $\overrightarrow{h} := \overrightarrow{\nabla} \times \overrightarrow{f} \in A^{\oplus 3}$. It is easy to see that $\overrightarrow{\nabla} \cdot \overrightarrow{h} = 0$ and $\overrightarrow{h} \cdot \overrightarrow{\nabla}\Omega = 0$. So $\overrightarrow{h} = \overrightarrow{\nabla} g \times \overrightarrow{\nabla}\Omega = \overrightarrow{\nabla} \times (g\overrightarrow{\nabla}\Omega)$ for some $g \in A$. This implies that $\overrightarrow{\nabla} \times (\overrightarrow{f} - g\overrightarrow{\nabla}\Omega) = 0$. By the exactness of (6.10), we can write $\overrightarrow{f} - g\overrightarrow{\nabla}\Omega = \overrightarrow{\nabla} m$ for some $m \in A$. So (3) holds.

Conversely, suppose $\overrightarrow{\nabla} \cdot \overrightarrow{f} = 0$ and $\overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega = 0$ for some $\overrightarrow{f} \in A^{\oplus 3}$. By the exactness of the de Rham complex for A (6.10), we can write $\overrightarrow{f} = \overrightarrow{\nabla} \times \overrightarrow{h}$ for some $\overrightarrow{h} \in A^{\oplus 3}$. Hence, $(\overrightarrow{\nabla} \times \overrightarrow{h}) \cdot \overrightarrow{\nabla}\Omega = 0$. So, by (3), we can write $\overrightarrow{h} = \overrightarrow{\nabla} m + g\overrightarrow{\nabla}\Omega$ for some $g, m \in A$. Thus, $\overrightarrow{f} = \overrightarrow{\nabla} \times (\overrightarrow{\nabla} m + g\overrightarrow{\nabla}\Omega) = \overrightarrow{\nabla} g \times \overrightarrow{\nabla}\Omega$ and (2) follows.

(3) $\Leftrightarrow$ (4) We use the identification $\mathfrak{X}^2(A) \xrightarrow{\sim} A^{\oplus 3}$ described in (2.5) via

$$\delta \mapsto \overrightarrow{\delta} = (\delta(y, z), \delta(z, x), \delta(x, y)),$$

which is different from the notation $\overrightarrow{\delta}$ used in the proof of part (1). One can check that π_g and $f\pi_\Omega$ correspond to $\overrightarrow{\nabla} g$ and $f\overrightarrow{\nabla}\Omega$, respectively. Moreover, by (2.9), we have $-[\delta, \pi_\Omega]_S = -(\overrightarrow{\nabla} \times \overrightarrow{\delta}) \cdot \overrightarrow{\nabla}\Omega$. Then (3) and (4) are equivalent by reinterpreting the conditions through the above identification. ∎

The result below links the K_1-sealness of a potential Ω to the H-ozoneness of its associated Poisson algebra, A_Ω.

Lemma 6.11. *Let Ω be a homogeneous polynomial of positive degree n. If Ω is K_1-sealed, then A_Ω is H-ozone.*

Proof. It suffices to show that A satisfies condition (3) in Lemma 6.10. Suppose that $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$ for some positive integers a, b, c. Note that Ω is homogeneous with a positive degree n, which is not necessarily equal to $a + b + c$. As a result, the Poisson bracket π_Ω on A is homogeneous of degree $n - a - b - c$. The following diagram combines the Koszul complex (6.1) and the de Rham complex (6.10).

$$\begin{array}{ccccc} & & (A[-a] \oplus A[-b] \oplus A[-c])[-n] & & \\ & & \Big\downarrow \scriptstyle{\overrightarrow{\nabla}\Omega\times} & & \\ A[-a] \oplus A[-b] \oplus A[-c] & \xrightarrow{\overrightarrow{\nabla}\times} & A[-b-c] \oplus A[-a-c] \oplus A[-a-b] & \xrightarrow{\overrightarrow{\nabla}\cdot} & A[-a-b-c] \\ & & \Big\downarrow \scriptstyle{\overrightarrow{\nabla}\Omega\cdot} & & \\ & & A[n-a-b-c]. & & \end{array}$$

Without loss of generality, assume that $\overrightarrow{f} \in (A[-a] \oplus A[-b] \oplus A[-c])_\ell$ for some $\ell \in \mathbb{Z}$ such that $(\overrightarrow{\nabla} \times \overrightarrow{f}) \cdot \overrightarrow{\nabla}\Omega = 0$. The condition $(\overrightarrow{\nabla} \times \overrightarrow{f}) \cdot \overrightarrow{\nabla}\Omega = 0$ implies that $\overrightarrow{\nabla} \times \overrightarrow{f} \in Z_1(K_\bullet(\overrightarrow{\nabla}\Omega))$. So we can write

$$\overrightarrow{\nabla} \times \overrightarrow{f} = \overrightarrow{f_1} \times \overrightarrow{\nabla}\Omega + \overrightarrow{h}$$

for some $\overrightarrow{f_1} \in (A[-a] \oplus A[-b] \oplus A[-c])_{\ell-n}$ and $\overrightarrow{h} \in Z_1(K_\bullet(\overrightarrow{\nabla}\Omega)) \setminus B_1(K_\bullet(\overrightarrow{\nabla}\Omega))$ or 0. If $\overrightarrow{h} \neq 0$, then

$$\begin{aligned}\overrightarrow{\nabla} \cdot \overrightarrow{h} &= \overrightarrow{\nabla} \cdot (\overrightarrow{\nabla} \times \overrightarrow{f} - \overrightarrow{f_1} \times \overrightarrow{\nabla}\Omega)\\ &= -\overrightarrow{\nabla} \cdot (\overrightarrow{f_1} \times \overrightarrow{\nabla}\Omega)\\ &= -(\overrightarrow{\nabla} \times \overrightarrow{f_1}) \cdot \overrightarrow{\nabla}\Omega\\ &= 0 \quad \text{in } A_{\text{sing}}.\end{aligned}$$

The last equality follows from a direct computation. Since Ω is K_1-sealed, $\overrightarrow{h} \in B_1(K_\bullet(\overrightarrow{\nabla}\Omega))$, yielding a contradiction. Thus, $\overrightarrow{h} = \overrightarrow{0}$, and consequently, $(\overrightarrow{\nabla} \times \overrightarrow{f_1}) \cdot \overrightarrow{\nabla}\Omega = 0$ by the above calculation. Repeating this procedure with initial setting $f_0 = f$, we get

$$\overrightarrow{\nabla} \times \overrightarrow{f_{m-1}} = \overrightarrow{f_m} \times \overrightarrow{\nabla}\Omega \quad \text{and} \quad (\overrightarrow{\nabla} \times \overrightarrow{f_m}) \cdot \overrightarrow{\nabla}\Omega = 0$$

for some $\overrightarrow{f_m} \in (A[-a] \oplus A[-b] \oplus A[-c])_{\ell-mn}$ for $m \geq 1$. Since A is connected graded, $\overrightarrow{f_p} = 0$ for some $p \gg 0$, which implies that $\overrightarrow{\nabla} \times \overrightarrow{f_{p-1}} = 0$. By the exactness of the de Rham complex for A, we have $\overrightarrow{f_{p-1}} = \overrightarrow{\nabla} G_{p-1}$ for some $G_{p-1} \in A$. To complete the proof, it suffices to prove that, if we can write $\overrightarrow{f_q} = \overrightarrow{\nabla} G_q + H_q \overrightarrow{\nabla}\Omega$ for some $G_q, H_q \in A$, then so can we do the same for $\overrightarrow{f_{q-1}}$. An easy calculation yields that

$$\begin{aligned}\overrightarrow{\nabla} \times \overrightarrow{f_{q-1}} &= \overrightarrow{f_q} \times \overrightarrow{\nabla}\Omega\\ &= (\overrightarrow{\nabla} G_q + H_q \overrightarrow{\nabla}\Omega) \times \overrightarrow{\nabla}\Omega\\ &= \overrightarrow{\nabla} G_q \times \overrightarrow{\nabla}\Omega\\ &= \overrightarrow{\nabla} \times (G_q \overrightarrow{\nabla}\Omega).\end{aligned}$$

So $\overrightarrow{\nabla} \times (\overrightarrow{f_{q-1}} - G_q \overrightarrow{\nabla}\Omega) = 0$. Due to the exactness of the de Rham complex for A, we can write $\overrightarrow{f_{q-1}} - G_q \overrightarrow{\nabla}\Omega = \overrightarrow{\nabla} G_{q-1}$ for some $G_{q-1} \in A$. Now, our result is deduced by a downward induction. ■

By the above two lemmas (and Theorem 1.6), we shall have

$$\begin{aligned}\Omega \text{ is } K_1\text{-sealed} \Rightarrow A_\Omega \text{ is uPH}^2\text{-vacant} &\Leftrightarrow A_\Omega \text{ is } H\text{-ozone}\\ &\Leftrightarrow \Omega \text{ is irreducible and balanced.}\end{aligned} \tag{6.11}$$

However, it is not clear if "Ω is K_1-sealed $\Leftarrow A_\Omega$ is uPH^2-vacant". So, we ask the following question.

Question 6.12. Is K_1-sealing condition equivalent to H-ozone for Ω with $|\Omega| = |x| + |y| + |z|$?

7. Poisson cohomology of weighted Poisson algebras

In this last section, we investigate Poisson cohomology for a graded unimodular Poisson algebra A in dimension three assuming Hypothesis 1.1 unless mentioned otherwise. We aim to expand upon the results of van den Bergh [50] and Pichereau [40] to encompass a wider range of Poisson algebras. First, we show that A_Ω is H-ozone for all balanced irreducible Ω. When Ω has an isolated singularity, Ω is K_1-sealed and A_Ω is thus H-ozone. By Theorem 4.4 and Lemma 2.3, we can use the classification of Ω in Theorem 3.5 to further divide the rest of the irreducible ones (without isolated singularity) into the following four cases:

(1) Under a new grading, the associated graded Poisson algebra of A_Ω is $A_{\overline{\Omega}}$ for some $\overline{\Omega} = z^n + x^k y^l$ with $n \geq 2$ and $\gcd(k,l) = 1$ up to a permutation of x, y, z; that is, Ω is one of these irreducible potentials not included in Cases (2)–(4) below.

(2) $\Omega = xyz + x^{b'} + y^{a'}$, where $a', b' \geq 3$.

(3) Ω is non-balanced and irreducible; that is, $\Omega = z^2 + y^3$, or $z^2 + x^{2+\frac{2b}{a}}$, or $x^{\frac{b}{\gcd(a,b)}} + y^{\frac{a}{\gcd(a,b)}}$.

(4) $\Omega = z^2 + y^3 + x^2y^2$ or $z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$.

7.1. Case (1)

We first verify that the algebra A_Ω is H-ozone in the case, where $\Omega = z^n + x^k y^l$ with $n \geq 2$ and $\gcd(k,l) = 1$. For any general Ω containing $z^n + x^k y^l$, we construct a w-filtration on A_Ω and consider the associated w-graded Poisson algebra instead. Applying a spectral sequence argument, we can establish that A_Ω is still H-ozone for any such general Ω.

Lemma 7.1. *Let $\Omega = z^n + x^k y^l$ for some positive integers n, k, l such that $n \geq 2$ and* $\gcd(k,l) = 1$. *Then A_Ω is H-ozone.*

Proof. It follows from Lemmas 6.5 (1) and 6.6 that Ω is K_1-sealed. Then the assertion follows from Lemma 6.11. ∎

Now suppose Ω is as in Case (1). We will consider some Poisson w-filtration on A_Ω, whose associated w-graded Poisson algebra is defined by a potential of the form $z^n + x^k y^l$.

For the rest of this subsection, it is more convenient to use the filtration $\mathbb{F} = \{F_i \mid i \in \mathbb{Z}\}$ on A consisting of an increasing chain of $\Bbbk$-subspaces $F_i \subseteq F_{i+1}$ (which is different from the one given in Definition 2.5). Accordingly, we have to modify other parts of Definition 2.5. In particular, we will change w to $-w$ when we use a Poisson w-filtration.

Lemma 7.2. *Let A be a connected graded polynomial Poisson algebra and $\mathbb{F}$ be a w-filtration of A with w being not necessarily zero. Suppose the following hold:*

(1) *The Poisson center $Z_P(A) = \Bbbk[\chi]$ with $\deg(\chi) > 0$.*

(2) *The Euler derivation E of A preserves the w-filtration $\mathbb{F}$.*

(3) *The associated graded algebra $\mathrm{gr}_{\mathbb{F}} A$ is a connected w-graded polynomial Poisson algebra.*

(4) *The Poisson center $Z_P(\mathrm{gr}_{\mathbb{F}} A)$ is $\Bbbk[\overline{\chi}]$ with $s := \deg_{\mathrm{new}}(\overline{\chi}) > 0$ here $\deg_{\mathrm{new}}$ is the new grading associated to the filtration $\mathbb{F}$.*

(5) *$\mathrm{gr}_{\mathbb{F}} A$ has no nonzero Poisson derivation of degree $-s$.*

(6) *$\mathrm{gr}_{\mathbb{F}} A$ is H-ozone.*

Then A is both H-ozone and PH^1-minimal.

Proof. Let E be the Euler derivation of A. By (2), we have the induced graded Poisson derivation, denoted by E^{ind}, on $\mathrm{gr}_{\mathbb{F}} A$. So we can write $E^{\mathrm{ind}}(\overline{\chi}) = m\overline{\chi}$, where $m = \deg(\chi)$ is under the original grading of A. Denote $Z = Z_P(\mathrm{gr}_{\mathbb{F}} A)$. We claim that $\mathrm{PH}^1(\mathrm{gr}_{\mathbb{F}} A) \cong ZE^{\mathrm{ind}}$ by following the argument in [49, Lemma 7.7]. Note that the induced Poisson bracket on $\mathrm{gr}_{\mathbb{F}} A$ is homogeneous of degree w. So, it suffices to consider all homogeneous Poisson derivations. Say ϕ is one of degree i. One can check that $\phi(\overline{\chi}) \in Z$. By (5), we get $\phi(\overline{\chi}) = a\overline{\chi}^n$ for some $a \in \Bbbk$ and $n > 0$ (and further $\phi(\overline{\chi}) = 0$ if $s \nmid i$). Write $\phi' = \phi - \frac{a}{m}\overline{\chi}^{n-1}E^{\mathrm{ind}}$. Then $\phi'(\overline{\chi}) = 0$, whence ϕ' is ozone. Now (6) implies that $\phi = \frac{a}{m}\overline{\chi}^{n-1}E^{\mathrm{ind}} + \phi'$ for some Hamiltonian derivation ϕ'. This means that $\mathrm{Pd}(\mathrm{gr}_{\mathbb{F}} A) = ZE^{\mathrm{ind}} + \mathrm{Hd}(\mathrm{gr}_{\mathbb{F}} A)$. It remains to show that $ZE^{\mathrm{ind}} \cap \mathrm{Hd}(\mathrm{gr}_{\mathbb{F}} A) = 0$. Let $\phi = fE^{\mathrm{ind}}$ be Hamiltonian for some $f \in Z$. So $\phi(\overline{\chi}) = fE^{\mathrm{ind}}(\overline{\chi}) = mf\overline{\chi} = 0$. Hence, $f = 0$ and $\phi = 0$ since Z is an integral domain. This proves our claim.

Next, we use the w-filtration $\mathbb{F} = \{F_i A \mid i \in \mathbb{Z}\}$ of A to filter the cochain complex $(\mathfrak{X}^\bullet(A), d_\pi^\bullet)$ and compute $\mathrm{PH}^1(A)$ by spectral sequence. For each $p, i \in \mathbb{Z}$ with $i \geq 0$, we define a $\Bbbk$-subspace $F_p\mathfrak{X}^i(A)$ of $\mathfrak{X}^i(A)$

$$F_p\mathfrak{X}^i(A) = \{f \in \mathfrak{X}^i(A) \mid f(a_1, \ldots, a_i) \in F_{l_1+\cdots+l_i-p+iw}A \quad \text{for any } a_j \in F_{l_j}A,\ 1 \leq j \leq i\}.$$

Applying the differential formula (2.1), for any $f \in F_p\mathfrak{X}^i(A)$ we have

$$\begin{aligned} d_\pi^i(f)(a_0, \ldots, a_i) &= \sum_{j=0}^{i} (-1)^j \{a_j, f(a_0, \ldots, \widehat{a_j}, \ldots, a_i)\} \\ &\quad + \sum_{0\leq j<k\leq i} (-1)^{j+k} f(\{a_j, a_k\}, a_0, \ldots, \widehat{a_j}, \ldots, \widehat{a_k}, \ldots, a_i) \\ &\in F_{l_0+\cdots+l_i-p+(i+1)w}A, \end{aligned}$$

where $a_j \in F_{l_j}A$ for $0 \leq j \leq i$. So $d_\pi^i : F_p\mathfrak{X}^i(A) \to F_p\mathfrak{X}^{i+1}(A)$. Then $\{F_p\mathfrak{X}^\bullet(A) \mid p \in \mathbb{Z}\}$ is a (decreasing) filtration on the cochain complex $(\mathfrak{X}^\bullet(A), d_\pi^\bullet)$, which is exhaustive and bounded below since $\mathrm{gr}_{\mathbb{F}} A$ is connected graded. Thus, according to [54, Section 5.4], we have a cohomology spectral sequence with

$$E_0^{p,q} = F_p\mathfrak{X}^{p+q}(A)/F_{p+1}\mathfrak{X}^{p+q} \cong \mathfrak{X}^{p+q}(\mathrm{gr}_{\mathbb{F}} A)_{-p+(p+q)w}$$

and

$$E_1^{p,q} = \mathrm{PH}^{p+q}(\mathrm{gr}_{\mathbb{F}} A)_{-p+(p+q)w} \Rightarrow \mathrm{PH}^{p+q}(A).$$

By (4), we know $\mathrm{PH}^0(\mathrm{gr}_{\mathbb{F}} A) = \Bbbk[\overline{\chi}]$. Since every cocycle $(\overline{\chi})^n$ in E_1-page can be lifted to a Poisson central element χ^n in $\mathrm{PH}^0(A)$, the elements in $Z = \Bbbk[\overline{\chi}]$ are all permanent cocycles and survive to E_∞-page. As a consequence, the differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are all zero whenever $p+q=0$. So $E_1^{p,q} = E_\infty^{p,q}$ when $p+q=0$. By our previous claim, we know $\mathrm{PH}^1(\mathrm{gr}_{\mathbb{F}} A) = ZE^{\mathrm{ind}}$. Similarly, every cocycle $\overline{\chi}^n E^{\mathrm{ind}}$ can be lifted to a Poisson derivation $\chi^n E$ in $\mathrm{PH}^1(A)$. Hence, $E_1^{p,q} = E_\infty^{p,q}$ when $p+q=1$. This implies that $\mathrm{PH}^1(A) = \Bbbk[\chi]E$ and hence A is PH^1-minimal. Finally, A is H-ozone by [49, Lemma 7.5]. ∎

By combining Lemmas 7.1 and 7.2, we obtain the following consequence.

Proposition 7.3. *If Ω is of the form in Case (1), then A_Ω is H-ozone.*

Proof. We first show that any such A_Ω satisfies the assumptions in Lemma 7.2, and then the result follows. We use $\Omega = z^2 + x^2y^2 + x^4y$ as an illustration. Note that the original grading on A is given by $\deg(x) = 1, \deg(y) = 2, \deg(z) = 3$. Now we set $\deg_{\mathrm{new}}(x) = 3, \deg_{\mathrm{new}}(y) = 2, \deg_{\mathrm{new}}(z) = 7$ and consider the following algebra filtration $\mathbb{F} = \{F_i \mid i \in \mathbb{N}\}$, where F_i are spanned by all monomials $x^j y^k z^l$ satisfying $3j + 2k + 7l \le i$. By [24, Lemma 2.9], it is easy to check that $\{F_i, F_j\} \subseteq F_{i+j+2}$ for all possible $i, j \in \mathbb{N}$. So $\mathbb{F}$ is a w-filtration for the Poisson algebra A_Ω with $w = 2$. It is clear that $\mathrm{gr}_{\mathbb{F}} A = \Bbbk[\overline{x}, \overline{y}, \overline{z}]$ with the new grading $\deg_{\mathrm{new}}(\overline{x}) = 3, \deg_{\mathrm{new}}(\overline{y}) = 2, \deg_{\mathrm{new}}(\overline{z}) = 7$. One can further verify that $\mathrm{gr}_{\mathbb{F}} A$ is still unimodular with a homogeneous potential $\overline{\Omega} = \overline{z}^2 + \overline{x}^4\overline{y}$. By Lemma 7.1, $(\mathrm{gr}_{\mathbb{F}} A)_{\overline{\Omega}}$ is H-ozone. Moreover, it is routine to check that $(\mathrm{gr}_{\mathbb{F}} A)_{\overline{\Omega}}$ satisfies all the requirements in Lemma 7.2. So A_Ω is H-ozone. ∎

7.2. Case (2)

Proposition 7.4. *If Ω is of the form in Case (2), then A_Ω is H-ozone.*

Proof. If Ω is of the form $xyz + x^{b'} + y^{a'}$, by Lemma 6.3, Ω is K_1-sealed. The assertion follows from Lemma 6.11. ∎

7.3. Case (3)

In this subsection, we show that A_Ω is not H-ozone for each non-balanced irreducible Ω.

Lemma 7.5. *Let Ω be non-balanced and irreducible. Then A is not H-ozone.*

Proof. By a choice of (x, y, z), we may assume that $\Omega = h(x, y) = \sum_{ai+bj=n} \alpha_{ij} x^i y^j$, where $n = a + b + c$. Since Ω is irreducible, α_{0k} and α_{l0} are nonzero, where $kb = n = al$. Since $c = n - a - b = kb - a - b$, the assumption $\gcd(a, b, c) = 1$ implies that $\gcd(a, b) = 1$. Then the equation $kb = al$ implies that $k = ag$ and $l = bg$, where $g = \gcd(k, l)$. If $\alpha_{ij} \neq 0$ for some (i, j), then $ai + bj = n = abg$. Hence, $a \mid j$ and $b \mid i$. This means that $h(x, y) = f(x^b, y^a)$ for some homogeneous polynomial $f(s, t)$. Since Ω is irreducible, so is $f(s, t)$. The only possibility is when $f(s, t)$ is linear, or equivalently, $h(x, y) = x^b + y^a$. Since $ab = n = a + b + c$, $a, b \geq 2$.

To show A being not H-ozone, it suffices to show that Ω does not satisfy condition (2) in Lemma 6.10. Considering $\overrightarrow{f} = (0, 0, 1) \in A^{\oplus 3}$. Hence, $\overrightarrow{\nabla} \cdot \overrightarrow{f} = \overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega = 0$. Suppose $\overrightarrow{f} = \overrightarrow{\nabla} g \times \overrightarrow{\nabla}\Omega$ for some $g \in A$. Write $\overrightarrow{\nabla} g = (g_1, g_2, g_3) \in A^{\oplus 3}$. Then we must have $1 = ag_1 y^{a-1} - bg_2 x^{b-1}$, which is impossible since $a, b \geq 2$. ■

7.4. Case (4)

Finally, we deal with the two exceptional potentials: $\Omega = z^2 + y^3 + x^2y^2$ or $z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$.

Proposition 7.6. *If Ω is of one of the above forms, then A_Ω is H-ozone.*

Proof. By Lemmas 6.5 (2) and (3) and 6.6, Ω is K_1-sealed. The assertion follows from Lemma 6.11. ■

Corollary 7.7. *Let Ω be an irreducible potential in the classification that is neither $x^k + y^l$ nor $x^k + z^l$ nor $y^k + z^l$. Then Ω is balanced.*

Proof. By Propositions 7.3, 7.4 and 7.6, for such an Ω, A_Ω is H-ozone. The assertion follows from Lemma 7.5. ■

7.5. Main results on Poisson cohomology

We compute the Hilbert series of the Poisson cohomology groups of A_Ω when it is H-ozone. Our result shows that their Hilbert series only depends on the grading of A_Ω.

Theorem 7.8. *Let $A = \Bbbk[x, y, z]$ be a connected graded algebra such that* $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$ *with unimodular Poisson structure given by some homogeneous polynomial Ω of degree $n > 0$. Here, n may not be necessarily equal to $a + b + c$. Suppose the following statements hold:*

(a) $Z_P(A) = \Bbbk[\Omega]$.

(b) *A is H-ozone.*

(c) *A has a degree 0 Poisson derivation that is not ozone.*

(d) *A has no nonzero Poisson derivation of degree $-n$.*

Then the Hilbert series of the Poisson cohomology groups of A is given as follows:

(1) $h_{\mathrm{PH}^0(A)}(t) = \frac{1}{1-t^n}$.

(2) $h_{\mathrm{PH}^1(A)}(t) = \frac{1}{1-t^n}$.

(3) $h_{\mathrm{PH}^2(A)}(t) = \frac{1}{t^{a+b+c}}\big(\frac{(1-t^{n-a})(1-t^{n-b})(1-t^{n-c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\big)$.

(4) $h_{\mathrm{PH}^3(A)}(t) = \frac{(1-t^{n-a})(1-t^{n-b})(1-t^{n-c})}{t^{a+b+c}(1-t^n)(1-t^a)(1-t^b)(1-t^c)}$.

Proof. Consider the cochain complex $(\mathfrak{X}^\bullet(A), d_\pi^\bullet)$ in (2.4) for computing $\mathrm{PH}^\bullet(A)$.

(1) By (a), it is clear that $h_{\mathrm{PH}^0(A)}(t) = h_{Z_P(A)}(t) = \frac{1}{1-t^n}$.

(2) By (c), we have a degree 0 Poisson derivation, denoted by δ, such that $\delta(\Omega) \neq 0$. Without loss of generality, we can assume $\delta(\Omega) = \Omega$ as δ has degree 0. By (d) and a similar argument of [49, Lemma 7.7] (also see the proof of Lemma 7.2), one can see that $\mathrm{PH}^1(A) \cong Z_P(A)\delta$. So $h_{\mathrm{PH}^1(A)}(t) = h_{Z_P(A)}(t) = \frac{1}{1-t^n}$.

(3) By (b) and Lemma 6.10, A is $\mathrm{uPH}^2(A)$-vacant, namely, $\mathrm{lPH}^2(A) = \mathrm{PH}^2(A)$. Then the result follows from Lemma 6.9 and parts (1) and (2).

(4) Notice $\ker(d_\pi^2) = M^2(A)$ since A is $\mathrm{uPH}^2(A)$-vacant. Considering (6.4) and (6.5) and the following exact sequence

$$0 \to \mathrm{Pd}(A)[w] \to \mathfrak{X}^1(A)[w] \to \ker(d_\pi^2)[2w] \to \mathrm{PH}^2(A)[2w] \to 0, \tag{7.1}$$

we obtain

$$\begin{aligned} h_{\mathrm{PH}^3(A)}(t) = h_{\ker(d_\pi^2)}(t)t^w - h_{\mathfrak{X}^1(A)}(t)t^{2w} + h_{\mathfrak{X}^0(A)}(t)t^{3w} \\ + \frac{1}{t^{a+b+c}}\frac{(1-t^{w+a})(1-t^{w+b})(1-t^{w+c})}{(1-t^a)(1-t^b)(1-t^c)} \end{aligned} \tag{7.2}$$

via (2.10). The rest can be deduced from (6.3) via another direct computation. ∎

As a consequence, we obtain the Hilbert series of Poisson cohomology for any connected graded unimodular Poisson algebra A_Ω for any balanced irreducible Ω (but not necessarily having isolated singularity).

Corollary 7.9. *Assume Hypothesis 1.1. If Ω is an irreducible potential in the classification that is neither $x^k + y^l$ nor $x^k + z^l$ nor $y^k + z^l$, then the Hilbert series of Poisson cohomology of A is given as follows:*

(1) $h_{\mathrm{PH}^0(A)(t)} = \frac{1}{1-t^n}$.

(2) $h_{\mathrm{PH}^1(A)}(t) = \frac{1}{1-t^n}$.

(3) $h_{\mathrm{PH}^2(A)}(t) = \frac{1}{t^n}\big(\frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\big)$.

(4) $h_{\mathrm{PH}^3(A)}(t) = \frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{t^n(1-t^n)(1-t^a)(1-t^b)(1-t^c)}$.

Proof. It suffices to check that A_Ω satisfies all the requirements in Theorem 7.8 when Ω is an irreducible potential mentioned above – (a), (c) and (d) are obvious. So it remains to show all such A_Ω are H-ozone. If Ω has isolated singularity, A_Ω is H-ozone. As discussed before Lemma 7.1, the rest of the irreducible Ω classified in Theorem 3.5 are divided into four classes. Note that Proposition 7.3 covers class (1), Proposition 7.4 covers class (2), Proposition 7.6 covers class (4) and Lemma 7.5 covers class (3). So our result follows immediately by letting $n = a + b + c$. ■

The Hilbert series of the Poisson cohomology only depends on the weights of x, y, z when Ω is balanced. Thus, these connected graded unimodular Poisson algebras exhibit the same homological behavior, making it impossible to distinguish irreducible potentials solely by using the Hilbert series. It would be interesting to see if additional structures in Poisson cohomology can distinguish different irreducible potentials. See the next question.

Question 7.10. Can we use $\Bbbk[\Omega]$-module structures on $\mathrm{PH}^\bullet(A)$ to distinguish between singular and smooth curves and identify types of singularity for $\Omega = 0$?

Finally, we generalize [49, Theorem 0.6] by removing the restriction that generators are in degree 1 and including more equivalent conditions involving the second Poisson cohomological group. The result is stated as Theorem 1.6.

Proof of Theorem 1.6. (1) $\Rightarrow$ (2): Since $\mathrm{rgt}(A) = 0$, every degree 0 semi-Poisson derivation δ for A is of the form αE for some $\alpha \in \Bbbk$, where E is the Euler derivation. Thus, $E \wedge \delta = 0$. By (2.3), $\pi_{\mathrm{new}} = \pi$. So $A = A^\delta$. The assertion follows. In particular, A is unimodular by [49, Corollary 0.3].

(2) $\Rightarrow$ (1): By [49, Corollary 0.3], there is a Poisson derivation δ of degree 0 such that A^δ is unimodular. Since $A^\delta \cong A$ for all δ, A is unimodular. Suppose, to the contrary that A is not rigid, that is $\mathrm{rgt}(A) \neq 0$. Then there is a Poisson derivation δ of degree 0 not in $\Bbbk E$. Thus, by [49, Theorem 0.2], the modular derivation of A^δ is

$$\mathbf{n} = 0 + n\delta - \mathrm{div}(\delta)E$$

which cannot be zero as $\mathrm{div}(\delta) \in \Bbbk$ [49, Lemma 1.2 (3)]. Therefore, A^δ is not isomorphic to A, yielding a contradiction.

(1) $\Leftrightarrow$ (8): In the proof of (1) $\Leftrightarrow$ (2), (1) implies that A is unimodular with a potential Ω. By Theorem 5.2, Ω is irreducible. According to Theorem 3.5 and Corollary 7.7, the non-balanced potentials are $\Omega = z^2 + y^3$, $z^2 + x^{2+\frac{2b}{a}}$ or $\Omega = f(x, y)$ in Lemma 3.2, up to graded isomorphisms of A. It is easy to check that these corresponding Poisson algebras have Poisson derivations of negative degree. For example, one can define a Poisson derivation δ such that $\delta(z) = \delta(y) = 0, \delta(x) = 1$ when $\Omega = z^2 + y^3$, and $\delta(z) = \delta(x) = 0, \delta(y) = 1$ when $\Omega = z^2 + x^{2+\frac{2b}{a}}$, and $\delta(x) = \delta(y) = 0, \delta(z) = 1$ when $\Omega = f(x, y)$. So we get (8). The other direction follows from the calculation of $\mathrm{PH}^1(A)$ in Corollary 7.9 and [49, Remark 5.2].

(3) $\Leftrightarrow$ (5): This follows from [49, Proposition 7.4].

(5) $\Leftrightarrow$ (6): Under the hypothesis (5), A is PH^1-minimal. One implication follows from [49, Lemma 7.5] and the other direction is clear.

(6) $\Rightarrow$ (7): See the proof of Lemma 7.5 of [49].

(7) $\Rightarrow$ (1): Note that A is unimodular in this case. From the classification of Ω in Theorem 3.5, it is easy to see that $Z_P(A) = \Bbbk[\chi]$ with $\deg(\chi) > 0$. Thus, $\mathrm{rgt}(A) = 0$ from [49, Lemma 7.7 (2)]. In particular, Ω is irreducible and $Z_P(A) = \Bbbk[\Omega]$. Moreover, suppose $\delta \in \mathrm{Pd}(A)_{<0}$. If $\delta(\Omega) = 0$, then $\delta = H_f$ for some $f \in A$. This is impossible since $\deg(f) = \deg(\delta) < 0$ and A is connected graded. If $\delta(\Omega) \in \Bbbk^\times$, then $\deg(\delta) = -n$. Then $\deg(\delta(x)) = -b - c$, $\deg(\delta(y)) = -a - c$, $\deg(\delta(z)) = -a - b$, which are all negative. Hence, $\delta = 0$. So $\mathrm{Pd}(A)_{<0} = 0$.

(8) $\Rightarrow$ (4), (5), (10), (11): This assertion follows from Corollary 7.9.

(4) $\Rightarrow$ (1): The assertion $\mathrm{rgt}(A) = 0$ follows from [49, Remark 5.2]. Moreover, since A is connected graded, each nonzero Hamiltonian derivation is of positive degree. As a result, we get $\mathrm{Pd}(A)_{<0} = \mathrm{PH}^1(A)_{<0} = 0$.

(5) $\Leftrightarrow$ (9): It follows from (2.10) that $h_{\mathrm{PH}^3(A)}(t) - h_{\mathrm{PH}^2(A)}(t) = h_{\mathrm{PH}^0(A)}(t) - h_{\mathrm{PH}^1(A)}(t) + t^{-n}$. We know that $\mathrm{PH}^0(A) = Z_P(A)$. So, the assertion follows from the fact that $h_{\mathrm{PH}^1(A)}(t) = h_{\mathrm{PH}^0(A)}(t) = h_Z(t)$ if and only if $h_{\mathrm{PH}^3(A)}(t) - h_{\mathrm{PH}^2(A)}(t) = t^{-n}$.

(10) $\Rightarrow$ (7): Note that the subspace $M^2(A) = \{f\pi_\Omega + \pi_g \mid f, g \in A\}$ of $\mathfrak{X}^2(A)$ lies in $\ker(d_\pi^2)$ (Definition-Lemma 6.7). For any two locally finite graded vector spaces M and N, we use the notation $h_M(t) \geq h_N(t)$ to mean $\dim M_i \geq \dim N_i$ for each $i \in \mathbb{Z}$. By (6.3)–(6.5) and (7.1) with $w = 0$, we get

$$\begin{aligned}
h_{\mathrm{PH}^2(A)}(t) &= h_{\ker(d_\pi^2)}(t) - h_{\mathfrak{X}^1(A)}(t) + h_{\mathfrak{X}^0(A)}(t) + (h_{\mathrm{PH}^1(A)}(t) - h_{\mathrm{PH}^0(A)}(t)) \\
&\geq h_{M^2(A)}(t) - h_{\mathfrak{X}^1(A)}(t) + h_{\mathfrak{X}^0(A)}(t) + (h_{Z_P(A)E}(t) - h_{\mathrm{PH}^0(A)}(t)) \\
&\geq h_{M^2(A)}(t) - h_{\mathfrak{X}^1(A)}(t) + h_{\mathfrak{X}^0(A)}(t) \\
&= \frac{1}{t^n}\Big(\frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\Big),
\end{aligned}$$

where the last equality follows from the exact sequence (6.3) together with a direct computation. By the assumption, one can obtain $M^2(A) = \ker(d_\pi^2)$ and so A is uPH^2-vacant. This is equivalent to A being H-ozone by Lemma 6.10.

(11) $\Rightarrow$ (7): The argument is similar to the proof of (10) $\Rightarrow$ (7) by using (7.2).

(7) $\Leftrightarrow$ (12): It follows from Lemma 6.10. ■

A. Classification of weight polynomials Ω of degree $|x| + |y| + |z|$ in $\Bbbk[x, y, z]$

The classification of Ω when $\deg(x) = \deg(y) = \deg(z) = 1$ is well known; see [6, 11, 12, 25, 29] and the first table below.

Irreducible Ω	**Reducible Ω**
$x^3 + y^2z$ ⓦ, $x^3 + x^2z + y^2z$ ⓠ	x^3, x^2y, xyz
$x^3 + y^3 + z^3 + 3\lambda xyz$ ($\lambda^3 \neq -1$, ⓘ)	$xy(x + y), xyz + x^3, xy^2 + x^2z$

Table A.1. $(a, b, c) = (1, 1, 1)$.

Irreducible Ω	**Reducible Ω**
$z^2 + x^3y$ ⓦ, $z^2 + x^2y^2 + x^3y$ ⓠ	$x^4, x^3y, x^2y^2, x^2y^2 + x^3y$
$z^2 + xy^3 + \lambda x^2y^2 + x^3y$	$z^2 + x^2y^2, z^2, z^2 + x^4,$
($\lambda \neq \pm 2$, ⓘ), ($\lambda = \pm 2$, ⓠ)	$xy^3 + \lambda x^2y^2 + x^3y$
$x^2z + y^4$ ⓦ, $x^2z + xy^3 + y^4$ ⓦ	$x^2z, x^2z + xy^3$
$xyz + x^4 + y^4$ ⓠ	$xyz, xyz + x^4$

Table A.2. $(a, b, c) = (1, 1, 2)$.

$k \in \mathbb{N}$	**Irreducible Ω**	**Reducible Ω**
$c \neq ka$		xyz, x^2z
	$x^2z + xy^{k+1} + y^{k+2}$ ⓦ	$x^2z, x^2z + xy^{k+1}$
$c = ka$	$x^2z + y^{k+2}$ ⓦ	$xyz, xyz + x^{k+2}$
$k \neq 2$	$xyz + x^{k+2} + y^{k+2}$ ⓠ	$h(x, y)$ of degree $(k + 2)a$

Table A.3. $(a = b < c) \neq (1, 1, 2)$.

Reducible Ω
$xyz, xy^2, xz^2 + x^{1+\frac{2b}{a}}, xyz + x^{1+\frac{2b}{a}}, x^{1+\frac{b}{a}}y + xz^2, x^{1+\frac{2b}{a}}, x^{1+\frac{b}{a}}y$

Table A.4. $(a < b = c)$.

Irreducible Ω	**Reducible Ω**
$z^2 + y^3$ ⓦ	$z^2, y^3, x^6, y^3 + x^2y^2$
$z^2 + y^3 + x^2y^2$ ⓠ	$x^2y^2, x^2y^2 + x^4y$
$z^2 + y^3 + \lambda x^2y^2 + x^4y$	$y^3 + \lambda x^2y^2 + x^4y$
($\lambda \neq \pm 2$, ⓘ), ($\lambda = \pm 2$, ⓠ)	x^4y
$z^2 + x^4y$ ⓦ	$z^2 + x^6$
$z^2 + x^2y^2 + x^4y$ ⓠ	$z^2 + x^2y^2$
$x^3z + y^3 + x^2y^2$ ⓦ, $x^3z + y^3$ ⓦ	$x^3z, x^3z + x^2y^2$
$xyz + x^6 + y^3$ ⓠ	$xyz, xyz + x^6, xyz + y^3$

Table A.5. $(a, b, c) = (1, 2, 3)$.

$m, n \in \mathbb{N} \cup \{-1\}$	**Irreducible Ω**	**Reducible Ω**
$c \neq ma + nb$		$xyz, x^{1+\frac{b}{a}}z$
$c = a + b$	$z^2 + x^{2+\frac{2b}{a}}$ ⓦ	$z^2, z^2 + x^2y^2$
$a \nmid b$	$z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$ ⓠ	$xyz, xyz + x^{2+\frac{2b}{a}}$
		$h(x, y)$ of degree $(2a + 2b)$
		$z^2, z^2 + x^{2+\frac{2b}{a}}$
$c = a + b$	$z^2 + x^2y^2 + x^{2+\frac{b}{a}}y$ ⓠ	$z^2 + x^2y^2, x^{1+\frac{b}{a}}z$
$b \neq 2a, a \mid b$	$z^2 + x^{2+\frac{b}{a}}y$ ⓦ	$x^{1+\frac{b}{a}}z + x^2y^2, xyz$
		$xyz + x^{2+\frac{2b}{a}}$
		$h(x, y)$ of degree $(2a + 2b)$
$c = ma + nb$		$xyz + x^{m+1+k}$
$c \neq a + b, a \nmid b$	$xyz + x^{m+1+k} + y^{n+1+l}$ ⓠ	$xyz + y^{n+1+l}$
$\frac{m+1}{k} = \frac{l}{n+1}$	$x^{\frac{b}{\gcd(a,b)}} + y^{\frac{a}{\gcd(a,b)}}$ ⓦ	xyz
		$h(x, y)$ of degree $(m + 1)a + (n + 1)b$
$c = ma + nb$	$xyz + x^{m+1+k} + y^{n+1+l}$ ⓠ	$x^{1+\frac{b}{a}}z, xyz + x^{m+1+k}$
$c \neq a + b, a \mid b$	$x^{1+\frac{b}{a}}z + y^{1+n+l}$ ⓦ	$xyz, xyz + y^{n+1+l}$
$\frac{m+1}{k} = \frac{l}{n+1}$	$x^{1+\frac{b}{a}}z + x^{\frac{b}{a}}y^{n+l} + y^{1+n+l}$ ⓦ	$x^{1+\frac{b}{a}}z + x^{\frac{b}{a}}y^{n+l}$
		$h(x, y)$ of degree $(m + 1)a + (n + 1)b$

Table A.6. $(a < b < c) \neq (1, 2, 3)$.

Ω	rgt	GK	Ω	rgt	GK
$z^2 + x^3y$	0	1	$xz^2 + x^{1+\frac{2b}{a}}$	$-2(a\mid b)$ $-1(a\nmid b)$	2
$z^2 + x^2y^2 + x^3y$	0	1	$xyz + x^{1+\frac{2b}{a}}$	-1	1
$z^2 + xy^3 + \lambda x^2y^2 + x^3y$	0	$0(\lambda \neq \pm 2)$ $1(\lambda = \pm 2)$	$x^{1+\frac{b}{a}}y + xz^2$	-1	1
$x^2z + y^4$	0	1	$x^{1+\frac{2b}{a}}$	$-5(a\mid b)$ $-3(a\nmid b)$	2
$x^2z + xy^3 + y^4$	0	1	$x^{1+\frac{b}{a}}y(*)$	-3	2
$xyz + x^4 + y^4$	0	1	$x^2z + xy^{k+1} + y^{k+2}$	0	1
x^4	-5	2	$x^2z + y^{k+2}$	0	1
x^3y	-4	2	$xyz + x^{k+2} + y^{k+2}$	0	1
x^2y^2	-4	2	xyz	-2	1
$x^2y^2 + x^3y$	-3	2	x^2z	-2	2
$z^2 + x^2y^2$	-1	1	x^2z	-2	2
z^2	-3	2	$x^2z + xy^{k+1}$	-1	1
$z^2 + x^4$	-1	1	xyz	-2	1
$xy^3 + \lambda x^2y^2 + x^3y$	-3	1	$xyz + x^{k+2}$ $(*)$	-1	1
x^2z	-2	2	$z^2 + x^{2+\frac{2b}{a}}$	$0(a \nmid b)$	1
$x^2z + xy^3$	-1	1	$z^2 + x^2y^2 + x^{2+\frac{2b}{a}}$	0	1

Table A.7. $\mathrm{rgt} := \mathrm{rgt}(A)$ and $\mathrm{GK} := \mathrm{GKdim}(A_{\mathrm{sing}})$ for Ω in Tables A.2–A.6.

Ω	**rgt**	**GK**	**Ω**	**rgt**	**GK**
xyz	-2	1	$z^2+x^2y^2+x^{2+\frac{b}{a}}y$	0	1
$xyz+x^4$ $(*)$	-1	1	$z^2+x^{2+\frac{b}{a}}y$	0	1
z^2+y^3	0	1	$xyz+x^{m+1+k}+y^{n+1+l}(a\nmid b)$	0	1
$z^2+y^3+x^2y^2$	0	1	$xyz+x^{m+1+k}+y^{n+1+l}(a\mid b)$	0	1
$z^2+y^3+\lambda x^2y^2+x^4y$	0	$0(\lambda\neq\pm 2)$ $1(\lambda=\pm 2)$	$x^{1+\frac{b}{a}}z+y^{1+n+l}$	0	1
z^2+x^4y	0	1	$x^{1+\frac{b}{a}}z+x^{\frac{b}{a}}y^{n+l}+y^{1+n+l}$	0	1
$z^2+x^2y^2+x^4y$	0	1	xyz	-2	1
$x^3z+y^3+x^2y^2$	0	1	$x^{1+\frac{b}{a}}z$	-2	2
x^3z+y^3	0	1	z^2	$-1(a\nmid b)$	2
$xyz+x^6+y^3$	0	1	$z^2+x^2y^2$	-1	1
z^2	-2	2	xyz	-2	1
y^3	-3	2	$xyz+x^{2+\frac{2b}{a}}$	-1	1
x^6	-4	2	z^2	$-2(a\mid b)$	2
$y^3+x^2y^2$	-2	2	$z^2+x^{2+\frac{2b}{a}}$	$-1(a\mid b)$	1
x^2y^2	-3	2	$z^2+x^2y^2$	-1	1
$x^2y^2+x^4y$	-2	2	$x^{1+\frac{b}{a}}z$	-2	2
$y^3+\lambda x^2y^2+x^4y$	-2	$1(\lambda\neq\pm 2)$ $2(\lambda=\pm 2)$	$x^{1+\frac{b}{a}}z+x^2y^2$	-1	2
x^4y	-3	2	xyz	-2	1
z^2+x^6	-1	1	$xyz+x^{2+\frac{2b}{a}}$	-1	1
$z^2+x^2y^2$	-1	1	$xyz+x^{m+1+k}$	-1	1
x^3z	-2	2	$xyz+y^{n+1+l}$	-1	1
$x^3z+x^2y^2$	-1	2	xyz	-2	1
xyz	-2	1	$x^{1+\frac{b}{a}}z$	-2	2
$xyz+x^6$	-1	1	$xyz+x^{m+1+k}$	-1	1
$xyz+y^3$ $(*)$	-1	1	xyz	-2	1
xyz	-2	1	$xyz+y^{n+1+l}$	-1	1
xy^2	-2	2	$x^{1+\frac{b}{a}}z+x^{\frac{b}{a}}y^{n+l}$	-1	2
$x^{\frac{b}{\gcd(a,b)}}+y^{\frac{a}{\gcd(a,b)}}$	0	1	Reducible $h(x,y)$	≤ -1	$\{1,2\}$

Table A.7. *(Continued)*
The '$*$' indicates the last Ω from Tables A.2–A.6 accordingly.

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