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Bott-integrability of overtwisted contact structures

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Abstract. We show that an overtwisted contact structure on a closed and oriented 3-manifold can be defined by a contact form having a Bott-integrable Reeb flow if and only if the Poincaré dual of its Euler class is represented by a graph link.

1. Introduction

In this paper, we continue our systematic study of Bott-integrable Reeb flows on 3-manifolds begun in [6]. Slightly streamlining the terminology used in our earlier paper, we speak of Bott integrability of contact forms and contact structures in the following sense. Our 3-manifolds are always understood to be oriented, and our contact structures are coorientable and positive, that is, $\xi = \ker \alpha$, with $\alpha \wedge d\alpha > 0$.

Definition 1.1. A contact form α on a compact 3-manifold M is *Bott integrable* if there is a Morse–Bott function $f: M \to \mathbb{R}$ invariant under the Reeb flow of α, that is, $df(R) = 0$ for the Reeb vector field R of α.

A contact structure ξ on M is *Bott integrable* if there is some Bott-integrable contact form α with $\xi = \ker \alpha$.

Our main result in the present paper gives a characterisation of all Bott-integrable contact structures amongst the overtwisted ones on any closed 3-manifold. To formulate it, we need to explain the concept of a graph link.

Recall that a *graph manifold* is a 3-manifold that can be cut along tori into Seifert fibred pieces. In Theorem 1.3 of [6], we showed that a closed 3-manifold admits a Bott-integrable contact structure if and only if it is a graph manifold. We also gave examples of non-integrable contact structures on graph manifolds, and examples of 3-manifolds (the 3-sphere S^3, the 3-torus T^3, and $S^1 \times S^2$) where every contact structure is Bott integrable.

A link in a 3-manifold is called a *graph link* if its exterior is a graph manifold. In particular, the ambient manifold containing a graph link must be a graph manifold, since it is obtained by gluing solid tori to the link exterior.

Theorem 1.2. *An overtwisted contact structure on a closed* 3*-manifold is Bott integrable if and only if the Poincaré dual of its Euler class can be represented by a graph link.*

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Remark 1.3. Here is a brief comment on the dynamical significance of this theorem. By a result of Paternain [18], a Hamiltonian flow (on a 3-dimensional energy hypersurface) with an integral whose critical set consists of submanifolds has vanishing topological entropy. In particular, this applies to Bott-integrable Reeb flows. Macarini–Schlenk [16], by contrast, have shown the existence of contact structures that force entropy, in the sense that any associated Reeb flow has positive topological entropy. Other results of this type are due to Alves [1] and Alves–Colin–Honda [2]. In all cases, the relevant property of the contact structure that forces entropy also implies tightness of the contact structure.

In Question 1.5 of [2], the authors ask whether there exists an *overtwisted* contact structure in dimension 3 that forces entropy. As shown by Yau [24], the contact homology of overtwisted contact structures vanishes. This suggests that the orbit structure of overtwisted Reeb flows may not be rich enough to establish forced entropy by the methods of [1].

In combination with the result of Paternain, our Theorem 1.2 gives a criterion for an overtwisted contact structure *not* to force entropy. As an example, our results in Section 4 show that for Seifert manifolds the answer to Question 1.5 in [2] is negative: no overtwisted contact structure on a Seifert manifold forces entropy.

On the other hand, Theorem 1.2 provides examples of overtwisted contact structures on graph manifolds that force any associated Reeb flow at least to be non-integrable (in the Bott sense); see Section 6. Now, one of two things can happen with these examples: either (i) entropy is forced, providing a positive answer to Question 1.5 in [2], or (ii) there is a Reeb flow of zero topological entropy that cannot be approximated by Bott-integrable Reeb flows. Case (ii) would go some way towards answering a question of Katok (see Problem 1 in [15]), who wondered about the existence of such approximations by Liouville-integrable systems.

Example 1.4. Here is an example that the 'if' part of Theorem 1.2 fails, in general, for tight contact structures. By combining the results of [16] and [18], one sees that the canonical contact structure ξ on the unit cotangent bundle $ST^*\Sigma$ of a closed, oriented surface Σ of genus at least 2 is not Bott integrable; see Proposition 1.7 in [6]. The contact structure ξ is fillable (by the standard symplectic form on the unit disc cotangent bundle) and hence tight, and its Euler class is trivial, since the fibre direction of $ST^*\Sigma$ provides a global nowhere vanishing section. Since the ambient manifold $ST^*\Sigma$, as an S^1-bundle, is a graph manifold, the empty link representing the Euler class is a graph link.

In Sections 2 and 3, we prove a few results that will be of more general interest for the study of Bott-integrable Reeb flows: neighbourhood theorems for surfaces contained in the critical set of the Bott integral, perturbation results for such critical surfaces, and a formula for the Euler class of a Bott-integrable contact structure in terms of the critical periodic Reeb orbits.

This leads to a proof of the 'only if' part of Theorem 1.2, which is in fact a general statement about Bott-integrable contact structures, not only the overtwisted ones. Given a Bott-integrable contact form with Bott integral f, write L_f for the link of critical Reeb orbits. We then show that L_f is a graph link, and with the components oriented depending on their elliptic or hyperbolic type, L_f represents the dual of the Euler class. The fact that this remains true even in the presence of critical surfaces rests in an essential way on the neighbourhood and perturbation theorems.

Given a compact graph manifold M, there is a characterisation due to Yano [22, 23] of the homology classes in $H_1(M)$ that can be represented by a graph link. This is instrumental for the proof of the 'if' part of Theorem 1.2. In Section 5, we provide the necessary background on 3-manifold topology, notably the JSJ decomposition, for understanding Yano's result.

In order to apply Yano's result in our setting, in Section 4 we show how to realise any 1-dimensional homology class on a given compact Seifert fibred manifold by a sublink of the critical link of a suitable Bott-integrable contact form. This also provides an alternative contact geometric proof of an auxiliary result in [22]. For Seifert manifolds over a non-orientable base, the proof of this H_1-completeness, as we call it, again uses the perturbation result for critical surfaces. Furthermore, the argument requires the fibre connected sum of Bott-integrable contact forms, which is another general construction we introduce here. A consequence of this discussion is a direct proof (not involving Yano's results or any advanced 3-manifold topology used to establish Theorem 1.2) that every overtwisted contact structure on any closed Seifert manifold is Bott integrable (Corollary 4.4).

By Eliashberg's classification [4], every homotopy class of tangent 2-plane fields on a closed, oriented 3-manifold contains a unique overtwisted contact structure up to isotopy. Thus, together with the results of Yano, Theorem 1.2 allows one to give a complete homotopy theoretic characterisation, on a given 3-manifold, of the Bott-integrable overtwisted contact structures. As an instructive example, in Section 6 we classify the Bott-integrable contact structures on the mapping torus of Arnold's cat map, including the tight ones. This serves to illustrate Yano's results and the 3-manifold topology behind them, and it extends our classification results from [6]. In particular, it gives further examples of contact structures (on a graph manifold) that do *not* admit Bott-integrable Reeb flows.

2. The critical set

We consider a pair (M, α) consisting of a closed, oriented 3-manifold M and a positive contact form α. The Reeb vector field of α will be denoted by R.

In Theorem 2.2 of [6], we proved the following Liouville type theorem: if f is an integral for the Reeb flow, that is, if $\mathrm{d}f(R) = 0$, then any regular level set of f is a 2-torus T^2, and in terms of circle coordinates x_1, x_2 on T^2 and a suitable transverse coordinate r, the contact form α is of Lutz type $\alpha = h_1(r)\,\mathrm{d}x_1 + h_2(r)\,\mathrm{d}x_2$, and the integral f a function of r.

For the case when the integral f is a Morse–Bott function, we also established a neighbourhood theorem for isolated elliptic Reeb orbits in the critical set $\mathrm{Crit}(f)$, that is, an orbit where the Bott integral has a minimum or a maximum; see Theorem 2.4 in [6]: in terms of suitable coordinates $(\theta; r, \varphi)$ on a neighbourhood $S^1 \times D^2_\delta$ of the critical orbit $S^1 \times \{0\}$, the Bott integral f equals $f(r) = c \pm r^2$, and up to an isotopic deformation of the contact form in a neighbourhood of the critical orbit – changing the Reeb flow but not the Bott integral –, the contact form is given by $\mathrm{d}\theta + r^2\,\mathrm{d}\varphi$. Also, by Gray stability, the contact structure $\xi = \ker\alpha$ remains unchanged, up to a diffeotopy of M, under this deformation.

In the present section, we extend the arguments for achieving those results in order to prove neighbourhood theorems for tori and Klein bottles contained in $\mathrm{Crit}(f)$. We then reason as in Section 10.2 of [6] to remove those surfaces from the critical set by a C^∞-small perturbation of the Bott integral f, at the cost of introducing an elliptic and a parallel hyperbolic Reeb orbit into the critical set.

Taken together, these results will imply the following genericity result.

Theorem 2.1. *Let (M,α) be a closed 3-dimensional contact manifold whose Reeb flow has a Bott integral f. Then, after an isotopic deformation of α and a C^∞-small deformation of f, both localised near the surfaces contained in* $\mathrm{Crit}(f)$, *we may assume that the Reeb flow has a Bott integral whose critical set consists of isolated periodic Reeb orbits only.*

2.1. Neighbourhood theorems for critical surfaces

Since the Reeb vector field R is tangent to the level sets of a Bott integral f, those level sets have to be periodic Reeb orbits, tori, or Klein bottles. The vector field Y on M defined by

$$\alpha(Y)=0,\quad i_Y\mathrm{d}\alpha=-\mathrm{d}f, \tag{2.1}$$

is non-singular and not collinear with R along regular level sets, which in consequence have to be tori. As critical level sets, all three types just mentioned can occur. We now prove neighbourhood theorems for critical tori and Klein bottles.

Theorem 2.2 (Neighbourhood theorem for critical tori). *Let f be a Bott integral for the Reeb flow of (M,α), and let $T^2\cong\Sigma\subset M$ be a component of a level set $\{f=c\}$ along which the Bott integral has a local minimum or maximum. Then a neighbourhood of Σ in M is diffeomorphic to $[-\delta,\delta]\times T^2$ such that with coordinates $r\in[-\delta,\delta]$ and x_1,x_2 on $T^2=(\mathbb{R}/2\pi\mathbb{Z})^2$, we have*

(i) $f(r,x_1,x_2)=c\pm r^2$;

(ii) $\alpha=h_1(r)\,\mathrm{d}x_1+h_2(r)\,\mathrm{d}x_2$.

Recall that the contact condition for a contact form of Lutz type as in (ii) is

$$\Delta:=\begin{vmatrix} h_1 & h_1' \\ h_2 & h_2' \end{vmatrix}<0,$$

and the Reeb vector field

$$R=\frac{h_2'\partial_{x_1}-h_1'\partial_{x_2}}{\Delta}$$

is then tangent to the T^2-factors.

Proof of Theorem 2.2. Statement (i) follows from the generalised Morse–Bott lemma (see Lemma 1.7 in [3]). In order to prove (ii), we slightly modify the argument for proving Theorem 2.2 in [6]. There we constructed covering maps $\mathbb{R}^2\to\{r\}\times T^2$ by using the flow of R and Y. In the present situation, Y vanishes along the critical set $\{r=0\}$. We circumvent this problem by working with the vector field Y' defined by

$$\alpha(Y')=0,\quad i_Y\mathrm{d}\alpha=\mp 2\,\mathrm{d}r;$$

in other words, we replace $\mathrm{d}f$ in the defining equations (2.1) for Y by $\mathrm{d}f/r = \pm 2\,\mathrm{d}r$ for $r \neq 0$. This is well defined also in $r = 0$, since r is a smooth coordinate on $(-\delta, \delta)$. So Y' coincides with Y/r for $r \neq 0$ and extends smoothly as a non-vanishing vector field over $r = 0$. The remainder of the argument is now as in the proof of Theorem 2.2 in [6]. ∎

Next we describe the neighbourhood of a critical Klein bottle K. Recall from [7] or Section 10.1 of [6] that a closed tubular neighbourhood νK of a Klein bottle K embedded in an orientable 3-manifold, which is the unique non-trivial interval bundle over K, can be described by

$$\nu K = \big([-\delta, \delta] \times \mathbb{R} \times S^1\big)/(r, s, \theta) \sim (-r, s-1, -\theta).$$

We write the class of (r, s, θ) as $[r, s, \theta]$.

In [6], we already worked with the following local model as an example, but we did not provide a proof that this model is universal. In fact, it is only universal up to a rescaling of the contact form and the Bott integral.

Theorem 2.3 (Neighbourhood theorem for critical Klein bottles). *Let f be a Bott integral for the Reeb flow of (M, α), and let $K \cong \Sigma \subset M$ be a component of a level set $\{f = c\}$ along which the Bott integral has a local minimum or maximum. Then, up to a rescaling of the Bott integral, as well as an isotopy and a rescaling of the contact form, all supported in a neighbourhood of Σ, a smaller neighbourhood of Σ in M looks like νK, with*

(i) $f([r, s, \theta]) = c \pm r^2$;

(ii) $\alpha = \mathrm{d}s - r\,\mathrm{d}\theta$.

Proof. Again, statement (i) follows from the generalised Morse–Bott lemma.

The orientable double cover of νK is given by

$$[-\delta, \delta] \times T^2 = ([-\delta, \delta] \times \mathbb{R} \times S^1)/(r, s, \theta) \sim (r, s-2, \theta).$$

The contact form α and the Bott integral f lift to this double cover. In this lift, $\{0\} \times T^2 = \{r = 0\}$ is a component of the critical set. Now, we cannot directly quote the previous result, since we need to have a $\mathbb{Z}_2$-equivariant version of that neighbourhood theorem, and we want to achieve a more specific Lutz form. So we need to dig a little deeper into the proof of the neighbourhood theorem for elliptic critical orbits (Theorem 2.4 in [6]), which contains the relevant ideas. At each step in the following argument, we deal with a $\mathbb{Z}_2$-equivariant isotopy or a $\mathbb{Z}_2$-invariant rescaling of the contact form, all of which are supported in a neighbourhood of $\{r = 0\}$. So the actual normal form we finally arrive at is defined on a neighbourhood of K defined by a smaller δ than the one we started with.

First of all, as before one shows that the lifted contact form $\tilde{\alpha}$ can be written as

$$\tilde{\alpha} = h_0(r)\,\mathrm{d}r + h_1(r)\,\mathrm{d}s + h_2(r)\,\mathrm{d}\theta.$$

Notice that the Reeb vector field $\widetilde{R}$ of $\tilde{\alpha}$ is tangent to the tori $\{r = \text{constant}\}$. (In the proof of the previous theorem, the coordinates x_1, x_2 on the torus factor are determined by a choice of basis of the lattice defining the covering $\mathbb{R}^2 \to T^2$; by applying a suitable element of $\mathrm{SL}(2, \mathbb{Z})$, we may assume these coordinates to be $(s, \theta) \in (\mathbb{R}/2\mathbb{Z}) \times S^1$.)

The invariance under the $\mathbb{Z}_2$-action

$$(r, s, \theta) \longmapsto (-r, s-1, -\theta)$$

on $[-\delta, \delta] \times T^2$ translates into

$$h_0(-r) = -h_0(r), \quad h_1(-r) = h_1(r), \quad \text{and} \quad h_2(-r) = -h_2(r).$$

In particular, we have $h_0(0) = 0$. As in Step 1 of the proof of Theorem 2.4 in [6], we first deform h_0, relative to a neighbourhood of $\{\pm\delta\} \times T^2$, so that it becomes identically zero near $r = 0$; this can be done within the class of odd functions. The convex linear interpolation between the old and the deformed h_0 gives rise to a strict contact isotopy, defined by a flow along the level surfaces $\{r = \text{constant}\}$, and stationary on $\{r = 0\}$, just like in the proof of Theorem 2.2 in [6]; see equation (3) in that proof and the argument following it.

The $\mathbb{Z}_2$-invariance of $\widetilde{R}$, which depends on r only, forces the Reeb vector field to be a multiple of ∂_t along $\{r = 0\}$. This implies that $h_1(0) \neq 0$, and by choosing the signs of the coordinates r, s appropriately, we may assume that $h_1(0) > 0$ and that the ambient orientation is given by $\mathrm{d}r \wedge \mathrm{d}s \wedge \mathrm{d}\theta$. Then as in Step 2 of the proof of Theorem 2.4 in [6], we can make a further deformation such that the contact form looks like $\mathrm{d}s + h_2(r)\,\mathrm{d}\theta$ (with a modified, but still odd h_2) near $\{r = 0\}$: for this we need to divide the contact form by an even function $\chi(r)$ equal to $h_1(r)$ near $r = 0$, and identically equal to 1 outside a small neighbourhood of $r = 0$; so the contact structure remains unchanged. This deformation changes the Reeb vector field, keeping it tangent to the tori $\{r = \text{constant}\}$. Thanks to the contact condition, h_2 satisfies $h_2' < 0$.

Finally, as Step 3, we choose an odd function $h_2^*(r)$ (with negative derivative) that equals $-r$ near $r = 0$, and h_2 outside a small neighbourhood of $r = 0$. Then the convex linear interpolation between h_2 and h_2^* defines a $\mathbb{Z}_2$-invariant deformation of $\tilde{\alpha}$ into the desired normal form, which descends to the $\mathbb{Z}_2$-quotient νK. One may check that this deformation of contact forms is induced by the flow of a vector field $X = a(r)\partial_r$, with $a(r)$ an odd function. In particular, this flow is stationary on $\{r = 0\}$ and sends level sets of the Bott integral to level sets. ∎

2.2. Removing critical surfaces

For simplicity of notation, we assume that we are dealing with a critical torus or Klein bottle along which f has an isolated local minimum, and that the value of f at the minimum is 0. In other words, we are dealing with $f(r, x_1, x_2) = r^2$ in the case of the torus, and with $f\big([r, s, \theta]\big) = r^2$ in the case of a Klein bottle.

For the Klein bottle, we explained in Section 10.2 of [6] how to perturb the Bott integral f into one that has two isolated critical Reeb orbits, one elliptic and one hyperbolic, in place of the critical Klein bottle.

We briefly recall the construction for the Klein bottle, and then show how to adapt it to the torus case. We choose $\varepsilon > 0$ such that $2\varepsilon < \delta$. Let $\chi\colon [-\delta, \delta] \to [0, \varepsilon^2]$ be a smooth function identically equal to ε^2 on the interval $[-\varepsilon, \varepsilon]$, and identically equal to 0 on the intervals $[-\delta, -2\varepsilon]$ and $[2\varepsilon, \delta]$. We may assume that $|\chi'(r)| < |2r|$.

Recall that the Reeb vector field R equals ∂_s on the neighbourhood νK of the critical Klein bottle, so the function

$$[r, s, \theta] \longmapsto r^2 + \chi(r)\cos\theta$$

is still an integral of R. A straightforward computation, see [6], shows that this function is a Bott integral with an elliptic critical orbit at $(r, \theta) = (0, \pi)$, and a critical hyperbolic orbit at $(r, \theta) = (0, 0)$.

In the torus case, we modify the contact form $\alpha = h_1(r)\,\mathrm{d}x_1 + h_2(r)\,\mathrm{d}x_2$ near $\{r = 0\}$ in a way described in the proof of Proposition 2.6 in [6]. By a small reparametrisation of the curve $r \mapsto (h_1(r), h_2(r))$, we may assume that $h_1(0)$ and $h_2(0)$ are rationally dependent. Then, up to an $\mathrm{SL}(2, \mathbb{Z})$-transformation, we may assume that $h_1(0) > 0$ and $h_2(0) = 0$, which implies $h_2'(0) < 0$. After a small deformation and a further reparametrisation of the curve $r \mapsto (h_1(r), h_2(r)$, up to a global constant we may finally assume that h_1 is identically equal to 1 near $r = 0$, and $h_2(r) = -r$ near $r = 0$. These modifications change the Reeb vector field, keeping it tangent to the tori $\{r = \text{constant}\}$, so f is still a Bott integral. Moreover, near $\{r = 0\}$ the new Reeb vector field equals ∂_{x_1}, so the new Bott integral

$$(r, x_1, x_2) \longmapsto r^2 + \chi(r)\cos x_2$$

has the properties analogous to the previous case.

3. The Euler class

In this section, we prove the easier direction of Theorem 1.2. Given a contact form α with Bott integral f on a closed 3-manifold M, we write $L_f \subset \mathrm{Crit}(f)$ for the link made up of the critical periodic Reeb orbits. We show that the Poincaré dual of the Euler class of the contact structure $\ker\alpha$ is represented by L_f, and that $L_f \subset M$ is a graph link.

3.1. Euler class and critical orbits

For the case that $L_f = \mathrm{Crit}(f)$, that is, in the absence of critical surfaces, the following result was stated implicitly in the proof of Lemma 7.2 in [6]. Here and throughout this paper, (co-)homology groups are understood to be with $\mathbb{Z}$-coefficients.

Lemma 3.1. *Let f be a Bott integral for (M, α). Write $E \subset L_f$ for the set of elliptic orbits, and $H \subset L_f$ for the set of hyperbolic orbits. We orient the critical orbits by the Reeb flow. Then the Euler class of $\xi = \ker\alpha$ is given by*

$$e(\xi) = \sum_{\gamma \in E} \mathrm{PD}([\gamma]) - \sum_{\gamma \in H} \mathrm{PD}([\gamma]) \in H^2(M).$$

Proof. The vector field Y defined in (2.1) can be read as a section $Y\colon M \to \xi$ of the 2-plane bundle ξ. The intersection of $Y(M)$ with the zero section 0_ξ of ξ equals $\mathrm{Crit}(f)$.

(i) We first consider the case that there are no critical surfaces. From the normal form of the Bott integral f in terms of transverse coordinates x, y near critical orbits, $\pm(x^2 + y^2)$ near elliptic orbits and xy near hyperbolic ones, it follows that $Y(M)$ intersects 0_ξ transversely. Both $Y(M)$ and 0_ξ carry the orientation induced by M, and ξ the orientation given by $\mathrm{d}\alpha$. This defines an orientation on $L_f = Y(M) \cap 0_\xi$. As in the 2-dimensional Poincaré–Hopf theorem one then sees that for $\gamma \in E$, this orientation coincides with the direction of the Reeb flow; for $\gamma \in H$ it is the opposite orientation. Regarding this last statement, observe that the R-invariance of f allows us to reduce the reasoning to the consideration of a local surface of section of the Reeb flow near the critical Reeb orbit, oriented by the area form $\mathrm{d}\alpha$.

(ii) Now suppose that $\mathrm{Crit}(f)$ does contain critical surfaces. As we have seen in Section 2, any surface contained in $\mathrm{Crit}(f)$ can be replaced by a pair of mutually isotopic critical Reeb orbits, one elliptic and one hyperbolic, so in sum they do not contribute to the Euler class. This means that the formula for $e(\xi)$ remains true in this general case. ■

3.2. The link of critical orbits

With the next lemma we complete the proof of the 'only if' part of Theorem 1.2. Note that this is a general statement about Bott-integrable contact forms; overtwistedness of the contact structure is not needed here.

Lemma 3.2. *Let f be a Bott integral for (M, α). Then $L_f \subset M$ is a graph link.*

Proof. In Section 3.2 of [6], we gave a summary of the results from [3] concerning the (singular) Liouville foliation defined by the level sets of a Bott integral. The key statement is that up to a diffeomorphism preserving the Liouville foliation, any Bott-integrable Reeb flow can be constructed from some simple building blocks, which are products of a surface (with boundary) with S^1, or Seifert bundles with singular fibres of order 2, coming from fixed points of an involution on the surface. The Liouville foliation on these building blocks is defined by the level sets of a Morse–Bott function lifted from a Morse–Bott function on the base, with all fixed points of the involution contained in the critical set of that Morse function.

From this description of the building blocks, one sees that the complement of tubular neighbourhoods of the critical orbits is fibred by circles. So the exterior of L_f, which is the manifold obtained by gluing these complements, is a graph manifold. ■

Remark 3.3. In Section 3.2.3 of [6], we wrote that the Morse–Bott functions on the Seifert fibred building blocks are lifted from Morse functions on the base, when in fact we should have written "Morse–Bott functions on the base". The neighbourhood νK of a critical Klein bottle is a case in point. As described in detail in [7], νK may be thought of as the Seifert bundle over a disc with two singular fibres of order 2. This corresponds to starting with the annulus $[-\delta, \delta] \times S^1$ with Morse–Bott function $f(r, \theta) = r^2$ and involution $\tau(r, \theta) = (-r, -\theta)$ having two fixed points. One then obtains νK with the described Seifert fibration as the mapping torus of τ.

Critical tori may also be produced by the gluing of the building blocks along torus boundaries.

4. Seifert manifolds

In Theorem 1.8 of [6], we showed that every contact structure on a closed Seifert manifold invariant under the fixed-point free S^1-action defining the Seifert fibration is Bott integrable. In the present section, we deal with overtwisted (but not necessarily S^1-invariant) contact structures on Seifert manifolds.

Our aim will be to find a Bott-integrable contact structure on any given Seifert manifold that is sufficiently 'rich' in the following sense. When we speak of a Bott-integrable contact structure on a manifold with boundary, it is understood that the boundary components are tori, and the contact structure is of Lutz type on collar neighbourhoods of those boundary components. In particular, the boundary is a regular level set of the Bott integral, and the Reeb flow is tangent to the boundary. By the sewing lemma (see Lemma 3.4 in [6]), the gluing along such boundaries produces again a Bott-integrable contact structure.

Definition 4.1. A Bott-integrable contact structure ξ on a compact 3-manifold M is called *H_1-complete* if, given any homology class $u \in H_1(M)$, one can find a Bott-integrable contact form α defining ξ with a Bott integral f whose critical link L_f contains a sublink of elliptic periodic Reeb orbits representing u.

The methods developed in [6] allow us to prove the following key lemma.

Lemma 4.2. *If a closed 3-manifold M admits an H_1-complete Bott-integrable contact structure, then every overtwisted contact structure on M is Bott integrable.*

Proof. By Eliashberg's classification [4] of overtwisted contact structures, we need only show that every homotopy class of tangent 2-plane fields on M contains a Bott-integrable overtwisted contact structure. Write ξ_0 for the given H_1-complete Bott-integrable contact structure on M. We may assume that ξ_0 is overtwisted; if it is not, we create an elliptic critical orbit (together with a hyperbolic one) as in Section 2.3 of [6], and then perform a full Lutz twist, which does not change the homotopy class of ξ_0 as a tangent 2-plane field.

Now, given any tangent 2-plane field η, there is an obstruction $d^2(\xi_0, \eta) \in H^2(M)$ for ξ_0 to be homotopic to η over the 2-skeleton of M. This obstruction is antisymmetric and additive; see Section 4.2 of [5]:

$$d^2(\eta_1, \eta_2) = -d^2(\eta_2, \eta_1) \quad \text{and} \quad d^2(\eta_1, \eta_2) + d^2(\eta_2, \eta_3) = d^2(\eta_1, \eta_3).$$

The relation of the obstruction class d^2 with the Euler class is given by

$$2d^2(\eta_1, \eta_2) = e(\eta_1) - e(\eta_2);$$

see Remark 4.3.4 in [5]. Thus, in the presence of 2-torsion in $H^2(M)$, the obstruction class is not detected by the Euler class.

If a contact structure ξ is modified by a Lutz twist along a positively transverse knot $K \subset (M, \xi)$ into a new contact structure ξ^K, then $d^2(\xi^K, \xi) = -\mathrm{PD}[K]$.

The H_1-completeness of ξ_0 allows us to choose a Bott-integrable contact form α_0^u with a Bott integral f_u whose critical link L_{f_u} contains a sublink L_u of elliptic Reeb orbits representing the class $u := \mathrm{PD}(d^2(\xi_0, \eta)) \in H_1(M)$. By performing Lutz twists along the components of L_u, we obtain a Bott-integrable contact form α_1^u (with the same Bott integral f_u) defining a contact structure $\xi^u = \ker \alpha_1^u$ with $d^2(\xi^u, \xi_0) = -\mathrm{PD}(u)$, and hence $d^2(\xi^u, \eta) = 0$.

Finally, in order to modify ξ^u into a Bott-integrable contact structure homotopic to η over all of M, it suffices to take a contact connected sum as in Section 6 of [6] with a suitable overtwisted contact structure on S^3, all of which are Bott integrable by Theorem 1.9 in [6]. ∎

Here is the central result of this section.

Proposition 4.3. *Every compact Seifert manifold (possibly with toral boundaries) admits an H_1-complete Bott-integrable contact structure.*

The following corollary is then immediate from Lemma 4.2 and Proposition 4.3.

Corollary 4.4. *Every overtwisted contact structure on any closed Seifert manifold is Bott integrable.* ∎

We now turn to the proof of Proposition 4.3. Most of the work goes into proving this result for a product bundle $\Sigma \times S^1$, where Σ is a compact orientable surface. The case of general Seifert bundles over orientable surfaces then follows without much effort. The case of a Seifert bundle over a non-orientable base (with orientable total space) requires an extra argument involving a construction of fibre connected sums for Bott-integrable contact structures.

4.1. Orientable base

We start by looking at the case of product bundles $\Sigma \times S^1$ with an orientable base surface Σ, which is allowed to have boundary. We recall from [6] how to construct a Bott-integrable contact form on $\Sigma \times S^1$ from an open book decomposition. We then modify this contact form by introducing homotopically inessential Lutz components, and turn some periodic Reeb orbits inside these Lutz components into critical ones by changing the Bott integral.

4.1.1. Open books. As discussed in Section 7 of [6], the manifold $\Sigma \times S^1$ may be thought of as an open book with page Σ_0 equal to Σ with two open discs removed, and monodromy a couple of Dehn twists (one right-handed, one left-handed) parallel to the two boundary components $\partial\Sigma_0 \setminus \partial\Sigma$ created by the removal of these discs. For the construction of an H_1-complete Bott-integrable contact structure on $\Sigma \times S^1$, we need the following mild extension of Lemma 3.3 in [6]. We formulate it for a single oriented, embedded circle $\gamma \subset \Sigma_0$, but it works equally well for a finite collection of pairwise disjoint embedded circles.

We choose 1-forms $\lambda_i = \rho_i(r_i)\,\mathrm{d}\varphi_i$, with $\rho_i' > 0$, on collar neighbourhoods of the finitely many boundary components $\partial_i\Sigma_0$, using collar coordinates $(r_i, \varphi_i) \in (-1, 0] \times S^1$. Furthermore – this is where we extend the statement from our previous paper –, we consider the 1-form $\lambda_\gamma = s\,\mathrm{d}\varphi$ in terms of coordinates $(s, \varphi) \in (-1, 1) \times S^1$ on a neighbourhood of $\gamma \equiv \{0\} \times S^1$. It will be understood that this neighbourhood of γ is disjoint from the support of the Dehn twists.

Lemma 4.5. *If $\sum_i \rho_i(0) > 0$, there is an exact area form $\omega = \mathrm{d}\lambda$ on Σ_0 with $\lambda = \lambda_i$ near $\partial_i\Sigma$ and $\lambda = \lambda_\gamma$ near γ.*

Proof. Thanks to the assumption in the lemma, for $\varepsilon > 0$ sufficiently small we still have $\sum_i \rho_i(-\varepsilon) > 0$. Hence

$$\sum_i \int_{\partial_i \Sigma_0} \lambda_i = 2\pi \sum_i \rho_i(0) > 2\pi \sum_i \big(\rho_i(0) - \rho_i(-\varepsilon)\big) = \sum_i \int_{(-\varepsilon,0]\times\partial_i \Sigma_0} \mathrm{d}\lambda_i .$$

This allows us to choose an area form ω on Σ_0 that coincides with $\mathrm{d}\lambda_i$ near $\partial_i\Sigma_0$ and with $\mathrm{d}\lambda_\gamma$ near γ, and which satisfies

$$\int_{\Sigma_0} \omega = \sum_i \int_{\partial_i \Sigma_0} \lambda_i .$$

Now apply the argument from Lemma 3.3 in [6], involving compactly supported cohomology, to the surface obtained by cutting open Σ_0 along γ in order to find a primitive λ of ω with the desired properties. ■

Write $\psi\colon \Sigma_0 \to \Sigma_0$ for the diffeomorphism given by the two boundary-parallel Dehn twists. As shown in Section 7.2 of [6], an area form $\omega = \mathrm{d}\lambda$ as just constructed gives rise to a Bott-integrable contact form on $\Sigma \times S^1$, where the Bott integral on the mapping torus $\Sigma_0(\psi)$ can be taken to be one induced from any Morse function on Σ_0 that only depends on the radial coordinate in the collar neighbourhoods of the boundary (and hence is invariant under ψ). On the complement $\Sigma_0' \subset \Sigma_0$ of two collars containing the annuli where the Dehn twists are performed, the mapping torus looks like a product $\Sigma_0' \times S^1$ with contact form $\lambda + C\,\mathrm{d}\theta$, where C is some large positive constant.

The boundary of this mapping torus is $\partial\Sigma_0(\psi) = \partial\Sigma_0 \times S^1$, and the manifold $\Sigma \times S^1$ is obtained by gluing two solid tori $S^1 \times D^2$ to this mapping torus along the boundary components $(\partial\Sigma_0 \setminus \partial\Sigma) \times S^1$. Then extend the Morse–Bott function on $\Sigma_0(\psi)$ over these two solid tori as a function in the radial coordinate on the D^2-factor, with an isolated non-degenerate maximum at the centre. This Morse–Bott function is an integral for the Giroux contact form on the open book decomposition of $\Sigma \times S^1$.

The next lemma says that the Morse function on Σ_0 can be chosen to be adapted to any embedded circle $\gamma \subset \Sigma_0$ (or a collection of such circles).

Lemma 4.6. *Given $\gamma \subset \Sigma_0$, we can find a Morse function on Σ_0 which is a strictly monotone function of the radial coordinate in the collar neighbourhoods of the boundary, and a strictly monotone function of s in the coordinates $(s,\varphi) \in (-1,1) \times S^1$ around γ.*

Proof. Let Σ_0^γ be the surface obtained by cutting open Σ_0 along γ and compactifying it by including the new boundary circles $\pm\gamma$. The desired Morse function is given by the height function (say, the z-coordinate) on a copy of Σ_0^γ embedded in $\mathbb{R}^3$ in such a way that each boundary component of Σ_0^γ lies in a plane $\{z = \text{constant}\}$, with the boundary components $\pm\gamma$ placed at the same height, one as a lower and one as an upper boundary. All other boundary components may be chosen to be local maxima of the height function. ■

Note 4.7. (1) The isolated minima of the Morse function on Σ_0 give rise to elliptic critical Reeb orbits. By performing a full Lutz twist along such an orbit, we obtain an overtwisted Bott-integrable contact structure in the same homotopy class of tangent 2-plane fields.

(2) We may assume that the 1-form λ we construct for any given simple closed loop $\gamma \subset \Sigma_0$ does not depend on γ near $\partial\Sigma_0$. Then the convex linear interpolation between the 1-forms λ and λ' corresponding to two curves $\gamma, \gamma' \subset \Sigma_0$ induces a homotopy of contact forms relative to a neighbourhood of $\partial\Sigma_0$. This means that the isotopy class of the contact structure we constructed on $\Sigma \times S^1$ does not depend on γ.

4.1.2. Introducing Lutz components and critical orbits. Let $\gamma \subset \Sigma_0$, its neighbourhood $(-1,1)_s \times S^1_\varphi$, and the adapted Morse function on Σ_0 be as before. On the thickened torus $(-\varepsilon,\varepsilon)_s \times S^1_\varphi \times S^1_\theta$, for $\varepsilon > 0$ sufficiently small, the contact form $\lambda + C\,\mathrm{d}\theta$ equals $s\,\mathrm{d}\varphi + C\,\mathrm{d}\theta$. We now perform a full Lutz twist, that is, we replace this contact form by $h_1(s)\mathrm{d}\varphi + h_2(s)\mathrm{d}\theta$, with $(-\varepsilon,\varepsilon) \ni s \mapsto (h_1(s), h_2(s))$ describing the planar curve as shown in Figure 1, subject to the boundary conditions $h_1(s) = s$ and $h_2(s) = C$ near $s = \pm\varepsilon$. In this way, we construct three smaller thickened tori where, up to irrelevant positive constants, the contact form is given by $h_1(s)\,\mathrm{d}\varphi \pm \mathrm{d}\theta$ or $\mathrm{d}\varphi + h_2(s)\mathrm{d}\theta$, whose Reeb vector field is a constant positive multiple of $\pm\partial_\theta$ or ∂_φ, respectively.

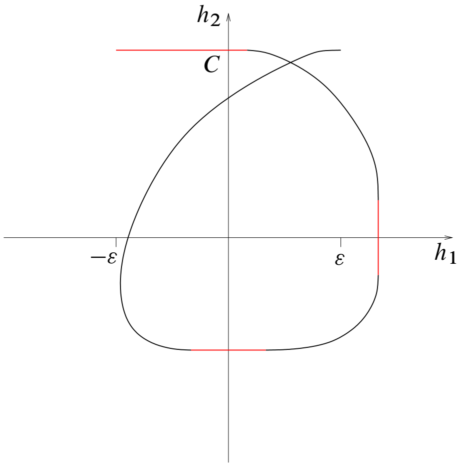

Figure 1. Introducing a Lutz component.

This modified contact form still has the same Bott integral (a monotone function of s), and the homotopy class of the contact structure as a tangent 2-plane field is unchanged by the full Lutz twist; see Lemma 4.5.3 in [5].

In a second step, we modify the Bott integral by a finite number of 'reverse Giroux eliminations' in the (s,φ)- or the (s,θ)-plane transverse to the Reeb flow in the direction of $\pm\partial_\theta$ or ∂_φ, respectively; see Section 2.3 of [6] for the details. In this way, we can introduce any finite number of pairs of elliptic and hyperbolic critical orbits into the Reeb flow, lying in the isotopy class of $\pm S^1_\theta$ (the fibre class) or S^1_φ (the class represented by γ).

4.1.3. Proof of Proposition 4.3 for orientable base. Any Seifert manifold M with orientable base is obtained from a product manifold $\Sigma \times S^1$, where Σ may always be assumed to have at least one boundary component, by gluing solid tori to some of the boundary components. The contact form we constructed on $\Sigma \times S^1$ is of Lutz type near the boundary, and so it extends over the solid tori as a Bott-integrable contact form with an elliptic critical Reeb orbit at each spine.

Write g for the genus of Σ, and $n+1 \geq 1$ for the number of boundary components, so that Σ is obtained from the closed, orientable surface Σ_g of genus g by removing $n+1$ discs. The homology group $H_1(\Sigma \times S^1) \cong \mathbb{Z}^{2g+n+1}$ is freely generated by the standard generators $a_1, b_1, \dots, a_g, b_g$ of $H_1(\Sigma_g)$, boundary parallel loops $q_1, \dots, q_n$ to n of the $n+1$ boundary components, and the fibre class $h = [* \times S^1]$. The gluing of solid tori adds relations to give us $H_1(M)$, but no new generators. It therefore suffices to show that the contact structure on $\Sigma \times S^1$ is H_1-complete.

Any class $u \in H_1(\Sigma \times S^1)$ can be written as

$$u = kc + \sum_{i=1}^{n} \ell_i q_i + mh,$$

with c an indivisible class in the subgroup generated by $a_1, b_1, \dots, a_g, b_g$, the coefficient k a non-negative integer, and $\ell_1, \dots, \ell_n, m \in \mathbb{Z}$. By [17], the class c can be represented by a simple closed loop $\gamma \subset \Sigma_0$. The sum $\sum_i \ell_i q_i$ can be represented by a disjoint union of boundary parallel curves in the collars of $\partial\Sigma \subset \Sigma_0$, where these curves are level sets of the Bott integral. The class mh is represented by m parallel S^1-fibres of $\Sigma_0 \times S^1$. Thus, our arguments above show that u can be represented by a collection of some elliptic critical Reeb orbits of a suitably modified Bott-integrable contact form on $\Sigma \times S^1$.

Remark 4.8. In order to obtain a Seifert manifold, the solid tori have to be attached to $\Sigma \times S^1$ by sending the meridian of the solid torus to a simple closed curve on the boundary torus winding non-trivially around the corresponding boundary component of Σ. If instead the meridian is glued to a curve representing the S^1-fibre of $\Sigma \times S^1$, one speaks of a *generalised* Seifert manifold; see [13]. Our argument applies equally to those.

4.2. Non-orientable base

Any Seifert manifold over a non-orientable base can be obtained from a Seifert manifold over an orientable base by a fibre connected sum (along a regular fibre) with the unique non-trivial S^1-bundle over $\mathbb{R}P^2$. When we think of $\mathbb{R}P^2$ as being obtained by gluing a Möbius band and a 2-disc, the restriction of the non-trivial S^1-bundle to the Möbius band gives us precisely the Klein bottle neighbourhood νK described earlier. The only new homological contribution comes from the spine of the Möbius band.

4.2.1. Modification near a Klein bottle. On νK we take the standard Bott-integrable contact form $\alpha = \mathrm{d}s - r\,\mathrm{d}\theta$ as in Theorem 2.3. Inside this neighbourhood, we can introduce a Lutz component as in Section 4.1.2 near a regular torus $\{r = r_0\}$ for some small positive r_0, and then introduce an elliptic critical Reeb orbit in the class of the S^1-fibre. Also, by a perturbation as in Section 2.2, we create an elliptic critical Reeb orbit inside the Klein bottle – thought of as the non-orientable S^1-bundle over the spine of the Möbius band – in the class of the spine.

The latter elliptic orbit gives us the H_1-completeness; the former allows us to perform the fibre connected sum.

4.2.2. Fibre connected sum. The general construction of fibre connected sums of contact manifolds along two diffeomorphic contact submanifolds with opposite normal bundles has been described in Section 7.4 of [5].

In our situation, an *ad hoc* construction is much simpler and yields the Bott-integrability to boot. Indeed, the fibre connected sum along two elliptic critical Reeb orbits can be defined by removing standard neighbourhoods as in Theorem 2.4 of [6] around the two orbits, and then identifying the toral boundaries of the complements by a diffeomorphism that sends the fibre circle on one side to the fibre circle on the other, and glues the meridional circles with a flip of the orientation. Notice that the outer normal on one complement glues with the inner normal on the other, so the gluing of the torus boundary should indeed be done with an orientation-reversing diffeomorphism.

Since the contact form near the boundary of a tubular neighbourhood of an elliptic orbit is of Lutz type, the Bott-integrable contact form on the fibre connected sum can now be constructed by an application of the sewing lemma (Lemma 3.4 in [6]).

This concludes the proof of Proposition 4.3.

5. Proof of the main theorem

We now want to prove the 'if' part of Theorem 1.2 for a fixed closed, orientable 3-dimensional graph manifold M. Our task is to show that any overtwisted contact structure ξ on M with $\mathrm{PD}(e(\xi)) \in H_1(M)$ representable by a graph link is Bott integrable. Similar to the proof of this statement for Seifert manifolds in Corollary 4.4, we need to find a Bott-integrable overtwisted contact structure ξ_0 on M which is H_1-complete within the range of homology classes representable by graph links.

This strategy is based on a homological characterisation of graph links, which has been established by Yano [22]. We recall some of the relevant 3-manifold topology; for more details, we recommend Hatcher's notes [10].

5.1. Graph manifolds and the JSJ decomposition

Graph manifolds, as introduced by Waldhausen [20,21], are the closed, orientable 3-manifolds that can be decomposed along a disjoint collection of embedded 2-tori into Seifert fibred pieces.

Given a graph manifold M, we consider a prime decomposition

$$M = M_1 \# \cdots \# M_n$$

of M as a connected sum of prime 3-manifolds $M_1, \dots, M_n$. Recall that M_i being prime means that it is not diffeomorphic to the 3-sphere S^3, and that in any connected sum decomposition $M_i = M_i' \# M_i''$, one of the summands M_i', M_i'' must be S^3. In this prime decomposition, the summands $M_1, \dots, M_n$ are uniquely determined by M, but the isotopy class of the collection of 2-spheres defining the splitting into prime summands is not.

The summands $M_1, \dots, M_n$ in the prime decomposition of a graph manifold M are also graph manifolds; see Theorems 6.3 and 7.1 in [21]. Conversely, the connected sum of graph manifolds is a graph manifold; see the remark on page 91 of [21] or Lemma 3.1 in [6].

A 3-manifold is called irreducible if every embedded 2-sphere bounds a ball. In particular, an irreducible 3-manifold is prime. The manifold $S^1 \times S^2$ plays a special role in 3-manifold topology as the only prime manifold that is not irreducible.

A 2-torus embedded in a 3-manifold is said to be incompressible if the inclusion map induces a monomorphism on fundamental groups. A 3-manifold is called atoroidal if the only incompressible tori it contains are boundary parallel.

If M is an irreducible manifold, it can be cut along a disjoint collection of incompressible tori into pieces that are either atoroidal or Seifert fibred. Rather amazingly, the minimal (with respect to inclusion) collection of such tori is unique up to isotopy; see Theorem 1.9 in [10]. This unique decomposition is known as the JSJ decomposition of M, named after Jaco–Shalen [12] and Johannson [14].

It turns out that amongst irreducible 3-manifolds, graph manifolds are characterised as those whose JSJ decomposition only contains Seifert fibred pieces. One direction of this statement is obvious; the other is the content of Theorem 6.3 in [21]. In fact, in the contemporary literature this characterisation is often taken as the definition of a graph manifold.

5.2. Graph links

A graph link is a link in a closed, orientable 3-manifold whose exterior, that is, the complement of an open tubular neighbourhood, is a graph manifold. Observe that any sublink of a graph link, including the empty one, is graph; in particular, any manifold containing a graph link is a graph manifold.

5.2.1. Graph manifolds prime to $S^1 \times S^2$. A closed, orientable 3-manifold M is called prime to $S^1 \times S^2$ if it does not contain any summands $S^1 \times S^2$ in its prime decomposition, so that M is the connected sum of irreducible manifolds.

In [22, 23], Yano associates with a graph manifold M prime to $S^1 \times S^2$ its JSJ complex $\mathcal{C}_M$, which is a 1-dimensional complex built as follows. Write M as the connected sum $M = M_1 \# \cdots \# M_n$ of irreducible manifolds $M_1, \dots, M_n$. For each M_i, we consider its JSJ decomposition into Seifert fibred pieces. Thicken the incompressible tori defining this decomposition into disjointly embedded copies of $T^2 \times [-1, 1]$; the closure of their complement is then a finite collection $\{S_{ij}\}$ of Seifert manifolds with toral boundaries.

Now, for each irreducible summand M_i, we build the 1-complex consisting of one vertex for each S_{ij} and a connecting edge for the gluing of two boundary components. In the JSJ decomposition, it may well happen that one toral boundary component of a Seifert fibred piece is glued to another boundary component of the same piece, as we shall see in an example in Section 6, so some 1-cells in the complex may form loops.

We then introduce $n - 1$ further edges, connecting a vertex in the graph for M_i with one in the graph for M_{i+1}, $i = 1, \dots, n-1$. The resulting complex $\mathcal{C}_M$ does not depend, up to homotopy equivalence, on the choices in forming the prime decomposition of M and in connecting the n constituent parts.

There is a natural map $\rho\colon M \to \mathcal{C}_M$, defined by collapsing the Seifert fibred pieces (possibly with an open 3-ball removed if the connected sum is performed in that piece) of the prime summands to the corresponding vertex, and the thickened tori as well as the thickened 2-spheres where the connected sums are formed to the corresponding edge. In other words, the neighbourhoods $T^2 \times [-1,1]$ and $S^2 \times [-1,1]$ of those tori and spheres are projected onto the interval $[-1,1]$, regarded as the edge connecting the vertices in $\mathcal{C}_M$ representing the adjacent Seifert fibred piece(s).

Here is the main theorem from [22].

Theorem 5.1 (Yano). *Let M be a graph manifold prime to $S^1 \times S^2$. A homology class $u \in H_1(M)$ can be represented by a graph link if and only if $\rho_*(u) = 0$ in $H_1(\mathcal{C}_M)$.*

As a crucial preliminary step for this theorem, Yano establishes the particular case that every element in the first homology group of a Seifert manifold can be represented by a graph link, see Proposition 2.2 in [22]. Our argument in Section 4 gives an alternative proof of that part.

5.2.2. Graph manifolds containing ($S^1 \times S^2$)-summands. If the graph manifold M is not prime to $S^1 \times S^2$, one fixes a description of M as a connected sum

$$M = N\#k(S^1 \times S^2),$$

with N a graph manifold prime to $S^1 \times S^2$. This induces a splitting $H_1(M) = H_1(N) \oplus \mathbb{Z}^k$, and we write elements of $H_1(M)$ as pairs (a,b) with respect to this splitting. For $b \neq 0$, we write $m(b)$ for the largest natural number dividing $b \in \mathbb{Z}^k$, and we set $m(0) = 0$. The JSJ complex $\mathcal{C}_N$ and the map $\rho^N \colon N \to \mathcal{C}_N$ are constructed as before.

Here is Yano's homological characterisation of graph links (Theorem 3.3 in [22]) in this general setting.

Theorem 5.2 (Yano). *Let $M = N\#k(S^1 \times S^2)$ be a graph manifold, with N prime to $S^1 \times S^2$. A homology class $u = (a,b) \in H_1(M) = H_1(N) \oplus \mathbb{Z}^k$ can be represented by a graph link if and only if $\rho^N_*(a)$ is divisible by $m(b)$.*

5.3. Graph manifolds prime to $S^1 \times S^2$

We can now prove the 'if' part of Theorem 1.2 for any graph manifold M prime to $S^1 \times S^2$. Write M as a connected sum $M = M_1\#\cdots\#M_n$ of irreducible graph manifolds. Construct an overtwisted Bott-integrable contact structure ξ_0 on M as follows:

(i) First we construct a Bott-integrable contact structure on each M_i by sewing together H_1-complete contact structures on the Seifert pieces S_{ij} of the JSJ decomposition of M_i.

(ii) Form the connected sum of the M_i as in Section 6 of [6].

(iii) If necessary, perform a full Lutz twist along an elliptic critical orbit to make the contact structure overtwisted.

From Section 3, we know that the Poincaré dual of the Euler class $e(\xi_0)$ is represented by a graph link. Hence, by Theorem 5.1, we know that $\rho_{\mathrm{dual}}(e(\xi_0)) = 0$, where for ease of notation we set

$$\rho_{\mathrm{dual}} := \rho_* \circ \mathrm{PD} : H^2(M) \to H_1(\mathcal{C}_M).$$

Now let η be any tangent 2-plane field on M with an Euler class $e(\eta)$ whose Poincaré dual can be represented by a graph link, so that $\rho_{\mathrm{dual}}(e(\eta)) = 0$. Then for the obstruction class $d^2(\xi_0, \eta)$ we find

$$2\rho_{\text{dual}}(d^2(\xi_0, \eta)) = \rho_{\text{dual}}((e(\xi_0) - e(\eta)) = 0.$$

Since $H_1(\mathcal{C}_M)$ is a free abelian group, this implies $\rho_{\text{dual}}(d^2(\xi_0, \eta)) = 0$.

The class $\text{PD}(d^2(\xi_0, \eta))$ can be represented by an oriented link $L_\eta \subset M$ intersecting the neighbourhoods $S^2 \times [-1, 1]$ and $T^2 \times [-1, 1]$ of the spheres defining the connected sum and the tori defining the JSJ decomposition, respectively, along intervals $* \times [-1, 1]$, each of which is mapped to an edge of $\mathcal{C}_M$. The fact that the class $\rho_{\text{dual}}(d^2(\xi_0, \eta)) = [\rho(L_\eta)] \in H_1(\mathcal{C}_M)$ vanishes implies that the intersection number of the 1-cycle $\rho(L_\eta) \subset \mathcal{C}_M$ with any point in $\mathcal{C}_M$ is zero. For $L_\eta \subset M$, this means that the link traverses each neighbourhood $S^2 \times [-1, 1]$ or $T^2 \times [-1, 1]$ zero times, when counted with orientation. Thus, by link connected sums along pairs of positive and negative strands passing through such a neighbourhood, followed by an isotopy, we may assume that each component of L_η is contained entirely in one of the Seifert fibred components S_{ij}. The collection of all link components contained in a single S_{ij} defines a class in $H_1(S_{ij})$.

In summary, the homology class $\text{PD}(d^2(\xi_0, \eta)) \in H_1(M)$ can be written as a sum of homology classes coming from the inclusions $S_{ij} \subset M$. As above, for this inclusion to make sense, we have to remove an open ball from the S_{ij} where the connected sum with $M_{i\pm1}$ is performed.

Now the H_1-completeness of the S_{ij} (Proposition 4.3) allows us to modify ξ_0 into a Bott-integrable contact structure on M homotopic to η as a tangent 2-plane field as in the proof of Lemma 4.2. This completes the proof of Theorem 1.2 for the case under consideration.

Observe that, by Yano's criterion, we may assume the link L_η to be a graph link. It will then remain a graph link after the described modification, since (self) link connected sum preserves the graph link property. However, the sublink of critical elliptic orbits that H_1-completeness allows us to construct need not be equal to this graph link; in general, it merely represents the same class in homology.

5.4. Graph manifolds containing ($S^1 \times S^2$)-summands

Finally, we come to the general case of the 'if' direction in Theorem 1.2. Write $M = N\#k(S^1 \times S^2)$ with N a graph manifold prime to $S^1 \times S^2$.

Let η be a tangent 2-plane field on M with

$$\text{PD}(e(\eta)) = (a_\eta, b_\eta) \in H_1(N) \oplus \mathbb{Z}^k = H_1(M)$$

represented by a graph link. On N, as before, we choose a Bott-integrable contact structure that is H_1-complete on the Seifert fibred pieces coming from the JSJ decompositions of the irreducible summands of N. On $k(S^1 \times S^2)$ we may choose, by Theorem 1.9 in [6], a Bott-integrable contact structure whose Euler class has the Poincaré dual b_η. Write ξ_0 for the Bott-integrable contact structure on M obtained via the connected sum.

Then $\text{PD}(e(\xi_0)) = (a_0, b_\eta)$ for some $a_0) \in HH_1(N)$. Now, the class

$$(a_0 - a_\eta, 0) = \text{PD}(e(\xi_0) - e(\eta))$$

is representable by a graph link, and it equals $2\mathrm{PD}(d^2(\xi_0, \eta))$. This allows us to write

$$\mathrm{PD}(d^2(\xi_0, \eta)) = (a_{0\eta}, 0)$$

with $2a_{0\eta} = a_0 - a_\eta$. Moreover, by Theorem 5.2, the class $\rho_*^N(a_0 - a_\eta) \in H_1(\mathcal{C}_N)$ is divisible by $m(0) = 0$, which forces it to equal 0. As above, using the fact that $H_1(\mathcal{C}_N)$ is a free abelian group, we conclude that $\rho_*^N(a_{0\eta}) = 0$. As in the previous case, we then argue that the Poincaré dual $(a_{0\eta}, 0)$ of the obstruction class $d^2(\xi_0, \eta)$ can be represented by a link with each component contained in a Seifert fibred piece of N, and the proof concludes as before.

6. The mapping torus of Arnold's cat map

In this section, we illustrate many of the notions used in the proof of Theorem 1.2 with an instructive example.

The diffeomorphism ψ of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

known as Arnold's cat map, possesses several interesting dynamical features, in spite of its apparent simplicity. For instance, ψ is an Anosov diffeomorphism, an ergodic map, and mixing. The torus automorphism ψ is of hyperbolic type, with eigenvalues $\lambda := (3 + \sqrt{5})/2$ and $(3 - \sqrt{5})/2 = \lambda^{-1}$.

Let $M = M(\psi)$ be the mapping torus of ψ, that is, the torus bundle over S^1 defined as the quotient space of $T^2 \times \mathbb{R}$ under the $\mathbb{Z}$-action generated by

$$(x, y; t) \longmapsto (2x + y, x + y; t - 1).$$

The T^2-fibration of M is induced by the map $(x, y; t) \mapsto t$. This aspherical (and hence irreducible) 3-manifold carries Sol geometry and is not Seifert fibred [19].

6.1. Topological data of $M(\psi)$

The inclusion $T^2 \to M$ of a fibre is π_1-injective by the homotopy exact sequence of the fibration $M \to S^1$, so the fibre $T^2 \subset M$ is incompressible. In particular, M is not atoroidal. This implies that the JSJ decomposition requires cutting along at least one torus, and indeed one will suffice: when an open tubular neighbourhood of a fibre is removed, the complement $T^2 \times [\varepsilon, 1 - \varepsilon]$ is S^1-fibred. So the JSJ complex of M consists of a single 1-cell attached to a single 0-cell.

The cat map ψ has a single fixed point $(x, y) = (0, 0)$. So the standard cell decomposition of T^2 (with the zero cell taken as the fixed point of ψ) gives rise to a cell decomposition of M with an additional $(k + 1)$-cell for each k-cell of T^2. This allows one to read off the following presentation of the fundamental group:

$$\pi_1(M) \cong \langle a, b, c \mid [a, b],\ acb^{-1}a^{-2}c^{-1},\ bcb^{-1}a^{-1}c^{-1} \rangle.$$

The first homology group $H_1(M)$, as the abelianisation of $\pi_1(M)$, is then isomorphic to $\mathbb{Z}$, generated by the loop in $t \mapsto (0,0;t)$, $t \in [0,1]$, corresponding to c.

It follows that Yano's homomorphism $\rho_*\colon H_1(M) \to H_1(\mathcal{C}_M)$ is an isomorphism. By Theorem 5.1, every graph link in M must be homologically trivial. As a first consequence we note that, thanks to Lemmas 3.1 and 3.2, any Bott-integrable contact structure on $M(\psi)$ must have trivial Euler class.

6.2. A family of integrable contact structures on $M(\psi)$

Consider a contact form

$$\alpha = h_1(t)\,\mathrm{d}x + h_2(t)\,\mathrm{d}y$$

of Lutz type on $T^2 \times \mathbb{R}$. The condition for this to be invariant under the $\mathbb{Z}$-action with quotient $M(\psi)$ is that

$$\begin{cases} h_1(t) = 2h_1(t-1) + h_2(t-1), \\ h_2(t) = h_1(t-1) + h_2(t-1). \end{cases} \tag{6.1}$$

On the other hand, if one takes u, v to be coordinates on $\mathbb{R}^2$ with respect to a basis of eigenvectors of A for the eigenvalues λ and λ^{-1}, the invariant 1-form α should look like

$$\alpha = \lambda^t\,\mathrm{d}u + \lambda^{-t}\,\mathrm{d}v,$$

which is more obviously adapted to Sol geometry [19].

With some linear algebra, one arrives at the following proposition.

Proposition 6.1. *A countable family α_n, $n \in \mathbb{N}_0$, of Bott-integrable contact forms on $M(\psi)$ can be defined by the formula*

$$\begin{aligned}\alpha_n = \frac{\sqrt{2}}{\sqrt{5}}\Big[&\Big(\sin\Big(\frac{\pi}{4} + 2n\pi t\Big)\lambda^t - \cos\Big(\frac{\pi}{4} + 2n\pi t\Big)\lambda^{-t}\Big)\,\mathrm{d}x \\ &+ \Big(\frac{\sqrt{5}-1}{2}\,\sin\Big(\frac{\pi}{4} + 2n\pi t\Big)\lambda^t + \frac{\sqrt{5}+1}{2}\,\cos\Big(\frac{\pi}{4} + 2n\pi t\Big)\lambda^{-t}\Big)\,\mathrm{d}y\Big].\end{aligned}$$

Proof. The verification of the invariance condition (6.1) is straightforward. The factor $\sqrt{2}/\sqrt{5}$ has been chosen for purely aesthetic reasons; it ensures that $h_1(0) = 0$ and $h_2(0) = 1$, so that the sequence

$$h_2(1), h_1(1), h_2(2), h_1(2), h_2(3), h_1(3), \ldots$$

equals the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$

One computes that

$$h_1(t)h_2'(t) - h_1'(t)h_2(t) = -\frac{2}{\sqrt{5}}\,(2n\pi + \cos(4n\pi t)\log\lambda),$$

which is negative for all $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$, so the α_n are contact forms. A Bott integral for α_n is given by the lift of any Morse function in the variable $t \in \mathbb{R}/\mathbb{Z}$. ■

As tangent 2-plane fields, the contact structures $\ker\alpha_n$ are all homotopic to the trivial 2-plane bundle $\ker \mathrm{d}t$. The Reeb vector fields of the contact forms α_n are tangent to the fibres.

6.3. The classification of Bott-integrable contact structures on $M(\psi)$

The special linear group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix A defining the Arnold cat map can be factorised as

$$A = -ST^{-1}ST^{-2}S,$$

which is conjugate to $T^{-3}S$. Therefore, by Honda's classification (up to isotopy) of tight contact structures on torus bundles, see specifically the table on p. 90 of [11], all tight contact structures on $M(\psi)$ are universally tight, that is, their lift to $\mathbb{R}^3$ is tight. There is a unique tight structure for each value $2n\pi$, $n \in \mathbb{N}$, of its Giroux torsion (see Definition 1.2 in [9]), and two with zero Giroux torsion. For the relation between the Giroux torsion and the 'twisting in the S^1-direction' used by Honda, see p. 86 of [11].

Giroux's classification [9] of tight contact structures on torus bundles is up to diffeomorphism, and according to Theorem 1.3 in [9], there is a unique one for each value $2n\pi$ of the Giroux torsion, including the case $n = 0$.

The discrepancy between the classification up to isotopy and that up to diffeomorphism comes from the diffeomorphism of $M(\psi)$ which acts as minus the identity on the torus fibres and changes the coorientation of the contact structure. For positive torsion, this can be effected by an isotopy (a shift transverse to the fibres), but not for zero torsion.

Given this information, we can now formulate the complete classification of the Bott-integrable contact structures on $M(\psi)$.

Proposition 6.2. *The* $\ker\alpha_n$, $n \in \mathbb{N}_0$, *constitute a complete list of the tight contact structures on* $M(\psi)$ *and, as we have seen, they are all Bott integrable. An overtwisted contact structure on* $M(\psi)$ *is Bott integrable if and only if its Euler class is trivial. There is a countable family of such overtwisted structures, distinguished by their* 3*-dimensional homotopy invariant.*

Proof. On $\mathbb{R}^3$, the 1-form α_n, just like any 1-form of Lutz type, defines the standard tight contact structure; see Lemme 1 in [8] for the simple argument. For $n = 0$, the coefficient function h_2 of $\mathrm{d}y$ is always positive, so the Giroux torsion of $\ker\alpha_0$ is zero. An increase of n by 1 introduces one additional full twist around the origin by the curve $t \mapsto (h_1(t), h_2(t))$ on the interval $t \in [0,1]$, which implies an additional 2π Giroux torsion.

As there is no 2-torsion in $H^2(M(\psi))$, the overtwisted contact structures are classified by their Euler class and their 3-dimensional homotopy invariant. As we have seen, the Bott-integrable ones must have trivial Euler class, so we can obtain all of them simply by taking the connected sum of $\big(M(\psi), \ker\alpha_0\big)$, say, with any of the overtwisted contact structures on S^3, which are Bott integrable by Theorem 1.9 in [6]. ∎

Notice that triviality of the Euler class means that the contact structures are trivial as abstract 2-plane bundles, but as tangent 2-plane fields they are distinguished by the 3-dimensional homotopy invariant.

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