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# Criterion on $L^p$-Boundedness for a class of Oscillatory Singular Integrals with rough Kernels

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## 1. Introduction.

It was known that the following form of oscillatory singular integrals defined for smooth $f$ with compact support had been studied by F. Ricci and E. M. Stein in [1]:

$$\text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy,$$

where $p(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, and $K(x)$ is a standard Calderón-Zygmund kernel. That means, $K$ satisfies:

(1.1) $K(x)$ is $C^1$-continuous away from the origin,

(1.2) $K(x) = \Omega(x')/|x|^n$ with $\Omega$ homogeneous of degree 0 on $S^{n-1}$,

(1.3) $$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

The following theorem is the main result in [1]:

**Theorem A.** *Suppose $p(x, y)$ is a real valued polynomial. If $K(x)$ satisfies* (1.1)-(1.3), *then the operator*

$$f \mapsto \text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy$$

*can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of this operator depends only on the total degree of $p(x, y)$, but not on the coefficients of $p(x, y)$.*

Evidently (1.1) shows that the kernel $K(x)$ requires certain smoothness. In this paper, we shall discuss a class of oscillatory singular integrals with rough kernel. Precisely, the kernel $K(x)$ satisfies (1.2), and

(1.4) $\qquad \Omega(x') \in L^q(S^{n-1}), \qquad$ for some $q$, $1 < q \leq +\infty$ .

In this case, the kernel $K(x)$ can be very rough on $S^{n-1}$, and $K(x)$ is not necessary to be a standard Calderón-Zygmund kernel. Since (1.1) implies $\Omega(x') \in L^\infty(S^{n-1})$, the following result can be regarded as an improvement of Theorem A.

**Theorem 1.** *Suppose $p(x, y)$ is a real valued polynomial. If $K(x)$ satisfies* (1.2)-(1.4), *then the operator*

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy$$

*can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of this operator depends only on the total degree of $p(x, y)$, but not on the coefficients of $p(x, y)$.*

In fact, Theorem 1 is an immediate consequence of the following stronger result.

**Theorem 2.** *Suppose $p(x, y)$ is a real valued polynomial, $K(x)$ satisfies* (1.2)-(1.4), *and $b(r)$ is a bounded variation function on $[0, \infty)$. If the operator*

$$f \longmapsto \text{p.v.} \int b(|\, x-y \,|) K(x-y) f(y)\, dy$$

*is a bounded operator on $L^p(\mathbb{R}^n)$ to itself with $1 < p < +\infty$, then the oscillatory integral operator*

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} b(|\, x - y \,|) K(x - y) f(y)\, dy$$

*is a bounded operator on $L^p(\mathbb{R}^n)$ to itself, and the norm of $T$ depends only on the total degree of $p(x, y)$, but not on the coefficients of $p(x, y)$.*

Let us introduce two concepts before we formulate our main result of this paper.

**Definition 1.** *A real valued polynomial $p(x, y)$ is called non-trivial if $p(x, y)$ does not take the form of $p_0(x) + p_1(y)$, where $p_0$ and $p_1$ are polynomials defined on $\mathbb{R}^n$.*

**Definition 2.** *We will say that the non-trivial polynomial $p(x, y)$ has property $\mathcal{P}$, if $p$ satisfies*

$$p(x + h, y + h) = p(x, y) + R_0(x, h) + R_1(y, h)$$

*where $R_0$ and $R_1$ are real polynomials.*

The main result in this paper is

**Theorem 3.** *Suppose $1 < p < +\infty$. If $K(x)$ satisfies* (1.2) *and* (1.4), *then the following three facts are equivalent:*

(i) *If $p(x, y)$ is a non-trivial polynomial, then the operator*

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} K(x - y) f(y)\, dy$$

*can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself.*

(ii) *If $Q(x, y)$ has the property $\mathcal{P}$, then the operator*

$$Gf(x) = \text{p.v.} \int e^{iQ(x,y)} K(x - y) f(y)\, dy$$

*can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself.*

(iii) *The truncated operator*

$$Sf(x) = \int\limits_{|x-y|<1} K(x-y)f(y)dy$$

*can be extended to be a bounded operator on* $L^p(\mathbb{R}^n)$ *to itself.*

Let us give out a simple application of Theorem 3. S. Chanillo, D. S. Kurtz and G. Sampson investigated the following oscillatory integrals in [2]:

$$f \longmapsto \text{p.v.} \int_{\mathbb{R}} e^{i|x-y|^a} \frac{f(y)}{1+|x-y|} dy, \qquad a > 0. \tag{1.5}$$

Since the truncated operator

$$f \longmapsto \int_{|x-y|<1} \frac{f(y)}{1+|x-y|} dy$$

is a bounded operator on $L^p(\mathbb{R})$ to itself, it follows from Theorem 3 that if $a$ is even, then the operator defined by (1.5) is a bounded operator on $L^p(\mathbb{R})$ to itself. Besides, we have the following conclusion. The operator defined by

$$f \longmapsto \int_{\mathbb{R}} e^{ip(x,y)} \frac{f(y)}{1+|x-y|} dy$$

is a bounded operator on $L^p(\mathbb{R})$ to itself, where $p(x,y)$ is a nontrivial polynomial. Let us make two explanations on the above conclusion. First, the above conclusion is not contained in [2]. Secondly, since the operator

$$f \longmapsto \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{1+|x-y|} dy$$

is not bounded on $L^p(\mathbb{R})$, the above conclusion can not be obtained from [1].

Finally, we shall get a similar result for the maximal operator corresponding to $T$.

**Theorem 4.** *Suppose* $p(x,y)$ *is a real valued polynomial. If* $K(x)$ *satisfies* (1.2)-(1.4), *then the maximal operator*

$$T_*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y)f(y)dy \right|$$

*is a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of $T_*$ depends only on the total degree of $p(x, y)$, but not on the coefficients of $p(x, y)$.*

## 2. Proof of the theorems.

First, we state several lemmas.

**Lemma 1.** *Suppose $\Omega(x')$ is homogeneous of degree 0 on $S^{n-1}$, and $\Omega(x') \in L^q(S^{n-1})$ with $1 < q \leq +\infty$. If*

$$Tf(x) = \text{p.v.} \int K(x, y) f(y)\, dy$$

*is a $(L^p, L^p)$ type operator with $1 < p < +\infty$, and $K(x, y)$ satisfies*

$$|K(x, y)| \leq \frac{|\Omega[(x - y)']|}{|x - y|^n},$$

*then the operators*

$$T_\varepsilon f(x) = \int_{|x-y|<\varepsilon} K(x, y) f(y)\, dy$$

*are $(L^p, L^p)$ type operators, and $\|T_\varepsilon\| \leq C(\|T\| + A)$, where $C$ is independent of $T$, $\varepsilon$ and $A$ depends only on $\Omega(x')$.*

PROOF: We split $f$ into three parts $f(y) = f_1(y) + f_2(y) + f_3(y)$ for $h \in \mathbb{R}^n$. Here

$$f_1(y) = f(y)\chi_{\{|y-h|<\varepsilon/2\}}(y),$$
$$f_2(y) = f(y)\chi_{\{\varepsilon/2\leq|y-h|<5\varepsilon/4\}}(y),$$
$$f_3(y) = f(y)\chi_{\{|y-h|\geq 5\varepsilon/4\}}(y).$$

When $|x - h| < \varepsilon/4$, it is easy to see $T_\varepsilon f_1(x) = Tf_1(x)$. So we have

$$\begin{aligned} \int_{|x-h|<\varepsilon/4} |T_\varepsilon f_1(x)|^p\, dx &\leq \int_{\mathbb{R}^n} |Tf_1(x)|^p\, dx \\ &\leq \|T\|^p \int_{|y-h|<\varepsilon/2} |f(y)|^p\, dy. \end{aligned} \tag{2.1}$$

If $|x-h| < \varepsilon/4, \varepsilon/2 \le |y-h| < 5\varepsilon/4$, then $\varepsilon/4 < |x-y| < 3\varepsilon/2$. So we have

$$|T_\varepsilon f_2(x)| \le \int_{\varepsilon/4<|y|\le\varepsilon} \frac{|\Omega(y')|}{|y|^n} |f_2(x-y)|\, dy.$$

By the Minkowski's inequality, it follows that

$$\begin{aligned}
&\left(\int_{|x-h|<\varepsilon/4} |T_\varepsilon f_2(x)|^p\, dx\right)^{1/p} \\
&\le \int_{\varepsilon/4<|y|\le\varepsilon} \frac{|\Omega(y')|}{|y|^n} \left(\int_{|x-h|<\varepsilon/4} |f_2(x-y)|^p\, dx\right)^{1/p} dy \\
&\le C\left(\int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy\right)^{1/p} \int_{S^{n-1}} |\Omega(y')|\, d\sigma(y') \\
&\le C\, \|\Omega\|_{L^q(S^{n-1})} \left(\int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy\right)^{1/p}.
\end{aligned} \tag{2.2}$$

If $|x-h| < \varepsilon/4,\ |y-h| \ge 5/4,$ then $|x-y| > \varepsilon$. So we have

$$T_\varepsilon f_3(x) = 0. \tag{2.3}$$

From (2.1), (2.2) and (2.3) it follows that the estimate

$$\int_{|x-h|<\varepsilon/4} |T_\varepsilon f(x)|^p\, dx \le C(\|T\|+A)^p \int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy$$

holds uniformly in $h \in \mathbb{R}^n$.

The above estimates imply

$$\|T_\varepsilon f\|_p \le C(\|T\|+A)\|f\|_p.$$

**Lemma 2.** (Van der Corput [3]) *Suppose $\phi \in C^{(k)}[a,b]$ and $|\phi^{(k)}(t)| \ge 1$ (when $t \in (a,b)$), then we have*

$$\left|\int_a^b e^{i\lambda\phi(t)}\, dt\right| \le C\, |\lambda|^{-1/k}$$

*where $\lambda \in \mathbb{R}$, and $C$ is independent of $a, b$ and $\phi$.*

**Lemma 3.** (See [1]) *Suppose $p(x) = \sum_{|\alpha| \le d} a_\alpha x^\alpha$ is a polynomial of degree $d$, and $\varepsilon < 1/d$. Then*

$$\sup_{y \in \mathbb{R}^n} \int_{|x| \le 1} |p(x-y)|^{-\varepsilon}\, dx \le A_\varepsilon \left( \sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon}.$$

*The bound $A_\varepsilon$ depends on $\varepsilon$ (and the dimension $n$), but not on the coefficients $\{a_\alpha\}$.*

**Lemma 4.** (See [1]) *Suppose $p(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ is a homogeneous polynomial of degree $d$ in $\mathbb{R}^n$, and $\varepsilon \le 1/d$. Then*

$$\int_{S^{n-1}} |p(x)|^{-\varepsilon}\, d\sigma(x) \le A_\varepsilon \left( \sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon}.$$

*The bound $A_\varepsilon$ depends on $\varepsilon$, but not on the coefficients $\{a_\alpha\}$.*

Now, let us turn to prove the theorems.

PROOF OF THEOREM 2. We shall carry out the argument by a double induction on the degrees in $x$ and $y$ of the polynomial $p$ as follows. We assume the theorem is known for all polynomials which are sums of monomials degree less than $k$ in $x$ times monomials of any degree in $y$, together with monomials which are of degree $k$ in $x$ times monomials which are of degree less than $l$ in $y$. Our inductive step will be to add to this all the monomials which have degree $k$ in $x$ and degree $l$ in $y$.

For general $p(x,y)$, we may write

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R_0(x,y)$$

where $R_0(x,y)$ satisfies the above induction assumption.

For $k = 0$ and $l$ arbitrary, the theorem is known. Let us now prove that Theorem 2 holds for arbitrary $k > 0$ and $l > 0$ by induction.

Without loss of generality, we may assume $\sum_{\substack{|\alpha|=k \\ |\beta|=l}} |a_{\alpha\beta}| > 0$.

*Case* 1. $\sum_{\substack{|\alpha|=k \\ |\beta|=l}} |a_{\alpha\beta}| = 1$.

We write

$$\begin{aligned} Tf(x) &= \int_{|x-y|\le 1} e^{ip(x,y)} b(|x-y|)K(x-y)f(y)\,dy \\ &\quad + \int_{|x-y|>1} e^{ip(x,y)} b(|x-y|)K(x-y)f(y)\,dy \\ &= T_0 f(x) + T_\infty f(x)\,. \end{aligned}$$

Take $h \in \mathbb{R}^n$, and write

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta + R(x,y,h)\,,$$

where the polynomial $R(x,y,h)$ satisfies the induction assumption, and the coefficients of $R(x,y,h)$ depend on $h$.

We have

$$\begin{aligned} |T_0 f(x)| \le {} & \Big|\int_{|x-y|\le 1} \exp\{i\,[R(x,y,h) + \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\} \\ & \cdot b(|x-y|)K(|x-y|)f(y)\,dy\Big| \\ & + \Big|\int_{|x-y|\le 1} \{\exp(ip(x,y)) - \exp(i[R(x,y,h) + \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}])\} \\ & \cdot b(|x-y|)K(x-y)f(y)\,dy\Big| \\ = {} & |T_{01}f(x)| + |T_{02}f(x)|\,. \end{aligned}$$

Note that $\|b\|_\infty < +\infty$, from the induction assumption and Lemma 1 we obtain that $T_{01}$ is a $(L^p, L^p)$ type operator, and the norm of $T_{01}$ depends on $\|b\|_\infty$, but not on the coefficients of $p(x,y)$ and $h$.

When $|x-h|<1/4, |x-y|<1$, we have

$$\begin{aligned}&|\exp\{ip(x,y)\}-\exp\{i[R(x,y,h)+\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}(y-h)^{\alpha+\beta}]\}\,|\\&\le C\sum_{\substack{|\alpha|=k\\|\beta|=l}}|a_{\alpha\beta}||x-y|\le C|x-y|.\end{aligned}$$

Thus

$$\begin{aligned}|T_{02}f(x)|&\le\int_{|x-y|\le 1}\frac{C\|b\|_\infty|\Omega[(x-y')]|}{|x-y|^{n-1}}|f(y)|\,dy\\&\le C\|b\|_\infty\int_{|y|\le 1}\frac{|\Omega(y')|}{|y|^{n-1}}|f(x-y)\chi_{B(h,5/4)}(x-y)|\,dy.\end{aligned}$$

By the Minkowski's inequality, we obtain

$$\int_{|x-h|<1/4}|T_{02}f(x)|^p\,dx\le C\|b\|_\infty^p\|\Omega\|_{L^q(S^{n-1})}^p\int_{|y-h|<5/4}|f(y)|^p\,dy.$$

Thus $\qquad \|T_{02}f\|_p\le C\|b\|_\infty\,\|\Omega\|_{L^q(S^{n-1})}\,\|f\|_p.$
Hence

$$\|T_0f\|_p\le C\,\|f\|_p,\tag{2.4}$$

where $C$ depends on $\|b\|_\infty$, but not on the coefficients of $p(x,y)$.
We write

$$\begin{aligned}T_\infty f(x)&=\sum_{j-1}^{+\infty}\int_{2^{j-1}<|x-y|\le 2^j}e^{ip(x,y)}b(|x-y|)K(x-y)f(y)\,dy\\&=\sum_{j=1}^{+\infty}T_jf(x).\end{aligned}$$

We have

$$\begin{aligned}T_jf(x)&=\int_{2^{j-1}<|y|\le 2^j}e^{ip(x,x-y)}b(|y|)K(y)f(x-y)\,dy\\&=\int_{S^{n-1}}\Omega(y')\int_{2^{j-1}<r\le 2^j}e^{ip(x,x-ry')}\frac{b(r)f(x-ry')}{r}\,dr\,d\sigma(y').\end{aligned}$$

For a fixed $y' \in S^{n-1}$, Let $Y$ be the hyperplane through the origin orthogonal to $y'$. We have, for $x \in \mathbb{R}^n, x = z + sy'$, with $s \in \mathbb{R}$, $z \in Y$, and so

$$\begin{aligned}\int_{2^{j-i}<r\leq 2^j} & e^{ip(x,x-ry')}\frac{b(r)f(x-ry')}{r}\,dr \\ &= \int_{2^{j-1}<r\leq 2^j} e^{ip(z+sy',z+(s-r)y')}\frac{b(r)f(z+(s-r)y')}{r}\,dr \\ &= \int_{2^{j-1}<s-t\leq 2^j} e^{ip(z+sy',z+ty')}\frac{b(s-t)}{s-t}f(z+ty')\,dt \\ &= N_j[f(z+\cdot\, y')](s)\end{aligned}$$

where $N_j$ is a linear operator defined on $L^2(\mathbb{R})$. Denote $N_j^*$ be its adjoint operator. Let us now consider the operator $N_j^*N_j$ with the kernel

$$\begin{aligned}M_j(u,v) = \int_{2^{j-1}<r-v,r-u\leq 2^j} & e^{i[p(z+ry',z+vy')-p(z+ry',z+uy')]} \\ & \cdot\frac{b(r-v)b(r-u)}{(r-v)(r-u)}\,dr \\ = \int_{\substack{\frac{1}{2}<r\leq 1 \\ 2^{j-1}<2^j r+v-u\leq 2^j}} & e^{i[p(2^j ry'+z+vy',z+vy')-p(2^j ry'+z+vy',z+uy')]} \\ & \cdot\frac{b(2^jr)b(2^jr+v-u)}{r(2^jr+v-u)}\,dr\,.\end{aligned}$$

It is easy to see

$$|M_j(u,u)| \leq \frac{C}{2^j}\,\chi_{[0,2^{j-1}]}(|v-u|). \tag{2.5}$$

Now we write $p(x,y)$ as follows

$$p(x,y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x,y),$$

where $R(x,y)$ is a polynomial with $x$-degree less than $k$, and $Q_\alpha(y)$ is a polynomial with degree $l$. So we can write

$$M_j(u,v) = \int_{\frac{1}{2}<r\leq 1,2^{j-1}<2^j r+v-u\leq 2^j} e^{i(E+F)}\psi\,dr\,,$$

where

$$E = (2^j r)^k \sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')],$$

and $F$ with $r$-degree less than $k$, and

$$\psi(r) = \frac{b(2^j r)b(2^j r + v - u)}{r(2^j r + v - u)}\,.$$

From Lemma 2, we have

$$\left|\int_{1/2}^{t} e^{i(E+F)}dr\right| \leq C\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]\right|\right)^{-\frac{1}{k}}$$

From integration by parts, we have

$$\begin{aligned}
|M_j(u,v)| &\leq C\{2^{jk}\,|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\}^{-1/k} \\
&\qquad \cdot \{|\psi(1)| + \int_{\substack{\frac{1}{2}<r\leq 1 \\ 2^{j-1}<2^j r+v-u\leq 2^j}} |d\psi(r)|\} \\
&\leq C\{2^{jk}\,|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\}^{-1/k} \\
&\qquad \cdot \left[\frac{\|b\|_\infty^2}{2^j} + \frac{\|b\|_\infty V_0^{+\infty}(b)}{2^j}\right] \\
&\leq C(b)2^{-j}\{2^{jk}|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\,\}^{-1/k}.
\end{aligned}$$

From (2.5) and the above inequality, we get that the estimate

$$\begin{aligned}
|M_{j(u,v)}| &\leq C(b)2^{-j}\{2^{jk}|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - (Q_\alpha(z+uy')]|\}^{-\delta/k} \\
&\qquad \cdot \chi_{[0,z^{j-1}]}(|v-u|)
\end{aligned}$$

holds uniformly in $\delta \in (0,1]$.

Thus

$$\int |M_j(u,v)|\,dv = \int_{|v-u|<2^j} |M_j(u,v)|\,dv$$

$$\leq C(b)2^{-j}2^{-j\delta}\int_{|v-u|<2^j}|\sum_{|\alpha|=k}y'^{\alpha}[Q_\alpha(z+vy')-Q_\alpha(z+uy')]|^{-\delta/k}\,dv$$

$$\leq C(b)2^{-j\delta}\int_{|v|<1}|\sum_{|\alpha|=k}y'^{\alpha}[Q_\alpha(z+2^j(v+\frac{u}{2^j})y')-Q_\alpha(z+uy')]|^{-\delta/k}dv.$$

Now, we take $\delta\in(0,1]$ such that $\delta/k<1/l$, then from Lemma 3 it follows

$$\begin{aligned}\int|M_j(u,v)|dv &\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}2^{-jl\delta/k}\\ &\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.\end{aligned}$$

Thus

$$\|N_j^*N_j\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

Similarly, we have

$$\|N_j^*N_j\|_{L^1(\mathbb{R})\to L^1(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

By the Riesz-Thorin's interpolation theorem, we obtain

$$\|N_j^*N_j\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

Hence, we have

$$\|N_j\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq C(b)2^{-j\delta/2}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/2k}.$$

Since

$$|N_jg(s)|\leq\|b\|_\infty\int_{2^{j-1}<s-t\leq 2^j}\frac{|g(t)|}{s-t}\,dt\leq C(b)\,HL(g)(s)$$

we have

$$\|N_j\|_{L^{p_0}(\mathbb{R})\to L^{p_0}(\mathbb{R})} \le C(b,p_0), \quad \text{with } 1<p_0<+\infty.$$

By the Riesz-Thorin's interpolation theorem, we obtain

$$\|N_j\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \le C(b)2^{-\theta j\delta/2}\Big|\sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}{y'}^{\alpha+\beta}\Big|^{-\theta\delta/2k} \tag{2.6}$$

where $0<\theta<1$.

From (2.6) and the Minkowski's inequality, we have

$$\begin{aligned}\|T_jf\|_p &\le \int_{S^{n-1}}|\Omega(y')|\Big(\int_Y\int_{\mathbb{R}}|N_j[f(z+\cdot y')](s)|^p\,dsdz\Big)^{1/p}d\sigma(y')\\ &\le C(b)2^{-\theta j\delta/2}\|f\|_p\int_{S^{n-1}}|\Omega(y')|\Big|\sum_{\substack{|az|=k\\|bz|=l}} a_{\alpha\beta}{y'}^{\alpha+\beta}\Big|^{-\theta\delta/2k}d\sigma(y')\\ &\le C(b)2^{-\theta j\delta/2}\,\|f\|_p\,\|\Omega\|_{L^q(S^{n-1})}\\ &\qquad\cdot\Big(\int_{S^{n-1}}\Big|\sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}{y'}^{\alpha+\beta}\Big|^{-\theta\delta q'/(2k)}d\sigma(y')\Big)^{1/q'}.\end{aligned}$$

We take $\delta\in(0,1]$, such that $\delta<\min\{k/l,\,2k/((k+l)\,q')\}$. Then from Lemma 4, we get

$$\|T_jf\|_p \le C(b)2^{-\theta j\delta/2}\,\|f\|_p\,. \tag{2.7}$$

Thus

$$\|T_\infty f\|_p \le C(b)\,\|f\|_p\,. \tag{2.8}$$

From (2.4) and (2.8), we obtain

$$\|Tf\|_p \le C(b)\,\|f\|_p \tag{2.9}$$

where $C(b)$ depends on the total degree of $p(x,y)$, $\|b\|_\infty$ and $V_0^\infty(b)$, but not on the coefficients of $p(x,y)$.

*Case* 2. $\sum_{\substack{|\alpha|=k\\|\beta|=l}}|a_{\alpha\beta}|\ne 1$

Denote $A = (\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}|)^{1/(k+l)}$. We can write $p(x,y)$ as follows

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} \frac{a_{\alpha\beta}}{A^{k+l}} (Ax)^\alpha (Ay)^\beta + R_0(\frac{Ax}{A}, \frac{Ay}{A}) \stackrel{\text{def}}{=} Q(Ax, Ay).$$

Thus

$$\begin{aligned} Tf(x) &= \int e^{iQ(Ax,Ay)} b(|x-y|) K(x-y) f(y)\, dy \\ &= \int e^{iQ(Ax,y)} b(\frac{|Ax-y|}{A}) K(Ax-y) f(\frac{y}{A})\, dy. \end{aligned}$$

Since $\|b(\cdot/A)\|_\infty = \|b\|_\infty$ and $V_0^\infty(b(\cdot/A)) = V_0^\infty(b)$, from the result in Case 1, we obtain

$$\|Tf\|_p \leq C\|f\|_p\,,$$

where $C$ depends on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$. So Theorem 2 holds for any polynomial $p(x,y)$ by induction principle.

THE PROOF OF THEOREM 1. When $K(x)$ satisfies (1.2)-(1.4), from the result in [4], we know that the operator

$$f \longmapsto \text{p.v.} \int K(x-y) f(y)\, dy$$

is a $(L^p, L^p)$ type operator with $1 < p < +\infty$. So Theorem 1 follows from Theorem 2.

THE PROOF OF THEOREM 3.
(i) *implies* (ii). This step is obvious.
(ii) *implies* (iii). Set

$$\begin{aligned} Gf(x) &= \int_{|x-y|<1} e^{iQ(x,y)} K(x-y) f(y)\, dy \\ &\quad + \int_{|x-y|\geq 1} e^{iQ(x,y)} K(x-y) f(y)\, dy \\ &= G_0 f(x) + G_\infty f(x). \end{aligned}$$

From the method similar to the proof of (2.8), we know that $G_\infty$ is a $(L^p, L^p)$ type operator. So $G_0$ is a $(L^p, L^p)$ type operator.

We take $h \in \mathbb{R}^n$. For $|x-h|<1$, we have

$$G_0 f(x) = G_0[f(\cdot)\chi_{B(h,2)}(\cdot)](x).$$

Thus

$$(2.10) \qquad (\int_{|x-h|<1} |G_0 f(x)|^p \, dx)^{1/p} \le C(\int_{|y-h|<2} |f(y)|^p \, dy)^{1/p}$$

where $C$ is independent of $h$.

Since $Q(x,y)$ has property $\mathcal{P}$, we have

$$Q(x,y) = Q(x-h, y-h) + R_0(x,h) + R_1(y,h)$$

where $R_0, R_1$ are real polynomials.

It follows that

$$\begin{aligned} Sf(x) &= \int_{|x-y|<1} K(x-y) f(y) \chi_{B(h,2)}(y) \, dy \\ &= e^{-iR_0(x,h)} \int_{|x-y|<1} e^{iQ(x,y)} K(x-y) e^{-iQ(x-h,y-h)} \\ &\qquad \cdot e^{-iR_1(y,h)} f(y) \chi_{B(h,2)}(y) \, dy \,. \end{aligned}$$

Note that the Taylor's expression of $e^{-iQ(x-h,y-h)}$ is

$$\begin{aligned} e^{-iQ(x-h,y-h)} &= \sum_{m=0}^{+\infty} \frac{i^m}{m!} [\sum_{\alpha,\beta} a_{\alpha\beta}(x-h)^{\alpha}(y-h)^{\beta}]^m \\ &= \sum_{m=0}^{+\infty} \frac{i^m}{m!} \sum_{l} C_{m,l} b_{\alpha\beta l}(x-h)^{u(\alpha,\beta,l)}(y-h)^{v(\alpha,\beta,l)} \end{aligned}$$

where $u$ and $v$ are multi-index.

Thus, if we set $a = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$ and $b = (\frac{3}{2}, \cdots, \frac{3}{2}) \in \mathbb{R}^n$, then we have

$$
\begin{aligned}
&\left(\int_{|x-h|<1} |Sf(x)|^p \, dx\right)^{1/p} \\
&\quad \le \sum_{m=0}^{+\infty} \sum_{l} \frac{|C_{ml} b_{\alpha\beta l}|}{m!} \left[ \int_{|x-h|<1} |(x-h)^u|^p \right. \\
&\qquad\qquad \left. |G_0[e^{-iR_1(\cdot,h)} f(\cdot)\chi_{B(h,2)}(\cdot)(\cdot - h)^v](x)|^p \, dx \right]^{1/p} \\
&\quad \le \sum_{m=0}^{+\infty} \sum_{l} \frac{|C_{ml} b_{\alpha\beta l}| a^u}{m!} C \left[ \int_{|y-h|<2} |f(y)|^p |(y-h)^v|^p \, dy) \right]^{1/p} \\
&\quad \le \sum_{m=0}^{+\infty} \sum_{l} \frac{|C_{ml} b_{\alpha\beta l}| a^u b^v}{m!} C \left[ \int_{|y-h|<2} |f(y)|^p \, dy \right]^{1/p} \\
&\quad = C \sum_{m=0}^{+\infty} \frac{1}{m!} \left(\sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta\right)^m \left[ \int_{|y-h|<2} |f(y)|^p \, dy \right]^{1/p} \\
&\quad = C \exp\left\{ \sum_{\alpha,\beta} |a_{\alpha\beta}| a^\alpha b^\beta \right\} \left[ \int_{|y-h|<2} |f(y)|^p \, dy \right]^{1/p} .
\end{aligned}
$$

Thus

$$\|Sf\|_p \le C \|f\|_p .$$

(iii) *implies* (i). Set

$$
b(r) = \begin{cases} 1, & r \in [0,1), \\ 0, & r \in [1,+\infty). \end{cases}
$$

It is easy to see that $b(r)$ is a bounded variation function on $[0,+\infty)$.

Since the truncated operator

$$Sf(x) = \text{p.v.} \int b(|x-y|) K(x-y) f(y) \, dy$$

is a $(L^p, L^p)$ type operator, from Theorem 2 we know that the operator

$$
\begin{aligned}
T_0 f(x) &= \text{p.v.} \int e^{ip(x,y)} b(|x-y|) K(x-y) f(y) \, dy \\
&= \int_{|x-y|<1} e^{ip(x,y)} K(x-y) f(y) \, dy
\end{aligned}
$$

is a $(L^p, L^p)$ type operator.

Since $p(x,y)$ is a nontrivial polynomial, by the methods similar to the proof of (2.8), we can prove that the operator

$$T_\infty f(x) = \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy$$

is a $(L^p, L^p)$ type operator.

Thus $T$ is a $(L^p, L^p)$ type operator.

THE PROOF OF THEOREM 4. We shall carry out the argument by a double induction on the degrees in $x$ and $y$ of the polynomial $p$ as in the proof of Theorem 2.

As in the proof of Theorem 2, we write

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R(x,y).$$

Since our conclusion is clearly invariant under dialation, we may assume that

$$\sum_{\substack{|\alpha|=k \\ |\beta|=l}} |a_{\alpha\beta}| = 1\,.$$

If $k = 0$, we know that the conclusion holds from the result in [4]. For general $p(x,y)$, we have

$$\begin{aligned}
T_* f(x) &\leq \sup_{0<\varepsilon<1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\
&\quad + \sup_{\varepsilon\geq 1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\
&\leq \sup_{0<\varepsilon<1} \Big| \int_{\varepsilon<|x-y|<1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\
&\quad + \Big| \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\
&\quad + \sup_{\varepsilon\geq 1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\
&= T_{*0} f(x) + \Big| \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| + T_{*\infty} f(x)
\end{aligned}$$

Now, it suffices to prove that $T_{*0}$ and $T_{*\infty}$ are $(L^p, L^p)$ type operators.

By the method similar to proving (2.4), we can easily prove that $T_{*0}$ is a $(L^p, L^p)$ type operator, and the norm of $T_{*0}$ depends on the total degree of $p(x,y)$, but on the coefficients of $p(x,y)$.

We have unique $J \in \mathbb{Z}^+$ such that $2^{J-1} \leq \varepsilon < 2^J$. Thus

$$\begin{aligned} T_{*\infty}f(x) \leq & \sup_{J\in\mathbb{Z}^+} \int_{2^{J-1}\leq|y|<2^J} \frac{|\Omega(y')|}{|y|^n}|f(x-y)|\,dy \\ & + \sup_{J\in\mathbb{Z}^+} \sum_{j=J+1} \Big| \int_{2^{j-1}\leq|x-y|<2^j} e^{ip(x,y)}K(x-y)f(y)\,dy\Big| \\ \leq & \sup_{J\in\mathbb{Z}^+} \int_{2^{J-1}\leq|y|<2^J} \frac{|\Omega(y')|}{|y|^n}|f(x-y)|\,dy \\ & + \sum_{j=1}^{+\infty} \Big| \int_{2^{j-1}\leq|x-y|<2^j} e^{ip(x,y)}K(x-y)f(y)\,dy\Big| . \end{aligned}$$

From the Minkowski's inequality and the method similar to proving (2.8), we get

$$\|T_{*\infty}f\|_p \leq C\,\|f\|_p\,.$$

where $C$ depends on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$. So we have finished the proof of the theorem.

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