Rev. Mat. Iberoam. 42 (2026), no. 4, 1331–1362
DOI 10.4171/RMI/1571

On the self-similar stability of the parabolic-parabolic Keller–Segel equation

Frank Alvarez Borges, Kleber Carrapatoso and Stéphane Mischler

Abstract. We consider the parabolic-parabolic Keller–Segel equation in the plane and prove the nonlinear exponential stability of the self-similar profile in a quasi-parabolic-elliptic regime. We first perform a perturbation argument in order to obtain exponential stability for the semigroup associated to part of the first component of the linearized operator, by exploiting the exponential stability of the linearized operator for the parabolic-elliptic Keller–Segel equation. We finally employ a purely semigroup analysis to prove linear, and then nonlinear, exponential stability of the system in appropriated functional spaces.

1. Introduction

In this paper we are concerned with the parabolic-parabolic Keller–Segel system in self-similar variables in the plane:

$$\begin{cases} \partial_t f = \Delta f + \operatorname{div}(\mu x f - f \nabla u) \\ \partial_t u = \frac{1}{\varepsilon}(\Delta u + f) + \mu x \cdot \nabla u, \end{cases} \tag{1.1}$$

with fixed drift parameter $\mu > 0$ and with small time scale parameter $\varepsilon > 0$, which aims to give the time evolution of the collective motion of cells (described by the *cells density* $f = f(t,x)$) that are attracted by a chemical substance (described by the *chemo-attractant concentration* $u = u(t,x)$) they are able to emit (see [18, 26]). Here $t \geq 0$ is the time variable and $x \in \mathbb{R}^2$ stands for the space variable. We refer to the work [7] as well as to the reviews [16, 28] and the references cited therein for the biological motivation and mathematical background.

In this work, we establish in a convenient weighted Sobolev space the exponential stability of the *normalized self-similar profile* in the quasi-parabolic-elliptic regime, that is, for small values of the time scale $\varepsilon > 0$, without assuming any radial symmetry property on the initial datum. This extends similar results obtained in [9] in a radially symmetric framework. As in that last reference, the proof of the stability is based on a perturbation

---

Mathematics Subject Classification 2020: 35K55 (primary); 35B35, 35B25, 35K45, 35Q92 (secondary).
Keywords: Keller–Segel equation, self-similar variables, large-time behavior, nonlinear stability.

argument which takes advantage of the exponential stability of the self-similar profile for the parabolic-elliptic Keller–Segel equation established in [8, 12]. The proof, however, differs from [9] because it uses, among other things, (1) a different, and somehow more standard, perturbation argument performed at the level of the main part of the first component of the linearized operator instead of at the level of the whole linearized system, and (2) a purely semigroup analysis of the linear and nonlinear stability of the system.

Our result implies that in a quasi-parabolic-elliptic regime and for some class of initial data, without assuming any radial symmetry property, the associated solution to the parabolic-parabolic KS system in standard variables (corresponding to $\mu = 0$) has a self-similar long-time behavior, which, in particular, means that no concentration occurs in large time and thus the diffusion mechanism is really the dominant phenomenon all along the time evolution.

It is worth mentioning that as far as the existence problem is concerned, an alternative possible approach has been developed in [7], where weak solutions have been proved to exist for a very general class of initial data, see also [21, 22]. The associated uniqueness result has been solved in [9], see also [3, 10], but the accurate analysis of the long-time behavior of these solutions is still lacking. On the other hand, mild solutions have been proved to exist under a smallness condition in the initial datum, for instance, in [4, 11, 13], with associated self-similar behavior result in the longtime asymptotic in [11, 23, 24] or under a large time scale parameter, for instance, in [6, 11].

The two, and only two, general properties satisfied (at least formally) by the solutions of the parabolic-parabolic Keller–Segel equation are the positivity preservation of the cells density, i.e.,

$$f(t,\cdot) \ge 0 \quad \text{if } f(0,\cdot) \ge 0,$$

and a similar positivity property for the chemo-attractant concentration u, as well as the mass conservation of the cells density, namely,

$$\langle\!\langle f(t,\cdot)\rangle\!\rangle = \langle\!\langle f(0,\cdot)\rangle\!\rangle \quad \text{for all } t \ge 0, \quad \text{where} \quad \langle\!\langle h\rangle\!\rangle := \int_{\mathbb{R}^2} h\,dx. \tag{1.2}$$

That mass conservation (1.2) is known to be violated in some *supercritical mass* situation. However, we will only be concerned in this paper with a *subcritical mass framework* that we describe now.

We denote by $Q = Q^\mu_\varepsilon$ and $P = P^\mu_\varepsilon$ the normalized stationary solutions to the Keller–Segel system (1.1), that is,

$$\begin{cases} 0 = \Delta Q + \operatorname{div}(\mu x Q - Q\nabla P), & Q(0) = 8, \\ 0 = \Delta P + Q + \varepsilon\mu x\cdot\nabla P, & P(0) = 0, \end{cases} \tag{1.3}$$

whose existence, uniqueness, radially symmetric property and smoothness have all been established in [5, 11, 24]. It is worth emphasizing that we adopt here the normalizing convention of [15] motivated by the fact that in the vanishing drift limit

$$Q^\mu_\varepsilon \to Q^0, \quad \nabla P^\mu_\varepsilon \to \nabla P^0 \quad \text{as } \mu \to 0,$$

where (Q^0, P^0) is defined by

$$Q^0(x) := \frac{8}{(1+|x|^2)^2}, \quad -\Delta P^0 = Q^0,$$

and thus Q^0 is the well-known 8π critical mass solution to the parabolic-elliptic Keller–Segel equation in standard variables (corresponding thus to $\mu = \varepsilon = 0$). Because for any $\varepsilon \geq 0$, there exists a one-to-one mapping

$$\mathcal{M}_\varepsilon : [0,\infty) \to (0, 8\pi], \quad \mu \mapsto \mathcal{M}_\varepsilon(\mu) := \langle\!\langle Q_\varepsilon^\mu \rangle\!\rangle,$$

another possible (and more standard) normalization convention should be to fix the drift term $\mu := 1$ and to normalize the stationary solution by its subcritical mass in the interval $(0, 8\pi)$.

We next introduce the perturbation (g, v) of the stationary state (Q, P), defined by

$$f = Q + g, \quad \langle\!\langle g \rangle\!\rangle = 0, \quad u = P + v,$$

in such a way that the mass compatibility condition $\langle\!\langle f \rangle\!\rangle = \langle\!\langle Q \rangle\!\rangle$ is satisfied. If (f, u) is a solution to (1.1), then (g, v) satisfies the system

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}(\mu x g - g \nabla P - Q \nabla v) - \operatorname{div}(g \nabla v), \\ \partial_t v = \frac{1}{\varepsilon}(\Delta v + g) + \mu x \cdot \nabla v, \end{cases} \tag{1.4}$$

and reciprocally. Instead of working with solutions (g, v) to (1.4), we shall rather work with the unknown (g, w) defined by

$$w := v - \kappa * g,$$

where κ is the Laplace kernel in the plane:

$$\kappa(z) := -\frac{1}{2\pi} \log|z|, \quad \nabla\kappa(z) = -\frac{1}{2\pi}\frac{z}{|z|^2},$$

so that $\kappa * \Omega$ is a solution to the Laplace equation $-\Delta(\kappa * \Omega) = \Omega$ in $\mathbb{R}^2$. We will therefore consider the modified system

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}(\mu x g - g \nabla P - Q \nabla\kappa * g - Q \nabla w) \\ \qquad\quad - \operatorname{div}(g \nabla\kappa * g) - \operatorname{div}(g \nabla w), \\ \partial_t w = \frac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + g \\ \qquad\quad + \nabla\kappa * [g \nabla P + Q \nabla\kappa * g + Q \nabla w] + \nabla\kappa * [g \nabla w + g \nabla\kappa * g], \end{cases} \tag{1.5}$$

satisfied by (g, w), that we complement with an initial condition (g_0, w_0). It is worth mentioning that working with the unknown w is convenient in order to get some nice information (improving estimates) in the limit $\varepsilon \to 0$, which is not the case when considering the unknown v because of the singular term g/ε on the right-hand side of the second equation of (1.4) in the limit $\varepsilon \to 0$.

We introduce the Banach spaces $\mathcal{X} := L^2_k \times (\dot{H}^s \cap \dot{H}^1)$ and $\mathcal{Y} = H^1_k \times (\dot{H}^s \cap \dot{H}^2)$, with $s \in (0, 1)$, endowed with the norms

$$\begin{cases} \|(g, w)\|_{\mathcal{X}} := \|g\|_{L^2_k} + \|w\|_{\dot{H}^s} + \|w\|_{\dot{H}^1}, \\ \|(g, w)\|_{\mathcal{Y}} := \|g\|_{H^1_k} + \|w\|_{\dot{H}^s} + \|w\|_{\dot{H}^2}, \end{cases} \tag{1.6}$$

where the weighted Lebesgue space $L^p_k(\mathbb{R}^2)$, for $1 \le p \le \infty$ and $k \in \mathbb{R}$, is defined by

$$L^p_k(\mathbb{R}^2) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \|f\|_{L^p_k} := \|\langle x\rangle^k f\|_{L^p} < \infty\}, \quad \langle x\rangle := (1+|x|^2)^{1/2},$$

the norm of the Sobolev space $H^1_k(\mathbb{R}^2)$ is defined by

$$\|f\|^2_{H^1_k} := \sum_{|\alpha|\le 1} \|\langle x\rangle^k \partial^\alpha f\|^2_{L^2}.$$

and, for any $\sigma \in \mathbb{R}_+$, the homogeneous seminorm $\|\cdot\|_{\dot{H}^\sigma}$ and the associated homogeneous Sobolev space $\dot{H}^\sigma$ are defined by

$$\dot{H}^\sigma = \{f \in \mathcal{S}'(\mathbb{R}^2) : \widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^2),\ \|f\|_{\dot{H}^\sigma} := \||\xi|^\sigma \widehat{f}\|_{L^2} < \infty\}.$$

We recall that, on the plane, $\dot{H}^\sigma$ is a Hilbert space for $\sigma < 1$ (see Proposition 1.34 in [2]). We also denote by H^{-1}_k the duality space of H^1_k for the scalar product $\langle\cdot,\cdot\rangle_{L^2_k}$, namely,

$$\|\phi\|_{H^{-1}_k} = \sup_{\|f\|_{H^1_k}\le 1} \langle \phi, f\rangle_{L^2_k} = \sup_{\|g\|_{H^1}\le 1} \langle \langle x\rangle^k \phi, g\rangle_{L^2} = \|\langle x\rangle^k \phi\|_{H^{-1}}, \tag{1.7}$$

so that we may identify $H^{-1}_k = \{F_0 + \operatorname{div} F_1 : F_i \in L^2_k\}$. For $k > 1$, so that $L^2_k \subset L^1$, we finally denote

$$L^2_{k,0} := \{f \in L^2_k : \langle\!\langle f \rangle\!\rangle = 0\}.$$

We may now state our main result.

Theorem 1.1. *Let us fix $\mu \in (0,\infty)$, $k > 3$ and $s \in (0,1)$. There are $\varepsilon_0, \eta_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and any initial data $(g_0, w_0) \in \mathcal{X}$ with $\langle\!\langle g_0 \rangle\!\rangle = 0$ and $\|(g_0,w_0)\|_{\mathcal{X}} \le \eta_0$, there exists a unique global solution $(g,w) \in L^\infty_t(\mathcal{X}) \cap L^2_t(\mathcal{Y})$ to (1.5) which satisfies*

$$\|(g,w)\|_{L^\infty_t(\mathcal{X})} + \|(g,w)\|_{L^2_t(\mathcal{Y})} \lesssim \|(g_0,w_0)\|_{\mathcal{X}}. \tag{1.8}$$

Moreover, for any $\lambda \in (0, \mu(1-s))$, we have the exponential decay estimate

$$\|(g(t),w(t))\|_{\mathcal{X}} \lesssim e^{-\lambda t}\|(g_0,w_0)\|_{\mathcal{X}} \quad \text{for all } t \ge 0. \tag{1.9}$$

This result improves Theorem 1.4 in [9] where similar estimates are established with the restriction that the initial datum is radially symmetric (and satisfies additional regularity and confinement conditions) and also improves [11, 23, 24], which deal with small initial data and an arbitrary time scale parameter $\varepsilon > 0$. It is worth emphasizing that because of the one-to-one mapping $\mu \mapsto \mathcal{M}_\varepsilon(\mu)$, the choice of a given drift parameter $\mu \in (0,\infty)$ and its associated steady state Q^μ_ε here is equivalent to the choice of a given subcritical mass in $(0, 8\pi)$ for the initial datum in [9,11,23,24]. It is also worth underlining that Theorem 1.1 implies that the corresponding solution (F,U) to the parabolic-parabolic Keller–Segel equation in standard variables (i.e., $\mu = 0$) satisfies

$$F(t,x) \sim \frac{1}{R(t)^2} Q\Big(\frac{x}{R(t)}\Big), \quad U(t,x) \sim P\Big(\frac{x}{R(t)}\Big) \quad \text{as } t \to \infty,$$

with $R(t) := (1+\mu t)^{1/2}$, and we refer to Section 1 of [9] for further discussions.

As said above, we shall always work with the unknown (g, w) and it is worth stressing why the corresponding evolution system is given by (1.5). We may indeed observe that if (g, v) satisfies (1.4), then the function

$$w := v - \kappa * g$$

straightforwardly satisfies

$$\begin{aligned}\partial_t w = {} & \frac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + \mu x \cdot \nabla \kappa * g \\ & - \nabla \kappa * [\nabla g + \mu x g - g \nabla P - Q \nabla \kappa * g - Q \nabla w] + \nabla \kappa * [g \nabla w + g \nabla \kappa * g].\end{aligned}$$

Using that

$$\begin{aligned} x \cdot \nabla \kappa * g - \nabla \kappa * (xg) &= -\frac{1}{2\pi} \int \frac{(x-y)}{|x-y|^2} \{x\, g(y) - y\, g(y)\}\, dy \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} g(y)\, dy = 0, \end{aligned}$$

because of the mass vanishing condition on g, the equation on w simplifies and thus (g, w) satisfies (1.5). Equivalently, defining the operator

$$\mathcal{L}(g, w) = (\mathcal{L}_1(g, w), \mathcal{L}_2(g, w))$$

by

$$\mathcal{L}_i(g, w) = \mathcal{L}_{i,1} g + \mathcal{L}_{i,2} w, \quad i = 1, 2,$$

with

$$\tag{1.10} \begin{cases} \mathcal{L}_{1,1} g = \Delta g + \operatorname{div}(\mu x g - g \nabla P - Q \nabla \kappa * g), \\ \mathcal{L}_{1,2} w = -\operatorname{div}(Q \nabla w), \\ \mathcal{L}_{2,1} g = g + \nabla \kappa * [g \nabla P + Q \nabla \kappa * g], \\ \mathcal{L}_{2,2} w = \frac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + \nabla \kappa * [Q \nabla w], \end{cases}$$

the system (1.5) on (g, w) rewrites as

$$\tag{1.11} \begin{cases} \partial_t (g, w) = \mathcal{L}(g, w) \\ \qquad\qquad + (-\operatorname{div}(g \nabla \kappa * g) - \operatorname{div}(g \nabla w), \nabla \kappa * [g \nabla w + g \nabla \kappa * g]), \\ (g, w)_{|t=0} = (g_0, w_0). \end{cases}$$

In Sections 2 and 3, we present some estimates on the family of steady states (Q, P) and some functional inequalities that will be useful throughout the paper. In Section 4, we establish the dissipativity of the operator $\mathcal{L}_{1,1}$ and next the exponential decay of the associated semigroup for small enough values of $\varepsilon > 0$, thanks to a perturbation argument and by taking advantage of the dissipativity of the limit operator for $\varepsilon = 0$ corresponding to the usual linearized parabolic-elliptic Keller–Segel operator. In Section 5, we prove in a more direct way the dissipativity of the operator $\mathcal{L}_{2,2}$. In Section 6, we deduce then the decay of the semigroup $S_{\mathcal{L}}$ associated to $\mathcal{L}$ by writing in a proper accurate enough semigroup way the two decay estimates of $S_{\mathcal{L}_{i,i}}$ and by showing that both out of the diagonal contributions $\mathcal{L}_{i,j}$, $i \neq j$, are small enough. The above two arguments significantly differ from those used in the proof of Theorem 1.4 in [9]. In Section 7, we finally present the proof of Theorem 1.1, which is based on a classical nonlinear stability trick.

In the sequel, for two functions S and T defined on $\mathbb{R}_+$, we define the convolution $S * T$ by

$$(S * T)(t) = \int_0^t S(t-s)T(s)\,\mathrm{d}s \quad \text{for all } t \geq 0,$$

so that in particular the Duhamel formula associated to an evolution equation

$$\partial_t g = \Lambda g + G, \quad g(0) = g_0,$$

writes

$$g = S_\Lambda g_0 + S_\Lambda * G.$$

Moreover, for $\lambda \in \mathbb{R}$, we denote $\mathbf{e}_\lambda : t \mapsto e^{\lambda t}$. We also write $A \simeq B$ if $A = cB$ for a numerical constant c and $A \lesssim B$ when $A \leq cB$ for a numerical constant $c > 0$ and $A, B \geq 0$.

2. Estimates over Q and P

We present some estimates on the steady states $Q = Q_\varepsilon^\mu$ and $P = P_\varepsilon^\mu$, that we recall satisfy (1.3), which will be useful in the next sections.

Proposition 2.1. *There exist $\varepsilon_0 > 0$ and $\alpha_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (\alpha_0, 1)$, the following hold*:

(1) (*Bounds over P*) *For all $x \in \mathbb{R}^2$,*

$$P^0(x) - \frac{\mu\alpha|x|^2}{2} < P(x) - \frac{\mu\alpha|x|^2}{2} < P^0(x) < P(x) < 0 \tag{2.1}$$

and

$$x \cdot \nabla P(x) - \mu\alpha|x|^2 < x \cdot \nabla P^0(x) < x \cdot \nabla P(x) < 0. \tag{2.2}$$

(2) (*Bounds over Q*) *For all $x \in \mathbb{R}^2$,*

$$Q^0(x)\, e^{-\mu|x|^2/2} < Q(x) < Q^0(x)\, e^{-\mu(1-\alpha)|x|^2/2}. \tag{2.3}$$

Proof of Proposition 2.1. In order to prove (2.1) and (2.2), we follow the same ideas as in the proof of Proposition 4.1 in [15], but including the necessary modifications to handle the terms depending on ε that appear for this new problem.

First notice that P^0 and P are radial functions solving the equations

$$\Delta P^0(x) = -Q^0(x) = -8e^{P^0(x)}, \qquad P^0(0) = 0,$$
$$\Delta P(x) + \mu\varepsilon x \cdot \nabla P(x) = -Q(x) = -8e^{P(x) - \mu|x|^2/2}, \qquad P(0) = 0.$$

In polar variables, these equations read as

$$(P^0)''(r) + \frac{1}{r}(P^0)'(r) = -8e^{P^0(r)}, \qquad P^0(0) = (P^0)'(0) = 0,$$

$$(P)''(r) + \Big(\frac{1}{r} + \mu\varepsilon r\Big)(P)'(r) = -8e^{P(r)-\mu|r|^2/2}, \quad P(0) = P'(0) = 0. \tag{2.4}$$

Solving these two equations, we get

$$P^0(r) = -8\int_0^r \frac{1}{\rho}\int_0^\rho \tau e^{P^0(\tau)}\, d\tau\, d\rho, \tag{2.5}$$

$$P(r) = -8\int_0^r \frac{e^{-\mu\varepsilon\rho^2/2}}{\rho}\int_0^\rho \tau e^{P(\tau)-\mu(1-\varepsilon)\tau^2/2}\, d\tau\, d\rho. \tag{2.6}$$

Plugging an expansion in powers of r up to order 4 for $P(r)$ in (2.4), the coefficients of such expansion can be computed, which gives

$$\begin{aligned} P(r) - \mu\alpha\frac{r^2}{2} &= -\Big(2 + \frac{\mu\alpha}{2}\Big)r^2 + \Big(1 + \frac{\mu}{4}(1+\varepsilon)\Big)r^4 + o(r^4),\\ P^0(r) &= -2r^2 + r^4 + o(r^4),\\ P(r) &= -2r^2 + \Big(1 + \frac{\mu}{4}(1+\varepsilon)\Big)r^4 + o(r^4), \end{aligned}$$

for any given $\alpha \in (0,1)$. This implies that there exists $r_0 = r_0(\alpha) > 0$ such that the following relation holds true:

$$P^0(r) - \frac{\mu\alpha|r|^2}{2} < P(r) - \frac{\mu\alpha|r|^2}{2} < P^0(r) < P(r) < 0 \quad \text{for all } r \in (0, r_0). \tag{2.7}$$

Set $\alpha = 1 - \varepsilon$ and assume now, by contradiction, that there exists $r_1 > 0$ such that

$$P^0(r_1) = P(r_1) \quad \text{and} \quad P(r) - \frac{\mu(1-\varepsilon)|r|^2}{2} < P^0(r) < P(r) \quad \text{for all } r \in (0, r_1).$$

Using (2.5) and (2.6), we get

$$\begin{aligned} 0 = P^0(r_1) - P(r_1) &= -8\int_0^{r_1}\frac{1}{\rho}\int_0^\rho \tau\big(e^{P^0(\tau)} - e^{P(\tau)-\mu(1-\varepsilon)\tau^2/2-\mu\varepsilon\rho^2/2}\big)\, d\tau\, d\rho\\ &\le -8\int_0^{r_1}\frac{1}{\rho}\int_0^\rho \tau\big(e^{P^0(\tau)} - e^{P(\tau)-\mu(1-\varepsilon)\tau^2/2}\big)\, d\tau\, d\rho < 0, \end{aligned}$$

the last strict inequality being due to the second inequality in (2.7). This is a contradiction and therefore $P_0(r) < P(r)$ for all $r > 0$.

On the other hand, suppose by contradiction again that there exist $\alpha \in (0,1)$ and $r_\alpha > 0$ such that

$$\Psi_\varepsilon^\mu(\alpha, r_\alpha) := P(r_\alpha) - \frac{\mu\alpha|r_\alpha|^2}{2} = P^0(r_\alpha)$$

and

$$P(r) - \frac{\mu\alpha|r|^2}{2} < P^0(r) < P(r) \quad \text{for all } r \in (0, r_\alpha).$$

Using (2.6), we have

$$\Psi^\mu_\varepsilon(\alpha,r_\alpha) = -8\int_0^{r_\alpha}\frac{1}{\rho}\int_0^\rho \tau e^{P(\tau)-\mu(1-\varepsilon)\tau^2/2-\mu\varepsilon\rho^2/2}\,d\tau\,d\rho - \frac{\mu\alpha|r_\alpha|^2}{2}$$
$$< -8\int_0^{r_\alpha}\frac{1}{\rho}\int_0^\rho \tau e^{P^0(\tau)-\mu(1-\varepsilon)\tau^2/2-\mu\varepsilon\rho^2/2}\,d\tau\,d\rho - \frac{\mu\alpha|r_\alpha|^2}{2} =: \tilde\Psi(\alpha,r_\alpha,\mu).$$

Notice that $\tilde\Psi(\alpha,r_\alpha,0) = P^0(r_\alpha)$. If we prove that there exist values of α such that $\partial_\mu\tilde\Psi(\alpha,r_\alpha,0)<0$, then, in a neighbourhood of $\mu=0$, we would have

$$\Psi^\mu_\varepsilon(\alpha,r_\alpha) < \tilde\Psi(\alpha,r_\alpha,\mu) < \tilde\Psi(\alpha,r_\alpha,0) = P^0(r_\alpha),$$

which would be a contradiction. Since there exist $\varepsilon_0>0$ and $\alpha_0>0$ such that for all $(\varepsilon,\alpha)\in[0,\varepsilon_0]\times[\alpha_0,1]$, the function

$$\partial_r\partial_\mu\tilde\Psi(\alpha,r,0) = \frac{4(1-\varepsilon)}{r}\int_0^r \tau^3 e^{P^0(\tau)}\,d\tau + 4\varepsilon r\int_0^r \tau e^{P^0(\tau)}\,d\tau - \alpha r$$
$$= \frac{2(1-\varepsilon)}{r}\Big(\ln(1+r^2)+\frac{1}{1+r^2}-1\Big) + 4\varepsilon r\Big(1-\frac{1}{1+r^2}\Big) - \alpha r$$

is less than 0 for all $r>0$, we deduce that

$$\partial_\mu\tilde\Psi(\alpha,r_\alpha,0) = \int_0^{r_\alpha}\partial_r\partial_\mu\tilde\Psi(\alpha,r,0)\,dr < 0,$$

which leads to the desired contradiction and finishes the proof of (2.1). We may establish (2.2) in a very similar way, but using the expressions for $x\cdot\nabla P^0$ and $x\cdot\nabla P$. We finally prove (2.3) by taking the exponential of the estimate (2.1). ■

Lemma 2.2. *There exist some constants $C_i>0$, $i=0,\dots,3$, $\varepsilon_0>0$ and $\vartheta\in(0,1)$, such that for any $\mu\in(0,\infty)$ and any $\varepsilon\in(0,\varepsilon_0]$,*

$$0\le Q(x)\le C_0\,e^{-\mu\vartheta|x|^2/2}\langle x\rangle^{-4}, \tag{2.8}$$

$$\sup_{x\in\mathbb{R}^2}\Big(\frac{1}{|x|}+\langle x\rangle\Big)|\nabla P(x)|\le C_1 \tag{2.9}$$

and

$$|\Delta P|\le C_2\mu\varepsilon + C_3\langle x\rangle^{-1}. \tag{2.10}$$

Proof of Lemma 2.2. Consider the values of $\varepsilon_0>0$ and $\alpha_0>0$ given in Proposition 2.1, so that from its proof, $\vartheta := 1-\alpha_0\in(0,1)$ is independent from μ and ε. Estimate (2.8) is then nothing but (2.3). Computing the explicit expression for ∇P gives

$$\nabla P = -e^{-\mu\varepsilon|x|^2/2}\frac{x}{|x|^2}\int_0^{|x|}Q(r)\,e^{\mu\varepsilon r^2/2}\,r\,dr,$$

and hence

$$|\nabla P|\le\frac{1}{|x|}\int_0^{|x|}Q(r)e^{\mu\varepsilon r^2/2}\,dr \le \frac{1}{|x|}\int_0^{|x|}\langle r\rangle^{-4}\,e^{-\mu(\vartheta-\varepsilon)r^2/2}\,r\,dr.$$

Taking ε small enough, we deduce

$$|\nabla P| \le \frac{1}{|x|}\int_0^{|x|} \langle r\rangle^{-4}\, r\, dr \le |x|\,\langle x\rangle^{-2},$$

which directly implies (2.9). Finally, writing

$$\Delta P = -\mu\varepsilon x\cdot\nabla P - Q,$$

we conclude to (2.10) thanks to (2.2) and (2.8). ∎

Lemma 2.3. *There exist some constants $\vartheta\in(0,1)$, $C_i>0$, $i=1,\dots,4$, such that for any $\mu\in(0,\infty)$, any $\varepsilon\in(0,\varepsilon_0]$ and any $x\in\mathbb{R}^2$,*

$$|\nabla P_\varepsilon^\mu - \nabla P_0^\mu| \le \sqrt{\mu\varepsilon}\, C_1, \tag{2.11}$$

$$|\Delta P_\varepsilon^\mu - \Delta P_0^\mu| \le \mu\varepsilon C_2, \tag{2.12}$$

$$|Q_\varepsilon^\mu - Q_0^\mu| \le \mu\varepsilon\, C_3\, e^{-\vartheta\mu|x|^2/2}, \tag{2.13}$$

$$|\nabla Q_\varepsilon^\mu - \nabla Q_0^\mu| \le \mu\varepsilon\, C_4\, e^{-\vartheta\mu|x|^2/2}. \tag{2.14}$$

Proof of Lemma 2.3. We recall that in radial variables, we have the expressions

$$P_0^\mu(r) = -8\int_0^r \frac{1}{\rho}\int_0^\rho e^{P_0^\mu-\mu\tau^2/2}\,\tau\, d\tau\, d\rho,$$

$$P_\varepsilon^\mu(r) = -8\int_0^r \frac{e^{-\mu\varepsilon\rho^2/2}}{\rho}\int_0^\rho e^{P_\varepsilon^\mu-\mu(1-\varepsilon)\tau^2/2}\,\tau\, d\tau\, d\rho,$$

which imply

$$\begin{aligned} P_\varepsilon^\mu - P_0^\mu &= \Big(P_\varepsilon^\mu - \int_0^r (P_\varepsilon^\mu)' e^{\mu\varepsilon\tau^2/2}\, d\tau\Big) + \Big(\int_0^r (P_\varepsilon^\mu)' e^{\mu\varepsilon\tau^2/2}\, d\tau - P_0^\mu\Big) \\ &= -\int_0^r \frac{e^{-\mu\varepsilon\rho^2/2}-1}{\rho}\int_0^\rho Q_\varepsilon^\mu(\tau)\, e^{\mu\varepsilon\tau^2/2}\,\tau\, d\tau\, d\rho \\ &\quad - 8\int_0^r \frac{1}{\rho}\int_0^\rho (e^{P_\varepsilon^\mu-\mu(1-\varepsilon)\tau^2/2} - e^{P_0^\mu-\mu\tau^2/2})\,\tau\, d\tau. \end{aligned}$$

Directly from Proposition 2.1, we know that

$$\int_0^\rho Q_\varepsilon^\mu(\tau)\, e^{\mu\varepsilon\tau^2/2}\,\tau\, d\tau \le \int_0^\rho Q^0(\tau)\,\tau\, d\tau \le 8\pi,$$

and the mean value theorem gives us that

$$e^{P_\varepsilon^\mu-\mu(1-\varepsilon)\tau^2/2} - e^{P_0^\mu-\mu\tau^2/2} = \Big(P_\varepsilon^\mu - P_0^\mu + \mu\varepsilon\,\frac{\tau^2}{2}\Big)\, e^{h(\tau)},$$

with $h(\tau)$ satisfying

$$h(\tau) \le \max\Big\{P_\varepsilon^\mu - \mu(1-\varepsilon)\,\frac{\tau^2}{2},\, P_0^\mu - \mu\,\frac{\tau^2}{2}\Big\} \le P^0 - \frac{\vartheta\mu\tau^2}{2}.$$

Thanks to Proposition 2.1, we deduce

$$8e^{h(\tau)} \le Q^0(\tau)\, e^{-\vartheta\mu\tau^2/2}.$$

Putting everything together, we get

$$\begin{aligned} |P^\mu_\varepsilon - P^\mu_0| &\le 8\pi \int_0^r \frac{1-e^{-\mu\varepsilon\rho^2/2}}{\rho}\, d\rho + \int_0^r \frac{1}{\rho} \int_0^\rho \left| P^\mu_\varepsilon - P^\mu_0 + \mu\varepsilon \frac{\tau^2}{2} \right| Q^0(\tau)\,\tau\, d\tau\, d\rho \\ &\le \mu\varepsilon K r^2 + \int_0^r \int_0^\rho |P^\mu_\varepsilon - P^\mu_0|\, Q^0(\tau)\, d\tau\, d\rho, \end{aligned}$$

where K is a constant independent of μ and ε. Integrating by parts the integral term, we get

$$|P^\mu_\varepsilon - P^\mu_0| \le \mu\varepsilon K r^2 + r \int_0^r |P^\mu_\varepsilon - P^\mu_0| Q^0(\tau)\, d\tau,$$

which thanks to Grönwall's Lemma gives

$$|P^\mu_\varepsilon - P^\mu_0| \le \mu\varepsilon C_0 |x|^2.$$

This estimate together with a similar manipulation on the gradients of P^μ_ε and P^μ_0 gives

$$|\nabla P^\mu_\varepsilon - \nabla P^\mu_0| \le \mu\varepsilon C_0 |x|,$$

which interpolated with (2.9) gives (2.11).

On the other hand, we have

$$\begin{aligned} |Q^\mu_\varepsilon - Q^\mu_0| = 8\,|e^{P^\mu_\varepsilon - \mu(1-\varepsilon)r^2/2} - e^{P^\mu_0 - \mu r^2/2}| &= \left| P^\mu_\varepsilon - P^\mu_0 + \mu\varepsilon \frac{r^2}{2} \right| 8e^{h(r)} \\ &\le \mu\varepsilon C_2 |r|^2\, Q^0(r)\, e^{-\vartheta\mu r^2/2}, \end{aligned}$$

which is nothing but (2.13). Repeating the same process for $\nabla(Q - Q_\mu)$ gives (2.14).

Finally, using the equations for P^μ_ε and P^μ_0, we have

$$\Delta(P^\mu_\varepsilon - P^\mu_0) = -(Q^\mu_\varepsilon - Q^\mu_0) - \mu\varepsilon x \cdot \nabla P^\mu_\varepsilon.$$

We conclude (2.12) from (2.13) for the first term and from (2.2) for the second. ■

3. Functional inequalities

We gather in this section some functional inequalities that we shall use through the paper. First, we provide some estimates over the solution for the Poisson problem in the spirit of Lemma B.2 in [9].

Lemma 3.1. *For any $\sigma \in (0,1)$,*

$$\|\nabla\kappa * g\|_{\dot H^\sigma} \lesssim \|g\|_{L^1\cap L^2} \quad \textit{for all } g \in L^1 \cap L^2. \tag{3.1}$$

Moreover, for any $k > 2$,

$$\|\nabla\kappa * g\|_{L^2} \lesssim \|g\|_{L^2_k} \quad \textit{for all } g \in L^2_{k,0}. \tag{3.2}$$

Proof of Lemma 3.1. From the definition $-\Delta\kappa * g = g$, we have

$$|\xi|^2 \mathcal{F}(\kappa * g)(\xi) = \hat{g},$$

or, equivalently,

$$|\xi|^\sigma |\widehat{(\nabla\kappa * g)}(\xi)| = |\xi|^{\sigma-1} |\widehat{g}(\xi)| \quad \text{for any } \sigma \in [0,1].$$

For $\sigma \in (0,1)$, we have

$$\begin{aligned}\|\nabla\kappa * g\|_{\dot{H}^\sigma}^2 &= \int_{|\xi|\le 1} |\xi|^{2\sigma-2}|\widehat{g}|^2 + \int_{|\xi|\ge 1} |\xi|^{2\sigma-2}|\widehat{g}|^2 \\ &\le \|\widehat{g}\|_{L^\infty}^2 \int_{|\xi|\le 1} |\xi|^{2\sigma-2} + \int |\widehat{g}|^2 \lesssim \|g\|_{L^1}^2 + \|g\|_{L^2}^2,\end{aligned}$$

where we have used the definition of the norm $\|\cdot\|_{\dot{H}^\sigma}$ in the first line and some elementary Fourier estimates (including the Plancherel identity) in the last inequality. This completes the proof of (3.1).

Similarly, we have

$$\|\nabla\kappa * g\|_{L^2}^2 = \int \mathbf{1}_{|\xi|\le 1} \frac{|\widehat{g}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi + \int \mathbf{1}_{|\xi|>1} \frac{|\widehat{g}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi =: \mathrm{I}_1 + \mathrm{I}_2.$$

For the second term, we write

$$\mathrm{I}_2 \le \int |\widehat{g}(\xi)|^2\,\mathrm{d}\xi = \|g\|_{L^2}^2.$$

For the first term, using that $\widehat{g}(0) = 0$ because $\langle\!\langle g \rangle\!\rangle = 0$, we write

$$\widehat{g}(\xi) = \xi \cdot \int_0^1 D_\xi \widehat{g}(\theta\xi)\,\mathrm{d}\theta,$$

and we thus obtain

$$\mathrm{I}_1 \le \Big(\sup_{|\xi|\le 1} |D_\xi \widehat{g}(\xi)|^2\Big) \int \mathbf{1}_{|\xi|\le 1}\,\mathrm{d}\xi \lesssim \|\widehat{xg}\|_{L^\infty}^2 \lesssim \|xg\|_{L^1}^2 \lesssim \|g\|_{L^2_k}^2,$$

by using some classical and elementary Fourier identity and estimate as well as the continuous embedding $L^2_k(\mathbb{R}^2) \subset L^1_1(\mathbb{R}^2)$. Altogether, we have established (3.2). ■

We give now a possible variant of (3.2).

Lemma 3.2. *For $p > 2$, $2 \le q \le p$ and $k > 2 - 2/q$, we have*

$$\|\nabla\kappa * g\|_{L^p} \lesssim \|g\|_{L^q_k} \quad \textit{for all } g \in L^q_k.$$

We emphasize that the same estimate is not true for $p = 2$. From the proof of (3.2), we indeed have

$$\|\nabla\kappa * g\|_{L^2} = \||\xi|^{-1}\widehat{g}\|_{L^2},$$

and the right-hand side term is infinite if $\widehat{g}(0) \ne 0$ or, equivalently, if the mass of g does not vanish, regardless of its L^q_k norm.

Proof of Lemma 3.2. We split $|\nabla\kappa| := K_1 + K_2$, with

$$K_1 := \frac{1}{|x|}\mathbf{1}_{|x|\le 1} \in L^{r_1} \quad \text{for all } r_1 < 2, \qquad K_2 := \frac{1}{|x|}\mathbf{1}_{|x|\ge 1} \in L^{r_2} \quad \text{for all } r_2 > 2,$$

that we use with $r_1 := (1 + 1/p - 1/q)^{-1} \in (1,2)$ and $r_2 := p > 2$. We then have

$$\begin{aligned} \|\nabla\kappa * f\|_{L^p} &\le \|K_1 * f\|_{L^p} + \|K_2 * f\|_{L^p} \\ &\le \|K_1\|_{L^{r_1}}\|f\|_{L^q} + \|K_2\|_{L^p}\|f\|_{L^1} \\ &\lesssim \|f\|_{L^q_k}, \end{aligned}$$

where we have used the convolution embeddings $L^{r_1} * L^q \subset L^p$ and $L^1 * L^p \subset L^p$ in the second line as well as the Hölder inequality in the last line in order to prove $L^q_k \subset L^1$. ∎

We finally recall two standard interpolation inequalities, namely, a particular case of the Gagliardo–Nirenberg interpolation theorem in dimension 2 (see, for instance, p. 125 of [25]) and an interpolation between homogeneous Sobolev spaces (see Proposition 1.32 in [2]).

Lemma 3.3. (1) *The following Ladyzhenskaya's inequality holds*:

$$\|f\|_{L^4} \lesssim \|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \quad \textit{for all } f \in H^1. \tag{3.3}$$

(2) *Consider real numbers $s_0 < s < s_1$; then*

$$\|f\|_{\dot H^s} \lesssim \|f\|_{\dot H^{s_0}}^{1-\theta}\|f\|_{\dot H^{s_1}}^{\theta} \quad \textit{for all } f \in \dot H^{s_0} \cap \dot H^{s_1}, \tag{3.4}$$

where $\theta = (s - s_0)/(s_1 - s_0)$.

4. Estimates for $\mathscr{L}_{1,1}$

In this section, we establish some dissipativity estimates and related semigroup decay estimates successively on the operators $\mathscr{L}_{1,1}$ and related operators.

4.1. Dissipativity estimates related to $\mathscr{L}_{1,1}$

We recall that this one is defined by

$$\mathscr{L}_{1,1}g := \Delta g + \operatorname{div}(\mu x g - g\nabla P - Q\nabla\kappa * g).$$

We start with a first fundamental dissipativity estimate.

Lemma 4.1. *For any $k > 3$, there exist constants $\varepsilon_0 > 0$ (small enough) and $C_0, \varrho_0 > 0$ (large enough) such that*

$$\langle \mathscr{L}_{1,1}g, g\rangle_{L^2_k} \le -\mu(k-2)\|g\|^2_{L^2_k} - \frac{1}{2}\|\nabla g\|^2_{L^2_k} + C_0\|g\|^2_{L^2(B_{\varrho_0})},$$

for any $\varepsilon \in (0, \varepsilon_0)$ and $g \in H^1_k$.

Proof of Lemma 4.1. We briefly repeat the proof of Lemma 4.4 in [9]. We compute

$$\begin{aligned}\langle \mathcal{L}_{1,1} g, g\rangle_{L^2_k} &= \int \Delta g g \langle x\rangle^{2k} + \int \operatorname{div}(\mu x g) g \langle x\rangle^{2k} \\ &\quad - \int \operatorname{div}(g \nabla P) g \langle x\rangle^{2k} - \int \operatorname{div}(Q \nabla \kappa * g) g \langle x\rangle^{2k} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,\end{aligned}$$

and estimate each term separately. For the first two terms, we have

$$\mathrm{I}_1 + \mathrm{I}_2 = -\int |\nabla g|^2 \langle x\rangle^{2k} + \int \psi_1 g^2 \langle x\rangle^{2k},$$

where

$$\begin{aligned}\psi_1 &= \frac{|\nabla \langle x\rangle^k|^2}{\langle x\rangle^{2k}} + \frac{\Delta \langle x\rangle^k}{\langle x\rangle^k} + \mu - \mu x \cdot \frac{\nabla \langle x\rangle^k}{\langle x\rangle^k} \\ &= k(2k+\mu)\langle x\rangle^{-2} - \mu(k-1) - 2k\langle x\rangle^{-4}.\end{aligned}$$

Moreover, for the third term, we compute

$$\begin{aligned}\mathrm{I}_3 &= \int g \nabla P \cdot \nabla g \langle x\rangle^{2k} + 2\int g^2 \langle x\rangle^{2k} \nabla P \cdot \frac{\nabla \langle x\rangle^k}{\langle x\rangle^k} \\ &= -\frac{1}{2}\int \Delta P g^2 \langle x\rangle^{2k} + \int \nabla P \cdot \frac{\nabla \langle x\rangle^k}{\langle x\rangle^k} g^2 \langle x\rangle^{2k}.\end{aligned}$$

Thanks to the uniform estimates (2.9) and (2.10) on P, we observe that

$$\left|\nabla P \cdot \frac{\nabla \langle x\rangle^k}{\langle x\rangle^k}\right| \le \|x \cdot \nabla P\|_{L^\infty} \langle x\rangle^{-1} \le C_1 \langle x\rangle^{-1}$$

and

$$|\Delta P| \le C_2 \mu\varepsilon + C_3 \langle x\rangle^{-1}$$

for some constants $C_i > 0$, which imply

$$\mathrm{I}_3 \le \frac{\varepsilon\mu C_2}{2} \|g\|^2_{L^2_k} + \left(C_1 + \frac{C_3}{2}\right) \|\langle x\rangle^{-1/2} g\|^2_{L^2_k}.$$

For the last term, we write

$$\mathrm{I}_4 = \int Q(\nabla\kappa * g)\nabla g \langle x\rangle^{2k} + 2\int Q(\nabla\kappa * g)\frac{\nabla \langle x\rangle^k}{\langle x\rangle^k} g \langle x\rangle^{2k}.$$

Since

$$\|Q\langle x\rangle^{2k}\|_{L^\infty} \le C_4 \quad \text{and} \quad \|Q\langle x\rangle^k \nabla \langle x\rangle^k\|_{L^\infty} \le C_4,$$

thanks to estimate (2.8), we obtain

$$\begin{aligned}\mathrm{I}_4 &\le C_4 \|\nabla\kappa * g\|_{L^2} \|\nabla g\|_{L^2} + 2C_4 \|\nabla\kappa * g\|_{L^2} \|g\|_{L^2} \\ &\le C_4' \|\nabla\kappa * g\|^2_{L^2} + \frac{1}{2}\|\nabla g\|^2_{L^2} + C_4'\|g\|^2_{L^2} \\ &\le C_4'' \|\langle x\rangle^{-1} g\|^2_{L^2_k} + \frac{1}{2}\|\nabla g\|^2_{L^2},\end{aligned}$$

where we have used Lemma 3.1 and Young's inequality. Gathering the previous estimates, we get

$$\langle \mathcal{L}_{1,1} g, g\rangle_{L^2_k} \le -\frac{1}{2}\int |\nabla g|^2 \langle x\rangle^{2k} + \int \bar\psi_1 g^2 \langle x\rangle^{2k},$$

with

$$\begin{aligned}\bar\psi_1 &= -\mu\Big(k-1-\frac{\varepsilon C_2}{2}\Big) + \Big(C_4'' + C_1 + \frac{C_3}{2}\Big)\langle x\rangle^{-1} + k(2k+\mu)\langle x\rangle^{-2} - k^2\langle x\rangle^{-4} \\ &\le -\mu\Big(k-1-\frac{\varepsilon C_2}{2}\Big) + C_5\langle x\rangle^{-1}.\end{aligned}$$

We remark that, for any $\varrho_0 \ge 1$, we have

$$\langle x\rangle^{-1}\langle x\rangle^{2k} \le \varrho_0^{2k-1}\mathbf{1}_{\langle x\rangle\le\varrho_0} + \frac{1}{\varrho_0}\langle x\rangle^{2k},$$

thus we obtain

$$\langle \mathcal{L}_{1,1} g, g\rangle_{L^2_k} \le -\frac{1}{2}\|\nabla g\|^2_{L^2_k} - \mu\Big(k-1-\frac{\varepsilon C_2}{2} - \frac{C_5}{\mu\varrho_0}\Big)\|g\|^2_{L^2_k} + C_0\|g\|^2_{L^2(B_{\varrho_0})},$$

where $C_0 = C_5\varrho_0^{2k-1}$. We therefore choose $\varepsilon_0 > 0$ small enough such that $\varepsilon_0 C_2 \le 1$ and $\varrho_0 \ge 1$ large enough such that $\frac{C_5}{\mu\varrho_0} \le 1/2$, which concludes the proof. ■

4.2. Splitting of the operator $\mathcal{L}_{1,1}$

We introduce the splitting

$$\mathcal{L}_{1,1} = \mathcal{A} + \mathcal{B}_\varepsilon, \quad \mathcal{A} := M\chi_\varrho,$$

with $\chi_\varrho(x) := \chi(x/\varrho)$, $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$, and constants $M, \varrho > 0$. We immediately deduce from Lemma 4.1 that $\mathcal{B}_\varepsilon$ is dissipative, more precisely, the following holds.

Corollary 4.2. *For any $k > 3$, any $\varepsilon \in (0, \varepsilon_0)$ and any constants $M \ge C_0$ and $\varrho \ge \varrho_0$,*

$$\langle \mathcal{B}_\varepsilon g, g\rangle_{L^2_k} \le -\mu(k-2)\|g\|^2_{L^2_k} - \frac{1}{2}\|\nabla g\|^2_{L^2_k} \le -\lambda\|g\|^2_{L^2_k} - \sigma\|g\|^2_{H^1_k} \tag{4.1}$$

for any $0 \le \lambda < \mu(k-2)$, with $\sigma = \min(1/2, \mu - \lambda)$, and where $\varepsilon_0, C_0, \varrho_0 > 0$ are taken from Lemma 4.1.

Remark 4.3. We shall fix hereafter the parameters $M \ge C_0$ and $\varrho \ge \varrho_0$ in the definition of $\mathcal{B}_\varepsilon$ such that Corollary 4.2 holds.

In order to work at the level of the semigroup, we reformulate (4.1) in the following way, recalling that we define H^{-1}_k through the dual norm (1.7).

Lemma 4.4. *For any $k > 3$, $\varepsilon \in (0, \varepsilon_0)$, $M \ge C_0$ and $\varrho \ge \varrho_0$, the following hold:*

(1) *For all $0 \le \lambda < \mu(k-2)$ and all $g \in L^2_k$, we have*

$$\|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot) g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot) g\|_{L^2_t H^1_k} \lesssim \|g\|_{L^2_k}.$$

(2) *For all* $0 \le \lambda < \mu(k-2)$ *and all* $\mathbf{e}_\lambda R \in L^2_t H^{-1}_k$, *we have*

$$\|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.$$

Proof of Lemma 4.4. Let $0 \le \lambda < \mu(k-2)$. For $g \in L^2_k$, we first consider

$$f := \mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot) g,$$

the solution to the evolution equation

$$\partial_t f = \mathcal{B}_\varepsilon f + \lambda f, \quad f(0) = g.$$

Because of (4.1), we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_k} = \langle \mathcal{B}_\varepsilon f, f\rangle_{L^2_k} + \lambda \|f\|^2_{L^2_k} \le -\sigma \|f\|^2_{H^1_k},$$

from which we deduce (1) thanks to Grönwall's lemma.

For R such that $\mathbf{e}_\lambda R \in L^2_t H^{-1}_k$, we next consider

$$f := \mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R),$$

the solution to the evolution equation

$$\partial_t f = \mathcal{B}_\varepsilon f + \lambda f + \mathbf{e}_\lambda R, \quad f(0) = 0.$$

Because of (4.1) and Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_k} &= \langle \mathcal{B}_\varepsilon f, f\rangle_{L^2_k} + \lambda \|f\|^2_{L^2_k} + \langle \mathbf{e}_\lambda R, f\rangle_{L^2_k} \\ &\le -\sigma \|f\|^2_{H^1_k} + \|\mathbf{e}_\lambda R\|_{H^{-1}_k} \|f\|_{H^1_k} \le -\frac{\sigma}{2}\|f\|^2_{H^1_k} + C\|\mathbf{e}_\lambda R\|^2_{H^{-1}_k},\end{aligned}$$

for some constant $C = C(\mu, \lambda) > 0$. We deduce (2) thanks to Grönwall's lemma again. ■

4.3. Spectral analysis of $\mathcal{L}_{1,1}$

We deduce a nice localization of the spectrum of $\mathcal{L}_{1,1}$ from the previous estimates and a perturbation argument. In order to keep track of the $\varepsilon \ge 0$ dependence, let us denote

$$\Lambda_\varepsilon := \mathcal{L}_{1,1},$$

so that Λ_0 is the linearized operator of the parabolic-elliptic Keller–Segel equation, which is given by

$$\Lambda_0 g = \Delta g + \operatorname{div}(\mu x g - g \nabla P_0 - Q_0 \nabla \kappa * g),$$

where (Q_0, P_0) is a solution to (1.3) with $\varepsilon = 0$. From [8, 12], we know that for $k > 3$ and $0 < \lambda < \mu$, there exists a constant $C = C(\lambda, \mu, k) \ge 1$ such that

$$\|S_{\Lambda_0}(t) f\|_{L^2_k} \le C e^{-\lambda t} \|f\|_{L^2_k} \quad \text{for all } f \in L^2_{k,0},$$

and the spectrum satisfies

$$\Sigma(\Lambda_0) \cap \Delta_{-\mu} = \{0\}, \tag{4.2}$$

where $\Delta_{-\mu} := \{z \in \mathbb{C} : \operatorname{Re} z > -\mu\}$.

By a perturbation argument similar to the one used in [19] (see also [17, 29]), we are able to obtain a similar picture for the operator $\mathcal{L}_{1,1} = \Lambda_\varepsilon$.

Proposition 4.5. *Let $k > 3$. For any $0 < \lambda < \mu$, there exists $\varepsilon_* > 0$ small enough such that the operator $\mathcal{L}_{1,1}$ on L^2_k satisfies*

$$\Sigma(\mathcal{L}_{1,1}) \cap \Delta_{-\mu} = \{0\} \quad \textit{for all } \varepsilon \in (0, \varepsilon_*).$$

Furthermore, 0 belongs to the discrete spectrum of $\mathcal{L}_{1,1}$, and denoting by Π the projector on the associated eigenspace, we have $\Pi f = 0$ if and only if $\langle\langle f \rangle\rangle = 0$.

Proof of Proposition 4.5. We split the proof into several steps.

Step 1. We claim that

$$\mathcal{U}_\varepsilon(z) := \mathcal{R}_{\mathcal{B}_\varepsilon}(z) - \mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$$

is uniformly bounded in $\mathscr{B}(L^2_k)$ and $\mathscr{B}(H^{-1}_k, H^1_k)$ for any $z \in \Omega := \Delta_{-\lambda} \setminus B(0, r/2)$, any $\varepsilon \geq 0$ and $0 < r < \lambda < \mu$. On the one hand, $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathscr{B}(L^2_k)$ is just an immediate consequence of the growth estimate on $S_{\mathcal{B}_\varepsilon}$ established in Lemma 4.4 (1). For proving $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathscr{B}(H^{-1}_k, H^1_k)$, we consider first $g \in L^2_k$, $z \in \Delta_{-\mu}$ and we define

$$f := \mathcal{R}_{\mathcal{B}_\varepsilon}(z)g,$$

so that $(z - \mathcal{B}_\varepsilon)f = g$. Using (4.1) and the fact that $\mu(k-2) \geq \mu$, we deduce

$$\frac{1}{2}\|\nabla f\|^2_{L^2_k} + (\operatorname{Re} z + \mu)\|f\|^2_{L^2_k} \leq \langle (z - \mathcal{B}_\varepsilon)f, f\rangle_{L^2_k} = \langle f, g\rangle_{L^2_k} \leq \|f\|_{H^1_k}\|g\|_{H^{-1}_k},$$

and thus

$$\|\nabla f\|_{L^2_k} \leq \max(2, \mu^{-1})\|g\|_{H^{-1}_k}.$$

By a density argument, the same holds for any $g \in H^{-1}_k$. From (4.2), we also have that $\mathcal{R}_{\Lambda_0}(z) \in \mathscr{B}(L^2_k)$ is uniformly bounded in $\mathscr{B}(L^2_k)$ for any $z \in \Omega := \Delta_{-\lambda} \setminus B(0, r/2)$. Moreover, the proof of the bound in $\mathscr{B}(L^2_k, H^1_k)$ is exactly the same as for $\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$. Indeed, arguing as in Lemma 4.1, we first obtain

$$\langle \Lambda_0 f, f\rangle_{L^2_k} \leq -\mu\|f\|^2_{L^2_k} - \frac{1}{2}\|\nabla f\|^2_{L^2_k} + C_0\|f\|^2_{L^2(B_{\varrho_0})}.$$

Defining $f := \mathcal{R}_{\Lambda_0}(z)g$, we deduce

$$\begin{aligned}(\operatorname{Re} z + \mu)\|f\|^2_{L^2_k} + \frac{1}{2}\|\nabla f\|^2_{L^2_k} - C_0\|f\|^2_{L^2(B_{\varrho_0})} &\leq \langle (z - \Lambda_0)f, f\rangle_{L^2_k} \\ &= \langle g, f\rangle_{L^2_k} \leq C\|g\|^2_{L^2_k}.\end{aligned}$$

Step 2. We claim that the family of operators (Λ_ε) converges in the sense

$$\|\Lambda_\varepsilon - \Lambda_0\|_{\mathcal{B}(H^1_k, L^2_k)} \le \eta_1(\varepsilon) \to 0.$$

We may indeed write

$$\begin{aligned}(\Lambda_\varepsilon - \Lambda_0)g = (\mathcal{B}_\varepsilon - \mathcal{B}_0)g &= -\operatorname{div}(g\nabla(P_\varepsilon - P_0)) - \operatorname{div}((Q_\varepsilon - Q_0)\nabla\kappa * g)\\ &= -\nabla g \cdot \nabla(P_\varepsilon - P_0) + g\Delta(P_\varepsilon - P_0)\\ &\quad + \nabla(Q_\varepsilon - Q_0)\cdot\nabla\kappa * g - (Q_\varepsilon - Q_0)g,\end{aligned}$$

so that

$$\begin{aligned}\|(\Lambda_\varepsilon - \Lambda_0)g\|_{L^2_k} \le{}& \|\nabla(P_\varepsilon - P_0)\|_{L^\infty}\|\nabla g\|_{L^2_k} + \|\Delta(P_\varepsilon - P_0)\|_{L^\infty}\|g\|_{L^2_k}\\ &+ \|\langle x\rangle^k \nabla(Q_\varepsilon - Q_0)\|_{L^\infty}\|g\|_{L^2_{1+0}} + \|Q_\varepsilon - Q_0\|_{L^\infty}\|g\|_{L^2_k}.\end{aligned}$$

We immediately conclude since we are able to prove (see Lemma 2.3)

$$\nabla(P_\varepsilon - P_0) \to 0, \quad \Delta(P_\varepsilon - P_0) \to 0, \quad \langle x\rangle^k\nabla(Q_\varepsilon - Q_0) \to 0, \quad Q_\varepsilon - Q_0 \to 0$$

uniformly in $L^\infty(\mathbb{R}^2)$.

Step 3. We claim that $\Sigma(\Lambda_\varepsilon) \cap \Delta_a \subset B(0, r/2)$ for any $\varepsilon \in (0, \varepsilon_0)$, choosing $\varepsilon_0 > 0$ small enough. On the one hand, we write the two resolvent equations

$$\mathcal{R}_{\Lambda_\varepsilon} = \mathcal{R}_{\mathcal{B}_\varepsilon} - \mathcal{R}_{\Lambda_\varepsilon}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}, \quad \mathcal{R}_{\Lambda_\varepsilon} = \mathcal{R}_{\Lambda_0} - \mathcal{R}_{\Lambda_\varepsilon}(\Lambda_\varepsilon - \Lambda_0)\mathcal{R}_{\Lambda_0},$$

from which we deduce

$$\mathcal{R}_{\Lambda_\varepsilon} = \mathcal{R}_{\mathcal{B}_\varepsilon} - \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon} + \mathcal{R}_{\Lambda_\varepsilon}(\Lambda_\varepsilon - \Lambda_0)\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}$$

or, equivalently,

$$\mathcal{R}_{\Lambda_\varepsilon}(I + \mathcal{K}_\varepsilon) = \mathcal{U}_\varepsilon,$$

with

$$\mathcal{K}_\varepsilon := (\Lambda_0 - \Lambda_\varepsilon)\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}.$$

On the other hand, from step 1, we have $\mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$ is bounded in $\mathcal{B}(L^2_k, H^1_k)$ uniformly in $z \in \Omega := \Delta_{-\lambda} \setminus B(0, r/2)$ and $\Lambda_0 - \Lambda_\varepsilon$ is small in $\mathcal{B}(H^1_k, L^2_k)$ for $\varepsilon > 0$ small, so that both estimates together imply

$$\sup_{z \in \Delta_{-\lambda}\setminus B(0,r/2)} \|\mathcal{K}_\varepsilon(z)\|_{\mathcal{B}(L^2_k)} < 1,$$

for any $0 < r < \lambda < \mu$ and $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(r, \lambda) > 0$ small enough. This implies that $I + \mathcal{K}_\varepsilon$ is invertible on $\Omega := \Delta_{-\lambda} \setminus B(0, r/2)$, so that

$$\mathcal{R}_{\Lambda_\varepsilon} = \mathcal{U}_\varepsilon(I + \mathcal{K}_\varepsilon)^{-1}$$

is bounded on Ω, which proves our claim.

Step 4. We now define

$$\Pi_\varepsilon := \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_\varepsilon}(z)\,dz, \quad \Gamma := \{z \in \mathbb{C} : |z| = r\},$$

the Dunford projector on the eigenspace associated to eigenvalues of Λ_ε which belong to the ball $B(0,r)$. We write

$$\begin{aligned}\Pi_\varepsilon &= \frac{i}{2\pi}\int_\Gamma \mathcal{U}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz\\ &= \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{B}_\varepsilon}\{I-\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\}\,dz - \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\varepsilon}\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz\\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{B}_\varepsilon}\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz - \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\varepsilon}\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz\end{aligned}$$

and

$$\begin{aligned}\Pi_0 &= \frac{i}{2\pi}\int_\Gamma \{\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\}\,dz\\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\{(I+\mathcal{K}_\varepsilon)^{-1} + \mathcal{K}_\varepsilon(I+\mathcal{K}_\varepsilon)^{-1}\}\,dz.\end{aligned}$$

We deduce

$$\begin{aligned}\Pi_\varepsilon - \Pi_0 &= \frac{i}{2\pi}\int_\Gamma (\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\mathcal{B}_\varepsilon})\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz\\ &\quad + \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\{\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\mathcal{B}_\varepsilon}\}(I+\mathcal{K}_\varepsilon)^{-1}\,dz\\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{U}_\varepsilon\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz\\ &\quad + \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\{\mathcal{B}_0 - \mathcal{B}_\varepsilon\}\mathcal{R}_{\mathcal{B}_\varepsilon}(I+\mathcal{K}_\varepsilon)^{-1}\,dz.\end{aligned}$$

We conclude that

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathscr{B}(L^2_k)} = \mathcal{O}(\varepsilon) < 1$$

for $\varepsilon > 0$ small enough, by taking advantage of the estimates established in Steps 1 and 2. By classical operator theory (see, for instance, the arguments presented in [17] in order to prove (4.43) in Chapter 1 of [17]), one deduces that $\dim \Pi_\varepsilon = \dim \Pi_0 = 1$.

Step 5. In the Banach space $(L^2_k)' = L^2_{-k}$, the dual operator Λ^*_ε is defined by

$$\Lambda^*_\varepsilon \psi := \Delta\psi + (\nabla P - \mu x)\cdot\nabla\psi + \kappa * (\mathrm{div}(Q\nabla\psi)).$$

At first glance, we have $1 \in L^2_{-k}$ and $\Lambda^*_\varepsilon 1 = 0$, so that $0 \in \Sigma(\Lambda^*_\varepsilon) = \Sigma(\Lambda_\varepsilon)$, and 0 is the only spectral value of Λ_ε in the ball $B(0,r)$. Denoting by $G_0 \in L^2_k$ the first eigenvector of Λ_0 such that $\langle\!\langle G_0\rangle\!\rangle = 1$, we have

$$|\langle\!\langle \Pi_\varepsilon G_0\rangle\!\rangle - 1| = |\langle(\Pi_\varepsilon - \Pi_0)G_0, 1\rangle| \le \|(\Pi_\varepsilon - \Pi_0)G_0\|_{L^2_k}\|1\|_{L^2_{-k}} \lesssim \|\Pi_\varepsilon - \Pi_0\| < 1.$$

We may thus define $G_\varepsilon := \Pi_\varepsilon G_0 / \langle\!\langle \Pi_\varepsilon G_0 \rangle\!\rangle$, which satisfies $G_\varepsilon \in L^2_k$, $\Lambda_\varepsilon G_\varepsilon = 0$ and $\langle\!\langle G_\varepsilon \rangle\!\rangle = 1$. We finally observe that we may also define the first eigenprojector by the more explicit formula

$$\Pi_\varepsilon f = G_\varepsilon \langle\!\langle f \rangle\!\rangle,$$

so that, in particular, $\Pi_\varepsilon f = 0$ if and only if $\langle\!\langle f \rangle\!\rangle = 0$. ∎

4.4. Semigroup decay estimates for $\mathcal{L}_{1,1}$

We are now able to deduce a nice semigroup decay estimate on $S_{\mathcal{L}_{1,1}}$ from the previous estimates on the resolvent.

Proposition 4.6. *With the notation of Proposition 4.5, for $k > 3$, all $0 < \lambda < \mu$ and all $\varepsilon \in (0, \varepsilon_*)$, we have, for any $g \in L^2_{k,0}$,*

$$\|S_{\mathcal{L}_{1,1}}(t) g\|_{L^2_k} \lesssim e^{-\lambda t} \|g\|_{L^2_k}.$$

Proof of Proposition 4.6. It is a consequence of Proposition 4.5 and of the splitting structure of the operator Λ_ε. More precisely, we may, for instance, apply the quantitative mapping theorem, Theorem 2.1 in [20], where it is worth emphasizing that $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) : L^2_k \to H^1_k \subset D(\Lambda_\varepsilon^{1/2})$ with uniform bound in $z \in \Delta_{-\lambda}$, which is a strong enough information in order to establish (2.23) of [20] without checking (H2) of [20], and to obtain

$$\|(I - \Pi) S_{\mathcal{L}_{1,1}}(t) g\|_{L^2_k} \lesssim e^{-\lambda t} \|g\|_{L^2_k} \quad \text{for all } g \in L^2_k.$$

Alternatively, one can use the Gearhart–Prüss–Greiner theorem [1, 14, 27] in order to get the same conclusion. ∎

Thanks to the previous estimate for $S_{\mathcal{L}_{1,1}}$ and the estimates for $S_{\mathcal{B}_\varepsilon}$ in Lemma 4.4, we are able to deduce semigroup estimates for $S_{\mathcal{L}_{1,1}}$ (in Propositions 4.7 below) similar to those satisfied by $S_{\mathcal{B}_\varepsilon}$.

We start by observing that, thanks to Duhamel's formula, we have

$$S_{\mathcal{L}_{1,1}} = S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}} \quad \text{and} \quad S_{\mathcal{L}_{1,1}} = S_{\mathcal{B}_\varepsilon} + S_{\mathcal{L}_{1,1}} * \mathcal{A} S_{\mathcal{B}_\varepsilon}. \tag{4.3}$$

Recalling the definition of the projector Π introduced in Proposition 4.5 and denoting $\Pi^\perp := I - \Pi$, we have

$$\begin{cases} S_{\mathcal{L}_{1,1}} \Pi^\perp = S_{\mathcal{B}_\varepsilon} \Pi^\perp + (S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}} \Pi^\perp), \\ S_{\mathcal{L}_{1,1}} \Pi^\perp = \Pi^\perp S_{\mathcal{B}_\varepsilon} + (S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon}). \end{cases} \tag{4.4}$$

Using that $S_{\mathcal{L}_{1,1}} \Pi^\perp = \Pi^\perp S_{\mathcal{L}_{1,1}}$ and iterating the formulas yields

$$S_{\mathcal{L}_{1,1}} \Pi^\perp = S_{\mathcal{B}_\varepsilon} \Pi^\perp + S_{\mathcal{B}_\varepsilon}\mathcal{A} * \Pi^\perp S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon}$$

and

$$S_{\mathcal{L}_{1,1}} \Pi^\perp = \Pi^\perp S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon}. \tag{4.5}$$

We deduce, by combining some results of Section 4.2 and the above Proposition 4.6, some additional estimates on the semigroup $S_{\mathcal{L}_{1,1}}$.

Proposition 4.7. *Let $0 \le \lambda < \mu$ and $k > 3$. There exists $\varepsilon_* > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_*)$, the following hold:*

(1) *For all $g \in L^2_{k,0}$, we have*

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^2_t H^1_k} \lesssim \|g\|_{L^2_k}.$$

(2) *For all $\mathbf{e}_\lambda R \in L^2_t H^{-1}_k$ with $\langle\!\langle R \rangle\!\rangle = 0$, we have*

$$\|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R)\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R)\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.$$

Proof of Proposition 4.7. (1) Remark that $g = \Pi^\perp g$, since $g \in L^2_{k,0}$. The first estimate is nothing but Proposition 4.6. In particular, we deduce from this one that

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^1_t \mathcal{B}(L^2_k)} \lesssim \|\mathbf{e}_{\lambda'} S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^\infty_t \mathcal{B}(L^2_k)} \lesssim 1, \tag{4.6}$$

by choosing $\lambda < \lambda' < 1$. On the other hand, thanks to Lemma 4.4 (1) and the first estimate, we have

$$\|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot)g\|_{L^2_t H^1_k} \lesssim \|g\|_{L^2_k}$$

and

$$\|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * \mathcal{A} S_{\mathcal{L}_{1,1}})(\cdot)g\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda \mathcal{A} S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^2_t H^{-1}_k} \lesssim \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^2_t L^2_k} \lesssim \|g\|_{L^2_k},$$

where we have used that $\mathcal{A} \in \mathcal{B}(L^2_k)$, from which together with the first identity in (4.3), we immediately obtain the second estimate.

(2) Remarking that $R(t) = \Pi^\perp R(t)$, for all $t \ge 0$, and using the second identity in (4.4), we may write

$$S_{\mathcal{L}_{1,1}} * R = S_{\mathcal{L}_{1,1}} \Pi^\perp * R = \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R,$$

and thus

$$\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R) = \mathbf{e}_\lambda \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + (\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * \mathcal{A}[\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)].$$

We deduce

$$\begin{aligned}
&\|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R)\|_{L^\infty_t L^2_k} \\
&\quad \le \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} + \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * \mathcal{A}[\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)]\|_{L^\infty_t L^2_k} \\
&\quad \le \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^1_t(\mathcal{B}(L^2_k))} \|\mathcal{A}\|_{\mathcal{B}(L^2_k)} \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} \\
&\quad \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k},
\end{aligned}$$

where we have used Lemma 4.4 (2) and (4.6) in the last line. We have established the first estimate in (2).

For the second term, using (4.5), we may write

$$S_{\mathcal{L}_{1,1}} * R = \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + S_{\mathcal{B}_\varepsilon} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R + S_{\mathcal{B}_\varepsilon} \mathcal{A} * S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R,$$

and thus

$$\mathbf{e}_\lambda(S_{\mathcal{L}_{1,1}} * R) = \Pi^\perp \mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R) + \mathbf{e}_\lambda[S_{\mathcal{B}_\varepsilon} * (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)] + \mathbf{e}_\lambda[(S_{\mathcal{B}_\varepsilon}\mathcal{A}) * (S_{\mathcal{L}_{1,1}}\Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)].$$

We now estimate each term separately. From Lemma 4.4 (2), we have

$$\|\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.$$

From Lemma 4.4 (2) again, we also have

$$\begin{aligned}
\|\mathbf{e}_\lambda[S_{\mathcal{B}_\varepsilon} * (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^1_k} &\lesssim \|\mathbf{e}_\lambda(\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t H^{-1}_k} \\
&\lesssim \|\mathbf{e}_\lambda(\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t L^2_k} \\
&\lesssim \|\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t L^2_k} \\
&\lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.
\end{aligned}$$

We finally have

$$\begin{aligned}
&\|\mathbf{e}_\lambda[(S_{\mathcal{B}_\varepsilon}\mathcal{A}) * (S_{\mathcal{L}_{1,1}}\Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^1_k} \\
&\quad\lesssim \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}\Pi^\perp) * [\mathcal{A}\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^{-1}_k} \\
&\quad\lesssim \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}\Pi^\perp) * [\mathcal{A}\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t L^2_k} \\
&\quad\lesssim \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}\Pi^\perp\|_{L^1_t(\mathscr{B}(L^2_k))} \|\mathcal{A}\|_{\mathscr{B}(L^2_k)} \|[\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t L^2_k} \\
&\quad\lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k},
\end{aligned}$$

where we have used Lemma 4.4 (2) in the second line as well as Proposition 4.6 and Lemma 4.4 (2) in the last line. Putting the three last estimates together, we conclude the second estimate in (2). ∎

5. Estimates for $\mathcal{L}_{2,2}$

In this section, we are concerned with establishing estimates on $S_{\mathcal{L}_{2,2}}$, where we recall that $\mathcal{L}_{2,2}$ is defined by

$$\mathcal{L}_{2,2} w = \frac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + \nabla\kappa * [Q\nabla w].$$

Before presenting our key estimate, we make several observations.

On the one hand, we have

$$\nabla \mathcal{L}_{2,2} w = \frac{1}{\varepsilon}\Delta(\nabla w) + \mu x \cdot \nabla(\nabla w) + \mu\nabla w + \nabla^2\kappa * [Q\nabla w],$$

where we denote

$$(x \cdot \nabla\Phi)_i = x_\ell \partial_\ell \Phi_i, \quad (\nabla^2\kappa * \Phi)_i = \partial_{i\ell}\kappa * \Phi_\ell,$$

for any vector Φ. A straightforward computation then gives

$$\begin{aligned}\langle \nabla \mathcal{L}_{2,2} w, \nabla w\rangle_{L^2} &= \int \Big(\frac{1}{\varepsilon}\,\Delta \nabla w + \mu \nabla w + \mu x\cdot \nabla^2 w + \nabla^2 \kappa * [Q\nabla w]\Big)\nabla w \\ &= -\frac{1}{\varepsilon}\int |\nabla^2 w|^2 - \int Q|\nabla w|^2,\end{aligned}$$

where we have performed two integrations by parts for the last term and we have used the identity $\Delta\kappa = -\delta$. That nice estimate tells us that $\mathcal{L}_{2,2}$ is dissipative in $\dot H^1$, but it does not provide a spectral gap. Namely, recalling that

$$\langle \nabla w, \nabla z\rangle_{L^2} = \frac{1}{(2\pi)^2}\langle w, z\rangle_{\dot H^1} \quad\text{and}\quad \|\nabla^2 w\|^2_{L^2} = \frac{1}{(2\pi)^2}\|w\|^2_{\dot H^2},$$

we have established the following.

Lemma 5.1. *For any $w \in \dot H^1 \cap \dot H^2$,*

$$\langle \mathcal{L}_{2,2} w, w\rangle_{\dot H^1} = -\frac{1}{\varepsilon}\|w\|^2_{\dot H^2} - (2\pi)^2\|Q^{1/2}\nabla w\|^2_{L^2}.$$

We now establish that $\mathcal{L}_{2,2}$ has a stronger dissipativity property in $\dot H^s$ for $s \in (0,1)$.

Lemma 5.2. *For any $s \in (0,1)$ and $\vartheta < \mu(1-s)$, there exists $\varepsilon > 0$ small enough such that*

$$\langle \mathcal{L}_{2,2} w, w\rangle_{\dot H^s} \le -\frac{1}{2\varepsilon}\|w\|^2_{\dot H^{s+1}} - \vartheta\|w\|^2_{\dot H^s},$$

for any $w \in H^{s+1}$.

Proof of Lemma 5.2. We split the operator $\mathcal{L}_{2,2} = \mathcal{L}_{2,2,1} + \mathcal{L}_{2,2,2}$ with

$$\mathcal{L}_{2,2,1} w := \frac{1}{\varepsilon}\,\Delta w + \mu x\cdot\nabla w \quad\text{and}\quad \mathcal{L}_{2,2,2} w := \nabla\kappa * [Q\nabla w].$$

For the first term, we write

$$\begin{aligned}\langle \mathcal{L}_{2,2,1} w, w\rangle_{\dot H^s} &:= \Big\langle \frac{1}{\varepsilon}\Delta w + \mu x\cdot\nabla w, w\Big\rangle_{\dot H^s} \\ &= \int |\xi|^{2s}\Big\{-\frac{1}{\varepsilon}|\xi|^2\widehat w - \mu\,\mathrm{div}(\xi\widehat w)\Big\}\,\overline{\widehat w} \\ &= -\frac{1}{\varepsilon}\int |\xi|^{2+2s}|\widehat w|^2 - \mu\int\Big[2|\xi|^{2s} - \frac{1}{2}\,\mathrm{div}(\xi|\xi|^{2s})\Big]|\widehat w|^2, \\ &= -\frac{1}{\varepsilon}\,\|w\|^2_{\dot H^{1+s}} - \mu(1-s)\,\|w\|^2_{\dot H^s},\end{aligned}$$

which gives the contribution of the first part of the operator $\mathcal{L}_{2,2}$ in $\dot H^s$.

For the second part of the operator $\mathcal{L}_{2,2}$ in $\dot H^s$, we shall take advantage of the fact that, roughly speaking, the operator $\nabla\kappa *$ behaves as taking a primitive. More precisely, we have

$$\begin{aligned}\langle \mathcal{L}_{2,2,2} w, w\rangle_{\dot H^s} &:= \langle \nabla\kappa * [Q\nabla w], w\rangle_{\dot H^s}\\ &\le \|\nabla\kappa * [Q\nabla w]\|_{\dot H^s}\|w\|_{\dot H^s}\\ &\lesssim \|Q\nabla w\|_{L^1\cap L^2}\|w\|_{\dot H^s}\\ &\lesssim \|\nabla w\|_{L^2}\|w\|_{\dot H^s}\\ &\lesssim \|w\|_{\dot H^s}^{1+s}\|w\|_{\dot H^{s+1}}^{1-s},\end{aligned}$$

where we have used the Cauchy–Schwarz inequality in the second line, estimate (3.1) of Lemma 3.1 in the third one, the exponential decay of Q in Lemma 2.2 in the fourth one, and finally the interpolation inequality (3.4) in the last one.

We can conclude that $\mathcal{L}_{2,2}$ is dissipative in $\dot H^s$. Fix $0 \le \vartheta < \mu(1-s)$. Then, by gathering previous estimates, it follows that

$$\langle \mathcal{L}_{2,2} w, w\rangle_{\dot H^s} \le -\frac{1}{\varepsilon}\|w\|_{\dot H^{s+1}}^2 - \mu(1-s)\|w\|_{\dot H^s}^2 + C\|w\|_{\dot H^s}^{1+s}\|w\|_{\dot H^{s+1}}^{1-s},$$

and we conclude by applying Young's inequality to the last term and choosing $\varepsilon > 0$ small enough. ∎

As a consequence of previous estimates, we obtain the following decay and regularization estimates for the semigroup $S_{\mathcal{L}_{2,2}}$ in $\dot H^s \cap \dot H^1$. Roughly speaking, in that space the semigroup $S_{\mathcal{L}_{2,2}}$ inherits both the good dissipativity property of $S_{\mathcal{L}_{2,2}}$ in $\dot H^s$ and the good regularity property of $S_{\mathcal{L}_{2,2}}$ in $\dot H^1$; that last point being crucial for dealing with the nonlinear term coming from the first equation.

Lemma 5.3. *Let $s \in (0,1)$ and $0 \le \vartheta < \mu(1-s)$. There exists $\varepsilon_2 > 0$ small enough such that for any $\varepsilon \in (0,\varepsilon_2)$, the following hold*:

(1) *For all $w \in \dot H^s \cap \dot H^1$, we have*

$$\|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^\infty_t(\dot H^s\cap\dot H^1)} + \|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^2_t\dot H^s} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^2_t(\dot H^{s+1}\cap\dot H^2)} \lesssim \|w\|_{\dot H^s\cap\dot H^1}.$$

(2) *For all $\mathbf{e}_\vartheta S \in L^2_t(\dot H^s \cap \dot H^1)$, we have*

$$\|\mathbf{e}_\vartheta (S_{\mathcal{L}_{2,2}} * S)\|_{L^\infty_t(\dot H^s\cap\dot H^1)} + \|\mathbf{e}_\vartheta (S_{\mathcal{L}_{2,2}} * S)\|_{L^2_t\dot H^s} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{e}_\vartheta (S_{\mathcal{L}_{2,2}} * S)\|_{L^2_t(\dot H^{s+1}\cap\dot H^2)} \lesssim \|\mathbf{e}_\vartheta S\|_{L^2_t(\dot H^s\cap\dot H^1)}.$$

Proof of Lemma 5.3. Let us denote $\phi = \mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w$, the solution to the evolution equation

$$\partial_t\phi = \mathcal{L}_{2,2}\phi + \vartheta\phi, \quad \phi(0) = w.$$

Fix $\vartheta < \vartheta' < \mu(1-s)$. Thanks to Lemma 5.1 and Lemma 5.2, we have

$$\frac{1}{2}\frac{d}{dt}\{\|\phi\|_{\dot H^s}^2 + \|\phi\|_{\dot H^1}^2\} \le -\frac{1}{2\varepsilon}\{\|\phi\|_{\dot H^{s+1}}^2 + \|\phi\|_{\dot H^2}^2\} - (\vartheta'-\vartheta)\|\phi\|_{\dot H^s}^2 + \vartheta\|\phi\|_{\dot H^1}^2.$$

Thanks to the interpolation inequality (3.4) together with Young's inequality, we observe that

$$\vartheta \|\phi\|^2_{\dot H^1} \le \frac{(\vartheta'-\vartheta)}{2} \|\phi\|^2_{\dot H^s} + C\|\phi\|^2_{\dot H^{s+1}}$$

for some $C > 0$. Gathering previous estimates and taking $\varepsilon > 0$ small enough yields

$$\frac{1}{2}\frac{d}{dt}\{\|\phi\|^2_{\dot H^s} + \|\phi\|^2_{\dot H^1}\} \le -\frac{1}{4\varepsilon}\{\|\phi\|^2_{\dot H^{s+1}} + \|\phi\|^2_{\dot H^2}\} - \frac{(\vartheta'-\vartheta)}{2}\|\phi\|^2_{\dot H^s},$$

from which (1) follows by Grönwall's lemma.

We now consider $\phi = \mathbf{e}_\vartheta(S_{\mathcal{L}_{2,2}} * S)$, the solution to the evolution equation

$$\partial_t \phi = \mathcal{L}_{2,2}\phi + \vartheta\phi + \mathbf{e}_\vartheta S, \quad \phi(0) = 0.$$

Arguing as above, we have, with $\vartheta < \vartheta' < \mu(1-s)$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\phi\|^2_{\dot H^s} + \|\phi\|^2_{\dot H^1}\} \le{}& -\frac{1}{4\varepsilon}\{\|\phi\|^2_{\dot H^{s+1}} + \|\phi\|^2_{\dot H^2}\} - \frac{(\vartheta'-\vartheta)}{2}\|\phi\|^2_{\dot H^s}\\ &+ \langle \phi, \mathbf{e}_\vartheta S\rangle_{\dot H^s} + \langle \phi, \mathbf{e}_\vartheta S\rangle_{\dot H^1}.\end{aligned}$$

Writing

$$\langle \phi, \mathbf{e}_\vartheta S\rangle_{\dot H^s} \le \frac{(\vartheta'-\vartheta)}{8}\|\phi\|^2_{\dot H^s} + C\|\mathbf{e}_\vartheta S\|^2_{\dot H^s}$$

for some $C > 0$, and

$$\langle \phi, \mathbf{e}_\vartheta S\rangle_{\dot H^1} \le \|\phi\|^2_{\dot H^1} + \|\mathbf{e}_\vartheta S\|^2_{\dot H^1},$$

we then use (3.4) again to get

$$\|\phi\|^2_{\dot H^1} \le \frac{(\vartheta'-\vartheta)}{8}\|\phi\|^2_{\dot H^s} + C'\|\phi\|^2_{\dot H^{s+1}}$$

for some $C' > 0$. Taking $\varepsilon > 0$ small enough then yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\phi\|^2_{\dot H^s} + \|\phi\|^2_{\dot H^1}\} \le{}& -\frac{1}{8\varepsilon}\{\|\phi\|^2_{\dot H^{s+1}} + \|\phi\|^2_{\dot H^2}\} - \frac{(\vartheta'-\vartheta)}{4}\|\phi\|^2_{\dot H^s}\\ &+ C(\|\mathbf{e}_\vartheta S\|^2_{\dot H^s} + \|\mathbf{e}_\vartheta S\|^2_{\dot H^1}),\end{aligned}$$

and we conclude (2) by applying Grönwall's lemma again. ■

6. Semigroup estimates for the linearized system

We start with some estimates on the out of the diagonal operators $\mathcal{L}_{1,2}$ and $\mathcal{L}_{2,1}$ which we recall that are defined in (1.10) by

$$\mathcal{L}_{1,2} w = -\operatorname{div}(Q\nabla w), \quad \mathcal{L}_{2,1} g = g + \nabla\kappa * [g\nabla P + Q\nabla\kappa * g].$$

Lemma 6.1. *Let $s \in (0,1)$. For $k > 3$,*

$$\|\mathcal{L}_{1,2} w\|_{H_k^{-1}} \lesssim \|w\|_{\dot{H}^s}^{1-\theta} \|w\|_{\dot{H}^2}^{\theta} \quad \text{for all } w \in \dot{H}^s \cap \dot{H}^2,$$

with $\theta = (1-s)/(2-s)$, and

$$\|\mathcal{L}_{2,1} g\|_{\dot{H}^s} + \|\mathcal{L}_{2,1} g\|_{\dot{H}^1} \lesssim \|g\|_{H_k^1} \quad \text{for all } g \in H_k^1.$$

Proof of Lemma 6.1. For the first estimate, we write

$$\begin{aligned}\| \operatorname{div}(Q\nabla w)\|_{H_k^{-1}} &= \sup_{\|\psi\|_{H_k^1} \le 1} \int \operatorname{div}(Q\nabla w)\, \psi \langle x\rangle^{2k} \\ &= \sup_{\|\psi\|_{H_k^1} \le 1} \int Q\nabla w(\langle x\rangle^{2k} \nabla\psi + \psi\nabla\langle x\rangle^{2k}) \lesssim \|\nabla w\|_{L^2},\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the uniform bound of Q (see Lemma 2.2) in the last line. Using the interpolation inequality (3.4), we then obtain

$$\|\operatorname{div}(Q\nabla w)\|_{H_k^{-1}} \lesssim \|w\|_{\dot{H}^s}^{1/(2-s)} \|w\|_{\dot{H}^2}^{(1-s)/(2-s)},$$

which is nothing but the first claimed estimate.

For the second estimate, we write

$$\|\mathcal{L}_{2,1} g\|_{\dot{H}^s \cap \dot{H}^1} \le \|g\|_{\dot{H}^s \cap \dot{H}^1} + \|\nabla\kappa * [g\nabla P]\|_{\dot{H}^s \cap \dot{H}^1} + \|\nabla\kappa * [Q\nabla\kappa * g]\|_{\dot{H}^s \cap \dot{H}^1},$$

and we estimate the three terms at the right-hand side separately. On the one hand, we have

$$\|\nabla\kappa * [g\nabla P]\|_{\dot{H}^s \cap \dot{H}^1} \lesssim \|g\nabla P\|_{L^1 \cap L^2} \lesssim \|g\|_{L_k^2},$$

where we have used Lemma 3.1 in the first inequality, and Lemma 2.2 in the second one together with the embedding $L_k^2(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$. Similarly, for some $p > 2$, we have

$$\|\nabla\kappa * [Q\nabla\kappa * g]\|_{\dot{H}^s \cap \dot{H}^1} \lesssim \|Q\nabla\kappa * g\|_{L^1 \cap L^2} \lesssim \|\nabla\kappa * g\|_{L^p} \lesssim \|g\|_{L_k^2}$$

by using successively Lemmas 3.1, 2.2 and 3.2. We conclude by putting together the two previous estimates and observing that $\|g\|_{\dot{H}^s \cap \dot{H}^1} \lesssim \|g\|_{H_k^1}$. ∎

As a consequence of Proposition 4.7, Lemma 5.3 and Lemma 6.1, recalling the definition of the spaces $\mathcal{X}$ and $\mathcal{Y}$ in (1.6), we obtain the following semigroup estimates on the linearized problem.

Proposition 6.2. *Let $s \in (0,1)$, $0 \le \lambda < \mu(1-s)$ and $k > 3$. There exists $\varepsilon_* > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_*)$, the following hold:*

(1) *For any $(g_0, w_0) \in L_{k,0}^2 \times (\dot{H}^s \cap \dot{H}^1)$, we have*

$$\|\mathbf{e}_\lambda S_{\mathfrak{L}}(\cdot)(g_0, w_0)\|_{L_t^\infty(\mathcal{X})} + \|\mathbf{e}_\lambda S_{\mathfrak{L}}(\cdot)(g_0, w_0)\|_{L_t^2(\mathcal{Y})} \lesssim \|(g_0, w_0)\|_{\mathcal{X}}.$$

(2) *For any* $\mathbf{e}_\lambda \mathcal{R} = \mathbf{e}_\lambda(\mathcal{R}_1, \mathcal{R}_2) \in L^2_t(H^{-1}_k \times (\dot H^s \cap \dot H^1))$, *with* $\langle\!\langle \mathcal{R}_1 \rangle\!\rangle = 0$, *we have*

$$\|\mathbf{e}_\lambda (S_{\mathscr{L}} * \mathcal{R})\|_{L^\infty_t(\mathcal{X})} + \|\mathbf{e}_\lambda (S_{\mathscr{L}} * \mathcal{R})\|_{L^2_t(\mathcal{Y})} \lesssim \|\mathbf{e}_\lambda \mathcal{R}\|_{L^2_t(H^{-1}_k \times (\dot H^s \cap \dot H^1))}.$$

Proof of Proposition 6.2. (1) Let us denote

$$(g(t), w(t)) = S_{\mathscr{L}}(t)(g_0, w_0),$$

so that

$$g(t) = S_{\mathscr{L}_{1,1}}(t) g_0 + (S_{\mathscr{L}_{1,1}} * \mathscr{L}_{1,2} w)(t)$$

and

$$w(t) = S_{\mathscr{L}_{2,2}}(t) w_0 + (S_{\mathscr{L}_{2,2}} * \mathscr{L}_{2,1} g)(t).$$

We observe that $\langle\!\langle \mathscr{L}_{1,2} w \rangle\!\rangle = 0$, so that we can then hereafter apply the results of Proposition 4.7 to $S_{\mathscr{L}_{1,1}} * \mathscr{L}_{1,2} w$. From Proposition 4.7 (1), we have

$$\|\mathbf{e}_\lambda S_{\mathscr{L}_{1,1}}(\cdot) g_0\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathscr{L}_{1,1}}(\cdot) g_0\|_{L^2_t H^1_k} \lesssim \|g_0\|_{L^2_k}.$$

On the other hand, from Proposition 4.7 (2) , we have

$$\begin{aligned}\|\mathbf{e}_\lambda (S_{\mathscr{L}_{1,1}} * \mathscr{L}_{1,2} w)\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda (S_{\mathscr{L}_{1,1}} * \mathscr{L}_{1,2} w)\|_{L^2_t H^1_k} &\lesssim \|\mathbf{e}_\lambda \mathscr{L}_{1,2} w\|_{L^2_t H^{-1}_k} \\ &\lesssim \|\mathbf{e}_\lambda w\|^{1-\theta}_{L^2_t \dot H^s} \|\mathbf{e}_\lambda w\|^{\theta}_{L^2_t \dot H^2},\end{aligned}$$

with $\theta = (1-s)/(2-s)$, where we have used the first estimate in Lemma 6.1. We have thus established

$$\|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \le C_1 \|g_0\|_{L^2_k} + C_2 \|\mathbf{e}_\lambda w\|^{1-\theta}_{L^2_t \dot H^s} \|\mathbf{e}_\lambda w\|^{\theta}_{L^2_t \dot H^2}, \tag{6.1}$$

for some constants $C_1, C_2 > 0$.

We now come to the estimate of w. We recall that from Lemma 5.3 (1), we have

$$\begin{aligned}&\|\mathbf{e}_\lambda S_{\mathscr{L}_{2,2}}(\cdot) w_0\|_{L^\infty_t(\dot H^s \cap \dot H^1)} + \|\mathbf{e}_\lambda S_{\mathscr{L}_{2,2}}(\cdot) w_0\|_{L^2_t \dot H^s} \\ &\qquad + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda S_{\mathscr{L}_{2,2}}(\cdot) w_0\|_{L^2_t \dot H^2} \lesssim \|w_0\|_{\dot H^s \cap \dot H^1}.\end{aligned}$$

Moreover, from Lemma 5.3 (2), we have

$$\begin{aligned}&\|\mathbf{e}_\lambda (S_{\mathscr{L}_{2,2}} * \mathscr{L}_{2,1} g)\|_{L^\infty_t(\dot H^s \cap \dot H^1)} + \|\mathbf{e}_\lambda (S_{\mathscr{L}_{2,2}} * \mathscr{L}_{2,1} g)\|_{L^2_t \dot H^s} \\ &\qquad + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda (S_{\mathscr{L}_{2,2}} * \mathscr{L}_{2,1} g)\|_{L^2_t \dot H^2} \lesssim \|\mathbf{e}_\lambda \mathscr{L}_{2,1} g\|_{L^2_t(\dot H^s \cap \dot H^1)} \lesssim \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k},\end{aligned}$$

where we have used the second estimate in Lemma 6.1 for obtaining the last inequality. Combining the last two estimates together, we have thus established

$$\begin{aligned}&\|\mathbf{e}_\lambda w\|_{L^\infty_t(\dot H^s \cap \dot H^1)} + \|\mathbf{e}_\lambda w\|_{L^2_t \dot H^s} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda w\|_{L^2_t \dot H^2} \\ &\qquad \le C_3 \|w_0\|_{\dot H^s \cap \dot H^1} + C_4 \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k},\end{aligned} \tag{6.2}$$

for some constants $C_3, C_4 > 0$.

Coming back to (6.1) and using Young's inequality, we deduce that for any $\beta > 0$, there exists $C_\beta > 0$ such that

$$\|\mathbf{e}_\lambda g\|_{L_t^\infty L_k^2} + \|\mathbf{e}_\lambda g\|_{L_t^2 H_k^1} \le C_1 \|g_0\|_{L_k^2} + \beta \|\mathbf{e}_\lambda w\|_{L_t^2 \dot{H}^s} + C_\beta \|\mathbf{e}_\lambda w\|_{L_t^2 \dot{H}^2}.$$

Combining the last estimate with (6.2) yields

$$\begin{aligned}\|\mathbf{e}_\lambda g\|_{L_t^\infty L_k^2} + \|\mathbf{e}_\lambda g\|_{L_t^2 H_k^1} \le C_1 \|g_0\|_{L_k^2} &+ \beta C_3 \|w_0\|_{\dot{H}^s \cap \dot{H}^1} + \beta C_4 \|\mathbf{e}_\lambda g\|_{L_t^2 H_k^1} \\ &+ \sqrt{\varepsilon} C_\beta C_3 \|w_0\|_{\dot{H}^s \cap \dot{H}^1} + \sqrt{\varepsilon} C_\beta C_4 \|\mathbf{e}_\lambda g\|_{L_t^2 H_k^1}.\end{aligned}$$

Choosing first $\beta > 0$ small enough and then $\varepsilon > 0$ small enough, we obtain

$$\|\mathbf{e}_\lambda g\|_{L_t^\infty L_k^2} + \|\mathbf{e}_\lambda g\|_{L_t^2 H_k^1} \le C_5 \|g_0\|_{L_k^2} + C_6 \|w_0\|_{\dot{H}^s \cap \dot{H}^1},$$

for some constants $C_5, C_6 > 0$. We then conclude part (1) by gathering this last estimate with (6.2).

(2) Let us denote $(G, W)(t) = (S_{\mathcal{L}} * \mathcal{R})(t)$, so that

$$G(t) = (S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} W)(t) + (S_{\mathcal{L}_{1,1}} * \mathcal{R}_1)(t)$$

and

$$W(t) = (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} G)(t) + (S_{\mathcal{L}_{2,2}} * \mathcal{R}_2)(t).$$

Observing that $\langle\!\langle \mathcal{R}_1 \rangle\!\rangle = \langle\!\langle \mathcal{L}_{1,2} W \rangle\!\rangle = 0$, we may use Proposition 4.7 (2) as well as Lemma 6.1 for handling $S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} W$, and we obtain

$$\|\mathbf{e}_\lambda G\|_{L_t^\infty L_k^2} + \|\mathbf{e}_\lambda G\|_{L_t^2 H_k^1} \le C_2 \|\mathbf{e}_\lambda W\|_{L_t^2 \dot{H}^s}^{1-\theta} \|\mathbf{e}_\lambda W\|_{L_t^2 \dot{H}^2}^{\theta} + C_1' \|\mathbf{e}_\lambda \mathcal{R}_1\|_{L_t^2 H_k^{-1}},$$

for some constants $C_1', C_2 > 0$ and with $\theta = (1-s)/(2-s)$.

Similarly, using Lemma 5.3 (2) and additionally Lemma 6.1 for handling the term $S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} G$, we have

$$\begin{aligned}\|\mathbf{e}_\lambda W\|_{L_t^\infty(\dot{H}^s \cap \dot{H}^1)} + \|\mathbf{e}_\lambda W\|_{L_t^2 \dot{H}^s} &+ \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda W\|_{L_t^2 \dot{H}^2} \\ &\le C_4 \|\mathbf{e}_\lambda G\|_{L_t^2 H_k^1} + C_3' \|\mathbf{e}_\lambda \mathcal{R}_2\|_{L_t^2(\dot{H}^s \cap \dot{H}^1)},\end{aligned}$$

for some constants $C_3', C_4 > 0$. We can then conclude (2) by arguing as in (1). ∎

7. Proof of the nonlinear stability theorem

This section is devoted to the proof of Theorem 1.1. First we fix $s \in (0,1)$, $k > 3$ and $\lambda \in [0, \mu(1-s))$. We next choose $\varepsilon_0 \in (0, \varepsilon_*)$, where $\varepsilon_* > 0$ is the small scale time provided by Proposition 6.2, and recall that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are defined in (1.6).

Consider the space

$$\mathcal{Z} = \big\{(g, w) \in L_t^\infty(L_{k,0}^2 \times (\dot{H}^s \cap \dot{H}^1)) \cap L_t^2(H_k^1 \times (\dot{H}^s \cap \dot{H}^2)) : \|(g,w)\|_{\mathcal{Z}} < \infty\big\},$$

with

$$\|(g,w)\|_{\mathcal{Z}} := \|\mathbf{e}_\lambda (g,w)\|_{L^\infty_t(\mathcal{X})} + \|\mathbf{e}_\lambda (g,w)\|_{L^2_t(\mathcal{Y})}.$$

For a fixed initial datum $(g_0, w_0) \in \mathcal{X}$ with $\langle\!\langle g_0 \rangle\!\rangle = 0$, define next the mapping $\Phi\colon \mathcal{Z} \to \mathcal{Z}$, $(g,w) \mapsto \Phi[g,w]$, given by, for all $t \ge 0$,

$$\Phi[g,w](t) = S_{\mathcal{L}}(t)(g_0,w_0) + (S_{\mathcal{L}} * \mathcal{R}[(g,w),(g,w)])(t),$$

where

$$\begin{aligned}\mathcal{R}[(g,w),(g,w)] = \big(&R_1[(g,w),(g,w)] + S_1[(g,w),(g,w)],\\ &R_2[(g,w),(g,w)] + S_2[(g,w),(g,w)]\big),\end{aligned}$$

with

$$\begin{aligned} R_1[(g,w),(\bar g,\bar w)] &= -\operatorname{div}(g\nabla \bar w), \quad & S_1[(g,w),(\bar g,\bar w)] &= -\operatorname{div}(g\nabla\kappa * \bar g),\\ R_2[(g,w),(\bar g,\bar w)] &= \nabla\kappa * [g\nabla \bar w], \quad & S_2[(g,w),(\bar g,\bar w)] &= \nabla\kappa * [g\nabla\kappa * \bar g].\end{aligned}$$

We observe here that the first component of $\Phi[g,w](t)$ belongs to $L^2_{k,0}$ since

$$\langle\!\langle R_1[(g,w),(g,w)]\rangle\!\rangle = \langle\!\langle S_1[(g,w),(g,w)]\rangle\!\rangle = 0,$$

thus in the sequel we can apply the results of Proposition 6.2.

Thanks to Proposition 6.2, we have

$$\|S_{\mathcal{L}}(\cdot)(g_0,w_0)\|_{\mathcal{Z}} \lesssim \|(g_0,w_0)\|_{\mathcal{X}},$$

as well as

$$\begin{aligned}\|S_{\mathcal{L}} * \mathcal{R}\|_{\mathcal{Z}} &\lesssim \|\mathbf{e}_\lambda R_1[(g,w),(g,w)]\|_{L^2_t H^{-1}_k} + \|\mathbf{e}_\lambda S_1[(g,w),(g,w)]\|_{L^2_t H^{-1}_k}\\ &\quad + \|\mathbf{e}_\lambda R_2[(g,w),(g,w)]\|_{L^2_t(\dot H^s\cap \dot H^1)} + \|\mathbf{e}_\lambda S_2[(g,w),(g,w)]\|_{L^2_t(\dot H^s\cap \dot H^1)},\end{aligned}$$

and we now estimate each term separately.

For the term associated to R_1, we first have

$$\begin{aligned}\|\operatorname{div}(g\nabla w)\|_{H^{-1}_k} &\lesssim \|g\nabla w\|_{L^2_k}\\ &\lesssim \|g\|_{L^4_k}\|\nabla w\|_{L^4}\\ &\lesssim \|g\|^{1/2}_{L^2_k}\|g\|^{1/2}_{H^1_k}\|\nabla w\|^{1/2}_{L^2}\|\nabla^2 w\|^{1/2}_{L^2}\\ &\lesssim \|(g,w)\|_{\mathcal{X}}\|(g,w)\|_{\mathcal{Y}},\end{aligned}$$

where we have used Hölder's inequality in the second line, and twice the Ladyzhenskaya's inequality (3.3) in the last one. We hence obtain

$$\begin{aligned}\|\mathbf{e}_\lambda R_1[(g,w),(g,w)]\|_{L^2_t H^{-1}_k} &\lesssim \|\mathbf{e}_\lambda g\|^{1/2}_{L^\infty_t L^2_k}\|\nabla w\|^{1/2}_{L^\infty_t L^2}\|\mathbf{e}_\lambda g\|^{1/2}_{L^2_t H^1_k}\|\nabla^2 w\|^{1/2}_{L^2_t L^2}\\ &\lesssim \|(g,w)\|^2_{\mathcal{Z}}.\end{aligned}\tag{7.1}$$

For the term associated to S_1, arguing similarly, as above, we get

$$\begin{aligned}\|\operatorname{div}(g\nabla\kappa * g)\|_{H_k^{-1}} &\lesssim \|g\nabla\kappa * g\|_{L_k^2}\\ &\lesssim \|g\|_{L_k^4}\|\nabla\kappa * g\|_{L^4}\\ &\lesssim \|g\|_{L_k^2}^{1/2}\|g\|_{H_k^1}^{1/2}\|\nabla\kappa * g\|_{L^2}^{1/2}\|\nabla\kappa * g\|_{\dot H^1}^{1/2}\\ &\lesssim \|g\|_{L_k^2}\|g\|_{H_k^1},\end{aligned}$$

where we have used Hölder's inequality in the second line, twice the Ladyzhenskaya's inequality (3.3) in the third line and finally Lemma 3.1 in the last line.

We hence obtain

$$\|\mathbf{e}_\lambda S_1[(g,w),(g,w)]\|_{L_t^2H_k^{-1}} \lesssim \|\mathbf{e}_\lambda g\|_{L_t^\infty L_k^2}\|g\|_{L_t^2H_k^1} \lesssim \|(g,w)\|_{\mathcal{Z}}^2. \tag{7.2}$$

For the term associated to R_2, thanks to Lemma 3.1, we have

$$\|\nabla\kappa * (g\nabla w)\|_{\dot H^s\cap\dot H^1} \lesssim \|g\nabla w\|_{L^1\cap L^2} \lesssim \|g\nabla w\|_{L_k^2},$$

where we have used the embedding $L_k^2(\mathbb{R}^2)\subset L^1(\mathbb{R}^2)$ in the last inequality. We can thus argue as above for obtaining (7.1), and we deduce

$$\|\mathbf{e}_\lambda R_2[(g,w),(g,w)]\|_{L_t^2(\dot H^s\cap\dot H^1)} \lesssim \|(g,w)\|_{\mathcal{Z}}^2. \tag{7.3}$$

Finally, for the term associated to S_2, thanks to Lemma 3.1, similarly, we have

$$\|\nabla\kappa * (g\nabla\kappa * g)\|_{\dot H^s\cap\dot H^1} \lesssim \|g\nabla\kappa * g\|_{L^1\cap L^2} \lesssim \|g\nabla\kappa * g\|_{L_k^2},$$

and therefore arguing as in the derivation of (7.2) yields

$$\|\mathbf{e}_\lambda S_2[(g,w),(g,w)]\|_{L_t^2(\dot H^s\cap\dot H^1)} \lesssim \|(g,w)\|_{\mathcal{Z}}^2. \tag{7.4}$$

Putting together (7.1)–(7.4), we have hence obtained a first estimate:

$$\|\Phi[g,w]\|_{\mathcal{Z}} \le C_0\|(g_0,w_0)\|_{\mathcal{X}} + C_1\|(g,w)\|_{\mathcal{Z}}^2, \tag{7.5}$$

for some constants $C_0, C_1 > 0$.

Now, for $(g,w),(\bar g,\bar w)\in\mathcal{Z}$, we remark that

$$\Phi[g,w]-\Phi[\bar g,\bar w] = S_{\mathfrak{L}} * (R_1^*+S_1^*, R_2^*+S_2^*),$$

with

$$\begin{aligned}R_1^* &= R_1[(g,w),(g,w)] - R_1[(\bar g,\bar w),(\bar g,\bar w)]\\ &= R_1[(g,w),(g,w)-(\bar g,\bar w)] + R_1[(g,w)-(\bar g,\bar w),(\bar g,\bar w)],\\ S_1^* &= S_1[(g,w),(g,w)] - S_1[(\bar g,\bar w),(\bar g,\bar w)]\\ &= S_1[(g,w),(g,w)-(\bar g,\bar w)] + S_1[(g,w)-(\bar g,\bar w),(\bar g,\bar w)],\\ R_2^* &= R_2[(g,w),(g,w)] - R_2[(\bar g,\bar w),(\bar g,\bar w)]\\ &= R_2[(g,w),(g,w)-(\bar g,\bar w)] + R_2[(g,w)-(\bar g,\bar w),(\bar g,\bar w)],\\ S_2^* &= S_2[(g,w),(g,w)] - S_2[(\bar g,\bar w),(\bar g,\bar w)]\\ &= S_2[(g,w),(g,w)-(\bar g,\bar w)] + S_2[(g,w)-(\bar g,\bar w),(\bar g,\bar w)].\end{aligned}$$

Arguing exactly as above, we may establish a second estimate:

$$\|\Phi(g,w) - \Phi(\bar g,\bar w)\|_{\mathcal Z} \le C_2(\|(g,w)\|_{\mathcal Z} + \|(\bar g,\bar w)\|_{\mathcal Z})\|(g,w) - (\bar g,\bar w)\|_{\mathcal Z}, \tag{7.6}$$

for some constant $C_2 > 0$.

As a consequence of the estimates (7.5) and (7.6), we can find $\eta_0, \eta_1 > 0$ small enough such that $C_0\eta_0 + C_1\eta_1^2 \le \eta_1$ and $2C_2\eta_1 < 1$, in such a way that Φ is a contraction on $B_{\mathcal Z}(0,\eta_1)$ for any $(g_0,w_0) \in B_{\mathcal X}(0,\eta_0)$. By a standard Banach fixed-point argument, one can construct a unique global mild solution $(g,w) \in \mathcal Z$ to (1.11) for any $(g_0,w_0) \in \mathcal X$ such that $\|(g_0,w_0)\|_{\mathcal X} \le \eta_0$. More specifically, choosing $\eta_1 := 2C_0\eta_0$ and $4C_0\max(C_1,C_2)\eta_0 < 1$, the above solution, in particular, satisfies the energy estimate

$$\|(g,w)\|_{L^\infty_t(\mathcal X)} + \|(g,w)\|_{L^2_t(\mathcal Y)} \le 2C_0\,\|(g_0,w_0)\|_{\mathcal X},$$

which is nothing but (1.8), as well as the decay estimate

$$\|\mathbf e_\lambda(g,w)\|_{L^\infty_t(\mathcal X)} + \|\mathbf e_\lambda(g,w)\|_{L^2_t(\mathcal Y)} \le 2C_0\,\|(g_0,w_0)\|_{\mathcal X},$$

which is nothing but (1.9).

References

[1] Arendt, W., Grabosch, A., Greiner, G., Groh, U., Lotz, H. P., Moustakas, U., Nagel, R., Neubrander, F. and Schlotterbeck, U.: *One-parameter semigroups of positive operators*. Lecture Notes in Mathematics 1184, Springer, Berlin, 1986. Zbl 0585.47030 MR 839450

[2] Bahouri, H., Chemin, J.-Y. and Danchin, R.: *Fourier analysis and nonlinear partial differential equations*. Grundlehren Math. Wiss. 343, Springer, Heidelberg, 2011. Zbl 1227.35004 MR 2768550

[3] Biler, P.: Local and global solvability of some parabolic systems modelling chemotaxis. *Adv. Math. Sci. Appl.* **8** (1998), no. 2, 715–743. Zbl 0913.35021 MR 1657160

[4] Biler, P. and Brandolese, L.: On the parabolic-elliptic limit of the doubly parabolic Keller–Segel system modelling chemotaxis. *Studia Math.* **193** (2009), no. 3, 241–261. Zbl 1167.35316 MR 2515582

[5] Biler, P., Corrias, L. and Dolbeault, J.: Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model of chemotaxis. *J. Math. Biol.* **63** (2011), no. 1, 1–32. Zbl 1230.92011 MR 2806487

[6] Biler, P., Guerra, I. and Karch, G.: Large global-in-time solutions of the parabolic-parabolic Keller–Segel system on the plane. *Commun. Pure Appl. Anal.* **14** (2015), no. 6, 2117–2126. Zbl 1326.35398 MR 3411100

[7] Calvez, V. and Corrias, L.: The parabolic-parabolic Keller–Segel model in $\mathbb R^2$. *Commun. Math. Sci.* **6** (2008), no. 2, 417–447. Zbl 1149.35360 MR 2433703

[8] Campos, J. F. and Dolbeault, J.: Asymptotic estimates for the parabolic-elliptic Keller–Segel model in the plane. *Comm. Partial Differential Equations* **39** (2014), no. 5, 806–841. Zbl 1297.35032 MR 3196188

[9] Carrapatoso, K. and Mischler, S.: Uniqueness and long time asymptotics for the parabolic-parabolic Keller–Segel equation. *Comm. Partial Differential Equations* **42** (2017), no. 2, 291–345. Zbl 1377.35134 MR 3615547

[10] Carrillo, J. A., Lisini, S. and Mainini, E.: Uniqueness for Keller–Segel-type chemotaxis models. *Discrete Contin. Dyn. Syst.* **34** (2014), no. 4, 1319–1338. Zbl 1277.35009 MR 3117843

[11] Corrias, L., Escobedo, M. and Matos, J.: Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller–Segel system in the plane. *J. Differential Equations* **257** (2014), no. 6, 1840–1878. Zbl 1297.35033 MR 3227285

[12] Fernández, G. E. and Mischler, S.: Uniqueness and long time asymptotic for the Keller–Segel equation: the parabolic-elliptic case. *Arch. Ration. Mech. Anal.* **220** (2016), no. 3, 1159–1194. Zbl 1334.35358 MR 3466844

[13] Ferreira, L. C. F. and Precioso, J. C.: Existence and asymptotic behaviour for the parabolic-parabolic Keller–Segel system with singular data. *Nonlinearity* **24** (2011), no. 5, 1433–1449. Zbl 1221.35205 MR 2785976

[14] Gearhart, L.: Spectral theory for contraction semigroups on Hilbert space. *Trans. Amer. Math. Soc.* **236** (1978), 385–394. Zbl 0326.47038 MR 461206

[15] Ghoul, T.-E. and Masmoudi, N.: Minimal mass blowup solutions for the Patlak–Keller–Segel equation. *Comm. Pure Appl. Math.* **71** (2018), no. 10, 1957–2015. Zbl 1404.35457 MR 3861072

[16] Hillen, T. and Painter, K. J.: A user's guide to PDE models for chemotaxis. *J. Math. Biol.* **58** (2009), no. 1-2, 183–217. Zbl 1161.92003 MR 2448428

[17] Kato, T.: *Perturbation theory for linear operators*. Classics in Mathematics, Springer, Berlin, 1995. Zbl 0836.47009 MR 1335452

[18] Keller, E. F. and Segel, L. A.: Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.* **26** (1970), no. 3, 399–415. Zbl 1170.92306 MR 3925816

[19] Mischler, S. and Mouhot, C.: Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres. *Comm. Math. Phys.* **288** (2009), no. 2, 431–502. Zbl 1178.82056 MR 2500990

[20] Mischler, S. and Scher, J.: Spectral analysis of semigroups and growth-fragmentation equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33** (2016), no. 3, 849–898. Zbl 1357.47044 MR 3489637

[21] Mizoguchi, N.: Global existence for the Cauchy problem of the parabolic-parabolic Keller–Segel system on the plane. *Calc. Var. Partial Differential Equations* **48** (2013), no. 3-4, 491–505. Zbl 1288.35286 MR 3116019

[22] Nagai, T. and Ogawa, T.: Brezis–Merle inequalities and application to the global existence of the Cauchy problem of the Keller–Segel system. *Commun. Contemp. Math.* **13** (2011), no. 5, 795–812. Zbl 1254.35111 MR 2847229

[23] Naito, Y.: Asymptotically self-similar solutions for the parabolic system modelling chemotaxis. In *Self-similar solutions of nonlinear PDE*, pp. 149–160. Banach Center Publ. 74, Polish Acad. Sci. Inst. Math., Warsaw, 2006. Zbl 1115.35059 MR 2295185

[24] Naito, Y., Suzuki, T. and Yoshida, K.: Self-similar solutions to a parabolic system modeling chemotaxis. *J. Differential Equations* **184** (2002), no. 2, 386–421. Zbl 1016.35037 MR 1929883

[25] Nirenberg, L.: On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **13** (1959), 115–162. Zbl 0088.07601 MR 109940

[26] Patlak, C. S.: Random walk with persistence and external bias. *Bull. Math. Biophys.* **15** (1953), 311–338. Zbl 1296.82044 MR 81586

[27] Prüss, J.: On the spectrum of C_0-semigroups. *Trans. Amer. Math. Soc.* **284** (1984), no. 2, 847–857. Zbl 0572.47030 MR 743749

[28] Tindall, M. J., Porter, S. L., Maini, P. K., Gaglia, G. and Armitage, J. P.: Overview of mathematical approaches used to model bacterial chemotaxis. I. The single cell. *Bull. Math. Biol.* **70** (2008), no. 6, 1525–1569. Zbl 1166.92008 MR 2430317

[29] Tristani, I.: Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting. *J. Funct. Anal.* **270** (2016), no. 5, 1922–1970. Zbl 1330.76124 MR 3452720

Received February 19, 2025; revised May 13, 2025.

Frank Alvarez Borges
Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université UPMC
4 Place Jussieu, 75005 Paris, France;
frank.alvarez_borges@sorbonne-universite.fr
Author IDs: MR 1300936 ORCID 0000-0002-6651-7374

Kleber Carrapatoso
Centre de Mathématiques Laurent Schwartz, École Polytechnique, Institut Polytechnique de Paris
91128 Palaiseau, France;
kleber.carrapatoso@polytechnique.edu
Author IDs: MR 1043725 ORCID 0009-0005-6416-3567

Stéphane Mischler
Centre de Recherche en Mathématiques de la Décision (CEREMADE), Universités PSL & Paris-Dauphine
Place de Lattre de Tassigny, 75775 Paris;
Institut Universitaire de France (IUF)
103 boulevard Saint-Michel, 75005 Paris, France;
mischler@ceremade.dauphine.fr
Author IDs: MR 607483 ORCID 0000-0002-8312-0985