J. Noncommut. Geom. 20 (2026), 953–993
DOI 10.4171/JNCG/661

Crystalline representations and p-adic Hodge theory for non-commutative algebraic varieties

Keiho Matsumoto

Abstract. In this paper, we study p-adic Hodge theory for non-commutative algebraic varieties. Firstly, we propose a conjecture that for K a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and $\mathcal{T}$ an $\mathcal{O}_K$-linear idempotent-complete, small smooth proper stable ∞-category, there exists a B_{crys}-coefficient isomorphism preserving additional structures between the $K(1)$-local K-theory on the generic fiber of $\mathcal{T}$ and the topological periodic cyclic homology on the special fiber. This conjecture can be regarded as a non-commutative analog of the crystalline comparison theorem. We then proceed to prove the following results: the topological negative cyclic homology $\pi_i \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ admits a Breuil–Kisin module structure, and the non-commutative analog of Bhatt–Morrow–Scholze's comparison theorems holds. Additionally, we demonstrate that the $\mathbb{Z}_p[G_K]$-module obtained from the topological negative cyclic homology is a $\mathbb{Z}_p$-lattice of a crystalline representation. Finally, we show that when the generic fiber of $\mathcal{T}$ admits a geometric realization in the sense of Orlov, the non-commutative analog of the crystalline comparison theorem proposed by the author holds.

1. Introduction

The aim of this paper is to study p-adic Hodge theory for non-commutative algebraic varieties. Specifically, we propose a conjecture and prove several results related to the relationship between $K(1)$-local K-theory and topological periodic homology in this non-commutative setting.

Notation 1.1. Fix a prime p. Let K be a complete discretely valued nonarchimedean extension K of $\mathbb{Q}_p$ with perfect residue field k. Here, $\mathcal{O}_K$ is the ring of integers of K and $\pi \in \mathcal{O}_K$ is a uniformizer. We write $\mathcal{C}$ to denote the completion $\widehat{\overline{K}}$ of $\overline{K}$ endowed with its unique absolute value extending the given absolute value on K, let W be the Witt ring of k, and let K_0 be the fraction field of W. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_K$. For a spectrum S, let $L_{K(1)}S$ be the Bousfield localization of complex K-theory at prime p.

Cohomology theories such as de Rham cohomology $R\Gamma_{\mathrm{dR}}(-/K)$, Hodge cohomology $R\Gamma_{\mathrm{Zar}}(-, \Omega^*_{-/K})$, l-adic cohomology $R\Gamma_{\text{ét}}(-, \mathbb{Z}_l)$ and crystalline cohomology

Mathematics Subject Classification 2020: 14F30 (primary); 14A22 (secondary).
Keywords: p-adic representation, non-commutative algebraic geometry.

$R\Gamma_{\mathrm{cry}}(-/W(k))$ are important tools in the study of algebraic geometry and arithmetic geometry. On the other hand, homological invariants such as (topological) periodic homology, (topological) cyclic homology and K-theory play an important role in the study of C^*-algebra, Lie algebra and non-commutative geometry. There are deep and subtle links between cohomology invariants and homological invariants. One of the most well-known examples is the Atiyah–Hirzebruch spectral sequence. For a finite-dimensional CW-complex M, Atiyah–Hirzebruch [4] prove that there is a spectral sequence

$$E_2^{i,j} = \begin{cases} H^i_{\mathrm{Sing}}(M,\mathbb{Z}), & j \text{ even} \\ 0, & j \text{ odd} \end{cases} \Longrightarrow K^{\mathrm{top}}_{j-i}(M). \tag{1.1}$$

The Thomason spectral sequence is an arithmetic-geometrical analog of the Atiyah–Hirzebruch spectral sequence. For a smooth variety X over a field of characteristic 0, Thomason [36] shows that there exists a spectral sequence

$$E_2^{i,j} = \begin{cases} H^i_{\text{ét}}(X,\mathbb{Z}_p(l)), & j = 2l \\ 0, & j \text{ odd} \end{cases} \Longrightarrow \pi_{j-i} L_{K(1)} K(X), \tag{1.2}$$

and this spectral sequence degenerates after tensoring $\mathbb{Q}_p$. Besides, Hesselholt [21] shows a close relation between p-adic cohomology theory and topological cyclic homology.

Bondal–Kapranov [10], Orlov [34] and Kontsevich [27] introduce *non-commutative algebraic geometry* in which a dg-category (or a stable ∞-category) is studied as a *non-commutative space*. Nowadays, non-commutative algebraic geometry plays an important role in research on mirror symmetry, mathematical physics and algebraic geometry. Besides, homological invariants are, in general, well defined for dg-categories and stable ∞-categories. It has been known that some comparison theorems between cohomology groups can be formulated naturally for dg-categories and stable ∞-categories: instead of cohomology theories, one can consider homological invariants. For a smooth proper dg-category $\mathcal{T}$ over $\mathbb{C}$, Kaledin [25] proves that there is an isomorphism $\mathrm{HP}_n(\mathcal{T}/\mathbb{C}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{HH}_{n+2i}(\mathcal{T}/\mathbb{C})$, which has been conjectured by Kontsevich–Soibelman [28] and can be regarded as a *non-commutative Hodge decomposition* via Connes [12], Feigin–Tsygan [18] and Hochschild–Kostant–Rosenberg [24]. Besides, Blanc [9] conjectured that there is an equivalence $\mathrm{Ch}^{\mathrm{top}} \wedge_{\mathbb{S}} H\mathbb{C} : K_{\mathrm{top}}(\mathcal{T}) \wedge_{\mathbb{S}} H\mathbb{C} \to \mathrm{HP}(\mathcal{T}/\mathbb{C})$, which can be regarded as the *non-commutative de Rham comparison theorem*, and in some cases, the equivalence is proved by A. A. Khan [26]. For a smooth proper stable ∞-category $\mathcal{T}$ over W, Scholze proves that there is an isomorphism of W-modules $\pi_n\,\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p) \simeq \pi_i\,\mathrm{HP}(\mathcal{T}/W)$ (it has been unpublished yet), and Petrov–Vologodsky obtain the same result for any stable ∞-category [35].

In the study of p-adic cohomology theories, crystalline comparison theory [16] states that for a smooth proper variety X over $\mathcal{O}_K$, the p-adic étale cohomology $H^i_{\text{ét}}(X_{\mathcal{C}},\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$ is isomorphic to $H^i_{\mathrm{cry}}(X_k/W) \otimes_W B_{\mathrm{crys}}$, and the isomorphism is compatible with G_K-action, Frobenius endomorphism and filtration. We study a non-commutative version of the crystalline comparison theorem. For a commutative ring R, we will refer to R-linear idempotent-complete, small stable ∞-categories simply as R-linear categories. For an $\mathcal{O}_K$-linear category $\mathcal{T}$, G_K acts continuously on $\mathbb{Z}_p$-module $\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})$, and there is a Frobenius operator $\mathrm{Fr} : \pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right] \overset{\mathrm{can}}{\simeq} \pi_i \operatorname{TC}^-(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right] \xrightarrow{\varphi} \pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right]$ (see [2]). Inspired by Petrov and Vologodsky's work on non-commutative crystalline cohomology theory [35], and the study of motivic filtration of the $K(1)$-local K-theory [36] and the topological periodic homology [8], we predict the following conjecture.

Conjecture 1.2 (Non-commutative version of crystalline comparison theorem [16, 37]). *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there is an isomorphism of B_{crys}-module*

$$\pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p) \otimes_W B_{\mathrm{crys}} \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}$$

which is compatible with G_K-action and Frobenius endomorphism. In particular, the p-adic representation $\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline.

Remark 1.3. Assume $\mathcal{O}_K = W$. Then the isomorphism of W-modules

$$\pi_n \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p) \xrightarrow{\sim} \pi_n \operatorname{HP}(\mathcal{T}/W)$$

has been proved independently by Scholze (unpublished) and by Petrov–Vologodsky [35] for $p > 2$. For all primes p and for smooth proper dg-categories, this result has since been established by Devalapurkar–Raksit [13] and Mao [32]. Consequently, the non-commutative Hodge–de Rham filtration on $\pi_n \operatorname{HP}(\mathcal{T}/W)$ induces a natural filtration on $\operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)$. It is further expected that for a general complete DVR $\mathcal{O}_K$ of mixed characteristic, the base change

$$\pi_n \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p) \otimes_W K$$

inherits a filtration arising from the non-commutative Hodge–de Rham filtration on $\pi_n \operatorname{HP}(\mathcal{T}/\mathcal{O}_K)$. Moreover, we expect the isomorphism in Conjecture 1.2 to be compatible with these filtrations.

For a smooth proper variety X over $\mathcal{O}_K$, by [36] one has a G_K-equivariant isomorphism $\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \bigoplus_{n \in \mathbb{Z}} H^{i+2n}_{\text{ét}}(X_{\mathcal{C}}, \mathbb{Z}_p(n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and similarly by [7] one has an isomorphism $\pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right] \simeq \bigoplus_{n \in \mathbb{Z}} H^{i+2n}_{\mathrm{cry}}\left(X_k / W\left[\frac{1}{p}\right]\right)(n)$ of isocrystals, thus the conjecture holds for $\mathcal{T} = \operatorname{perf}(X)$ via crystalline comparison theorem [37]. In this paper, we approach this conjecture via K-theoretical version of Bhatt–Morrow–Scholze's comparison theorem [8]. We will use the language of stable ∞-categories, following Lurie [31].

Definition 1.4. For an $\mathbb{E}_\infty$-ring R, we let $\mathrm{Cat}^{\mathrm{perf}}_\infty(R)$ denote the ∞-category of R-linear categories, where the morphisms are exact functors.

Definition 1.5. For an $\mathbb{E}_\infty$-ring R, we let $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(R)$ denote the ∞-category of smooth proper R-linear categories, where the morphisms are exact functors.

Definition 1.6 (See also [34, Section 4.1]). For an $\mathbb{E}_\infty$-ring R, an R-linear category $\mathcal{T}$ *admits a geometric realization* if there is a derived scheme X over R such that the truncation $\pi_0(X)$ is separated scheme of finite type over $\pi_0(R)$ and there is a fully faithful admissible inclusion $\mathcal{T} \subset \mathrm{perf}(X)$ of R-linear categories (see [1, Definition 3.1]).

In many cases, a stable ∞-category is known to admit a geometric realization. For example, the derived Fukaya category of a symplectic manifold is known or expected to admit a geometric realization from the study of mirror symmetry. Besides, Lunts–Bergh–Schnürer [5] proved that the stable infinity category of perfect complexes on a smooth proper Deligne–Mumford admits a geometric realization.

Remark 1.7. Assume R is an algebraically closed field of characteristic 0. In [34, Question 4.4], Orlov asked if there exist R-linear idempotent-complete, small smooth proper stable ∞-categories which do not admit a geometric realization. This is still an important open problem.

Remark 1.8. If a smooth proper R-linear category $\mathcal{T}$ admits a geometric realization $\mathcal{T} \hookrightarrow \mathrm{perf}(X)$, the dual $\mathcal{T}^{\mathrm{op}} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}$ also admits a geometric realization $\mathcal{T}^{\mathrm{op}} \hookrightarrow \mathrm{perf}(X)^{\mathrm{op}} = \mathrm{perf}(X)$.

First, we fix a sequence of elements $\zeta_n \in \overline{K}$ inductively such that $\zeta_0 = 1$ and $(\zeta_{n+1})^p = \zeta_n$. Given this sequence, let $\varepsilon = (\zeta_0, \zeta_1, \zeta_2, \dots) \in \varprojlim_{\mathrm{Frob}} \mathcal{O}_{\mathbb{C}}/p$ be the corresponding element in the tilt, and let $[\varepsilon] \in \mathbb{A}_{\mathrm{inf}}$ be its Teichmüller lift. Let $\mathfrak{S} = W[\![z]\!]$, and let $\widetilde{\theta} : \mathfrak{S} \to \mathcal{O}_K$ be the usual map whose kernel is generated by Eisenstein polynomial E of π. Let $\phi : \mathfrak{S} \to A_{\mathrm{inf}}$ be the W-linear map that sends z to $[\pi^\flat]$. We write $\xi = \phi(E)$. We write $\mu = [\varepsilon] - 1$. Let $\varphi : \mathfrak{S} \to \mathfrak{S}$ be a Frobenius endomorphism which is Frobenius on W and sends z to z^p. We prove a non-commutative version of Bhatt–Morrow–Scholze's Breuil–Kisin cohomology theory $R\Gamma_{\mathfrak{S}}(-)$ [8].

Theorem 1.9 (Theorem 2.17, non-commutative version of [8]). *Let $\mathcal{T}$ be an $\mathcal{O}_K$-linear smooth proper category. Then there is a natural number n such that the following holds:*

(1) *For any $i \geq n$, $\pi_i\, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ has a natural structure of a Breuil–Kisin module.*

(1′) *For any $i \geq n$, $\pi_i\, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee$ has a natural structure of a Breuil–Kisin module of finite E-height.*

(2) *($K(1)$-local K-theory comparison) Assume $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization. For any $i \geq n$, after scalar extension along $\overline{\phi} : \mathfrak{S} \to A_{\mathrm{inf}}$ which sends z to $[\pi^\flat]^p$ and is the Frobenius on W, one recovers $K(1)$-local K-theory of the generic fiber*

$$\pi_i\, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathfrak{S},\overline{\phi}} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]^\wedge_p \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]^\wedge_p.$$

(3) *(Topological periodic homology theory comparison) For any $i \geq n$, after scalar extension along the map $\widetilde{\phi} : \mathfrak{S} \to W$ which is the Frobenius on W and sends z to 0, one recovers topological periodic homology theory of the special fiber*

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\widetilde{\phi}} K_0 \simeq \pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right].$$

Bhatt–Morrow–Scholze's Breuil–Kisin cohomology theory [8, Theorem 1.2] implies the crystalline comparison theorem. On the other hand, Theorem 1.9 does not imply Conjecture 1.2. This difference arises as follows. On Breuil–Kisin cohomology theory, there is a G_K-equivariant isomorphism (see [7, Theorem 1.8 (iii)])

$$R\Gamma_{A_{\mathrm{inf}}}(X_{\mathcal{O}_{\mathcal{C}}}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cry}} \simeq R\Gamma_{\mathrm{cry}}(X_{\mathcal{O}_{\mathcal{C}}/p}/A_{\mathrm{cry}}). \tag{1.3}$$

By combining (1.3) with [7, Theorem 1.8 (iv)], we obtain a canonical (G_K, φ)-equivariant isomorphism

$$R\Gamma_{\mathrm{\acute{e}t}}(X_{\mathcal{C}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{cry}}\left[\frac{1}{p\mu}\right] \simeq R\Gamma_{\mathrm{cry}}(X_{\mathcal{O}_{\mathcal{C}}/p}/A_{\mathrm{cry}})\left[\frac{1}{p\mu}\right]. \tag{1.4}$$

The isomorphism induces the crystalline comparison theorem (see [7, Theorem 14.4]). On the other hand, the non-commutative analog of (1.3) becomes the following G_K-equivariant isomorphism:

$$\pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p) \otimes_{A_{\mathrm{inf}}} \widehat{A_{\mathrm{cry}}} \simeq \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}/p}; \mathbb{Z}_p), \tag{1.5}$$

where $\widehat{A_{\mathrm{cry}}}$ is the completion of A_{cry} with respect to the Nygaard filtration (see [8, Definition 8.9]). The problem is that μ is a 0-divisor in $\widehat{A_{\mathrm{cry}}}$ (see [6, Corollaries 2.11 and 2.12]). Thus the non-commutative analog of (1.4) becomes the trivial equation. Therefore, we will study $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ in more detail. In Section 3, we will show that the dual Breuil–Kisin module $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee$ admits a Breuil–Kisin G_K-module structure in the sense of Gao [19]. In Section 4, using Du–Liu's work on $(\varphi, \widehat{G})$-module [14], we will prove the following.

Remark 1.10. As pointed out by the referee, the issue that only the Nygaard-completed A_{crys} arises naturally from THH has recently been resolved by Mao [32, Proposition 1.8], who constructs a decompleted version of $\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}/p}; \mathbb{Z}_p)$. Although we have not reworked the present arguments in that formalism, we expect that Mao's approach could simplify some of the proofs.

Theorem 1.11. *Let $\mathcal{T}$ be a smooth and proper $\mathcal{O}_K$-linear category. Then there exists an integer $n \geq 0$ such that, for all $i \geq n$, the following statements hold:*

(1) *The $\mathbb{Z}_p[G_K]$-module $T_{A_{\mathrm{inf}}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee)$ is a $\mathbb{Z}_p$-lattice of a crystalline representation, and there is an isomorphism of B_{crys}-module*

$$\pi_i \operatorname{TP}(\mathcal{T}_k; \mathbb{Z}_p)^\vee \otimes_W B_{\mathrm{crys}} \simeq T_{A_{\mathrm{inf}}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}$$

which is compatible with G_K-action and Frobenius endomorphism.

(2) *If $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then there is a G_K-equivariant isomorphism*

$$T_{A_{\mathrm{inf}}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee) \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^\vee$$

of $\mathbb{Z}_p$-modules.

We prove the following as a corollary.

Theorem 1.12 (Main Theorem, Theorem 4.19). *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. If $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then Conjecture 1.2 holds for $\mathcal{T}$.*

Remark 1.13. In the case $\mathcal{T}$ embeds fully faithfully into $\operatorname{perf}(X)$ with X smooth proper over $\mathcal{O}_K$, Theorem 1.11 can be deduced from the crystalline comparison for smooth proper schemes via a Fourier–Mukai kernel argument.

2. Non-commutative version of Breuil–Kisin cohomology

2.1. Breuil–Kisin modules and Breuil–Kisin cohomology theory $R\Gamma_{\mathfrak{S}}$

Let us start by recalling the theory of Breuil–Kisin modules.

Definition 2.1. A Breuil–Kisin module is a finitely generated $\mathfrak{S}$-module M equipped with an $\mathfrak{S}$-linear isomorphism

$$\varphi_M : M \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right] \simeq M\left[\frac{1}{E}\right].$$

For a Breuil–Kisin module $\mathfrak{M}$, let us denote $\mathfrak{M}^\vee = \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{S})$. We note $\mathfrak{M}^\vee = \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{S})$ is a Breuil–Kisin module whose Frobenius map $\varphi_{\mathfrak{M}^\vee}$ is given by

$$\begin{aligned}\mathfrak{M}^\vee \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right] &\simeq \operatorname{Hom}_{\mathfrak{S}[\frac{1}{E}]}\left(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right], \mathfrak{S}\left[\frac{1}{E}\right]\right)\\ &\overset{\varphi_{\mathfrak{M}}^\vee}{\simeq} \operatorname{Hom}_{\mathfrak{S}[\frac{1}{E}]}\left(\mathfrak{M}\left[\frac{1}{E}\right], \mathfrak{S}\left[\frac{1}{E}\right]\right) \simeq \mathfrak{M}^\vee\left[\frac{1}{E}\right], \end{aligned} \tag{2.1}$$

where we use facts that $\varphi : \mathfrak{S} \to \mathfrak{S}$ and $\mathfrak{S} \to \mathfrak{S}[\frac{1}{E}]$ are flat. We note that $\mathfrak{M}^\vee$ is finite free $\mathfrak{S}$-module; this follows from the facts that $\operatorname{gl.dim} \mathfrak{S} = 2$.

Lemma 2.2 ([17, Corollaire 11.1.14] and [7, Lemma 4.27]). *Let $\overline{\phi} : \mathfrak{S} \to A_{\mathrm{inf}}$ be the map that sends z to $[\pi^\flat]^p$ and is Frobenius on W. Let M be a Breuil–Kisin module, and let $M_{A_{\mathrm{inf}}} = M \otimes_{\mathfrak{S},\overline{\phi}} A_{\mathrm{inf}}$. Then $M'[\frac{1}{p}] = M_{A_{\mathrm{inf}}}[\frac{1}{p}] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} K_0$ is a finite free K_0-module equipped with a Frobenius automorphism. Fix a section $k \to \mathcal{O}_{\mathcal{C}}/p$, then there is a (noncanonical) φ-equivariant isomorphism*

$$M_{A_{\mathrm{inf}}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{crys}} \simeq M'\left[\frac{1}{p}\right] \otimes_{K_0} B_{\mathrm{crys}}.$$

In [8], Bhatt–Morrow–Scholze constructed a cohomology theory valued in Breuil–Kisin modules for smooth proper formal schemes over $\mathcal{O}_K$. Let $\varphi_{A_{\mathrm{inf}}} : A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ be the Frobenius endomorphism of A_{inf}. Let $\phi : \mathfrak{S} \to A_{\mathrm{inf}}$ be the W-linear map that sends z to $[\pi^\flat]$. Note that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "\varphi"] \arrow[d, "\phi"'] \arrow[dr, "\overline{\phi}"] & \mathfrak{S} \arrow[d, "\phi"] \\ A_{\mathrm{inf}} \arrow[r, "\simeq", "\varphi_{A_{\mathrm{inf}}}"'] & A_{\mathrm{inf}} \end{tikzcd} \tag{2.2}$$

Theorem 2.3 ([8]). *Let $X/\mathcal{O}_K$ be a smooth proper formal scheme. Then there exists an $\mathfrak{S}$-linear cohomology theory $R\Gamma_{\mathfrak{S}}(X)$ equipped with a φ-semi-linear map, with the following properties:*

(1) *All $H^i_{\mathfrak{S}}(X) := H^i(R\Gamma_{\mathfrak{S}}(X))$ are Breuil–Kisin modules.*

(2) *(étale comparison) After scalar extension along $\overline{\phi} : \mathfrak{S} \to A_{\mathrm{inf}}$, one recovers étale cohomology of the generic fiber*

$$R\Gamma_{\mathfrak{S}}(X) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]^{\wedge}_{p} \simeq R\Gamma_{\text{ét}}(X_{\mathcal{C}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]^{\wedge}_{p}.$$

(3) *(Crystalline comparison) After scalar extension along the map $\mathfrak{S} \to W$, which is the Frobenius on W and sends z to 0, one recovers crystalline cohomology of the special fiber*

$$R\Gamma_{\mathfrak{S}}(X) \otimes^{\mathbb{L}}_{\mathfrak{S}} W \simeq R\Gamma_{\mathrm{cry}}(X_k/W).$$

(4) *(de Rham comparison) After scalar extension along the map $\widetilde{\theta} \circ \varphi : \mathfrak{S} \to \mathcal{O}_K$, one recovers de Rham cohomology*

$$R\Gamma_{\mathfrak{S}}(X) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(X/\mathcal{O}_K).$$

2.2. Perfect modules and Künneth formula

Let $(\mathcal{A}, \otimes, 1_{\mathcal{A}})$ be a symmetric monoidal, stable ∞-category with biexact tensor product. Firstly, we recall from [2, Section 1].

Definition 2.4 ([2, Definition 1.2]). An object $X \in A$ is perfect if it belongs to the thick subcategory generated by the unit.

For a lax symmetric monoidal, exact ∞-functor $F : \mathcal{A} \to \mathrm{Sp}$, $F(1_{\mathcal{A}})$ is naturally an $\mathbb{E}_\infty$-ring. For any $X, Y \in \mathcal{A}$, we have a natural map

$$F(X) \otimes_{F(1_{\mathcal{A}})} F(Y) \to F(X \otimes Y). \tag{2.3}$$

Since F is exact, if X is perfect, then the map (2.3) is an equivalence, and $F(X)$ is a perfect $F(1_{\mathcal{A}})$-module.

We regard $\mathcal{O}_K$ as an $\mathbb{S}[z]$-algebra via $z \mapsto \pi$. There is a symmetric monoidal ∞-functor

$$\mathrm{THH}(-/\mathbb{S}[z];\mathbb{Z}_p) : \mathrm{Cat}^{\mathrm{perf}}_\infty(\mathcal{O}_K) \to \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1}).$$

Let us study this functor from [8, Section 11]. By the base change along $\mathbb{S}[z] \to \mathbb{S}[z^{1/p^\infty}] \mid z \mapsto z$, there is a natural equivalence

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z]) \otimes_{\mathbb{S}[z]} \mathbb{S}[z^{1/p^\infty}] \simeq \mathrm{THH}(\mathcal{O}_K[\pi^{1/p^\infty}]/\mathbb{S}[z^{1/p^\infty}]). \tag{2.4}$$

Since the natural map $\mathrm{THH}(\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \to \mathbb{S}[z^{1/p^\infty}]^\wedge_p$ is an equivalence (see [8, Proposition 11.7]), we obtain equivalences

$$\begin{aligned}\mathrm{THH}(\mathcal{O}_K[\pi^{1/p^\infty}];\mathbb{Z}_p) &\simeq \mathrm{THH}(\mathcal{O}_K[\pi^{1/p^\infty}]/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)\\ &\overset{(2.4)}{\simeq} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathbb{S}[z]^\wedge_p} \mathbb{S}[z^{1/p^\infty}]^\wedge_p.\end{aligned}$$

Since a morphism of homotopy groups $\pi_*(\mathbb{S}[z]^\wedge_p) \to \pi_*(\mathbb{S}[z^{1/p^\infty}]^\wedge_p)$ is faithfully flat and there is an isomorphism $\pi_* \mathrm{THH}(\mathcal{O}_K[\pi^{1/p^\infty}];\mathbb{Z}_p) \simeq \mathcal{O}_K[\pi^{1/p^\infty}][u]$, where $\deg u = 2$ (see [8, Section 6]), we obtain an isomorphism

$$\pi_* \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p) \simeq \mathcal{O}_K[u], \tag{2.5}$$

where u has degree 2 (see [8, Proposition 11.10]).

Proposition 2.5. *Any dualizable object in* $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$ *is perfect.*

Proof. By the isomorphism (2.5), $\pi_* \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$ is a regular noetherian ring of finite Krull dimension concentrated in even degrees. We obtain the claim by [2, Theorem 2.15]. ∎

Proposition 2.6. *Suppose $\mathcal{T}_1, \mathcal{T}_2$ are $\mathcal{O}_K$-linear categories, and suppose $\mathcal{T}_1$ is smooth and proper. Then* $\mathrm{THH}(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p)$ *is perfect in* $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$*, and* $\mathrm{TC}^-(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p)$ *(resp.* $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$*) is a perfect* $\mathrm{TC}^-(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p)$*-module (resp.* $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$*-module) and the natural map*

$$\begin{aligned}&\mathrm{TC}^-(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathrm{TC}^-(\mathcal{T}_2/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TC}^-(\mathcal{T}_1 \otimes_{\mathcal{O}_K} \mathcal{T}_2/\mathbb{S}[z];\mathbb{Z}_p)\\ &(\textit{resp. } \mathrm{TP}(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathrm{TP}(\mathcal{T}_2/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}_1 \otimes_{\mathcal{O}_K} \mathcal{T}_2/\mathbb{S}[z];\mathbb{Z}_p))\end{aligned}$$

is an equivalence.

Proof. Note that ∞-functors

$$(-)^{hS^1} : \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_{-\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp})$$

and

$$(-)^{tS^1} : \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_{-\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp})$$

are lax symmetric monoidal and exact (see [33, Corollary I.4.3]). It is enough to show that $\mathrm{THH}(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p)$ is perfect in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. Since the ∞-functor

$$\mathrm{THH}(-/\mathbb{S}[z];\mathbb{Z}_p):\mathrm{Cat}_\infty^{\mathrm{perf}}(\mathcal{O}_K)\to \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$$

is symmetric monoidal, and $\mathcal{T}_1$ is dualizable in $\mathrm{Cat}_\infty^{\mathrm{perf}}(\mathcal{O}_K)$ (cf. [31, Chapter 11]), it follows that $\mathrm{THH}(\mathcal{T}_1/\mathbb{S}[z];\mathbb{Z}_p)$ is dualizable in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. By Proposition 2.5, we obtain the claim. ∎

2.3. Breuil–Kisin module of stable ∞-categories

Let us recall Antieau–Mathew–Nikolaus's comparison theorem of symmetric monoidal ∞-functors from [2].

Proposition 2.7 ([2, Proposition 4.6]). *Let $\mathcal{T},\hat{\mathcal{T}}$ be symmetric monoidal ∞-categories. Let $F_1,F_2:\mathcal{T}\to\hat{\mathcal{T}}$ be symmetric monoidal functors, and let $t:F_1\Rightarrow F_2$ be a symmetric monoidal natural transformation. Suppose every object of $\mathcal{T}$ is dualizable. Then t is an equivalence.*

In [8, Proposition 11.10], Bhatt–Morrow–Scholze showed the following: on homotopy groups, there exist isomorphisms

$$\pi_*\,\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\simeq \mathfrak{S}[u,v]/(uv-E),$$

where u is of degree 2 and v is of degree -2, and

$$\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\simeq \mathfrak{S}[\sigma^{\pm}],$$

where σ is of degree 2. Let φ be the endomorphism of $\mathfrak{S}$ determined by the Frobenius on W and $z\mapsto z^p$. Nikolaus–Scholze [33] construct two maps

$$\mathrm{can},\varphi_{\mathcal{T}}^{hS^1}:\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\rightrightarrows \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p).$$

In [8, Proposition 11.10], Bhatt–Morrow–Scholze also showed the morphism

$$\mathrm{can}_{\mathcal{O}_K}:\pi_*\,\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\to\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$$

is $\mathfrak{S}$-linear and sends u to $E\sigma$ and v to σ^{-1}, and the morphism

$$\varphi_{\mathcal{O}_K}^{hS^1}:\pi_*\,\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\to\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$$

is φ-linear and sends u to σ and v to $\varphi(E)\sigma^{-1}$. Since σ is an invertible element in $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$, $\varphi_{\mathcal{O}_K}^{hS^1}$ induces a map

$$\tilde{\varphi}_{\mathcal{T}}^{hS^1}:\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]\to\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \tag{2.6}$$

for an $\mathcal{O}_K$-linear stable ∞-category $\mathcal{T}$. We note that on homotopy groups, the morphism

$$\tilde{\varphi}^{hS^1}_{\mathcal{O}_K} : \pi_* \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right] \to \pi_* \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$$

is given by $\mathfrak{S}[u^{\pm}] \to \mathfrak{S}[\sigma^{\pm}]$ which is φ-semi-linear and sends u to σ.

Lemma 2.8. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then the following statements hold:*

(1) *$\pi_* \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is a finitely generated $\pi_* \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$-module, thus $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is a finitely generated $\mathfrak{S}$-module for all i.*

(2) *There is a natural number n satisfying that for any $j \geq n$, $\pi_j \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \xrightarrow{u\cdot} \pi_{j+2} \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is an isomorphism.*

Proof. Claim (1) directly follows from Proposition 2.6. Moreover, for any $j \geq 0$, by the calculation of $\pi_* \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$ (see [8, Proposition 11.10]), we know that the map $\pi_j \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p) \xrightarrow{u\cdot} \pi_{j+2} \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$ is an isomorphism. This yields claim (2) by claim (1). ∎

Theorem 2.9. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there is a natural number n such that the homotopy group $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ has a natural structure of a Breuil–Kisin module for any $i \geq n$, and the dual $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ also has a natural structure of a Breuil–Kisin module of finite E-height for any $i \geq n$.*

Proof. The morphism $\tilde{\varphi}^{hS^1}_{\mathcal{T}}$ induces a morphism of $\operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$-modules

$$\begin{aligned}\operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right] &\otimes_{\operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{u}],\tilde{\varphi}^{hS^1}_{\mathcal{O}_K}} \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\\ &\to \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p). \end{aligned} \tag{2.7}$$

By Proposition 2.6, both sides of the map (2.7) yield symmetric monoidal functors from $\operatorname{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\operatorname{Mod}_{-\operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp})$, and the map (2.7) yields a symmetric monoidal natural transformation between them. By Proposition 2.7, the morphism (2.7) is an equivalence. On 0-th homotopy group, $\tilde{\varphi}^{hS^1}_{\mathcal{O}_K}$ is given by $\varphi : \mathfrak{S} \to \mathfrak{S}$. Since φ is flat, one has an isomorphism

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right] \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \simeq \pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \tag{2.8}$$

for any i. By Lemma 2.8 (2), one obtains an isomorphism

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \simeq \pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]$$

for any $i \geq n$, thus we have an $\mathfrak{S}$-linear isomorphism on homotopy groups

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \simeq \pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \tag{2.9}$$

for any $i \geq n$.

After inverting $E \in \mathfrak{S} \simeq \pi_0 \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)$, the morphism $\operatorname{can}_{\mathcal{T}}$ induces a morphism of $\operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{E}]$-module

$$\begin{aligned}\operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right] &\otimes_{\operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{E}],\operatorname{can}_{\mathcal{O}_K}} \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right] \\ &\to \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right].\end{aligned} \tag{2.10}$$

Both sides of the map (2.10) yield symmetric monoidal functors from $\operatorname{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\operatorname{Mod}-_{\operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{E}]}(\mathrm{Sp})$, and the map (2.10) yields a symmetric monoidal natural transformation between them. Thus the morphism (2.10) is an equivalence. Note that the morphism

$$\operatorname{can}_{\mathcal{O}_K}\left[\frac{1}{E}\right] : \pi_* \operatorname{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right] \to \pi_* \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right]$$

is an isomorphism (see [8, Proposition 11.10]). This yields an isomorphism

$$\operatorname{can}_{\mathcal{T}}\left[\frac{1}{E}\right] : \pi_* \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right] \simeq \pi_* \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right]. \tag{2.11}$$

Combining (2.11) with (2.9), we obtain an isomorphism

$$\begin{aligned}\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right] &\overset{(2.9)}{\simeq} \pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right] \\ &\overset{(2.11)}{\simeq} \pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{E}\right]\end{aligned} \tag{2.12}$$

for any $i \geq n$. Let us study the dual of (2.12). We have a morphism

$$\begin{aligned}\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\varphi} \mathfrak{S} &= \operatorname{Hom}_{\mathfrak{S}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}, \mathfrak{S}) \\ &\overset{(2.9)^\vee}{\simeq} \operatorname{Hom}_{\mathfrak{S}}(\pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p), \mathfrak{S}) \\ &\xrightarrow{\operatorname{can}_{\mathcal{T}}^\vee} \operatorname{Hom}_{\mathfrak{S}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p), \mathfrak{S}).\end{aligned} \tag{2.13}$$

After localization by E, the morphism (2.13) coincides with the dual of (2.12). ∎

2.4. The comparison between $\operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ and $\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$

In this section, for a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, we will carefully compare $\operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ with $\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$. Denote $K_\infty = K(\pi^{1/p^\infty})$, and let $\mathcal{O}_{K_\infty}$ be the ring of integers of K_∞.

Lemma 2.10 ([8, Corollary 11.8]). *For any $\mathcal{O}_K$-linear stable ∞-category $\mathcal{T}$, the natural map*

$$\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) \to \mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$$

is an equivalence which is compatible with S^1-action and G_{K_∞}-action. In particular, the natural map

$$\mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$$

is an equivalence which is compatible with G_{K_∞}-action.

Proof. The morphism $\mathbb{S}[z^{1/p^\infty}] \to \mathcal{O}_{K_\infty} \mid z^{1/p^n} \mapsto \pi^{1/p^n}$ fits into the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] & \mathcal{O}_{K_\infty} \arrow[r] & \mathcal{O}_\mathcal{C} \\ \mathbb{S} \arrow[r] & \mathbb{S}[z] \arrow[u] \arrow[r] & \mathbb{S}[z^{1/p^\infty}] \arrow[u] \end{tikzcd} \tag{2.14}$$

The diagram yields a map

$$\begin{aligned}\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) &\to \mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)\\ &\simeq \mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)} \mathbb{S}[z^{1/p^\infty}]^\wedge_p\end{aligned}$$

which is compatible with S^1-action. According to [8, Proposition 11.7], the natural map $\mathrm{THH}(\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \to \mathbb{S}[z^{1/p^\infty}]^\wedge_p$ is an equivalence. Since π^{1/p^n} is in K_∞ for any n, thus the equivalence is G_{K_∞}-equivariant. ■

Proposition 2.11. *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$,* $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$ *is perfect in* $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$*. In particular,* $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$ *is perfect in* $\mathrm{Mod}_{-\mathrm{TP}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp})$.

Proof. We already know $\mathrm{THH}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is perfect in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$ (see Proposition 2.6). Besides, the functor

$$\begin{aligned}- \bigotimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z];\mathbb{Z}_p) &: \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})\\ &\to \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})\end{aligned}$$

is exact and sends $\mathrm{THH}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ to $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z];\mathbb{Z}_p)$; $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z];\mathbb{Z}_p)$ is a perfect object in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. Since the functor

$$\begin{aligned}- \bigotimes_{\mathrm{THH}(\mathbb{S}[z^{1/p^\infty}]/\mathbb{S}[z];\mathbb{Z}_p)} \mathbb{S}[z^{1/p^\infty}]^\wedge_p &: \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})\\ &\to \mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})\end{aligned}$$

is exact and sends $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z];\mathbb{Z}_p)$ to $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$, it follows that $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$ is a perfect object in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_\mathcal{C}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. ■

Lemma 2.10 and Proposition 2.11 imply the following.

Proposition 2.12. *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$ is a perfect object in* $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$.

The following is a non-commutative version of the comparison theorem between $R\Gamma_{\mathfrak{S}}$ and $R\Gamma_{A_{\mathrm{inf}}}$ in [8, Theorem 1.2 (1)].

Theorem 2.13. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there is a natural number n satisfying that there is a G_{K_∞}-equivariant isomorphism*

$$\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathfrak{S},\overline{\phi}}A_{\mathrm{inf}}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$$

for any $i\geq n$, where $\overline{\phi}$ is the map which sends z to $[\pi^\flat]^p$ and is the Frobenius on W, and $g\in G_{K_\infty}$ acts on $1\otimes g$ on left-hand side.

Proof. For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, consider the following morphism:

$$\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\xrightarrow{\varphi_{\mathcal{T}}^{hS^1}}\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\to\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p),$$

where the second map is given by diagram (2.14). The map sends u to σ and u is an invertible element in $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$, we have a morphism

$$\underline{\varphi}_{\mathcal{T}}^{hS^1}:\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]\to\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p).\tag{2.15}$$

Let us prove that the map $\underline{\varphi}_{\mathcal{T}}^{hS^1}$ yields an equivalence of $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$-module

$$\begin{aligned}&\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]\otimes_{\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{u}],\underline{\varphi}_{\mathcal{O}_K}^{hS^1}}\mathrm{TP}(\mathcal{O}_{\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)\\&\quad\simeq\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)\end{aligned}\tag{2.16}$$

which is compatible with G_{K_∞}-action. By Proposition 2.6, the left side of the map (2.16) yields a symmetric monoidal functor from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}_{-\mathrm{TP}(\mathcal{O}_{\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp})$. By Proposition 2.11, the right side of the map (2.16) also yields a symmetric monoidal functor from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}_{-\mathrm{TP}(\mathcal{O}_{\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)}(\mathrm{Sp})$. By [8, Section 11], on homotopy groups, the morphism

$$\underline{\varphi}_{\mathcal{O}_K}^{hS^1}:\pi_*\,\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]\to\pi_*\,\mathrm{TP}(\mathcal{O}_{\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$$

is given by $\mathfrak{S}[u^{\pm}]\to A_{\mathrm{inf}}[\sigma^{\pm}]$ which is $\overline{\phi}$-linear and sends u to σ, thus it is flat by [7, Lemma 4.30]. We now have a G_{K_∞}-equivariant isomorphism

$$\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{u}\right]\otimes_{\mathfrak{S},\overline{\phi}}A_{\mathrm{inf}}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p)$$

for any i. By Lemma 2.8 (2), we obtain the claim. ∎

2.5. The comparison between $\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ and $\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)$

In this section, for a smooth and proper $\mathcal{O}_K$-linear category $\mathcal{T}$, we will carefully compare $\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ with $\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)$. There exists a Cartesian diagram of $\mathbb{E}_\infty$-ring below.

$$\begin{tikzcd} k & \mathcal{O}_K \arrow[l] \\ \mathbb{S} \arrow[u] & \mathbb{S}[z] \arrow[l] \arrow[u] \end{tikzcd}$$

For an $\mathcal{O}_K$-linear stable ∞-category $\mathcal{T}$, the diagram yields morphisms

$$\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TC}^-(\mathcal{T}_k;\mathbb{Z}_p)$$

and

$$\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TC}^-(\mathcal{T}_k;\mathbb{Z}_p) \xrightarrow{\varphi_k^{hS^1}} \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p). \tag{2.17}$$

Theorem 2.14. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there is a natural number n such that the morphism (2.17) is an isomorphism*

$$\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\tilde{\phi}} K_0 \to \pi_i \,\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right]$$

for any $i \geq n$, where $\tilde{\phi}$ is the map which sends z to 0 and Frobenius on W.

Proof. The morphism (2.17) induces a commutative diagram

$$\begin{tikzcd} \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[r] \arrow[d, "\varphi_{\mathcal{T}}^{hS^1}"'] & \mathrm{TC}^-(\mathcal{T}_k;\mathbb{Z}_p) \arrow[d, "\varphi_k^{hS^1}"] \\ \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[r] & \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p). \end{tikzcd}$$

Firstly, we prove that the morphism $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)$ induces an isomorphism

$$\pi_i \,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\gamma} K_0 \to \pi_i \,\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right]$$

for any i, where γ is a W-linear and sends z to 0. By Theorem 2.9 and [7, Proposition 4.3], there is n such that for any $i \geq n$, the homotopy group $\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is a Breuil–Kisin module, and $\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{p}]$ is a finite free $\mathfrak{S}[\frac{1}{p}]$-module. By the isomorphism (2.9) and the fact that $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is 2-periodic, we obtain that

$\pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{p}]$ is a finite free $\mathfrak{S}[\frac{1}{p}]$-module for any i. Thus, on homotopy groups, one obtains an isomorphism

$$\begin{aligned}
&\pi_* \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \\
&\quad\simeq \pi_0 \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}]} \pi_* \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \\
&\qquad \oplus \pi_1 \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}]} \pi_* \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right],
\end{aligned} \tag{2.18}$$

and we see that $\pi_* \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{p}]$ is a flat graded $\pi_* \operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{p}]$-module. Let us prove the morphism

$$\operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\operatorname{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)[\frac{1}{p}]} \operatorname{TP}(k;\mathbb{Z}_p)\left[\frac{1}{p}\right] \to \operatorname{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right] \tag{2.19}$$

is an equivalence. Both sides of the map (2.19) yield symmetric monoidal functors from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}-_{\operatorname{TP}(k;\mathbb{Z}_p)[\frac{1}{E}]}(\mathrm{Sp})$, thus by Proposition 2.7, we know the map (2.19) is an equivalence. By the isomorphism (2.18) and the fact that $\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$ is 2-periodic, on homotopy groups, we have an isomorphism

$$\pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\gamma} K_0 \to \pi_i \operatorname{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right] \tag{2.20}$$

for any i. Combining the isomorphism (2.9) with an equality $\gamma \circ \varphi = \widetilde{\phi}$, we obtain an isomorphism

$$\begin{aligned}
&\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\widetilde{\phi}} K_0 \\
&\quad\simeq \left(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\phi} \mathfrak{S}\left[\frac{1}{p}\right]\right) \otimes_{\mathfrak{S}[\frac{1}{p}],\tau} K_0 \\
&\quad\overset{(2.9)}{\simeq} \pi_i \operatorname{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\tau} K_0 \\
&\quad\overset{(2.20)}{\simeq} \pi_i \operatorname{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right].
\end{aligned}$$

for any $i \geq n$. ■

2.6. The comparison theorem between $\operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ and $L_{K(1)}K(\mathcal{T}_{\mathcal{C}})$

Firstly, we prove Künneth formula of $K(1)$-local K-theory for $\mathcal{C}$-linear categories that admit a geometric realization.

Proposition 2.15. *For $\mathcal{C}$-linear categories $\mathcal{T}_1$, $\mathcal{T}_2$ which admit geometric realization, the natural map*

$$L_{K(1)}K(\mathcal{T}_1) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}K(\mathcal{T}_2) \to L_{K(1)}(\mathcal{T}_1 \otimes_{\mathcal{C}} \mathcal{T}_2) \tag{2.21}$$

is an equivalence.

Proof. At first, we prove the claim in the case that there exist smooth proper varieties X_1 and X_2 so that $\mathcal{T}_1 = \mathrm{perf}(X_1)$ and $\mathcal{T}_2 = \mathrm{perf}(X_2)$. As a preliminary reduction, we first establish the claim over $\mathcal{C} = \mathbb{C}_p$, the completion of $\overline{\mathbb{Q}}_p$ with respect to $|\cdot|_p$. We fix an isomorphism of fields $\sigma : \mathcal{C} \simeq \mathbb{C}$. We denote $X_i \times_{\mathrm{Spec}\,\mathcal{C}} \mathrm{Spec}\,\mathbb{C}$ by $X_{i,\mathbb{C}}$. Due to Thomason [36], we have an equivalence $L_{K(1)}K(X_i) \simeq K_{\mathrm{top}}(X^{\mathrm{an}}_{i,\mathbb{C}})^{\wedge}_p$. Due to Atiyah [3], since a CW complex which comes from a compact complex manifold is finite, we know a natural map

$$K_{\mathrm{top}}(X^{\mathrm{an}}_{1,,\mathbb{C}}) \otimes_{K_{\mathrm{top}}(\mathrm{pt})} K_{\mathrm{top}}(X^{\mathrm{an}}_{2,\mathbb{C}}) \to K_{\mathrm{top}}(X^{\mathrm{an}}_{1,\mathbb{C}} \times X^{\mathrm{an}}_{2,\mathbb{C}})$$

is an equivalence. The isomorphism of fields $\sigma : \mathcal{C} \simeq \mathbb{C}$ induces an equivalence of ring spectra $K_{\mathrm{top}}(\mathrm{pt})^{\wedge}_p \simeq L_{K(1)}K(\mathbb{C})$. We see that there exists a commutative diagram

$$\begin{tikzcd}
L_{K(1)}K(X_1) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}K(X_2) \arrow[r] \arrow[d, "\simeq"'] & L_{K(1)}K(X_1 \times_{\mathrm{Spec}\,\mathcal{C}} X_2) \arrow[d, "\simeq"] \\
K_{\mathrm{top}}(X^{\mathrm{an}}_{1,,\mathbb{C}}) \otimes_{K_{\mathrm{top}}(\mathrm{pt})} K_{\mathrm{top}}(X^{\mathrm{an}}_{2,\mathbb{C}}) \arrow[r, "\simeq"'] & K_{\mathrm{top}}(X^{\mathrm{an}}_{1,\mathbb{C}} \times X^{\mathrm{an}}_{2,\mathbb{C}})
\end{tikzcd}$$

and we obtain the claim. We now treat the general case. We fix an embedding of fields $\mathbb{C}_p \hookrightarrow \mathcal{C}$. There exists a subfield $\mathbb{C}_p \subset L \subset \mathcal{C}$ such that L is finitely generated over $\mathbb{C}_p$ and X and Y are defined over L. Take a variety T over $\mathbb{C}_p$ such that the function field $K(T)$ is isomorphic to L. Then we can take varieties X', Y' over $\mathbb{C}_p$ and morphisms $X' \to T, Y' \to T$ over $\mathbb{C}_p$ such that the generic fibers of X', Y' tensored with $\mathcal{C}$ are X, Y, respectively. Then, by shrinking T, we may assume X', Y' are proper and smooth over T. Choose a $\mathbb{C}_p$-value point t on T. In that case, by using proper smooth base change theorems in étale cohomology and Thomason's spectral sequence [36], we have canonical equivalences

$$\begin{aligned}
L_{K(1)}K(X'_t) &\simeq L_{K(1)}(X), \\
L_{K(1)}K(Y'_t) &\simeq L_{K(1)}(Y), \\
L_{K(1)}K(X'_t \times_{\mathbb{C}_p} Y'_t) &\simeq L_{K(1)}(X \times_{\mathcal{C}} Y).
\end{aligned}$$

The claim follows from the $\mathcal{C} = \mathbb{C}_p$-case.

Secondly, we prove the claim in the case when $\mathcal{T}_1 = \mathrm{perf}(X)$ and $\mathcal{T}_2 = \mathrm{perf}(Y)$ for a smooth variety X and a smooth proper variety Y. Choose a good compactification $(\overline{X}, \Sigma^n_{i=1} D_i)$ of X where each D_i is irreducible. For $n \leq i$, we write $X_i = X - \bigcup^i_{j=1} D_j$. The Verdier quotient $\mathrm{perf}_{D_i \cap X_{i-1}}(X_{i-1}) \subset \mathrm{perf}(X_{i-1}) \to \mathrm{perf}(X_i)$ induces a fiber sequence

$$L_{K(1)}K(Y) \to L_{K(1)}K(Y) \to L_{K(1)}K(Y), \tag{2.22}$$

and the Verdier quotient $\mathrm{perf}_{(D_i \cap X_{i-1}) \times Y}(X_{i-1} \times Y) = \mathrm{perf}_{D_i \cap X_{i-1}}(X_{i-1}) \otimes \mathrm{perf}(Y) \subset \mathrm{perf}(X_{i-1}) \otimes \mathrm{perf}(Y) \to \mathrm{perf}(X_i) \otimes \mathrm{perf}(Y)$ induces a fiber sequence

$$L_{K(1)}K((D_i \cap X_{i-1}) \times Y) \to L_{K(1)}K(X_{i-1} \times Y) \to L_{K(1)}K(X_i \times Y). \tag{2.23}$$

There is a morphism of fiber sequence $L_{K(1)}K(D_i \cap X_{i-1}) \otimes_{L_{K(1)}K(\mathcal{C})}$ (2.22) $\to$ (2.23). Note that $D_i \cap X_{i-1}$ have a good compactification $(D_i, \Sigma_{l=1}^{i-1} D_i \cap D_l)$. By the induction on i, we obtain the claim. The same argument as above proves in the case that X and Y are smooth varieties.

Thirdly, we prove the claim in the case that $\mathcal{T}_1 = \mathrm{perf}(X)$ and $\mathcal{T}_2 = \mathrm{perf}(Y)$ for a separated and finite type scheme X over $\mathcal{C}$ and a smooth variety Y. Choose a sequence of closed subschemes of X

$$\emptyset = Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_n = X$$

such that $Z_i \backslash |Z_{i-1}|$ is smooth over $\mathcal{C}$ for any i. For any variety Z over $\mathcal{C}$, since $\mathcal{C}$ is characteristic 0, we have an equivalence $L_{K(1)}K(Z) \simeq L_{K(1)}\,\mathrm{HK}(Z)$ (see [38]), where $\mathrm{HK}(Z)$ is the homotopy K-theory of Z. Using Quillen's localization theorem, we have fiber sequences

$$L_{K(1)}K(Z_{i-1}) \to L_{K(1)}K(Z_i) \to L_{K(1)}K(Z_i \backslash |Z_{i-1}|) \tag{2.24}$$

and

$$L_{K(1)}K(Z_{i-1} \times Y) \to L_{K(1)}K(Z_i \times Y) \to L_{K(1)}K((Z_i \backslash |Z_{i-1}|) \times Y), \tag{2.25}$$

and there is a morphism of fiber sequences $L_{K(1)}K(Y) \otimes_{L_{K(1)}K(\mathcal{C})}$ (2.24) $\to$ (2.25). By the induction on i, we obtain the claim. The same argument as above shows the claim in the case that X and Y are smooth varieties.

For a $\mathcal{C}$-dg-algebra B concentrated in non-positive degrees, using the Dundas–Goodwillie–McCarthy theorem [15], we have a homotopy pullback square

$$\begin{array}{ccc} K(B) & \longrightarrow & K(H^0(B)) \\ \downarrow & & \downarrow \\ \mathrm{TC}(B) & \longrightarrow & \mathrm{TC}(H^0(B)) \end{array}$$

Since $\mathcal{C}$ has characteristic 0, we have $L_{K(1)}\,\mathrm{TC}(B) = L_{K(1)}\,\mathrm{TC}(H^0(B)) = 0$. Thus the natural map $L_{K(1)}K(B) \to L_{K(1)}K(H^0(B))$ is an equivalence. For affine derived schemes X and Y whose truncations are separated and of finite type over $\mathcal{C}$, the claim holds for $\mathcal{T}_1 = \mathrm{perf}(X)$ and $\mathcal{T}_2 = \mathrm{perf}(Y)$. For derived schemes X and Y whose truncations are smooth over $\mathcal{C}$, the claim follows from the case that X and Y are affine derived schemes by [11, Theorem A.4]. For $\mathcal{C}$-linear categories $\mathcal{T}_1$, $\mathcal{T}_2$ which admit geometric realization $\mathcal{T}_1 \hookrightarrow \mathrm{perf}(X)$ and $\mathcal{T}_2 \hookrightarrow \mathrm{perf}(Y)$, the claim follows from the fact that $L_{K(1)}K(\mathcal{T}_1)$ (resp. $L_{K(1)}K(\mathcal{T}_2)$) is a retract of $L_{K(1)}K(X)$ (resp. $L_{K(1)}K(Y)$) (see [1, Proposition 3.4]). ∎

We let $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)_{\mathrm{geom}}$ denote the ∞-category of smooth proper $\mathcal{O}_K$-linear categories $\mathcal{T}$ such that $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization.

Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Using [6, Theorem 2.16], one obtains an equivalence $L_{K(1)}K(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}) \simeq L_{K(1)}K(\mathcal{T}_{\mathcal{C}})$ of ring spectra. We recall some results about topological Hochschild homology theory from [6,8,22]. On homotopy groups, there is an isomorphism

$$\pi_* \operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p) \simeq A_{\inf}[\sigma,\sigma^{-1}], \tag{2.26}$$

where σ is a generator $\sigma \in \operatorname{TP}_2(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)$. Let $\beta \in K_2(\mathcal{C})$ be the Bott element. For any $\mathcal{O}_{\mathcal{C}}$-linear stable ∞-category $\mathcal{D}$, we have an identification $L_{K(1)}K(\mathcal{D}) \simeq L_{K(1)}K(\mathcal{D}_{\mathcal{C}})$ (see [6, Theorem 2.16]). The cyclotomic trace map $\operatorname{tr}: K(\mathcal{O}_{\mathcal{C}}) \to \operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)$ sends β to a $\mathbb{Z}_p^*$-multiple of $([\varepsilon]-1)\sigma$ (see [22] and [23, Theorem 1.3.6]); the cyclotomic trace map induces a morphism

$$L_{K(1)}\operatorname{tr}: L_{K(1)}K(\mathcal{D}_{\mathcal{C}}) \simeq L_{K(1)}K(\mathcal{D}) \to L_{K(1)}\operatorname{TP}(\mathcal{D}) \simeq \operatorname{TP}(\mathcal{D};\mathbb{Z}_p)\left[\frac{1}{[\varepsilon]-1}\right]_p^\wedge$$

for any $\mathcal{O}_{\mathcal{C}}$-linear category $\mathcal{D}$.

Theorem 2.16. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Assume that $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization; then the trace map induces an equivalence*

$$L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}\operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p) \to L_{K(1)}\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p). \tag{2.27}$$

In particular, there is a G_K-equivariant isomorphism

$$\pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} A_{\inf}\left[\frac{1}{[\varepsilon]-1}\right]_p^\wedge \simeq \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{[\varepsilon]-1}\right]_p^\wedge$$

for any i.

Proof. By applying Propositions 2.12 and 2.15, we see that both sides of the map (2.27) naturally yield symmetric monoidal functors from $\operatorname{Cat}^{\mathrm{perf}}_{k,\mathrm{sat}}$ to $\operatorname{Mod}_{L_{K(1)}\operatorname{TP}(C/\mathbb{Z}_p)}(\mathrm{Sp})$, and thus the map (2.27) indeed induces an equivalence. On homotopy groups, the morphism $L_{K(1)}\operatorname{tr}: \pi_* L_{K(1)}K(\mathcal{C}) \to \pi_* L_{K(1)}\operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)$ is given by $\mathbb{Z}_p[\beta^{\pm}] \to A_{\inf}\left[\frac{1}{[\varepsilon]-1}\right]_p^\wedge[\sigma^{\pm}]$ which sends β to $([\varepsilon]-1)\sigma$, and is flat. Thus we obtain the claim. ∎

2.7. Non-commutative version of Bhatt–Morrow–Scholze's comparison theorem

In this section, as a summary of the previous several sections, we prove a non-commutative version of Theorem 2.3.

Theorem 2.17. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there is n satisfying the following statements:*

(1) *For any $i \geq n$, $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is a Breuil–Kisin module.*

(1′) *For any $i \geq n$, $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ is a Breuil–Kisin module of finite E-height.*

(2) *($K(1)$-local K-theory comparison) Assume $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization. For any $i \geq n$, after scalar extension along $\overline{\phi}: \mathfrak{S} \to A_{\mathrm{inf}}$, one recovers $K(1)$-local K-theory of generic fiber*

$$\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathfrak{S},\overline{\phi}} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]_p^\wedge \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]_p^\wedge.$$

(3) *(Topological periodic homology theory comparison) For any $i \geq n$, after scalar extension along the map $\widetilde{\phi}: \mathfrak{S} \to W$ which is the Frobenius on W and sends z to 0, one recovers topological periodic homology theory of the special fiber*

$$\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathfrak{S}[\frac{1}{p}],\widetilde{\phi}} K_0 \simeq \pi_i \,\mathrm{TP}(\mathcal{T}_k; \mathbb{Z}_p)\left[\frac{1}{p}\right].$$

Proof. Claims (1) and (1′) follow from Theorem 2.9. Claim (2) follows from Theorems 2.13 and 2.16. Claim (3) follows from Theorem 2.14. ∎

Corollary 2.18. *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$ which admits a geometric realization and an integer i, there is an isomorphism of B_{crys}-modules*

$$\pi_i \,\mathrm{TP}(\mathcal{T}_k; \mathbb{Z}_p) \otimes_W B_{\mathrm{crys}} \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}.$$

Proof. The claim follows from Theorem 2.17 and Lemma 2.2. ∎

3. Breuil–Kisin module and semi-stable representation

3.1. Breuil–Kisin G_K-module

Let us recall the work of H. Gao on Breuil–Kisin G_K-module [19]. Look at the following diagram:

$$\begin{array}{ccc} A_{\mathrm{inf}} & \xrightarrow{\varphi_{A_{\mathrm{inf}}}} & A_{\mathrm{inf}} \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\phi} \\ \mathfrak{S} & \xrightarrow[\varphi]{} & \mathfrak{S} \end{array}$$

We note that ϕ sends E to ξ. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a Breuil–Kisin module. We write $\widehat{\mathfrak{M}}$ to denote $\mathfrak{M} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$. Upon taking the tensor product $- \otimes_{\mathcal{S},\phi} \mathbb{A}_{\mathrm{inf}}$, the induced $\mathcal{S}[\frac{1}{E}]$-linear isomorphism $\mathfrak{M} \otimes_{\mathcal{S},\varphi} \mathcal{S}[\frac{1}{E}] \xrightarrow{\varphi_{\mathfrak{M}}} \mathfrak{M}[\frac{1}{E}]$ then gives rise to the isomorphism

$$\widehat{\varphi}_{\mathfrak{M}} : \widehat{\mathfrak{M}} \otimes_{A_{\mathrm{inf}},\varphi_{A_{\mathrm{inf}}}} A_{\mathrm{inf}}\left[\frac{1}{\xi}\right] \simeq \widehat{\mathfrak{M}}\left[\frac{1}{\xi}\right]. \tag{3.1}$$

Definition 3.1 ([19, Definition 7.1.1]). Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a finite free Breuil–Kisin module of finite E-height, equipped with a continuous A_{inf}-semi-linear G_K-action on $\widehat{\mathfrak{M}} = \mathfrak{M} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$. We call $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ a Breuil–Kisin G_K-module if this G_K-action satisfies the following conditions:

(1) G_K commutes with $\widehat{\varphi}_{\mathfrak{M}}$.
(2) $\mathfrak{M} \subset \widehat{\mathfrak{M}}^{G_{K_\infty}}$ via the embedding $\mathfrak{M} \subset \widehat{\mathfrak{M}}$.
(3) $\mathfrak{M}/z\mathfrak{M} \subset \big(\widehat{\mathfrak{M}} \otimes_{A_{\mathrm{inf}}} W(\overline{k})\big)^{G_K}$ via the embedding $\mathfrak{M}/z\mathfrak{M} \subset \widehat{\mathfrak{M}} \otimes_{A_{\mathrm{inf}}} W(\overline{k})$.

The G_K-equivariant surjective map of rings $\mathcal{O}_{\mathcal{C}} \to \overline{k}$ induces a morphism of ring spectra

$$\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}_{\overline{k}}; \mathbb{Z}_p). \tag{3.2}$$

Combining an equivalence $\mathrm{THH}(\mathcal{T}_{\overline{k}}; \mathbb{Z}_p) \simeq \mathrm{THH}(\overline{k}; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)} \mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)$ with Proposition 2.12, we obtain the following proposition.

Proposition 3.2. *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, $\mathrm{THH}(\mathcal{T}_{\overline{k}}; \mathbb{Z}_p)$ is perfect in $\mathrm{Mod}\text{-}_{\mathrm{THH}(\overline{k};\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$.*

Let us recall the calculation of $\mathrm{TP}(\mathcal{O}_{\mathcal{C}}; \mathbb{Z}_p)$ [8]. On homotopy groups,

$$\varphi^{hS^1}_{\mathcal{O}_{\mathcal{C}}} : \pi_* \mathrm{TC}^-(\mathcal{O}_{\mathcal{C}}; \mathbb{Z}_p) \to \pi_* \mathrm{TP}(\mathcal{O}_{\mathcal{C}}; \mathbb{Z}_p) \tag{3.3}$$

is given by

$$A_{\mathrm{inf}}[u, v]/(uv - \xi) \to A_{\mathrm{inf}}[\sigma^{\pm}]$$

which is $\varphi_{A_{\mathrm{inf}}}$-linear and sends u to σ and v to $\varphi_{A_{\mathrm{inf}}}(\xi)\sigma^{-1}$, where u and σ have degree 2 and v has degree -2. Similarly, on homotopy groups,

$$\varphi^{hS^1}_{\overline{k}} : \pi_* \mathrm{TC}^-(\overline{k}; \mathbb{Z}_p) \to \pi_* \mathrm{TP}(\overline{k}; \mathbb{Z}_p) \tag{3.4}$$

is given by

$$W(\overline{k})[u, v]/(uv - p) \to W(\overline{k})[\sigma^{\pm}]$$

which is $\varphi_{W(\overline{k})}$-linear and sends u to σ and v to $p\sigma^{-1}$.

Theorem 3.3. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. The morphism (3.2) induces an isomorphism*

$$\pi_i \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} W(\overline{k})\left[\frac{1}{p}\right] \simeq \pi_i \mathrm{TP}(\mathcal{T}_{\overline{k}}; \mathbb{Z}_p)\left[\frac{1}{p}\right]$$

for any i. In particular, $\pi_i \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)[\frac{1}{p}] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} W(\overline{k})[\frac{1}{p}]$ is fixed by $G_{K^{\mathrm{ur}}}$.

Proof. The morphism (3.2) yields a morphism

$$\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\mathrm{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)[\frac{1}{p}]} \mathrm{TP}(\overline{k}; \mathbb{Z}_p)\left[\frac{1}{p}\right] \to \mathrm{TP}(\mathcal{T}_{\overline{k}}; \mathbb{Z}_p)\left[\frac{1}{p}\right]. \tag{3.5}$$

By Propositions 2.12 and 3.2, both sides of the map (3.2) yield symmetric monoidal functors from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}\text{-}_{\mathrm{TP}(\overline{k};\mathbb{Z}_p)}(\mathrm{Sp})$, and the map (3.5) yields a symmetric monoidal natural transformation between them. Thus the morphism (3.5) is an equivalence. By Theorem 2.9 and [7, Proposition 4.3], there is n such that for any $i \geq n$, the homotopy group $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ is a Breuil–Kisin module, and $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)[\frac{1}{p}]$ is a finite free $\mathfrak{S}[\frac{1}{p}]$-module. By Theorem 2.13 and the fact that $\operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ is 2-periodic, $\pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)[\frac{1}{p}]$ is a finite free $A_{\mathrm{inf}}[\frac{1}{p}]$-module for any i. Thus $\pi_* \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)[\frac{1}{p}]$ is a flat graded $\pi_* \operatorname{TP}(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)[\frac{1}{p}]$-module, and we obtain the claim. ∎

Lemma 3.4. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then the following statements hold:*

(1) *$\pi_* \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ is a finitely generated $\pi_* \operatorname{TC}^-(\mathcal{O}_\mathcal{C}/\mathbb{S}[z]; \mathbb{Z}_p)$-module, thus $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ is a finitely generated A_{inf}-module for any i.*

(2) *There is a natural number n such that for any $j \geq n$, $\pi_j \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \xrightarrow{u\cdot} \pi_{j+2} \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ is an isomorphism.*

Proof. Claim (1) directly follows from Proposition 2.12. For any $j \geq 0$, by the calculation of $\pi_* \operatorname{TC}^-(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)$ (see [8, Proposition 11.10]), we know that $\pi_j \operatorname{TC}^-(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p) \xrightarrow{u\cdot} \pi_{j+2} \operatorname{TC}^-(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)$ is an isomorphism. By claim (1), we see claim (2). ∎

Theorem 3.5. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. There is a natural number n such that the cyclotomic Frobenius morphism $\varphi^{hS^1}_{\mathcal{T}_{\mathcal{O}_\mathcal{C}}} : \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \to \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ induces a G_K-equivariant isomorphism*

$$\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \otimes_{A_{\mathrm{inf}}, \varphi_{A_{\mathrm{inf}}}} A_{\mathrm{inf}} \simeq \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \tag{3.6}$$

for any $i \geq n$.

Proof. Cyclotomic Frobenius map $\varphi^{hS^1}_{\mathcal{T}_{\mathcal{O}_\mathcal{C}}} : \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \to \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$ yields a morphism

$$\underline{\varphi}^{hS^1}_{\mathcal{T}_{\mathcal{O}_\mathcal{C}}} : \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)\left[\frac{1}{u}\right] \to \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p).$$

The morphism $\underline{\varphi}^{hS^1}_{\mathcal{T}_{\mathcal{O}_\mathcal{C}}}$ induces a morphism of $\operatorname{TP}(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)$-module

$$\operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)\left[\frac{1}{u}\right] \otimes_{\operatorname{TC}^-(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)[\frac{1}{u}], \underline{\varphi}^{hS^1}_{\mathcal{O}_\mathcal{C}}} \operatorname{TP}(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p) \to \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p). \tag{3.7}$$

By Proposition 2.12, both sides of the map (3.7) yield symmetric monoidal functors from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}_{-\operatorname{TP}(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)}(\mathrm{Sp})$, thus the map (3.7) is an equivalence. On homotopy groups, $\underline{\varphi}^{hS^1}_{\mathcal{O}_\mathcal{C}} : \pi_* \operatorname{TC}^-(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)[\frac{1}{u}] \to \pi_* \operatorname{TP}(\mathcal{O}_\mathcal{C}; \mathbb{Z}_p)$ is given by $\varphi_{A_{\mathrm{inf}}}$-linear map

$$A_{\mathrm{inf}}[u^{\pm}] \to A_{\mathrm{inf}}[\sigma^{\pm}],$$

thus this is an isomorphism. Thus we obtain the isomorphism

$$\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)\left[\frac{1}{u}\right] \otimes_{A_{\mathrm{inf}}, \varphi_{A_{\mathrm{inf}}}} \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p)$$

for any i. By Lemma 3.4, there is a natural number n such that $\pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p) \simeq \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\big[\frac{1}{u}\big]$ for any $i \geq n$. ■

For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, we will prove that the free Breuil–Kisin module $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ is a Breuil–Kisin G_K-module.

Theorem 3.6. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. There is an n such that for any $i \geq n$, the following statements hold:*

(1) *There is a continuous A_{inf}-semi-linear G_K-action on*

$$\widehat{\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee} = \pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}.$$

(2) *G_K commutes with $\widehat{\varphi}_{\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee}$.*

(3) *One has the inclusion $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \subset \big(\widehat{\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee}\big)^{G_{K_\infty}}$ induced via the embedding $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \subset \widehat{\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee}$.*

(4) *$\pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee\big[\frac{1}{p}\big] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} W(\overline{k})\big[\frac{1}{p}\big]$ is fixed by $G_{K^{\mathrm{ur}}}$.*

In particular, $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ is a Breuil–Kisin G_K-module.

Proof. After localization by $\xi \in A_{\mathrm{inf}} \simeq \pi_0\,\mathrm{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)$, the morphism $\mathrm{can}_{\mathcal{T}}$ induces a morphism of $\mathrm{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\big[\frac{1}{\xi}\big]$-module

$$\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right] \otimes_{\mathrm{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)[\frac{1}{\xi}],\mathrm{can}_{\mathcal{O}_{\mathcal{C}}}} \mathrm{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right] \to \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]. \tag{3.8}$$

Both sides of the map (3.8) yield symmetric monoidal functors from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}_{-\mathrm{TP}(\mathcal{O}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)[\frac{1}{\xi}]}(\mathrm{Sp})$, and the map (3.8) yields a symmetric monoidal natural transformation between them. Thus the morphism (3.8) is an equivalence. Note that the morphism

$$\mathrm{can}_{\mathcal{O}_{\mathcal{C}}} : \pi_*\,\mathrm{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right] \to \pi_*\,\mathrm{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]$$

is an isomorphism (see [8, Proposition 11.10]). This yields a G_K-equivariant isomorphism

$$\mathrm{can}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}\left[\frac{1}{\xi}\right] : \pi_*\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right] \simeq \pi_*\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]. \tag{3.9}$$

Combined with (3.6), this yields a G_K-equivariant isomorphism

$$\begin{aligned}\pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p) \otimes_{A_{\mathrm{inf}},\varphi_{A_{\mathrm{inf}}}} A_{\mathrm{inf}}\left[\frac{1}{\xi}\right] &\overset{(3.6)}{\simeq} \pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right] \\ &\overset{(3.9)}{\simeq} \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]\end{aligned} \tag{3.10}$$

for any $i \geq n$. Since $\varphi_{A_{\mathrm{inf}}}$ is an isomorphism, by Theorems 2.13 and 3.5, we have a G_{K_∞}-equivariant isomorphism

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \simeq \pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p), \tag{3.11}$$

and we see the isomorphism (3.10) is the same as $\widehat{\varphi}_{\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)}$. The group G_K naturally acts on $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$, which is $\pi_0 \operatorname{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p) \simeq A_{\mathrm{inf}}$-semi-linear. Since $\varphi : \mathfrak{S} \to A_{\mathrm{inf}}$ is flat, the dual of (3.11) becomes the following G_{K_∞}-equivariant isomorphism:

$$\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \simeq \pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee. \tag{3.12}$$

Via this isomorphism, $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$ admits a continuous A_{inf}-semi-linear G_K-action.

Since (3.10) is G_K-equivariant, G_K commutes with $\widehat{\varphi}_{\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee}$, yielding claims (1) and (2).

Consider the G_{K_∞}-equivariant isomorphism (3.12). Since $g \in G_{K_\infty}$ acts on $1 \otimes g$ on $\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$, we obtain claim (3).

Since $\varphi_{A_{\mathrm{inf}}} : A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ is flat, we have G_K-equivariant isomorphism

$$\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \otimes_{A_{\mathrm{inf}},\varphi_{A_{\mathrm{inf}}}} A_{\mathrm{inf}} \simeq \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \tag{3.13}$$

by Theorem 3.5. Combined with Theorem 3.3, we see that $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee\left[\frac{1}{p}\right] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} W(\bar{k})\left[\frac{1}{p}\right]$ is fixed by $G_{K^{\mathrm{ur}}}$.

Since $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee$ is finite free $\mathfrak{S}$-module, there is a natural inclusion $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \hookrightarrow \pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee\left[\frac{1}{p}\right]$, and it induces a natural inclusion

$$\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \otimes_{A_{\mathrm{inf}}} W(\bar{k}) \hookrightarrow \pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee\left[\frac{1}{p}\right] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} W(\bar{k})\left[\frac{1}{p}\right].$$

Thus $\pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \otimes_{A_{\mathrm{inf}}} W(\bar{k})$ is fixed by $G_{K^{\mathrm{ur}}}$. The last claim follows from claims (1)–(4) and [19, Lemma 7.1.2]. ∎

For a Breuil–Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, let $\overline{\varphi_{\mathfrak{M}}}$ denote the composite

$$\overline{\varphi_{\mathfrak{M}}} : \mathfrak{M} \otimes_{\mathfrak{S}} W(\mathcal{C}^\flat) \xrightarrow{\varphi'_{\mathfrak{M}} \otimes \varphi_{\mathcal{C}}} \mathfrak{M}\left[\frac{1}{E}\right] \otimes_{\mathfrak{S}[\frac{1}{E}]} W(\mathcal{C}^\flat) = \mathfrak{M} \otimes_{\mathfrak{S}} W(\mathcal{C}^\flat),$$

where $\varphi'_{\mathfrak{M}}$ is the composite $\mathfrak{M} \xrightarrow{\varphi} \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S}\left[\frac{1}{E}\right] \xrightarrow{\varphi_{\mathfrak{M}}} \mathfrak{M}\left[\frac{1}{E}\right]$, and $\varphi_{\mathcal{C}}$ is the Frobenius of $W(\mathcal{C}^\flat)$. Let us define a $\mathbb{Z}_p$-module

$$T_{A_{\mathrm{inf}}}(\mathfrak{M}) := \left(\widehat{\mathfrak{M}} \otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat)\right)^{\overline{\varphi_{\mathfrak{M}}}=1}.$$

If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is equipped with a Breuil–Kisin G_K-module structure, then $T_{A_{\mathrm{inf}}}(\mathfrak{M})$ inherits a natural structure of a $\mathbb{Z}_p[G_K]$-module.

Theorem 3.7. *The following statements hold:*

(1) *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$ and a sufficiently large i, the $\mathbb{Z}_p[G_K]$-module $T_{A_{\mathrm{inf}}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee)$ is a $\mathbb{Z}_p$-lattice of a semi-stable representation.*

(2) *We assume $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization. Then there is a G_K-equivariant isomorphism*

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^\vee \simeq T_{A_{\mathrm{inf}}}(\pi_i \operatorname{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee).$$

In particular, $\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable representation.

Proof. (1) By Theorem [19, Theorem 7.1.7], we obtain the claim.

(2) Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. We assume $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization. By the same discussion as in the proof of Theorem 2.16, we have an equivalence

$$\begin{aligned} &L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}\left(\operatorname{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]\right) \\ &\quad \to L_{K(1)}\left(\operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]\right). \end{aligned} \tag{3.14}$$

By the functionality of cyclotomic Frobenius $\varphi_-^{hS^1} : \operatorname{TC}^-(-;\mathbb{Z}_p) \to \operatorname{TP}(-;\mathbb{Z}_p)$, we have the following commutative diagram:

$$\begin{tikzcd} L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}\left(\operatorname{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]\right) \arrow[r, "(3.14)", "\simeq"'] \arrow[d, "\mathrm{id}\otimes\varphi^{hS^1}_{\mathcal{O}_{\mathcal{C}}}"'] & L_{K(1)}\left(\operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\xi}\right]\right) \arrow[d, "\varphi^{hS^1}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}"] \\ L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{L_{K(1)}K(\mathcal{C})} L_{K(1)}\left(\operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)\left[\frac{1}{\tilde{\xi}}\right]\right) \arrow[r, "\simeq", "(2.27)"'] & L_{K(1)}\left(\operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\tilde{\xi}}\right]\right) \end{tikzcd}$$

We note $\xi | \mu$ in A_{inf}. Since $\varphi^{hS^1}_{\mathcal{O}_{\mathcal{C}}} : \pi_0 L_{K(1)} \operatorname{TC}^-(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)[\frac{1}{\xi}] \to \pi_0 L_{K(1)} \operatorname{TP}(\mathcal{O}_{\mathcal{C}};\mathbb{Z}_p)[\frac{1}{\tilde{\xi}}]$ is given by the Frobenius $\varphi_{A_{\mathrm{inf}}} : A_{\mathrm{inf}}[\frac{1}{\mu}]^\wedge_p \to A_{\mathrm{inf}}[\frac{1}{\varphi(\mu)}]^\wedge_p$, on homotopy groups, we have G_K-equivariant diagram

$$\begin{tikzcd} \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\frac{1}{\mu}\right]^\wedge_p \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes\varphi_{A_{\mathrm{inf}}}"'] & \pi_i \operatorname{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\mu}\right]^\wedge_p \arrow[d, "\varphi^{hS^1}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}"] \\ \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\frac{1}{\varphi(\mu)}\right]^\wedge_p \arrow[r, "\simeq"'] & \pi_i \operatorname{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\left[\frac{1}{\varphi(\mu)}\right]^\wedge_p \end{tikzcd}$$

After the base change along $A_{\mathrm{inf}}[\frac{1}{\mu}]^{\wedge}_p \to W(\mathcal{C}^\flat)$, the above diagram becomes the following G_K-equivariant diagram:

$$\begin{tikzcd}
\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} W(\mathcal{C}^\flat) \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes\varphi_{\mathcal{C}}"'] & \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat) \arrow[d, "\varphi^{hS^1}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}\otimes\varphi_{\mathcal{C}}"] \arrow[dr, "\widehat{\varphi}\otimes\varphi_{\mathcal{C}}"] & \\
\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} W(\mathcal{C}^\flat) \arrow[r, "\simeq"'] & \pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat) & \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat) \arrow[l, "\mathrm{can}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}"', "\simeq"]
\end{tikzcd} \tag{3.15}$$

Thus the isomorphism $\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} W(\mathcal{C}^\flat) \simeq \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat)$ is Frobenius equivariant. Since $\mathbb{Z}_p \to A_{\mathrm{inf}}$ and $A_{\mathrm{inf}} \to W(\mathcal{C}^\flat)$ are flat and compatible with Frobenius automorphisms, the dual of the isomorphism becomes the Frobenius-equivariant isomorphism

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^\vee \otimes_{\mathbb{Z}_p} W(\mathcal{C}^\flat) \simeq \pi_i\,\mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee \otimes_{A_{\mathrm{inf}}} W(\mathcal{C}^\flat),$$

and we obtain the claim. ∎

Remark 3.8. Similarly, we can obtain that there is an isomorphism of $\mathbb{Z}_p[G_K]$-modules

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^{\vee\vee} \simeq T_{A_{\mathrm{inf}}}(\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee\vee}).$$

4. $(\varphi, \widehat{G})$-modules and crystalline representations

In this section, we will prove Theorem 1.12. We would like to thank H. Gao [20] for sharing the strategy and key constructions to prove Theorem 1.12.

4.1. Breuil–Kisin G_K-modules and $(\varphi, \widehat{G})$-modules

Let us recall $(\varphi, \widehat{G})$-module from [14]. Let $L := \bigcup_{n=1}^\infty K_\infty(\zeta_n)$, $\widehat{G} := \mathrm{Gal}(L/K)$ and $H_K := \mathrm{Gal}(L/K_\infty)$. Du–Liu construct $\mathfrak{S}$-algebras $\mathfrak{S}^{(2)}$ and $\mathfrak{S}^{(2)}_{\mathrm{st}}$ as the following [14, Sections 2.1 and 2.3]. We set $\mathfrak{S}^{\widehat{\otimes}2} := \mathfrak{S}[\![y-z]\!] = W[\![z, y-z]\!]$. Then $\mathfrak{S}^{\widehat{\otimes}2}$ is $\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathfrak{S}$-algebra via $z\otimes 1 \mapsto z$, $1\otimes z \mapsto y = (y-z)+z$; in this way, one can extend Frobenius action on $\mathfrak{S}$ to $\mathfrak{S}^{\widehat{\otimes}2}$ which is Frobenius on W and sends z to z^p and $y-z$ to $y^p - z^p$. Let $\mathfrak{S}^{(2)}$ be $\mathfrak{S}[\![y-z]\!]\{\frac{y-z}{E}\}^{\wedge}_{\delta}$, where $\{-\}^{\wedge}_{\delta}$ means freely adjoining elements in the category of (p, E)-completed δ-$\mathfrak{S}$-algebras [14, Sections 2.1 and 4.1]. Then $\mathfrak{S}^{(2)}$ is an $\mathfrak{S}$-algebra via $u \mapsto u$. The structure map is flat (see [14, Proposition 2.2.7]), and $\mathfrak{S}^{(2)}$ is a sub-$\mathfrak{S}$-algebra of A_{inf}, where we regard A_{inf} as an $\mathfrak{S}$-algebra via ϕ (see [14, Section 2.4]). Let us recall the definition of $\mathfrak{S}^{(2)}_{\mathrm{st}}$ from [14, Section 2.3]. Define a Frobenius action on $W[\![z, \mathfrak{y}]\!]$ which sends x to x^p and $\mathfrak{y}$ to $(1+\mathfrak{y})^p - 1$ and set $\mathfrak{w} = \frac{\mathfrak{y}}{E}$. Let $\mathfrak{S}^{(2)}_{\mathrm{st}} = \mathfrak{S}[\![z, \mathfrak{y}]\!]\{\mathfrak{w}\}^{\wedge}_{\delta}$, and it is a sub-$\mathfrak{S}$-algebra of A_{inf}, where we regard A_{inf} as an $\mathfrak{S}$-algebra via ϕ (see [14, Section 2.4]). There is a natural inclusion of $\mathfrak{S}$-algebra $W[\![z, y-z]\!] \subset W[\![z, \mathfrak{y}]\!]$ which sends y to $z(\mathfrak{y}+1)$; it is a δ-ring map. By construction, we thus obtain an inclusion of sub-$\mathfrak{S}$-algebras $\mathfrak{S}^{(2)} \hookrightarrow \mathfrak{S}^{(2)}_{\mathrm{st}}$ of A_{inf}. Let us denote $\theta_0 : \mathfrak{S} \to \mathfrak{S}^{(2)} \mid z \mapsto z$, and denote $\theta_1 : \mathfrak{S} \to \mathfrak{S}^{(2)} \mid z \mapsto y$. We note that there is a commutative diagram (see [14, Corollary 2.4.5])

$$\begin{tikzcd}
\mathfrak{S} \arrow[d, hook, "\theta_0"'] \arrow[rrd, hook, "\phi"] & & \\
\mathfrak{S}^{(2)} \arrow[r, hook] \arrow[rr, bend right, "\iota"'] & \mathfrak{S}^{(2)}_{\mathrm{st}} \arrow[r, hook] & A_{\mathrm{inf}}
\end{tikzcd} \tag{4.1}$$

where ι sends z to $[\pi^\flat]$ and y to $[\varepsilon] \cdot [\pi^\flat]$. We regard $\mathfrak{S}^{(2)}$ and $\mathfrak{S}^{(2)}_{\mathrm{st}}$ as $\mathfrak{S}$-algebras via the diagram unless otherwise stated.

Definition 4.1 ([14, Definition 3.3.2]). Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a finite free Breuil–Kisin module of finite E-height. We call $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ $(\varphi, \hat{G})$-module if it satisfies the following conditions:

(1) There is a continuous $\mathfrak{S}^{(2)}_{\mathrm{st}}$-semi-linear $\hat{G}$-action on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}}$.

(2) $\hat{G}$ commutes with φ on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}}$.

(3) $\mathfrak{M} \subset (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}})^{H_K}$.

(4) $\hat{G}$ acts on $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}}) \otimes_{\mathfrak{S}^{(2)}_{\mathrm{st}}} W(k)$ trivially.

Remark 4.2. For a Breuil–Kisin G_K-module $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K \curvearrowright \mathfrak{M} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}})$, the p-adic representation $T_{A_{\mathrm{inf}}}(\mathfrak{M})$ is a $\mathbb{Z}_p$-lattice of semi-stable representation of non-negative Hodge–Tate weight [19, Theorem 7.1.7]. Du–Liu proved that the sub-modules $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}} \subset \mathfrak{M} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \subset T_{A_{\mathrm{inf}}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ are G_K-stable [14, Theorem 3.3.3] (see also [30, Proposition 3.1.3]). By Definition 3.1 (3), $G_L = \mathrm{Gal}(\overline{K}/L) (\subset G_{K_\infty})$ acts on $\mathfrak{M} \subset \mathfrak{M} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$ trivially, thus G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}}$ factors through $\hat{G}$. The $\hat{G}$-action on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)}_{\mathrm{st}}$ induces the $(\varphi, \hat{G})$-module structure on $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ [14, Theorem 3.3.3].

4.2. The comparison theorem between $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ and $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$

We regard $\mathcal{O}_K$ as an $\mathbb{S}[z_0, z_1]$-algebra via the map $z_0, z_1 \mapsto \pi$. In [29], Liu–Wang revealed the structure of $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$: there is an isomorphism

$$\pi_0 \, \mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \simeq \mathfrak{S}^{(2)},$$

where we identify $z_0 = z$ and $z_1 = y$ (see [29, Theorem 1.3]). In this section, we will compare $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$ with $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$ for an $\mathcal{O}_K$-linear category $\mathcal{T}$. We note that the functor

$$\mathrm{TP}(-/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) : \mathrm{Cat}^{\mathrm{perf}}_\infty(\mathcal{O}_K) \xrightarrow{\mathrm{THH}(-/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)} \mathrm{Sp}^{BS^1} \xrightarrow{(-)^{tS^1}} \mathrm{Sp}$$

is a lax symmetric monoidal; $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$ admits an $\mathbb{E}_\infty$-ring structure, and $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$ also admits an $\mathbb{E}_\infty$-ring structure. For each $i = 0, 1$, we consider the map of ring spectra $\mathbb{S}[z] \xrightarrow{z\mapsto z_i} \mathbb{S}[z_0, z_1]$, which in turn induces a morphism of $\mathbb{E}_\infty$-ring spectra

$$e_i : \mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{z\mapsto z_i} \mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p). \tag{4.2}$$

For $\heartsuit \in \{\mathrm{THH}, \mathrm{TP}, \mathrm{TC}^-\}$ and an $\mathcal{O}_K$-linear category $\mathcal{T}$, the left unit $e_{0,\mathcal{T}}$ and right unit $e_{1,\mathcal{T}}$ are the maps

$$\heartsuit(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \to \heartsuit(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$$

induced by $z \mapsto z_0$ and $z \mapsto z_1$, respectively. Let us denote by $\mathrm{THH}(\mathbb{S}[z_0, z_1]/_0\mathbb{S}[z])$ (resp. $\mathrm{THH}(\mathbb{S}[z_0, z_1]/_1\mathbb{S}[z])$) the relative topological Hochschild homology of $\mathbb{S}[z] \to \mathbb{S}[z_0, z_1] \mid z \to z_0$ (resp. $\mathbb{S}[z] \to \mathbb{S}[z_0, z_1] \mid z \to z_1$). There is a commutative diagram in $\mathrm{Mod}\text{–}_{\mathrm{THH}(\mathbb{S}[z])}(\mathrm{Sp}^{BS^1})$

$$\begin{tikzcd}
 & \mathbb{S}[z_0, z_1]^\wedge_p \arrow[r] & \mathrm{THH}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \\
\mathbb{S}[z]^\wedge_p \arrow[r, "z\mapsto z_i"] & \mathrm{THH}(\mathbb{S}[z_0, z_1]/_i\mathbb{S}[z]; \mathbb{Z}_p) \arrow[u] \arrow[r] & \mathrm{THH}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \arrow[u, "e_{i,\mathcal{T}}"'] \\
\mathrm{THH}(\mathbb{S}[z]; \mathbb{Z}_p) \arrow[u] \arrow[r, "e_i"'] & \mathrm{THH}(\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \arrow[u] \arrow[r] & \mathrm{THH}(\mathcal{T}; \mathbb{Z}_p) \arrow[u]
\end{tikzcd}$$

Since the bottom-left square and the bottom square are pushout squares (see [29, Lemma 2.3]), and similarly, since the right square is a pushout square, the top-right square is also a pushout square. Thus, by the transitivity property of relative THH, we obtain the following equivalence:

$$\mathrm{THH}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \simeq \mathrm{THH}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{S}[z_0,z_1]/_i\mathbb{S}[z];\mathbb{Z}_p)} \mathbb{S}[z_0, z_1]^\wedge_p. \tag{4.3}$$

We also have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{THH}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \arrow[r, "e_{i,\mathcal{T}}"] & \mathrm{THH}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \\
\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z]; \mathbb{Z}_p) \arrow[u] \arrow[r, "e_i"] & \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \arrow[u] \\
\mathrm{THH}(\mathbb{S}[z_0, z_1]/_i\mathbb{S}[z]; \mathbb{Z}_p) \arrow[u] \arrow[r] & \mathbb{S}[z_0, z_1]^\wedge_p \arrow[u]
\end{tikzcd}$$

Applying $\mathcal{T} = \mathrm{perf}(\mathcal{O}_K)$ to the equivalence (4.3), we see that the bottom and total squares are pushout squares. We know that the top square is a pushout square. Thus we have the following S^1-equivariant equivalence:

$$\mathrm{THH}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \simeq \mathrm{THH}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p). \tag{4.4}$$

Notice an exact symmetric monoidal ∞-functor

$$- \bigotimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p),e_i} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$$

from $\mathrm{Mod}_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$ to $\mathrm{Mod}_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. In the case where the $\mathcal{O}_K$-linear category $\mathcal{T}$ is smooth and proper, it follows that $\mathrm{THH}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is dualizable and perfect in $\mathrm{Mod}_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$. Since the base change functor

$$- \bigotimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$$

is an exact symmetric monoidal functor, we consequently obtain the following lemma:

Lemma 4.3. *For a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$, $\mathrm{THH}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$ is dualizable and perfect in $\mathrm{Mod}_{-\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)}(\mathrm{Sp}^{BS^1})$.*

Proposition 4.4. *Let $\mathcal{T}_1, \mathcal{T}_2$ be smooth proper $\mathcal{O}_K$-linear categories. There are equivalences*

$$\begin{aligned}
&\mathrm{TC}^-(\mathcal{T}_1/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \otimes_{\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)} \mathrm{TC}^-(\mathcal{T}_2/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\\
&\qquad \simeq \mathrm{TC}^-(\mathcal{T}_1 \otimes_{\mathcal{O}_K} \mathcal{T}_2/\mathbb{S}[z_0,z_1];\mathbb{Z}_p),\\
&\mathrm{TP}(\mathcal{T}_1/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \otimes_{\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)} \mathrm{TP}(\mathcal{T}_2/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\\
&\qquad \simeq \mathrm{TP}(\mathcal{T}_1 \otimes_{\mathcal{O}_K} \mathcal{T}_2/\mathbb{S}[z_0,z_1];\mathbb{Z}_p).
\end{aligned}$$

Proof. By Lemma 4.3, one can prove using the same argument as in the proof of Proposition 2.6. ∎

Proposition 4.5. *Let $\mathcal{T}_1, \mathcal{T}_2$ be smooth proper $\mathcal{O}_K$-linear categories. There are equivalences*

$$\begin{aligned}
&\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p),e_i} \mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\\
&\qquad \simeq \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\\
&\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p),e_i} \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\\
&\qquad \simeq \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)
\end{aligned}$$

for $i = 0, 1$.

Proof. For $i = 0, 1$ and $\heartsuit \in \{\mathrm{TC}^-, \mathrm{TP}\}$, there is a commutative diagram

$$\begin{tikzcd}
\heartsuit(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[r, "e_{i,\mathcal{T}}"] \arrow[d] & \heartsuit(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \arrow[d] \\
\heartsuit(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p) \arrow[r, "e_i"'] & \heartsuit(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)
\end{tikzcd}$$

This diagram induces a morphism of $\heartsuit(\mathcal{O}_K/\mathbb{S}[z_0,z_1])$-module spectra

$$\heartsuit(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\heartsuit(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p),e_i}\heartsuit(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\to\heartsuit(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p). \tag{4.5}$$

By Propositions 2.6 and 4.4, both sides of the map (4.5) yield symmetric monoidal functors from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}(\mathcal{O}_K)$ to $\mathrm{Mod}_{-\heartsuit(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)}(\mathrm{Sp})$. Since every smooth proper $\mathcal{O}_K$-linear category is dualizable in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\mathrm{sat}}$, by [2, Proposition 4.6] we obtain the claim. ∎

There is an equivalence of $\mathbb{E}_\infty$-ring spectra $\mathrm{ex}:\mathbb{S}[z_0,z_1]\xrightarrow{\simeq}\mathbb{S}[z_0,z_1]\mid z_0\mapsto z_1, z_1\mapsto z_0$, and it induces a commutative diagram

$$\begin{tikzcd}
\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p) \arrow[d, "e_0"'] \arrow[dr, "e_1"] & \\
\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \arrow[r, "\mathrm{ex}"'] & \mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)
\end{tikzcd}$$

and an isomorphism of $\mathfrak{S}$-algebra (see also [14, Section 4])

$$\mathfrak{S}^{(2)}\otimes_{\theta_0,\mathfrak{S}}\mathfrak{S}\simeq\mathfrak{S}^{(2)}\otimes_{\theta_1,\mathfrak{S}}\mathfrak{S}. \tag{4.6}$$

By [29, Corollary 3.7], we see that $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$ is 2-periodic and there is an isomorphism $\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)=\mathfrak{S}^{(2)}[\sigma,\sigma^{-1}]$ by choosing a generator $\sigma\in\pi_2\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$. For $i=0,1$, on homotopy groups, the morphism

$$\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z];\mathbb{Z}_p)\xrightarrow{e_i}\pi_*\,\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$$

is given by

$$\mathfrak{S}[u,u^{-1}]\to\mathfrak{S}^{(2)}[\sigma,\sigma^{-1}] \tag{4.7}$$

which is a θ_i-linear map sending u to $a_i\cdot\sigma$ for some units $a_i\in(\mathfrak{S}^{(2)})^*$.

Proposition 4.6. *There is an isomorphism*

$$\pi_j\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathfrak{S},\theta_i}\mathfrak{S}^{(2)}\simeq\pi_j\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \tag{4.8}$$

for any j and $i=0,1$.

Proof. If $i=0$, by [14, Proposition 2.2.7] the graded-ring morphism (4.7) is flat. We obtain the claim by Proposition 4.5. By the isomorphism (4.6), θ_1 is also flat. Thus the morphism (4.7) is also flat for $i=1$. ∎

4.3. The comparison theorem between $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$ and $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)$

We regard $\mathcal{O}_{\mathcal{C}}$ as an $\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}]$-algebra via the map $z_0^{1/p^n}\mapsto \pi^{1/p^n}$, $z_1^{1/p^n}\mapsto \zeta_n\pi^{1/p^n}$. We will prove the comparison theorem between $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$ and $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)$ for a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$. The following lemma is inspired by H. Gao in [20].

Lemma 4.7. *The natural map* $\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\to\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)$ *is an equivalence which is compatible with S^1-action. In particular, the natural map*

$$\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\to\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)$$

is an equivalence.

Proof. There is a diagram

$$\begin{aligned}\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)&\to\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)\\&\simeq\mathrm{THH}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)}\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}]^\wedge_p\end{aligned}$$

which is compatible with S^1-action. For $i=0,1$, we have an S^1-equivariant equivalence $\mathrm{THH}(\mathbb{S}[z_i^{1/p^\infty}];\mathbb{Z}_p)\simeq\mathbb{S}[z_i^{1/p^\infty}]^\wedge_p$ [8, Proposition 11.7], and we obtain the claim. ■

Lemma 4.8. *The natural map* $\mathbb{S}[z_0,z_1]\to\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}]$ *induces an isomorphism*

$$\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p)\simeq\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)\otimes_{\mathfrak{S}^{(2)},\iota}A_{\mathrm{inf}}$$

for any i.

Proof. The morphism of $\mathbb{E}_\infty$-ring spectra $e_0:\mathbb{S}[z]\to\mathbb{S}[z_0,z_1]$ naturally induces an isomorphism $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathcal{S},\theta_0}\mathcal{S}_{z_0,z_1}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p)$. Similarly, the composite map $\mathbb{S}[z]\xrightarrow{e_0}\mathbb{S}[z_0,z_1]\to\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}]$ also induces an isomorphism

$$\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathfrak{S},\phi}A_{\mathrm{inf}}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}];\mathbb{Z}_p).$$

Since there is an equation of morphisms $\phi=\iota\circ\theta_0$, we obtain the claim. ■

4.4. τ-action on $\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)$

The construction in this section is inspired by H. Gao [20]. Let $\tau\in\mathrm{Gal}(L/\bigcup_{n=1}^\infty K(\zeta_n))$ be a topological generator and $\tilde{\tau}\in G_K$ be the lift of τ satisfying $\tilde{\tau}(\pi^{1/p^n})=\zeta_n\pi^{1/p^n}$. Consider a map $\eta:\mathbb{S}[z_0^{1/p^\infty},z_1^{1/p^\infty}]\to\mathcal{O}_{\mathcal{C}}$ which sends z_0^{1/p^n} to π^{1/p^n} and z_1^{1/p^n} to $\zeta_n\pi^{1/p^n}$.

We will use the following notations associated with a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$:

- $e_0^{\infty} : \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^{\infty}}];\mathbb{Z}_p) \xrightarrow{\simeq} \mathrm{TP}\big(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big];\mathbb{Z}_p\big)$ the morphism given by $z^{1/p^{\infty}} \mapsto z_0^{1/p^{\infty}}$
- $a : \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^{\infty}}];\mathbb{Z}_p)$ the morphism given by the commutative diagram

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] & \mathcal{O}_{\mathcal{C}} \\ \mathbb{S}[z] \arrow[u] \arrow[r] & \mathbb{S}[z^{1/p^{\infty}}] \arrow[u] \end{tikzcd}$$

- $b : \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{TP}\big(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big];\mathbb{Z}_p\big)$ the morphism given by commutative diagram

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] & \mathcal{O}_{\mathcal{C}} \\ \mathbb{S}[z] \arrow[u] \arrow[r, "z\mapsto z_0"'] & \mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big] \arrow[u, "\eta"'] \end{tikzcd}$$

- $c : \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1];\mathbb{Z}_p) \to \mathrm{TP}\big(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big];\mathbb{Z}_p\big)$ the morphism given by commutative diagram

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] & \mathcal{O}_{\mathcal{C}} \\ \mathbb{S}[z_0, z_1] \arrow[u] \arrow[r, "z_i\mapsto z_i"'] & \mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big] \arrow[u, "\eta"'] \end{tikzcd}$$

- $\tilde{\tau}_1 : \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^{\infty}}];\mathbb{Z}_p) \to \mathrm{TP}\big(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big];\mathbb{Z}_p\big)$ the morphism given by the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\mathcal{C}} \arrow[r, "\tilde{\tau}"] & \mathcal{O}_{\mathcal{C}} \\ \mathbb{S}[z^{1/p^{\infty}}] \arrow[u] \arrow[r, "z^{1/p^n}\mapsto z_1^{1/p^n}"'] & \mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big] \arrow[u] \end{tikzcd} \tag{4.9}$$

Lemma 4.9. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p) \arrow[r, "\tilde{\tau}", "\simeq"'] \arrow[d, "\simeq"] & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p) \arrow[d, "\simeq"'] \\ \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^{\infty}}];\mathbb{Z}_p) \arrow[r, "\tilde{\tau}_1"'] & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}\big[z_0^{1/p^{\infty}}, z_1^{1/p^{\infty}}\big];\mathbb{Z}_p) \end{tikzcd}$$

Proof. Since diagram (4.9) is compatible with $\mathbb{S}$-ring spectrum structure, we obtain the claim. ∎

The following theorem, inspired by H. Gao [20], plays the most important role in the proof of Theorem 1.12.

Proposition 4.10. *The following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{E}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \arrow[r, "\tilde{\tau}_1", "\simeq"'] & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{E}}}/\mathbb{S}[z_0^{1/p^\infty}, z_1^{1/p^\infty}];\mathbb{Z}_p) & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{E}}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \arrow[l, "e_0^\infty"', "\simeq"] \\
\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[u, "a"] \arrow[r, "e_{1,\mathcal{T}}"'] & \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1];\mathbb{Z}_p) \arrow[u, "c"'] & \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[u, "a"] \arrow[l, "e_{0,\mathcal{T}}"]
\end{tikzcd}
\tag{4.10}
$$

Proof. Diagram (4.9) fits into the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{O}_{\mathcal{E}} \arrow[rr, "\tilde{\tau}"] & & \mathcal{O}_{\mathcal{E}} & \\
 & \mathbb{S}[z^{1/p^\infty}] \arrow[ul] \arrow[rr, "z^{1/p^\infty} \mapsto z_1^{1/p^\infty}"] & & \mathbb{S}\big[z_0^{1/p^\infty}, z_1^{1/p^\infty}\big] \arrow[ul, "\eta"'] \\
\mathcal{O}_K \arrow[uu] \arrow[rr, "\mathrm{id}_{\mathcal{O}_K}"] & & \mathcal{O}_K \arrow[uu] & \\
 & \mathbb{S}[z] \arrow[ul] \arrow[uu] \arrow[rr, "z \mapsto z_1"'] & & \mathbb{S}[z_0, z_1] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$

Thus we see that the left square of (4.10) commutes. Look at the following diagram:

$$
\begin{tikzcd}
\mathcal{O}_{\mathcal{E}} \arrow[rr, "\mathrm{id}_{\mathcal{O}_{\mathcal{E}}}"] & & \mathcal{O}_{\mathcal{E}} & \\
 & \mathbb{S}[z^{1/p^\infty}] \arrow[ul] \arrow[rr, "z^{1/p^\infty} \mapsto z_0^{1/p^\infty}"] & & \mathbb{S}\big[z_0^{1/p^\infty}, z_1^{1/p^\infty}\big] \arrow[ul, "\eta"'] \\
\mathcal{O}_K \arrow[uu] \arrow[rr, "\mathrm{id}_{\mathcal{O}_K}"] & & \mathcal{O}_K \arrow[uu] & \\
 & \mathbb{S}[z] \arrow[ul] \arrow[uu] \arrow[rr, "z \mapsto z_0"'] & & \mathbb{S}[z_0, z_1] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$

then we see that the right square of (4.10) commutes. ∎

Fix an integer $i \gg 0$. By Proposition 2.6, $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ is a finitely generated $\mathfrak{S}$-module. Choose an $\mathfrak{S}$-generator $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ of $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$. We have an equation $\overline{\phi} = \phi \circ \varphi : \mathfrak{S} \to A_{\mathrm{inf}}$ (see diagram (2.2)); combining isomorphism (2.9) with Theorem 2.13, we see that a induces an isomorphism $\pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \simeq \pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}]; \mathbb{Z}_p)$. For any element $x \in \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$, let us write $\overline{x}$ for the image of x under the sequence of maps $\pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{a} \pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathbb{C}}/\mathbb{S}[z^{1/p^\infty}]; \mathbb{Z}_p) \xleftarrow{\simeq} \pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathbb{C}}; \mathbb{Z}_p)$. Then it follows that the elements $\overline{\mathfrak{m}_1}, \overline{\mathfrak{m}_2}, \dots, \overline{\mathfrak{m}_n}$ form a set of $\mathbb{A}_{\mathrm{inf}}$-generators for $\pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathbb{C}}; \mathbb{Z}_p)$. Choose elements $a_{x1}, a_{x2}, \dots, a_{xn}$ of A_{inf} so that

$$\widetilde{\tau}(\overline{x}) = a_{x1}\overline{\mathfrak{m}_1} + a_{x2}\overline{\mathfrak{m}_2} + \cdots + a_{xn}\overline{\mathfrak{m}_n}.$$

Proposition 4.11. *For any x, any choice of $\mathcal{S}$-generators $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, and any choice of elements $a_{xl} \in \mathbb{A}_{\mathrm{inf}}$, it follows that a_{xl} is contained in $\mathcal{S}_{z_0,z_1}$ via the inclusion $\iota : \mathcal{S}_{z_0,z_1} \hookrightarrow \mathbb{A}_{\mathrm{inf}}$.*

Proof. Combining Proposition 4.10 with Lemma 4.9, we see that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) \arrow[r, "\widetilde{\tau}", "\simeq"'] \arrow[d, "\simeq"] & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}};\mathbb{Z}_p) \arrow[d, "\simeq"'] \arrow[dr, "\simeq"] & \\
\mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \arrow[r, "\widetilde{\tau}_1", "\simeq"'] & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z_0^{1/p^\infty}, z_1^{1/p^\infty}];\mathbb{Z}_p) & \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z^{1/p^\infty}];\mathbb{Z}_p) \arrow[l, "e_0^\infty"', "\simeq"] \\
\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[u, "a"] \arrow[r, "e_{1,\mathcal{T}}"'] & \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \arrow[u, "c"'] & \mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \arrow[u, "a"] \arrow[l, "e_{0,\mathcal{T}}"]
\end{tikzcd} \tag{4.11}$$

We note that the map $\pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}; \mathbb{Z}_p) \to \pi_i\, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_\mathcal{C}}/\mathbb{S}[z_0^{1/p^\infty}, z_1^{1/p^\infty}]; \mathbb{Z}_p)$ is $\mathbb{A}_{\mathrm{inf}}$-linear. According to the results in Proposition 4.6 and Lemma 4.8, the morphism $c : \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p) \to \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}\big[z_0^{1/p^\infty}, z_1^{1/p^\infty}\big]; \mathbb{Z}_p)$ is explicitly given by the map

$$\pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathcal{S},\theta_0} \mathcal{S}_{z_0,z_1} \xrightarrow{\mathrm{id}\otimes\iota} \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathcal{S},\phi} \mathbb{A}_{\mathrm{inf}}.$$

By Proposition 4.6, the morphism $e_{0,\mathcal{T}} : \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \to \mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$ is given by the natural map $\pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \to \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathcal{S},\theta_0} \mathcal{S}_{z_0,z_1}$, and consequently, the set $\{e_{0,\mathcal{T}}(\mathfrak{m}_1), e_{0,\mathcal{T}}(\mathfrak{m}_2), \dots, e_{0,\mathcal{T}}(\mathfrak{m}_n)\}$ forms an $\mathcal{S}_{z_0,z_1}$-generator set of $\mathrm{TP}(\mathcal{T}/\mathbb{S}[z_0, z_1]; \mathbb{Z}_p)$. Therefore, for $x \in \pi_i\, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)$, there are $a'_{x1}, a'_{x2}, \dots, a'_{xn} \in \mathfrak{S}^{(2)} \overset{\iota}{\hookrightarrow} A_{\mathrm{inf}}$ such that

$$e_{1,\mathcal{T}}(x) = \Sigma_{l=1}^n a'_{xl} e_{0,\mathcal{T}}(\mathfrak{m}_l).$$

Look at the diagram, then we see an equation $\widetilde{\tau}_1(a(x)) = \Sigma a'_{xl} c(e_{0,\mathcal{T}}(\mathfrak{m}_l))$, and we obtain the claim. ∎

Remark 4.12. We can apply the same procedure to $\widetilde{\tau}^{-1}$, and we obtain that there are elements b_{xj} of $\mathfrak{S}^{(2)}$ so that $\widetilde{\tau}^{-1}(\overline{x}) = b_{x1}\overline{\mathfrak{m}_1} + b_{x2}\overline{\mathfrak{m}_2} + \cdots + b_{xn}\overline{\mathfrak{m}_n}$, where instead of η we use a morphism $\mathbb{S}\big[z_0^{1/p^\infty}, z_1^{1/p^\infty}\big] \to \mathcal{O}_\mathcal{C}$ which sends ${z_0}^{1/p^n}$ to π^{1/p^n} and z_1^{1/p^n} to $\zeta_n^{-1}\pi^{1/p^n}$.

Let $\tilde{\tau}^*$ denote the dual action of $\tilde{\tau}$ on the dual module $\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathbb{C}}};\mathbb{Z}_p)^\vee$, which is defined as $\mathrm{Hom}_{\mathbb{A}_{\mathrm{inf}}}(\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathbb{C}}};\mathbb{Z}_p),\mathbb{A}_{\mathrm{inf}})$. Choose an $\mathfrak{S}$-basis $\mathfrak{n}_1,\mathfrak{n}_2,\ldots,\mathfrak{n}_d$ of $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$. Combining (2.9) with Theorem 2.13, we obtain an isomorphism $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee\otimes_{\mathfrak{S},\phi}A_{\mathrm{inf}}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}};\mathbb{Z}_p)^\vee$. For any element $y\in\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathbb{C}}};\mathbb{Z}_p)^\vee$, let us denote by $\overline{y}$ the image of y under the natural inclusion $\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee\hookrightarrow\pi_i\,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee\otimes_{\mathcal{S},\phi}\mathbb{A}_{\mathrm{inf}}\simeq\pi_i\,\mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathbb{C}}};\mathbb{Z}_p)^\vee$. Choose elements $b_{y1},b_{y2},\ldots,b_{yd}$ of A_{inf} so that

$$\tilde{\tau}^*(\overline{y})=b_{y1}\overline{\mathfrak{n}_l}+b_{y2}\overline{\mathfrak{n}_2}+\cdots+b_{yd}\overline{\mathfrak{n}_d}.$$

Remark 4.12 directly induces the following.

Corollary 4.13. *For any y, $b_{y1},b_{y2},\ldots,b_{yd}$ are contained in $\mathfrak{S}^{(2)}$ under the inclusion $\mathfrak{S}^{(2)}\overset{\iota}{\hookrightarrow}A_{\mathrm{inf}}$.*

4.5. $\tilde{\tau}$-action and crystalline representations

Let $\phi_1:\mathfrak{S}\to A_{\mathrm{inf}}$ be a W-linear map which sends z to $[\varepsilon][\pi^\flat]$. The following diagram commutes:

$$\begin{tikzcd}
A_{\mathrm{inf}} \arrow[r, "\tilde{\tau}"] & A_{\mathrm{inf}} \\
\mathfrak{S} \arrow[u, "\phi"] \arrow[ur, "\phi_1"] \arrow[r, "\theta_1"'] & \mathfrak{S}^{(2)} \arrow[u, "\iota"']
\end{tikzcd}$$

Let E_1 denote $\theta_1(E)$, E_0 denote $\theta_0(E)$ and ξ_1 denote $\tilde{\tau}(\xi)$. We note that $\mathfrak{S}^{(2)}$ is an integral domain (see [14, Proof of Lemma 2.3.2]), and $\iota(E_1)=\xi_1$. Firstly, we prove the following lemma.

Lemma 4.14. *As sub-rings of $A_{\mathrm{inf}}[\frac{1}{\xi_1}]$, there is an equation*

$$\mathfrak{S}^{(2)}=\mathfrak{S}^{(2)}\left[\frac{1}{E_1}\right]\cap A_{\mathrm{inf}}$$

of sub-algebra $A_{\mathrm{inf}}[\frac{1}{\xi_1}]$, where we regard $\mathfrak{S}^{(2)}[\frac{1}{E_1}]$ as a sub-ring of $A_{\mathrm{inf}}[\frac{1}{\xi_1}]$ via $\iota[\frac{1}{E_1}]:\mathfrak{S}^{(2)}[\frac{1}{E_1}]\hookrightarrow A_{\mathrm{inf}}[\frac{1}{\xi_1}]$.

Proof. It suffices to show that $\mathfrak{S}^{(2)}$ contains $\mathfrak{S}^{(2)}[\frac{1}{E_1}]\cap A_{\mathrm{inf}}$. The map ι induces a morphism of W-algebra $\bar{\iota}:\mathfrak{S}^{(2)}/(E_1)\to A_{\mathrm{inf}}/(\xi_1)$ and θ_1 induces a morphism of W-algebra $\overline{\theta_1}:\mathfrak{S}/(E)\to\mathfrak{S}^{(2)}/(E_1)$. The isomorphism (4.6) induces an isomorphism of $\mathfrak{S}$-algebra $\mathfrak{S}\otimes_{\mathfrak{S},\theta_0}\mathfrak{S}^{(2)}\simeq\mathfrak{S}\otimes_{\mathfrak{S},\theta_1}\mathfrak{S}^{(2)}$ which sends E_0 to E_1. By [14, Lemma 2.2.8 (2)], θ_0 induces an isomorphism $\mathcal{O}_K\simeq\mathfrak{S}/(E)\simeq\mathfrak{S}^{(2)}/(E_0)\simeq\mathfrak{S}^{(2)}/(E_1)$ of W-algebras. Thus we obtain that $\bar{\iota}$ is given by natural W-algebra $\mathcal{O}_K\hookrightarrow\mathcal{O}_{\mathcal{C}}$. We assume that there exists an element x of $\mathfrak{S}^{(2)}[\frac{1}{E_1}]\cap A_{\mathrm{inf}}\subset A_{\mathrm{inf}}[\frac{1}{\xi_1}]$ such that x is not contained in $\mathfrak{S}^{(2)}$. From the fact that $x\in\mathcal{S}_{z_0,z_1}[\frac{1}{E_1}]$, it follows that, there exists a natural number $n\geq 1$ such that $E_1^n\cdot x$ is in $\mathfrak{S}^{(2)}$ and $E_1^{n-1} \cdot x$ is not in $\mathfrak{S}^{(2)}$. Since $\mathfrak{S}^{(2)}$ is an integral domain, $E_1^n \cdot x \notin (E_1) \subset \mathfrak{S}^{(2)}$. Thus the class $\overline{E_1^n \cdot x}$ is not zero in $\mathfrak{S}^{(2)}/(E_1)$. On the other hand, since $x \in A_{\mathrm{inf}}$, $E_1^n \cdot x \in (\xi_1) \subset A_{\mathrm{inf}}$. Thus the class $E_1^n \cdot x$ is zero in $A_{\mathrm{inf}}/(\xi_1)$. This is contradictory to the fact that $\bar{\iota} : \mathfrak{S}^{(2)}/(E_1) \to A_{\mathrm{inf}}/(\xi_1)$ is injective. ■

Fix a smooth proper $\mathcal{O}_K$-linear category $\mathcal{T}$ and an integer $i \gg 0$. The finite free $\mathfrak{S}$-module $\pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee$ carries a natural $(\varphi, \widehat{G})$-module structure by Theorem 3.6 and Remark 4.2. By functoriality of $\mathrm{can}_{(-)}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
\pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \arrow[r, "(3.12)", "\simeq"'] & \pi_i \, \mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}]; \mathbb{Z}_p)^\vee & \pi_i \, \mathrm{TC}^-(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)^\vee \arrow[l, "\simeq"'] \\
\pi_i \, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \arrow[r, "\simeq"] \arrow[u, "\mathrm{can}_{\mathcal{T}}^\vee \otimes \mathrm{id}_{A_{\mathrm{inf}}}"'] & \pi_i \, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}]; \mathbb{Z}_p)^\vee \arrow[u, "\mathrm{can}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}/\mathbb{S}[z^{1/p^\infty}]}^\vee"'] & \pi_i \, \mathrm{TP}(\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}; \mathbb{Z}_p)^\vee \arrow[l, "\simeq"'] \arrow[u, "\mathrm{can}_{\mathcal{T}_{\mathcal{O}_{\mathcal{C}}}}^\vee"']
\end{tikzcd}$$

Hence,

$$\begin{aligned} \mathrm{can}_{\mathcal{T}}^\vee \otimes \mathrm{id}_{A_{\mathrm{inf}}} \colon & \pi_i \, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \\ & \longrightarrow \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \end{aligned} \tag{4.12}$$

is a G_K-equivariant morphism.

We obtain the following commutative diagram:

$$\begin{tikzcd}
\pi_i \, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \arrow[r, "\mathrm{can}_{\mathcal{T}}^\vee"] \arrow[d] & \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \arrow[d] \\
\pi_i \, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \arrow[r, "\mathrm{can}_{\mathcal{T}}^\vee \otimes \mathrm{id}_{A_{\mathrm{inf}}}"] \arrow[d, "\bar{\tau}^*"'] & \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \arrow[d, "\bar{\tau}^*"'] \\
\pi_i \, \mathrm{TP}(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}} \arrow[r, "\mathrm{can}_{\mathcal{T}}^\vee \otimes \mathrm{id}_{A_{\mathrm{inf}}}"] & \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}
\end{tikzcd} \tag{4.13}$$

Note that $\mathrm{can}_{\mathcal{T}}^\vee$ is $\mathfrak{S}$-linear, and $\mathrm{can}_{\mathcal{T}}^\vee \otimes \mathrm{id}_{A_{\mathrm{inf}}}$ is A_{inf}-linear.

We choose an $\mathcal{S}$-basis $\{\mathfrak{n}_1, \ldots, \mathfrak{n}_d\}$ for the module $\pi_i \, \mathrm{TC}_n(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee$. For any element x in this module, we let $\overline{x}$ denote its image under the natural inclusion

$$\pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \hookrightarrow \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}.$$

Since $\{\overline{\mathfrak{n}_1}, \ldots, \overline{\mathfrak{n}_d}\}$ is an A_{inf}-basis of $\pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$, for each $x \in \pi_i \, \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z]; \mathbb{Z}_p)^\vee$ there exist elements $a_{x1}, \ldots, a_{xd} \in A_{\mathrm{inf}}$ such that

$$\bar{\tau}^*(\overline{x}) = a_{x1}\overline{\mathfrak{n}_1} + a_{x2}\overline{\mathfrak{n}_2} + \cdots + a_{xd}\overline{\mathfrak{n}_d}.$$

Proposition 4.15. *For any x, the elements $a_{x1}, a_{x2}, \ldots, a_{xd}$ lie in $\mathfrak{S}^{(2)}$ under the inclusion $\mathfrak{S}^{(2)} \overset{\iota}{\hookrightarrow} A_{\mathrm{inf}}$.*

Proof. After inverting E, both $\mathrm{can}_{\mathcal{T}}^{\vee}$ and the inclusion

$$\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee} \hookrightarrow \pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$$

become isomorphisms (see (2.11)). Hence, for any $x \in \pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$, there exists $y \in \pi_i \,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$ such that

$$f(y) = E^m \cdot x \quad \text{in } \pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}. \tag{4.14}$$

Choose an $\mathfrak{S}$-basis $\mathfrak{n}_1, \dots, \mathfrak{n}_d$ of $\pi_i \,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$. There exist elements $c_{jl} \in \mathfrak{S}$ (for $1 \le j \le d$, $1 \le l \le n$) such that

$$\mathrm{can}_{\mathcal{T}}^{\vee}(\mathfrak{n}_j) = c_{j1}\mathfrak{m}_1 + c_{j2}\mathfrak{m}_2 + \cdots + c_{jn}\mathfrak{m}_n \tag{4.15}$$

in $\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$. Since the upper square of (4.13) is $\mathfrak{S}$-linear, we have

$$(\mathrm{can}_{\mathcal{T}}^{\vee} \otimes \mathrm{id}_{A_{\mathrm{inf}}})(\overline{\mathfrak{n}_j}) = \phi(c_{j1})\overline{\mathfrak{m}_1} + \phi(c_{j2})\overline{\mathfrak{m}_2} + \cdots + \phi(c_{jn})\overline{\mathfrak{m}_n} \tag{4.16}$$

in $\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee} \otimes_{\mathfrak{S},\phi} A_{\mathrm{inf}}$. Choose elements $b_{y1}, \dots, b_{yd} \in A_{\mathrm{inf}}$ such that

$$\tilde{\tau}^*(\overline{y}) = b_{y1}\overline{\mathfrak{n}_1} + b_{y2}\overline{\mathfrak{n}_2} + \cdots + b_{yd}\overline{\mathfrak{n}_d} \tag{4.17}$$

in $\pi_i \,\mathrm{TP}(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee}$. By the commutativity of (4.13), we obtain

$$\begin{aligned}
&\overline{\tau}^*(\overline{E^m \cdot x}) \\
&\quad= (\mathrm{can}_{\mathcal{T}}^{\vee} \otimes \mathrm{id}_{A_{\mathrm{inf}}})(\tilde{\tau}^*(\overline{y})) \\
&\quad\overset{(4.17)}{=} (\mathrm{can}_{\mathcal{T}}^{\vee} \otimes \mathrm{id}_{A_{\mathrm{inf}}})\big(b_{y1}\overline{\mathfrak{n}_1} + b_{y2}\overline{\mathfrak{n}_2} + \cdots + b_{yd}\overline{\mathfrak{n}_d}\big) \\
&\quad\overset{(4.16)}{=} \Big(\sum_{j=1}^{d} b_{yj}\phi(c_{j1})\Big)\overline{\mathfrak{m}_1} + \Big(\sum_{j=1}^{d} b_{yj}\phi(c_{j2})\Big)\overline{\mathfrak{m}_2} + \cdots + \Big(\sum_{j=1}^{d} b_{yj}\phi(c_{jn})\Big)\overline{\mathfrak{m}_n}.
\end{aligned}$$

Since $\overline{\tau}^*(\overline{E^m \cdot x}) = E_1^m \cdot \overline{\tau}^*(\overline{x})$, we have

$$E_1^m \cdot a_{xl} = \sum_{j=1}^{d} b_{yj}\phi(c_{jl})$$

in A_{inf}. By diagram (4.1) and Corollary 4.13, the right-hand side lies in $\mathfrak{S}^{(2)}$. Hence $a_{xl} \in \mathfrak{S}^{(2)}[1/E_1] \cap A_{\mathrm{inf}} = \mathfrak{S}^{(2)}$. ∎

Corollary 4.16. *The p-adic representation $T_{A_{\mathrm{inf}}}(\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee})$ is a $\mathbb{Z}_p$-lattice of a crystalline representation.*

Proof. By [14, Corollary 3.3.4], it suffices to show that $\tilde{\tau}^*(\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee})$ is contained in $\pi_i \,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee} \otimes_{\mathfrak{S},\theta_0} \mathfrak{S}^{(2)}$. By Proposition 4.5, we obtain the claim. ∎

Remark 4.17. We expect a more geometric proof of the corollary by showing that, for each i, the group $\pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)$ underlies a prismatic F-crystal. In particular, the associated descent datum $\pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \rightrightarrows \pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)} \substack{\rightarrow\\\rightarrow\\\rightarrow} \pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S}} \mathfrak{S}^{(3)}$ should be induced by the canonical maps

$$\pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \rightrightarrows \pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z_0,z_1];\mathbb{Z}_p) \substack{\rightarrow\\\rightarrow\\\rightarrow} \pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z_0,z_1,z_2];\mathbb{Z}_p).$$

However, we do not currently know whether the natural base change map

$$\pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathfrak{S}} \mathfrak{S}^{(3)} \xrightarrow{\sim} \pi_i\ \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z_0,z_1,z_2];\mathbb{Z}_p)$$

is an isomorphism. This result depends on the structure of $\pi_*\ \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z_0,z_1,z_2];\mathbb{Z}_p)$, which is currently not fully understood.

4.6. The proof of Main Theorems

The following theorem is the summary of Sections 3 and 4.

Theorem 4.18. *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. Then there exists an integer $n \geq 0$ such that, for all $i \geq n$, the following statements hold:*

1. *Breuil–Kisin module $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ has a Breuil–Kisin G_K-module structure in the sense of* [19].
2. *Breuil–Kisin module $\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee$ has a $(\varphi,\widehat{G})$-module structure in the sense of* [14].
3. *The $\mathbb{Z}_p[G_K]$-module $T_{A_{\mathrm{inf}}}(\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee)$ is a $\mathbb{Z}_p$-lattice of a crystalline representation.*
4. *If $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then there is a G_K-equivariant isomorphism*

$$T_{A_{\mathrm{inf}}}(\pi_i\,\mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^\vee) \simeq \pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}})^\vee$$

of $\mathbb{Z}_p$-modules.

Proof. (1) follows from Theorem 3.6. (2) follows from Remark 4.2. (3) follows from Corollary 4.16. (4) follows from Theorem 3.7. ∎

Theorem 4.19 (Main Theorem). *Let $\mathcal{T}$ be a smooth proper $\mathcal{O}_K$-linear category. If $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then Conjecture 1.2 holds for $\mathcal{T}$, that is, there is an isomorphism of B_{crys}-module*

$$\pi_i\,\mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p) \otimes_W B_{\mathrm{crys}} \simeq \pi_i L_{K(1)}K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}$$

which is compatible with G_K-action and Frobenius endomorphism.

Proof. Firstly, we will prove the claim when i is large enough. By Remark 3.8, we have a G_K-equivariant isomorphism

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^{\vee\vee} \simeq T_{A_{\mathrm{inf}}}(\pi_i \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee\vee}),$$

thus we have an isomorphism

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^{\vee\vee} \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} \simeq \pi_i \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee\vee} \otimes_{\mathfrak{S}} B_{\mathrm{crys}}$$

which is G_K-equivariant and compatible with Frobenius endomorphism, and there is an identification of rational Dieudonné modules (see [7, Remark 4.5])

$$D_{\mathrm{crys}}(\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}})^{\vee\vee} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \pi_i \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee\vee} \otimes_{\mathfrak{S},\tilde{\phi}} W\left[\frac{1}{p}\right].$$

By Theorem 2.14, we have an isomorphism $\pi_i \mathrm{TC}^-(\mathcal{T}/\mathbb{S}[z];\mathbb{Z}_p)^{\vee\vee} \otimes_{\mathfrak{S},\tilde{\phi}} W[\frac{1}{p}] \simeq \pi_i \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)[\frac{1}{p}]$. Thus we obtain the claim.

Now we prove the general case. For any $i \in \mathbb{Z}$, since $L_{K(1)} K(\mathcal{T})$ is 2-periodic, there is an isomorphism

$$\pi_i L_{K(1)} K(\mathcal{T}) \otimes_{\pi_0 L_{K(1)} K(\mathcal{C})} \pi_2 L_{K(1)} K(\mathcal{C}) \simeq \pi_{i+2} L_{K(1)} K(\mathcal{T}).$$

Moreover, we have an isomorphism $\pi_2 L_{K(1)} K(\mathcal{C}) \simeq \mathbb{Z}_p(1)$ (see [36]), hence we obtain

$$\pi_i L_{K(1)} K(\mathcal{T})(1) \simeq \pi_{i+2} L_{K(1)} K(\mathcal{T}).$$

Similarly, there is an isomorphism

$$\pi_i \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right] \otimes_{\pi_0 \mathrm{TP}(k;\mathbb{Z}_p)[\frac{1}{p}]} \pi_2 \mathrm{TP}(k;\mathbb{Z}_p)\left[\frac{1}{p}\right] \simeq \pi_{i+2} \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right],$$

and we have an isomorphism $\pi_2 \mathrm{TP}(k;\mathbb{Z}_p)[\frac{1}{p}] \simeq W[\frac{1}{p}](1)$ (see [8]), where the twist refers to twisting the Frobenius. Thus we obtain

$$\pi_i \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right](1) \simeq \pi_{i+2} \mathrm{TP}(\mathcal{T}_k;\mathbb{Z}_p)\left[\frac{1}{p}\right].$$

Since we already proved the claim when i is large enough, we obtain the claim. ∎

Acknowledgments. The author would like to thank Federico Binda, Lars Hesselholt, Buntaro Kakinoki, Shunsuke Kano, Hyungseop Kim and Hiroyasu Miyazaki for helpful discussions related to this subject. The author would also like to thank Ryomei Iwasa for comments on an earlier draft, Alexender Petrov for useful comments on a draft, Tasuki Kinjo for helpful discussions about derived schemes and Isamu Iwanari, Atsushi Takahashi and Shinnosuke Okawa for helpful discussions about non-commutative algebraic geometry. The author is deeply grateful to Hui Gao for helpful discussions about

$(\varphi, \widehat{G})$-modules and for sharing his ideas about Theorems 1.11 and 1.12 with us. The author is also grateful to the anonymous referee for a careful reading of the manuscript and for many insightful comments and suggestions, which greatly improved the paper.

Funding. This work was supported by JSPS KAKENHI Grant Numbers 22K13898 and 25KJ0210.

References

[1] B. Antieau and E. Elmanto, Descent for semiorthogonal decompositions. *Adv. Math.* **380** (2021), article no. 107600, 37 pp. Zbl 1467.14049 MR 4205113

[2] B. Antieau, A. Mathew, and T. Nikolaus, On the Blumberg–Mandell Künneth theorem for TP. *Selecta Math. (N.S.)* **24** (2018), no. 5, 4555–4576 Zbl 1410.14017 MR 3874698

[3] M. F. Atiyah, Vector bundles and the Künneth formula. *Topology* **1** (1962), no. 3, 245–248 Zbl 0108.17801 MR 0150780

[4] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pp. 7–38, American Mathematical Society, Providence, RI, 1961 Zbl 0108.17705 MR 0139181

[5] D. Bergh, V. A. Lunts, and O. M. Schnürer, Geometricity for derived categories of algebraic stacks. *Selecta Math. (N.S.)* **22** (2016), no. 4, 2535–2568 Zbl 1360.14058 MR 3573964

[6] B. Bhatt, D. Clausen, and A. Mathew, Remarks on $K(1)$-local K-theory. *Selecta Math. (N.S.)* **26** (2020), no. 3, article no. 39, 16 pp. Zbl 1454.19004 MR 4110725

[7] B. Bhatt, M. Morrow, and P. Scholze, Integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.* **128** (2018), no. 1, 219–397 Zbl 1446.14011 MR 3905467

[8] B. Bhatt, M. Morrow, and P. Scholze, Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), no. 1, 199–310 Zbl 1478.14039 MR 3949030

[9] A. Blanc, Topological K-theory of complex noncommutative spaces. *Compos. Math.* **152** (2016), no. 3, 489–555 Zbl 1343.14003 MR 3477639

[10] A. I. Bondal and M. M. Kapranov, Enhanced triangulated categories (in Russian). *Mat. Sb.* **181** (1990), no. 5, 669–683. English translation: *Math. USSR-Sb.* **70** (1991), no. 1, 93–107 Zbl 0729.18008 MR 1055981

[11] D. Clausen, A. Mathew, N. Naumann, and J. Noel, Descent in algebraic K-theory and a conjecture of Ausoni–Rognes. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 4, 1149–1200 Zbl 1453.18011 MR 4071324

[12] A. Connes, Cohomologie cyclique et foncteurs Ext^n. *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 23, 953–958 Zbl 0534.18009 MR 0777584

[13] S. K. Devalapurkar and A. Raksit, THH(Z) and the image of J. 2025, arXiv:2505.02218v1

[14] H. Du and T. Liu, A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals. *J. Eur. Math. Soc. (JEMS)* **28** (2026), no. 6, 2581–2636 Zbl 08176145 MR 5045694

[15] B. I. Dundas, T. G. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*. Algebr. Appl. 18, Springer, London, 2013, 435 pp. Zbl 1272.55002 MR 3013261

[16] G. Faltings, Crystalline cohomology and p-adic Galois-representations. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pp. 25–80, Johns Hopkins University Press, Baltimore, MD, 1989 Zbl 0805.14008 MR 1463696

[17] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique (with a preface by Pierre Colmez). *Astérisque* (2018), no. 406, 382 pp. Zbl 1470.14001 MR 3917141

[18] B. L. Feigin and B. L. Tsygan, Cohomology of Lie algebras of generalized Jacobi matrices (in Russian). *Funktsional. Anal. i Prilozhen.* **17** (1983), no. 2, 86–87. English translation: *Funct. Anal. Appl.* **17** (1983), no. 2, 153–155 Zbl 0544.17011 MR 0705056

[19] H. Gao, Breuil–Kisin modules and integral p-adic Hodge theory (with appendix A by Yoshiyasu Ozeki, and appendix B by Hui Gao and Tong Liu). *J. Eur. Math. Soc. (JEMS)* **25** (2023), no. 10, 3979–4032 Zbl 1537.11079 MR 4634688

[20] H. Gao, Letter to Keiho, private communication (6/9/2023)

[21] L. Hesselholt, On the p-typical curves in Quillen's K-theory. *Acta Math.* **177** (1996), no. 1, 1–53 Zbl 0892.19003 MR 1417085

[22] L. Hesselholt, On the topological cyclic homology of the algebraic closure of a local field. In *An alpine anthology of homotopy theory*, pp. 133–162, Contemp. Math. 399, American Mathematical Society, Providence, RI, 2006 Zbl 1217.19002 MR 2222509

[23] L. Hesselholt and T. Nikolaus, Topological cyclic homology. In *Handbook of homotopy theory*, pp. 619–656, CRC Press/Chapman Hall Handb. Math. Ser., CRC Press, Boca Raton, FL, 2020 Zbl 1473.14038 MR 4197995

[24] G. Hochschild, B. Kostant, and A. Rosenberg, Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.* **102** (1962), 383–408 Zbl 0102.27701 MR 0142598

[25] D. Kaledin, Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie. *Pure Appl. Math. Q.* **4** (2008), no. 3, 785–875 Zbl 1189.14013 MR 2435845

[26] A. A. Khan, The lattice property for perfect complexes on singular stacks. 2023, arXiv:2308.01617v1

[27] M. Kontsevich, Noncommutative motives. Talk at the conference on Pierre Deligne's 61st birthday

[28] M. Kontsevich and Y. Soibelman, Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In *Homological mirror symmetry*, pp. 153–219, Lecture Notes in Phys. 757, Springer, Berlin, 2009 Zbl 1202.81120 MR 2596638

[29] R. Liu and G. Wang, Topological cyclic homology of local fields. *Invent. Math.* **230** (2022), no. 2, 851–932 MR 4493328

[30] T. Liu, A note on lattices in semi-stable representations. *Math. Ann.* **346** (2010), no. 1, 117–138 Zbl 1208.14017 MR 2558890

[31] J. Lurie, Spectral algebraic geometry. Last update: Feb 2018, Preprint

[32] Z. Mao, Equivariant aspects of de-completing cyclic homology. [v1] 2024, [v3] 2025, arXiv:2410.05994v3

[33] T. Nikolaus and P. Scholze, On topological cyclic homology. *Acta Math.* **221** (2018), no. 2, 203–409 Zbl 1457.19007 MR 3904731

[34] D. Orlov, Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.* **302** (2016), 59–105 Zbl 1368.14031 MR 3545926

[35] A. Petrov and V. Vologodsky, On the periodic topological cyclic homology of DG categories in characteristic p. [v1] 2019, [v2] 2023, arXiv:1912.03246v2

[36] R. W. Thomason, Erratum: "Algebraic K-theory and etale cohomology" [*Ann. Sci. Éc. Norm. Super. (4)* **18** (1985), no. 3, 437–552 Zbl 0596.14012 MR 0826102]. *Ann. Sci. Éc. Norm. Super. (4)* **22** (1989), no. 4, 675–677 Zbl 0714.14006 MR 1026753

[37] T. Tsuji, p-Adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Invent. Math.* **137** (1999), no. 2, 233–411 Zbl 0945.14008 MR 1705837

[38] C. A. Weibel, Homotopy algebraic K-theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, pp. 461–488, Contemp. Math. 83, American Mathematical Society, Providence, RI, 1989 Zbl 0669.18007 MR 0991991

Received 5 August 2024; revised 17 October 2025.

Keiho Matsumoto
Department of Mathematics, Osaka University, 1-1 Yamadaoka, Suita-shi, Osaka-Fu 565-0871, Japan; matsumoto.keihou.sci@osaka-u.ac.jp, lightkun0526@gmail.com