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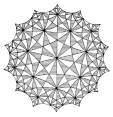

Transfer of homological objects in exact categories via adjoint triples. Applications to functor categories

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Abstract. For a given family $\{(\mathrm{q}_i, \mathrm{t}_i, \mathrm{p}_i)\}_{i\in I}$ of adjoint triples between exact categories $\mathcal{C}$ or $\mathcal{D}$, we show that any cotorsion pair in $\mathcal{C}$ and $\mathcal{D}$ yields two canonical cotorsion pairs providing a concrete description of objects without using any injectives/projectives object hypothesis. We firstly apply this result for the evaluation functor on the functor category $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ equipped with an exact structure $\mathcal{E}$. Under mild conditions on $\mathcal{A}$, we introduce the stalk functor at any object of $\mathcal{A}$, and subsequently, we investigate cotorsion pairs induced by stalk functors. Finally, we use them to present an intrinsic characterization of projective/injective objects in $(\mathrm{Add}(\mathcal{A}, R\text{-Mod}); \mathcal{E})$.

Introduction

The lifting of particular homological objects to functor categories has been of interest for a long time as it provides a kind of control for globally defined objects through their local information. The emblematic examples come up in the category of chain complexes. Gillespie in [5] shows that a complete hereditary cotorsion pair $(\mathcal{F}, \mathcal{G})$ in R-Mod, the category of left R-modules, induces a pair of hereditary complete cotorsion pairs $(\mathrm{dg}\,\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ and $(\tilde{\mathcal{F}}, \mathrm{dg}\,\tilde{\mathcal{G}})$ in $\mathbf{C}(R)$, the category of chain complexes of left R-modules, and provides an intrinsic characterization of chain complexes in $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$. Combining this with the well-known result in [13], those induced cotorsion pairs yield a model structure on $\mathbf{C}(R)$ whose homotopy category is just the derived category $\mathbf{D}(R)$. There are several advantages of this lifting mechanism. Firstly, it generates several non-trivial model structures on $\mathbf{C}(R)$, as well as recovering the well-known injective and projective ones. Secondly, the classes $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ are mostly the classes of categorically defined homological objects in $\mathbf{C}(R)$ such as projective, injective, flat and absolutely pure chain complexes. Those results have been generalized to a category of chain complexes $\mathbf{C}(\mathcal{C})$ over a sufficiently nice exact category $\mathcal{C}$; see Section 7 in [25]. In the recent work [11], Holm and Jørgensen focus on the problem of how to obtain projective and injective model structures on k-Lin$(\mathcal{A}, R\text{-Mod})$, the category of R-module valued k-linear functors on a more general k-linear category $\mathcal{A}$, where k is a commutative ring and R is a k-algebra.

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Several authors have applied a similar lifting process to modules over a formal triangular matrix ring $L := \left[\begin{smallmatrix} T & 0 \\ N & S \end{smallmatrix}\right]$. For instance, Mao in [15] lifts two cotorsion pairs $(\mathcal{F}_T, \mathcal{G}_T)$ and $(\mathcal{F}_S, \mathcal{G}_S)$ in T-Mod and S-Mod, respectively, to several cotorsion pairs in L-Mod providing an explicit description of their modules. As a particular case, the trivial cotorsion pairs $(\mathcal{F}_T, \mathcal{G}_T)$ and $(\mathcal{F}_S, \mathcal{G}_S)$ (i.e., projective/injective modules) lead to the trivial cotorsion pairs in L-Mod. In this case, the characterization of projective/injective left L-modules is the same as in [8] and [9]. The subtle difference with the case of chain complexes is that the classes used to lift come from the module categories over two rings T and S. However, the standard tool used in both cases is the adjoint functors. So this paper aims to provide a systematic study of how to lift cotorsion pairs through general adjoint functors between exact categories with applications in functor categories so that it covers the categories of chain complexes and modules over a formal triangular matrix ring. Specifically, we give a very intrinsic characterization of projective and injective additive functors under mild conditions.

The paper starts by proposing a general question: for a given integer $n \geq 1$, and a pair (q, t) of adjoint functors between exact categories $\mathcal{C}$ and $\mathcal{D}$, under which conditions does there exist a canonical natural isomorphism $\mathrm{Ext}^n(\mathrm{q}(-), -) \cong \mathrm{Ext}^n(-, \mathrm{t}(-))$? Of course, this isomorphism is immediate when both functors are exact between exact categories. However, this is not the case in most situations of interest (for instance, see Example 2.13). In Section 2, we address this question in full generality, fixing one of the objects in components, that is, for given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, to guarantee natural isomorphisms

$$(*) \qquad \mathrm{Ext}^n(\mathrm{q}(C), -) \cong \mathrm{Ext}^n(C, \mathrm{t}(-)) \quad \text{and} \quad \mathrm{Ext}^n(\mathrm{q}(-), D) \cong \mathrm{Ext}^n(-, \mathrm{t}(D)).$$

The case $n = 1$ is fully answered in Proposition 2.10. For it, we introduce two important notions: q-flat and t-coflat objects (see Definition 2.6). These notions are crucial since they help remove the hypothesis of having enough projective/injective objects required in most situations with similar constructions such as [10] and [15] (see also Remark 3.2). We also show that the notion of acyclicity for chain complexes over any exact category can be expressed in terms of (co)-flatness to specific functors (see Example 2.13).

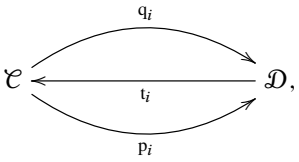
When working with a general $n \geq 1$, lengthy technical conditions will be required if one aims to avoid projective/injective objects. Instead, we firstly rethink the problem $(*)$ not for fixed objects, but for fixed classes $\mathcal{F} \subseteq \mathcal{C}$ and $\mathcal{G} \subseteq \mathcal{D}$. So under conditions given in Lemma 2.12, we prove that there exist the following natural isomorphisms

$$\mathrm{Ext}^n(\mathrm{q}(\mathcal{F}), -) \cong \mathrm{Ext}^n(\mathcal{F}, \mathrm{t}(-)) \quad \text{and} \quad \mathrm{Ext}^n(\mathrm{q}(-), \mathcal{G}) \cong \mathrm{Ext}^n(-, \mathrm{t}(\mathcal{G})).$$

This result is meritorious on its own. The classic example representing this phenomenon is the adjoint triple induced from sphere chain complexes and acyclic chain complexes (see Example 2.13). Lemma 2.12 also provides us the tool of the relatively deriving for functors q and t to answer the question $(*)$ for any $n \geq 1$ (see Proposition 2.15).

In Section 3, we apply results from Section 2 for the lifting problem of cotorsion pairs through adjoint functors. Corollary 3.3, a special case of Theorem 3.1, is the main form of the situations of interest, and is used frequently in the rest of the paper. To explain, if

there is a family $\{(\mathrm{q}_i, \mathrm{t}_i, \mathrm{p}_i)\}_{i\in I}$ of adjoint triples between exact categories



then any cotorsion pairs $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ in $\mathcal{C}$ and $\mathcal{D}$, respectively, yield two canonical cotorsion pairs

$$({}^{\perp}(\mathrm{q}_*(\mathcal{F})^{\perp}), \mathrm{q}_*(\mathcal{F})^{\perp}) \quad \text{and} \quad ({}^{\perp}\mathrm{p}_*(\mathcal{G}), ({}^{\perp}\mathrm{p}_*(\mathcal{G}))^{\perp}),$$
$$({}^{\perp}(\mathrm{t}_*(\mathcal{F}')^{\perp}), \mathrm{t}_*(\mathcal{F}')^{\perp}) \quad \text{and} \quad ({}^{\perp}\mathrm{t}_*(\mathcal{G}'), ({}^{\perp}\mathrm{t}_*(\mathcal{G}'))^{\perp})$$

in $\mathcal{D}$ and $\mathcal{C}$, respectively. In Corollary 3.3, we provide a concrete description of objects in $\mathrm{q}_*(\mathcal{F})^{\perp}$, ${}^{\perp}\mathrm{p}_*(\mathcal{G})$, $\mathrm{t}_*(\mathcal{F}')^{\perp}$ and ${}^{\perp}\mathrm{t}_*(\mathcal{G}')$ without using any injectives/projectives objects. Applications of this result are presented in the next sections.

Starting from Section 4, we focus on $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$, the category of R-module-valued additive functors on an R-R-bimodule enriched small category $\mathcal{A}$ (see Hypothesis 4.3). Our approach is based on consideration of module categories attached locally to $\mathcal{A}$, that is, the module category over the endomorphism ring $R_A := \mathrm{End}_{\mathcal{A}}(A)$ of any object A in $\mathcal{A}$. We assume that the product category $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$ is equipped with a fixed exact sructure $\prod_{A\in\mathcal{A}} \mathcal{E}_A$, and we consider a cotorsion pair $(\prod_{A\in\mathcal{A}} \mathcal{F}_A, \prod_{A\in\mathcal{A}} \mathcal{G}_A)$ in $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$.

In Section 4, we also study the adjoint triple induced from the evaluation functor $\mathrm{ev}_A\colon \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \to R\text{-Mod}$, $\mathrm{ev}_A(X) := X(A)$. As $\mathcal{A}$ is an R-R-bimodule enriched category, ev_A factors through R_A-Mod. So the fixed exact structure $\prod_{A\in\mathcal{A}} \mathcal{E}_A$ on the product category $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$ induces an exact structure $\mathcal{E}$ on $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ defined as follows: $\mathbb{E} \in \mathcal{E}$ if and only in $\mathrm{ev}_A(\mathbb{E}) \in \mathcal{E}_A$ for every $A \in \mathcal{A}$ (see Hypothesis 4.11). So it makes the functor

$$\mathrm{ev} := (\mathrm{ev}_A)_{A\in\mathcal{A}} : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \longrightarrow \prod_{A\in\mathcal{A}} R_A\text{-Mod}$$

an exact functor. Applying Corollary 3.3 for the adjoint triple $(\mathrm{q}, \mathrm{ev}, \mathrm{p})$, we have:

Proposition 4.13. *If for every $A \in \mathcal{A}$, any object in $\mathcal{F}_A$ and $\mathcal{G}_A$ is q_A-flat and p_A-coflat, respectively, then*

$$\mathrm{q}\Big(\prod_{A\in\mathcal{A}} \mathcal{F}_A\Big)^{\perp} = \{X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A) \in \mathcal{G}_A \text{ for every } A \in \mathcal{A}\},$$
$${}^{\perp}\mathrm{p}\Big(\prod_{A\in\mathcal{A}} \mathcal{G}_A\Big) = \{X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A) \in \mathcal{F}_A \text{ for every } A \in \mathcal{A}\}.$$

In Section 5, we impose further conditions (see Hypothesis 5.1) on the category $\mathcal{A}$ in order to define $\mathrm{s}_A\colon R_A\text{-Mod} \longrightarrow \mathrm{Add}(\mathcal{A}, R\text{-Mod})$, the 'stalk' functor at an object A in $\mathcal{A}$ (see (5.3) and Proposition 5.4). Our stalk functor differs subtly from that of Definition 7.9

in [11], and the one given in the category of representations of quivers (see Remark 5.5). This difference provides us an advantage in Section 6, when we characterize projective and injective functors in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ for more general categories $\mathcal{A}$ than the ones considered in the literature.

The stalk functor is a generalization of the n-sphere chain complex functor (see Example 2.13) in the category of chain complexes, which is crucial in defining the classes $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$. In Proposition 5.8, we show that the stalk functors induce an adjoint triple $(\mathrm{c}, \mathrm{s}, \mathrm{k})$ between $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ and $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$. In Proposition 5.18, we describe c-flat and k-coflat functors in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ in terms of specific short exact sequences. This concrete characterization, as well as providing us a better understanding of c-flat and k-coflat functors, is the key result in proving Theorem 6.5. In the case of chain complexes, the short exact sequences given in Proposition 5.18 correspond to the canonical short exact sequences

$$\mathrm{c}_{i+1}(X) \longrightarrow X_i \longrightarrow \mathrm{c}_{i+1}(X) \quad \text{and} \quad \mathrm{k}_i(X) \longrightarrow X_i \longrightarrow \mathrm{k}_{i-1}(X)$$

(for functors c_i and k_i on chain complexes, see Example 2.13). Using Corollary 3.3 and Proposition 5.18, we obtain the following result.

Theorem 5.19. (i) *If $\prod_{A\in\mathcal{A}} \mathcal{F}_A$ is a generating class in $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$, then*

$$\mathrm{s}\Big(\prod_{A\in\mathcal{A}} \mathcal{F}_A\Big)^{\perp} = \Big\{X\colon \mathcal{A} \to R\text{-Mod} \;\Big|\; \begin{array}{l} \mathrm{k}_A(X) \in \mathcal{G}_A,\ X(A) \longrightarrow \ker(\mu_{A,X}) \\ \text{is an admissible epic in } R_A\text{-Mod},\ \forall A \in \mathcal{A} \end{array}\Big\}.$$

(ii) *If $\prod_{A\in\mathcal{A}} \mathcal{G}_A$ is a cogenerating class in $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$, then*

$${}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}} \mathcal{G}_A\Big) = \Big\{X\colon \mathcal{A} \to R\text{-Mod} \;\Big|\; \begin{array}{l} \mathrm{c}_A(X) \in \mathcal{F}_A,\ \mathrm{coker}(\phi_{A,X}) \hookrightarrow X(A) \\ \text{is an admissible monic in } R_A\text{-Mod},\ \forall A \in \mathcal{A} \end{array}\Big\}.$$

In Section 6, we focus on two trivial cotorsion pairs,

$$\Big(\prod_{A\in\mathcal{A}} \mathcal{E}_A\text{-Proj}, \prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big) \quad \text{and} \quad \Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}, \prod_{A\in\mathcal{A}} \mathcal{E}_A\text{-Inj}\Big),$$

to provide an intrinsic characterization of projective and injective objects in the exact category $(\mathrm{Add}(\mathcal{A}, R\text{-Mod}); \mathcal{E})$. Such intrinsic characterizations of those homological objects provide a crucial tool for their analysis and application to relevant questions.

Theorem 6.5. (i) *If Condition 6.1 is satisfied, then*

$$\mathcal{E}\text{-Proj} = {}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big).$$

(ii) *If Condition 6.2 is satisfied, then*

$$\mathcal{E}\text{-Inj} = \mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big)^{\perp}$$

Combining Theorem 5.19 together with Theorem 6.5, we obtain a characterization of relative projective and injective functors.

Corollary. (i) *If for every* $A \in \mathcal{A}$, R_A-Mod *has enough* $\mathcal{E}_A$*-projectives, and Condition* 6.1 *is satisfied, then*

$$\mathcal{E}\text{-Proj} = \left\{ X \colon \mathcal{A} \to R\text{-Mod} \;\middle|\; \begin{array}{l} \mathrm{c}_A(X) \in \mathcal{E}_A\text{-Proj}, \operatorname{coker}(\phi_{A,X}) \hookrightarrow X(A) \\ \textit{is an admissible monic in } R_A\text{-Mod}, \forall A \in \mathcal{A} \end{array} \right\}.$$

(ii) *If for every* $A \in \mathcal{A}$, R_A-Mod *has enough* $\mathcal{E}_A$*-injectives, and Condition* 6.2 *is satisfied, then*

$$\mathcal{E}\text{-Inj} = \left\{ X \colon \mathcal{A} \to R\text{-Mod} \;\middle|\; \begin{array}{l} \mathrm{k}_A(X) \in \mathcal{E}_A\text{-Inj}, X(A) \to \ker(\mu_{A,X}) \\ \textit{is an admissible epic in } R_A\text{-Mod}, \forall A \in \mathcal{A} \end{array} \right\}.$$

It is worth pointing out that no such treatment exists in the literature for relative projective and injective functors, and even in the absolute exact structure case, the previous corollary applies to categories $\mathcal{A}$, which are more general than those considered in Theorem 7.29 of [11] (see Example 6.4). In particular, it covers both Morita context rings with zero trace ideals – including, as a special case, formal triangular matrix rings – and left- and right-rooted categories. Consequently, our result extends those of [8,9] and Theorem 1 in [3].

We conclude this introduction with a final remark. The results obtained from Section 4 onward have been developed within the framework of R-module valued additive functors on a small preadditive category $\mathcal{A}$. So this setting excludes categories such as the category of chain complexes of quasi-coherent sheaves over any scheme X. However, we think that under mild conditions on a enriched category $\mathcal{V}$ (for example, being a Grothendieck cosmos; see (2.6) in [12]), it is possible to extend these results, not just for the category Add($\mathcal{A}$, R-Mod), but for $\mathcal{V}$-[$\mathcal{U}$, ${}_R\mathcal{V}$]. This latter category involves the category of $\mathcal{V}$-enriched functors from a small $\mathcal{V}$-enriched category $\mathcal{U}$ to the category ${}_R\mathcal{V}$ of R-left-objects (see [17]), where R is a monoid object in $\mathcal{V}$. To develop the theory in this setting, as done in Section 4 onward, one essentially applies the same ideas but employs more technical tools, such as (co)end or weighted (co)limits from the enriched category theory.

It is worth noting that $\mathcal{V}$ is a special case of ${}_R\mathcal{V}$, since the unit object I in $\mathcal{V}$ is a commutative monoid in $\mathcal{V}$ with ${}_I\mathcal{V} = \mathcal{V}$ (see Corollary 2.3 in [17]). Besides, for a given monoid R in $\mathcal{V}$, and a usual small category (not necessarily preadditive) $\mathcal{A}$, there exists the so-called *the free* $\mathcal{V}$*-category* $\mathcal{A}_\mathcal{V}$ *on* $\mathcal{A}$ (see Section 2.5 of [14]) satisfying

$$\mathrm{Fun}(\mathcal{A}, {}_R\mathcal{V}) \cong \mathcal{V}\text{-}[\mathcal{A}_\mathcal{V}, {}_R\mathcal{V}],$$

where Fun($\mathcal{A}$, ${}_R\mathcal{V}$) denotes the category of usual functors from $\mathcal{A}$ to ${}_R\mathcal{V}$. A non-trivial example of a Grothendieck cosmos is the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves over any scheme (see Example 2.4 in [12]). Consequently, this framework allows us to obtain results for Fun($\mathcal{A}$, $\mathfrak{Qco}(X)$), including, as a special case, the category $\mathbf{C}(X)$ of chain complexes of quasi coherent sheaves over any scheme X.

1. Preliminares

In this section, we review certain basic notions and results that are essential for our study.

1.1. Product category. Let $\{\mathcal{C}_i\}_{i\in I}$ be a family of additive categories. *The product category*, denoted by $\prod_{i\in I}\mathcal{C}_i$, is the category with objects

$$\mathrm{Ob}\Big(\prod_{i\in I}\mathcal{C}_i\Big) := \prod_{i\in I}\mathrm{Ob}(\mathcal{C}_i),$$

and morphisms

$$\mathrm{Hom}_{\prod_{i\in I}\mathcal{C}_i}((C_i)_{i\in I},(C_i')_{i\in I}) := \prod_{i\in I}\mathrm{Hom}_{\mathcal{C}_i}(C_i,C_i').$$

For any $i\in I$, π_i and ι_i denote the canonical ith projection and inclusion functors, respectively:

$$\pi_i : \prod_{i\in I}\mathcal{C}_i \longrightarrow \mathcal{C}_i \quad \text{and} \quad \iota_i : \mathcal{C}_i \longrightarrow \prod_{i\in I}\mathcal{C}_i.$$

1.2. Exact categories. An *exact structure on an additive category* $\mathcal{C}$ is an isomorphism-closed collection $\mathcal{E}$ of distinguished kernel-cokernel pairs (i,p) in $\mathcal{C}$,

$$\mathbb{E}:\ A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C\ ,$$

whose components are called *admissible monic* and *admissible epic* of $\mathbb{E}$, respectively, satisfying:

(i) for every object $A\in\mathcal{C}$, the identity morphism id_A is both an admissible monic and an admissible epic;
(ii) admissible monics and admissible epics are closed under compositions;
(iii) the class $\mathcal{E}$ is closed under pushout and pullback along a morphism in $\mathcal{C}$.

$\mathcal{C}$ equipped with an exact structure $\mathcal{E}$, denoted by $(\mathcal{C};\mathcal{E})$, is called an *exact category*. A kernel-cokernel pair (i,p), or $\mathbb{E}$, in $\mathcal{E}$ is called a *short exact sequence* in $(\mathcal{C};\mathcal{E})$.

A sequence in $\mathcal{C}$ of the form

$$\mathbb{E}:\ A \rightarrowtail B_n \longrightarrow \cdots \longrightarrow B_1 \twoheadrightarrow C$$

is said to be *exact* if there exist short exact sequences $\mathbb{E}_i:\ Z_i \rightarrowtail B_i \twoheadrightarrow Z_{i-1}$ in $(\mathcal{C};\mathcal{E})$, $1\le i\le n$, with $Z_0=C$ and $Z_n=A$ in such a way that the morphism $B_{i+1}\longrightarrow B_i$ is factorized as $B_{i+1}\twoheadrightarrow Z_i \rightarrowtail B_i$ for every $1\le i\le n-1$. In other words, $\mathbb{E}=\mathbb{E}_n\circ\cdots\circ\mathbb{E}_1$. We refer to [2] for a detailed treatement on exact categories.

We recall the following result, which will be used in Section 4.

1.3. Proposition. (Corollary 2.14 in [2]). *Let $(\mathcal{C};\mathcal{E})$ be an exact category. For any morphism $f:A\to B$ in $\mathcal{C}$, the following holds.*

(i) *If $g:C\to B$ is an admissible epic in $(\mathcal{C};\mathcal{E})$, then so is the induced morphism $(f,g):A\oplus C\to B$.*
(ii) *If $f:A\to C$ is an admissible monic in $(\mathcal{C};\mathcal{E})$, then so is the induced morphism $(f,g):A\to B\oplus C$.*

1.4. Cotorsion pairs. For any integer $n \geq 1$, the exact structure $\mathcal{E}$ on $\mathcal{C}$ leads to the so-called *nth Yoneda extension functor*

$$\mathrm{Ext}^n_{\mathcal{E}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab},$$

where $\mathrm{Ext}^n_{\mathcal{E}}(C, A)$ denotes the abelian group of equivalence classes of all exact sequences in $(\mathcal{C}; \mathcal{E})$ of the form

$$\mathbb{E} : \ A \rightarrowtail B_n \longrightarrow \cdots \longrightarrow B_1 \twoheadrightarrow C \ .$$

Recall from Theorem 4.2 in Chapter VII of [16] that two exact sequences $\mathbb{E}$ and $\mathbb{E}'$ in $\mathrm{Ext}^n_{\mathcal{E}}(C, A)$ are equivalent if and only if there is a chain of morphisms between exact sequences $\mathbb{E} \to \mathbb{E}_1 \leftarrow \mathbb{E}_2 \to \mathbb{E}'$ with fixed ends.

For given morphisms $i : C' \to C$ and $j : A \to A'$ in $\mathcal{C}$, we have the canonical morphisms

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{E}}(i, A) : \mathrm{Ext}^n_{\mathcal{E}}(C, A) \to \mathrm{Ext}^n_{\mathcal{E}}(C', A), \quad \mathbb{E} \rightsquigarrow \mathbb{E}i, \\ \mathrm{Ext}^n_{\mathcal{E}}(C, j) : \mathrm{Ext}^n_{\mathcal{E}}(C, A) \to \mathrm{Ext}^n_{\mathcal{E}}(C, A'), \quad \mathbb{E} \rightsquigarrow j\mathbb{E}, \end{aligned}$$

where $\mathbb{E}i$ and $j\mathbb{E}$ denote the pullback and pushout of $\mathbb{E}$ along i and j, respectively. So we have

$$\mathrm{Ext}^n_{\mathcal{E}}(i, j) = \mathrm{Ext}^n_{\mathcal{E}}(i, A') \circ \mathrm{Ext}^n_{\mathcal{E}}(C, j) \cong \mathrm{Ext}^n_{\mathcal{E}}(C', j) \circ \mathrm{Ext}^n_{\mathcal{E}}(i, A),$$

that is, $(j\mathbb{E})i \equiv j(\mathbb{E}i)$.

Recall also, from Lemma 4.1 in Chapter VII of [16], that an exact sequence $\mathbb{E} = \mathbb{E}_n \circ \cdots \circ \mathbb{E}_1$ in $\mathrm{Ext}^n_{\mathcal{E}}(C, A)$ is trivial if and only if there exist a morphism φ and a short exact sequence $\mathbb{E}'_{n-1}$ and such that $\mathbb{E}_n\varphi$ is a split short exact sequence and $\varphi\mathbb{E}'_{n-1} \equiv \mathbb{E}_{n-1}$.

For a given class $\mathcal{S}$ of objects in $\mathcal{C}$, we let

$$\begin{aligned} \mathcal{S}^{\perp} &= \{A \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{E}}(S, A) = 0, \text{ for every } S \in \mathcal{S}\}, \\ {}^{\perp}\mathcal{S} &= \{A \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{E}}(A, S) = 0, \text{ for every } S \in \mathcal{S}\}. \end{aligned}$$

A pair $(\mathcal{F}, \mathcal{G})$ of classes of objects in $\mathcal{C}$ is said to be *cotorsion pair* if $\mathcal{F}^{\perp} = \mathcal{G}$ and $\mathcal{F} = {}^{\perp}\mathcal{G}$. It is said to be *hereditary* if $\mathrm{Ext}^n_{\mathcal{E}}(F, G) = 0$ for every $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $n \geq 1$.

For a class $\mathcal{S}$ of objects in $\mathcal{C}$, the cotorsion pair $(\mathcal{F}, \mathcal{G})$ is said to be *generated by* $\mathcal{S}$ if $\mathcal{S}^{\perp} = \mathcal{G}$; is said to be *cogenerated by* $\mathcal{S}$ if $\mathcal{F} = {}^{\perp}\mathcal{S}$.

The cotorsion pair $(\mathcal{F}, \mathcal{G})$ is said to *have enough injectives* if for every object A of $\mathcal{C}$, there exists a short exact sequence in $(\mathcal{C}; \mathcal{E})$ of the form

$$A \rightarrowtail G \twoheadrightarrow F$$

with $G \in \mathcal{G}$ and $F \in \mathcal{F}$; it is said to *have enough projectives* if for every object A of $\mathcal{C}$, there exists a short exact sequence in $(\mathcal{C}; \mathcal{E})$ of the form

$$G' \rightarrowtail F' \twoheadrightarrow A \ ,$$

where $G' \in \mathcal{G}$ and $F' \in \mathcal{F}$; it is said to be *complete* if it has enough projectives and injectives.

1.5. Projective and injective objects. An object P in $(\mathcal{C};\mathcal{E})$ is said to be *projective* if for every short exact sequence $\mathbb{E}$ in $\mathcal{E}$, the sequence $\mathrm{Hom}(P,\mathbb{E})$ of abelian groups is exact. The class of all projective objects in $(\mathcal{C};\mathcal{E})$ is denoted by $\mathcal{E}$-Proj.

We say that $(\mathcal{C};\mathcal{E})$ has *enough projectives* if $(\mathcal{E}\text{-Proj},\mathcal{C})$ is a complete cotorsion pair, equivalently, for every object A in $\mathcal{C}$ there exists an admissible epic $P \twoheadrightarrow A$ with $P \in \mathcal{E}$-Proj.

Dually, an *injective object* in $(\mathcal{C};\mathcal{E})$ is defined, and the class of injective objects in $(\mathcal{C};\mathcal{E})$ is denoted by $\mathcal{E}$-Inj. The category $\mathcal{C}$ is said to have *enough injectives* if for every object A in $\mathcal{C}$ there exists an admissible monic $A \hookrightarrow E$ with $E \in \mathcal{E}$-Inj.

1.6. Left and right derived functors. (Remark 12.12 in Section 10 of [2]). Let $\mathcal{F}$ be an extension closed class of objects in $(\mathcal{C};\mathcal{E})$ satisfying

(i) $\mathcal{F}$ is a *generating class*, that is, for every object A in $\mathcal{C}$, there exists an admissible epic $F \twoheadrightarrow A$ with F in $\mathcal{F}$;

(ii) $\mathcal{F}$ is closed under kernels of admissible epics, that is, if $F' \rightarrowtail F \twoheadrightarrow F''$ is a short exact sequence in $(\mathcal{C};\mathcal{E})$ with objects F and F'' in $\mathcal{F}$, then so is F'.

So for any object A in $\mathcal{C}$ there exists an exact sequence in $(\mathcal{C};\mathcal{E})$ of the form

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \twoheadrightarrow A,$$

with $F_i \in \mathcal{F}$ for every $i \geq 0$. The deleted chain complex

$$F_*^A : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is called a $\mathcal{F}$*-resolution* of A. If an additive functor $\mathrm{p}\colon \mathcal{C} \longrightarrow \mathcal{D}$, where $\mathcal{D}$ is an abelian category, preserves short exact sequences in $(\mathcal{C};\mathcal{E})$ with objects in $\mathcal{F}$, then for any integer $i \geq 0$, the so-called ith *left derived functor of* p *relative to* $\mathcal{F}$, denoted by $\mathrm{L}_i^{\mathcal{F}}\mathrm{p}$, exists and is the functor from $\mathcal{C}$ to $\mathcal{D}$ defined by $\mathrm{L}_i^{\mathcal{F}}\mathrm{p}(A) := \mathrm{H}_i(\mathrm{p}(F_*^A))$, the ith homology of the chain complex $\mathrm{p}(F_*^A)$. It is well known that any short exact sequence in $(\mathcal{C};\mathcal{E})$

$$\mathbb{E} : A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$$

induces the following long exact sequence in $\mathcal{D}$:

$$\cdots \longrightarrow \mathrm{L}_1^{\mathcal{F}}\mathrm{p}(B) \longrightarrow \mathrm{L}_1^{\mathcal{F}}\mathrm{p}(C) \longrightarrow \mathrm{L}_0^{\mathcal{F}}\mathrm{p}(A) \longrightarrow \mathrm{L}_0^{\mathcal{F}}\mathrm{p}(B) \longrightarrow \mathrm{L}_0^{\mathcal{F}}\mathrm{p}(C) \longrightarrow 0,$$

and if p is right exact, then $\mathrm{L}_0^{\mathcal{F}}\mathrm{p} \cong \mathrm{p}$.

If $\mathcal{F}$ satisfies dual properties, then every object in $\mathcal{C}$ has a $\mathcal{F}$*-coresolution*, and the ith *right derived functor*, denoted by $\mathrm{R}^i_{\mathcal{F}}\mathrm{p}\colon \mathcal{C} \longrightarrow \mathcal{D}$, of p exists and is defined dually.

1.7. Returning to the product categories, for a family $\{(\mathcal{C}_i;\mathcal{E}_i)\}_{i\in I}$ of exact categories, *the product exact structure* $\prod_{i\in I}\mathcal{E}_i$ on the product category $\prod_{i\in I}\mathcal{C}_i$ consists of I-tuples $(\mathbb{E}_i)_{i\in I}$ of short exact sequences $\mathbb{E}_i$ in $(\mathcal{C}_i;\mathcal{E}_i)$. So we have

$$\mathrm{Ext}^n_{\prod_{i\in I}\mathcal{E}_i}((C_i)_{i\in I},(A_i)_{i\in I}) := \prod_{i\in I}\mathrm{Ext}^n_{\mathcal{E}_i}(C_i,A_i). \tag{1.7.1}$$

The following is an easy observation, but for the reader's convenience, we provide its proof.

1.8. Lemma. *Any (hereditary) cotorsion pair in* $(\prod_{i\in I}\mathcal{C}_i;\prod_{i\in I}\mathcal{E}_i)$ *is of the form*

$$\Big(\prod_{i\in I}\mathcal{F}_i,\prod_{i\in I}\mathcal{G}_i\Big),$$

where $(\mathcal{F}_i,\mathcal{G}_i)$ *is a (hereditary) cotorsion pair in* $(\mathcal{C}_i;\mathcal{E}_i)$ *for every* $i\in I$.

Proof. Let $\{(\mathcal{F}_i,\mathcal{G}_i)\}_{i\in I}$ be a family of (hereditary) cotorsion pairs in $(\mathcal{C}_i;\mathcal{E}_i)$. From (1.7.1), we already have

$$\prod_{i\in I}\mathcal{F}_i\subseteq {}^{\perp}\Big(\prod_{i\in I}\mathcal{G}_i\Big)\quad\text{and}\quad\prod_{i\in I}\mathcal{G}_i\subseteq\Big(\prod_{i\in I}\mathcal{F}_i\Big)^{\perp}.$$

Let $(C_i)_{i\in I}$ be an object in ${}^{\perp}(\prod_{i\in I}\mathcal{G}_i)$. It is clear that for any object G_{i_0} in $\mathcal{G}_i$, we have that $\iota_{i_0}(G_{i_0})\in\prod_{i\in I}\mathcal{G}_i$, and by assumption, we have

$$0=\operatorname{Ext}^1_{\prod_{i\in I}\mathcal{E}_i}(\,(C_i)_{i\in I},\iota_{i_0}(G_{\iota_0}))=\operatorname{Ext}^1_{\mathcal{E}_{i_0}}(C_{i_0},G_{i_0}).$$

Therefore, for every $i\in I$, $C_i\in{}^{\perp}\mathcal{G}_i=\mathcal{F}_i$. Similarly, we have $(\prod_{i\in I}\mathcal{F}_i)^{\perp}\subseteq\prod_{i\in I}\mathcal{G}_i$.

Now suppose that $(\mathcal{F},\mathcal{G})$ is a cotorsion pair in $(\prod_{i\in I}\mathcal{C}_i;\prod_{i\in I}\mathcal{E}_i)$. For every $i\in I$, we let $\mathcal{F}_i:=\pi_i(\mathcal{F})$ and $\mathcal{G}_i:=\pi_i(\mathcal{G})$. From (1.7.1), it is easy to observe that $\mathcal{F}_i\subseteq{}^{\perp}\mathcal{G}_i$ and $\mathcal{G}_i\subseteq\mathcal{F}_i^{\perp}$ for every $i\in I$. Now, let $C_{i_0}\in{}^{\perp}\mathcal{G}_{i_0}$ for some $i_0\in I$. For every $(G_i)_{i\in I}\in\mathcal{G}$, we have $G_{i_0}\in\mathcal{G}_{i_0}$ and

$$\operatorname{Ext}^1_{\prod_{i\in I}\mathcal{E}_i}(\iota_{i_0}(C_{i_0}),(G_i)_{i\in I})=\operatorname{Ext}^1_{\mathcal{E}_{i_0}}(C_{i_0},G_{i_0})=0,$$

therefore, $\iota_{i_0}(C_{i_0})\in\mathcal{F}$. This implies that $C_{i_0}=\pi_{i_0}(\iota_{i_0}(C_{i_0}))\in\mathcal{F}_{i_0}$. Similarly, we have $\mathcal{G}_i=\mathcal{F}_i^{\perp}$ for every $i\in I$.

The hereditary condition follows easily. ∎

2. Ext isomorphisms

Throughout, $\mathcal{C}$ and $\mathcal{D}$ denote weakly idempotent complete exact categories equipped with exact structures $\mathcal{E}$ and $\mathcal{E}'$, respectively. So we omit the subscript in the corresponding nth Yoneda extension functors. We also fix an adjoint pair (q,t) of functors between $\mathcal{C}$ and $\mathcal{D}$,

$$\mathcal{C}\underset{\mathrm{t}}{\overset{\mathrm{q}}{\rightleftarrows}}\mathcal{D}.$$

We let $\eta\colon\mathrm{id}_{\mathcal{C}}\Rightarrow\mathrm{tq}$ and $\xi\colon\mathrm{qt}\Rightarrow\mathrm{id}_{\mathcal{D}}$ denote the unit and the counit of the adjunction (q,t). It is well known that q preserves all colimits while t preserves all limits.

In this section, we investigate sufficient and necessary conditions for which the functors q and t interact together inside the functor Ext^n in the same way as they do inside of the Hom functor. The following presents a more general form of Lemma 5.1 in [10] (see also Lemma 1.7 in [18]).

2.1. Lemma. *For any objects C and D in $\mathcal{C}$ and $\mathcal{D}$, respectively, and an integer $n \geq 1$, the following hold.*

(i) *If* t *preserves exact sequences in* $\mathrm{Ext}^n(\mathrm{q}(C), D)$*, then there exists a canonical morphism*

$$\Theta^n_{CD} : \mathrm{Ext}^n(\mathrm{q}(C), D) \longrightarrow \mathrm{Ext}^n(C, \mathrm{t}(D)).$$

If $n = 1$, then Θ^1_{CD} is a monomorphism.

(ii) *If* q *preserves exact sequences in* $\mathrm{Ext}^n(C, \mathrm{t}(D))$*, then there exists a canonical morphism*

$$\Omega^n_{CD} : \mathrm{Ext}^n(C, \mathrm{t}(D)) \longrightarrow \mathrm{Ext}^n(\mathrm{q}(C), D).$$

If $n = 1$, then Ω^1_{CD} is a monomorphism.

(iii) *If the functors* q *and* t *satisfy the conditions given in* (i)-(ii)*, then*

$$\Theta^n_{CD} : \mathrm{Ext}^n(\mathrm{q}(C), D) \longrightarrow \mathrm{Ext}^n(C, \mathrm{t}(D))$$

is an isomorphism with the inverse Ω^n_{CD}.

Proof. (i) Consider an exact sequence $\mathbb{E}$ in $\mathrm{Ext}^n(\mathrm{q}(C), D)$. By assumption, $\mathrm{t}(\mathbb{E})$ is an exact sequence in $\mathrm{Ext}^n(tq(C), \mathrm{t}(D))$. Taking the pullback of $\mathrm{t}(\mathbb{E})$ along the unit morphism $\eta_C : C \longrightarrow tq(X)$, we obtain an exact sequence $\mathrm{t}(\mathbb{E})\eta_C$ in $\mathrm{Ext}^n(C, \mathrm{t}(D))$ together with the pullback diagram $\mathrm{t}(\mathbb{E})\eta_C \to \mathrm{t}(\mathbb{E})$.

Since taking pullback and t are additive functors, the assignment

$$\Theta^n_{CD} : \mathrm{Ext}^n(\mathrm{q}(C), D) \longrightarrow \mathrm{Ext}^n(C, \mathrm{t}(D)), \quad \mathbb{E} \rightsquigarrow \mathrm{t}(\mathbb{E})\eta_C,$$

is well defined.

Now, we show that Θ^1_{CD} is a monomorphism. Suppose that for some short exact sequence $\mathbb{E}$ in $\mathrm{Ext}^1(\mathrm{q}(C), D)$, the short exact sequence $\mathrm{t}(\mathbb{E})\eta_C$ is trivial in $\mathrm{Ext}^1(C, \mathrm{t}(D))$. As q is additive, $\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C)$ is trivial, as well. On the other hand, we consider the composition of the following canonical morphisms between sequences:

$$\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \mathrm{qt}(\mathbb{E}) \longrightarrow \mathbb{E}, \tag{2.1.1}$$

where $qt(\mathbb{E}) \longrightarrow \mathbb{E}$ is the morphism induced from the counit natural transformation ξ. The last column of the composition given in (2.1.1) is the composition

$$\xi_{\mathrm{q}(C)} \circ \mathrm{q}(\eta_C) : \mathrm{q}(C) \longrightarrow \mathrm{qtq}(C) \longrightarrow \mathrm{q}(C),$$

which is identity. Hence, the short exact sequence $\mathbb{E}$ is in fact a pushout of the trivial short exact sequence $\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C)$. As a consequence, $\mathbb{E}$ is a trivial short exact sequence.

(ii) Similar to the proof of the first statement.

(iii) From (i) and (ii), we have assignments Θ^n_{CD} and Ω^n_{CD}. We claim that

$$\Omega^n_{CD} \circ \Theta^n_{CD} = \mathrm{id} \quad \text{and} \quad \Theta^n_{CD} \circ \Omega^n_{CD} = \mathrm{id}.$$

By duality of arguments, we only show the equality $\Omega^n_{CD} \circ \Theta^n_{CD} = \mathrm{id}$.

Let $\mathbb{E}$ be any exact sequence in $\mathrm{Ext}^n(\mathrm{q}(C), D)$. By construction, we have

$$\Theta^n_{CD}(\mathbb{E}) = \mathrm{t}(\mathbb{E})\eta_C,$$

together with the pullback diagram $\mathrm{t}(\mathbb{E})\eta_C \longrightarrow \mathrm{t}(\mathbb{E})$. Applying q to the aforementioned pullback diagram, we still have a commutative diagram $\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \mathrm{qt}(\mathbb{E})$ with the rightmost vertical arrow $\mathrm{q}(\mathrm{t}(\eta_C))$. By assumption, $\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C)$ is an exact sequence in $\mathcal{D}$. Composing it with the counit morphism, we have a commutative diagram with canonical morphisms

$$\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \mathrm{qt}(\mathbb{E}) \longrightarrow \mathbb{E},$$

The last column of the composition is the composition

$$\xi_{\mathrm{q}(C)} \circ \mathrm{q}(\eta_C) : \mathrm{q}(C) \longrightarrow \mathrm{qtq}(C) \longrightarrow \mathrm{q}(C),$$

which is the identity. And the left vertical map is ξ_D. By the universal property of pushouts, the commutative diagram $\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \mathbb{E}$ is factorized as

$$\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \xi_D\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C) \longrightarrow \mathbb{E},$$

where both $\Omega^n_{CD}\,\Theta^n_{CD}(\mathbb{E}) = \xi_D\mathrm{q}(\mathrm{t}(\mathbb{E})\eta_C)$ and $\mathbb{E}$ belongs to $\mathrm{Ext}^n(\mathrm{q}(C), D)$. So

$$\Omega^n_{CD}\Theta^n_{CD}(\mathbb{E}) \equiv \mathbb{E}.$$ ∎

As an aside, it is easy to verify that if $1 \leq m \leq n$ and t preserves exact sequences in $\mathrm{Ext}^n(\mathrm{q}(C), D)$, then it preserves also exact sequences in $\mathrm{Ext}^m(\mathrm{q}(C), D)$, and therefore, there exists a morphism $\Theta^m_{CD} : \mathrm{Ext}^m(\mathrm{q}(C), D) \longrightarrow \mathrm{Ext}^m(C, \mathrm{t}(D))$. The dual affirmation is valid for q.

As pullback and pushout diagrams satisfy the universal property, and the (co)unit of an adjunction is a natural transformation, we have the following result.

2.2. Corollary. *Let $n \geq 1$. We have the following.*

(i) *If* t *preserves exact sequences in* $\mathrm{Ext}^n(\mathrm{q}(-), -)$*, then* $\Theta^n := \{\Theta^n_{CD}\}_{C\in\mathcal{C}, D\in\mathcal{D}}$ *is a natural transformation*

$$\Theta^n : \mathrm{Ext}^n(\mathrm{q}(-), -) \Longrightarrow \mathrm{Ext}^n(-, \mathrm{t}(-)).$$

(ii) *If* q *preserves exact sequences in* $\mathrm{Ext}^n(-, \mathrm{t}(-))$*, then* $\Omega^n := \{\Omega^n_{CD}\}_{C\in\mathcal{C}, D\in\mathcal{D}}$ *is a natural transformation*

$$\Omega^n : \mathrm{Ext}^n(-, \mathrm{t}(-)) \Longrightarrow \mathrm{Ext}^n(\mathrm{q}(-), -).$$

(iii) *If the functors* q *and* t *satisfy the conditions given in* (i)-(ii)*, then the natural transformation*

$$\Theta^n : \mathrm{Ext}^n(\mathrm{q}(-), -) \Longrightarrow \mathrm{Ext}^n(-, \mathrm{t}(-))$$

is an isomorphism with the inverse Ω^n.

2.3. Remark. The condition that t preserves exact sequences in $\mathrm{Ext}^n(\mathrm{q}(-),-)$ for some $n \geq 1$ is equivalent to the exactness of t; see Lemma 2.8. Therefore, if both functors t and q are exact, then the natural transformation

$$\Theta^n : \mathrm{Ext}^n(\mathrm{q}(-),-) \Longrightarrow \mathrm{Ext}^n(-,\mathrm{t}(-))$$

is an isomorphism for every $n \geq 1$.

2.4. Example. We let $(\mathbf{C}(\mathcal{C});\mathbf{C}(\mathcal{E}))$ denote the category of chain complexes of objects in $\mathcal{C}$ equipped with the degreewise exact structure. For every $i \in \mathbb{Z}$, there is an adjoint triple $(\mathrm{D}^i, \mathrm{ev}_i, \mathrm{D}^{i+1})$,

$$\mathcal{C} \underset{\mathrm{ev}_i}{\overset{\mathrm{D}^i}{\underset{\mathrm{D}^{i+1}}{\rightleftarrows}}} \mathbf{C}(\mathcal{C}),$$

where ev_i denotes the ith evaluation functor, and $\mathrm{D}^i(C)$ is the chain complex with C at ith and $(i-1)$th positions, and 0 elsewhere. For every $i \in \mathbb{Z}$, the functors D^i and ev_i are known to be exact, and therefore, we have natural isomorphisms

$$\mathrm{Ext}^n(\mathrm{D}^i(-),-) \cong \mathrm{Ext}^n(-,\mathrm{ev}_i(-)) \quad \text{and} \quad \mathrm{Ext}^n(\mathrm{ev}_i(-),-) \cong \mathrm{Ext}^n(-,\mathrm{D}^{i+1}(-))$$

for every $n \geq 1$.

2.5. Example. Let X be a semi-separated scheme. For any affine subscheme $\iota : U \hookrightarrow X$, we have the adjoint pair (res, ι_*) of functors

$$\mathfrak{Qco}(X) \underset{\iota_*}{\overset{\mathrm{res}}{\rightleftarrows}} \mathfrak{Qco}(U)$$

between categories of quasi-coherent sheaves on X and U. Both the restriction res and the direct image functor ι_* are exact, and therefore, for every $n \geq 1$,

$$\mathrm{Ext}^n(\mathrm{res}(-),-) \cong \mathrm{Ext}^n(-,\iota_*(-)).$$

From now on, even though we present statements together with their dual versions, we only prove the first statement, as their arguments can be dualized easily. We also use notations presented in Lemma 2.1.

Now we investigate conditions for which Θ^n_C and Ω^n_D are isomorphisms for fixed objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$. As a first step, we study the case $n = 1$. For it, we introduce the following definition.

2.6. Definition. We call an object C in $\mathcal{C}$ q-*flat* if every short exact sequence in $\mathcal{C}$ ending with C remains exact under q. Dually, a t-*coflat* object in $\mathcal{D}$ is defined.

2.7. Remark. In [20], for given two proper classes $\mathcal{P}$ and $\mathcal{R}$ in an abelian category $\mathcal{B}$, the notions of $\mathcal{P}$-$\mathcal{R}$-flat and divisible object have been introduced. Namely, an object B in $\mathcal{B}$ is called $\mathcal{P}$-$\mathcal{R}$-flat if every short exact sequence in $\mathcal{P}$ ending at B belongs to $\mathcal{R}$. $\mathcal{P}$-$\mathcal{R}$ divisibles are defined dually. In our case, q is a left adjoint functor, so by Lemma 3.3 in Chapter 1 of [22], the class $\overline{\mathcal{E}}$ consisting of short exact sequences in $\mathcal{E}$ which remains exact under q is an exact substructure of $\mathcal{E}$. So an object C in $\mathcal{C}$ is q-flat if and only if it is $\mathcal{E}$-$\overline{\mathcal{E}}$-flat in the sense of [20]. The dual statement holds for p-coflat and divisible objects.

The following lemma will be used for proving Proposition 2.10.

2.8. Lemma. *For given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the following hold.*

(i) *The functor* q *preserves short exact sequences in* $\mathrm{Ext}^1(C, \mathrm{t}(-))$ *if and only if C is* q-flat.

(ii) *The functor* t *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{q}(-), D)$ *if and only if D is* t-coflat.

Proof. The "if" condition of the statement (i) is immediate. For the "only if" part, suppose that q preserves short exact sequences in $\mathrm{Ext}^1(C, \mathrm{t}(-))$. Consider a short exact sequence in $\mathcal{C}$,

$$\mathbb{E}: \ A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C.$$

By taking pushout of $\mathbb{E}$ along the unit morphism η_A, we have a short exact sequence in $\mathcal{C}$

$$\eta_A\mathbb{E}: \ \mathrm{tq}(A) \rightarrowtail B' \twoheadrightarrow C$$

together with the pushout diagram $\mathbb{E} \longrightarrow \eta_A\mathbb{E}$. By assumption, $\mathrm{q}(\eta_A\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. Taking pushout of $\mathrm{q}(\eta_A\mathbb{E})$ along the morphism $\xi_{\mathrm{q}(A)}$, we obtain a short exact sequence $\xi_{\mathrm{q}(A)}\mathrm{q}(\eta_A\mathbb{E})$ together with the commutative diagram

$$\mathrm{q}(\mathbb{E}) \longrightarrow \mathrm{q}(\eta_A\mathbb{E}) \rightarrow \xi_{\mathrm{q}(A)}\mathrm{q}(\eta_A\mathbb{E}),$$

$$\begin{array}{ccccc}
\mathrm{q}(\mathbb{E}): & & \mathrm{q}(A) & \xrightarrow{\mathrm{q}(i)} & \mathrm{q}(B) \xrightarrow{\mathrm{q}(p)} \mathrm{q}(C) \\
\downarrow & & \| & & \downarrow \quad \| \\
\xi_{\mathrm{q}(A)}\mathrm{q}(\eta_A\mathbb{E}): & & \mathrm{q}(A) & \rightarrowtail & X \twoheadrightarrow \mathrm{q}(C).
\end{array}$$

Since q preserves cokernels, $\mathrm{q}(p)$ is the cokernel of $\mathrm{q}(i)$, and by Proposition 2.16 in [2], $\mathrm{q}(i)$ is an admissible monic in $\mathcal{D}$, and therefore, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. ∎

Next lemma, which will be used to prove Theorem 3.1, illustrates that (co)generating classes are sufficient for showing (co)flatness of an object.

2.9. Lemma. *Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in $\mathcal{C}$ and $\mathcal{D}$, respectively. The following hold.*

(i) *If $\mathcal{G}$ is a cogenerating class in $\mathcal{D}$, and if the functor* q *preserves short exact sequences in* $\mathrm{Ext}^1(C, \mathrm{t}(\mathcal{G}))$, *then C is a* q*-flat object.*

(ii) *If $\mathcal{F}$ is a generating class in $\mathcal{C}$, and if the functor* t *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{q}(\mathcal{F}), D)$, *then D is a* t*-coflat object.*

Proof. Let $\mathbb{E}\colon C'' \rightarrowtail C' \twoheadrightarrow C$ be a short exact sequence in $\mathcal{C}$. By assumption, there exists an admissible monic $f\colon \mathrm{q}(C'') \hookrightarrow G$ in $\mathcal{D}$ with $G \in \mathcal{G}$. Let

$$\bar{f} := \mathrm{t}(f) \circ \eta_{C''} : C'' \to \mathrm{t}(G).$$

Consider the pushout diagram $\mathbb{E} \longrightarrow \bar{f}\mathbb{E}$. We have that $\bar{f}\mathbb{E}$ is a short exact sequence in $\mathrm{Ext}^1(C, \mathrm{t}(G))$, and by assumption, $\mathrm{q}(\bar{f}\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. Again, taking pushout of the short exact sequence $\mathrm{q}(\bar{f}\mathbb{E})$ along the counit morphism $\xi_G\colon \mathrm{qt}(G) \to G$, we obtain a short exact sequence $\xi_G\mathrm{q}(\bar{f}\mathbb{E})$. Consider the natural composition of sequences

$$\mathrm{q}(\mathbb{E}) \longrightarrow \mathrm{q}(\bar{f}\mathbb{E}) \longrightarrow \xi_G\mathrm{q}(\bar{f}\mathbb{E}).$$

Note that the morphism on the most left column of the above composite is $\xi_G \circ \mathrm{q}(\bar{f}) = f$, which is an admissible monic in $\mathcal{D}$. Since q preserves cokernels, the morphism $\mathrm{q}(C') \to \mathrm{q}(C)$ is the cokernel of the morphism $\mathrm{q}(C'') \to \mathrm{q}(C')$, and by Proposition 2.16 in [2], $\mathrm{q}(C'') \to \mathrm{q}(C')$ is an admissible monic in $\mathcal{D}$, and therefore, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. ∎

2.10. Proposition. *For given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the following hold.*

(i) *Suppose that* t *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{q}(C), -)$. *Then the following are equivalent*:

  (a) *The natural transformation*

$$\Theta^1_C : \mathrm{Ext}^1(\mathrm{q}(C), -) \Longrightarrow \mathrm{Ext}^1(C, \mathrm{t}(-))$$

  is an isomorphism.

  (b) *C is a* q*-flat object in $\mathcal{C}$.*

(ii) *Suppose that* q *preserves short exact sequences in* $\mathrm{Ext}^1(-, \mathrm{t}(D))$. *Then the following are equivalent*:

  (a) *The natural transformation*

$$\Omega^1_D : \mathrm{Ext}^1(-, \mathrm{t}(D)) \Longrightarrow \mathrm{Ext}^1(\mathrm{q}(-), D)$$

  is an isomorphism.

  (b) *D is a* t*-coflat object in $\mathcal{D}$.*

Proof. The implication (b)⇒(a) in (i) follows from Lemma 2.1-(iii) and Lemma 2.8-(i). Now suppose that the natural transformation Θ^1_C is an isomorphism. Consider a short exact sequence in $\mathcal{C}$,

$$\mathbb{E}: \ A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \ .$$

By taking pushout of $\mathbb{E}$ along the unit morphism η_A, we have a short exact sequence in $\mathcal{C}$

$$\eta_A\mathbb{E}: \ \mathrm{tq}(A) \rightarrowtail B' \twoheadrightarrow C$$

together with the pushout diagram $\mathbb{E} \longrightarrow \eta_A\mathbb{E}$. By assumption, there exists a short exact sequence $\mathbb{E}'$: $\mathrm{q}(A) \rightarrowtail X \twoheadrightarrow \mathrm{q}(C)$ in $\mathcal{D}$ such that $\mathrm{t}(\mathbb{E}')\eta_C \equiv \eta_A\mathbb{E}$. Note that, by

assumption, $\mathrm{t}(\mathbb{E}')$ is a short exact sequence in $\mathcal{C}$. Applying q, we have a commutative diagram

$$\mathrm{q}(\mathbb{E}) \longrightarrow \mathrm{q}(\eta_A \mathbb{E}) \equiv \mathrm{q}(\mathrm{t}(\mathbb{E}')\eta_C) \to qt(\mathbb{E}').$$

Composing it with the canonical diagram $qt(\mathbb{E}') \longrightarrow \mathbb{E}'$ induced from the counit natural transformation ξ, we have a commutative diagram

$$\begin{array}{ccccccc} \mathrm{q}(\mathbb{E}): & & \mathrm{q}(A) & \xrightarrow{\mathrm{q}(i)} & \mathrm{q}(B) & \xrightarrow{\mathrm{q}(p)} & \mathrm{q}(C) \\ \downarrow & & \| & & \downarrow & & \| \\ \mathbb{E}': & & \mathrm{q}(A) & \rightarrowtail & X & \twoheadrightarrow & \mathrm{q}(C). \end{array}$$

Since q preserves cokernels, $\mathrm{q}(p)$ is the cokernel of $\mathrm{q}(i)$, and by Proposition 2.16 in [2], $\mathrm{q}(i)$ is an admissible monic in $\mathcal{D}$, and therefore, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. ∎

The previous result covers the one proven in Theorem 4.2 of [1] for $\mathbf{I}$-directed colimit functor, as shown in the following example.

2.11. Example. Let $\mathrm{Fun}(\mathbf{I}, \mathcal{C})$ be the category of functors from $\mathbf{I}$ to $\mathcal{C}$ equipped with the degreewise exact structure $\mathrm{Fun}(\mathbf{I}, \mathcal{E})$. We let $\kappa_{\mathbf{I}} : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathbf{I}, \mathcal{C})$ denote the constant functor, that is, for a given object C in $\mathcal{C}$,

$$\kappa_{\mathbf{I}}(C)(i) := C \quad \text{and} \quad \kappa_{\mathbf{I}}(C)(f) := \mathrm{id}_C$$

for every object i and a morphism f in $\mathbf{I}$. It is a well-known fact that $\kappa_{\mathbf{I}}$ is an exact functor, and that $\kappa_{\mathbf{I}}$ has a left adjoint if and only if $\mathcal{C}$ has $\mathbf{I}$-directed colimits. We assume that $\mathcal{C}$ has $\mathbf{I}$-directed colimits. For a given functor $F : \mathbf{I} \to \mathcal{C}$, $\mathrm{colim}_{\mathbf{I}}(F)$ denotes its colimit. So the canonical natural transformation

$$\mathrm{Ext}^1(\mathrm{colim}_{\mathbf{I}}(F), -) \Longrightarrow \mathrm{Ext}^1(F, \kappa_{\mathbf{I}}(-))$$

is an isomorphism if and only if for every short exact sequence $\mathbb{E}$ in $\mathrm{Fun}(\mathbf{I}, \mathcal{C})$ ending with F, $\mathrm{colim}_{\mathbf{I}}(\mathbb{E})$ is a short exact sequence in $\mathcal{C}$, as well. In particular, if $\mathcal{C}$ is an abelian category with $\mathbf{I}$-directed colimits, then the functor of $\mathbf{I}$-directed colimits in $\mathcal{C}$ is exact if and only if $\mathrm{Ext}^1(\mathrm{colim}_{\mathbf{I}}(-), -) \Longrightarrow \mathrm{Ext}^1(-, \kappa_{\mathbf{I}}(-))$ is an isomorphism; see Theorem 4.2 in [1].

As a particular case, if $\mathbf{I}$ is a discrete category, then

$$\mathrm{Fun}(\mathbf{I}, \mathcal{C}) \cong \prod_{i \in \mathbf{I}} \mathcal{C} \quad \text{and} \quad \mathrm{colim}_{\mathbf{I}}(-) \cong \bigoplus_{i \in \mathbf{I}}.$$

So $\mathcal{C}$ has exact $\mathbf{I}$-indexed coproducts if and only if

$$\mathrm{Ext}^1\left(\bigoplus_{\mathbf{I}}(-), -\right) \cong \prod_{i \in \mathbf{I}} \mathrm{Ext}^1(-, -).$$

To obtain similar results for Θ_C^n and Ω_D^n for any $n \geq 1$ in such generality, we would need to consider some lengthy technical conditions, such as being all syzygies of C q-flat. Instead, we prefer to use left and right derived functors (see Proposition 2.15). For it, we prove the following two lemmas.

2.12. Lemma. *Let $\mathcal{F}$ and $\mathcal{G}$ be extension closed-classes of objects in $\mathcal{C}$ and $\mathcal{D}$ satisfying conditions* (i)–(ii) *and their duals given in* 1.6*, respectively. The following holds.*

(i) *Suppose* t *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{q}(\mathcal{F}), -)$. *The functor* q *restricted on $\mathcal{F}$ is exact if and only if Θ^n_F is an isomorphism for every $F \in \mathcal{F}$ and $n \geq 1$.*

(ii) *Suppose* q *preserves short exact sequences in* $\mathrm{Ext}^1(-, \mathrm{t}(\mathcal{G}))$. *The functor* t *restricted on $\mathcal{G}$ is exact if and only if Ω^n_D is an isomorphism for every $G \in \mathcal{G}$ and $n \geq 1$.*

Proof. Firstly, note that the functor t is an exact functor by Lemma 2.9. So the natural transformation $\Theta^n_F : \mathrm{Ext}^n(\mathrm{q}(F), -) \Longrightarrow \mathrm{Ext}^n(F, \mathrm{t}(-))$ exists for every $F \in \mathcal{F}$ and $n \geq 1$.

The "if" statement now follows from Proposition 2.10. For the "only if" part, suppose that q preserves short exact sequences in $\mathcal{C}$ ending with objects in $\mathcal{F}$. We firstly prove that q preserves all short exact sequences in $\mathcal{C}$ ending with an object in $\mathcal{F}$. Let

$$\mathbb{E} : \; A \hookrightarrow B \twoheadrightarrow F$$

be a short exact sequence in $\mathcal{C}$ with an object F in $\mathcal{F}$.

By assumption, there exists an admissible epic $F_0 \twoheadrightarrow B$ with $F_0 \in \mathcal{F}$, which induces the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{E}' : & & F_1 & \hookrightarrow & F_0 & \twoheadrightarrow & F \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| \\ \mathbb{E} : & & A & \hookrightarrow & B & \twoheadrightarrow & F \end{array}$$

with exact rows, and therefore, $f\mathbb{E}' = \mathbb{E}$. Again by assumption, $F_1 \in \mathcal{F}$, and hence $\mathrm{q}(\mathbb{E}')$ is a short exact sequence in $\mathcal{D}$. Since q preserves all colimits, it preserves cokernels and pushout diagrams. So we have

$$\mathrm{q}(\mathbb{E}) \cong \mathrm{q}(f\mathbb{E}') \cong \mathrm{q}(f)\,\mathrm{q}(\mathbb{E}'),$$

that is, $\mathrm{q}(\mathbb{E})$ is a pushout of $\mathrm{q}(\mathbb{E}')$ along $\mathrm{q}(f)$. This implies that $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$, and by Proposition 2.10, Θ^1_F is an isomorphism for every F in $\mathcal{F}$.

Now, we suppose that the statement is true for some $n \geq 1$, and we prove it for $n+1$. The natural transformation

$$\Theta^{n+1}_F : \mathrm{Ext}^{n+1}(\mathrm{q}(F), -) \Longrightarrow \mathrm{Ext}^{n+1}(F, \mathrm{t}(-))$$

is an epimorphism for every $F \in \mathcal{F}$. Indeed, if $\overline{\mathbb{E}} = \overline{\mathbb{E}}_2 \circ \overline{\mathbb{E}}_1$ is an exact sequence in $\mathrm{Ext}^{n+1}(F, \mathrm{t}(D))$, where $\overline{\mathbb{E}}_1$ is a short exact sequence in $\mathcal{C}$, then by the same argument as above, there exist a morphism f and a short exact sequence $\mathbb{E}_1 \colon F_1 \hookrightarrow F_0 \twoheadrightarrow F$ in $\mathcal{C}$ with $F, F_0, F_1 \in \mathcal{F}$ such that $f\mathbb{E}_1 \equiv \overline{\mathbb{E}}_1$. So

$$\overline{\mathbb{E}} = \overline{\mathbb{E}}_2 \circ \overline{\mathbb{E}}_1 \equiv \overline{\mathbb{E}}_2 \circ (f\mathbb{E}_1) \equiv (\overline{\mathbb{E}}_2 f) \circ \mathbb{E}_1 =: \mathbb{E},$$

$$\begin{array}{ccccccccccc} \mathbb{E} : & \mathrm{t}(D) & \hookrightarrow & X_n & \to & \cdots & \to & X_1 & \longrightarrow & F_0 & \twoheadrightarrow F. \\ & & & & & & & & \searrow & \nearrow & \\ & & & & & & & & F_1 & & \end{array} \tag{2.12.1}$$

We let $\mathbb{E}_2 := \bar{\mathbb{E}}_2 f$, which belongs to $\mathrm{Ext}^n(F_1, \mathrm{t}(D))$. Since $\Theta^n_{F_1 D}$ is an isomorphism, there exists an exact sequence $\mathbb{E}'_2$ in $\mathrm{Ext}^n(\mathrm{q}(F_1), D)$ such that

$$\Theta^n_{F_1 D}(\mathbb{E}'_2) = \mathrm{t}(\mathbb{E}'_2)\,\eta_{F_1} \equiv \mathbb{E}_2.$$

So we have

$$\mathbb{E} \equiv \mathbb{E}_2 \circ \mathbb{E}_1 \equiv (\mathrm{t}(\mathbb{E}'_2)\,\eta_{F_1}) \circ \mathbb{E}_1 \equiv \mathrm{t}(\mathbb{E}'_2) \circ (\eta_{F_1}\mathbb{E}_1).$$

Note that $\eta_{F_1}\mathbb{E}_1$ is a short exact sequence in $\mathrm{Ext}^1(F, \mathrm{tq}(F_1))$. Similarly, as $\Theta^1_{F\mathrm{q}(F_1)}$ is an isomorphism, there exists a short exact sequence $\mathbb{E}'_1$ in $\mathrm{Ext}^1(\mathrm{q}(F), \mathrm{q}(F_1))$ such that $\mathrm{t}(\mathbb{E}'_1)\eta_F \equiv \eta_{F_1}\mathbb{E}_1$.

Let $\mathbb{E}' := \mathbb{E}'_2 \circ \mathbb{E}'_1$. Notice that $\mathbb{E}'$ is an exact sequence in $\mathrm{Ext}^{n+1}(\mathrm{q}(F), D)$, and

$$\mathrm{t}(\mathbb{E}')\eta_F = \mathrm{t}(\mathbb{E}'_2) \circ (\mathrm{t}(\mathbb{E}'_1)\eta_F) \equiv \mathrm{t}(\mathbb{E}'_2) \circ (\eta_{F_1}\mathbb{E}_1) \equiv \mathbb{E}.$$

Now we prove that Θ^{n+1}_{FD} is injective. Let $\mathbb{E}''$ be an exact sequence in $\mathrm{Ext}^{n+1}(\mathrm{q}(F), D)$ such that $\Theta^{n+1}_{FD}(\mathbb{E}'') = \mathrm{t}(\mathbb{E}'')\eta_F$ is trivial. Using Theorem 4.2 in [16] and similar strategy applied above, there exist an exact sequence $\mathbb{E} = \mathbb{E}_2 \circ \mathbb{E}_1$ of the form as given in (2.12.1) and a morphism $\mathbb{E} \to \mathrm{t}(\mathbb{E}'')$ with morphisms $\mathrm{id}_{\mathrm{t}(D)}$ and η_F on the most left and right ending objects, respectively, such that $\mathbb{E}_2$ is a trivial exact sequence in $\mathrm{Ext}^n(F_1, \mathrm{t}(D))$. Notice that $\mathrm{q}(\mathbb{E}) = \mathrm{q}(\mathbb{E}_2) \circ \mathrm{q}(\mathbb{E}_1)$ is exact, and that $\mathrm{q}(\mathbb{E}_2)$ is a trivial exact sequence. We consider the composition of morphism of sequences

$$\mathrm{q}(\mathbb{E}) \longrightarrow \mathrm{qt}(\mathbb{E}'') \longrightarrow \mathbb{E}'',$$

where the last morphism is the canonical counit morphism of the adjunction. The composite has the morphisms ξ_D and $\mathrm{id}_{\mathrm{q}(F)}$ on the most left and right objects, respectively. So it can be factorized as

$$\mathrm{q}(\mathbb{E}) \longrightarrow \xi_D \mathrm{q}(\mathbb{E}) \longrightarrow \mathbb{E}''.$$

Notice that $\xi_D \mathrm{q}(\mathbb{E})$ is a trivial exact sequence which is equivalent to $\mathbb{E}''$. As a consequence, Θ^{n+1}_F is an isomorphism for every F in $\mathcal{F}$. By mathematical induction, Θ^n_F is an isomorphism for every $n \geq 1$. ∎

2.13. Example. We assume that $\mathbf{C}(\mathcal{C})$ is equipped with the exact structure $\mathbf{C}(\mathcal{E})$ (see Example 2.4). If $\mathcal{C}$ has (co)kernels, then for every $i \in \mathbb{Z}$, there is an adjoint triple $(\mathrm{c}_i, \mathrm{s}^i, \mathrm{k}_i)$

$$\mathbf{C}(\mathcal{C}) \;\underset{\mathrm{k}_i}{\overset{\mathrm{c}_i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}}\; \mathcal{C}, \qquad \mathrm{s}^i \text{ in the middle},$$

where $\mathrm{s}^i(C)$ denotes the ith sphere chain complex of C, and $\mathrm{c}_i(X)$ and $\mathrm{k}_i(X)$ are the cokernel and kernel of differentials $X_{i+1} \longrightarrow X_i$ and $X_i \longrightarrow X_{i-1}$, respectively. Note that for every $i \in \mathbb{Z}$, the stalk functor s^i is exact. So for any chain complex X in $\mathbf{C}(\mathcal{C})$ and $n \geq 1$, there are natural transformations

$$\Theta^n_X : \mathrm{Ext}^n(\mathrm{c}_i(X), -) \Longrightarrow \mathrm{Ext}^n(X, \mathrm{s}_i(-)),$$
$$\Omega^n_X : \mathrm{Ext}^n(-, \mathrm{k}_i(X)) \Longrightarrow \mathrm{Ext}^n(\mathrm{s}_i(-), X).$$

In general, the functors c_i and k_i are not exact (see Example 3.6), and therefore, Θ^n_X and Ω^n_X may not be an isomorphism for an arbitrary X. Recall that a chain complex X in $\mathbf{C}(\mathcal{C})$ is said to be *acyclic* if every differential factors as $X_n \twoheadrightarrow \mathrm{k}_{i-1}(X) \hookrightarrow X_{i-1}$ in such a way that each sequence

$$\mathrm{k}_i(X) \longrightarrow X_i \longrightarrow \mathrm{k}_{i-1}(X)$$

is a short exact sequence in $\mathcal{C}$; see Definition 10.1 in [2]. We let Acyc denote the class of acyclic chain complexes in $\mathbf{C}(\mathcal{C})$. By Proposition 7.5 in [25], we already know that Acyc satisfies all conditions and their dual forms given in 1.6. As it will be shown in Example 3.6, the functors c_i and k_i restricted on Acyc are exact, and by Lemma 2.12, the natural transformations Θ^n_X and Ω^n_X are isomorphisms for every acyclic chain complex X.

2.14. Lemma. *Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in $\mathcal{C}$ and $\mathcal{D}$, respectively. For any integer $n \geq 1$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the followings hold.*

(i) *Suppose that* t *is exact, $\mathcal{D}$ is an abelian category and $\mathcal{F}$ and* q *satisfy conditions given in 1.6. If for every $1 \leq k \leq n$, $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$, then the natural transformation*

$$\Theta^{n+1}_C : \mathrm{Ext}^{n+1}(\mathrm{q}(C), -) \Longrightarrow \mathrm{Ext}^{n+1}(C, \mathrm{t}(-))$$

is a monomorphism.

(ii) *Suppose that* q *is exact, $\mathcal{C}$ is an abelian category and $\mathcal{G}$ and* t *satisfies dual of conditions given in 1.6. If for every $1 \leq k \leq n$, $\mathrm{R}^k_{\mathcal{G}}\mathrm{t}(D) = 0$, then the natural transformation*

$$\Omega^{n+1}_D : \mathrm{Ext}^{n+1}(-, \mathrm{t}(D)) \Longrightarrow \mathrm{Ext}^{n+1}(\mathrm{q}(-), D)$$

is a monomorphism.

Proof. As pointed out in 1.6, using $\mathcal{F}$-resolutions, we have the kth derived functor $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}\colon \mathcal{C} \longrightarrow \mathcal{D}$ satisfying $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(F) = 0$ for every $F \in \mathcal{F}$ and $k \geq 1$.

Consider a short exact sequence in $\mathcal{C}$ of the form

$$\mathbb{E} : K \hookrightarrow F \twoheadrightarrow C,$$

with F in $\mathcal{F}$. It induces the following long exact sequence in $\mathcal{D}$:

$$0 \longrightarrow \mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) \longrightarrow \mathrm{q}(K) \longrightarrow \mathrm{q}(F) \longrightarrow \mathrm{q}(C) \longrightarrow 0. \tag{2.14.1}$$

Suppose that $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$ for every $1 \leq k \leq n$. Then the exact sequence (2.14.1) turns out to be of the form

$$0 \longrightarrow \mathrm{q}(K) \longrightarrow \mathrm{q}(F) \longrightarrow \mathrm{q}(C) \longrightarrow 0.$$

For a given object D in $\mathcal{D}$, applying the contravariant functor $\mathrm{Hom}(-, D)$, and subsequently, using the adjoint pair (q, t) and Lemma 2.12, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\mathrm{Ext}^n(\mathrm{q}(F), D) & \longrightarrow & \mathrm{Ext}^n(\mathrm{q}(K), D) & \longrightarrow & \mathrm{Ext}^{n+1}(\mathrm{q}(C), D) & \longrightarrow & \mathrm{Ext}^{n+1}(\mathrm{q}(F), D) \\
\cong \downarrow \Theta^n_{FD} & & \downarrow \Theta^n_{KD} & & \downarrow \Theta^{n+1}_{CD} & & \cong \downarrow \Theta^{n+1}_{FD} \\
\mathrm{Ext}^n(F, \mathrm{t}(D)) & \longrightarrow & \mathrm{Ext}^n(F, \mathrm{t}(D)) & \longrightarrow & \mathrm{Ext}^{n+1}(C, \mathrm{t}(D)) & \longrightarrow & \mathrm{Ext}^{n+1}(F, \mathrm{t}(D)).
\end{array}$$

Now, we apply mathematical induction on n. If $n = 1$, then by Lemma 2.1, Θ^1_{KD} is a monomorphism, and therefore, so is Θ^2_{CD}.

Now, suppose that the statement is true for some $n \geq 1$, and $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$ for every $1 \leq k \leq n+1$. Then $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(K) = 0$ for every $1 \leq k \leq n$. By induction hypothesis, Θ^{n+1}_{KD} is a monomorphism, then so is Θ^{n+2}_{CD} ∎

2.15. Proposition. *Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects in $\mathcal{C}$ and $\mathcal{D}$, respectively. For any integer $n \geq 1$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the followings hold.*

(i) *If* t *is exact, $\mathcal{D}$ is an abelian category, and $\mathcal{F}$ and* q *satisfy conditions given in 1.6, then the following are equivalent*:
   (a) *for every $1 \leq k \leq n$, $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$;*
   (b) *for every $1 \leq k \leq n$, the natural transformation*

$$\Theta^k_C : \mathrm{Ext}^k(\mathrm{q}(C), -) \Longrightarrow \mathrm{Ext}^k(C, \mathrm{t}(-)),$$

   is an isomorphism.

(ii) *If* q *is exact, $\mathcal{C}$ is an abelian category, and $\mathcal{G}$ and* t *satisfy dual conditions given in 1.6, then the followings hold.*
   (a) *For every $1 \leq k \leq n$, $\mathrm{R}^k_{\mathcal{G}} \mathrm{t}(D) = 0$.*
   (b) *For every $1 \leq k \leq n$, the natural transformation*

$$\Omega^k_D : \mathrm{Ext}^k(-, \mathrm{t}(D)) \Longrightarrow \mathrm{Ext}^k(\mathrm{q}(-), D),$$

   is an isomorphism.

Proof. Consider a short exact sequence in $\mathcal{C}$ of the form

$$\mathbb{E} : \ K \hookrightarrow F \twoheadrightarrow C,$$

with F in $\mathcal{F}$. It induces the following long exact sequence in $\mathcal{D}$:

$$0 \longrightarrow \mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) \longrightarrow \mathrm{q}(F) \longrightarrow \mathrm{q}(P) \longrightarrow \mathrm{q}(C) \longrightarrow 0. \tag{2.15.1}$$

We apply mathematical induction on n to prove the equivalence of the statements (a) and (b). If $\mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) = 0$, then the exact sequence (2.15.1) is turned out being of the form

$$0 \longrightarrow \mathrm{q}(K) \longrightarrow \mathrm{q}(F) \longrightarrow \mathrm{q}(C) \longrightarrow 0,$$

which by Lemma 2.12, yields the following commutative diagram of exact sequences of functors:

$$\begin{array}{ccccccccc}
\mathrm{Hom}(\mathrm{q}(F), -) & \Longrightarrow & \mathrm{Hom}(\mathrm{q}(K), -) & \Longrightarrow & \mathrm{Ext}^1(\mathrm{q}(C), -) & \Longrightarrow & \mathrm{Ext}^1(\mathrm{q}(F), -) & \Longrightarrow & \mathrm{Ext}^1(\mathrm{q}(K), -) \\
\cong\downarrow & & \cong\downarrow & & \downarrow\Theta^1_C & & \cong\downarrow\Theta^1_F & & \downarrow\Theta^1_K \\
\mathrm{Hom}(F, \mathrm{t}(-)) & \Longrightarrow & \mathrm{Hom}(F, \mathrm{t}(-))] & \Longrightarrow & \mathrm{Ext}^1(C, \mathrm{t}(-)) & \Longrightarrow & \mathrm{Ext}^1(F, \mathrm{t}(-)) & \Longrightarrow & \mathrm{Ext}^1(K, \mathrm{t}(-))
\end{array}$$

Using the fact that Θ^1_{CD} and Θ^1_{KD} are monomorphisms, one can easily show that Θ^1_{CD} is in fact an isomorphism.

For the converse, suppose that the natural transformation

$$\Theta^1_C : \mathrm{Ext}^1(\mathrm{q}(C), -) \Longrightarrow \mathrm{Ext}^1(C, \mathrm{t}(-))$$

is an isomorphism. By Proposition 2.10, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$, and hence, $\mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) = 0$.

Now, suppose that the equivalence (a)$\Leftrightarrow$(b) is true for any $n \geq 1$ and any object in $\mathcal{C}$. Assume that $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$ for every $1 \leq k \leq n+1$. By induction step, the natural transformation

$$\Theta^k_C : \mathrm{Ext}^k(\mathrm{q}(C), -) \Longrightarrow \mathrm{Ext}^k(C, \mathrm{t}(-))$$

is an isomorphism for every $1 \leq k \leq n$, and by Lemma 2.14, Θ^{n+1}_C is a monomorphism. And from the long exact sequence of derived functors $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(-)$ obtained through the short exact sequence $\mathbb{E}$, we have

$$0 = \mathrm{L}^{\mathcal{F}}_{k+1} \mathrm{q}(C) \cong \mathrm{L}^{\mathcal{F}}_k \mathrm{q}(K)$$

for every $1 \leq k \leq n$. By induction step, Θ^k_K is an isomorphism for every $1 \leq k \leq n$, and by Lemma 2.14, Θ^{n+1}_K is a monomorphism.

As $\mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) = 0$, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$, which induces the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Ext}^n(\mathrm{q}(K), -) & \Longrightarrow & \mathrm{Ext}^{n+1}(\mathrm{q}(C), -) & \Longrightarrow & \mathrm{Ext}^{n+1}(\mathrm{q}(F), -) & \Longrightarrow & \mathrm{Ext}^{n+1}(K, \mathrm{t}(-)) \\
\cong \Big\Vert \Theta^n_K & & \Big\Vert \Theta^{n+1}_C & & \cong \Big\Vert \Theta^{n+1}_F & & \Big\Vert \Theta^{n+1}_K \\
\mathrm{Ext}^n(F, \mathrm{t}(-))] & \Longrightarrow & \mathrm{Ext}^{n+1}(C, \mathrm{t}(-)) & \Longrightarrow & \mathrm{Ext}^{n+1}(F, \mathrm{t}(-)) & \Longrightarrow & \mathrm{Ext}^{n+1}(K, \mathrm{t}(-))
\end{array}$$

with exact rows. From the diagram, it is easy to verify that Θ^{n+1}_C is an epimorphism.

For the converse, suppose that the natural transformation Θ^k_C is an isomorphism for every $1 \leq k \leq n+1$. By induction hypothesis, $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(C) = 0$ for every $1 \leq k \leq n$. As a particular case, $\mathrm{L}^{\mathcal{F}}_1 \mathrm{q}(C) = 0$, and therefore, $\mathrm{q}(\mathbb{E})$ is a short exact sequence in $\mathcal{D}$. Applying five lemma, one can show that Θ^k_K is an isomorphism for every $1 \leq k \leq n$. Again, by induction hypothesis, $\mathrm{L}^{\mathcal{F}}_k \mathrm{q}(K) = 0$ for every $1 \leq k \leq n$. By shifting dimension, we have $\mathrm{L}^{\mathcal{F}}_{n+1} \mathrm{q}(C) \cong L^{\mathcal{F}}_n \mathrm{q}(K) = 0$. ∎

2.16. Example. Let $f : R \to S$ a ring homomorphism. Consider the adjoint pair

$$R\text{-Mod} \underset{\mathrm{res}}{\overset{S \otimes_R -}{\rightleftarrows}} S\text{-Mod},$$

where $\mathrm{res} : S\text{-Mod} \longrightarrow R\text{-Mod}$ denotes the functor of restriction of scalars through the ring homomorphism f. Note that res is an exact functor, and

$$\mathrm{L}_k(S \otimes_R -) = \mathrm{Tor}^k_R(S, -).$$

By Proposition 2.15-(i), for a left R-module M, $\mathrm{Tor}^k_R(S, M) = 0$ for every $1 \leq k \leq n$ if and only if $\Theta^k_M : \mathrm{Ext}^k(S \otimes_R M, -) \Longrightarrow \mathrm{Ext}^k(M, \mathrm{res}(-))$, is an isomorphism for every $1 \leq k \leq n$.

Similarly, consider the adjoint pair

$$S\text{-Mod} \underset{\mathrm{Hom}(S,-)}{\overset{\mathrm{res}}{\rightleftarrows}} R\text{-Mod}.$$

By Proposition 2.15, for a left R-module N, $\mathrm{Ext}^k(S,N)=0$ for every $1\le k\le n$ if and only if $\Omega^k_N\colon \mathrm{Ext}^k(-,\mathrm{Hom}(S,N)) \Longrightarrow \mathrm{Ext}^k(\mathrm{res}(-),N)$ is an isomorphism for every $1\le k\le n$.

3. Cotorsion pairs induced from adjoint pairs

In this section, we apply the results from Section 2 to investigate cotorsion pairs arising through an adjoint pair. We also provide characterizations of objects in the right/left perpendicular classes that cover well-known examples.

We keep assuming that $\mathcal{C}$ and $\mathcal{D}$ are exact categories. For a given family $\{x_i\}_{i\in I}$ of functors from $\mathcal{C}$ to $\mathcal{D}$, and classes $\mathcal{F}$ and $\mathcal{G}$ of objects in $\mathcal{C}$ and $\mathcal{D}$, respectively, we let

$$\begin{aligned} x_*(\mathcal{F}) &:= \{x_i(F) \mid F\in\mathcal{F} \text{ and } i\in I\},\\ x_*^{-1}(\mathcal{G}) &:= \{C\in\mathcal{C} \mid x_i(C)\in\mathcal{G}, \text{ for every } i\in I\}. \end{aligned}$$

An object in $\mathcal{C}$ will be said to be x_*-(co)flat if it is x_i-(co)flat for every $i\in I$.

3.1. Theorem. *Let $\{(\mathrm{q}_i,\mathrm{t}_i)\}_{i\in I}$ be a family of adjoint pairs*

$$\mathcal{C} \underset{\mathrm{t}_i}{\overset{\mathrm{q}_i}{\rightleftarrows}} \mathcal{D}.$$

For given cotorsion pairs $(\mathcal{F},\mathcal{G})$ and $(\mathcal{F}',\mathcal{G}')$ in $\mathcal{C}$ and $\mathcal{D}$, respectively, the following statements hold.

(i) *Suppose that for every $i\in I$, every object in $\mathcal{F}$ is q_i-flat.*

(a) *If for every $i\in I$, t_i preserves short exact sequences in $\mathrm{Ext}^1(\mathrm{q}_i(\mathcal{F}),-)$, then*

$$(\mathrm{q}_*(\mathcal{F}))^{\perp} = \mathrm{t}_*^{-1}(\mathcal{G}).$$

(b) *If $\mathcal{F}$ is a generating class in $\mathcal{C}$, then*

$$(\mathrm{q}_*(\mathcal{F}))^{\perp} = \{D\in \mathrm{t}_*^{-1}(\mathcal{G}) \mid D \text{ is a } \mathrm{t}_*\text{-coflat object}\}.$$

(ii) *Suppose that for every $i\in I$, every object in $\mathcal{G}'$ is t_i-coflat.*

(a) *If for every $i\in I$, q_i preserves short exact sequences in $\mathrm{Ext}^1(-,\mathrm{t}_i(\mathcal{G}'))$, then*

$${}^{\perp}(\mathrm{t}_*(\mathcal{G}')) = \mathrm{q}_*^{-1}(\mathcal{F}').$$

(b) *If $\mathcal{G}'$ is a cogenerating class in $\mathcal{D}$, then*

$${}^{\perp}(\mathrm{t}_*(\mathcal{G}')) = \{C\in \mathrm{q}_*^{-1}(\mathcal{F}') \mid C \text{ is a } \mathrm{q}_*\text{-flat object}\}.$$

Proof. We only prove the first statement, as the other can be proved by dual arguments. Firstly, as Ω^1_F is a monomorphism for every $F \in \mathcal{F}$, by Lemma 2.1-(ii),

$$(\mathrm{q}_*(\mathcal{F}))^{\perp} \subseteq \mathrm{t}_*^{-1}(\mathcal{G}).$$

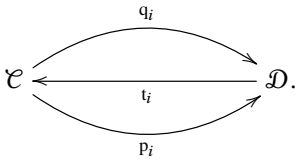
If for every $i \in I$, t_i preserves short exact sequences in $\mathrm{Ext}^1(\mathrm{q}_i(\mathcal{F}), -)$, and then by Lemma 2.1-(iii),

$$(\mathrm{q}_*(\mathcal{F}))^{\perp} = \mathrm{t}_*^{-1}(\mathcal{G}).$$

On the other hand, if $\mathcal{F}$ is a generating class, then by Lemma 2.9-(ii), any object in $(\mathrm{q}_*(\mathcal{F}))^{\perp}$ is a t_i-coflat object for every $i \in I$. The converse inclusion $\supseteq$ in (i)-(b) is immediate by Lemma 2.1-(i). ∎

3.2. Remark. Note that if for every $i \in I$, q_i is exact, $\mathcal{C}$ is an abelian category, and the kth right derived functor $\mathrm{R}^k_{\mathcal{G}}\mathrm{t}_i$ exists as described in 1.6, then by Proposition 2.10-(ii) and Proposition 2.15-(ii), for every $i \in I$, an object D in $\mathcal{D}$ being t_i-coflat is equivalent to $\mathrm{R}^1_{\mathcal{G}}\mathrm{t}_i(D) = 0$. In particular, if $\mathcal{D}$ is an abelian category with enough injectives, and $\mathcal{C}$ has enough projectives then the classes $\Psi(\mathcal{G})$ and $\Phi(\mathcal{F}')$ defined in Definition 1.5 of [10] are exactly the classes $\mathrm{q}_*(\mathcal{F})^{\perp}$ and ${}^{\perp}\mathrm{t}_*(\mathcal{G}')$ described in Theorem 3.1-(i)-(b) and (ii)-(b), respectively.

3.3. Corollary. *Let $\{(\mathrm{q}_i, \mathrm{t}_i, \mathrm{p}_i)\}_{i\in I}$ be a family of adjoint triples*



For given cotorsion pairs $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ in $\mathcal{C}$ and $\mathcal{D}$, respectively, the following hold.

(i) *Suppose that for every $i \in I$, q_i preserves short exact sequences in $\mathrm{Ext}^1(\mathcal{F}, \mathrm{t}_i(-))$.*

(ii) *If for every $i \in I$, t_i preserves short exact sequences in $\mathrm{Ext}^1(\mathrm{q}_i(\mathcal{F}), -)$, then*

$$\mathrm{q}_*(\mathcal{F})^{\perp} = \mathrm{t}_*^{-1}(\mathcal{G});$$

(a) *If $\mathcal{F}$ is a generating class in $\mathcal{C}$, then*

$$\mathrm{q}_*(\mathcal{F})^{\perp} = \{D \in \mathrm{t}_*^{-1}(\mathcal{G}) \mid D \text{ is a } \mathrm{t}_*\text{-coflat object}\}.$$

(iii) *Suppose that for every $i \in I$, p_i preserves short exact sequences in $\mathrm{Ext}^1(\mathrm{t}_i(-), \mathcal{G})$.*

(a) *If for every $i \in I$, t_i preserves short exact sequences in $\mathrm{Ext}^1(-, \mathrm{p}_i(\mathcal{G}))$, then*

$${}^{\perp}\mathrm{p}_*(\mathcal{G}) = \mathrm{t}_*^{-1}(\mathcal{F});$$

(b) *If $\mathcal{G}$ is a cogenerating class in $\mathcal{C}$, then*

$${}^{\perp}\mathrm{p}_*(\mathcal{G}) = \{D \in \mathrm{t}_*^{-1}(\mathcal{F}) \mid D \text{ is a } \mathrm{t}_*\text{-flat object}\}.$$

(iii) *Suppose that for every* $i \in I$, t_i *preserves short exact sequences in* $\mathrm{Ext}^1(\mathcal{F}', \mathrm{p}_i(-))$.

(a) *If for every* $i \in I$, p_i *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{t}_i(\mathcal{F}'), -)$, *then*

$$\mathrm{t}_*(\mathcal{F}')^{\perp} = \mathrm{p}_*^{-1}(\mathcal{G}').$$

(b) *If* $\mathcal{F}'$ *is a generating class in* $\mathcal{D}$, *then*

$$\mathrm{t}_*(\mathcal{F}')^{\perp} = \{C \in \mathrm{p}_*^{-1}(\mathcal{G}') \mid C \text{ is a } \mathrm{p}_*\text{-coflat object}\}.$$

(iv) *Suppose that for every* $i \in I$, t_i *preserves short exact sequences in* $\mathrm{Ext}^1(\mathrm{q}_i(-), \mathcal{G}')$.

(a) *If for every* $i \in I$, q_i *preserves short exact sequences in* $\mathrm{Ext}^1(-, \mathrm{t}_i(\mathcal{G}'))$, *then*

$${}^{\perp}\mathrm{t}_*(\mathcal{G}') = \mathrm{q}_*^{-1}(\mathcal{F}').$$

(b) *If* $\mathcal{G}'$ *is a cogenerating class in* $\mathcal{D}$, *then*

$${}^{\perp}\mathrm{t}_*(\mathcal{G}') = \{C \in \mathrm{q}_*^{-1}(\mathcal{F}') \mid C \text{ is a } \mathrm{q}_*\text{-flat object}\}.$$

Proof. Apply Theorem 3.1. ∎

3.4. Remark. Since for every $i \in I$, the middle functor t_i given in Corollary 3.3 is both right and left adjoint functor, it preserves all (co)limits, but it is not necessarily an exact functor between exact categories. However, it is exact whenever $\mathcal{C}$ is an abelian category.

3.5. Example. Consider Example 2.4. For a given cotorsion pair $(\mathcal{F}, \mathcal{G})$ in $\mathcal{C}$ and $i \in \mathbb{Z}$, we let

$$\mathbf{C}(\mathcal{F})_i := \{X \in \mathbf{C}(\mathcal{C}) \mid X_i \in \mathcal{F}\} \quad \text{and} \quad \mathbf{C}(\mathcal{G})_i := \{X \in \mathbf{C}(\mathcal{C}) \mid X_i \in \mathcal{G}\}.$$

As D^i and ev_i are exact functors, by Corollary 3.3, there are two cotorsion pairs in $\mathbf{C}(\mathcal{C})$,

$$({}^{\perp}\mathbf{C}(\mathcal{G})_i, \mathbf{C}(\mathcal{G})_i) \quad \text{and} \quad (\mathbf{C}(\mathcal{F})_i, \mathbf{C}(\mathcal{F})_i^{\perp}),$$

generated by $\mathrm{D}^i(\mathcal{F})$ and cogenerated by $\mathrm{D}^{i+1}(\mathcal{G})$, respectively. If we consider all $i \in \mathbb{Z}$, we have the following cotorsion pairs:

$$({}^{\perp}\mathbf{C}(\mathcal{G}), \mathbf{C}(\mathcal{G})) \quad \text{and} \quad (\mathbf{C}(\mathcal{F}), \mathbf{C}(\mathcal{F})^{\perp});$$

see Proposition 3.2 in [6].

If $(\mathcal{F}', \mathcal{G}')$ is a cotorsion pair in $\mathbf{C}(\mathcal{C})$, then

$$\mathrm{ev}_*(\mathcal{F}')^{\perp} = \{C \in \mathcal{C} \mid \mathrm{D}^i(C) \in \mathcal{G}' \text{ for every } i \in \mathbb{Z}\};$$
$${}^{\perp}\mathrm{ev}_*(\mathcal{G}') = \{C \in \mathcal{C} \mid \mathrm{D}^i(C) \in \mathcal{F}' \text{ for every } i \in \mathbb{Z}\};$$

see Lemma 3.5 in [6].

3.6. Example. Consider Example 2.13. Let $(\mathcal{F}, \mathcal{G})$ be a cotorsion pair in $\mathcal{C}$. Note that k_i does not preserve all short exact sequences in $\mathrm{Ext}^1(\mathrm{s}^i(\mathcal{F}), -)$ for any $i \in \mathbb{Z}$; for example, consider the canonical short exact sequence

$$\mathrm{s}^{i-1}(F) \hookrightarrow \mathrm{D}^i(F) \twoheadrightarrow \mathrm{s}^i(F),$$

where F is a non-zero object in $\mathcal{F}$. On the other hand, if $\mathcal{F}$ is a generating class in $\mathcal{C}$, then

$$\mathrm{s}^i(\mathcal{F})^{\perp} = \{X \in \mathbf{C}(\mathcal{C}) \mid \mathrm{k}_i(X) \in \mathcal{G} \text{ and } X \text{ is a } \mathrm{k}_i\text{-coflat object}\}.$$

We now show that a chain complex X in $\mathbf{C}(\mathcal{C})$ is a k_i-coflat object if and only if the canonical sequence

$$\mathrm{k}_i(X) \longrightarrow X_i \longrightarrow \mathrm{k}_{i-1}(X)$$

is a short exact sequence in $\mathcal{C}$.

Suppose that X is a k_i-coflat object for some $i \in \mathbb{Z}$; then consider the canonical morphism $f_{i-1} : \mathrm{s}^{i-1}(\mathrm{k}_{i-1}(X)) \longrightarrow X$, which induces the degreewise split short exact sequence in $\mathbf{C}(\mathcal{C})$

$$\mathbb{E} : X \hookrightarrow \mathrm{cone}(f_{i-1}) \twoheadrightarrow \mathrm{s}^i(\mathrm{k}_{i-1}(X)),$$

where $\mathrm{cone}(f_{i-1})$ is the cone of the morphism f_i. By assumption, $\mathrm{k}_i(\mathbb{E})$ is a short exact sequence in $\mathcal{C}$, and it is of the form

$$\mathrm{k}_i(\mathbb{E}) : \mathrm{k}_i(X) \hookrightarrow X_i \twoheadrightarrow \mathrm{k}_{i-1}(X).$$

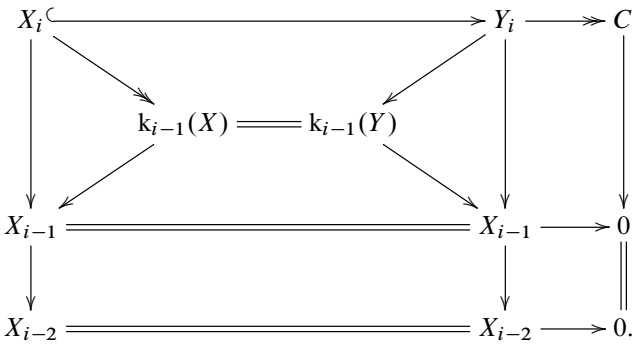
Conversely, suppose that the canonical sequence

$$\mathrm{k}_i(\mathbb{E}) : \mathrm{k}_i(X) \hookrightarrow X_i \twoheadrightarrow \mathrm{k}_{i-1}(X)$$

is a short exact sequence in $\mathcal{C}$. By Lemma 2.8, it is enough to show that k_i preserves all short exact sequences in $\mathrm{Ext}^1(\mathrm{s}^i(-), X)$. Consider a short exact sequence in $\mathbf{C}(\mathcal{C})$ of the form

$$\mathbb{E} : X \hookrightarrow Y \twoheadrightarrow \mathrm{s}^i(C).$$

Without loss of generality, we can assume that $X_j = Y_j$ for every $i \neq j \in \mathbb{Z}$. The short exact sequence $\mathbb{E}$ induces the following commutative diagram:



By assumption, $\mathcal{C}$ has kernels, and therefore, the morphism $Y_i \longrightarrow \mathrm{k}_{i-1}(Y)$ is in fact an admissible epic. Applying the snake lemma (Corollary 8.13 in [2]) to the diagram

$$\begin{array}{ccccc} X_i & \rightarrowtail & Y_i & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{k}_{i-1}(X) & = & \mathrm{k}_{i-1}(Y) & \longrightarrow & 0, \end{array}$$

we obtain the short exact sequence in $\mathcal{C}$

$$\mathrm{k}_i(\mathbb{E}) : \mathrm{k}_i(X) \rightarrowtail \mathrm{k}_i(Y) \twoheadrightarrow C = \mathrm{k}_i(\mathrm{s}^i(C)).$$

If we consider all integers i, we have

$$\begin{aligned} \mathrm{s}^*(\mathcal{F})^{\perp} &= \{X \in \mathbf{C}(\mathcal{C}) \mid \mathrm{k}_*(X) \in \mathcal{G} \text{ and } X \text{ is a } \mathrm{k}_*\text{-coflat object}\} \\ &= \{X \in \mathbf{C}(\mathcal{C}) \mid \mathrm{k}_*(X) \in \mathcal{G} \text{ and } X \text{ is acyclic}\}. \end{aligned}$$

Similarly, c_i does not preserve all short exact sequences in $\mathrm{Ext}^1(-, \mathrm{s}^i(\mathcal{G}))$ for any $i \in \mathbb{Z}$, and if $\mathcal{G}$ is a cogenerating class in $\mathcal{C}$, then

$$\begin{aligned} {}^{\perp}\mathrm{s}^*(\mathcal{G}) &= \{X \in \mathbf{C}(\mathcal{C}) \mid \mathrm{c}_*(X) \in \mathcal{F} \text{ and } X \text{ is a } \mathrm{c}_*\text{-flat object}\} \\ &= \{X \in \mathbf{C}(\mathcal{C}) \mid \mathrm{c}_*(X) \in \mathcal{F} \text{ and } X \text{ is acyclic}\}. \end{aligned}$$

3.7. Example. Consider the following adjoint triple:

$$(R\text{-Mod})^{\mathrm{op}} \underset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\overset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\overset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\rightleftarrows}}} \text{Mod-}R.$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor. If $(\mathcal{F}, \mathcal{G})$ is a cotorsion pair in R-Mod, then

$$\begin{aligned} \mathrm{Hom}(\mathcal{G}, \mathbb{Q}/\mathbb{Z})^{\perp} &= \{M \in \text{Mod-}R \mid \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{F}\}, \\ {}^{\perp}\mathrm{Hom}(\mathcal{F}, \mathbb{Q}/\mathbb{Z}) &= \{M \in \text{Mod-}R \mid \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{G}\}. \end{aligned}$$

3.8. Example. Consider the degreewise pure and split exact structures on $\mathbf{C}(R)$ and $\mathbf{C}(R^{\mathrm{op}})$, respectively. The adjoint triple of Example 3.7 can be extended to the following adjoint triple of functors:

$$\mathbf{C}(R)^{\mathrm{op}} \underset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\overset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\overset{\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})}{\rightleftarrows}}} \mathbf{C}(R^{\mathrm{op}}),$$

which are exact. We let $\mathrm{Acyc}(R^{\mathrm{op}})$ and $\text{K-Inj}(R^{\mathrm{op}})$ denote the classes of acyclic and k-injective chain complexes (see [23]) of R^{op}-modules, respectively. It can be easily verifed that $(\mathrm{Acyc}(R^{\mathrm{op}}), \text{K-Inj}(R^{\mathrm{op}}))$ is a cotorsion pair in $\mathbf{C}(R^{\mathrm{op}})$. So we have

$$\begin{aligned} {}^{\perp}\operatorname{Hom}(\mathrm{Acyc}(R^{\mathrm{op}}), \mathbb{Q}/\mathbb{Z}) &= \{X \in \mathbf{C}(R) \mid \operatorname{Hom}(X, \mathbb{Q}/\mathbb{Z}) \in \text{K-Inj}(R^{\mathrm{op}})\} \\ &= \{X \in \mathbf{C}(R) \mid X \text{ is K-flat}\}. \end{aligned}$$

The last equality follows from the isomorphism

$$\mathrm{H}_i(\operatorname{Hom}(Y \otimes_R X, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbf{K}(R^{\mathrm{op}})}(Y, \operatorname{Hom}(X, \mathbb{Q}/\mathbb{Z})),$$

where $Y \in \mathbf{C}(R^{\mathrm{op}})$ and $Y \otimes_R X$ denotes the total tensor product of chain complexes.

3.9. Example. Let $\{f_i \colon R \to S_i\}_{i \in I}$ be a family of ring homomorphisms. For every $i \in I$, consider the adjoint triple $(S_i \otimes_R -, \mathrm{res}_i, \operatorname{Hom}_R(S_i, -))$ as given in Example 2.16, where $\mathrm{res}_i \colon S_i\text{-Mod} \longrightarrow R\text{-Mod}$ denotes the functor of restriction of scalars through f_i. They induce a family of adjoint triples

$$R\text{-Mod} \;\underset{\iota_i \circ \operatorname{Hom}_R(S_i, -)}{\overset{\iota_i \circ (S_i \otimes_R -)}{\rightleftarrows}}\; \prod_{i \in I} S_i\text{-Mod}, \qquad \mathrm{res}_i \circ \pi_i \colon \prod_{i \in I} S_i\text{-Mod} \to R\text{-Mod}.$$

Note that $\mathrm{res}_i \circ \pi_i$ is an exact functor. Let $(\mathcal{F}, \mathcal{G})$ be a cotorsion pair in R-Mod.

(i) By Lemma 2.15, for every $i \in I$, $(S_i \otimes_R -)$ preserves short exact sequences in $\operatorname{Ext}^1(\mathcal{F}, \mathrm{res}_i(-))$ if and only if $\operatorname{Tor}^1_R(S_i, F) = 0$ for every $F \in \mathcal{F}$. In such case,

$$(\iota_* \circ (S_* \otimes_R \mathcal{F}))^{\perp} = \Big\{(N_i)_{i \in I} \in \prod_{i \in I} S_i\text{-Mod} \mid \mathrm{res}_i(N_i) \in \mathcal{G} \text{ for every } i \in I\Big\}.$$

(ii) By Lemma 2.15, for every $i \in I$, the functor $\operatorname{Hom}_R(S_i, -)$ preserves short exact sequences in $\operatorname{Ext}^1(\mathrm{res}_i(-), \mathcal{G})$ if and only if $\operatorname{Ext}^1_R(S_i, G) = 0$ for every $G \in \mathcal{G}$, that is, $\mathrm{res}_i(S_i) \in \mathcal{F}$. In such case,

$${}^{\perp}(\iota_* \circ \operatorname{Hom}_R(S_*, \mathcal{G})) = \Big\{(N_i)_{i \in I} \in \prod_{i \in I} S_i\text{-Mod} \mid \mathrm{res}_i(N_i) \in \mathcal{F} \text{ for every } i \in I\Big\}.$$

On the other hand, by Lemma 1.8, any cotorsion pair in $\prod_{i \in I} S_i$-Mod is just a family $\{(\mathcal{F}_i, \mathcal{G}_i)\}_{i \in I}$ of cotorsion pairs $(\mathcal{F}_i, \mathcal{G}_i)$ in S_i-Mod.

(i) As for every $i \in I$, $\mathcal{F}_i$ is a generating class in S_i-Mod,

$$\mathrm{res}_*(\mathcal{F}_*)^{\perp} = \{N \in R\text{-Mod} \mid \operatorname{Hom}_R(S_i, N) \in \mathcal{G}_i, \operatorname{Ext}^1_R(S_i, N) = 0 \text{ for every } i \in I\}.$$

(ii) As for every $i \in I$, $\mathcal{G}_i$ is a cogenerating class in S_i-Mod,

$${}^{\perp}\mathrm{res}_*(\mathcal{G}_*) = \{N \in R\text{-Mod} \mid S_i \otimes_R N \in \mathcal{F}_i, \operatorname{Tor}^1_R(S_i, N) = 0 \text{ for every } i \in I\}.$$

3.10. Example. We now consider a very particular case of Example 3.9. Let R be a commutative ring. For a given element r of R, consider the localization of R at r, denoted by $R[r^{-1}]$, together with the ring homomorphism $R \to R[r^{-1}]$. As $R[r^{-1}]$ is a flat R-R-bimodule, the functor $R[r^{-1}] \otimes_R -$ is exact, and therefore, $\mathrm{Tor}^1_R(R[r^{-1}], -) = 0$. Note that for any $M \in R$-Mod,

$$R[r^{-1}] \otimes_R N \cong N[r^{-1}].$$

For a given cotorsion pair $(\mathcal{F}_r, \mathcal{G}_r)$ in $R[r^{-1}]$-Mod, we have

$$\begin{aligned} {}^{\perp}(\mathrm{res}(\mathcal{G}_r)) &= \{N \in R\text{-Mod} \mid N[r^{-1}] \in \mathcal{F}_r\}, \\ (\mathrm{res}(\mathcal{F}_r))^{\perp} &= \{N \in R\text{-Mod} \mid \mathrm{Hom}_R(R[r^{-1}], N) \in \mathcal{G}_r \text{ and } \mathrm{Ext}^1_R(R[r^{-1}], N) = 0\}. \end{aligned}$$

As a particular case, let us consider the trivial cotorsion pair $(\mathrm{Proj}_r, R[r^{-1}]\text{-Mod})$ in $R[r^{-1}]$-Mod, where Proj_r denotes the class of projective $R[r^{-1}]$-modules. So we have

$$\begin{aligned} {}^{\perp}(\mathrm{res}(R[r^{-1}]\text{-Mod})) &= \{N \in R\text{-Mod} \mid N[r^{-1}] \in \mathrm{Proj}_r\}, \\ (\mathrm{res}(\mathrm{Proj}_r))^{\perp} &= \{N \in R\text{-Mod} \mid \mathrm{Ext}^1_R(R[r^{-1}], N) = 0\} \\ &= \{r\text{-contraadjusted } R\text{-modules}\}. \end{aligned}$$

Consider the family of adjoint triples

$$R\text{-Mod} \;\underset{\iota_r \circ \mathrm{Hom}_R(R[r^{-1}], -)}{\overset{\iota_r \circ (R[r^{-1}] \otimes_R -)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\mathrm{res}_r \circ \pi_r}}}} \prod_{r \in R} R[r^{-1}]\text{-Mod},$$

and the family $\{(\mathrm{Proj}_r, R[r^{-1}]\text{-Mod})\}_{r \in R}$ of the trivial cotorsion pairs. We have

$$\begin{aligned} (\mathrm{res}_*(\mathrm{Proj}_*))^{\perp} &= \{N \in R\text{-Mod} \mid \mathrm{Ext}^1_R(R[r^{-1}], N) = 0 \text{ for every } r \in R\} \\ &= \{\text{contraadjusted } R\text{-modules}\}; \end{aligned}$$

R-modules in ${}^{\perp}((\mathrm{res}_*(\mathrm{Proj}_*))^{\perp})$ are called very flat R-modules; see Section 1.1 in [19].

4. Functor categories and evaluation functor

In this and upcoming sections, we center on functor categories in order to obtain further results on cotorsion pairs induced by adjoint triples.

Throughout, R denotes a ring with identity, and $\mathcal{A}$ denotes a small preadditive category. For any object A in $\mathcal{A}$, we let R_A denote the endomorphism ring $\mathrm{Hom}(A, A)$. Note that for any two objects A and B in $\mathcal{A}$, $\mathrm{Hom}(A, B)$ has a canonical R_B-R_A-bimodule structure induced from the composition rule

$$\mathrm{Hom}(B, B) \otimes_{\mathbb{Z}} \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathrm{Hom}(A, A) \longrightarrow \mathrm{Hom}(A, B).$$

Since for every morphisms $g\colon A \to B$, $f\colon B \to B$ and $h\colon B \to C$ in $\mathcal{A}$,

$$h \circ (f \circ g) = h \circ f \circ g = (h \circ f) \circ g,$$

the composition is canonically factorized over $\mathrm{Hom}(B,C) \otimes_{R_B} \mathrm{Hom}(A,B)$:

$$\begin{array}{ccc} \mathrm{Hom}(B,C) \otimes_{\mathbb{Z}} \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A,C) \\ & \searrow \quad \nearrow & \\ & \mathrm{Hom}(B,C) \otimes_{R_B} \mathrm{Hom}(A,B). & \end{array}$$

4.1. Evaluation functor. We let $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ denote the category of additive functors from $\mathcal{A}$ to R-Mod. The evaluation functor

$$\mathrm{ev} : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \otimes \mathcal{A} \to R\text{-Mod}$$

is the functor defined by $\mathrm{ev}(X, A) = X(A) =: \mathrm{ev}_A(X)$.

For a given additive functor $X\colon \mathcal{A} \longrightarrow R$-Mod and objects A and B in $\mathcal{A}$, $X(A)$ has a canonical left R_A-module structure defined by $f \cdot a := X(f)(a)$, where $f \in R_A$ and $a \in X(A)$. So it is easy to verify that the associated homomorphism of abelian groups

$$X_{AB} : \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}_R(X(A), X(B))$$

induces the following left R_B and R_A-module homomorphisms, respectively:

$$\overline{X}_{AB} : \mathrm{Hom}(A,B) \otimes_{R_A} X(A) \longrightarrow X(B), \quad f \otimes a \rightsquigarrow X(f)(x),$$
$$\underline{X}_{AB} : X(A) \longrightarrow \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)), \quad \underline{X}_{AB}(a)(g) = X(g)(a).$$

Besides, as for every $f \in R_A$, $X(f)$ is an R-module homomorphism, we have

$$f \cdot (r \cdot a) = r \cdot (f \cdot a)$$

for every $a \in X(A)$ and $r \in R$.

If $\alpha = \{\alpha_A\}_{A \in \mathcal{A}}\colon X \Rightarrow Y$ is a natural transformation of additive functors $X, Y\colon \mathcal{A} \to R$-Mod, then for any $f \in R_A$ and $a \in X(A)$, we have

$$\alpha_A(f \cdot a) = \alpha_A(X(f)(a)) = Y(f)(\alpha_A(a)) = f \cdot \alpha_A(a).$$

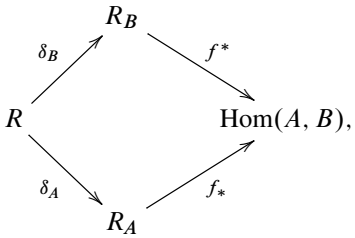
In other words, for every object A in $\mathcal{A}$, $\alpha_A\colon X(A) \to Y(A)$ is an R_A-module homomorphism. So the evaluation functor $\mathrm{ev}_A\colon \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \to R$-Mod has also image in R_A-Mod,

(4.1.1)
$$\begin{array}{ccc} \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \otimes \mathcal{A} & \xrightarrow{\mathrm{ev}_A} & R\text{-Mod} \\ & \searrow^{\mathrm{ev}_A} & \\ & R_A\text{-Mod}. & \end{array}$$

4.2. Remark. Similarly, the evaluation functor $\mathrm{ev}_A\colon \mathrm{Add}(\mathcal{A}^{\mathrm{op}}, R\text{-Mod}) \to R$-Mod has image in R_A^{op}-Mod $=$ Mod-R_A.

From now on, we assume that $\mathcal{A}$ satisfies the following condition.

4.3. Hypothesis. For every object A in $\mathcal{A}$, there exists a ring morphism $\delta_A\colon R \to R_A$ such that for every morphism $f\colon A \to B$ in $\mathcal{A}$, the following diagram commutes:



where $f^* = \mathrm{Hom}(f, B)$ and $f_* = \mathrm{Hom}(A, f)$.

Hypothesis 4.3 implies that for every object A in $\mathcal{A}$,

$$\mathrm{Hom}(A, -) \in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \quad \text{and} \quad \mathrm{Hom}(-, A) \in \mathrm{Add}(\mathcal{A}^{op}, \text{Mod-}R).$$

Indeed, the R_B-R_A-bimodule $\mathrm{Hom}(A, B)$ is clearly R-R-bimodule by restriction of scalars. Besides, it satisfies the following equality:

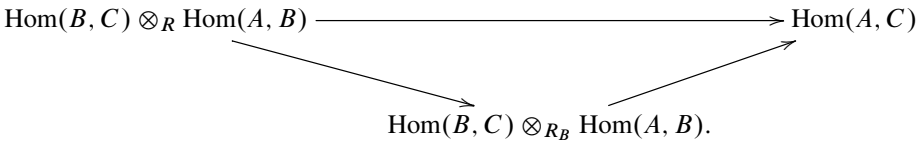
$$r \cdot f = \delta_B(r) \circ f = f \circ \delta_A(r) = f \cdot r, \tag{4.3.1}$$

for every $r \in R$ and $f \in \mathrm{Hom}(A, B)$. Besides, we have that, for any morphisms $f\colon A \to B$ and $g\colon B \to B'$ in $\mathcal{A}$ and for any element r in R,

$$g_*(r \cdot f) = g_*(f \circ \delta_A(r)) = g \circ f \circ \delta_A(r) = \delta_{B'}(r) \circ (g \circ f) = r \cdot (g_*(f)).$$

In other words, $g_*\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ is a left R-module homomorphism. Similarly, $g^*\colon \mathrm{Hom}(B', A) \to \mathrm{Hom}(B, A)$ is a right R-module homomorphism.

Furthermore, the evaluation functor ev_A given in (4.1.1) is factorized over the forgetful functor $U\colon R_A\text{-Mod} \to R\text{-Mod}$, and the composition rule is canonically factorized as



4.4. Remark. The conditions imposed on $\mathcal{A}$ in Hypothesis 4.3 is equivalent to being $\mathcal{A}$ an R-R-bimodule enriched category.

4.5. Remark. As observed by the referee, Hypothesis 4.3 implies that for every object A in $\mathcal{A}$, the image of the ring homomorphism δ_A is a subring of R_A contained within its center. Consequently, the induced ring homomorphism $\delta\colon R \longrightarrow \prod_{A\in\mathcal{A}} R_A$, defined by $\delta(r) := (\delta_A(r))_{A\in\mathcal{A}}$, yields a conmmutative ring $\overline{R} := \mathrm{Im}(\delta)$ so that $\mathcal{A}$ satisfies Hypothesis 4.3 with respect to $\overline{R}$. Furthermore, with careful calculations, one can verify that we have the equivalence of categories

$$\mathrm{Add}(\mathcal{A}, R\text{-Mod}) \cong \mathrm{Add}(\mathcal{A}, \overline{R}\text{-Mod}).$$

On the other hand, for any additive category $\mathcal{A}$ without satisfying Hypothesis 4.3, one can make an adjustment. Namely, consider the extension of $\mathcal{A}$ by R, denoted by $R \otimes \mathcal{A}$: It has the same objects as $\mathcal{A}$, but its hom-sets are defined as $R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A, B)$, and the composition rule is given by $(s \otimes g) \circ (r \otimes f) := (sr) \otimes (g \circ f)$; see p. 1122 in [4]. One can show that the additive category $R \otimes \mathcal{A}$ satisfies Hypothesis 4.3 with respect to $\mathbb{Z}$, and

$$\mathrm{Add}(\mathcal{A}, R\text{-Mod}) \cong \mathrm{Add}(R \otimes \mathcal{A}, \mathrm{Ab}).$$

4.6. Example. We let $\mathcal{A}$ be the category whose set of objects is $\mathbb{Z}$, and

$$\mathrm{Hom}(i, j) := \begin{cases} R, & \text{if } i = j \text{ or } i = j + 1, \\ 0, & \text{otherwise} \end{cases}$$

It is easy to verify that $\mathrm{Add}(\mathcal{A}, \mathrm{Ab})$ is equivalent to the category $\mathbf{C}(R)$ of chain complexes of left R-modules.

4.7. Example. Consider a Morita context ring $\left[\begin{smallmatrix} T & M \\ N & S \end{smallmatrix}\right]$, that is, T and S are rings, M and N are T-S and S-T-bimodules, respectively, together with bimodule homomorphisms $M \otimes_S N \to T$ and $N \otimes_T M \to S$.

We let $\mathcal{A}$ be the category with two objects A and B and with morphisms $R_A := T$, $R_B := S$, $\mathrm{Hom}(A, B) := N$, $\mathrm{Hom}(B, A) := M$ together with the canonical composition rule induced from the ring multiplications and bimodule homomorphisms of the given Morita context ring. It is immediate that

$$\mathrm{Add}(\mathcal{A}, \mathrm{Ab}) \cong \begin{bmatrix} T & M \\ N & S \end{bmatrix}\text{-Mod}.$$

4.8. For any object A in $\mathcal{A}$ and a left R_A-module M, Cortés-Izurdiaga and Torrecillas in Section 2 of [3] define the following Ab-valued functors:

$$\mathrm{q}_A(M)(-) := \mathrm{Hom}(A, -) \otimes_{R_A} M,$$
$$\mathrm{p}_A(M)(-) := \mathrm{Hom}_{R_A}(\mathrm{Hom}(-, A), M).$$

By the hypothesis imposed on $\mathcal{A}$ in 4.3, they are indeed R-Mod-valued functors, and hence, induce the functors

$$\mathrm{q}_A, \mathrm{p}_A : R_A\text{-Mod} \longrightarrow \mathrm{Add}(\mathcal{A}, R\text{-Mod}).$$

4.9. Proposition. *For every object A in $\mathcal{A}$, $(\mathrm{q}_A, \mathrm{ev}_A, \mathrm{p}_A)$ is an adjoint triple.*

Proof. By Proposition 1 in [3], viewing any functor $X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod})$ as an Ab-valued functor, we already know that the natural transformation

$$\mathrm{Nat}(\mathrm{q}_A(M), X) \to \mathrm{Hom}_{R_A}(M, X(A)), \quad \alpha \rightsquigarrow M \cong R_A \otimes_{R_A} M \xrightarrow{\alpha_A} X(A),$$

is injective. Surjectivity follows from the fact that the induced morphism

$$\overline{X}_{AB} : \mathrm{Hom}(A, B) \otimes_{R_A} X(A) \to X(B)$$

is a left R-module homomorphism.

By dual arguments, it is easy to verify that $(\mathrm{ev}_A, \mathrm{p}_A)$ is an adjoint pair, as well. ∎

We let

$$\mathrm{q}, \mathrm{p} : \prod_{A \in \mathcal{A}} R_A\text{-Mod} \to \mathrm{Add}(\mathcal{A}, R\text{-Mod})$$

denote the functors $\bigoplus_{A \in A} \mathrm{q}_A$ and $\prod_{A \in \mathcal{A}} \mathrm{p}_A$, respectively.

4.10. Proposition. $(\mathrm{q}, (\mathrm{ev}_A)_{A \in \mathcal{A}}, \mathrm{p})$ *is an adjoint triple*:

$$\prod_{A \in \mathcal{A}} R_A\text{-Mod} \underset{\mathrm{p}}{\overset{\mathrm{q}}{\rightleftarrows}} \mathrm{Add}(\mathcal{A}, R\text{-Mod}), \qquad (\mathrm{ev}_A)_{A \in \mathcal{A}} : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \to \prod_{A \in \mathcal{A}} R_A\text{-Mod}.$$

4.11. Hypothesis. From this point onwards, $\prod_{A \in \mathcal{A}} R_A$-Mod is assumed to be equipped with a fixed exact structure $\prod_{A \in \mathcal{A}} \mathcal{E}_A$, unless otherwise is stated. Accordingly, we assume that $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ is equipped with the exact structure $\mathcal{E}$, which consists of the short exact sequences $\mathbb{E}$ such that $\mathrm{ev}_A(\mathbb{E})$ belongs to $\mathcal{E}_A$ for every $A \in \mathcal{A}$. Clearly, $\mathcal{E}$ is a weakly idempotent complete exact structure on $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$.

4.12. Proposition. *For any object $A \in \mathcal{A}$, the following statements hold.*

(i) ev_A *is an exact functor for every object A in $\mathcal{A}$.*

(ii) *If $\prod_{A \in \mathcal{A}} \mathcal{F}_A$ is a generating class in $\prod_{A \in \mathcal{A}} R_A$-Mod, then so is $\{\mathrm{q}_A(\mathcal{F}_A)\}_{A \in \mathcal{A}}$ in* $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$.

(iii) *If $\prod_{A \in \mathcal{A}} \mathcal{G}_A$ is a cogenerating class in $\prod_{A \in \mathcal{A}} R_A$-Mod, then so is $\{\mathrm{p}_A(\mathcal{F}_A)\}_{A \in \mathcal{A}}$ in* $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$.

Proof. The first statement is immediate. It also implies that for any $\mathcal{E}_A$-projective object M in R_A-Mod, $\mathrm{q}_A(M)$ is projective in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$, as well.

Now we prove the second statement. The dual arguments can be applied for the third statement. Let $X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod})$. For any object A in $\mathcal{A}$, we consider an admissible epic $F_A \twoheadrightarrow X(A)$ in R_A-Mod with $F_A \in \mathcal{F}_A$. By adjunction given in Proposition 4.10, there exists a natural transformation $\alpha : \bigoplus_{A \in \mathcal{A}} \mathrm{q}_A(F_A) \Rightarrow X$. It is in fact an admissible epic in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$. Indeed, for every $B \in \mathcal{A}$, the morphism α_B is the coproduct of the morphism $\bigoplus_{B \neq A \in \mathcal{A}} \mathrm{q}_A(F_A)(B) \longrightarrow X(B)$ and the admissible epic $\mathrm{q}_B(F_B)(B) \cong F_B \twoheadrightarrow X(B)$ in R_B-Mod. By Proposition 1.3, α_B is an admissible epic in $(R_B\text{-Mod}; \mathcal{E}_B)$. ∎

As a particular case of Proposition 4.12, for any functor $X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod})$ the counit morphism $\mathrm{q}((X(A))_{A \in \mathcal{A}}) \Rightarrow X$ and the unit morphism $X \Rightarrow \mathrm{p}((X(A))_{A \in \mathcal{A}})$ are an admissible epic and monic, respectively.

Applying Corollary 3.3 for the adjoint triple given in Proposition 4.10, we obtain a more general form of Example 3.5.

4.13. Proposition. *Let* $(\prod_{A\in\mathcal{A}}\mathcal{F}_A, \prod_{A\in\mathcal{A}}\mathcal{G}_A)$ *be a cotorsion pair in* $\prod_{A\in\mathcal{A}} R_A$-Mod.

(i) *If for every* $A\in\mathcal{A}$, *any object in* $\mathcal{F}_A$ *is* q_A*-flat, then we have*

$$\mathrm{q}\Big(\prod_{A\in\mathcal{A}}\mathcal{F}_A\Big)^{\perp} = \{X\in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A)\in\mathcal{G}_A \text{ for every } A\in\mathcal{A}\}.$$

(ii) *If for every* $A\in\mathcal{A}$, *every object in* $\mathcal{G}_A$ *is* p_A*-coflat, then*

$${}^{\perp}\mathrm{p}\Big(\prod_{A\in\mathcal{A}}\mathcal{G}_A\Big) = \{X\in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A)\in\mathcal{F}_A \text{ for every } A\in\mathcal{A}\}$$

4.14. Example. Consider the functor category representation of $\mathbf{C}(R)$ as given in Example 4.6. For a given $i\in\mathbb{Z}$, the functors q_i and p_i are precisely the disc chain complexes D^i and D^{i+1} functors (see Example 2.4). Since D^i is an exact functor, the cotorsion pairs in Example 3.5 are easily obtained by Proposition 4.13.

As a particular case, if we consider the absolute exact structure on $\prod_{A\in\mathcal{A}} R_A$-Mod, then we have the following result.

4.15. Corollary. *Let* $(\prod_{A\in\mathcal{A}}\mathcal{F}_A, \prod_{A\in\mathcal{A}}\mathcal{G}_A)$ *be a cotorsion pair in* $\prod_{A\in\mathcal{A}} R_A$-Mod *equipped with the absolute exact structure.*

(i) *If*

$$\mathrm{L}_1\mathrm{q}_A(\mathcal{F}_A) = \mathrm{Tor}^1_{R_A}(\mathrm{Hom}(A,-),\mathcal{F}_A) = 0$$

for every object A *in* $\mathcal{A}$, *then we have*

$$\mathrm{q}\Big(\prod_{A\in\mathcal{A}}\mathcal{F}_A\Big)^{\perp} = \{X\in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A)\in\mathcal{G}_A \text{ for every } A\in\mathcal{A}\}.$$

(ii) *If*

$$\mathrm{R}^1\mathrm{p}_A(\mathcal{G}_A) = \mathrm{Ext}^1_{R_A}(\mathrm{Hom}(-,A),\mathcal{G}_A) = 0$$

for every object A *in* $\mathcal{A}$, *then*

$${}^{\perp}\mathrm{p}\Big(\prod_{A\in\mathcal{A}}\mathcal{G}_A\Big) = \{X\in \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \mid X(A)\in\mathcal{F}_A \text{ for every } A\in\mathcal{A}\}.$$

4.16. Remark. The categories $\prod_{A\in\mathcal{A}} R_A$-Mod and $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$, equipped with the absolute exact structures, have enough injectives and projectives. This means that, if $(\prod_{A\in\mathcal{A}}\mathcal{F}_A, \prod_{A\in\mathcal{A}}\mathcal{G}_A)$ is hereditary, we infer (see, for instance, Lemma 6.17 in [25]) that, for each $A\in\mathcal{A}$, the class $\mathcal{G}_A$ is closed under quotients of monomorphisms. Therefore, under the conditions given in Corollary 4.15-(i), the class $\mathrm{q}(\prod_{A\in\mathcal{A}}\mathcal{F}_A)^{\perp}$ will be also closed under quotients of monomorphism. Thus, another application of Lemma 6.17 in [25], gives that the cotorsion pair generated by $\mathrm{q}(\prod_{A\in\mathcal{A}}\mathcal{F}_A)$ is hereditary, as well. By the same reason, under the conditions given in Corollary 4.15-(ii), the cotorsion pair cogenerated by $\mathrm{p}(\prod_{A\in\mathcal{A}}\mathcal{G}_A)$ is hereditary, as well.

5. Stalk functor in functor categories

We continue studying on $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$. Unless otherwise stated, we still assume that $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ is equipped with the exact structure $\mathcal{E}$ induced from a fixed exact structure $\prod_{A\in\mathcal{A}} \mathcal{E}_A$ on $\prod_{A\in\mathcal{A}} R_A\text{-Mod}$ as stated in 4.11. In this section, we investigate cotorsion pairs arising through 'stalk functor'. In order to do it, we need to impose further conditions on $\mathcal{A}$.

5.1. Hypothesis. We assume that $\mathcal{A}$ satisfies the following condition: for every two objects $A \neq B$ in $\mathcal{A}$, the composition rule

$$\mathrm{Hom}(A, B) \otimes_{R_A} \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(B, B)$$

is zero.

5.2. Example. Consider a Morita context ring as described in Example 4.7. Recall that the images of the bimodule homomorphisms $M \otimes_S N \to T$ and $N \otimes_T M \to S$ are two-sided ideals in T and S, respectively, known as the *trace ideals of the Morita context*. So a Morita context ring with zero trace ideals satisfies Hypothesis 5.1. In particular, when either $M = 0$ or $N = 0$, it is called *a formal triangular matrix ring*.

The associated category $\mathcal{A}$ to a Morita context ring with zero trace ideals as presented in Example 4.7 satisfies Hypothesis 5.1.

5.3. Stalk functor. For every objects A and B in $\mathcal{A}$ and $M \in R_A\text{-Mod}$, we define

$$\mathrm{s}_A(M)(B) := \begin{cases} 0, & \text{if } A \neq B, \\ M, & \text{if } A = B. \end{cases}$$

For a given morphism $f : B \to B'$ in $\mathcal{A}$, we define a morphism

$$\mathrm{s}_A(M)(f) : \mathrm{s}_A(M)(B) \to \mathrm{s}_A(M)(B')$$

as follows: if $B \neq A$ or $B' \neq A$, then $\mathrm{s}_A(M)(f) := 0$; if $B = A = B'$, then $\mathrm{s}_A(M)(f) := f \cdot (-)$, the left R_A-module scalar multiplication on M.

5.4. Proposition. *For every object A in $\mathcal{A}$, $\mathrm{s}_A : R_A\text{-Mod} \longrightarrow \mathrm{Add}(\mathcal{A}, R\text{-Mod})$ is a functor.*

Proof. The only thing that we need to show is that for every left R_A-module M, $\mathrm{s}_A(M)(-)$ belongs to $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$.

It is immediate that for every $B \in \mathcal{A}$, $\mathrm{s}_A(M)(B)$ is a left R-module. By 4.3.1, for a given morphism $f \in R_A$, the morphism $\mathrm{s}_A(M)(f)$ is an R-module homomorphism.

Clearly, for every $B \in B$, $\mathrm{s}_A(M)(\mathrm{id}_B) = \mathrm{id}$. It is left to prove that $\mathrm{s}_A(M)(-)$ preserves compositions. Consider two arbitrary morphisms $f : B \to B'$ and $g : B' \to B''$ in $\mathcal{A}$. It is immediate that

$$\mathrm{s}_A(M)(g \circ f) = \mathrm{s}_A(M)(g) \circ \mathrm{s}_A(M)(f)$$

whenever $B \neq A$ or $B'' \neq A$. Suppose that $B = A = B''$. If $B' = A$, from the associativity of the scalar multiplication,

$$\mathrm{s}_A(M)(g \circ f) := (g \circ f) \cdot (-) = (g \cdot f)(-) = g \cdot (f \cdot -) = \mathrm{s}_A(M)(g) \circ \mathrm{s}_A(M)(f).$$

If $B' \neq A$, by Hypothesis 5.1, $g \circ f = 0$, and hence,

$$\mathrm{s}_A(M)(g \circ f) = 0 = \mathrm{s}_A(M)(g) \circ \mathrm{s}_A(M)(f). \qquad \blacksquare$$

5.5. Remark. The stalk functor s_A introduced in (5.3) differs subtly from the one defined in Definition 7.9 of [11]. For instance, let $\mathcal{A}$ be the additive category freely generated by the quiver

Then $\mathcal{A}$ has only one object with the endomorphism ring $R_A = \mathbb{Z}[x]$. For every polynomial $p(x) \in \mathbb{Z}[x]$, we have

$$p(x) = p(0) + \overline{p}(x) \in \mathbb{Z} \oplus \langle x \rangle = \mathbb{Z}[x].$$

We let $S_A \colon \mathrm{Ab} \to \mathrm{Add}(\mathcal{A}, \mathrm{Ab})$ denote the functor defined in Proposition 7.15 of [11]:

$$\mathbb{Z}[x]\text{-Mod} \xrightarrow{\mathrm{s}_A} \mathrm{Add}(\mathbb{Z}, \mathrm{Ab}) \xleftarrow{S_A} \mathrm{Ab}\,.$$

If $M \in \mathbb{Z}[x]$-Mod, then

$$S_A(M)(p(x)) = p(0) \cdot (-) \neq \mathrm{s}_A(M)(p(x)).$$

In 5.6, we introduce several crucial constructions which will be used both for the left/right adjoints of s_A, as well as to induce certain short exact sequences in Proposition 5.18.

5.6. Let $X \colon \mathcal{A} \longrightarrow R$-Mod be an additive functor, and let A be an object in $\mathcal{A}$. By the universal property of coproducts, the family $\{\overline{X}_{BA}\}_{B \in \mathcal{A}, A \neq B}$ of R_A-module homomorphisms (see 4.1) induces a unique R_A-module homomorphism

$$\varphi_A^X : \bigoplus_{\substack{B \in \mathcal{A} \\ A \neq B}} \mathrm{Hom}(B, A) \otimes_{R_B} X(B) \longrightarrow X(A)\,.$$

We let

$$\mathrm{c}_A(X) := \mathrm{coker}(\varphi_A^X).$$

By the universal property of coproducts and cokernels, it in fact leads to a functor

$$\mathrm{c}_A : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \longrightarrow R_A\text{-Mod}\,.$$

On the other hand, there exist two canonical R_A-module homomorphisms $h^1_{A,X}$ and $h^2_{A,X}$ making the following upper and lower square commutative, respectively:

$$\begin{array}{ccc}
\operatorname{Hom}(B',A)\otimes_{R_{B'}}\operatorname{Hom}(B,B')\otimes_{R_B}X(B) & \xrightarrow{\mathrm{id}\otimes\overline{X}_{BB'}} & \operatorname{Hom}(B',A)\otimes_{R_{B'}}X(B') \\
\downarrow & & \downarrow \\
\bigoplus_{\substack{B,B'\in\mathcal{A}\\ A\neq B,B'}}\operatorname{Hom}(B',A)\otimes_{R_{B'}}\operatorname{Hom}(B,B')\otimes_{R_B}X(B) & \underset{h^2_{A,X}}{\overset{h^1_{A,X}}{\rightrightarrows}} & \bigoplus_{\substack{B\in\mathcal{A}\\ A\neq B}}\operatorname{Hom}(B,A)\otimes_{R_B}X(B) \\
\uparrow & & \uparrow \\
\operatorname{Hom}(B',A)\otimes_{R_{B'}}\operatorname{Hom}(B,B')\otimes_{R_B}X(B) & \xrightarrow[\circ\otimes\mathrm{id}]{} & \operatorname{Hom}(B,A)\otimes_{R_B}X(B),
\end{array}$$

where $\circ$ denotes the composition rule. Note that the objects B and B' in the above diagrams can be assumed to be different. Besides, we let

$$\nu_{A,X} := (\mathrm{id}\otimes\overline{X}_{AB})_{\substack{B\in\mathcal{A}\\ A\neq B}}$$

We have

$$\begin{array}{ccc}
\operatorname{Hom}(B,A)\otimes_{R_B}\operatorname{Hom}(A,B)\otimes_{R_A}X(A) & \xrightarrow{\mathrm{id}\otimes\overline{X}_{AB}} & \operatorname{Hom}(B,A)\otimes_{R_B}X(B) \\
\downarrow & & \downarrow \\
\bigoplus_{\substack{B\in\mathcal{A}\\ A\neq B}}\operatorname{Hom}(B,A)\otimes_{R_B}\operatorname{Hom}(A,B)\otimes_{R_A}X(A) & \xrightarrow[\nu_{A,X}]{} & \bigoplus_{\substack{B\in\mathcal{A}\\ A\neq B}}\operatorname{Hom}(B,A)\otimes_{R_B}X(B).
\end{array}$$

As $\mathcal{A}$ satisfies Hypothesis 5.1, and X is a functor, we have the following commutative diagrams:

(5.6.1)
$$\begin{array}{ccc}
\operatorname{Hom}(B,A)\otimes_{R_B}\operatorname{Hom}(A,B)\otimes_{R_A}X(A) & \xrightarrow{\mathrm{id}\otimes\overline{X}_{AB}} & \operatorname{Hom}(B,A)\otimes_{R_B}X(B) \\
\downarrow & & \downarrow{\scriptstyle \overline{X}_{BA}} \\
0 & \longrightarrow & X(A)
\end{array}$$

and

(5.6.2)
$$\begin{array}{ccc}
\operatorname{Hom}(B',A)\otimes_{R_{B'}}\operatorname{Hom}(B,B')\otimes_{R_B}X(B) & \xrightarrow{\mathrm{id}\otimes\overline{X}_{BB'}} & \operatorname{Hom}(B',A)\otimes_{R_{B'}}X(B') \\
{\scriptstyle \circ\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \overline{X}_{B'A}} \\
\operatorname{Hom}(B,A)\otimes_{R_B}X(B) & \xrightarrow[\overline{X}_{BA}]{} & X(A),
\end{array}$$

for every objects $A \neq B, B'$ in $\mathcal{A}$, and therefore, the morphism φ_A^X is factorized over the cokernel of the homomorphism

$$\phi_A^X := (h^1_{A,X} - h^2_{A,X}, \nu_{A,X}),$$

$$\begin{array}{ccc} \bigoplus\limits_{\substack{B\in\mathcal{A}\\ A\neq B}} \mathrm{Hom}(B,A)\otimes_{R_B} X(B) & \xrightarrow{\quad\varphi_A^X\quad} & X(A) \\ & {\scriptstyle\sigma_A}\searrow \quad \nearrow & \\ & \mathrm{coker}(\phi_A^X), & \end{array} \tag{5.6.3}$$

where σ_A denotes the canonical quotient homomorphism. Note that

$$\begin{aligned} &\mathrm{coker}(\phi_A^X) \\ &\quad\cong \Big(\bigoplus_{A\neq B\in\mathcal{A}} \mathrm{Hom}(B,A)\otimes_{R_B} X(B)\Big)/\langle g\circ f\otimes x - g\otimes X(f)(x), s\otimes X(l)(y)\rangle, \end{aligned}$$

where $x \in X(B)$, $y \in X(A)$, and $f\colon B \to B'$, $g\colon B' \to A$, $s\colon B \to A$ and $l\colon A \to B$ are any morphisms in $\mathcal{A}$ with $B \neq B'$ and $B, B' \neq A$.

Similarly, applying the universal property of products for the family $\{\underline{X}_{AB}\}_{B\in\mathcal{A}, B\neq A}$ of R_A-module homomorphisms, we have the canonical left R_A-module homomorphism

$$\psi_A^X : X(A) \longrightarrow \prod_{A\neq B\in\mathcal{A}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)).$$

We let $\mathrm{k}_A(X) := \ker(\psi_A^X)$, which leads to a functor

$$\mathrm{k}_A : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \longrightarrow R_A\text{-Mod}.$$

Again, there exist two canonical R_A-module homomorphisms $t^1_{A,X}$ and $t^2_{A,X}$ making the following upper and lower square commutative, respectively,

$$\begin{array}{ccc} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)) & \xrightarrow{\quad\underline{X}_{BB'*}\quad} & \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), \mathrm{Hom}_{R_{B'}}(\mathrm{Hom}(B,B'), X(B'))) \\ \uparrow & & \uparrow \\ \prod\limits_{A\neq B\in\mathcal{A}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)) & \underset{t^2_{A,X}}{\overset{t^1_{A,X}}{\rightrightarrows}} & \prod\limits_{\substack{A\neq B\\ B'\in\mathcal{A}}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), \mathrm{Hom}_{R_{B'}}(\mathrm{Hom}(B,B'), X(B'))) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)) & \xrightarrow[*]{\quad\quad} & \mathrm{Hom}_{R_{B'}}(\mathrm{Hom}(A,B'), \mathrm{Hom}_{R_B}(\mathrm{Hom}(B',B), X(B))) \end{array}$$

where $*$ denotes the composition of

$$\mathrm{Hom}_{R_B}(\mathrm{Hom}(A,B), X(B)) \xrightarrow{\circ^*} \mathrm{Hom}_{R_B}(\mathrm{Hom}(B',B)\otimes_{R_{B'}} \mathrm{Hom}(A,B'), X(B))$$

with the canonical isomorphism

$$\mathrm{Hom}_{R_B}(\mathrm{Hom}(B', B) \otimes_{R_{B'}} \mathrm{Hom}(A, B'), X(B)) \xrightarrow{\cong} \mathrm{Hom}_{R_{B'}}(\mathrm{Hom}(A, B'), \mathrm{Hom}_{R_B}(\mathrm{Hom}(B', B), X(B))).$$

Besides, consider the R_A-module homomorphism $\tau_{A,X} := \prod_{A \neq B \in \mathcal{A}} \underline{X}_{BA*}$:

$$\begin{array}{ccc} \prod\limits_{A\neq B\in\mathcal{A}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), X(B)) & \xrightarrow{\tau_{A,X}} & \prod\limits_{A\neq B\in\mathcal{A}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), \mathrm{Hom}_{R_A}(\mathrm{Hom}(B, A), X(A))) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), X(B)) & \xrightarrow[\underline{X}_{BA*}]{} & \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), \mathrm{Hom}_{R_A}(\mathrm{Hom}(B, A), X(A))). \end{array}$$

It is easy to verify that the R_A-module homomorphism ψ_A^X is factorized over the kernel of the homomorphism

$$\mu_{A,X} := (t^1_{A,X} - t^2_{A,X}, \tau_{A,X}),$$

$$\begin{array}{ccc} X(A) & \xrightarrow{\psi_A^X} & \prod\limits_{A\neq B\in\mathcal{A}} \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), X(B)) \\ & \searrow \quad \nearrow & \\ & \ker(\mu_{A,X}). & \end{array} \tag{5.6.4}$$

5.7. Proposition. *For any object A in $\mathcal{A}$, $(\mathrm{c}_A, \mathrm{s}_A, \mathrm{k}_A)$ is an adjoint triple*

$$\mathrm{Add}(\mathcal{A}, R\text{-Mod}) \underset{\mathrm{k}_A}{\overset{\mathrm{c}_A}{\rightleftarrows}} R_A\text{-Mod}, \qquad \mathrm{s}_A \colon R_A\text{-Mod} \to \mathrm{Add}(\mathcal{A}, R\text{-Mod}).$$

Proof. We construct a natural isomorphism

$$\gamma : \mathrm{Hom}_{R_A}(\mathrm{c}_A(-), -) \Longrightarrow \mathrm{Nat}_R(-, \mathrm{s}_A(-)) .$$

Let $z \in \mathrm{Hom}_{R_A}(\mathrm{c}_A(X), M)$. We let $\gamma(z)_A : X(A) \longrightarrow M = \mathrm{s}_A(M)(A)$ be the composition of z with the canonical quotient homomorphism $X(A) \to \mathrm{c}_A(X)$, and $\gamma(z)_B := 0$: $X(B) \to 0 = \mathrm{s}_A(M)(B)$ for $B \neq A$. Clearly, $\gamma(z)_B$ is an R-module homomorphism for every object B in $\mathcal{A}$. Now we show that $\gamma(z) := \{\gamma(z)_B\}_{B\in\mathcal{A}}$ is a natural transformation from X to $\mathrm{s}_A(M)$. For it, for a given morphism $f : B \to B'$ in $\mathcal{A}$, we need to show the commutativity of the following diagram:

$$\begin{array}{ccc} X(B) & \xrightarrow{X(f)} & X(B') \\ {\scriptstyle \gamma(z)_B}\downarrow & & \downarrow{\scriptstyle \gamma(z)_{B'}} \\ \mathrm{s}_A(M)(B) & \xrightarrow[\mathrm{s}_A(M)(f)]{} & \mathrm{s}_A(M)(B'). \end{array}$$

This is immediate whenever $B' \neq A$. If $B = A = B'$, then it follows from the fact that $\gamma(z)_A$ is an R_A-module homomorphism. Now suppose that $B \neq A$ and $B' = A$. $X(f)$ is factorized as

$$X(B) \xrightarrow{f \otimes \mathrm{id}} \mathrm{Hom}(B, A) \otimes_{R_B} X(B) \hookrightarrow \bigoplus_{A \neq B \in \mathcal{A}} \mathrm{Hom}(B, A) \otimes_{R_B} XB \longrightarrow X(A).$$

Therefore, $\gamma(z)_A \circ X(f) = 0$. As a consequence, $\gamma \colon X \Rightarrow \mathrm{s}_A(M)$ is a natural transformation. One can easily show the naturality and injectivity of γ.

Let $\alpha \colon X \Rightarrow \mathrm{s}_A(M)$ be a natural transformation. As pointed out in 4.1, α_A is an R_A-module homomorphism. Besides, for every $B \neq A$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(B, A) \otimes_{R_B} X(B) & \xrightarrow{\overline{X}_{BA}} & X(A) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \alpha_B} & & \downarrow{\scriptstyle \alpha_A} \\ 0 = \mathrm{Hom}(B, A) \otimes_{R_B} \mathrm{s}_A(M)(B) & \longrightarrow & M, \end{array}$$

So α_A is factorized over $c_A(X)$, which shows the surjectivity of γ.

Hence, $(\mathrm{c}_A, \mathrm{s}_A)$ is an adjoint pair. Similar arguments can be applied for $(\mathrm{s}_A, \mathrm{k}_A)$. ∎

We let

$$\mathrm{c}, \mathrm{k} : \mathrm{Add}(\mathcal{A}, R\text{-Mod}) \longrightarrow \prod_{A \in \mathcal{A}} R_A\text{-Mod}$$

and

$$\mathrm{s} : \prod_{A \in \mathcal{A}} R_A\text{-Mod} \longrightarrow \mathrm{Add}(\mathcal{A}, R\text{-Mod})$$

denote the functors $(\mathrm{c}_A)_{A \in \mathcal{A}}$, $(\mathrm{k}_A)_{A \in \mathcal{A}}$ and $\bigoplus_{A \in \mathcal{A}} \mathrm{s}_A$, respectively.

5.8. Proposition. (c, s, k) *is an adjoint triple*

$$\mathrm{Add}(\mathcal{A}, R\text{-Mod}) \underset{\mathrm{k}}{\overset{\mathrm{c}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\mathrm{s}}}}} \prod_{A \in \mathcal{A}} R_A\text{-Mod}.$$

Proof. Note that for every object B in $\mathcal{A}$,

$$\bigoplus_{A \in \mathcal{A}} \mathrm{s}_A(M_A)(B) = \mathrm{s}_B(M_B) = \prod_{A \in \mathcal{A}} \mathrm{s}_A(M_A)(B).$$

Therefore,

$$\mathrm{s}((M_A)_{A \in \mathcal{A}}) = \bigoplus_{A \in \mathcal{A}} \mathrm{s}_A(M_A) = \prod_{A \in \mathcal{A}} \mathrm{s}_A(M_A).$$

So we have

$$\begin{aligned}\mathrm{Hom}_{\prod_{A\in\mathcal{A}} R_A\text{-Mod}}((\mathrm{c}_A(X))_{A\in\mathcal{A}},(M_A)_{A\in\mathcal{A}}) &:= \prod_{A\in\mathcal{A}} \mathrm{Hom}_{R_A}(\mathrm{c}_A(X), M_A)\\ &\cong \prod_{A\in\mathcal{A}} \mathrm{Nat}_R(X, \mathrm{s}_A(M_A)) \cong \mathrm{Nat}_R(X, \prod_{A\in\mathcal{A}} \mathrm{s}_A(M_A)).\end{aligned}$$

Similarly, (s, k) is proved to be an adjoint pair. ∎

5.9. Remark. It is immediate that the functor s is an exact functor from $\prod_{A\in\mathcal{A}} R_A$-Mod to Add($\mathcal{A}$, R-Mod). However, neither c_A nor k_A preserve short exact sequences in $\mathrm{Ext}^1(-, \mathrm{s}_A(M))$ and $\mathrm{Ext}^1(\mathrm{s}_A(M), -)$, respectively, as required in Corollary 3.3-(iii-iv). The category of chain complexes of left R-modules, which is a very particular case of Add($\mathcal{A}$, R-Mod), can be considered as a counterexample (see Example 3.6). For a more general case, for example, assume that there exist two different objects A and B in $\mathcal{A}$ such that $\mathrm{Hom}(A, B)$ is a flat right R_A-module. For any non-trivial left R_A-module M, consider the short exact sequence

$$K_{M,A} \hookrightarrow \mathrm{q}_A(M) \twoheadrightarrow \mathrm{s}_A(M)$$

given in Remark 5.15. Note that the associated R_A-module homomorphism

$$\underline{\mathrm{q}_A(M)}_{AB} : \mathrm{q}_A(M)(A) \cong M \longrightarrow \mathrm{Hom}_{R_B}(\mathrm{Hom}(A, B), \mathrm{q}_A(M)(B)),$$

with

$$\mathrm{q}_A(M)(A)(m) = - \otimes m : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B) \otimes_{R_A} M,$$

is injective, and therefore,

$$\mathrm{k}_A(\mathrm{q}_A(M)) = 0.$$

Applying k_A to the aforementioned short exact sequence, we have $0 \to 0 \to M$.

Applying Corollary 3.3 to the adjoint triples given in Proposition 5.7 and Proposition 5.8, we obtain the following two corollaries.

5.10. Corollary. *Let A be an object in $\mathcal{A}$. For a given cotorsion pair $(\mathcal{F}_A, \mathcal{G}_A)$ in R_A-Mod, the following statements hold.*

(i) *If $\mathcal{F}_A$ is a generating class in R_A-Mod, then*

$$\mathrm{s}_A(\mathcal{F}_A)^{\perp} = \left\{ X : \mathcal{A} \to R\text{-Mod} \;\middle|\; \begin{array}{l} \mathrm{k}_A(X) \in \mathcal{G}_A \text{ and} \\ X \text{ is a } \mathrm{k}_A\text{-coflat object} \end{array} \right\}.$$

(ii) *If $\mathcal{G}_A$ is a cogenerating class in R_A-Mod, then*

$${}^{\perp}\mathrm{s}_A(\mathcal{G}_A) = \left\{ X : \mathcal{A} \to R\text{-Mod} \;\middle|\; \begin{array}{l} \mathrm{c}_A(X) \in \mathcal{F}_A \text{ and} \\ X \text{ is a } \mathrm{c}_A\text{-flat object} \end{array} \right\}.$$

5.11. Corollary. *For any cotorsion pair* $(\prod_{A\in\mathcal{A}}\mathcal{F}_A, \prod_{A\in\mathcal{A}}\mathcal{G}_A)$ *in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, the following statements hold.*

(i) *If* $\prod_{A\in\mathcal{A}}\mathcal{F}_A$ *is a generating class in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, then*

$$\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{F}_A\Big)^{\perp} = \left\{X\colon \mathcal{A}\to R\text{-Mod} \;\middle|\; \begin{array}{l}\mathrm{k}_A(X)\in\mathcal{G}_A \text{ and } X \text{ is a } \mathrm{k}_A\text{-coflat}\\ \text{object for every } A\in\mathcal{A}\end{array}\right\}.$$

(ii) *If* $\prod_{A\in\mathcal{A}}\mathcal{G}_A$ *is a cogenerating class in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, then*

$${}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{G}_A\Big) = \left\{X\colon \mathcal{A}\to R\text{-Mod} \;\middle|\; \begin{array}{l}\mathrm{c}_A(X)\in\mathcal{F}_A \text{ and } X \text{ is a } \mathrm{c}_A\text{-flat}\\ \text{object for every } A\in\mathcal{A}\end{array}\right\}.$$

5.12. Remark. From Lemma 2.1-(i-ii), it is immediate that $\mathrm{p}_B(G)\in \mathrm{s}_A(\mathcal{F}_A)^{\perp}$, and $\mathrm{q}_B(F)\in {}^{\perp}\mathrm{s}_A(\mathcal{G}_A)$ for every $A, B\in\mathcal{A}$, $F\in\mathcal{F}_B$ and $G\in\mathcal{G}_B$. By Proposition 4.12, if $\prod_{A\in\mathcal{A}}\mathcal{G}_A$ is a cogenerating class in $\prod_{A\in\mathcal{A}} R_A$-Mod, so is $\mathrm{s}(\prod_{A\in\mathcal{A}}\mathcal{F}_A)^{\perp}$; if $\prod_{A\in\mathcal{A}}\mathcal{F}_A$ is a generating class in $\prod_{A\in\mathcal{A}} R_A$-Mod, so is ${}^{\perp}\mathrm{s}(\prod_{A\in\mathcal{A}}\mathcal{G}_A)$.

5.13. Remark. If we consider the absolute exact structure on both categories $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ and $\prod_{A\in\mathcal{A}} R_A$-Mod, then by Proposition 2.10 and Proposition 2.15, we have

$$\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{F}_A\Big)^{\perp} = \{X\colon \mathcal{A}\to R\text{-Mod} \mid \mathrm{k}_A(X)\in\mathcal{G}_A \text{ and } \mathrm{R}^1\mathrm{k}_A(X)=0 \text{ for every } A\in\mathcal{A}\},$$
$${}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{G}_A\Big) = \{X\colon \mathcal{A}\to R\text{-Mod} \mid \mathrm{c}_A(X)\in\mathcal{F}_A \text{ and } \mathrm{L}_1\mathrm{c}_A(X)=0 \text{ for every } A\in\mathcal{A}\},$$

which are equal to the classes $\Psi(\prod_{A\in\mathcal{A}}\mathcal{G}_A)$ and $\Phi(\prod_{A\in\mathcal{A}}\mathcal{F}_A)$, respectively, introduced in Definition 1.5 and Theorem 1.7 of [10].

The notion of (co)flatness related to a functor is, in general, abstract and not easily visualized. To better understand c_A-flat and k_A-coflat objects in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$, we aim to describe them in terms of specific short exact sequences, similar to the one given in Example 3.6. To do this, we first need to prove the following auxiliary results.

5.14. Lemma. *For every object* A *in* $\mathcal{A}$ *and a left* R_A*-module* M*, the canonical natural transformations*

$$\mathrm{q}_A(M)\Longrightarrow \mathrm{s}_A(M) \quad \textit{and} \quad \mathrm{s}_A(M)\Longrightarrow \mathrm{p}_A(M)$$

are admissible epic and admissible monic, respectively, in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$.

Proof. Recall from 4.8 the constructions of functors q_A and p_A. We only prove that the first morphism is an admissible epic as the argument for the second one is similar. Let $\alpha\colon \mathrm{q}_A(M)\Rightarrow \mathrm{s}_A(M)$ be the counit of the adjunction $(\mathrm{q}_A, \mathrm{ev}_A)$ for the functor $\mathrm{s}_A(M)$. It is easy to verify that

$$\alpha_A : R_A\otimes_{R_A} M\to M$$

is the canonical isomorphism induced from the left scalar multiplication of M by elements in R_A, and $\alpha_B = 0 : \mathrm{Hom}(A, B) \otimes_{R_A} M \to 0$ for $B \neq A$. So for every object B in $\mathcal{A}$, the morphism α_B is an admissible epic in $\mathcal{E}_B$. By construction of $\mathcal{E}$ (see (4.11)), α is an admissible epic in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$. ∎

5.15. Remark. The canonical morphisms given in Lemma 5.14 induce the following canonical short exact sequences in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$

$$\mathrm{K}_{M,A} \hookrightarrow \mathrm{q}_A(M) \twoheadrightarrow \mathrm{s}_A(M) \quad \text{and} \quad \mathrm{s}_A(M) \hookrightarrow \mathrm{p}_A(M) \twoheadrightarrow \mathrm{C}_{M,A}.$$

From the construction, it is immediate that the functors $\mathrm{K}_{M,A}, \mathrm{C}_{M,A} : \mathcal{A} \longrightarrow R\text{-Mod}$ satisfy

$$\mathrm{K}_{M,A}(B) := \begin{cases} 0, & B = A, \\ \mathrm{q}_A(M)(B), & B \neq A; \end{cases} \quad \text{and} \quad \mathrm{C}_{M,A}(B) := \begin{cases} 0, & B = A, \\ \mathrm{p}_A(M)(B), & B \neq A. \end{cases}$$

5.16. Lemma. *For any functor $X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod})$ and a left R_A-module M, we have*

$$\mathrm{Nat}_R(X, \mathrm{C}_{M,A}) \cong \mathrm{Hom}_{R_A}(\mathrm{coker}(\phi_{A,X}), M),$$
$$\mathrm{Nat}_R(\mathrm{K}_{M,A}, X) \cong \mathrm{Hom}_{R_A}(M, \ker(\mu_{A,X})).$$

Proof. Recall from (5.6.3) and (5.6.4) the constructions of the homomorphisms $\phi_{A,X}$ and $\mu_{A,X}$. Due to the duality of arguments, we only prove the isomorphism $\mathrm{Nat}_R(X, \mathrm{C}_{M,A}) \cong \mathrm{Hom}_{R_A}(\mathrm{coker}(\phi_{A,X}), M)$. Let $\alpha : X \Rightarrow \mathrm{C}_{M,A}$ be a natural transformation. From 4.1, we already know that α_B is an R_B-module homomorphism for any object B. Now, for a given object B in $\mathcal{A}$ with $B \neq A$, we let $\bar{\alpha}_B$ denote the R_A-module homomorphisms in $\mathrm{Hom}_{R_A}(\mathrm{Hom}(B, A) \otimes_{R_B} X(B), M) \cong \mathrm{Hom}_{R_B}(X(B), \mathrm{p}_A(M)(B))$ corresponding to α_B under the usual adjunction. Note that $\bar{\alpha}_B(f \otimes x) = \alpha_B(x)(f)$. It is easy to verify that for every two objects B and B' in $\mathcal{A}$ different from A, we have

$$\bar{\alpha}_{B'} \circ (\mathrm{id} \otimes \overline{X}_{BB'}) = \bar{\alpha}_B \circ (\circ \otimes \mathrm{id})$$

as given in the diagram (5.6.2), and since α is a natural transformation, $\alpha_B \circ X(l) = 0$ for every morphism $l : A \to B$, and hence, $\bar{\alpha}_B \circ (\mathrm{id} \otimes \overline{X}_{AB}) = 0$ (see the diagram (5.6.2)). So there exists a unique R_A-module homomorphism $z_\alpha : \mathrm{coker}(\phi_{A,X}) \longrightarrow M$ satisfying

$$z_\alpha \circ \sigma_A = \sum_{A \neq B} \bar{\alpha}_B : \bigoplus_{A \neq B \in \mathcal{A}} \mathrm{Hom}(B, A) \otimes_{R_B} X(B) \longrightarrow M.$$

Clearly, the assignment

$$\mathrm{Nat}_R(X, \mathrm{C}_{M,A}) \longrightarrow \mathrm{Hom}_{R_A}(\mathrm{coker}(\phi_{A,X}), M), \quad \alpha \rightsquigarrow z_\alpha,$$

is well defined, and is natural in both X and M. Since σ_A is an epimorphism, it is also injective.

Now, let $z : \mathrm{coker}(\phi_{A,X}) \longrightarrow M$ be an R_A-module homomorphism. We let $\alpha_A := 0 : X(A) \longrightarrow 0$, and for an object $B \neq A$, we let α_B denote the R_B-module homomorphism in $\mathrm{Hom}_{R_B}(X(B), \mathrm{p}_A(M)(B)) \cong \mathrm{Hom}_{R_A}(\mathrm{Hom}(B, A) \otimes_{R_B} X(B), M)$ corresponding

to the R_A-module homomorphism $z \circ \sigma_A$ restricted to $\mathrm{Hom}(B, A) \otimes_{R_B} X(B)$ under the usual adjunction. Note that $\alpha_B(x) = z \circ \sigma_A(- \otimes x)$. By restriction of scalars, the family $\alpha := \{\alpha_B\}_{B \in \mathcal{A}}$ is clearly a family of R-module homomorphisms. We claim that α is a natural transformation from X to $\mathrm{C}_{M,A}$. To prove this, consider a morphism $f: B \to B'$ in $\mathcal{A}$. The case when $B' = A$ is immediate, as $\alpha_A \circ X(f) = 0 = \mathrm{C}_{M,A}(f) \circ \alpha_B$. Now suppose that $B' \neq A$. Then there are two cases to consider: $B \neq A$ and $B = A$.

Case $B \neq A$. For any morphism $g: B' \longrightarrow A$ and $x \in X(B)$, we have

$$\begin{aligned}(\mathrm{C}_{M,A}(f) \circ \alpha_B)(x)(g) &= (z \circ \sigma_A)((g \circ f) \otimes x) = (z \circ \sigma_A)(g \otimes X(f)(x)) \\ &= \alpha_{B'}(X(f)(x))(g) = (\alpha_{B'} \circ X(f))(x)(g),\end{aligned}$$

which implies $\mathrm{C}_{M,A}(f) \circ \alpha_B = \alpha_{B'} \circ X(f)$.

Case $B = A$. For any object x in $X(A)$, we have

$$(\alpha_{B'} \circ X(f))(x)(-) = z \circ \sigma_A(- \otimes X(f)(x)) : \mathrm{Hom}(B', A) \longrightarrow M.$$

But $\sigma_A(g \otimes X(f)(x)) = 0$ for any morphism $g \in \mathrm{Hom}(B', A)$. So

$$\alpha_{B'} \circ X(f) = 0 = \mathrm{C}_{M,A}(f) \circ \alpha_A.$$

As a consequence, α is a natural transformation, and therefore, the assignment $\mathrm{Nat}_R(X, \mathrm{C}_{M,A}) \longrightarrow \mathrm{Hom}_{R_A}(\phi_{A,X}), M)$ defined above is surjective. ∎

5.17. Lemma. *For every two objects $A \neq B$ in $\mathcal{A}$,*

(i) $\mathrm{c}_A \circ \mathrm{q}_A = \mathrm{id}$ *and* $\mathrm{c}_A \circ \mathrm{q}_B = 0$,

(ii) $\mathrm{k}_A \circ \mathrm{p}_A = \mathrm{id}$ *and* $\mathrm{k}_A \circ \mathrm{p}_B = 0$.

Proof. By adjunctions, we have

$$\begin{aligned}\mathrm{Hom}_{R_A}(\mathrm{c}_A(\mathrm{q}_A(-)), -) &\cong \mathrm{Nat}_R(\mathrm{q}_A(-), \mathrm{s}_A(-)) \\ &\cong \mathrm{Hom}_{R_A}(-, \mathrm{ev}_A(\mathrm{s}_A(-))) = \mathrm{Hom}_{R_A}(-, \mathrm{id}(-)), \\ \mathrm{Hom}_{R_A}(\mathrm{c}_A(\mathrm{q}_B(-)), -) &\cong \mathrm{Nat}_R(\mathrm{q}_B(-), \mathrm{s}_A(-)) \cong \mathrm{Hom}_{R_B}(-, \mathrm{ev}_B(\mathrm{s}_A(-))) = 0,\end{aligned}$$

So $\mathrm{c}_A \circ \mathrm{q}_A = \mathrm{id}$ and $\mathrm{c}_A \circ \mathrm{q}_B = 0$. By similar arguments, we have the equalities in (ii). ∎

Now we can provide a concrete description of c_A-flat and k_A-coflat functors, which will also be crucial in proving Theorem 6.5. Recall morphisms $\phi_{A,X}$ and $\ker(\mu_{A,X})$ introduced in 5.6.

5.18. Proposition. *Let A be an object in $\mathcal{A}$. For any additive functor $X: \mathcal{A} \longrightarrow R$-Mod, the following statements hold.*

(i) *X is c_A-flat in* $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ *if and only if the canonical sequence*

$$\mathrm{coker}(\phi_{A,X}) \longrightarrow X(A) \longrightarrow \mathrm{c}_A(X)$$

given in 5.6 is a short exact sequence in R_A-Mod.

(ii) *X is k_A-coflat in* $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ *if and only if the canonical sequence*

$$\mathrm{k}_A(X) \longrightarrow X(A) \longrightarrow \ker(\mu_{A,X})$$

given in 5.6 *is a short exact sequence in* R_A-Mod.

Proof. We only prove the first statement.

($\Rightarrow$) Suppose that X is a c_A-flat object in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$. We firstly show that the morphism $X(A) \to \mathrm{c}_A(X)$ is an admissible epic in R_A-Mod. For it, using the identity morphism $\mathrm{id}_{X(B)}$ for every object B in $\mathcal{A}$ and the adjoint pair $(\mathrm{q}, (\mathrm{ev}_B)_{B\in\mathcal{A}})$, we have the canonical natural transformation

$$\alpha : \bigoplus_{B\in\mathcal{A}} \mathrm{q}_B(X(B)) \Rightarrow X,$$

which is an admissible epic in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ just as pointed out in the proof of Proposition 4.12. By assumption, $\mathrm{c}_A(\alpha)$ is an admissible epic in R_A-Mod. As c_A is a left adjoint functor, it preserves coproducts, and using Lemma 5.17-(i), we have

$$\mathrm{c}_A\Big(\bigoplus_{B\in\mathcal{A}} \mathrm{q}_B(X(B))\Big) \cong \bigoplus_{B\in\mathcal{A}} \mathrm{c}_A(\mathrm{q}_B(X(B)) \cong X(A).$$

So $\mathrm{c}_A(\alpha)$ is isomorphic to the canonical quotient morphism $X(A) \to \mathrm{c}_A(X)$. In other words, it is an admissible epic in R_A-Mod.

On the other hand, let E be an injective cogenerator left R_A-module in the absolute sense (that is, an injective cogenerating object in R_A-Mod regarding to the absolute exact structure). It is clearly an injective object with respect to the exact structure $\mathcal{E}_A$, as well. Consider the following canonical short exact sequence in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ as given in Lemma 5.14:

$$\mathbb{E} : \ \mathrm{s}_A(E) \rightarrowtail \mathrm{p}_A(E) \twoheadrightarrow \mathrm{C}_{E,A} \ .$$

Applying the functor $\mathrm{Nat}_R(X, -)$, it induces the following long exact sequence of abelian groups:

$$\begin{aligned} 0 \longrightarrow \mathrm{Nat}_R(X, \mathrm{s}_A(E)) \longrightarrow \mathrm{Nat}_R(X, \mathrm{p}_A(E)) \longrightarrow \mathrm{Nat}_R(X, \mathrm{C}_{E,A}) \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(X, \mathrm{s}_A(E)) \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(X, \mathrm{p}_A(E)). \end{aligned}$$

Since ev_A is an exact functor, $\mathrm{p}_A(E)$ is an injective object in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$. So $\mathrm{Ext}^1_{\mathcal{E}}(X, \mathrm{p}_A(E)) = 0$. Besides, as X is a c_A-flat object in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$, by Proposition 2.10, we have

$$\mathrm{Ext}^1_{\mathcal{E}}(X, \mathrm{s}_A(E)) \cong \mathrm{Ext}^1_{\mathcal{E}_A}(\mathrm{c}_A(X), E) = 0,$$

and therefore, $\mathrm{Nat}_R(X, \mathbb{E})$ is a short exact sequence of abelian groups. By adjunctions and Lemma 5.16, it is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_{R_A}(\mathrm{c}_A(X), E) \longrightarrow \mathrm{Hom}_{R_A}(X(A), E) \longrightarrow \mathrm{Hom}_{R_A}(\mathrm{coker}(\phi_{A,X}), E) \longrightarrow 0.$$

As E is a cogenerating left R_A-module, $\mathrm{coker}(\phi_{A,X})$ is the kernel of the admissible epic $X(A) \to \mathrm{c}_A(X)$, and therefore,

$$\mathrm{coker}(\phi_{A,X}) \rightarrowtail X(A) \twoheadrightarrow \mathrm{c}_A(X)$$

is a short exact sequence in R_A-Mod.

($\Leftarrow$) Suppose that the canonical sequence

$$\operatorname{coker}(\phi_{A,X}) \hookrightarrow X(A) \twoheadrightarrow \mathrm{c}_A(X)$$

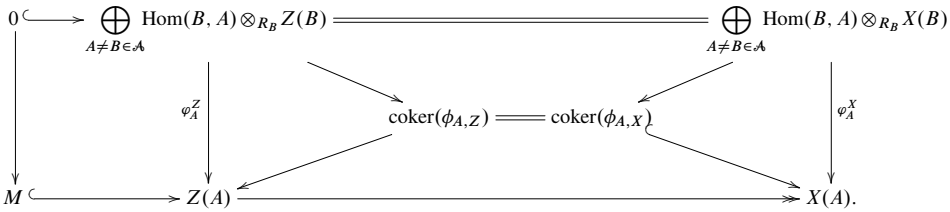
is a short exact sequence in R_A-Mod. By Lemma 2.8-(i), it is enough to show that c_A preserves short exact sequences in $\operatorname{Ext}^1(X, \mathrm{s}_A(-))$. Let $\mathbb{E}$ be a short exact sequence in $\operatorname{Add}(\mathcal{A}, R\text{-Mod})$ of the form

$$\mathrm{s}_A(M) \hookrightarrow Z \xrightarrow{\alpha} X \ ,$$

where M is a left R_A-module. Without lost of generality, we can assume that $X(B) = Z(B)$ for every object B in $\mathcal{A}$ with $A \neq B$. So we have $h^1_{A,X} = h^1_{A,Z}$ and $h^2_{A,X} = h^2_{A,Z}$. On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{A\neq B\in\mathcal{A}} \operatorname{Hom}(B,A)\otimes_{R_B}\operatorname{Hom}(A,B)\otimes_{R_A} Z(A) & \xrightarrow{\oplus \mathrm{id}\otimes\mathrm{id}\otimes\alpha_A} & \bigoplus_{A\neq B\in\mathcal{A}} \operatorname{Hom}(B,A)\otimes_{R_B}\operatorname{Hom}(A,B)\otimes_{R_A} X(A) \\
\downarrow{\scriptstyle \nu_{A,Z}} & & \downarrow{\scriptstyle \nu_{A,X}} \\
\bigoplus_{A\neq B\in\mathcal{A}} \operatorname{Hom}(B,A)\otimes_{R_B} Z(B) & = & \bigoplus_{A\neq B\in\mathcal{A}} \operatorname{Hom}(B,A)\otimes_{R_B} X(B),
\end{array}$$

whose upper horizontal arrow is an epimorphism. So the images of $\nu_{A,X}$ and $\nu_{A,Z}$ coincide, and therefore, we have $\operatorname{coker}(\phi_{A,X}) = \operatorname{coker}(\phi_{A,Z})$ together with the following commutative diagram:



By Proposition 2.16 in [2], the canonical morphism $\operatorname{coker}(\phi_{A,Z}) \longrightarrow Z(A)$ is an admissible monic in R_A, as well. Applying the snake lemma (see Corollary 8.13 in [2]) to the following commutative diagram,

$$\begin{array}{ccccc}
0 & \longrightarrow & \operatorname{coker}(\phi_{A,Z}) & = & \operatorname{coker}(\phi_{A,X}) \\
\downarrow & & \downarrow & & \downarrow \\
M & \hookrightarrow & Z(A) & \twoheadrightarrow & X(A),
\end{array}$$

we obtain the following short exact sequence in R_A-Mod:

$$M = \mathrm{c}_A(\mathrm{s}_A(M)) \hookrightarrow \mathrm{c}_A(Z) \twoheadrightarrow \mathrm{c}_A(X) \ . \qquad \blacksquare$$

5.19. Theorem. *For any cotorsion pair* $(\prod_{A\in\mathcal{A}}\mathcal{F}_A, \prod_{A\in\mathcal{A}}\mathcal{G}_A)$ *in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, the following statements hold.*

(i) *If* $\prod_{A\in\mathcal{A}}\mathcal{F}_A$ *is a generating class in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, then*

$$\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{F}_A\Big)^{\perp} = \Big\{X\colon \mathcal{A}\to R\text{-Mod} \;\Big|\; \begin{array}{l}\mathrm{k}_A(X)\in\mathcal{G}_A,\, X(A)\longrightarrow \ker(\mu_{A,X})\\ \text{is an admissible epic in } R_A\text{-Mod},\ \forall A\in\mathcal{A}\end{array}\Big\}.$$

(ii) *If* $\prod_{A\in\mathcal{A}}\mathcal{G}_A$ *is a cogenerating class in* $\prod_{A\in\mathcal{A}} R_A$-Mod*, then*

$${}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}}\mathcal{G}_A\Big) = \Big\{X\colon \mathcal{A}\to R\text{-Mod} \;\Big|\; \begin{array}{l}\mathrm{c}_A(X)\in\mathcal{F}_A,\, \mathrm{coker}(\phi_{A,X})\hookrightarrow X(A)\\ \text{is an admissible monic in } R_A\text{-Mod},\ \forall A\in\mathcal{A}\end{array}\Big\}.$$

6. Projective and injective functors

In this section, we apply Theorem 5.19 to provide an intrinsic characterization of projective and injective functors. We retain all the hypotheses assumed on $\mathcal{A}$ so far. Applying Theorem 5.19-(i) to the trivial cotorsion pair

$$\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}, \prod_{A\in\mathcal{A}}\mathcal{E}_A\text{-Inj}\Big),$$

we have

$$\mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big)^{\perp} = \left\{X\colon \mathcal{A}\to R\text{-Mod} \;\middle|\; \begin{array}{l}\mathrm{k}_A(X)\in\mathcal{E}_A\text{-Inj},\\ X(A)\longrightarrow \ker(\mu_{A,X}) \text{ is an admissible}\\ \text{epic in } R_A\text{-Mod},\ \forall A\in\mathcal{A}\end{array}\right\}.$$

Similarly, by Theorem 5.19-(ii), the trivial cotorsion pair

$$\Big(\prod_{A\in\mathcal{A}}\mathcal{E}_A\text{-Proj}, \prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big)$$

leads to

$${}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big) = \left\{X\colon \mathcal{A}\to R\text{-Mod} \;\middle|\; \begin{array}{l}\mathrm{c}_A(X)\in\mathcal{E}_A\text{-Proj},\\ \mathrm{coker}(\phi_{A,X})\hookrightarrow X(A) \text{ is an admissible}\\ \text{monic in } R_A\text{-Mod},\ \forall A\in\mathcal{A}\end{array}\right\}.$$

The following containments in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ can be easily verified:

$$\mathcal{E}\text{-Inj}\subseteq \mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big)^{\perp} \quad\text{and}\quad \mathcal{E}\text{-Proj}\subseteq {}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big).$$

The classes on the right side of the above inclusions have a precise description and are easier to work from a homological point of view. Furthermore, in the very well-known cases of functor categories (e.g., the category of chain complexes of left R-modules, the module category over a formal triangular matrix ring etc.), they give rise to injective and projective objects in functor categories. We need to impose further conditions on $\mathcal{A}$ to obtain the reverse containments.

6.1. Condition. There does not exist a chain of morphisms in $\mathcal{A}$

$$\cdots \longrightarrow A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1$$

satisfying $A_i \neq A_j$ for every $1 \leq i \neq j$, and $f_1 \circ \cdots \circ f_n \neq 0$ for every $n \geq 1$.

6.2. Condition. There does not exist a chain of morphisms in $\mathcal{A}$

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$$

satisfying $A_i \neq A_j$ for every $1 \leq i \neq j$, and $f_n \circ \cdots \circ f_1 \neq 0$ for every $n \geq 1$.

6.3. Lemma. *For any functor $X \in \mathrm{Add}(\mathcal{A}, R\text{-Mod})$, the following statements hold.*

(i) *If Condition 6.1 is satisfied, then $X = 0$ if and only if $\mathrm{c}_A(X) = 0$ for every $A \in \mathcal{A}$.*

(ii) *If Condition 6.2 is satisfied, then $X = 0$ if and only if $\mathrm{k}_A(X) = 0$ for every $A \in \mathcal{A}$.*

Proof. The necessity part is clear. Now suppose that $\mathrm{c}_A(X) = 0$ for every $A \in \mathcal{A}$. It implies that the R_A-module homomorphism φ_A^X is an epimorphism for every object A in $\mathcal{A}$. For the sake of contradiction, we assume that $X \neq 0$. Consider an object A_1 in $\mathcal{A}$ such that $X(A_1) \neq 0$. Let x_1 be a non-zero element in $X(A_1)$. By assumption, there exist a finite set I_2 of objects in $\mathcal{A}$ different from A_1, and a family $\{(f_B, x_B)\}_{B \in I_2}$, where $f_B \in \mathrm{Hom}(B, A_1)$ and $x_B \in X(B)$, such that

$$\sum_{B \in I_2} X(f_B)(x_B) = x_1.$$

Without loss of generality, we can assume that $X(f_B)(x_B) \neq 0$ for every $B \in I_2$. We choose any object in I_2, we denote it by A_2, and we write $f_1 := f_{A_2}$ and $x_2 := x_{A_2}$. Similarly, as $x_2 \neq 0$, there exist a finite family I_3 of objects in $\mathcal{A}$ different from A_2, and a family $\{(f_B, x_B)\}_{B \in I_3}$, where $f_B \in \mathrm{Hom}(B, A_2)$ and $x_B \in X(B)$, such that

$$\sum_{B \in I_3} X(f_B)(x_B) = x_2.$$

We claim that there is an object $B_0 \in I_3$ such that

$$X(f_1 \circ f_{B_0})(x_{B_0}) = (X(f_1) \circ X(f_{B_0}))(x_{B_0}) \neq 0.$$

Indeed, if $X(f_1 \circ f_B)(x_B) = 0$ for every $B \in I_3$, then we would have

$$X(f_1)(x_2) = X(f_1)\Big(\sum_{B \in I_3} X(f_B)(x_B)\Big) = \sum_{B \in I_3} X(f_1 \circ f_B)(x_B) = 0,$$

which is a contradiction. We let A_3 be any object in I_3 such that

$$(X(f_1 \circ f_{A_3}))(x_{A_3}) \neq 0.$$

We let $f_2 := f_{A_3}$ and $x_3 := x_{A_3}$. Note that $A_3 \neq A_2$. From Hypothesis 5.1, we also have that $A_3 \neq A_1$.

By induction, we can construct a sequence of morphisms in $\mathcal{A}$,

$$\cdots \longrightarrow A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1,$$

with $A_i \neq A_j$ for every $i \neq j \geq 1$, such that

$$X(f_1 \circ \ldots \circ f_n) \neq 0,$$

and therefore, $f_1 \circ \cdots \circ f_n \neq 0$ for every $n \geq 1$, which is a contradiction by assumption. So $X = 0$. ∎

The previous lemma is slightly different and more general than the result given in Theorem 7.19 of [11], as we have the following example.

6.4. Example. Let $\mathcal{A}$ be a category with objects $\mathbb{N}$ and morphisms

$$\mathrm{Hom}(n,m) := \begin{cases} R & \text{for } n \in \{m, \ldots, 2m\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\cdots \longrightarrow \bullet_3 \longrightarrow \bullet_2 \longrightarrow \bullet_1$$

It clearly satisfies Condition 6.1, but it is not left rooted nor fulfills the conditions given in Theorem 7.29 of [11].

Following a similar strategy used in the proof of Theorem 7.29 in [11], we have the following characterization of projective and injective additive functors.

6.5. Theorem.

(i) *If Condition 6.1 is satisfied, then*

$$\mathcal{E}\text{-Proj} = {}^{\perp}\mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big).$$

(ii) *If Condition 6.2 is satisfied, then*

$$\mathcal{E}\text{-Inj} = \mathrm{s}\Big(\prod_{A\in\mathcal{A}} R_A\text{-Mod}\Big)^{\perp}$$

Proof. We only prove the first statement. We already know the inclusion $\subseteq$. Now, let $X \in {}^{\perp}\mathrm{s}(\prod_{A\in\mathcal{A}} R_A\text{-Mod})$. Then for every $A \in \mathcal{A}$, we have the following canonical short exact sequence:

$$\mathrm{coker}(\phi_{A,X}) \hookrightarrow X(A) \xrightarrow{\pi_A} \mathrm{c}_A(X), \tag{6.5.1}$$

with an $\mathcal{E}_A$-projective left R_A-module $\mathrm{c}_A(X)$. So for every $A \in \mathcal{A}$, there exists an admissible monic $\iota_A : \mathrm{c}_A(X) \hookrightarrow X(A)$ such that $\pi_A \circ \iota_A = \mathrm{id}$. Using the counit of the adjunction

$(\mathrm{q}, (\mathrm{ev}_A)_{A\in\mathcal{A}})$, we obtain a natural transformation $\alpha\colon \bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(\mathrm{c}_A(X)) \Rightarrow X$. Note that α is the composition of the splitting admissible monic

$$\bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(\iota_A) : \bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(\mathrm{c}_A(X)) \Rightarrow \bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(X(A))$$

with the canonical natural transformation

$$\bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(X(A)) \Rightarrow X,$$

which is an admissible epic in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ as pointed out in the proof of Proposition 4.12. Note that $\bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(\mathrm{c}_A(X))$ is an $\mathcal{E}$-projective object in $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$

By Lemma 5.17, for any object B in $\mathcal{A}$, we have

$$\mathrm{c}_B\Big(\bigoplus_{A\in\mathcal{A}} \mathrm{q}_A(\mathrm{c}_A(X))\Big) \cong \bigoplus_{A\in\mathcal{A}} \mathrm{c}_B(\mathrm{q}_A(\mathrm{c}_A(X)) \cong \mathrm{c}_B(X),$$

and therefore, $\mathrm{c}_B(\alpha)$ is an isomorphism. Since c_B preserves all colimits, $\mathrm{c}_B(\mathrm{coker}(\alpha)) = \mathrm{coker}(\mathrm{c}_B(\alpha)) = 0$ for every object B in $\mathcal{A}$. By Lemma 6.3-(i), $\mathrm{coker}(\alpha) = 0$, that is, α is an epimorphism.

Notice that as R_A-Mod is an abelian category, the short exact sequence (6.5.1) in $(R_A\text{-Mod}; \mathcal{E}_A)$ is a usual short exact sequence in R_A-Mod with respect to the absolute exact structure. So by Proposition 5.18, for every object A in $\mathcal{A}$, X is a c_A-flat object whenever both categories $\mathrm{Add}(\mathcal{A}, R\text{-Mod})$ and R_A-Mod are considered to be equipped with the absolute exact structures. This implies that

$$\mathrm{c}_A(\ker(\alpha)) = \ker(\mathrm{c}_A(\alpha)) = 0$$

for every object A in $\mathcal{A}$, and hence, by Lemma 6.3-(i), $\ker(\alpha) = 0$. As a consequence, α is an isomorphism. ∎

6.6. Remark. If one starts with an exact structure $\mathcal{E}'$ on R-Mod, then by restriction of scalars, it leads to a family $\{\mathcal{E}_A\}_{A\in\mathcal{A}}$ of exact structures on R_A-Mod defined as follows: a short exact sequence $\mathbb{E}_A$ in R_A-Mod belongs to $\mathcal{E}_A$ if and only if it belongs to $\mathcal{E}'$ as a short exact sequence of R-modules. Note that $\mathcal{E}_A$-projective and injective objects are of the form $R_A \otimes_R M$ and $\mathrm{Hom}_R(R_A, N)$, respectively, where $M \in \mathcal{E}'\text{-Proj}$ and $N \in \mathcal{E}'\text{-Inj}$.

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