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On the S_4-Norm of a Hankel Form

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In the paper [2] a rather general theory of Hankel forms over domains Ω in $\mathbb{C}^n$ was developed, and then applied to various special cases, in the first place, to the Fock space [1]. In the latter case also a very curious result (Theorem 7.8 of [2]) was established, which however does *not* follow from the general theory. It states, roughly speaking, that the S_4-norm of a Hankel form in Fock space is exactly equal to a suitable L_4-norm of its symbol, not only equivalent to it, as predicted by the theory. This is a phenomenon peculiar to $p = 4$; it does not hold for any other value of p, except trivially for $p = 2$, of course.

Recently, Wallstén [6] (see also a forthcoming paper by Janson, Upmeier and Wallstén [3]) has published a similar result for the S_4-norm — and even stranger — for the S_6-norm of a «big» Hankel operator on a planar domain ($n = 1$) under rather general assumptions on the underlying measure. (Recall that the study of Hankel forms is essentially equivalent to the study of «small» Hankel operators.)

This has led us to reexamine Theorem 7.8. Essentially, what we do is that we give a new proof of this theorem, which is more transparent. In particular, it gives an indication of that the result is peculiar to Fock space. For instance, it is not true for the unit disk with the Drzhrabashyan (or weighted Bergman) measure

$$d\mu_\alpha(z) = (1 - |z|^2)^\alpha \, dx\, dy \qquad (\alpha > -1).$$

[1] Or Fock-Segal-Bargmann-Fischer-... space.

We shall throughout use the terminology and notation introduced in [2], with slight modifications [2]. Therefore we will assume that the reader has some previous acquaintance with that paper — it suffices that he has browsed through it once and, in particular, had a look at the appendices (Appendix 1 gives a quick summary of the whole theory).

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1. The S_4-Norm

Let Ω be a domain in $\mathbb{C}^n$ equipped with a measure $d\mu$, subject to certain assumptions, which we need not make precise, so that the main contentions of [2] are valid. Let b be an analytic function on Ω. The Hankel form H_b with symbol b is defined by the formula

$$H_b(f, g) = \int_\Omega \bar{b} f g \, d\nu \qquad (f, g \in A^2(\mu)),$$

where $A^2(\mu)$ denotes the Hilbert space of square integrable analytic functions on Ω with respect to $d\mu$. Here $d\nu$ is the measure «associated» with $d\mu$, *i.e.*

$$d\nu(z) = K(z, \bar{z})^{-1} \, d\mu(z),$$

where $K(z, \bar{w})$ is the reproducing kernel in $A^2(\mu)$. Let P denote orthogonal projection from $L^2(\mu)$ onto $A^2(\mu)$.

With the form H_b we juxtapose an integral operator $\bar{H}_b$ such that

$$H_b(Pf, Pg) = \langle \bar{H}_b f, \bar{g} \rangle \qquad (f, g \in L^2(\mu))$$

— it is the «small» Hankel operator corresponding to H_b; in symbols: $\bar{H}_b = \bar{P} M_b P$, where M_b is the multiplication operator and $\bar{P}$ is projection onto $\overline{A^2(\mu)}$.

It is clear that the kernel of $\bar{H}_b$ is

$$\int_\Omega \overline{K(z, \bar{w})} \, \overline{b(w) \frac{d\nu}{d\mu}(w)} \, K(w, \bar{\zeta}) \, d\mu(w),$$

[2] One exception is that the reproducing kernel is denoted $K(z, \bar{w})$; the bar is for book keeping reasons to indicate that the function is conjugate analytic in the second argument.

while the corresponding adjoint kernel is [3]

$$\int_\Omega K(\zeta, \bar{w}) b(w) \frac{d\nu}{d\mu}(w) K(w, \bar{z})\, d\mu(w).$$

(Here $d\nu/d\mu$ is the Radon-Nikodym derivative of $d\nu$ with respect to $d\mu$.) It follows that the operator $\bar{H}_b^* \bar{H}_b$, which may be viewed as an operator from $A^2(\mu)$ into itself, has the kernel [4]

$$\iint_{\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) \frac{d\nu}{d\mu}(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} \frac{d\nu}{d\mu}(w_2) K(w_2, \bar{\zeta})\, d\mu(w_1)\, d\mu(w_2)$$

$$= \iint_{\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} K(w_2, \bar{z})\, d\nu(w_1)\, d\nu(w_2).$$

From this we can obtain

$$\begin{aligned} \|H_b\|_{S_2}^2 &= \|\bar{H}_b\|_{S_2}^2 \\ &= \operatorname{trace} \bar{H}_b^* \bar{H}_b \\ &= \iint_{\Omega\times\Omega} K(w_2, \bar{w}_1)^2 b(w_1) \overline{b(w_2)}\, d\nu(w_1)\, d\nu(w_2). \end{aligned}$$

In particular, if $d\mu$ satisfies assumption (V) in [2], *i.e.* if

$$L(z, \bar{w}) = \text{const} \cdot K(z, \bar{w})^2,$$

where L is the reproducing kernel in $A^2(\nu)$, it follows that

$$\|H_b\|_{S_2}^2 = \|\bar{H}_b\|_{S_2}^2 = \text{const} \cdot \|b\|_{L^2(\nu)}^2.$$

This is of course in [2] (see *e.g.* Theorem 4.7).

Next we observe that the kernel of $(\bar{H}_b^* H_b)^2$ is

$$\iiiint_{\Omega\times\Omega\times\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} K(w_2, \bar{w}_3)$$
$$\times\, b(w_3) K(w_4, \bar{w}_3) \overline{b(w_4)} K(w_4, \bar{\zeta})\, d\nu(w_1)\, d\nu(w_2)\, d\nu(w_3)\, d\nu(w_4).$$

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[3] If $F(z, \zeta)$ is any kernel, then the adjoint kernel is $\overline{F(\zeta, z)}$.

[4] The composition $F(z, \zeta)$ of two kernels $F_1(z, \zeta)$ and $F_2(z, \zeta)$ is defined by the formula

$$F(z, \zeta) = \int_\Omega F_1(z, t) F_2(t, \zeta)\, d\mu(t).$$

This gives

$$\begin{aligned}\|H_b\|_{S_4}^4 &= \|\bar{H}_b\|_{S_4}^4 \\ &= \operatorname{trace}(\bar{H}_b^* H_b)^2 \\ &= \iiiint_{\Omega\times\Omega\times\Omega\times\Omega} K(w_2,\bar{w}_1)K(w_2,\bar{w}_3)K(w_4,\bar{w}_1)K(w_4,\bar{w}_3)b(w_1)b(w_3) \\ &\qquad\qquad \times \overline{b(w_2)}\;\overline{b(w_4)}\,d\nu(w_1)\,d\nu(w_2)\,d\nu(w_3)\,d\nu(w_4).\end{aligned} \tag{1}$$

The S_p-norm of H_b or $\bar{H}_b$, p an even integer, apparently, is given by an analogous expression (see Section 4).

We go on examining the S_4-norm, which seems to be most rewarding case beyond $p = 2$.

We shall compare the integral (1) with

$$\|b\|_{L_w^4}^4 = \int_\Omega |b(z)|^4 K^{-4}(z,\bar{z})\,d\lambda(z) = \int_\Omega |b(z)|^4\,d\rho(z). \tag{2}$$

Recall that $d\lambda(z) = K(z,\bar{z})\,d\mu(z)$ is the «invariant» measure (see [2]), so that, actually, $d\rho(z) = K^{-2}(z,\bar{z})\,d\mu(z)$.

We now «polarize», so that we get instead the integrals

$$\int_{\Omega^4} \prod_{\substack{j=1,3\\k=2,4}} K(w_k,\bar{w}_j)\cdot b_1(w_1)b_3(w_3)\overline{b_2(w_2)}\;\overline{b_4(w_4)}\cdot\prod_{j=1}^4 d\nu(w_j) \tag{1'}$$

respectively

$$\int_\Omega b_1(z)b_3(z)\overline{b_2(z)}\;\overline{b_4(z)}\,d\rho(z), \tag{2'}$$

with four different functions b_1, b_2, b_3, b_4. In operator terms this means that we are considering the operator $\bar{H}_{b_1}^*\bar{H}_{b_2}\bar{H}_{b_3}^*\bar{H}_{b_4}$.

Next we let b_1, b_2, b_3, b_4 be «atoms», that is, we take

$$b_j(z) = L(z,\bar{z}_j) \qquad (j = 1,2,3,4),$$

where z_1, z_2, z_3, z_4 are points of Ω. It is then clear that the question of proportionality of the forms (1) and (2) (or (1′) and (2′)) is equivalent to the identity

$$\begin{aligned}&\int_\Omega L(z_1,\bar{z})L(z_3,\bar{z})L(z,\bar{z}_2)L(z,\bar{z}_4)\,d\rho(z) \\ &\qquad\qquad = \text{const}\cdot K(z_1,\bar{z}_2)K(z_1,\bar{z}_4)K(z_3,\bar{z}_2)K(z_3,\bar{z}_4).\end{aligned} \tag{$*$}$$

2. Discussion of (*)

Some things are now obvious:

1. If hypothesis (V) or (V2) (see [2]) holds, for the measures $d\mu$ and $d\nu$, then (*) is true on the diagonal ($z_1 = z_3$, $z_2 = z_4$). It is then question of the formula

$$\int_\Omega L^2(z_1, \bar{z}) L^2(z, \bar{z}_2)\, d\rho(z) = \text{const} \cdot K^4(z_1, \bar{z}_2).$$

It is clear that both members are proportional to $L^2(z_1, \bar{z}_3)$.

2. On the other hand to verify (*) it suffices to do it on another diagonal ($z_1 = z_2$, $z_3 = z_4$). This is because both members are analytic in «odd» variables, conjugate analytic in «even» ones.

3. (*) is true in the Fock case. This we know ([2], Theorem 7.8), but here is the direct argument. In the Fock case[5] we have

$$K(z, \bar{w}) = e^{z\bar{w}}, \qquad L(z, \bar{w}) = e^{2z\bar{w}}.$$

So it is question of the formula

$$\iint_C e^{2(z_1 + z_3)\bar{z}} e^{2z\overline{(z_2 + z_4)}} e^{-4|z|^2}\, dx\, dy = \text{const} \cdot e^{(z_1 + z_3)\overline{(z_2 + z_4)}},$$

or, writing $a = z_1 + z_3$, $b = z_2 + z_4$,

$$\iint_C e^{2(a\bar{z} + \bar{b}z)} e^{-4|z|^2}\, dx\, dy = \text{const} \cdot e^{a\bar{b}}.$$

To prove this, we first consider the «real» integral ($z = x + iy$)

$$\iint_C e^{2kx + 2ly} e^{-4(x^2 + y^2)}\, dx\, dy.$$

As the variables in the integral separate, it is clear that it evaluates to

$$\text{const} \cdot e^{(k^2 + l^2)/4}.$$

Next we take $k = a + \bar{b}$, $l = i(b - \bar{a})$. Then

$$kx + ly = a\bar{z} + bz,$$

while the previous exponent becomes

$$\frac{1}{4}(k^2 + l^2) = \frac{1}{4}(a + \bar{b})^2 - \frac{1}{4}(a - \bar{b})^2 = a\bar{b}.$$

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[5] For simplicity (*i.e.* notational simplicity) we take $n = 1$ (and the parameter $\alpha = 1$).

3. The Case of a Bounded Symmetric Domain

Now we shall indicate why (*) cannot be true for a general bounded symmetric domain.

First we continue for a while the general discussion in Section 1. By 2. in Section 2 it suffices to work on the diagonal, that is we may take $z_1 = z_2$, $z_3 = z_4$. Next, we invoke the *Berezin kernel*, which is a general object known to be useful in Hankel theory (*cf.* [1], [4]). In our case, since we have also two measures μ and ν, we have two kernels, namely[6]

$$B_\mu(z, w) = \frac{|K(z, \bar{w})|^2}{K(z, \bar{z}) \cdot K(w, \bar{w})},$$

$$B_\nu(z, w) = \frac{|L(z, \bar{w})|^2}{L(z, \bar{z}) \cdot L(w, \bar{w})}.$$

Then, dividing with $K^2(z_1, \bar{z}_1)K^2(z_3, \bar{z}_3)$, we get from (*) the equivalent formula

$$(**) \qquad \int_\Omega B_\nu(z_1, z) B_\nu(z_3, z)\, d\lambda(z) = \text{const} \cdot B_\mu(z_1, z_3).$$

Putting ourselves in the homogeneous situation, it suffices, again, to establish (**) if one of the points z_1 and z_3 is «fixed», the other «variable». (That is, if Ω is a symmetric domain, we can take *e.g.* $z_3 = 0$.) We can then also say that, *a priori*, the integral to the left in (**) must be a function of $B_\mu(z_1, z_3)$, that is, we get a relation of the form

$$\int_\Omega B_\nu(z_1, z) B_\nu(z_3, z)\, d\lambda(z) = h(B_\mu(z_1, z_3)),$$

where h thus is a function of one real variable. In other words, *the question about the validity of* (*) *or* (**) *has been reduced to the question of deciding whether this function h possibly can be a multiple of the identity.*

Let us apply these considerations to the case of the unit disk, *i.e.* $\Omega = D \subset \mathbb{C}$. Then

$$d\mu(z) = \text{const} \cdot (1 - |z|^2)^\alpha\, dx\, dy,$$
$$K(z, \bar{w}) = (1 - z\bar{w})^{-(\alpha+2)},$$
$$B_\mu(z, w) = \frac{(1 - |z|^2)^{\alpha+2}(1 - |w|^2)^{\alpha+2}}{|1 - z\bar{w}|^{2(\alpha+2)}},$$

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[6] Since these kernels display now visible properties of analyticity, it is no point to put bars over the arguments.

and similarly for $d\nu(z)$, $L(z, \bar{w})$, $B_\nu(z, w)$, with α replaced by $\beta = 2\alpha + 2$ (if and only if $2\beta + 2 = 2(\alpha + 2)$). We have also

$$d\lambda(z) = \text{const} \cdot \frac{dx\,dy}{(1 - |z|^2)^2} \qquad \text{(Poincaré measure).}$$

So taking $z_3 = 0$ as our «base point», we are led to consider the integral

$$\int_D \frac{(1 - |z|^2)^{\beta+2}(1 - |w|^2)^{\beta+2}}{|1 - z\bar{w}|^{2(\beta+2)}} (1 - |z|^2)^{\beta+2} \frac{dx\,dy}{(1 - |z|^2)^2}.$$

Lemma. (*cf.* Rudin [5], p. 18).

$$(\gamma + 1) \int_D \frac{(1 - |z|^2)^\gamma}{(1 - |z|^2)^{2\delta}} dx\,dy = \pi F(\delta, \delta; \gamma + 2; |z|^2).$$

PROOF. Use the expansion

$$(1 - z\bar{w})^{-\delta} = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} (z\bar{w})^k \qquad (\text{with } (\delta)_k = \delta(\delta + 1) \cdots (\delta + k - 1))$$

together with the fact

$$(\gamma + 1) \int_D |z|^{2k} (1 - |z|^2)^\gamma \, dx\,dy = \frac{k!\,\pi}{(\gamma + 2)_k}.$$

In our case we have $\gamma = 2\beta + 2 = 4\alpha + 6$, while $\delta = \beta + 2$. If we write out the resulting formula for a general point z_3 we get

$$\int_D B_\nu(z_1, z) B_\nu(z_3, z) \frac{dx\,dy}{(1 - |z|^2)^2}$$

$$(\dagger) \qquad = \frac{\pi}{\gamma + 1} B_\nu(z_1, z_3) F\left(\beta + 2, \beta + 2; \gamma + 2; 1 - \frac{(1 - |z_1|^2)(1 - |z_3|^2)}{|1 - z_1\bar{z}_3|^2}\right).$$

(N.B. — B_ν, *not* B_μ!)

The hypergeometric function to the right in (†) is never a multiple of the identity. Therefore, from this formula it is seen that only in the limiting case $\alpha \to \infty$ (corresponding to Fock space) that (*) or (**) can hold. *So the claim made in Introduction is substantiated.*

Let us elaborate the above somewhat. To make the transition to the Fock case we have also to invoke the «radius of curvature» R and make $R \to \infty$ in such a way that $\alpha/R^2 \to 1$. In other words, we make the substitution $z \mapsto z/R$, so that the unit disk D gets replaced by a disk D_R of radius R, and at the same renormalize $d\mu$ and $d\nu$ so that we get probability measures. The left hand side of (†) then becomes

$$\int_{D_R} B_\nu(z_1, z) B_\nu(z_3, z) \frac{\alpha + 1}{R^2} \frac{dx\,dy}{(1 - |z|^2/R^2)^2}.$$

Now

$$B_\nu(z, w) = \frac{\left(1 - \dfrac{|z|^2}{R^2}\right)^{2(\beta+2)} \left(1 - \dfrac{|w|^2}{R^2}\right)^{2(\beta+2)}}{\left|1 - \dfrac{z\bar{w}}{R^2}\right|^{2(\beta+2)}},$$

which in the limit is

$$B_\nu(z, w) = e^{-2|z-w|^2},$$

leading to the integral

$$\int_{\mathbf{C}} B_\nu(z_1, z) B_\nu(z_3, z)\, dx\,dy = \int_{\mathbf{C}} e^{-2|z_1 - z|^2 - 2|z - z_3|^2}\, dx\,dy.$$

At the same time the right hand side is

$$\pi B_\nu(z_1, z_3) F\left[\beta + 2, \beta + 2; \gamma + 2; 1 - \frac{\left(1 - \dfrac{|z_1|^2}{R^2}\right)\left(1 - \dfrac{|z_3|^2}{R^2}\right)}{\left|1 - \dfrac{z_1 z_3}{R^2}\right|^2}\right]. \tag{3}$$

As in the proof of (†), it is now convenient to put temporarily $z_3 = 0$ and to write $z_1 = z$. The hypergeometric function in (3) is then

$$F\left(\beta + 2, \beta + 2; \gamma + 2; \frac{|z|^2}{R^2}\right)$$

with the series expansion

$$1 + \frac{(\beta + 2)^2}{(\gamma + 2)} \frac{\dfrac{|z|^2}{R^2}}{1!} + \frac{(\beta + 2)^2(\beta + 3)^2}{(\gamma + 2)(\gamma + 3)} \frac{\left(\dfrac{|z|^2}{R^2}\right)^2}{2!} + \cdots$$

It is clear that we in the limit then expect the contribution $e^{|z|^2}$[7]. On the other hand, we get $B_\nu(z, 0) = e^{-2|z|^2}$ for the first factor in (3). These two factors

[7] Although this is quite trivial, we have not been able to find in the literature any asymptotic formula of this kind for the hypergeometric function.

combine to $B_\mu(z,0) = e^{-|z|^2}$. Restoring the original variables z_1 and z_3, we get of course $B_\mu(z_1, z_3)$. *Thus we have effectively yet another «proof» of Theorem* 7.8!

So much for the unit disk. The case of the unit ball in $\mathbb{C}^n$ is quite parallel (see again [5], p. 18). It seems that the case of a general symmetric domain involves generalized hypergeometric functions encountered by Yan [8] in his recent thesis.

Remark. So far, we have not been able to extract any other information from a formula like (†).

4. On the S_6-Norm

Continuing the computation in Section 1, we are led to the following expressions for the S_6-norm (*cf.* formula (1) there)

$$\begin{aligned}\|H_b\|_{S_6}^6 &= \|\bar{H}_b\|_{S_6}^6\\ &= \operatorname{trace}(\bar{H}_b^* H_b)^3\\ &= \int_{\Omega^6} b(w_1)K(w_2,\bar{w}_1)\overline{b(w_2)}K(w_2,\bar{w}_3)b(w_3)K(w_4,\bar{w}_3)\overline{b(w_4)}K(w_4,\bar{w}_5)\\ &\qquad \times b(w_5)K(w_6,\bar{w}_5)\overline{b(w_6)}K(w_6,\bar{w}_1)\prod_{j=1}^{6} d\nu(w_j).\end{aligned}$$

(From this it is also completely clear how the general formula for the S_p-norm, p an even integer (thus the analogue of the preceding formula reads for $p = 8, 10, \ldots$), must look, but we do not bother to write it down.) This expression has to be compared with the integral

$$\int_\Omega |b(z)|^6 \, d\sigma(z),$$

where $d\sigma(z) = K^3(z,\bar{z})\,d\mu(z)$. Again we can polarize. The we obtain corresponding multilinear forms formed with six functions $b_1, \ldots, b_6$. In the next step we specialize to «atoms», thus taking $b_j = K(z, \bar{z}_j)$ $(j = 1, \ldots, 6)$. Even in the Fock case we have not been able to find any interesting relationship between the resulting expressions.

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