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Quantum symmetries of vertex-transitive graphs on 12 vertices

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Abstract. Recently, the work on quantum automorphism groups of graphs has seen renewed progress, which we expand in this paper. Quantum symmetry is a richer notion of symmetry than the classical symmetries of a graph. In general, it is non-trivial to decide whether a given graph does have quantum symmetries or not. For vertex-transitive graphs, the quantum symmetries have already been determined in earlier work on up to 11 and 13 vertices. This paper fills the gap by determining for all vertex-transitive graphs on 12 vertices, whether they have quantum symmetries and for most of these graphs, we also give their quantum automorphism group explicitly.

Introduction

Quantum automorphism groups of graphs were first studied by Bichon in [6] and by Banica in [1]. They are compact matrix quantum groups as introduced by Woronowicz in [22, 23]. In particular, they are quantum subgroups of the quantum symmetric group S_n^+, which was first described by Wang in [21]. If $n \geq 4$, the quantum symmetric group S_n^+ is bigger than the classical symmetric group S_n, and thus, for graphs on 4 or more vertices, also the quantum automorphism group is potentially bigger than the classical automorphism group; i.e., we have

$$G_{\mathrm{aut}}(\Gamma) \subseteq G_{\mathrm{aut}}^+(\Gamma)$$

for any graph Γ.

Previous work on quantum groups of graphs includes a computer-based approach by Eder et al. [16], where the authors compute for all connected, simple graphs on up to 6 vertices and all connected, simple graphs on 7 vertices with classical automorphism group of order less than 2, whether they have quantum symmetries. Moreover, in [13], van Dobben de Bruyn et al. give a complete characterisation of quantum automorphism groups of trees, including a polynomial-time algorithm to explicitly compute the quantum automorphism group of a given tree. Some of the same authors also give the first construction of a graph that has trivial automorphism group but non-trivial quantum automorphism group in [14].

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In [15], Gromada defines and investigates quantum automorphisms of Hadamard matrices and shows that these results also pass to the corresponding Hadamard graphs. The study of quantum symmetries of graphs has also been extended to the study of quantum symmetries of quantum graphs in [8, 11].

A particular class of graphs that is of interest are the vertex-transitive graphs. They are interesting, because the vertex-transitivity ensures that the graphs have at least a certain amount of (classical) symmetry. In [3], Banica and Bichon described the quantum automorphism groups of all vertex-transitive graphs on up to 11 vertices except for the Petersen graph, for which Schmidt showed in [18] that it does not have quantum symmetries. Moreover, Chassaniol described the quantum automorphism groups of all vertex-transitive graphs on 13 vertices in [9,10]. This paper fills the gap by investigating all vertex-transitive graphs on 12 vertices to see which of these have quantum symmetries and which do not, partially answering a question posed in the PhD thesis of Schmidt [19].

We sort the vertex-transitive graphs on 12 vertices in 5 subclasses: disconnected graphs, products of smaller graphs, circulant graphs, semicirculant graphs and special cases that do not fit into any of the other subclasses.

In total, there are 74 vertex-transitive graphs on 12 vertices. However, since the quantum automorphism group of a graph is the same as the one of its complement, one only needs to consider one of the two. Due to this, we study only 37 graphs.

Our main result is the continuation of the classification programme of quantum automorphism groups of graphs. We also observe that the existence of disjoint automorphisms is equivalent to a graph having quantum symmetries for vertex-transitive graphs on up to 13 vertices, which is not the case in general.

In Section 1, we give the basic definitions around quantum automorphism groups of graphs. In Section 2, we present some lemmas that are useful when computing the quantum automorphism group of a graph. In Section 3, we give an overview of the graphs that are considered in this paper and introduce notation to help with some of the proofs. In Sections 4, 5, and 6, we give the results on the existence of quantum symmetries for all the different groups of vertex transitive graphs and then compute the quantum automorphism group of two such graphs explicitly in Section 7. In Appendix A, we collect the proofs omitted in Section 5 about computing the quantum symmetries of the graphs.

Main results

The main result of this paper is that we have decided the existence of quantum symmetries of vertex-transitive graphs on 12 vertices. Combining this with previous results of Banica and Bichon in [3], Schmidt in [18], and Chassaniol in [9, 10], we get the following.

Main Theorem 1. *For vertex transitive graphs on up to* 13 *vertices, the existence of quantum symmetries has been completely determined.*

Of the 37 graphs we studied, 20 have quantum symmetries. Among those with quantum symmetries, there are all graphs that are disjoint copies of smaller graphs, some graph products, three circulant and three semi-circulant graphs.

Subclass	Total graphs	Graphs with quantum symmetries
Disconnected graphs	9	9
Products of smaller graphs	6	5
Circulant graphs	12	3
Semi-circulant graphs	5	3
Special cases	5	0

In Table 1, we recollect the previous results on quantum symmetries of vertex-transitive graphs by Banica and Bichon in [3] and by Chassaniol in [9,10]. Also included is the result that the Petersen graph has no quantum symmetries by Schmidt [18]. These are all vertex transitive graphs of the given order up to complements. We exclude the trivial cases of the full graphs and disjoint copies of smaller graphs, due to the fact that the full graph on n vertices always has quantum automorphism group S_n^+, which can be seen easily, and due to the result of Banica and Bichon in [2], that a graph consisting of n disjoint copies of a connected graph Γ has a quantum automorphism group

$$G^+_{\mathrm{aut}}(n\Gamma) = G^+_{\mathrm{aut}}(\Gamma) \wr_* S_n^+.$$

Next, we give a table of all vertex-transitive graphs on 12 vertices up to complements with their automorphism groups and their quantum automorphism groups, if it is known. For completeness sake, we include the trivial cases in this table. In particular, we have computed for all of the following graphs, whether they have quantum symmetries. Therefore, whenever the column with the quantum automorphism group is a question mark, the graph in question does have quantum symmetries, but we did not manage to find the actual quantum automorphism group.

Here, we call

$$\mathcal{A} := (\mathbb{Z}_2 \times ((((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2.$$

Looking at the data in Tables 1 and 2, one can make some observations.

Observation 1. For vertex transitive graphs on up to 13 vertices, the existence of quantum symmetries is equivalent to the existence of disjoint automorphisms.

It is already known that the statement from Observation 1 does not hold for all graphs, as has been shown in [19], but it is an open question for what families of graphs this is an alternative characterisation of the existence of quantum symmetries.

We moreover observe the following fact.

Observation 2. Let Γ be a vertex-transitive graph on up to 13 vertices. If the automorphism group of Γ is D_n with $n \neq 4$, then Γ does not have quantum symmetries.

Order	Graph	Automorphism group	Quantum automorphism group
$n \neq 4$	C_n	D_n	D_n
8	$C_8(4)$	D_8	D_8
8	$K_2 \square C_4$	$S_4 \times \mathbb{Z}_2$	$S_4^+ \times \mathbb{Z}_2$
9	$C_9(3)$	D_9	D_9
9	$K_3 \square K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$
10	$C_{10}(2)$	D_{10}	D_{10}
10	$C_{10}(5)$	D_{10}	D_{10}
10	$K_2 \square C_5$	D_{10}	D_{10}
10	Petersen	S_5	S_5
10	$K_2 \square K_5$	$S_5 \times \mathbb{Z}_2$	$S_5^+ \times \mathbb{Z}_2$
10	$C_{10}(4)$	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$
11	$C_{11}(2)$	D_{11}	D_{11}
11	$C_{11}(3)$	D_{11}	D_{11}
13	C_{13}	D_{13}	D_{13}
13	$C_{13}(2)$	D_{13}	D_{13}
13	$C_{13}(2,5)$	D_{13}	D_{13}
13	$C_{13}(2,6)$	D_{13}	D_{13}
13	$C_{13}(3)$	D_{13}	D_{13}
13	$C_{13}(5)$	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$
13	$C_{13}(3,4)$	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$

Table 1. Vertex-transitive graphs on ≤ 11 and 13 vertices.

Due to the result of van Dobben de Bruyn et al. [14] that there exists a graph with quantum symmetry but with trivial automorphism group, we know that a question of the form from the above theorem does not make sense for general graphs: calling the graph constructed in [14] Γ_0 and given a graph Γ with any automorphism group, one could simply take the disjoint union of Γ and would get a graph with the same automorphism group as Γ (as long as Γ is not the same as Γ_0) that now has quantum symmetries. In fact, in their paper, they even state a more general result, namely, that for any given group, there exist both graphs with that group as an automorphism group that do have quantum symmetries and graphs that do not have quantum symmetries. However, one could investigate this question only for certain classes of graphs, such as vertex-transitive graphs. Since Γ_0 is asymmetric, it is necessarily also not vertex-transitive, and thus, any construction

Graph	Automorphism group	Quantum automorphism group
$6K_2$	$\mathbb{Z}_2 \wr S_6$	$\mathbb{Z}_2 \wr_* S_6^+$
$4K_3$	$S_3 \wr S_4$	$S_3 \wr_* S_4^+$
$3K_4$	$S_4 \wr S_3$	$S_4^+ \wr_* S_3$
$3C_4$	$H_2 \wr S_3$	$H_2^+ \wr_* S_3$
$2K_6$	$S_6 \wr \mathbb{Z}_2$	$S_6^+ \wr_* \mathbb{Z}_2$
$2C_6$	$D_6 \wr \mathbb{Z}_2$	$D_6 \wr_* \mathbb{Z}_2$
$2(K_2 \square K_3)$	$D_6 \wr \mathbb{Z}_2$	$D_6 \wr_* \mathbb{Z}_2$
$2C_6(2)$	$(\mathbb{Z}_2 \wr S_3) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* S_3) \wr_* \mathbb{Z}_2$
$2C_6(3)$	$(S_3 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(S_3 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$
$K_6 \times K_2$	$S_6 \times \mathbb{Z}_2$	$S_6^+ \times \mathbb{Z}_2$
$K_3 \times K_4$	$S_3 \times S_4$	$S_3 \times S_4^+$
$C_4 \square C_3$	$H_2 \times S_3$	$H_2^+ \times S_3$
$K_2 \square C_6(3)$	$\mathbb{Z}_2 \times (S_3 \wr \mathbb{Z}_2)$	$\mathbb{Z}_2 \times (S_3 \wr_* \mathbb{Z}_2)$
$K_2 \square C_6$	$\mathbb{Z}_2 \times D_6$	$\mathbb{Z}_2 \times D_6$
$K_2 \square C_6(2)$	$\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr S_3)$	?
C_{12}	D_{12}	D_{12}
$C_{12}(3)$	D_{12}	D_{12}
$C_{12}(6)$	D_{12}	D_{12}
K_{12}	S_{12}	S_{12}^+
$C_{12}(5)$	$\mathcal{A}$	?
$C_{12}(4, 5)$	$H_2 \times S_3$	$H_2^+ \times S_3$
$C_{12}(5, 6)$	$\mathcal{A}$	?
$C_{12}(2)$	D_{12}	D_{12}
$C_{12}(4)$	D_{12}	D_{12}
$C_{12}(2, 6)$	D_{12}	D_{12}
$C_{12}(3, 6)$	D_{12}	D_{12}
$C_{12}(4, 6)$	D_{12}	D_{12}
$C_{12}(5^+)$	H_3	?
$C_{12}(3^+, 6)$	$H_2 \times S_3$	$H_2^+ \times S_3$
$C_{12}(5^+, 6)$	H_3	?
$C_{12}(2, 5^+)$	D_6	D_6
$C_{12}(4, 5^+)$	D_6	D_6
Icosahedral graph	$\mathbb{Z}_2 \times A_5$	$\mathbb{Z}_2 \times A_5$
$L(C_6(2))$	H_3	H_3
Trunc(K_4)	S_4	S_4
Antip(Trunc(K_4))	S_4	S_4
Cuboctahedral graph	H_3	H_3

Table 2. Vertex-transitive graphs on 12 vertices.

with Γ_0 is also not vertex-transitive. Inspired by Observation 2, one could, therefore, ask the question: Does any group G exist such that whenever a vertex-transitive graph Γ has automorphism group G, then Γ has no quantum symmetries?

In addition to computing whether the graphs have quantum symmetries and giving the explicit quantum automorphism group wherever it was known by some previous result, we also compute the quantum automorphism group of two graphs for which it was not yet known in Section 7.

1. Preliminaries

Definition 1.1. A *finite graph without multiple edges* Γ consists of a finite vertex set $V(\Gamma)$ and an edge set $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. It is called *undirected*, if for any edge $(u, v) \in E(\Gamma)$ it also holds that $(v, u) \in E(\Gamma)$. A *loop* is an edge of the form $(i, i) \in E(\Gamma)$.

The *adjacency matrix* A_Γ of a graph on n vertices is the $n \times n$-matrix with the (i, j)-entry being the number of edges form vertex i to vertex j. If Γ is undirected and without multiple edges, A_Γ is thus a symmetric matrix with $\{0, 1\}$-entries.

In this paper, we only consider undirected finite graphs without multiple edges and without loops. Since all graphs will be undirected, we will write $u \sim v$ to mean that $(u, v) \in E(\Gamma)$, and thus, also $(v, u) \in E(\Gamma)$.

Definition 1.2. Given a graph Γ on n vertices with adjacency matrix A_Γ, its automorphism group $G_{\mathrm{aut}}(\Gamma)$ is the group of all automorphisms of Γ and can be expressed as a group of permutation matrices in the following sense:

$$G_{\mathrm{aut}}(\Gamma) = \left\{P \in S_n \mid PA_\Gamma = A_\Gamma P\right\}.$$

Definition 1.3. A graph Γ is *vertex-transitive* if for any two vertices $i, j \in V(\Gamma)$ there exists an automorphism $\phi \in G_{\mathrm{aut}}(\Gamma)$ such that $\phi(i) = j$.

Definition 1.4. Given a graph Γ and two automorphisms ϕ and ψ of Γ, we say that ϕ and ψ are *disjoint* if we have

$$\phi(i) \neq i \Rightarrow \psi(i) = i \quad \text{and} \quad \psi(i) \neq i \Rightarrow \phi(i) = i$$

for all vertices $i \in V(\Gamma)$. We say that Γ *has disjoint automorphisms* if there are two non-trivial disjoint automorphisms in $G_{\mathrm{aut}}(\Gamma)$.

Definition 1.5. A *compact matrix quantum group* G as defined by Woronowicz [22, 23] in 1987 is a unital C^*-algebra $C(G)$ equipped with a $*$-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ and a unitary $u \in M_n(C(G))$, $n \in \mathbb{N}$, such that

(1) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j,

(2) $\bar{u}$ is an invertible matrix,

(3) the elements u_{ij} $(1 \leq i, j \leq n)$ generate $C(G)$ (as a C^*-algebra).

The unitary u is called the *fundamental corepresentation (matrix)* of $(C(G), \Delta, u)$.

Since (i) and (iii) uniquely determine Δ, one can also refer to the pair $(C(G), u)$ as a compact matrix quantum group.

If $G = (C(G), u)$ and $H = (C(H), v)$ are compact matrix quantum groups with $u \in M_n(C(G))$ and $v \in M_n(C(H))$, we say that G is a *compact matrix quantum subgroup* of H, if there is a surjective *-homomorphism from $C(H)$ to $C(G)$ mapping generators to generators. We then write $G \subseteq H$. If we have $G \subseteq H$ and $H \subseteq G$, they are said to be identical as compact matrix quantum groups.

Definition 1.6. The *quantum symmetric group*

$$S_n^+ = (C(S_n^+), u),$$

which was first defined by Wang [21] in 1998, is the compact matrix quantum group given by

$$C(S_n^+) := C^*\left(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1 \forall i, j = 1, \dots, n\right),$$

and the $*$-homomorphism Δ is given by

$$\Delta(u_{ij}) := u_{ij}^{'} := \sum_k u_{ik} \otimes u_{kj}.$$

It can be shown that the quotient of $C(S_n^+)$ by the relation that all u_{ij} commute is exactly $C(S_n)$. Moreover, S_n can be seen as a compact matrix quantum group $S_n = (C(S_n), u)$, where $u_{ij} : S_n \to \mathbb{C}$ are the evaluation maps of the matrix entries. We then have $S_n \subseteq S_n^+$ as compact matrix quantum groups, which justifies the name “quantum symmetric group”.

For $n < 4$, one can see that the relations of S_n^+ already imply the commutation of all the generators, which means that for $n < 4$, we have $S_n = S_n^+$ as compact matrix quantum groups. However, for $n \geq 4$, one can show that S_n^+ is non-commutative, and thus, strictly bigger than S_n.

Definition 1.7. The hyperoctahedral quantum group H_n^+ is a quantization of the hyperoctahedral group H_n and was first defined by Bichon in [7] and further studied by Banica, Bichon, and Collins in [4]. It is defined as the free wreath product of S_2 with S_n^+:

$$C(H_n^+) = C(S_2) \wr_* C(S_n^+).$$

Example 1.8. In [20], Schmidt and Weber showed that the hyperoctahedral quantum group H_2^+ is the quantum automorphism group of the 4-cycle:

$$G_{\text{aut}}^+(C_4) = H_2^+.$$

Definition 1.9. Given a graph Γ on n vertices, its quantum automorphism group $G_{\text{aut}}^+(\Gamma)$ is the compact matrix quantum group $(C(G_{\text{aut}}^+(\Gamma)), u)$, where $C(G_{\text{aut}}^+(\Gamma))$ is the universal

C^*-algebra with generators u_{ij}, $1 \leq i, j \leq j$, subject to the following relations:

$$u_{ij} = u_{ij}^* = u_{ij}^2, \qquad 1 \leq i, j \leq n, \tag{1.1}$$

$$\sum_{k=1}^{n} u_{ik} = 1 = \sum_{k=1}^{n} u_{ki}, \quad 1 \leq i \leq n, \tag{1.2}$$

$$u_{ij} u_{kl} = u_{kl} u_{ij} = 0, \quad i \sim k,\ j \not\sim l, \tag{1.3}$$

$$u_{ij} u_{kl} = u_{kl} u_{ij} = 0, \quad i \not\sim k,\ j \sim l. \tag{1.4}$$

To see the connection to the classical automorphism group of Γ, the relations above can equivalently be written as

$$C(G_{\mathrm{aut}}^{+}(\Gamma)) = C^*\left(u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_{k=1}^{n} u_{ik} = 1 = \sum_{k=1}^{n} u_{ki},\ 1 \leq i \leq n,\ A_\Gamma u = u A_\Gamma \right).$$

Remark 1.10. In [6], Bichon gave a slightly different version of a quantum automorphism group of a graph. His version of the quantum automorphism group $G_{\mathrm{aut}}^{*}(\Gamma)$ of a graph Γ is defined by the same relations (1.1)–(1.4) as above, with the additional relation

$$u_{ij} u_{kl} = u_{kl} u_{ij} \quad \text{for all } i \sim k,\ j \sim l.$$

This defines a quantum subgroup of Banica's quantum automorphism group. In this work, we only consider Banica's version.

The quantum automorphism group of a graph is a quantum subgroup of the quantum symmetric group S_n^+. Therefore, as soon as a graph has at least 4 vertices, the quantum automorphism group might not be commutative. Sometimes, however, the additional relations coming from the graph, i.e., relations (1.3) and (1.4) from the above definition, already are enough to get commutativity of the entire quantum group. In that case, we obtain that

$$C(G_{\mathrm{aut}}^{+}(\Gamma)) = C(G_{\mathrm{aut}}(\Gamma));$$

i.e., the quantum automorphism group is equal to the classical automorphism group as a compact matrix quantum group.

Definition 1.11. We say a graph Γ has quantum symmetries, whenever $G_{\mathrm{aut}}^{+}(\Gamma) \neq G_{\mathrm{aut}}(\Gamma)$, i.e., when $G_{\mathrm{aut}}^{+}(\Gamma)$ is not commutative. If $G_{\mathrm{aut}}^{+}(\Gamma)$ is commutative, we say that Γ has no quantum symmetries.

2. Useful lemmas

In this section, we collect some lemmas that are useful when computing whether a given graph has quantum symmetries or not. Some of these lemmas come from the PhD thesis of Schmidt [19] or are generalisations of lemmas from there, while some are new.

2.1. General lemmas

First, we give a few lemmas that hold in any graph. The following one makes use of the adjoint in the C^*-algebra defining the quantum automorphism group and is an essential part in most of the proofs of commutativity of the generators.

Lemma 2.1.1 ([19]). *Let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{\text{aut}}(\Gamma))$. If we have*

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij},$$

then u_{ij} and u_{kl} commute.

The next lemma has been shown by Schmidt in his PhD thesis [19] and is an essential tool to check whether a graph has quantum symmetries. For a lot of graphs of which we know that they have quantum symmetries, this lemma is the reason we know it. However, we also know that not all quantum symmetries in graphs come from disjoint automorphisms, as has been shown by Schmidt in [19].

Lemma 2.1.2 ([19, Theorem 3.1.2]). *Let Γ be a graph. If there exist two non-trivial, disjoint automorphisms in* $\mathrm{Aut}(\Gamma)$*, then Γ has quantum symmetries and its quantum automorphism group is infinite dimensional.*

The next lemma is a quantum version of the fact that any automorphism of a graph has to preserve the distance of any pair of vertices.

Lemma 2.1.3 ([19]). *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{\text{aut}}(\Gamma))$. If we have $d(i,k) \neq d(j,l)$, then $u_{ij}u_{kl} = 0$.*

The following two lemmas are quite technical. However, they allow to drastically simplify some computations by making the question whether certain products of generators are zero a question of simple properties of the graph in terms of the distance of certain sets of vertices to one another. This makes it possible to check, e.g., with a computer in an automated fashion, whether these products are zero. The two lemmas are used extensively in the proofs of this paper, which is why we introduce a special notation for certain applications of the lemmas in Notations 3.1 and 3.2.

Lemma 2.1.4 ([19]). *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{\text{aut}}(\Gamma))$. Let $d(i,k) = d(j,l) = m$. Let q be a vertex with $d(j,q) = s$, $d(q,l) = t$ and $u_{kl}u_{aq} = u_{aq}u_{kl}$ for all a with $d(a,k) = t$. Then,*

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=m,\\ d(p,q)=s}} u_{ip}.$$

In particular, if we have $m = 2$ and if $u_{ab}u_{cd} = u_{cd}u_{ab}$ holds for all vertices where $a \sim c$, $b \sim d$, then choosing $s = t = 1$ implies

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(p,l)=2\\ (p,q)\in E}} u_{ip}.$$

Lemma 2.1.5 ([19]). *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{\text{aut}}(\Gamma))$. Let $d(i,k) = d(j,l) = m$ and let $p \neq j$ be a vertex with $d(p,l) = m$. Let q be a vertex with $d(q,l) = s$ and $d(j,q) \neq d(q,p)$. Then,*

$$u_{ij}\Bigg(\sum_{\substack{t;d(t,j)=d(t,p)=m\\ d(t,q)=s}} u_{kt}\Bigg)u_{ip} = 0.$$

Especially, if l is the only vertex satisfying $d(l,q) = s$, $d(l,j) = m$ and $d(l,p) = m$, we obtain $u_{ij}u_{kl}u_{ip} = 0$.

The next lemma is a simple fact about a monomial that vanishes, but it appears a lot in the computations of this paper, and therefore, we include it as a lemma here.

Lemma 2.1.6. *Let Γ be a finite, undirected graph and let (u_{ij}) be the generators of $G^+_{\text{aut}}(\Gamma)$. Let $i,j,k,l,p,q \in V(\Gamma)$ be vertices such that $d(p,q) \neq d(j,q)$ and $u_{kl}u_{rq} = u_{rq}u_{kl}$ for any vertex $r \in V(\Gamma)$ with $d(i,r) = d(p,q)$. Then, it holds that*

$$u_{ij}u_{kl}u_{ip} = 0.$$

Proof. We observe that

$$\begin{aligned} u_{ij}u_{kl}u_{ip} &= u_{ij}u_{kl}\sum_{\substack{r\in V(\Gamma)\\ d(i,r)=d(p,q)}} u_{rq}u_{ip} \\ &= \underbrace{u_{ij}\sum_{\substack{r\in V(\Gamma)\\ d(i,r)=d(p,q)}} u_{rq}}_{=0}\, u_{kl}u_{ip} \\ &= 0. \end{aligned}$$

Here, the commutation of u_{kl} and u_{rq} is by assumption and the fact that $u_{ij}u_{rq} = 0$ is due to the fact that $d(i,r) = d(p,q) \neq d(j,q)$ by assumption. ∎

2.2. Lemmas for vertex-transitive graphs

The following lemma is similar to [19, Lemma 3.2.4]; however, there is a stronger version that is only applicable for distance-transitive graphs.

Lemma 2.2.1. *Let Γ be a graph and $(u_{ij})_{1\leq i,j\leq n}$ is the generators of $C(G^+_{\text{aut}}(\Gamma))$. Let j_1, l_1, j_2, l_2 be vertices in Γ such that $d(j_1,l_1) = d(j_2,l_2)$. If we have $u_{aj_1}u_{bl_1} = u_{bl_1}u_{aj_1}$ for all $a,b \in V(\Gamma)$ and there exists an automorphism $\phi \in G_{\text{aut}}(\Gamma)$ such that $\phi(j_1) = j_2$ and $\phi(l_1) = l_2$, then we get*

$$u_{aj_2}u_{bl_2} = u_{bl_2}u_{aj_2} \quad \text{for all } a,b \in V(\Gamma).$$

Corollary 2.2.2. *Let Γ be a vertex-transitive graph and $(u_{ij})_{1\le i,j\le n}$ is the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$. If there exist a vertex $j_0 \in V(\Gamma)$ and a distance m such that for all vertices $l \in V(\Gamma)$ with $d(j_0,l) = m$, we have commutation of u_{ij_0} and u_{kl} for any vertices $i,k \in V(\Gamma)$, then it already holds that for all vertices $i,j,k,l \in V(\Gamma)$, we have*

$$d(j,l) = m \Rightarrow u_{ij}u_{kl} = u_{kl}u_{ij}.$$

In particular, if u_{ij_0} commutes with all other generators u_{kl} of $C(G^+_{\mathrm{aut}}(\Gamma))$, then Γ does not have quantum symmetries.

Proof. Let $j', l' \in V(\Gamma)$ be vertices of Γ such that $d(j',l') = m$. Since Γ is vertex-transitive, there exists an automorphism ϕ of Γ that maps j_0 to j'. Since ϕ is an automorphism, the preimage of $l_0 := \phi^{-1}(l')$ of l' must satisfy

$$d(j_0,l_0) = d(j',l') = m.$$

By assumption, we know $u_{ij_0}u_{kl_0} = u_{kl_0}u_{ij_0}$ for any $i,k \in V(\Gamma)$. Applying Lemma 2.2.1, we get that also $u_{\phi(i)j'}$ and $u_{\phi(k)l'}$ commute. Since i and k are arbitrary, we also get $u_{ij'}u_{kl'} = u_{kl'}u_{ij'}$ for arbitrary $i,k \in V(\Gamma)$. ■

2.3. Lemmas about common neighbours

As seen in Lemma 2.1.3, the distance between vertices matters when looking at the generators of the quantum automorphism group of a graph. The simplest case to consider when looking at distances is the neighbour. Because of this, a few lemmas were found concerning common neighbours of vertices.

The following two lemmas provide an easy way to see the commutation of all generators for adjacent vertices by looking at the structure of the graph, namely, the existence of quadrangles and triangles.

Lemma 2.3.1 ([19]). *Let Γ be an undirected graph that does not contain any quadrangle. Then, for vertices $i \sim k$ and $j \sim l$, the generators u_{ij} and u_{kl} commute.*

Lemma 2.3.2 ([19]). *Let Γ be an undirected graph such that adjacent vertices have exactly one common neighbour. Then, for vertices $i \sim k$ and $j \sim l$ the generators u_{ij} and u_{kl} commute.*

The above lemma can be generalised a bit.

Lemma 2.3.3. *Let $\Gamma = (V,E)$ be an undirected graph and let $(u_{ij})_{1\le i,j\le n}$ be the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$. Let $i,j,k,l \in V$ be vertices of Γ such that $(i,k) \in E$ and $(j,l) \in E$. Let moreover i and k have exactly one common neighbour p and let it hold that for any two vertices from the set $\{i,k,p\}$ the third vertex from the set is the only common neighbour of the two. In other words, the only common neighbour of i and p is k and the only common neighbour of k and p is i. Let j and l also have exactly one common neighbour q and let the same as above hold for the set of vertices $\{j,l,q\}$. Then, $u_{ij}u_{kl} = u_{kl}u_{ij}$.*

Proof. The proof is the same as the one of Lemma 2.3.2 in the PhD thesis of Schmidt [19].

Using the relations of $C(G^+_{\mathrm{aut}}(\Gamma))$, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{a;(l,a)\in E} u_{ia}.$$

We want to show that $u_{ij}u_{kl}u_{ia} = 0$ for $a \neq j$, $(a,l) \in E$. First, we consider q, the unique common neighbour of j and l. We note that for any vertex $b \neq l$ we have $(j,b) \notin E$ or $(q,b) \notin E$, since l is also the only common neighbour of j and q. We deduce that

$$u_{ij}u_{kl}u_{iq} = u_{ij}\Big(\sum_b u_{kb}\Big)u_{iq} = u_{ij}1u_{iq} = u_{ij}u_{iq} = 0.$$

Let now $a \notin j,q$ and $(a,l) \in E$. Then, it holds that

$$u_{ij}u_{kl} = u_{ij}\Big(\sum_b u_{pb}\Big)u_{kl} = u_{ij}u_{pq}u_{kl},$$

since p is the only common neighbour of i and k while q is the only common neighbour of j and l. As j is the only common neighbour of l and q and we have that $(a,l) \in E$, we deduce that $(a,q) \notin E$. We thus get

$$0 = u_{pq}u_{ia} = u_{pq}\Big(\sum_c u_{cl}\Big)u_{ia} = u_{pq}u_{kl}u_{ia},$$

since $(i,p) \in E$ but $(a,q) \notin E$ and since k is the only common neighbour of p and i.

Using the two equations we deduced above, we get

$$u_{ij}u_{kl}u_{ia} = u_{ij}u_{pq}u_{kl}u_{ia} = 0.$$

We thus get for all $a \neq j$ that

$$u_{ij}u_{kl}u_{ia} = 0.$$

Finally, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\Big(\sum_a u_{ia}\Big) = u_{ij}u_{kl}u_{ij},$$

and thus, we get by Lemma 2.1.1 that $u_{ij}u_{kl} = u_{kl}u_{ij}$. ∎

In order to make some statements and computations more compact, we introduce the following definition.

Definition 2.3.4. Given a graph Γ and vertices i, j of Γ, we denote by $\mathrm{CN}(i,j)$ the set of all common neighbours of i and j, i.e., all vertices k of Γ such that both $i \sim k$ and $j \sim k$ hold.

The following lemma again yields an easy-to-check criterion for a certain product of generators to be zero. The way we used it in this paper was mainly by using the two corollaries that follow.

Lemma 2.3.5. *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\le i,j\le n}$ be the generators of $C(G_{\mathrm{aut}}^{+}(\Gamma))$. Let $i, j, k, l \in V$ be vertices in Γ.*

If $|\mathrm{CN}(i,k)| \neq |\mathrm{CN}(j,l)|$, then we already have

$$u_{ij}u_{kl} = 0.$$

Proof. We set $a = |\mathrm{CN}(i,k)|$ and $b = |\mathrm{CN}(j,l)|$ and observe the following:

$$\begin{aligned} a \cdot u_{ij}u_{kl} &= \sum_{p\in \mathrm{CN}(i,k)} \underbrace{u_{ij} \sum_{q\in \mathrm{CN}(j,l)} u_{pq}u_{kl}}_{=u_{ij}u_{kl}} \\ &= u_{ij} \sum_{p\in \mathrm{CN}(i,k)} \sum_{q\in \mathrm{CN}(j,l)} u_{pq}u_{kl} \\ &= \sum_{q\in \mathrm{CN}(j,l)} \underbrace{u_{ij} \sum_{p\in \mathrm{CN}(i,k)} u_{pq}u_{kl}}_{=u_{ij}u_{kl}} \\ &= b \cdot u_{ij}u_{kl}. \end{aligned}$$

Since by assumption we had $a \neq b$, we get that $u_{ij}u_{kl} = 0$. ∎

This lemma leads to the following corollaries.

Corollary 2.3.6. *If i, j, k, l, p are vertices in Γ and $|\mathrm{CN}(j,l)| \neq |\mathrm{CN}(l,p)|$, then*

$$u_{ij}u_{kl}u_{ip} = 0.$$

Proof. If $|\mathrm{CN}(i,k)| \neq |\mathrm{CN}(j,l)|$, then we have $u_{ij}u_{kl} = 0$, and thus, also $u_{ij}u_{kl}u_{ip} = 0$. Otherwise, we have

$$|\mathrm{CN}(i,k)| = |\mathrm{CN}(j,l)| \neq |\mathrm{CN}(l,p)|,$$

and thus, $u_{kl}u_{ip} = 0$, and thus, $u_{ij}u_{kl}u_{ip} = 0$. ∎

Corollary 2.3.7. *Let Γ be a finite, undirected graph and let $(u_{ij})_{1\le i,j\le n}$ be the generators of $C(G_{\mathrm{aut}}^{+}(\Gamma))$. If $i, j, k, l \in V$ are vertices of Γ with $(i,k) \in E$ and $(j,l) \in E$ such that i and k have at least one common neighbour, but j and l do not have a common neighbour, then*

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0.$$

In other words, if i and k are part of the same triangle but j and l are not, then the product of u_{ij} and u_{kl} is zero.

Proof. If i and k are part of the same triangle, then in particular $|\mathrm{CN}(i,k)| > 0$, while $|\mathrm{CN}(j,l)| = 0$, since they are not in the same triangle. Applying Lemma 2.3.5, we get $u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$. ∎

2.4. A strategy for deciding quantum symmetries

In this section, we want to give a short checklist of steps to take when one has a given graph Γ and wants to decide whether it has quantum symmetries.

(1) First, one should check whether Γ has disjoint automorphisms. If yes, then by Lemma 2.1.2 it has quantum symmetries.

(2) If Γ does not have disjoint automorphisms, one should first try to show that it does not have quantum symmetries, i.e., that $C(G^+_{\text{aut}}(\Gamma))$ is commutative.

   (a) By Lemma 2.1.3, it suffices to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ only for vertices that fulfill $d(i,k) = d(j,l)$. One can thus fix a distance $m = d(i,k) = d(j,l)$ and try to show commutativity for vertices in this distance.

   (b) For $m = 1$, the first thing to check is whether Lemma 2.3.1 or Lemma 2.3.2 are applicable because these already yield commutativity for this case.

   (c) In the next step, one can check whether using Lemmas 2.1.4 or 2.1.5 one can get $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ by showing that $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$. This is a task that can easily be delegated to a computer and can, therefore, be applied to all possible combinations of generators easily.

   (d) If the results of Lemmas 2.1.4 and 2.1.5 only yield partial results, i.e., they only show $u_{ij}u_{kl}u_{ip} = 0$ for some of the vertices p, one can try to use Corollary 2.3.6 to show it for the rest.

   (e) If at any point one has shown that

$$u_{ij}u_{kl} = u_{kl}u_{ij}$$

   for any generators, one can use Lemma 2.2.1 to propagate this to other pairs of generators.

   (f) Any remaining pairs of vertices will have to be considered individually, potentially using a combination of Lemmas 2.1.4, 2.1.5, 2.2.1 and Corollary 2.3.6.

3. Overview and notation

On twelve vertices, there are in total 74 vertex-transitive graphs. Since the quantum automorphism group of a graph is the same as the one of its complement, it suffices to only consider 37 of these graphs, since the rest can be obtained by taking their complements.

We will sort these graphs in five different subclasses: disconnected graphs, products of smaller graphs, circulant graphs, semicirculant graphs and special cases that do not fit into any of the other subclasses.

In total, we will consider the graphs in Table 3.

In the proofs that a graph has no quantum symmetries, we will often repeatedly apply Lemmas 2.1.4 and 2.1.5. In order to prevent this from taking up too much space, we will now introduce the following notation.

Subclass	Graphs
Disconnected	$6K_2$, $4K_3$, $3K_4$, $3C_4$, $2K_6$, $2C_6$,
	$2K_2\square K_3$, $2C_6(2)$, $2C_6(3)$
Products	$K_6 \times K_2$, $K_3 \times K_4$, $C_6\square K_2$,
	$C_4\square C_3$, $K_2\square C_6(2)$, $K_2\square C_6^+$
Circulant	C_{12}, K_{12}, C_{12}^+, $C_{12}(2)$, $C_{12}(3)$, $C_{12}(4)$,
	$C_{12}(5)$, $C_{12}(2,6)$, $C_{12}(4,6)$, $C_{12}(3,6)$,
	$C_{12}(4,5)$, $C_{12}(5,6)$
Semi-circulant	$C_{12}(5^+)$, $C_{12}(3^+,6)$, $C_{12}(5^+,6)$,
	$C_{12}(2,5^+)$, $C_{12}(4,5^+)$
Special cases	L(Cube), L($C_6(2)$), Icosahedron,
	Trunc(K_4), Antip(Trunc(K_4))

Table 3. Vertex-transitive graphs on 12 vertices up to complements.

Notation 3.1. Let Γ be a graph and j a fixed vertex in Γ. If we want to apply Lemma 2.1.4 to j and $l_1, l_2, \ldots$ and $q_1, q_2, \ldots$ to get that

$$u_{ij}u_{kl_a} = u_{ij}u_{kl_a} \sum_{p \in P_a},$$

we will write the following table:

Lemma 2.1.4.

j	l	q	p
j	l_1	q_1	P_1
	l_2	q_2	P_2
	$\vdots$	$\vdots$	$\vdots$

Here, the sets P_a will be the sets of vertices p such that $d(l_a, p) = d(j, l_a)$ and $d(p, q_a) = d(j, q_a)$. Often, we will use this when the sets P_a are singleton sets because this will mean that we have

$$u_{ij}u_{kl_a} = u_{ij}u_{kl_a}u_{ij}.$$

Notation 3.2. Let Γ be a graph and j a fixed vertex in Γ. If we want to apply Lemma 2.1.5 to j, l, p, and q, we often want to apply it in such a way that we get

$$u_{ij}u_{kl}u_{ip} = 0.$$

This is the case if l is the unique vertex in distance $d(l,q)$ from q, $d(l,j)$ from j and $d(l,p)$ from p. If that is the case, we will write the application of Lemma 2.1.5 to $j, l, p_1, p_2, \dots$ and $q_1, q_2, \dots$ as the following table:

Lemma 2.1.5.

j	l	p	q
j	l	p_1	q_1
		p_2	q_2
		$\vdots$	$\vdots$

A particular row of such a table containing j', l', p', q' then means that $u_{ij'}u_{kl'}u_{ip'} = 0$ for all vertices $i, k \in V(\Gamma)$.

To see how an application of the newly introduced notations might look in practice and how one might start working on a new graph in general, we give a simple example.

Example 3.3. We consider the 5-cycle C_5. We already know that it has no quantum symmetries, but we now want to show this using Lemmas 2.1.5 and 2.1.4. By Corollary 2.2.2, we only need to show that the generators u_{i1} commute with all generators u_{kl}. Note, moreover, that by Lemma 2.1.1, we only need to show that $u_{i1}u_{kl} = u_{i1}u_{kl}u_{i1}$. We do this by using the fact from the relations of $G_{\text{aut}}^{+}(C_5)$ that $\sum_r u_{ir} = 1$, and thus,

$$u_{i1}u_{kl} = u_{i1}u_{kl}\sum_{p} u_{ip}.$$

Using additionally Lemma 2.1.3, we get that

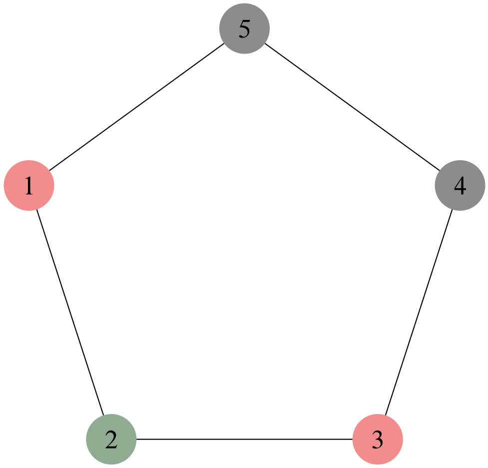
$$u_{i1}u_{kl}\sum_{p} u_{ip} = u_{i1}u_{kl}\sum_{\substack{p \\ d(p,l)=d(i,k)}} u_{ip}.$$

Now, we just need to show that $u_{i1}u_{kl}u_{ip} = 0$ for any vertex l and any $p \neq 1$ with $d(p,l) = d(i,k)$. Here, we can assume that $d(i,k) = d(1,l)$ without loss of generality, since whenever $d(i,k) \neq d(1,l)$, we have $u_{i1}u_{kl} = u_{kl}u_{i1} = 0$ and we are already done. Note that in the case of C_5 for any vertex l, only one vertex $p \neq 1$ is in the same distance to l as 1, and therefore, the above means that for any l, we only have to show

$$u_{i1}u_{kl}u_{ip} = 0$$

for a single vertex p.

In order to show that $u_{i1}u_{kl}u_{ip} = 0$ for the above specified vertices l and p, we will first use Lemma 2.1.5. To get that $u_{i1}u_{k2} = u_{k2}u_{i1}$, we will now use the lemma to show that $u_{i1}u_{k2}u_{i3} = 0$. Looking at the conditions for applying Lemma 2.1.5, we thus want to find a vertex q such that $d(1,q) \neq d(q,3)$ and such that 2 is the only vertex that is both in distance $s = d(2, q)$ from q and in distance 1 from both vertices 1 and 3. Choosing $q := 1$ satisfies this as can be seen by looking at the graph



We can therefore apply Lemma 2.1.5 to get that $u_{i1}u_{k2}u_{i3} = 0$, and therefore,

$$u_{i1}u_{k2} = u_{i1}u_{k2}u_{i1},$$

as desired. In a very similar manner, we can apply Lemma 2.1.5 to $l = 5$. Both of these applications are summarised in the following table, using the notation from Notation 3.2:

Lemma 2.1.5.

j	l	p	q
1	2	3	1
	5	4	1

This table thus already shows that the u_{i1} commute with all u_{kl}, where l is adjacent to 1. Using Lemma 2.2.1, we thus get that all u_{ij} and u_{kl} commute whenever $j \sim l$. We now want to use this information to apply Lemma 2.1.4 to show that $u_{i1}u_{k3} = u_{i1}u_{k3}u_{i1}$. We thus have to find a vertex q with $d(q, 3) = 1$ such that there is exactly one vertex p that satisfies

$$d(3, p) = 2 \quad \text{and} \quad d(p, q) = d(j, q).$$

Choosing $q := 2$ only leaves the vertex $p = 1$ satisfying these conditions, and therefore, Lemma 2.1.4 yields

$$u_{i1}u_{k3} = u_{i1}u_{k3}u_{i1},$$

and, therefore, also

$$u_{i1}u_{k3} = u_{k3}u_{i1}.$$

In a similar manner, we can apply Lemma 2.1.4 to get

$$u_{i1}u_{k4} = u_{i1}u_{k4}u_{i1},$$

and again, both of these applications are summarised in the following table according to Notation 3.1:

Lemma 2.1.5.

j	l	q	p
1	3	2	$\{1\}$
	4	3	$\{1\}$

All in all, we see that $u_{i1}u_{kl} = u_{kl}u_{i1}$ for all vertices i, k, l, and thus, by Lemma 2.2.1, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all generators of the quantum automorphism group of C_5.

4. Constructions from smaller graphs

In this section, we consider graphs that arise as constructions of smaller graphs. Often, their quantum automorphism groups can be computed in a simple manner from the quantum automorphism groups of the smaller graphs.

4.1. Disconnected graphs

Vertex-transitive graphs that are disconnected are always disjoint copies of a single connected graph. For such disjoint copies of a connected graph, it was shown in [2] how to compute the quantum automorphism groups.

Theorem 4.1.1 ([2]). *If Γ is a connected graph, then the graph consisting of n disjoint copies of Γ has quantum automorphism group*

$$G^+_{\mathrm{aut}}(n\Gamma) = G^+_{\mathrm{aut}}(\Gamma) \wr_* S_n^+.$$

We thus can compute the quantum automorphism groups of these graphs easily as shown in the following table:

Graph	Automorphism group	Quantum automorphism group
$6K_2$	$\mathbb{Z}_2 \wr S_6$	$\mathbb{Z}_2 \wr_* S_6^+$
$4K_3$	$S_3 \wr S_4$	$S_3 \wr_* S_4^+$
$3K_4$	$S_4 \wr S_3$	$S_4^+ \wr_* S_3$
$3C_4$	$H_2 \wr S_3$	$H_2^+ \wr_* S_3$
$2K_6$	$S_6 \wr \mathbb{Z}_2$	$S_6^+ \wr_* \mathbb{Z}_2$
$2C_6$	$D_6 \wr \mathbb{Z}_2$	$D_6 \wr_* \mathbb{Z}_2$
$2(K_2 \square K_3)$	$D_6 \wr \mathbb{Z}_2$	$D_6 \wr_* \mathbb{Z}_2$
$2C_6(2)$	$(\mathbb{Z}_2 \wr S_3) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* S_3) \wr_* \mathbb{Z}_2$
$2C_6(3)$	$(S_3 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(S_3 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$

4.2. Products of smaller graphs

For finite graphs, one can define the following product operations.

Definition 4.2.1. Let Γ and Γ' be finite graphs.

(1) The direct product $\Gamma \times \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and the following edges:

$$(i,k) \sim (j,l) \Leftrightarrow i \sim j \quad \text{and} \quad k \sim l.$$

(2) The Cartesian product $\Gamma \Box \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and the following edges:

$$(i,k) \sim (j,l) \Leftrightarrow i = j, \quad k \sim l \quad \text{or} \quad i \sim j,\ k = l.$$

In [1], the following theorem about quantum automorphism groups of graph products was proven.

Theorem 4.2.2. *Let Γ and Γ' be finite connected regular graphs. If their spectra $\{\lambda\}$ and $\{\mu\}$ do not contain 0 and satisfy*

$$\{\lambda_i/\lambda_j\} \cap \{\mu_k/\mu_l\} = \{1\},$$

then

$$C(G_{\text{aut}}^{+}(\Gamma \times \Gamma')) = C(G_{\text{aut}}^{+}(\Gamma)) \otimes C(G_{\text{aut}}^{+}(\Gamma')).$$

If the spectra satisfy

$$\{\lambda_i - \lambda_j\} \cap \{\mu_k - \mu_l\} = \{0\},$$

then

$$C(G_{\text{aut}}^{+}(\Gamma \Box \Gamma')) = C(G_{\text{aut}}^{+}(\Gamma)) \otimes C(G_{\text{aut}}^{+}(\Gamma')).$$

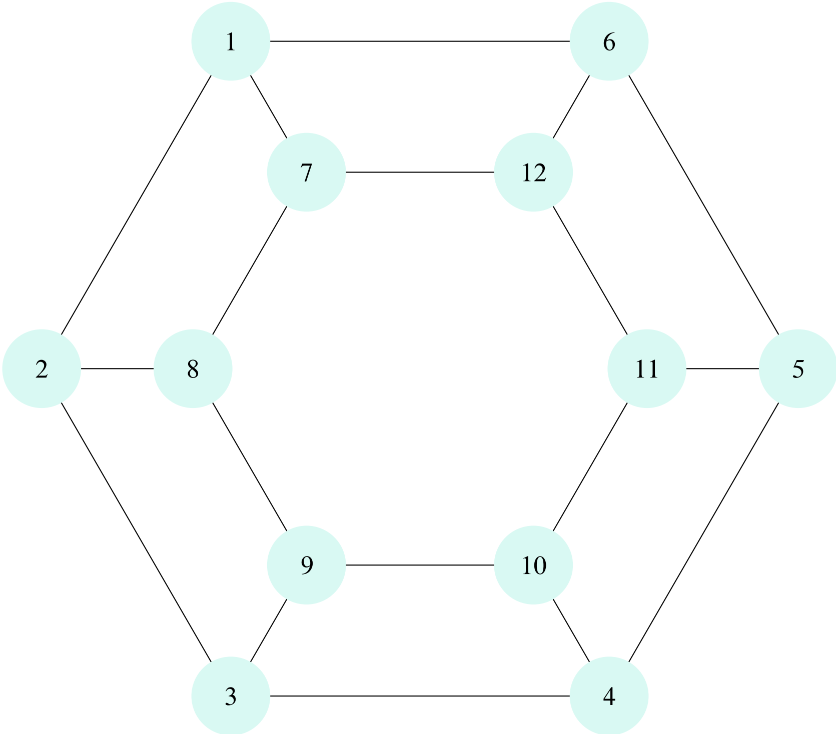
This theorem yields the quantum automorphism group for a few of the product graphs already given in the following table:

Graph	Automorphism group	Quantum automorphism group
$K_6 \times K_2$	$S_6 \times \mathbb{Z}_2$	$S_6^+ \times \mathbb{Z}_2$
$K_3 \times K_4$	$S_3 \times S_4$	$S_3 \times S_4^+$
$C_4 \Box C_3$	$H_2 \times S_3$	$H_2^+ \times S_3$
$K_2 \Box C_6(3)$	$\mathbb{Z}_2 \times (S_3 \wr \mathbb{Z}_2)$	$\mathbb{Z}_2 \times (S_3 \wr_* \mathbb{Z}_2)$

For the two product graphs $K_2 \Box C_6$ and $K_2 \Box C_6(2)$, the theorem does not apply, and they therefore have to be studied individually. We will first consider$K_2 \Box C_6$.

Proposition 4.2.3. *The graph $K_2 \Box C_6$, pictured below, does not have quantum symmetry.*

Proof. We will show that $u_{i1}u_{kl} = u_{kl}u_{i1}$ for all vertices i, k, l of $K_2 \Box C_6$. Then, the vertex transitivity will imply the commutativity of all generators.



We begin with distance 2. There are 4 vertices in distance 2 from 1: 3, 5, 8, and 12. First, considering vertex 3, we see that

$$u_{i1}u_{k3} = u_{i1}u_{k3}(u_{i1} + u_{i5} + u_{i8} + u_{i10}).$$

We will now apply Lemma 2.1.5 to 1 and 3 using the notation introduced in Notation 3.2:

Lemma 2.1.5.

j	l	p	q
1	3	5	2
		8	6

We thus get that $u_{i1}u_{k3}u_{i5} = 0 = u_{i1}u_{k3}u_{i8}$. Recall now that by Definition 2.3.4, we denote by $\mathrm{CN}(i, j)$ the set of common neighbours of the vertices i and j. We can now see that $|\mathrm{CN}(1, 3)| = 1$, but we have $|\mathrm{CN}(3, 10)| = 2$, and we thus get from Corollary 2.3.6 that $u_{i1}u_{k3}u_{i10} = 0$. We thus get $u_{i1}u_{k3} = u_{k3}u_{i1}$.

Next, we consider vertex 8 and apply Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	8	3	4
		12	2

Only $u_{i1}u_{k8}u_{i10}$ is the missing term, but seeing that $|\mathrm{CN}(1,8)| = 2 \neq |\mathrm{CN}(8,10) = 3$, we get again by Corollary 2.3.6 that $u_{i1}u_{k8}u_{i10} = 0$, and thus, $u_{i1}u_{k8} = u_{k8}u_{i1}$.

We observe that mirroring on the line that goes through 1, 7, 10, and 4 is an automorphism of $K_2 \square C_6$ that keeps 1 fixed and maps 3 to 5 and 8 to 12, we get by Lemma 2.2.1 that $u_{i1}u_{k5} = u_{k5}u_{i1}$ and also $u_{i1}u_{k12} = u_{k12}u_{i1}$. We thus have for all vertices j, l with $d(j,l) = 2$ that $u_{ij}u_{kl} = u_{kl}u_{ij}$.

For distance 1, we apply Lemma 2.1.4 to all neighbours of the vertex 1:

Lemma 2.1.4.

j	l	q	p
1	2	6	$\{1\}$
	6	2	$\{1\}$
	7	2	$\{1,8\}$

We see that $u_{i1}u_{k2} = u_{k2}u_{i1}$ and $u_{i1}u_{k6} = u_{k6}u_{i1}$. For $l = 7$, we moreover see that

$$u_{i1}u_{k7} = u_{i1}u_{k7}(u_{i1} + u_{i8}).$$

Looking at $u_{i1}u_{k7}u_{i8}$, we see that

$$\begin{aligned} u_{i1}u_{k7}u_{i8} &= u_{i1}u_{k7} \sum_{\substack{r;d(r,i)=1 \\ d(r,k)=2}} u_{r9}u_{i8} \\ &= u_{i1} \sum_{\substack{r;d(r,i)=1 \\ d(r,k)=2}} u_{r9}u_{k7}u_{i8} \quad \text{since } d(7,9) = 2 \\ &= 0 \quad \text{since } d(1,9) = 3 \neq d(r,i) = 1. \end{aligned}$$

For distance 3, it is enough to only apply Lemma 2.1.4:

Lemma 2.1.4.

j	l	q	p
1	4	10	$\{1\}$
	9	10	$\{1\}$
	11	10	$\{1\}$

Lastly, since 10 is the only vertex in distance 4 to 1, and thus, also vice versa, we get

$$u_{i1}u_{k10} = u_{i1}u_{k10} \sum_{r;d(r,10)=4} u_{ir} = u_{i1}u_{k10}u_{i1}.$$

From this, we get $u_{i1}u_{k10} = u_{k10}u_{i1}$, and therefore, $C(G^+_{\mathrm{aut}}(K_2 \square C_6))$ is commutative. ■

A lot of the proofs that a graph does not have quantum symmetries are similar to the one above, repeatedly applying Lemmas 2.1.5 and 2.1.4 and then using these results together with some properties of the specific graph to get the commutation of the rest of the generators. For this reason, we give the rest of such proofs in Appendix A.

Proposition 4.2.4. *The graph $K_2 \square C_6(2)$ has quantum symmetries.*

Proof. We have the following automorphisms of $K_2 \square C_6(2)$:

$$([1,1][1,4])([2,1][2,4]) \in \mathrm{Aut}(K_2 \square C_6(2)),$$
$$([1,2][1,5])([2,2][2,5]) \in \mathrm{Aut}(K_2 \square C_6(2)).$$

These are non-trivial and disjoint, and thus, by Lemma 2.1.2, $K_2 \square C_6(2)$ has quantum symmetries. ∎

Graph	Automorphism group	Quantum automorphism group
$K_2 \square C_6$	$\mathbb{Z}_2 \times D_6$	$\mathbb{Z}_2 \times D_6$
$K_2 \square C_6(2)$	$\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr S_3)$	?

5. Circulant and semi-circulant graphs

Two related families of vertex-transitive graphs are circulant and semi-circulant graphs. Roughly speaking they arise when taking a cycle-graph and adding edges between certain vertices.

Prior results yield the non-existence of quantum symmetries for a few of these graphs; however, most of them have to be considered individually.

5.1. Circulant graphs

Definition 5.1.1. For $1 < k_1 < \cdots < k_r \leq [\frac{n}{2}]$, we define the graph $C_n(k_1, \ldots, k_r)$, the *circulant graph* on n vertices with chords in distances $k_1, \ldots, k_r$, as the graph obtained by drawing the n-cycle C_n and then connecting all pairs of vertices in distance k_i for any i. In other words, the vertex set is $V = \{1, \ldots, n\}$, and for $i, j \in V$, we have

$$i \sim j \Leftrightarrow |i - j| \mod n \in \{1, k_1, \ldots, k_r\}.$$

For some circulant graphs [3, Theorem 3.1], shows that they do not have quantum symmetries.

Theorem 5.1.2 ([3]). *Given a circulant graph $C_n(k_1, \ldots, k_r)$ and putting $k_0 := 1$, we define the function $f : \{1, 2, \ldots, [\frac{n}{2}]\} \to \mathbb{R}$ as follows:*

$$f(s) = \sum_{i=0}^{r} \cos\left(\frac{2k_i s\pi}{n}\right).$$

If $n \neq 4$ and the function f is injective, then the graph $C_n(k_1, \ldots, k_r)$ does not have quantum symmetries.

Proposition 5.1.3. *The graphs C_{12}, $C_{12}(3)$ and $C_{12}(6)$ do not have quantum symmetries.*

Proof. We apply Theorem 5.1.2, and thus, we need to compute the values of the associated function f on the set $\{1, \dots, 6\}$:

	1	2	3	4	5	6
C_{12}	0.87	0.5	0	−0.5	−0.87	−1
$C_{12}(3)$	0.87	−0.5	−1.23	0.5	−0.87	−2
$C_{12}(6)$	−0.13	1.5	−1	0.5	−1.87	0

Here, the rounding error is less than 0.01, and thus, for all three graphs the function is injective. ∎

For some other graphs, we see that they have disjoint automorphisms, and therefore, they do have quantum symmetries.

Proposition 5.1.4. *The graphs K_{12}, $C_{12}(5)$, $C_{12}(4,5)$, and $C_{12}(5,6)$ have quantum symmetries.*

Proof. It is easy to see that the complete graph on n vertices has quantum automorphism group S_n^+, and thus, for $n \geq 4$ it does have quantum symmetries. For the other graphs, we can check that they have disjoint automorphisms, and thus, by Lemma 2.1.2, they have quantum symmetries.

Graph	Automorphism 1	Automorphism 2
$C_{12}(5)$	$(1,7)$	$(4,10)$
$C_{12}(4,5)$	$(1,7)\ (3,9)\ (5,11)$	$(2,8)\ (4,10)\ (6,12)$
$C_{12}(5,6)$	$(1,7)$	$(4,10)$

∎

For $C_{12}(4,5)$, we managed to compute the actual quantum automorphism group. We give the computation in Section 7.

Proposition 5.1.5. *The graphs $C_{12}(2)$, $C_{12}(4)$, $C_{12}(2,6)$, $C_{12}(3,6)$ and $C_{12}(4,6)$ do not have quantum symmetries.*

The proofs of the above statement are very similar to the one of Proposition 4.2.3, but for completeness they are given in Appendix A.

Summarising, we see that of the twelve circulant graphs on 12 points, four have quantum symmetries and eight do not.

Graph	Automorphism group	Quantum automorphism group
C_{12}	D_{12}	D_{12}
$C_{12}(3)$	D_{12}	D_{12}
$C_{12}(6)$	D_{12}	D_{12}
K_{12}	S_{12}	S_{12}^+
$C_{12}(5)$	$\mathcal{A}$	?
$C_{12}(4,5)$	$H_2 \times S_3$	$H_2^+ \times S_3$
$C_{12}(5,6)$	$\mathcal{A}$	?
$C_{12}(2)$	D_{12}	D_{12}
$C_{12}(4)$	D_{12}	D_{12}
$C_{12}(2,6)$	D_{12}	D_{12}
$C_{12}(3,6)$	D_{12}	D_{12}
$C_{12}(4,6)$	D_{12}	D_{12}

Here, we call

$$\mathcal{A} := (\mathbb{Z}_2 \times ((((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2.$$

5.2. Semi-circulant graphs

Definition 5.2.1. For $1 < k_1 < k_2 < \cdots < k_r \leq [\frac{n}{2}]$, we define the *semi-circulant* graph $C_n(k_1, \ldots, k_r, l_1^+, \ldots, l_s^+)$ as the graph that is constructed by taking the circulant graph $C_n(k_1, \ldots, k_r)$ and adding edges between $i, j \in V$ if i is even, $i < j$ and $j - i \in \{l_1^+, \ldots, l_s^+\}$.

As was the case with the circulant graphs, we see that some semi-circulant graphs have disjoint automorphisms, and therefore, quantum symmetries.

Proposition 5.2.2. *The graphs $C_{12}(5^+)$, $C_{12}(3^+, 6)$, and $C_{12}(5^+, 6)$ have quantum symmetries.*

Proof. The graphs have the following disjoint automorphisms, and thus, have quantum symmetries by Lemma 2.1.2.

Graph	Automorphism 1	Automorphism 2
$C_{12}(5^+)$	$(1,7)\ (2,8)$	$(3,9)\ (4,10)$
$C_{12}(3^+,6)$	$(2,7)\ (3,10)\ (6,11)$	$(1,8)\ (4,9)\ (5,12)$
$C_{12}(5^+,6)$	$(1,7)\ (2,8)$	$(3,9)\ (4,10)$

∎

For $C_{12}(3^+, 6)$, we managed to compute the actual quantum automorphism group. Again, we give the proof in Section 7.

Proposition 5.2.3. *The graphs $C_{12}(2, 5^+)$ and $C_{12}(4, 5^+)$ do not have quantum symmetries.*

The proofs of the above proposition are again very similar to the proof of Proposition 4.2.3, but for completeness they are given in Appendix A.

All in all, the following table summarises the results for semi-circulant graphs:

Graph	Automorphism group	Quantum automorphism group
$C_{12}(5^+)$	$\mathbb{Z}_2 \times S_4$	?
$C_{12}(3^+, 6)$	$H_2 \times S_3$	$H_2^+ \times S_3$
$C_{12}(5^+, 6)$	$\mathbb{Z}_2 \times S_4$	?
$C_{12}(2, 5^+)$	D_6	D_6
$C_{12}(4, 5^+)$	D_6	D_6

6. Special cases

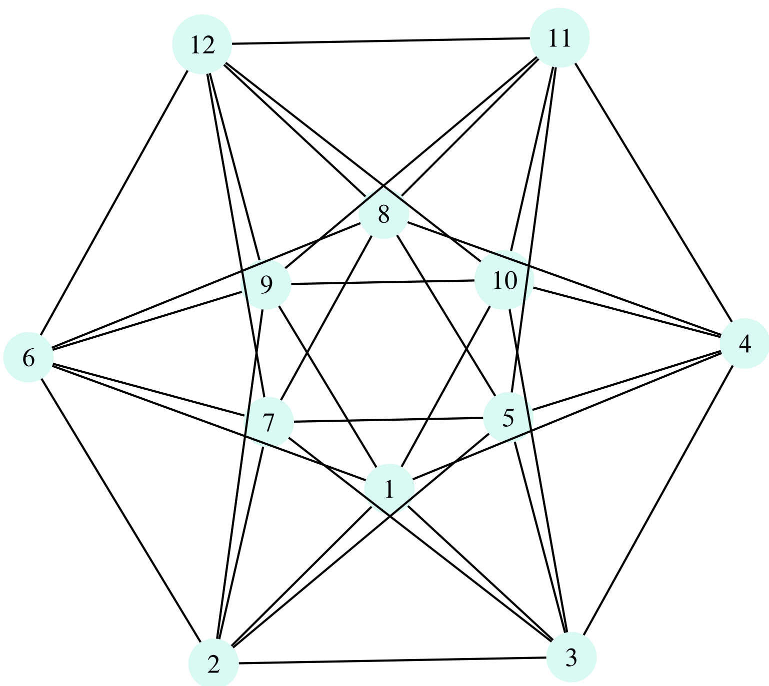
The following five graphs do not fall in any of the previous subclasses of graphs, and therefore, have to be considered individually.

6.1. The Icosahedron

It was shown by Schmidt in his PhD thesis that the Icosahedron does not have quantum symmetries.

Proposition 6.1.1 ([19]). *The Icosahedron does not have quantum symmetries.*

6.2. The line graph of $C_6(2)$



Proposition 6.2.1. *The graph $L(C_6(2))$ does not have quantum symmetries.*

Proof. The commutation of u_{i1} with u_{kl} for $l \in \{4, 5, 6, 7, 8, 11, 12\}$ can be seen by just applying Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	4	3	9
		5	7
		8	2
		10	2
		11	5
	5	6	3
		9	12
		10	6
		12	1

Lemma 2.1.5.

j	l	p	q
1	6	2	10
		7	5
		8	2
		9	3
		12	7
	7	4	2
		9	4
		10	11
		11	1

Lemma 2.1.5.

j	l	p	q
1	8	2	10
		3	9
		9	3
		10	2
	11	2	7
		3	6
		6	3
		7	1

Lemma 2.1.5.

j	l	p	q
1	12	2	4
		3	5
		4	2
		5	1

The only vertices l for which we still need to show commutativity of u_{i1} and u_{kl} are thus $l \in \{2, 3, 9, 10\}$. However, since we have the following three automorphisms in $G_{\mathrm{aut}}(L(C_6(2)))$, we only need to show it for one such l and the rest follows immediately by Lemma 2.2.1:

$$\phi = (2, 3)\ (4, 6)\ (5, 7)\ (9, 10)\ (11, 12),$$
$$\sigma = (2, 9)\ (3, 10)\ (5, 11)\ (7, 12),$$
$$\tau = (2, 10)\ (3, 9)\ (4, 6)\ (5, 12)\ (7, 11).$$

We will therefore now show commutation of u_{i1} and u_{k2}. We do this by first applying Lemma 2.1.4 and then using Lemma 2.1.5 to see that the remaining monomials vanish:

Lemma 2.1.4.

j	l	q	p
1	2	4	$\{1,3,5\}$

Lemma 2.1.5.

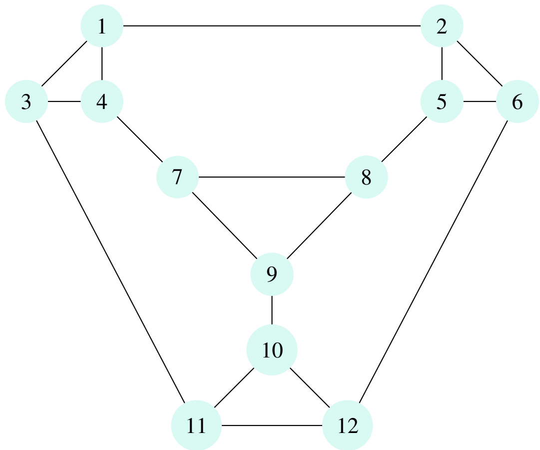
j	l	p	q
1	2	3	6
		5	6

All in all we thus get that $G^+_{\text{aut}}(L(C_6(2))) = G_{\text{aut}}(L(C_6(2)))$. ∎

6.3. The truncated tetrahedron

We want to show that the quantum automorphism group of $\text{Trunc}(K_4)$ is commutative, i.e., $G_{\text{aut}}(\text{Trunc}(K_4)) = G^+_{\text{aut}}(\text{Trunc}(K_4))$.

Theorem 6.3.1. *The truncated tetrahedron,* $\Gamma = \text{Trunc}(K_4)$*, shown below, does not have quantum symmetry, i.e.,* $G^+_{\text{aut}}(\Gamma)$ *is commutative.*



Before we prove the above theorem, we want to collect some properties of Γ.

Properties 6.3.2. We have the following properties of Γ.

- Γ has diameter 3, i.e., the largest distance between two vertices is 3.
- Γ is 3-regular, i.e., every vertex has exactly 3 neighbours.
- Γ consists of 4 triangles.
- Each vertex of Γ is connected exactly to two vertices in the same triangle and one vertex in a different triangle.

We now want to introduce some notation to make talking about the different triangles easier.

Definition 6.3.3. We have the following notation.

(i) We denote by $\mathcal{T}$ the set of the four triangles that make up Γ.

(ii) Let $v \in V(\Gamma)$. We denote by $T(v)$ the triangle containing v.

(iii) Let $v \in V(\Gamma)$. We denote by $AT(v)$ the unique triangle that is *adjacent* to v, i.e., that is connected to v by an edge but is different from $T(v)$.

(iv) Let $v \in V(\Gamma)$. We denote by $\mathcal{NT}(v) := \mathcal{T} \setminus \{T(v), AT(v)\}$ the set of *non-adjacent triangles* of v.

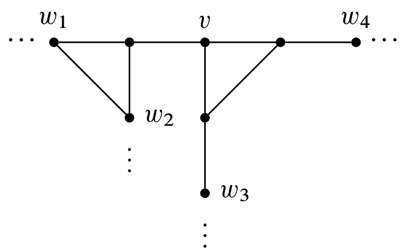
Lemma 6.3.4. *Let $v, w \in V(\Gamma)$ be two vertices. If they are in the same triangle, then their adjacent triangles are distinct:*

$$T(v) = T(w) \Rightarrow AT(v) \neq AT(w).$$

Lemma 6.3.5. *Let $v, w \in V(\Gamma)$ be two vertices with $d(v, w) = 2$. Then, it holds that either v is in the adjacent triangle of w or vice versa:*

$$d(v, w) = 2 \Rightarrow v \in AT(w) \vee w \in AT(v).$$

Proof. If v is not in $AT(w)$, then in order to reach v from w, one step has to be made inside $T(w)$. Since $d(v, w) = 2$, the next step, which is the first step out of $T(w)$, will have to reach v already. Thus, by the definition of $AT(v)$, the triangle we just left is $AT(v)$. Since it is also $T(w)$, we get $w \in AT(v)$. The proof can be visualized with the following sketch:



■

Lemma 6.3.6. *Let $v \in V(\Gamma)$ be a vertex. All vertices $w \in V(\Gamma)$ that fulfill $d(v, w) = 3$ are in the triangles in $\mathcal{NT}(v) = \mathcal{T} \setminus \{T(v), AT(v)\}$.*

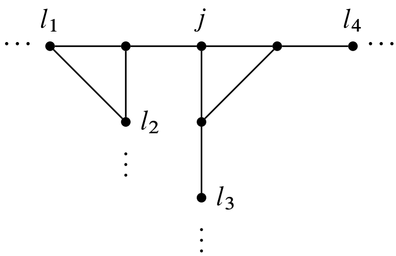
Proof. Indeed, if w was in $T(v)$, then the distance of w to v would be 1. On the other hand, if $w \in AT(v)$ holds, then either $d(v, w) = 1$, if w is the unique vertex in $AT(v)$ that shares an edge with v, or $d(v, w) = 2$, as w is connected to the vertex in $AT(v)$ that shares an edge with v. ■

We now come to the proof of Theorem 6.3.1.

Proof. Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$. We want to show that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $i, j, k, l \in V(\Gamma)$. By Lemma 2.1.3, it suffices to show the statement for i, j, k, l such that $d(i, k) = d(j, l)$.

Step 1: $d(i,k) = d(j,l) = 1$. Since Γ does not contain any quadrangles, by Lemma 2.3.1 we have $G^+_{\text{aut}}(\Gamma) = G^*_{\text{aut}}(\Gamma)$, and therefore, $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $d(i,k) = d(j,l) = 1$.

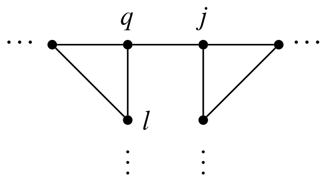
Step 2: $d(i,k) = d(j,l) = 2$. Let us fix $j \in V(\Gamma)$. Then, there are four vertices at distance 2 from j: two of them are in $AT(j)$, reached by following the edge into $AT(j)$ and then following either edge within the triangle, and one is in each of the triangles in $\mathcal{NT}(j)$, reached by following either edge within $T(j)$ and then taking the single edge that leads out of $T(j)$. Let us sketch the situation:



In this situation, l_1 and l_2 are the vertices at distance 2 that are in $AT(j)$ and l_3 and l_4 are each in one of the triangles of $\mathcal{NT}(j)$. Note that while l_3 and l_4 are not in $AT(j)$, it holds that $j \in AT(l_3)$ and $j \in AT(l_4)$ by Lemma 6.3.5. Let now l be one of l_1 and l_2, i.e., l is a vertex such that $d(j,l) = 2$ and $l \in AT(j)$. We will show that for all $i,k \in V(\Gamma)$ with $d(i,k) = 2$ the following holds:

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}.$$

Then, by Lemma 2.1.1, we have $u_{ij}u_{kl} = u_{kl}u_{ij}$. We want to use Lemma 2.1.4, with distance $m = 2$ and choosing $s = t = 1$, which we can do since $G^+_{\text{aut}}(\Gamma) = G^*_{\text{aut}}(\Gamma)$, we want to find a vertex q for which it holds that $d(j,q) = 1$ and $d(q,l) = 1$. If we choose q such that the only vertex p fulfilling $d(p,l) = 2$ and $d(p,q) = 1$ is j, then we get the desired result. Let us again take a look at the situation we have.



Choosing q as indicated in the sketch above, we see that there are three vertices adjacent to q, one of which is l and one is j. The third vertex is in the same triangle as l, and thus, is at distance 1 from l. Therefore, the only vertex adjacent to q and at distance 2 from l is in fact j. Lemma 2.1.4 yields

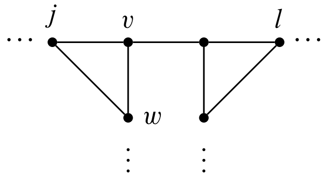
$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(p,l)=2 \\ (p,q)\in E(\Gamma)}} u_{ip} = u_{ij}u_{kl}u_{ij}.$$

Thus, for all $i, j, k, l \in V(\Gamma)$ with $d(i,k) = d(j,l) = 2$ and $l \in AT(j)$, u_{ij} and u_{kl} commute. As noted above, it holds that if $l \notin AT(j)$, we have $j \in AT(l)$. Therefore, we get that u_{ij} and u_{kl} commute for all i, j, k, l with $d(i,k) = d(i,j) = 2$.

Step 3: $d(i,k) = d(j,l) = 3$. Fixing again $j \in V(\Gamma)$, we know by Lemma 6.3.6 that all vertices l with $d(j,l)=3$ are in the triangles in $\mathcal{NT}(j)$, in other words $l \notin \{T(j), AT(j)\}$. Note, moreover, that $\mathcal{NT}(j) = \{AT(s), AT(t)\}$ if $T(j) = \{j, s, t\}$. Indeed, there are 4 triangles in total, and since $T(j) = T(s)$ implies $AT(j) \neq AT(s)$, we can write the set of all triangles as

$$\mathcal{T} = \big\{T(j), AT(j), AT(s), AT(k)\big\}.$$

Let now l be a fixed vertex at distance 3 from j and let $v \in T(j)$ be the vertex that satisfies $l \in AT(v)$. We are thus in the situation sketched below.



In this situation, we can use Lemma 2.1.4 with $s = 1, t = 2$, putting $q := v$, to get for all $i, k \in V(\Gamma)$, $d(i,k) = 3$ that

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p; d(p,l)=3 \\ (p,q)\in E(\Gamma)}} u_{ip}.$$

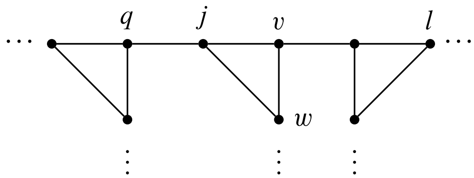
Note that we can apply Lemma 2.1.4 here as we have shown above that u_{xy} and $u_{x'y'}$ commute with $d(x,x') = d(y,y') = 2$, and thus, in particular for all a with $d(a,k) = 2$ it holds that u_{kl} and u_{aq} commute. If we now have $d(w,l) \neq 3$, then we are already done, since we then get by the above application of Lemma 2.1.4 that $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$, since then j is the only vertex connected to q and at distance 3 to l. Thus, u_{ij} and u_{kl} commute by Lemma 2.1.1.

If, however, $d(w,l) = 3$, then both j and w are connected to q and at distance 3 to l and Lemma 2.1.4 thus yields

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{iw}).$$

It remains to show that if $d(w,l) = 3$ we have $u_{ij}u_{kl}u_{iw} = 0$ for all $d(i,k) = 3$, since then $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$, and thus, u_{ij} and u_{kl} commute by Lemma 2.1.1. We want to apply Lemma 2.1.5 to i, j, k, l and w in order to show this. We thus have $m = 3$. The lemma is applicable, since we have $w \neq j$ and $d(w,l) = 3$. We want to find a vertex q and a distance s such that $d(q,l) = s$ and $d(j,q) \neq d(q,w)$ and that moreover fulfills that l is the only vertex satisfying $d(q,l) = s$, $d(l,j) = 3$ and $d(l,p) = 3$. If we find such a q and s, we are done, since then, by Lemma 2.1.5, $u_{ij}u_{kl}u_{iw} = 0$.

We put q as the unique vertex in $AT(j)$ that is adjacent to j and $s := d(q,l)$. We are thus in the situation sketched below.



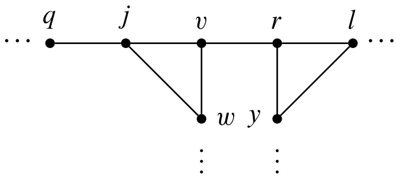
Then, obviously, $s = d(q,l)$ holds. Moreover, $1 = d(j,q) \neq d(w,q) = 2$ holds, since $T(q) \in \mathcal{NT}(w)$. It remains to show that for any vertex $x \in V(\Gamma)$ the following holds:

$$d(x,q) = s, \quad d(x,j) = 3, \quad d(x,w) = 3 \Rightarrow t = l.$$

Let, therefore, such a vertex x be given. We claim that $x \in AT(v)$.

Indeed, if $x \notin AT(v)$, then either x would have to be in $T(v)$ or $T(x)$ would have to be in $\mathcal{NT}(v)$. But if $x \in T(v) = T(j)$, then $d(x,j) \leq 1$ which is a contradiction to $d(x,j) = 3$. If on the other hand, we have $T(x) \in \mathcal{NT}(v) = \{AT(j), AT(w)\}$, then we have either $x \in AT(j)$ or $x \in AT(w)$. If $x \in AT(j)$, then $d(x,j) \leq 2$ as was argued above, which is again a contradiction to $d(x,j) = 3$. If $x \in AT(w)$, we have by the same argument that $d(x,w) \leq 2$, which is a contradiction to $d(x,w) = 3$.

We thus have $x \in AT(v) = T(l)$. Let us look at another sketch of the situation, in order to name the relevant vertices:



We know, so far, that $x \in \{r, y, l\}$. Since however $d(j,r) = d(w,r) = 2 \neq 3$ we can narrow it down to $x \in \{y, l\}$. It remains to show that $d(q,l) \neq d(q,y)$, since then we have shown that $x = l$. In the sketch above, we see that $T(q) \in \mathcal{NT}(r)$, and thus, we have either $q \in AT(l)$ or $q \in AT(y)$. We will show the claim for $q \in AT(l)$, the other situation can be shown similarly. If q is in $AT(l)$, then we have $d(q,l) \leq 2$, and since the unique vertex connected to q that is in a different triangle is already j we even know $d(q,l) = 2$. We will now show that $d(q,y) = 3$. Since Γ has diameter 3, we already know that $d(q,y) \leq 3$. Since a shortest path from l to q is of length 2, any path from y to q via l will have at least length 3 and there is also a path of length 3 via l. Moreover, any path from y to q via r will be at least of length 4, as can be seen in the sketch above. The last possibility to reach q from y is thus by first traversing $AT(y)$. But since $q \notin AT(y)$, which is the case as $q \in AT(l)$, any such path will consist of first taking the edge from y to $AT(y)$, then taking one edge within $AT(y)$ and then leaving $AT(y)$ via a third edge, and thus, such a path is at least of length 3 again. We thus conclude $d(q,l) = 2 \neq 3 = d(q,y)$.

Therefore, we can apply Lemma 2.1.5 and get

$$u_{ij}\Bigg(\sum_{\substack{x;d(x,j)=d(x,w)=3\\ d(q,x)=d(q,l)}} u_{kx}\Bigg)u_{iw} = u_{ij}u_{kl}u_{iw} = 0.$$

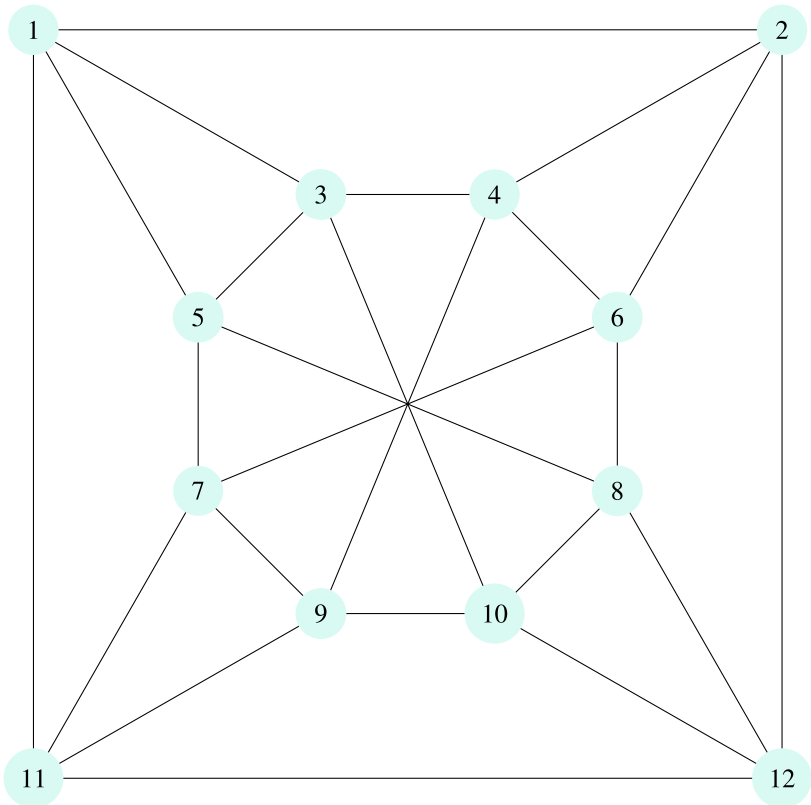
We thus conclude that u_{ij} and u_{kl} commute for any $i, j, k, l \in V(\Gamma)$ such that $d(i, k) = d(j, l) = 3$.

Since the diameter of Γ is 3, we now know that all generators of $G^+_{\text{aut}}(\Gamma)$ commute, and thus, $G^+_{\text{aut}}(\Gamma) = G_{\text{aut}}(\Gamma)$. ∎

6.4. The distance 3 graph of the truncated tetrahedron

We will show that the distance 3 graph of the truncated tetrahedron, Antip(Trunc(K_4)), does not have quantum symmetry.

Theorem 6.4.1. *The distance* 3 *graph of the truncated tetrahedron* $\Gamma = \text{Antip}(\text{Trunc}(K_4))$ *shown below, does not have quantum symmetry, i.e.,* $G^+_{\text{aut}}(\Gamma)$ *is commutative.*



We collect some properties about the graph before proving the theorem.

Properties 6.4.2. We have the following properties.

- Γ has diameter 2.
- Γ is 4-regular.
- The graph is made up of 4 disjoint triangles, i.e., for each vertex $v \in V$, it holds that there are two other vertices w and x such that $\{v, w, x\}$ forms a triangle and for all other vertices $y \in V \backslash \{v, w, x\}$ it holds that if $(v, y) \in E$ then v and y have no common neighbour.
- Every vertex v is also part of three distinct quadrangles. On the one hand, it is part of one containing vertices from all 4 triangles and on the other hand, it is part of two triangles each containing 2 vertices from $T(v)$ and 2 vertices from another triangle. Both of the latter vertices share a triangle however.
- All vertices that are not connected have either one or two common neighbours.
- For every pair of vertices that are in the same triangle, there is a unique quadrangle containing the same two vertices.
- For every pair of vertices that are in distance 2, there is a unique quadrangle containing the same two vertices.
- For every pair of adjacent vertices that do not share a triangle, there are 2 quadrangles containing both vertices.

Definition 6.4.3. (i) If $v \in V$ is a vertex, we denote the unique triangle containing v as $T(v)$.

(ii) If $v, w \in V$ and $d(v, w) = 2$, we want to count the number of distinct triangles connected by a possible quadrangle containing v and w. For this, we first note that if a quadrangle $\mathcal{Q}(v, w)$ containing v and w exists, then it is unique, i.e., there are no two distinct quadrangles containing both v and w, for $d(v, w) = 2$, as was noted in the Properties 6.4.2 above. We now put for an existing quadrangle $\mathcal{Q}(v, w)$ consisting of the vertices $\{v, w, x, y\}$ the set of triangles connected by this quadrangle as $\mathcal{T}(\mathcal{Q}(v, w)) := \{T(v), T(w), T(x), T(y)\}$. With this, we define

$$Q(v, w) = \begin{cases} 0, & \text{if } v \text{ and } w \text{ do not share a quadrangle,} \\ |\mathcal{T}(\mathcal{Q}(v, w))| & \text{otherwise.} \end{cases}$$

Note that by the properties of Γ listed above, we have that $Q(v, w) \in \{0, 2, 4\}$ for all vertices v and w at distance 2.

Lemma 6.4.4. *Let v and w be two vertices in Γ such that $d(v, w) = 2$ and $Q(v, w) = 0$. Then, the unique common neighbour p of v and w fulfills that we have either $p \in T(v)$ or $p \in T(w)$.*

Proof. The fact that p exists comes from the fact that $d(v, w) = 2$, and therefore, there is a path of length 2 from v to w. Since however there is no quadrangle containing both v and w, there cannot be two common neighbours.

The fact that $p \in T(v)$ or $p \in T(w)$ holds comes from the fact that each vertex has 4 neighbours, two of which are in the same triangle as the vertex itself, and the other two are in two distinct different triangles. Thus, if we take one edge incident to v that leads to a triangle different from $T(v)$, the vertex we reach has only one neighbour apart from v that is in a different triangle than itself. Therefore, there is only one option to take another edge to a completely new triangle. Repeating this twice more, we again reach v, and since each time there was only one choice, we have traversed the unique quadrangle containing v and connecting all 4 triangles. Since v and w do not share a quadrangle however, none of the vertices we traversed was w, and thus, on the path from v to w, at least one of the two steps has to be made within the same triangle. ∎

Lemma 6.4.5. *Let $v \neq w$ be two vertices in Γ such that $T(v) = T(w)$; in particular, we have $v \sim w$. There is at most one vertex $s \in V$ such that $d(v, s) = 2$, $Q(v, s) = 0$ and $w \sim s$ hold.*

Proof. Observe that both v and w have 4 neighbours, as Γ is 4-regular, and 2 of these neighbours are in the same triangle. Thus, each of v and w have two neighbours in triangles that are distinct from $T(v) = T(w)$. We note that as soon as v has a neighbour t in the same triangle as the neighbour s of w, we have that $\{v, w, s, t\}$ forms a quadrangle, as $s \sim t$ follows from the fact that they share a triangle. Note, moreover, that $s = t$ cannot be the case, since otherwise $\{v, w, s\}$ would form a triangle, which contradicts $T(s) \neq T(v)$. Therefore, we have that $d(v, t) = 2$ and $Q(v, t) > 0$.

If we have on the other hand that as soon as a neighbour s of w is in a triangle, in which v does not have a neighbour, then $d(v, s) = 2$ and $Q(v, s) = 0$.

However, as there are only 3 triangles in Γ distinct from $T(v)$, and as both v and w have each 2 neighbours in distinct triangles, at least one pair of neighbours has to be in the same triangle. Therefore, at most one neighbour s of v fulfills $d(v, s) = 2$ and $Q(v, s) = 0$. ∎

Proof of Theorem 6.4.1. Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{\text{aut}}(\Gamma))$. We will prove that $u_{ij}u_{kl} = u_{kl}u_{ij}$ in several steps. Recall that by Lemma 2.1.3, we only need to consider the case where $d(i, k) = d(j, l)$, since otherwise the product is already 0, and thus, the generators commute.

Step 1. Let first $d(i, k) = d(j, l) = 1$. Let now i and k be in the same triangle and let also j and l share a triangle. Then, by the Properties 6.4.2 of Γ, i, j, k and l fulfill the premise of Lemma 2.3.3. We thus get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 2.3.3.

If we have that one of the pairs of vertices $\{i, k\}$ or $\{j, l\}$ shares a triangle and the other does not, we are in the premise of Corollary 2.3.7, and therefore, we get $u_{ij}u_{kl} = u_{kl}u_{ij} = 0$ for these choices of i, j, k and l.

For $d(i, k) = d(j, l) = 1$, we now still need to show the case where neither i and k nor j and l share a triangle. However, in order to show the claim for this case, we will first need to consider the case of distance 2.

Next, we will consider vertices such that $d(i,k) = d(j,l) = 2$. Here, we will make a case distinction on $Q(i,k)$ and $Q(j,l)$ as defined above in Definition 6.4.3.

Step 2. We first consider the cases where $Q(i,k) \neq Q(j,l)$. In these cases, we will prove $u_{ij}u_{kl} = 0$. Let therefore $Q(i,k) = 2$, $Q(j,l) = 4$. Let q be a common neighbour of j and l. Then, $T(q) \notin \{T(j), T(l)\}$ since $Q(j,l) = 4$ and q lies in the quadrangle connecting j and l. For any common neighbour s of i and k however, we have either $s \in T(i)$ or $s \in T(k)$, since s is in the quadrangle containing i and k and as $Q(i,k) = 2$ there are vertices from only two distinct triangles in this quadrangle, which are $T(i)$ and $T(k)$. Since i and k have exactly 2 common neighbours, we call them s_1 and s_2, where $s_1 \in T(i)$ and $s_2 \in T(k)$. We thus get

$$u_{ij}u_{kl} = u_{ij} \sum_{s \in V} u_{sq}u_{kl} = u_{ij} \sum_{\substack{s \in V \\ s \sim i \\ s \sim k}} u_{sq}u_{kl} = u_{ij}u_{s_1q}u_{kl} + u_{ij}u_{s_2q}u_{kl}. \tag{6.1}$$

Since however, by Step 1, we know that $u_{ab}u_{cd} = 0$ if $T(a) = T(c)$ but $T(b) \neq T(d)$ or vice versa, we get

$$u_{ij}u_{s_1q} = 0$$

since $q \notin T(j)$ and

$$u_{s_2q}u_{kl} = 0$$

since $q \notin T(l)$. This together with (6.1) yields

$$u_{ij}u_{kl} = 0.$$

The same argument can be used to show that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 4$ and $Q(j,l) = 2$.

Step 3. Let now $Q(i,k) = 0$ and $Q(j,l) = 4$. Since $Q(i,k) = 0$, there is no quadrangle containing both i and k. Then, i and k have exactly one common neighbour, let us call it p. By Lemma 6.4.4, we have that either $p \in T(i)$ or $p \in T(k)$. Let now q be one of the common neighbours of j and l. As was argued in Step 2, we have that $T(q) \notin \{T(j), T(l)\}$. We get

$$u_{ij}u_{kl} = u_{ij} \sum_{s \in V} u_{sq}u_{kl} = u_{ij}u_{pq}u_{kl}$$

as q is a common neighbour of j and l and p is the only common neighbour of i and k. If now $q \in T(i)$, we get by Step 1 that

$$u_{ij}u_{pq} = 0$$

as $p \notin T(j)$ and similarly, if $q \in T(k)$, we get

$$u_{pq}u_{kl} = 0.$$

Thus, we get

$$u_{ij}u_{kl} = u_{ij}u_{pq}u_{kl} = 0.$$

We can show, similarly, that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 4$ and $Q(j,l) = 0$.

Step 4. Let $Q(i,k) = 0$ and $Q(j,l) = 2$ and let p be the unique common neighbour of i and k. We argue as in Step 3 that either $p \in T(i)$ or $p \in T(k)$ holds. We know moreover that j and l have two common neighbours, let us call them q_1 and q_2. We assume without loss of generality that $q_1 \in T(j)$ and $q_2 \in T(l)$. We know by Step 1 that $p \in T(i)$ implies $u_{ij}u_{pq_1} = u_{pq_1}u_{ij}$, while $p \in T(k)$ implies

$$u_{kl}u_{pq_2} = u_{pq_2}u_{kl}.$$

In both cases, we can apply Lemma 2.3.5 to get that $u_{ij}u_{kl} = 0$. Similarly, we get that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 2$ and $Q(j,l) = 0$ hold.

All in all, we see that

$$u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$$

holds whenever we have $Q(i,k) \neq Q(j,l)$.

Step 5. We now consider the three cases for $Q(i,k) = Q(j,l)$. Let first $Q(i,k) = Q(j,l) = 4$. Let now $p \in V\backslash\{j\}$ be a vertex that is distinct from j. We want to show that for any such p, we have

$$u_{ij}u_{kl}u_{ip} = 0$$

since then we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{v\in V} u_{iv} = u_{ij}u_{kl}u_{ij}$$

and then, by Lemma 2.1.1, u_{ij} and u_{kl} commute. First, we note that if $p = l$, we have $u_{kl}u_{il} = 0$, since $k \neq i$. Next, if $d(l,p) = 1$, we have $u_{kl}u_{ip} = 0$, since $d(k,i) = 2$, and then the statement follows from Lemma 2.1.3. The last case is thus $d(l,p) = 2$. We note that by the Properties 6.4.2 of Γ, l is part of exactly one quadrangle connecting all 4 triangles, i.e., there is only one vertex v such that $d(l,v) = 2$ and $Q(l,v) = 4$. Since j already fulfills both of these properties, we know that $Q(l,p) \neq 4$. Therefore, we have $Q(k,i) \neq Q(l,p)$, and thus, $u_{kl}u_{ip} = 0$.

Step 6. We now consider $Q(i,k) = Q(j,l) = 2$. Let p be again a vertex in $V\backslash\{j\}$. As above, we want to show

$$u_{ij}u_{kl}u_{ip} = 0.$$

Again, we already get $u_{kl}u_{ip} = 0$ if $d(l,p) \in \{0,1\}$. Let therefore $d(l,p) = 2$. If now $Q(l,p) \neq 2$, we get $u_{kl}u_{ip} = 0$ by Steps 2 and 4, as $Q(i,k) = 2$. We thus consider $Q(l,p) = 2$. Let now q be a common neighbour of i and k such that $q \in T(k)$ and let s and t be the common neighbours of j and l such that $s \in T(j)$ and $t \in T(l)$. In particular, we have that $q \notin T(i)$. We denote by $\mathcal{Q}(j,l)$ the unique quadrangle containing j and l and by $\mathcal{Q}(l,p)$ the unique quadrangle containing l and p. We now argue that $d(t,p) \neq 1$. If t and p were connected, there would be a quadrangle consisting of vertices $l \sim t \sim p \sim v \sim l$, with the fourth vertex being a common neighbour $v \neq t$ of l and p. Such a vertex v exists since $Q(l,p) = 2$. Since we know by the Properties 6.4.2 of Γ

that two vertices in the same triangle only share exactly one quadrangle, we thus get a contradiction, since we already have

$$t \in \mathcal{Q}(j,l) = l \sim t \sim j \sim s \sim l.$$

Therefore, $d(t,p) \neq 1$. We compute

$$\begin{aligned} u_{ij}u_{kl}u_{ip} &= u_{ij} \sum_{v \in V} u_{qv}u_{kl}u_{ip} = u_{ij} \sum_{\substack{v \in V \\ v \sim j \\ v \sim l}} u_{qv}u_{kl}u_{ip} \\ &= \underbrace{u_{ij}u_{qs}u_{kl}u_{ip}}_{\substack{=0 \text{ since } q \notin T(i) \\ \text{but } s \in T(j)}} + u_{ij}u_{qt}u_{kl}u_{ip}. \end{aligned}$$

Since $q \in T(k)$ and $t \in T(l)$ hold, however, we get by Step 1 that u_{qt} and u_{kl} commute, and we can continue the above calculation:

$$u_{ij}u_{qt}u_{kl}u_{ip} = u_{ij}u_{kl}u_{qt}u_{ip}.$$

However, q is a common neighbour of i and k, which means we have $d(i,q) = 1$, but as we argued above $d(t,p) \neq 1$, and thus, by Lemma 2.1.3, we have $u_{qt}u_{ip} = 0$, and thus,

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{kl}u_{qt}u_{ip} = 0.$$

Therefore, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 2.1.1.

Step 7. Before we can go on to the case where $Q(i,k) = Q(j,l) = 0$, we need to consider the last case of $d(i,k) = d(j,l) = 1$. Let therefore now be $d(i,k) = d(j,l) = 1$ and $T(i) \neq T(k)$ and $T(j) \neq T(l)$. We will now show that $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$. Observe that for $d(l,p) \neq 1$, we already get the statement. Moreover, if $T(p) = T(l)$, then we have $u_{kl}u_{ip} = 0$ by Step 1, since $T(k) \neq T(i)$ and we are done.

Let therefore $T(p) \neq T(l)$. We observe that there is a quadrangle, containing both l and j that connects 2 triangles. We denote by q the vertex in this quadrangle, that is, in the same triangle as t. Then, we have in particular $d(q,l) = 2$ and $Q(q,l) = 2$. Then, we have $d(p,q) \neq 1$: Assume $p \sim q$. Then, $j \sim q \sim p \sim l \sim j$ would form a quadrangle. However, if p were in $T(j)$, then p would be a common neighbour of j and l. Since such a common neighbour does not exist, we conclude $T(p) \neq T(j)$. Since we moreover have $T(j) = T(q)$, but $T(j) \neq T(l)$, $T(l) \neq T(p)$ and $T(j) \neq T(p)$, this quadrangle would connect 3 distinct triangles. Such a quadrangle does not exist in Γ however, and we conclude $d(p,q) \neq 1$. We can now compute

$$u_{ij}u_{kl}u_{ip} = u_{ij} \sum_{\substack{v \in V \\ v \sim i \\ d(k,v)=2 \\ Q(k,v)=2}} u_{vq}u_{kl}u_{ip} = u_{ij}u_{kl} \underbrace{\sum_{\substack{v \in V \\ v \sim i \\ d(k,v)=2 \\ Q(k,v)=2}} u_{vq}u_{ip}}_{\substack{=0 \text{ since} \\ d(i,v)=1 \text{ but} \\ d(q,p)=2}} = 0.$$

Here, the sum commutes with u_{kl} by Step 6, as $d(k,s) = 2 = d(q,l)$ and $Q(k,s) = 2 = Q(q,l)$ for all summands hold. We conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij},$$

and, therefore, u_{ij} and u_{kl} commute by Lemma 2.1.1.

Step 8. Let now $Q(i,k) = Q(j,l) = 0$. We show again that

$$u_{ij}u_{kl}u_{ip} = 0$$

for all vertices $p \neq j$. As above, we already get that $u_{kl}u_{ip} = 0$, whenever $d(l,p) \neq d(i,k) = 2$ or when $d(l,p) = 2$ but $Q(l,p) \neq Q(i,k) = 0$. We thus now consider $p \in V \backslash \{j\}$ such that $d(l,p) = 2$ and $Q(l,p) = 0$. We will denote by s the unique neighbour of i and k, and by t the unique neighbour of l and p. We compute

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{kl}\sum_{v\in V} u_{vt}u_{ip} = u_{ij}u_{kl}u_{st}u_{ip}.$$

We, moreover, denote by t' the unique neighbour of j and l. Then, in particular, $t \neq t'$, since $p \neq j$. Similar to above, we get

$$u_{ij}u_{kl}u_{st}u_{ip} = u_{ij}u_{st'}u_{kl}u_{st}u_{ip}.$$

Since, however, $d(s,k) = 1 = d(l,t')$, we know that $u_{st'}$ and u_{kl} commute, and thus,

$$u_{ij}u_{st'}u_{kl}u_{st}u_{ip} = u_{ij}u_{kl}u_{st'}u_{st}u_{ip} = 0$$

as $t \neq t'$. We thus have

$$u_{ij}u_{kl}u_{ip} = 0 \quad \text{for } p \neq j,$$

and thus,

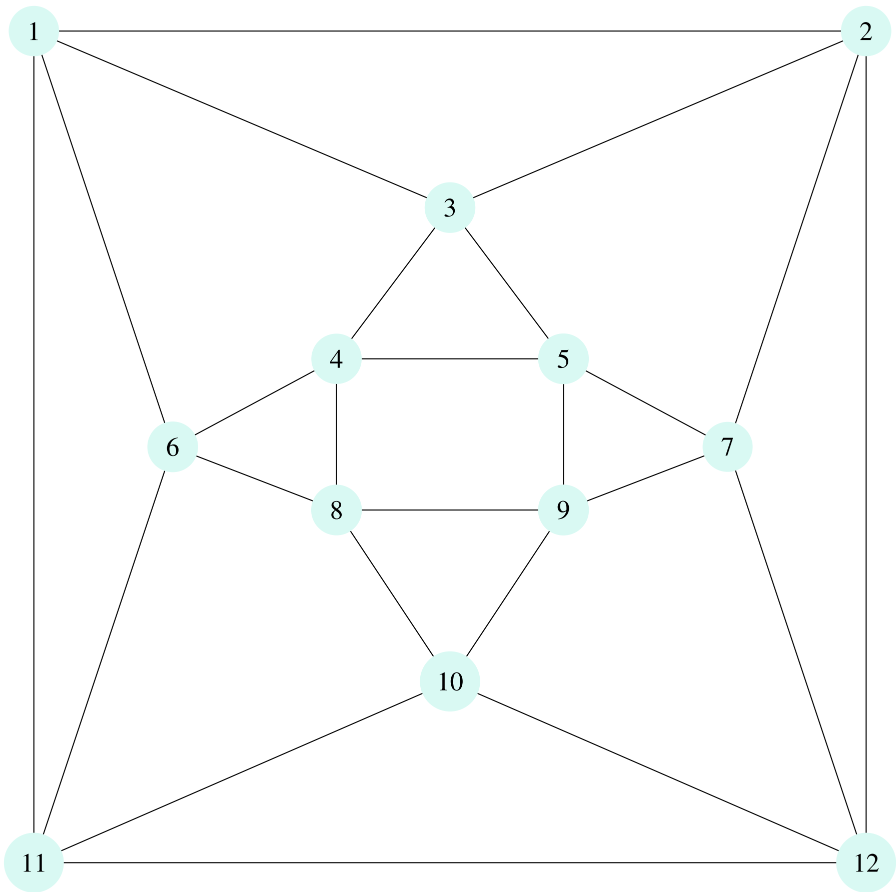
$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij},$$

and by Lemma 2.1.1, u_{ij} and u_{kl} commute. ■

6.5. The cuboctahedral graph

In this section, we show that the cuboctahedral graph, which is the line graph of the cube, has no quantum symmetries. By Γ we will denote the cuboctahedral graph throughout the entire section, even if it is sometimes not explicitly stated.

Theorem 6.5.1. *Let Γ be the cuboctahedral graph as shown below. It has no quantum symmetries, i.e., $G^+_{\mathrm{aut}}(\Gamma)$ is commutative.*



Before proving the statement, we will again first collect some properties of the graph in question.

Properties 6.5.2. We have the following properties.

- Γ has diameter 3, i.e., the largest distance between two vertices is 3.
- Γ is 4-regular, i.e., every vertex has exactly 4 neighbours.
- Γ contains 8 triangles.
- Each vertex of Γ is contained in exactly two triangles and every pair of triangles overlaps in at most one vertex.

Lemma 6.5.3. *Let $\Gamma = (V, E)$ be the cuboctahedral graph. Then, all vertices that are adjacent have exactly one common neighbour.*

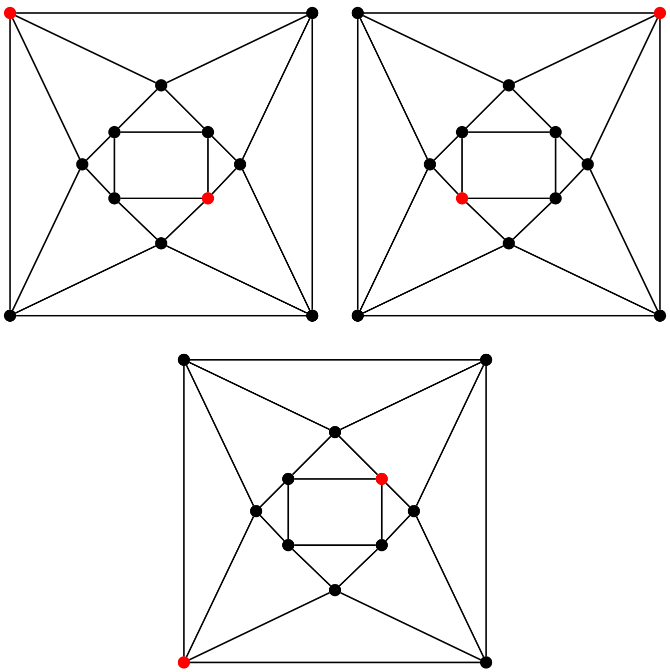
Proof. Γ is the line graph of the cube $C = (V', E')$, i.e., the vertices of Γ are the edges of C and two vertices of Γ are connected, if they share a vertex as edges of C. Let now

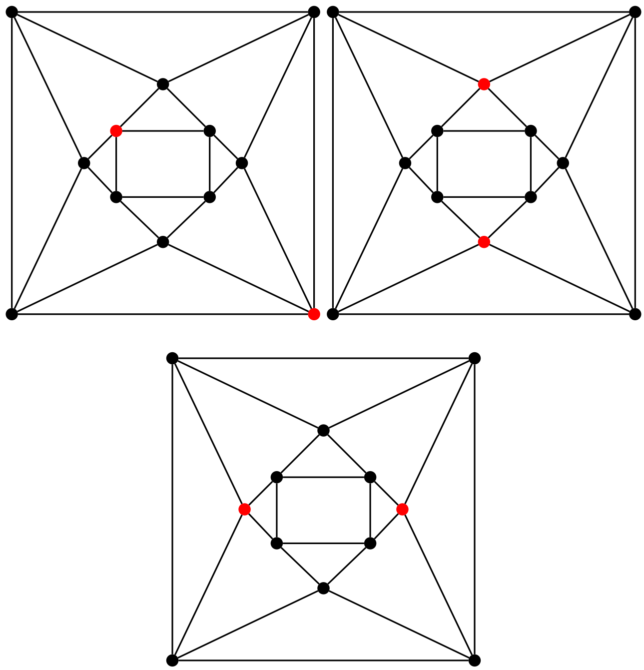
$d, e \in V = E'$ be adjacent vertices in Γ, i.e., there is a vertex $v \in V'$ of C such that $v \in d$ and $v \in e$. Since the cube is 3-regular, there is exactly one other edge, let us call it f that contains v. Then, f is a common neighbour of d and e in Γ.

It remains to show that d and e have no other common neighbour. For this, let $g \in E' \setminus \{d, e, f\}$ be another edge of C and let us assume that g is a common neighbour of d and e in Γ. It holds that $v \notin g$ since C is 3-regular and the three edges containing v are already d, e and f. Let us therefore fix $g = (x, y)$ for $x, y \in V' \setminus \{v\}$. Since we have $(g, d) \in E$ and $(g, e) \in E$ by assumption, there is a vertex in g for both d and e that they share with g. Since $(d, e) \in E$ we have $d \neq e$, and thus, the vertices they share with g are distinct. Thus, without loss of generality, we have $d = (v, x)$ and $e = (v, y)$. But then $\{x, y, v\}$ form a triangle. Since the cube does not contain any triangles, we have a contradiction, and thus, g cannot be a common neighbour of d and e. ∎

Lemma 6.5.4. *For each vertex v of Γ, there is exactly one vertex w of Γ such that $d(v, w) = 3$. We denote the map that maps v to w by a.*

Proof. We give the proof by giving illustrations of the 6 pairs of vertices in distance 3.





■

Lemma 6.5.5. *Let $v, w \in V$ be two vertices of Γ. If $d(v, w) = 2$, then there exists at least one vertex $x \in V$ that shares a triangle with both v and w.*

Proof. Consider the shortest path from v to w and name the vertex that is passed by x. Since all edges that contain vertex v only lead to vertices that share a triangle with v, x shares a triangle with v. By the same argument, x and w share a triangle. ■

Lemma 6.5.6. *Let $v \in V$ be a vertex in Γ. There are exactly two vertices that have distance 2 to both v and $a(v)$. Moreover, if w is one of these vertices, the other one is $a(w)$.*

Proof. We denote by $N(v)$ the neighbourhood of v, i.e., all vertices that are adjacent to v. Note that $\#(N(v) \cup N(a(v))) = 8$, i.e., 8 vertices are either adjacent to v or to $a(v)$. This is due to the fact that Γ has degree 4, and thus, all vertices have 4 neighbours and moreover,

$$N(v) \cap N(a(v)) = \{\}.$$

This holds, since if there was a vertex adjacent to both v and $a(v)$, this would yield a path from v to $a(v)$ of length 2, which is a contradiction to $d(v, a(v)) = 3$. We thus have

$$V \backslash (N(v) \cup N(a(v)) \cup \{v, a(v)\}) = \{w, x\}$$

for some vertices w, x. One can check by hand that $d(w, x) = 3$ for all choices of v. ■

We now come to the proof of Theorem 6.5.1.

Proof of Theorem 6.5.1. Let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$. We consider again vertices i, j, k, l in increasing order of $m = d(i,k) = d(j,l)$. Since Γ has diameter 3, m is maximally 3.

Step 1: $m = 1$. Since we know by Lemma 6.5.3 that in Γ all adjacent vertices have exactly one common neighbour, we can conclude by Lemma 2.3.2 that $G^+_{\mathrm{aut}}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma)$, and therefore, $u_{ij}u_{kl} = u_{kl}u_{ij}$.

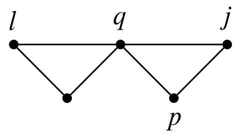
Step 2: $m = 2$. We consider the two following cases.

Step 2.1: $d(a(j),l) = 1$. If the unique vertex at distance 3 from j is adjacent to l, we can apply Lemma 2.1.4 by putting $q := a(j)$ and get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=2\\ d(a(j),p)=3}} u_{ip} = u_{ij}u_{kl}u_{ij}$$

since j is the only vertex at distance 3 to $a(j)$. We can apply the lemma, since we have $G^+_{\mathrm{aut}}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma)$ by Step 1. We thus can apply Lemma 2.1.1 to see that u_{ij} and u_{kl} commute.

Step 2.2: $d(a(j),l) = 2$. We apply again Lemma 2.1.4, and this time, we put q as a vertex that shares a triangle with both j and l, which exists by Lemma 6.5.5. Let us give a sketch of the situation:



This yields

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(p,l)=2\\ (p,q)\in E}} u_{ip}.$$

The vertex q has 4 neighbours, two of which are in the same triangle as l, and thus, they have a distance smaller than 2 to l. The remaining vertices are j and p and we thus get

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{ip}).$$

We now want to show that $d(a(p),l) = 1$. For this, first note that by Lemma 6.5.6 there are exactly 2 vertices that have distance 2 to both j and $a(j)$ and since l is one of those, the other one is $a(l)$. Moreover, this yields that the only vertices at distance 2 to both l and $a(l)$ are in turn j and $a(j)$. Therefore, we get

$$x \notin \{j, a(j), l, a(l)\} \quad \text{it holds that } d(x,l) = 1 \quad \text{or} \quad d(x,a(l)) = 1. \tag{6.2}$$

Since we have that $d(p, j) = 1$, and thus, $p \neq j$, and $d(p,l) = 2$ we thus get $d(p,a(l)) = 1$. But then, we get $d(a(p), a(l)) > 1$, since otherwise there would be a path from p to $a(p)$ of length 2 via $a(l)$, which cannot be. But then, we get, again applying (6.2) that $d(a(p), l) = 1$. Now, we can use our results from Step 2.1 to see that u_{kl} and u_{ip} commute. This yields

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{ip}u_{kl} = 0$$

since $j \neq p$. Therefore,

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij},$$

and again by Lemma 2.1.1, u_{ij} and u_{kl} commute.

Step 3: $m = 3$. If $d(i,k) = d(j,l) = 3$ holds, we get with Lemma 2.1.3 that

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{p\in V} u_{ip} = u_{ij}u_{kl}\sum_{p;d(l,p)=3} u_{ip}.$$

But since the only vertex at distance 3 from l is j this yields

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij},$$

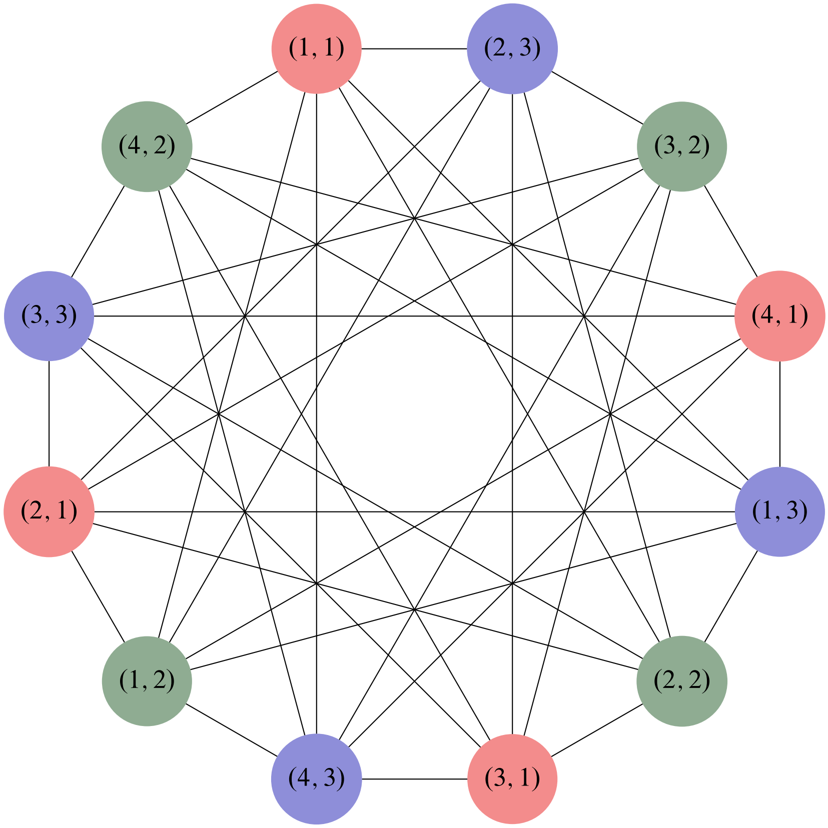
and by Lemma 2.1.1, u_{ij} and u_{kl} commute. ∎

7. Computation of quantum automorphism groups

In this section, we give the computation of the quantum automorphism groups of $C_{12}(4,5)$ and $C_{12}(3^+,6)$. For these computations, we used in some places a noncommutative Groebner basis implementation in the `OSCAR` [12, 17] package in the `julia` programming language [5].

Proposition 7.1. *The quantum automorphism group of* $C_{12}(4,5)$ *is* $H_2^+ \times S_3$.

Proof. For the purpose of this proof, we will call $C_{12}(4,5) =: \Gamma$. We first give an illustration of the graph that highlights the action of the automorphism group on its vertices, which will help in constructing the isomorphism between $G_{\mathrm{aut}}^+(\Gamma)$ and $H_2^+ \times S_3$. We also label the vertices of this graph differently than before to make it easier to talk about the construction of the $*$-isomorphism between the two quantum groups. Note that the colours in this illustration are only to make it easier to see the action of the automorphism group on Γ and are not an actual graph colouring.



The universal C^*-algebra defining $H_2^+ \times S_3$ is as follows:

$$C(H_2^+ \times S_3) = C^*\Bigg(v_{ij}, w_{kl}, 1 \leq i, j \leq 4, 1 \leq k, l \leq 3$$

$$\sum_{a=1}^{4} v_{ia} = \sum_{a=1}^{4} v_{aj} = 1 \ \forall 1 \leq i,\ j \leq 4$$

$$\sum_{b=1}^{3} w_{kb} = \sum_{b=1}^{3} w_{bl} = 1 \ \forall 1 \leq k,\ l \leq 3$$

$$v_{ij}^2 = v_{ij} = v_{ij}^*$$

$$w_{ij}^2 = w_{ij} = w_{ij}^*$$

$$A_{C_4} v = v A_{C_4}$$

$$v_{ij} w_{kl} = w_{kl} v_{ij} \Bigg).$$

Here, A_{C_4} is the adjacency matrix of the 4-cycle C_4 and v is the matrix with entries v_{ij} for $1 \leq i, j \leq 4$. We use the fact that $G^+_{\mathrm{aut}}(C_4) = H_2^+$, which was shown in [20].

We now claim the following.

(i) The matrix $\hat{u}$ with entries from $C(H_2^+ \times S_3)$ defined by

$$\hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}$$

satisfies the relations of $C(G^+_{\mathrm{aut}}(\Gamma))$.

(ii) The elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ in $C(G^+_{\mathrm{aut}}(\Gamma))$ defined by

$$\begin{aligned}\hat{v}_{ij} &= u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)},\\ \hat{w}_{kl} &= u_{(1,k)(1,l)} + u_{(1,k)(2,l)} + u_{(1,k)(3,l)} + u_{(1,k)(4,l)}\end{aligned}$$

satisfy the relations of $C(H_2^+ \times S_3)$.

Let us first prove (i).

For this, observe the following we have

$$\begin{aligned}&(i_1, i_2) \sim (k_1, k_2) \text{ in } \Gamma\\ \Leftrightarrow\, &i_2 \neq k_2 \wedge (i_1 = k_1 \vee i_1 \sim_{C_4} k_1).\end{aligned}$$

Let now $(i_1, i_2) \sim (k_1, k_2)$ and $(j_1, j_2) \not\sim (l_1, l_2)$, i.e., for (i_1, i_2) and (k_1, k_2) the above observation holds, while for (j_1, j_2) and (l_1, l_2) we have

$$j_2 = l_2 \vee (j_1 \neq l_1 \wedge j_1 \not\sim_{C_4} l_1).$$

In order to show that indeed $\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0$, we make the following case distinction.

- Case $j_2 = l_2$.
  Since we already know that $i_2 \neq k_2$, we get $w_{i_2 j_2} w_{k_2 j_2} = 0$, and thus,

$$\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = v_{i_1 j_1} w_{i_2 j_2} v_{k_1 l_1} w_{k_2 j_2} = v_{i_1 j_1} w_{i_2 j_2} w_{k_2 j_2} v_{k_1 l_1} = 0.$$

- Case $j_2 \neq l_2$.
  - Case $i_1 = k_1$.
    We get $v_{i_1 j_1} v_{i_1 l_1} = 0$, and thus,

$$\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0.$$

  - Case $i_1 \sim_{C_4} k_1$.
    Since $j_1 \not\sim_{C_4} l_1$, we get $v_{i_1 j_1} v_{k_1 l_1} = 0$, and thus,

$$\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0.$$

It is, moreover, easy to see that the relations of $H_2^+ \times S_3$ yield that all rows and columns of $\hat{u}$ sum up to 1 and that the entries of $\hat{u}$ are projections, as they are just products of commuting projections. From the universal property of $C(G_{\text{aut}}^+(\Gamma))$ we thus get a $*$-homomorphism

$$\begin{aligned}\phi : C(G_{\text{aut}}^+(\Gamma)) &\to C(H_2^+ \times S_3),\\ u_{(i_1,i_2)(j_1,j_2)} &\mapsto \hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}.\end{aligned}$$

It is also easy to see that all the generators of $C(H_2^+ \times S_3)$ are in the image of ϕ: for example, $v_{i_1 j_1}$ is the image of

$$u_{(i_1,1)(j_1,1)} + u_{(i_1,1)(j_1,2)} + u_{(i_1,1)(j_1,3)},$$

since

$$w_{11} + w_{12} + w_{13} = 1,$$

and a similar argument holds for $w_{i_2 j_2}$. Therefore, ϕ is surjective.

We will next prove (ii).

We first note that using the Groebner basis for the ideal generated by the relations of $C(G_{\text{aut}}^+(\Gamma))$, we see that

$$u_{(i,a)(k,b)} = u_{(i',a)(k',b)} \quad \text{for } i \not\sim_{C_4} i' \text{ and } k \not\sim_{C_4} k'. \tag{7.1}$$

Let $i \not\sim_{C_4} k$ and $j \sim_{C_4} l$. We then have $(i,1) \not\sim (k,1)$ but $(j,a) \sim (l,b)$ whenever $a \neq b$. Therefore, $u_{(i,1)(j,a)} u_{(k,1)(l,b)} = 0$ whenever $a \neq b$, and thus, in the product

$$\hat{v}_{ij} \hat{v}_{kl} = (u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)})(u_{(k,1)(l,1)} + u_{(k,1)(l,2)} + u_{(k,1)(l,3)}),$$

we only keep those products, where $a = b$ and are left with

$$\hat{v}_{ij} \hat{v}_{kl} = u_{(i,1)(j,1)} u_{(k,1)(l,1)} + u_{(i,1)(j,2)} u_{(k,1)(l,2)} + u_{(i,1)(j,3)} u_{(k,1)(l,3)}.$$

By (7.1), we know that $u_{(k,1)(l,a)} = u_{(i,1)(l',a)}$, where $l' \not\sim_{C_4} l$. Since $j \sim_{C_4} l$, we know in particular that $l' \neq j$ and get $u_{(i,1)(j,a)} u_{(i,1)(l',a)} = 0$, and thus,

$$\hat{v}_{ij} \hat{v}_{kl} = 0.$$

Let $i \sim_{C_4} k$ and $j \not\sim_{C_4} l$. Using a similar trick to above, we see that $u_{(i,1)(j,a)} u_{(k,1)(l,a)} = 0$, only this time using the fact that $u_{(k,1)(l,a)} = u_{(k',1)(j,a)}$, and get that all summands of that form in $\hat{v}_{ij} \hat{v}_{kl}$ disappear to get

$$\begin{aligned}\hat{v}_{ij} \hat{v}_{kl} = {} & u_{(i,1)(j,1)} u_{(k,1)(l,2)} + u_{(i,1)(j,1)} u_{(k,1)(l,3)} \\ & + u_{(i,1)(j,2)} u_{(k,1)(l,1)} + u_{(i,1)(j,2)} u_{(k,1)(l,3)} \\ & + u_{(i,1)(j,3)} u_{(k,1)(l,1)} + u_{(i,1)(j,2)} u_{(k,1)(l,2)}.\end{aligned}$$

Using again (7.1) to see that $u_{(k,1)(l,a)} = u_{(k',1)(j,a)}$, we get

$$\begin{aligned}\hat{v}_{ij}\hat{v}_{kl} &= u_{(i,1)(j,1)}u_{(k',1)(j,2)} + u_{(i,1)(j,1)}u_{(k',1)(j,3)}\\ &\quad + u_{(i,1)(j,2)}u_{(k',1)(j,1)} + u_{(i,1)(j,2)}u_{(k',1)(j,3)}\\ &\quad + u_{(i,1)(j,3)}u_{(k',1)(j,1)} + u_{(i,1)(j,2)}u_{(k',1)(j,2)}.\end{aligned}$$

However, we know that still $(i,1) \not\sim (k',1)$ holds but we also have $(j,a) \sim (j,b)$ for $a \neq b$. Therefore, we have

$$u_{(i,1)(j,a)}u_{(k',1)(j,b)} = 0,$$

and we get

$$\hat{v}_{ij}\hat{v}_{kl} = 0.$$

It is easy to see that the $\hat{v}_{(ij)}$ and the $\hat{w}_{kl}$ are self-adjoint, as they are just sums of self-adjoint elements. Moreover, since

$$u_{(i_1,i_2)(j_1,j_2)}u_{(i_1,i_2)(l_1,l_2)} = 0 \quad \text{for } (j_1,j_2) \neq (l_1,l_2),$$

seeing

$$\hat{v}_{ij}^2 = \hat{v}_{ij} \quad \text{and} \quad \hat{w}_{kl}^2 = \hat{w}_{kl}$$

is also straightforward.

It just remains to show that

$$\sum_a \hat{v}_{ia} = \sum_a \hat{v}_{aj} = 1 = \sum_b \hat{w}_{kb} = \sum_b \hat{w}_{bl}.$$

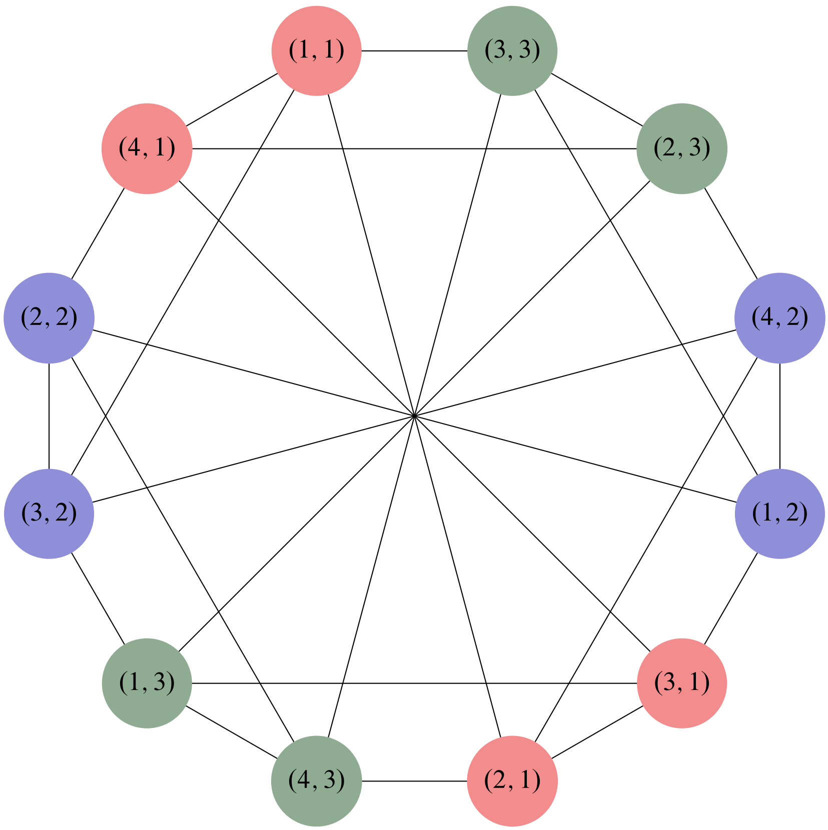
This is easy to see for $\sum_a \hat{v}_{ia}$ and $\sum_b \hat{w}_{bl}$, since these sums are just sums over one row of u. For $\sum_a \widehat{aj} = 1$ and $\sum_b \hat{w}_{kb}$, the Groebner basis yields that the sums are equal to 1. All in all, we get again by the universal property of $C(H_2^+ \times S_3)$ a $*$-homomorphism

$$\begin{aligned}\phi' : C(H_2^+ \times S_3) &\to C(G_{\text{aut}}^+(\Gamma)),\\ v_{ij} &\mapsto \hat{v}_{ij},\\ w_{ij} &\mapsto \hat{w}_{ij}.\end{aligned}$$

By (7.1), we see that all elements of u are hit by ϕ' and in particular, we have that ϕ' is the inverse of ϕ. We thus get that $C(H_2^+ \times S_3)$ and $C(G_{\text{aut}}^+(\Gamma))$ are $*$-isomorphic. ■

Proposition 7.2. *The quantum automorphism group of $C_{12}(3^+,6)$ is $H_2^+ \times S_3$.*

Proof. For this proof, we will write $C_{12}(3^+,6) =: \Gamma$. We will give a picture of the graph in question with vertices coloured in such a way that the action of the classical automorphism group $H_2 \times S_3$ is visible.



Recall from above that the universal C^*-algebra defining $H_2^+ \times S_3$ is as follows:

$$C(H_2^+ \times S_3) = C^*\Bigg(v_{ij}, w_{kl}, 1 \leq i,j \leq 4, 1 \leq k,l \leq 3$$

$$\sum_{a=1}^{4} v_{ia} = \sum_{a=1}^{4} v_{aj} = 1 \ \forall 1 \leq i,j \leq 4$$

$$\sum_{b=1}^{3} w_{kb} = \sum_{b=1}^{3} w_{bl} = 1 \ \forall 1 \leq k,l \leq 3$$

$$v_{ij}^2 = v_{ij} = v_{ij}^*$$

$$w_{ij}^2 = w_{ij} = w_{ij}^*$$

$$A_{C_4} v = v A_{C_4}$$

$$v_{ij} w_{kl} = w_{kl} v_{ij} \Bigg).$$

Here, A_{C_4} is the adjacency matrix of the 4-cycle C_4 and v is the matrix with entries v_{ij} for $1 \le i, j \le 4$.

We now claim the following.

(1) The matrix $\hat{u}$ with entries from $C(H_2^+ \times S_3)$ defined by

$$\hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}$$

satisfies the relations of $C(G_{\text{aut}}^+(\Gamma))$.

(2) The elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ in $C(G_{\text{aut}}^+(\Gamma))$ defined by

$$\begin{aligned}\hat{v}_{ij} &= u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)},\\ \hat{w}_{kl} &= u_{(1,k)(1,l)} + u_{(1,k)(2,l)} + u_{(1,k)(3,l)} + u_{(1,k)(4,l)}\end{aligned}$$

satisfy the relations of $C(H_2^+ \times S_3)$.

Let us first prove (1).

For this, observe the following, we have

$$\begin{aligned}&(i_1, i_2) \sim (k_1, k_2) \quad \text{in } \Gamma,\\ \Leftrightarrow &(i_2 = k_2 \wedge i_1 \sim_{C_4} k_1) \vee (i_2 \neq k_2 \wedge i_1 \neq k_1 \wedge i_1 \not\sim_{C_4} k_1).\end{aligned}$$

Let now $(i_1, i_2) \sim (k_1, k_2)$ and $(j_1, j_2) \not\sim (l_1, l_2)$, i.e., for (i_1, i_2) and (k_1, k_2) the above observation holds, while for (j_1, j_2) and (l_1, l_2) we have

$$(j_2 \neq l_2 \vee j_1 \not\sim_{C_4} l_1) \wedge (j_2 = l_2 \vee j_1 = l_1 \vee j_1 \sim_{C_4} l_1).$$

In order to show that $\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0$, we will make the following case distinction.

- Case $i_2 = k_2$.
  - Case $j_2 \neq l_2$.
    We have

    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} \hat{u}_{(k_1,k_2)(l_1,l_2)} = v_{i_1 j_1} w_{i_2 j_2} v_{k_1 l_1} w_{i_2 l_2} = v_{i_1 j_1} w_{i_2 j_2} w_{i_2 l_2} v_{k_1 l_1},$$

    but $w_{i_2 j_2} w_{i_2 l_2} = 0$ since $j_2 \neq l_2$, and thus,

    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0.$$

  - Case $j_2 = l_2$.
    We have $i_1 \sim_{C_4} k_1$ but $j_1 \not\sim_{C_4} l_1$, and thus,

    $$v_{i_1 j_1} v_{k_1 l_1} = 0,$$

    and therefore, also

    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0.$$

- Case $i_2 \neq k_2$.
  - Case $j_2 = l_2$.
    We have $w_{i_2 j_2} w_{k_2 j_2} = 0$ since $i_2 \neq k_2$, and thus,
    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0.$$
  - Case $j_2 \neq l_2$, $j_1 = l_1$.
    We have $v_{i_1 j_1} v_{k_1 j_1} = 0$ since $i_1 \neq k_1$, and thus,
    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0.$$
  - Case $j_2 \neq l_2$, $j_1 \sim_{C_4} l_1$.
    We have $v_{i_1 j_1} v_{k_1 l_1} = 0$ since $i_1 \not\sim_{C_4} k_1$ but $j_1 \sim_{C_4} l_1$, and thus,
    $$\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0.$$

A similar computation will show that $\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0$ if $(i_1, i_2) \not\sim (k_1, k_2)$ and $(j_1, j_2) \sim (l_1, l_2)$ hold.

It is immediate to see that $\hat{u}_{(i_1,i_2)(j_1,j_2)}$ is a projection again and that the rows and columns of $\hat{u}$ sum up to one from the relations of $C(H_2^+ \times S_3)$. We thus have that $\hat{u}$ satisfies the relations of $C(G_{\text{aut}}^+(\Gamma))$ and from the universal property of $C(G_{\text{aut}}^+(\Gamma))$ we get a $*$-homomorphism

$$\begin{aligned} \phi : C(G_{\text{aut}}^+(\Gamma)) &\to C(H_2^+ \times S_3), \\ u_{(i_1,i_2)(j_1,j_2)} &\mapsto \hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}. \end{aligned}$$

It is also easy to see that all the generators of $C(H_2^+ \times S_3)$ are in the image of ϕ: for example, $v_{i_1 j_1}$ is the image of

$$u_{(i_1,1)(j_1,1)} + u_{(i_1,1)(j_1,2)} + u_{(i_1,1)(j_1,3)}$$

since

$$w_{11} + w_{12} + w_{13} = 1,$$

and a similar argument holds for $w_{i_2 j_2}$. Therefore, ϕ is surjective.

We will now prove 2. Firstly, using the Groebner basis for the ideal generated by the relations of $C(G_{\text{aut}}^+(\Gamma))$, we can see that

$$\hat{v}_{i_1 j_1} \hat{w}_{i_2 j_2} = u_{(i_1,i_2)(j_1,j_2)} = \hat{w}_{i_2 j_2} \hat{v}_{i_1 j_1}. \tag{7.2}$$

In particular, we see that all the elements $\hat{v}_{ij}$ commute with all the $\hat{w}_{kl}$ as desired. It is moreover easy to see that $\hat{v}_{ij}^* = \hat{v}_{ij}$ and $\hat{w}_{kl}^* = \hat{w}_{kl}$ since all the entries of u are already selfadjoint.

Next, we compute

$$\begin{aligned}\hat{v}_{ij}^2 &= (u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)})^2 \\ &= u_{(i,1)(j,1)} + \underbrace{u_{(i,1)(j,1)}u_{(i,1)(j,2)} + u_{(i,1)(j,1)}u_{(i,1)(j,3)}}_{=0} \\ &\quad + u_{(i,1)(j,2)} + \underbrace{u_{(i,1)(j,2)}u_{(i,1)(j,1)} + u_{(i,1)(j,2)}u_{(i,1)(j,3)}}_{=0} \\ &\quad + u_{(i,1)(j,3)} + \underbrace{u_{(i,1)(j,3)}u_{(i,1)(j,1)} + u_{(i,1)(j,3)}u_{(i,1)(j,2)}}_{=0} \\ &= \hat{v}_{ij}.\end{aligned}$$

A similar computation can be done to see that $\hat{w}_{kl}^2 = \hat{w}_{kl}$, and thus, all elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ are projections.

Next, it is easy to see that

$$\sum_{a=1}^{4} \hat{v}_{ia} = \sum_{a=1}^{4}\sum_{j=1}^{3} u_{(i,1)(a,j)} = 1,$$
$$\sum_{b=1}^{3} \hat{w}_{kb} = \sum_{b=1}^{3}\sum_{l=1}^{4} u_{(k,1)(b,l)} = 1.$$

Using again the Groebner basis, we see that also

$$\sum_{a=1}^{4} \hat{v}_{ai} = 1 = \sum_{b=1}^{3} \widehat{w_{kb}}$$

holds.

Lastly, we now need to check that $\hat{v}$ satisfies

$$A_{C_4} v = v A_{C_4},$$

in other words, we need to show that

$$\hat{v}_{ij}\hat{v}_{kl} = 0 \quad \text{if } i \sim_{C_4} k,\ j \not\sim_{C_4} l \quad \text{or} \quad i \not\sim_{C_4} k, j \sim_{C_4} l.$$

Let therefore $i \sim_{C_4} k$ and $j \not\sim_{C_4} l$. Using the identity in (7.2), we write

$$\hat{v}_{ij} = \hat{v}_{ij}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3}$$

and compute

$$\begin{aligned}\hat{v}_{ij}\hat{v}_{kl} &= (\hat{v}_{ij}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3})(\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{kl}\hat{w}_{1,3}) \\ &= \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,3} \\ &\quad + \hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,3} \\ &\quad + \hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,3}.\end{aligned}$$

We have by (7.2) that $\hat{v}_{kl}$ and $\hat{w}_{ab}$ commute. Moreover, we have seen above that $\hat{w}_{ab}$ and $\hat{w}_{ac}$ are projections that sum up to one, and thus, $\hat{w}_{ab}\hat{w}_{ac} = 0$ for $b \neq c$. We thus get that any of the products in the above sum where both $\hat{w}_{1b}$ and $\hat{w}_{1c}$ appear for $b \neq c$ is 0 and we are only left with

$$\hat{v}_{ij}\hat{v}_{kl} = \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{2,2}\hat{v}_{kl}\hat{w}_{2,2} + \hat{v}_{ij}\hat{w}_{3,3}\hat{v}_{kl}\hat{w}_{3,3}.$$

Using again (7.2), we rewrite this as

$$\hat{v}_{ij}\hat{v}_{kl} = u_{(i,1)(j,1)}u_{(k,1)(l,1)} + u_{(i,1)(j,2)}u_{(k,1)(l,2)} + u_{(i,1)(j,3)}u_{(k,1)(l,3)}.$$

However, since by assumption $i \sim_{C_4} k$ but $j \not\sim_{C_4} l$, we have $(i,1) \sim (k,1)$ but $(j,a) \not\sim (l,a)$ for any $1 \leq a \leq 3$. We thus get

$$u_{(i,1)(j,a)}u_{(k,1)(l,a)} = 0,$$

and thus,

$$\hat{v}_{ij}\hat{v}_{kl} = 0.$$

A similar computation yields the same result whenever $i \not\sim_{C_4} k$ and $j \sim_{C_4} l$ hold. All in all, the universal property of $C(H_2^+ \times S_3)$ yields again a $*$-homomorphism

$$\begin{aligned} \phi' : C(H_2^+ \times S_3) &\to C(G^+_{\text{aut}}(\Gamma)), \\ v_{i_1 j_1} w_{i_2 j_2} &\mapsto u_{(i_1,i_2)(j_1,j_2)}. \end{aligned}$$

It is easy to see that all generators $u_{(i_1,i_2)(j_1,j_2)}$ of $C(G^+_{\text{aut}}(\Gamma))$ are in the image, and thus, ϕ' is surjective. We note, moreover, that ϕ' is exactly the inverse of ϕ and we thus get that $H_2^+ \times S_3$ and $G^+_{\text{aut}}(\Gamma)$ are $*$-isomorphic. ∎

A. Omitted proofs

In this section, we collect again all propositions that have not been proven above.

Proposition A.1. *The graph $C_{12}(2)$ does not have quantum symmetries.*

Proof. We will again show the commutation of the generator u_{i1} with all other generators of $G^+_{\text{aut}}(C_{12}(2))$.

First, we consider the vertices in distance 1. These are vertices 2, 3, 11 and 12. Since we can write $u_{i1}u_{kl} = u_{i1}u_{kl}\sum_{p\sim 1} u_{ip}$ for a vertex $l \sim 1$, we see that

$$\begin{aligned} u_{i1}u_{k2} &= u_{i1}u_{k2}(u_{i1} + u_{i3} + u_{i4} + u_{i12}), \\ u_{i1}u_{k3} &= u_{i1}u_{k3}(u_{i1} + u_{i2} + u_{i4} + u_{i5}), \\ u_{i1}u_{k11} &= u_{i1}u_{k11}(u_{i1} + u_{i9} + u_{i10} + u_{i12}), \\ u_{i1}u_{k12} &= u_{i1}u_{k12}(u_{i1} + u_{i2} + u_{i10} + u_{i11}). \end{aligned}$$

Applying Lemma 2.1.5 to these vertices, we see that on the right side, only the term with u_{i1} multiplied from the right does not vanish, and we thus get the commutation of u_{i1} and u_{kl} for $l \sim 1$ by Lemma 2.1.1.

Lemma 2.1.5.

j	l	p	q
1	2	3	12
		4	12
		12	10
	3	2	11
		4	12
		5	12

Lemma 2.1.5.

j	l	p	q
1	11	9	12
		10	9
		12	10
	12	2	11
		10	9
		11	11

Next, we consider vertices in distance 2, which are the vertices 4, 5, 9, and 10. For these, we can apply Lemma 2.1.4, since we already know that for all vertices $j \sim l$ that $u_{ij}u_{kl} = u_{kl}u_{ij}$:

Lemma 2.1.4.

j	l	q	p
1	4	3	$\{1\}$
	5	6	$\{1\}$
	9	8	$\{1\}$
	10	11	$\{1\}$

In distance 3, there are only three vertices, 6, 7, and 8. Applying Lemma 2.1.4 to them leads to the following table:

Lemma 2.1.4.

j	l	q	p
1	6	5	$\{1\}$
	7	5	$\{1,2\}$
	8	6	$\{1\}$

We thus have to consider $u_{i1}u_{k7}$ individually. By the above table, we know that

$$u_{i1}u_{k7} = u_{i1}u_{k7}(u_{i1} + u_{i2}).$$

Applying Lemma 2.1.6 with $q := 4$ yields that

$$u_{i1}u_{k7}u_{i2} = 0,$$

and we thus get

$$u_{i1}u_{k7} = u_{i1}u_{k7}u_{i1} = u_{k7}u_{i1}.$$

Lemma 2.1.6 is applicable since $d(7,4) = 2$ and we have shown above that for any vertices in distance 2, the generators commute.

All in all, we get that all generators commute, and thus, $G^+_{\mathrm{aut}}(C_{12}(2)) = G_{\mathrm{aut}}(C_{12}(2))$. ■

Proposition A.2. *The graph $C_{12}(4)$ does not have quantum symmetries.*

Proof. We first note that for each vertex there is only a single vertex in distance 3, and thus, for vertices i, j, k, l with $d(i,k) = 3 = d(j,l)$, we have

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij} = u_{kl}u_{ij}.$$

Next, we will see u_{i1} commutes with all u_{kl}, where $d(l,1) = 1$ by applying Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	2	3	1
		6	9
		10	5
	5	4	8
		6	9
		9	1

Lemma 2.1.5.

j	l	p	q
1	9	5	1
		8	4
		10	5
	12	4	8
		8	4
		11	1

For $d(l,1) = 2$, we can apply Lemma 2.1.4 and get the following:

Lemma 2.1.4.

j	l	q	p
1	3	7	$\{1\}$
	4	3	$\{1,6\}$
	6	7	$\{1\}$
	8	7	$\{1\}$
	10	2	$\{1,3\}$
	11	7	$\{1\}$

We thus already get commutation for all vertices except for $l = 4$ and $l = 10$. Applying Lemma 2.1.6 with $q := 10$ to 1 and 4 yields $u_{i1}u_{k4}u_{i6} = 0$, and thus, we have commutation of u_{i1} and u_{k4}. Similarly, Lemma 2.1.6 with $q := 4$ applied to 1 and 10 yields commutation of u_{i1} and u_{k10}.

All in all, we get that all generators commute, and thus,

$$G^+_{\text{aut}}(C_{12}(4)) = G_{\text{Aut}}(C_{12}(4)).$$ ■

Proposition A.3. *The graph $C_{12}(2,6)$ does not have quantum symmetries.*

Proof. As was the case with $C_{12}(4)$, we can show commutation for all l with $d(l,1) = 1$ by applying Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	2	3	1
		4	5
		8	3
		12	3
	3	2	4
		4	5
		5	2
		9	2

Lemma 2.1.5.

j	l	p	q
1	7	5	6
		6	2
		8	3
		9	8
	11	5	12
		9	10
		10	2
		12	3

Lemma 2.1.5.

j	l	p	q
1	12	2	4
		6	2
		10	2
		11	1

Using Lemma 2.1.5 is also sufficient for vertices 4, 6, 8, and 10:

Lemma 2.1.5.

j	l	p	q
1	4	7	2
		8	10
		9	2
		11	3
		12	7
	6	2	11
		3	5
		9	8
		10	3
		11	2

Lemma 2.1.5.

j	l	p	q
1	8	3	4
		4	6
		5	2
		11	5
		12	3
	10	2	7
		3	7
		5	4
		6	3
		7	2

The only remaining vertices are then 5 and 9. We thus have to consider

$$u_{i1}u_{k5} \quad \text{and} \quad u_{i1}u_{k9}.$$

Multiplying both of these by $1 = \sum_{r \in V(C_{12}(2,6))}$, we get

$$u_{i1}u_{k5} = u_{i1}u_{k5}(u_{i1} + u_{i2} + u_{i8} + u_{i9} + u_{i10} + u_{i12}),$$
$$u_{i1}u_{k9} = u_{i1}u_{k9}(u_{i1} + u_{i2} + u_{i4} + u_{i5} + u_{i6} + u_{i12}).$$

Applying Lemma 2.1.5 simplifies this

Lemma 2.1.5.

j	l	p	q
1	5	2	8
		8	10
		10	3
		12	10

Lemma 2.1.5.

j	l	p	q
1	9	2	4
		4	6
		6	3
		12	6

to

$$u_{i1}u_{k5} = u_{i1}u_{k5}(u_{i1} + u_{i9}),$$
$$u_{i1}u_{k9} = u_{i1}u_{k9}(u_{i1} + u_{i5}).$$

We would now like to apply Lemma 2.1.6 for the last step. However, so far, we only know that for any vertex q with $d(q,5) = 1$ the generators u_{k5} and u_{rq} commute. But for all of these vertices, we have $d(q,1) = d(q,9)$, and thus, the lemma is not applicable. A similar situation holds for any q with $d(q,9) = 1$.

Consider the following two automorphisms of $C_{12}(2,6)$:

$$\phi_1 = (1,5)\ (2,4)\ (6,12)\ (7,11)\ (8,10),$$
$$\phi_2 = (1,9)\ (2,8)\ (3,7)\ (4,6)\ (10,12).$$

It holds that $\phi_1(1) = 5$ and $\phi_1(4) = 2$. Since we already know by the above that $u_{i1}u_{k4} = u_{k4}u_{i1}$ for any $i,k \in V(C_{12}(2,6))$, we get by Lemma 2.2.1 that also $u_{i5}u_{k2}$ commute for any i,k. We thus can apply Lemma 2.1.6 with $q := 2$ to 1, 5 and 9 and get

$$u_{i1}u_{k5}u_{i9} = 0,$$

and therefore, u_{i1} and u_{k5} commute.

Similarly, we observe that $\phi_2(1) = 9$ and $\phi_2(6) = 4$. Again, we already know that u_{i1} and u_{k6} commute for any i,k, and thus, by Lemma 2.2.1 so do u_{i9} and u_{k4}. Then, applying Lemma 2.1.6 with $q := 4$ to 1, 9 and 5, we get that u_{i1} and u_{k9} commute.

We thus showed that u_{i1} commutes with every other generator, and since $C_{12}(2,6)$ is vertex-transitive this is enough to show that $G^+_{\mathrm{aut}}(C_{12}(2,6))$ is commutative. ■

Proposition A.4. *The graph $C_{12}(3,6)$ does not have quantum symmetries.*

Proof. We first note that the following permutation is an automorphism of $C_{12}(3,6)$:

$$\phi := (2,12)\ (3,11)\ (4,10)\ (5,9)\ (6,8).$$

This is the mirroring on the edge that goes through 1 and 7. In particular, we have that $\phi(1) = 1$, and thus, it suffices to show that u_{i1} and u_{kl} commute for $l \in \{2,3,4,5,6,7\}$, since the commutation with the rest of the generators then follows from Lemma 2.2.1 with ϕ.

We will now show that u_{i1} commutes with u_{k5}. To see this, we apply Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	5	3	6
		7	12
		9	4
		10	9
		12	4

It then immediately follows that u_{i1} also commutes with u_{k9} by Lemma 2.2.1 with ϕ.

Next, we consider all vertices l such that $d(l,1) = 1$, i.e., $l \in \{2,4,7\}$. We again apply Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	2	5	7
		8	4
		11	5
	4	3	7
		5	7
	7	6	3
		8	4

We get that $u_{i1}u_{k2}u_{i3} = 0$ by applying Lemma 2.1.6 with $q := 6$. It is applicable since applying Lemma 2.2.1 with the automorphism $v \mapsto v+1 \mod 12$ yields that u_{k2} and u_{r6} commute, since we already have commutation of u_{k1} and u_{r5}. We thus have commutation of u_{i1} and u_{k2}.

For $l = 4$, we still need to show that $u_{i1}u_{k4}u_{i7} = 0$ and $u_{i1}u_{k4}u_{i10} = 0$. We have commutation of u_{k4} with both u_{r8} and u_{r12} by Lemma 2.2.1 with the automorphism $v \mapsto v+3 \mod 12$, since we have commutation of u_{k1} with both u_{r5} and u_{r9}. We can thus apply Lemma 2.1.6 to both monomials, once with $q := 8$ and once with $q := 12$, and get that u_{i1} and u_{k5} commute.

For $l = 7$, we need to show that

$$u_{i1}u_{k7}u_{i4} = 0 \quad \text{and} \quad u_{i1}u_{k7}u_{i10} = 0.$$

Again, similar to above, we have commutation of u_{k7} with both u_{r3}, u_{r11} by Lemma 2.2.1 with the automorphism $v \mapsto v + 6$. We can then again apply Lemma 2.1.6 once with $q := 3$ and once with $q := 11$ and get that u_{i1} and u_{k7} commute.

We now have commutation of all generators u_{i1} and u_{kl}, where $d(l, 1) = 1$, and thus, we also have commutation of all generators u_{ij} and u_{kl}, where $d(i, k) = d(j, l) = 1$.

Only missing are now the vertices 3 and 6. For these, applying first Lemma 2.1.4 and then Lemma 2.1.5 shows the commutation of u_{i1} with u_{k3} and u_{k6}:

Lemma 2.1.4.

j	l	q	p
1	3	12	$\{1, 11\}$
	6	5	$\{1, 10\}$

Lemma 2.1.5.

j	l	p	q
1	3	11	4
	6	10	2

We thus have that all generators commute, and we get

$$G^+_{\text{aut}}(C_{12}(3, 6)) = G_{\text{aut}}(C_{12}(3, 6)). \qquad \blacksquare$$

Proposition A.5. *The graph $C_{12}(4, 6)$ does not have quantum symmetries.*

Proof. Again, we note that we have the following automorphism of $C_{12}(4, 6)$:

$$\phi := (2, 12)\ (3, 11)\ (4, 10)\ (5, 9)\ (6, 8),$$

i.e., the mirroring on the edge that goes through 1 and 7. As was the case for $C_{12}(3, 6)$, it suffices to show that u_{i1} and u_{kl} commute for $l \in \{2, 3, 4, 5, 6, 7\}$, since the commutation with the rest of the generators then follows from Lemma 2.2.1 with ϕ.

We now first consider u_{i1} and u_{k5}. Applying Lemma 2.1.5 already yields their commutation:

Lemma 2.1.5.

j	l	p	q
1	5	4	8
		6	9
		9	1
		11	9

Using this, we can show commutation for those vertices l that satisfy $d(1,l) = 2$, i.e., $l \in \{3,4,6\}$. We again apply Lemma 2.1.5 to these vertices to reduce the number of terms we have to consider:

Lemma 2.1.5.

j	l	p	q
1	3	10	11
		12	6
	4	2	7
		7	2
		9	7
		11	2
	6	3	4
		4	2
		9	2

Starting with vertex 3, we see that

$$u_{i1}u_{k3} = u_{i1}u_{k3}(u_{i1} + u_{i5} + u_{i6} + u_{i8}).$$

We note that the automorphism that rotates 1 onto 3 maps 5 to 7 and we thus get $u_{i3}u_{k7} = u_{k7}u_{i3}$ for any $i,k \in V(C_{12}(4,6))$ by Lemma 2.2.1. We can thus apply Lemma 2.1.6 with $q := 7$ to 1, 3 and 5, since $7 \sim 1$ but $5 \not\sim 7$ and get $u_{i1}u_{k3}u_{i5} = 0$.

Recall that by Definition 2.3.4, we denote by $\mathrm{CN}(i,j)$ the set of common neighbours of the vertices i and j. We now get by Lemma 2.3.5 that $u_{i1}u_{k3}u_{i6} = 0$ and that $u_{i1}u_{k3}u_{i8} = 0$, since $|\mathrm{CN}(1,3)| = 3$, $|\mathrm{CN}(3,6)| = 2$ and $|\mathrm{CN}(3,8)| = 4$. We thus have commutation of u_{i1} and u_{k3}.

Looking at vertex 4, we see that we have

$$u_{i1}u_{k4} = u_{i1}u_{k4}(u_{i1} + u_{i6}).$$

We can apply Lemma 2.3.5 to see that $u_{i1}u_{k4}u_{i6} = 0$, as $|\mathrm{CN}(1,4)| = 2$ and $|\mathrm{CN}(4,6)| = 6$. By this, we also get the commutation of u_{i1} and u_{k4}.

Lastly, looking at vertex 6, we see that we have

$$u_{i1}u_{k6} = u_{i1}u_{k6}(u_{i1} + u_{i8}u_{i9} + u_{i11}).$$

We will now show that u_{k6} commutes with u_{i8} and u_{i9}. If this is the case, then we get

$$u_{i1}u_{k6}u_{i8} = u_{i1}u_{i8}u_{k6} = 0,$$
$$u_{i1}u_{k6}u_{i9} = u_{i1}u_{i9}u_{k6} = 0.$$

Consider the automorphism

$$\sigma = (1,6)\,(2,5)\,(3,4)\,(7,12)\,(8,11)\,(9,10).$$

Since σ maps 1 to 6, we can translate the question of whether u_{k6} commutes with other generators to the question whether u_{v1} commutes with other generators for any vertex v by using Lemma 2.2.1. If we thus want to know whether u_{k6} and u_{i8} commute, we can instead ask whether u_{v1} and u_{w11} commute. Since we already know that u_{v1} and u_{w3} commute for any vertices v, w, and since the mirroring automorphism ϕ from above maps 1 to 1 and 3 to 11, we get that u_{v1} and u_{w11} commute for any v, w, again by Lemma 2.2.1. Therefore, also u_{k6} and u_{i8} commute.

Next, to see that u_{k6} and u_{i9} commute, we need to show that u_{v1} and u_{w10} commute. But we already know that u_{v1} and u_{w4} commute and since $\phi(4) = 10$, we get the commutation of u_{v1} and u_{w10}, and thus, also of u_{k6} and u_{i9}.

Lastly, we can apply Lemma 2.1.6 with $q := 2$ to see that

$$u_{i1}u_{k6}u_{i11} = 0.$$

To apply this lemma, we need that u_{k6} commute with u_{r2} for any vertex r. This can easily be seen by using Lemma 2.2.1 with the automorphism $\tau'(v) := v + 1 \mod 12$, as this maps 1 to 2 and 5 to 6.

We thus have that u_{i1} and u_{k6} commutes and thus have commutation of all u_{ij} and u_{kl}, where $d(i,k) = d(j,l) = 2$.

We still need to show commutation of u_{i1} with u_{k2} and with u_{k7}. Applying Lemma 2.1.4 with $q := 4$ to 1 and 2 yields

$$u_{i1}u_{k2} = u_{i1}u_{k2}(u_{i1} + u_{i6}).$$

Since we have already seen above that u_{k2} and u_{i6} commute, we get

$$u_{i1}u_{k2}u_{i6} = u_{i1}u_{i6}u_{k2} = 0,$$

and thus, we get the commutation of u_{i1} and u_{k2}.

Applying Lemma 2.1.4 with $q := 4$ to 1 and 7 yields

$$u_{i1}u_{k7} = u_{i1}u_{k7}(u_{i1} + u_{i6} + u_{i11}).$$

Since there is an automorphism that maps 1 to 6 and 2 to 7, namely, $v \mapsto v + 5 \mod 12$, we can use Lemma 2.2.1 to get that u_{k7} and u_{i6} commute, and thus,

$$u_{i1}u_{k7}u_{i6} = 0.$$

Similarly, the automorphism $v \mapsto v - 2 \mod 12$ maps 1 to 11 and 7 to 5, and we thus get by Lemma 2.2.1 that u_{k7} and u_{i11} commute, and thus,

$$u_{i1}u_{k7}u_{i11} = 0.$$

All in all we get that u_{i1} and u_{k7} commute and with that, all of the generators commute and we get $G^+_{\mathrm{aut}}(C_{12}(4,6)) = G_{\mathrm{aut}}(C_{12}(4,6))$. ∎

Proposition A.6. *The graph $C_{12}(2,5^+)$ does not have quantum symmetries.*

Proof. We begin to show that u_{i1} commutes with those generators u_{kl}, where $d(l,1)=1$, i.e., $l \in \{2,3,6,11,12\}$. As a first step, we apply Lemma 2.1.5 to these vertices:

Lemma 2.1.5.

j	l	p	q
1	2	3	1
		9	3
		12	3
	3	2	4
		5	2
		8	2

Lemma 2.1.5.

j	l	p	q
1	6	5	2
		7	2
		8	2
	11	9	3
		10	2
		12	3

Lemma 2.1.5.

j	l	p	q
1	12	2	4
		7	2
		10	2
		11	1

From the above, we already see that u_{i1} and u_{k12} commute. By Lemma 2.2.1, we get the commutation of all $u_{ij'}$ and $u_{kl'}$, where j' is uneven and $l' = j'-1 \mod 12$, using the automorphism $v \mapsto v+2 \mod 12$ repeatedly for the lemma.

Next, we note that, for the commutation of u_{i1} and u_{k2}, the only thing missing is that

$$u_{i1}u_{k2}u_{i4}=0.$$

This can be seen using Lemma 2.1.6 with $q:=3$ and the fact that u_{k2} and u_{r3} commute by the above.

Similarly, we only need

$$u_{i1}u_{k3}u_{i4}=0$$

to see that u_{i1} and u_{k3} commute and can see that again by Lemma 2.1.6 with $q:=2$.

In order to see that u_{i1} and u_{k6} commute, we need that

$$u_{i1}u_{k6}u_{i4}=0.$$

By Lemma 2.2.1 with the automorphism

$$\phi=(1,6)\,(2,5)\,(3,4)\,(7,12)\,(8,11)\,(9,10),$$

we get that u_{k6} and u_{i4} commute, since $\phi(1)=6$ and $\phi(3)=4$ and we have already shown commutation of u_{k1} and u_{i3}. We thus get

$$u_{i1}u_{k6}u_{i4}=u_{i1}u_{i4}u_{k6}=0,$$

as desired.

For the commutation of u_{i1} and u_{k11}, we need that

$$u_{i1}u_{k11}u_{i4}=0.$$

Since we already know that u_{k11} and u_{r10} commute, we can apply Lemma 2.1.6 with $q := 10$ to achieve this. We thus have commutation of all generators u_{ij} and u_{kl}, where $d(j,l) = 1$.

For some vertices l with $d(l,1) = 2$, it is enough to apply Lemma 2.1.4 first and then Lemma 2.1.5 to get that the missing monomial is 0:

Lemma 2.1.4.

j	l	q	p
1	4	3	$\{1,8\}$
	5	6	$\{1,8\}$
	10	5	$\{1,2\}$

Lemma 2.1.5.

j	l	p	q
1	4	8	2
	5	8	2
	10	2	6

For the other three vertices, we also first apply Lemma 2.1.4:

Lemma 2.1.4.

j	l	q	p
1	7	6	$\{1,4\}$
	8	6	$\{1,4,5\}$
	9	7	$\{1,3,4\}$

To get that $u_{i1}u_{k7}u_{i4} = 0$, we apply Lemma 2.1.6 with $q := 5$.

Next, we consider the automorphism

$$\sigma = (1,12)\,(2,11)\,(3,10)\,(4,9)\,(5,8)\,(6,7).$$

We see that $\sigma(1) = 12$ and $\sigma(5) = 8$, and thus, by Lemma 2.2.1, we get that u_{k8} and u_{r12} commute. To see now that both $u_{i1}u_{k8}u_{i4} = 0$ and $u_{i1}u_{k8}u_{i5} = 0$ hold, we can apply Lemma 2.1.6 with $q := 12$ to each of those monomials. With the same automorphism σ, Lemma 2.2.1 shows that u_{k9} and u_{r12} commute, and again, we see that

$$u_{i1}u_{k9}u_{i3} = 0 \quad \text{and} \quad u_{i1}u_{k9}u_{i4} = 0$$

hold by applying Lemma 2.1.6 with $q := 12$ to both monomials.

All in all, we see that all generators commute with u_{i1} and, thus, by Corollary 2.2.2, we get

$$G^+_{\text{aut}}(C_{12}(2,5^+)) = G_{\text{aut}}(C_{12}(2,5^+)). \qquad \blacksquare$$

Proposition A.7. *The graph $C_{12}(4,5^+)$ does not have quantum symmetries.*

Proof. For the vertices $l \in \{3, 10, 11, 12\}$, it is sufficient to apply Lemma 2.1.5 to show that u_{i1} and u_{kl} commute:

Lemma 2.1.5.

j	l	p	q
1	3	5	2
		6	12
		9	12
		10	12
		12	4
	10	3	1
		4	2
		7	2
		8	2
		12	4

Lemma 2.1.5.

j	l	p	q
1	11	2	10
		5	10
		6	10
		8	2
		9	8
	12	4	6
		7	2
		8	2
		11	1

We will now take a look at the rest of the vertices in distance 2 to the vertex 1, i.e., $l \in \{4, 7, 8\}$. To simplify, we will again first apply Lemma 2.1.5:

Lemma 2.1.5.

j	l	p	q
1	4	2	5
		6	7
		7	2
	7	4	2
		5	4
		9	6
	8	6	9
		10	11
		11	1

Next, we want to show that $u_{i1}u_{k4}u_{i9} = 0$ and $u_{i1}u_{k4}u_{i10} = 0$ hold. For this, we first observe that the permutation

$$\phi = (1, 4)\ (2, 3)\ (5, 12)\ (6, 11)\ (7, 10)\ (8, 9)$$

is an automorphism of $C_{12}(4, 5^+)$. By Lemma 2.2.1 with ϕ, we thus get that u_{k4} and u_{r2} commute. We can thus apply Lemma 2.1.6 to both monomials with $q := 2$ to get the desired result and see that u_{i1} and u_{k4} commute.

For $l = 7$, we want to show that $u_{i1}u_{k7}u_{i2} = 0 = u_{i1}u_{k7}u_{i10}$. Similar to above, we note that

$$\sigma = (1, 7)\ (2, 8)\ (3, 9)\ (4, 10)\ (5, 11)\ (6, 12)$$

is an automorphism of $C_{12}(4, 5^+)$ and, thus, by Lemma 2.2.1 we get commutation of u_{k7} and u_{r9}. Then, by Lemma 2.1.6 with $q := 9$, we get the desired result, and, thus u_{i1} and u_{k7} commute.

For $l = 8$, we need to show that $u_{i1}u_{k8}u_{i2} = 0 = u_{i1}u_{k8}u_{i5}$. In this case, we use the automorphism of $C_{12}(4, 5^+)$

$$\tau = (1, 8)\ (2, 7)\ (3, 6)\ (4, 5)\ (9, 12)\ (10, 11).$$

Using this with Lemma 2.2.1, we see that u_{k8} and u_{i6} commute and can thus apply Lemma 2.1.6 with $q := 6$ to get the desired outcome. We thus get that u_{i1} and u_{k8} commute and with this, all generators u_{ij} and u_{kl} commute where $d(j, l) = 2$.

For the remaining vertices l with $d(l, 1) = 1$, we now apply Lemma 2.1.4 and, where necessary, Lemma 2.1.5 to see that the remaining monomial also vanishes:

Lemma 2.1.4.

j	l	q	p
1	5	12	$\{1, 4\}$
	6	12	$\{1, 7\}$
	9	12	$\{1, 8\}$
	2	12	$\{1\}$

Lemma 2.1.5.

j	l	p	q
1	5	4	6
	6	7	2
	9	8	2

We thus see that all generators commute, and we get

$$G^+_{\text{aut}}(C_{12}(4, 5^+)) = G_{\text{aut}}(C_{12}(4, 5^+)). \qquad \blacksquare$$

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