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Convergence of Schrödinger operators on domains with scaled resonant potentials

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Abstract. We consider Schrödinger operators on a bounded, smooth domain of dimension $d \geq 2$ with Dirichlet boundary conditions and a properly scaled potential, which depends only on the distance to the boundary of the domain. Our aim is to analyse the convergence of these operators as the scaling parameter tends to zero. If the scaled potential is resonant, the limit in strong resolvent sense is a Robin Laplacian with boundary coefficient expressed in terms of the mean curvature of the boundary. A counterexample shows that norm resolvent convergence cannot hold in general in this setting. If the scaled potential is non-resonant and satisfies an explicit assumption on the smallness of the negative part, the limit in strong resolvent sense is the Dirichlet Laplacian. We conjecture that we can drop this additional assumption in the non-resonant case.

1. Introduction

1.1. Background and motivation

Convergence of Schrödinger operators with scaled potentials is a classical topic in mathematical physics. A collection of results in this area can be found in the monograph [4]. In many cases, it is observed that one obtains in the strong or norm resolvent limit an operator with an interaction supported on a set of measure zero, being defined via a boundary condition. The structure of the limiting operator often drastically depends on whether the scaled potential is resonant or not.

A typical example, where the limit depends on whether the scaled potential is resonant, is the approximation of a Schrödinger operator with a point δ-interaction in three dimensions by a family of Schrödinger operators with scaled regular potentials. First partial results on this approximation are obtained in [2, 15, 27]. To the best of our knowledge, approximations of δ-interactions were first addressed in full detail by Albeverio and Høegh-Krohn in [5], who proved the convergence in strong resolvent sense. The limit has a non-trivial point interaction if the scaled potential is resonant, otherwise the

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limit is the free Laplacian. Shortly thereafter, it was shown in [3] that under rather non-restrictive additional assumptions, *norm* instead of *strong* resolvent convergence holds in this approximation. By restricting to the class of radially symmetric potentials, one obtains upon separation of variables a model problem on the half-line, which was considered separately by Šeba in [33]. He proved that a family of half-line Schrödinger operators with Dirichlet boundary conditions and locally scaled potentials converges in norm resolvent sense to the Neumann (or Robin) Laplacian on the half-line if the scaled potential is resonant, and to the Dirichlet Laplacian otherwise. Further refinements and extensions of these results can be found in [12, 32].

Typically, in the analysis of such a convergence, the integral kernel of the resolvent of the unperturbed operator is used and the resolvent identity plays a significant role, even though, in certain approximation problems with a non-explicit integral kernel it suffices to know only its singular part; cf. the recent analysis [28] of approximation of point interactions on bounded domains. Our main motivation is to develop an approach to this class of problems, which does not use the integral kernels of the resolvent, and where the analysis is merely performed on the level of quadratic forms. The advantage of our method here is that it can be efficiently applied to settings in which the integral kernel of the resolvent of the unperturbed operator is not given in an explicit form. Our final goal and main motivation is to analyse the convergence of Schrödinger operators on bounded smooth domains with Dirichlet boundary conditions and scaled potentials which depend only on the distance to the boundary.

We remark that a similar phenomenon, in which the limit qualitatively depends on whether the scaled potential is resonant or not, was also observed in full line Schrödinger operators with scaled potentials [1, 10], motivated by the applications to curved quantum waveguides. An extension of these results to scaled potentials with zero mean was later performed in a series of papers [16–18], where the first two papers treat the one-dimensional case, while the last one deals with the two-dimensional case. In the latter papers, the scaled potentials converge to δ' in the sense of distributions. Thus, the resulting limit may then be interpreted as a rigorous definition of a Schrödinger operator with a δ'-potential.

Potentials dependent only on the distance to a hypersurface are also used in the approximation of Schrödinger operators with surface δ-interactions [7, 8, 13], and Dirac operators with δ-shell interactions [9, 11, 24]. In these settings, the effect of resonant potentials does not occur, and the choice of the potential merely manifests in the values of the parameters characterising the limiting operator. In a certain sense, the limit "continuously" depends on the approximating potential. Another important difference is that in those settings, the convergence typically holds in norm resolvent sense, while in the setting considered in the present paper, in general, only strong resolvent convergence can be proved, as a counter-example shows.

The proof of our main result for resonant potentials relies on the construction of a suitable identification operator between the form domains of the limiting operator and of the operators with scaled resonant potentials. The key idea is to employ multiplication with the scaled resonant solution as cut-off function in such identification operators. We analyse the non-resonant case only partially, and use a similar method in which we employ instead the derivative of the non-resonant solution in the construction of identification operators.

1.2. Resonant potentials in one dimension

We use the definition of resonant potentials borrowed from [33]. This class will be used throughout the whole paper.

Definition 1.1. A real-valued potential $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ is called *resonant* if the initial-value problem

$$\begin{cases} -\psi'' + V\psi = 0, & \text{on } \mathbb{R}_+, \\ \psi(0) = 0, \end{cases} \tag{1.1}$$

has a bounded non-trivial solution $\psi_0 \in C^\infty(\overline{\mathbb{R}}_+)$ (called *resonant solution*). For the sake of convenience, we assume that $\operatorname{supp} V \subset [0, a]$ with some $a > 0$, and normalise the solution ψ_0 so that $\psi_0(t) = 1$ for all $t > a$.

Remark 1.2. Several observations on resonant potentials are in order.

(a) It is not hard to see that $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ is resonant if and only if the Schrödinger operator with potential V on the interval $(0, a)$ with Dirichlet boundary condition at $t = 0$ and Neumann boundary condition at $t = a$ has eigenvalue zero. Indeed, the continuous extension of the corresponding eigenfunction by a constant for $t > a$ gives a bounded solution of (1.1). Conversely, the restriction of a bounded solution ψ_0 to $(0, a)$ is in the kernel of the aforementioned Schrödinger operator.

(b) A potential $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ not satisfying Definition 1.1 will be called *non-resonant*. By the observation in (a) of this remark, we immediately see that any non-negative potential $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ is non-resonant. For non-resonant potentials, we normalise the solution ψ_0 of the initial value problem (1.1) so that $\psi_0'(t) = 1$ for all $t > a$. We call such a solution ψ_0 *non-resonant*.

(c) If the potential V is non-positive, then there exists a sequence $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \ldots$, with $\alpha_n \to \infty$, such that the potential αV with $\alpha > 0$ is resonant if and only if $\alpha \in \{\alpha_1, \alpha_2, \ldots\}$. The assumption on the smoothness of the potential V is imposed for technical reasons only (e.g., when using V' in Lemma 4.4). In particular, for the characteristic function $\chi_{(0,1)}$ of the interval $(0, 1)$, the potential $V = -\alpha\chi_{(0,1)}$ is resonant if and only if $\alpha = (n + 1/2)^2\pi^2$ for some $n \in \mathbb{N}_0$ (cf. [33]).

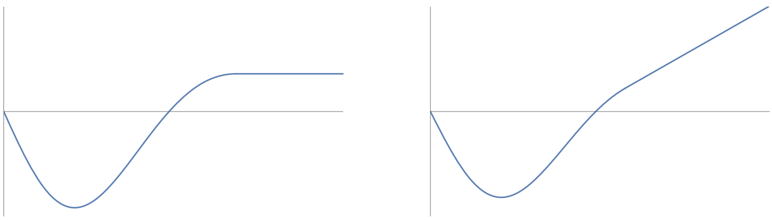

Figure 1.1. The solutions ψ_0 of (1.1) for resonant and non-resonant potentials on the left and on the right, respectively.

It was established in [33] that the family of self-adjoint Schrödinger operators in $L_2(\mathbb{R}_+)$ with Dirichlet boundary conditions

$$\mathsf{H}_\varepsilon\psi := -\psi'' + \frac{1}{\varepsilon^2} V\Big(\frac{\cdot}{\varepsilon}\Big)\psi, \quad \operatorname{dom} \mathsf{H}_\varepsilon := \mathring{H}^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$$

converges in norm resolvent sense (as $\varepsilon \to 0$) to the self-adjoint one-dimensional Laplace operator in $L_2(\mathbb{R}_+)$ with Neumann boundary conditions

$$\mathsf{H}\psi := -\psi'', \quad \operatorname{dom}\mathsf{H} := \{\psi \in H^2(\mathbb{R}_+) : \psi'(0) = 0\},$$

provided the potential V is resonant. However, if the potential V is non-resonant, the family of the operators H_ε converges in norm resolvent sense (as $\varepsilon \to 0$) to the Dirichlet Laplacian on the half-line

$$\mathsf{H}_0\psi := -\psi'', \quad \operatorname{dom}\mathsf{H}_0 := \mathring{H}^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+).$$

There are two main features in this approximation problem. Using appropriate test functions of the form $\psi_\varepsilon(t) = \varepsilon^{-1/2}\psi(t/\varepsilon)$, it can be checked that the operators in the approximating family are in general not uniformly bounded from below. Second, in the case of resonant potentials, the form domain $\mathring{H}^1(\mathbb{R}_+)$ of the approximating operators is a proper subspace of the form domain $H^1(\mathbb{R}_+)$ of the limit. These two features make the approximation problem difficult to treat with standard techniques based on comparison of quadratic forms; cf. Theorem VI.3.6 in [19].

Our aim in the present paper is to address a multi-dimensional counterpart of these convergence results, which we will describe below in detail. We remark that the result of Šeba in [33] is more general than we stated here. He has also shown how to approximate the one-dimensional Laplacian on the half-line with Robin boundary conditions by means of replacing $1/\varepsilon^2$ by $(1 + \beta\varepsilon)/\varepsilon^2$ in the operator family[1] H_ε. We will not address this more general case in our analysis of the multi-dimensional problem, as we already obtain a Robin-type boundary conditions by geometry.

1.3. Main results

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with C^∞-smooth connected boundary $\Sigma := \partial\Omega$. The C^∞-smoothness of the boundary and the boundedness of the domain are assumed for convenience, and most of our analysis extends to C^2-smooth domains with not necessarily compact boundaries under additional uniformity assumptions on the boundary (e.g., we have to require $\rho > 0$ in (2.4)). We denote by ν the outer unit normal vector to Ω. The differential of the Gauss map $\Sigma \ni s \mapsto \nu(s)$,

$$\mathsf{L}_s := \mathrm{d}\nu(s) : T_s\Sigma \to T_s\Sigma,$$

is called the *shape operator*; here $T_s\Sigma$ is the tangent space for Σ at $s \in \Sigma$. The eigenvalues $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{d-1}(s)\colon \partial\Omega \to \mathbb{R}$ of L_s are called the *principal curvatures* of $\partial\Omega$. In our sign convention for the shape operator, all the principal curvatures are non-negative if and only if Ω is convex. The mean curvature of Σ at $s \in \Sigma$ is defined as usual by

$$H(s) := \frac{1}{d-1}\sum_{j=1}^{d-1}\kappa_j(s).$$

[1] We remark that in the final formula in [33] for the boundary parameter in the limiting operator, the parameter β appears in the denominator, and formally the case $\beta = 0$ is excluded; but careful inspection of the proof shows that it applies also to the case $\beta = 0$, leading to Neumann boundary conditions in the limit.

Under our regularity assumptions on Ω, the mean curvature H is a C^∞-smooth function on Σ. Note that the mean curvature is non-negative for Ω being convex, while the converse is only true in two dimensions.

We adopt the notation $H^k(\Omega)$ for the L_2-based Sobolev space on Ω of order $k \in \mathbb{N}$, and $H^s(\partial\Omega)$ for the L_2-based Sobolev space on the boundary $\partial\Omega$ of order $s \in \mathbb{R}_+$. We denote by $\mathring{H}^k(\Omega)$ the closure of $C_c^\infty(\Omega)$ in the Sobolev space $H^k(\Omega)$ with respect to its norm. For a function $u \in H^2(\Omega)$, we use the notation $u\restriction_\Sigma \in H^{3/2}(\Sigma)$ for its trace on the boundary, and $\partial_\nu u\restriction_\Sigma \in H^{1/2}(\Sigma)$ for the normal derivative corresponding to the normal pointing outwards of Ω. The role of the one-dimensional Neumann Laplacian H from the previous subsection is played now by the self-adjoint Robin Laplacian in $L_2(\Omega)$

$$\mathsf{A}u := -\Delta u, \quad \operatorname{dom}\mathsf{A} := \Big\{u \in H^2(\Omega) : \partial_\nu u\restriction_\Sigma + \frac{d-1}{2} H u\restriction_\Sigma = 0\Big\}, \tag{1.2}$$

where the Robin coefficient is expressed in terms of the mean curvature. At the points where the mean curvature vanishes, we recover locally Neumann boundary conditions. The role of the family of one-dimensional Schrödinger operators H_ε is played by the self-adjoint Schrödinger operator in $L_2(\Omega)$ defined for $\varepsilon > 0$ by

$$\begin{aligned} \mathsf{A}_\varepsilon u &:= -\Delta u + V_\varepsilon u, \quad \operatorname{dom}\mathsf{A}_\varepsilon := \mathring{H}^1(\Omega) \cap H^2(\Omega),\\ &\text{where} \quad V_\varepsilon := \frac{1}{\varepsilon^2} V\Big(\frac{\operatorname{dist}(\cdot,\Sigma)}{\varepsilon}\Big), \end{aligned} \tag{1.3}$$

and where $V \in C_c^\infty(\overline{\mathbb{R}}_+)$. Our first main result concerns the class of resonant potential.

Theorem A (The resonant case). *Assume that $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ is resonant in the sense of Definition 1.1. Then the family of scaled Schrödinger operators A_ε converges to the Robin Laplacian A in strong resolvent sense as $\varepsilon \to 0$.*

In the proof of this result, we rely on a convenient representation of the sesquilinear form for the resolvent difference of the operators A and A_ε in terms of the resonant solution ψ_0. The technique shares common ideas with the abstract approach for proving norm resolvent convergence developed by the second-named author; see the monograph [31] and the references therein. Since the form domains of A and A_ε are different but the Hilbert spaces are the same, one only needs identification operators mapping from one form domain into the other. Thus, the analysis boils down to find a suitable identification operator which maps a function $H^1(\Omega)$ into a function $\mathring{H}^1(\Omega)$. In the construction of this operator, we use the resonant solution ψ_0 of (1.1). In Section 5, we construct a counterexample, which shows that the operators A_ε do not converge to A in *norm* resolvent sense. This counterexample relies on the analysis of the disk, where separation of variables is available. In particular, we cannot expect in general that norm resolvent convergence holds in Theorem A.

Remark 1.3 (Appearance of mean curvature terms in related problems).

(a) Note that the mean curvature term arises also in the large coupling asymptotics of the Robin Laplacian with a negative boundary parameter [14,20,29,30] and for the Robin Laplacian on a shell in the small thickness limit [21].

(b) The appearance of the curvature term in the boundary conditions of the limiting operator for scaled resonant potentials was also observed in [16] in two dimensions for a different approximation problem, in which the limit has transmission δ'-type boundary conditions. The result there is proved by a different technique, and is stated in terms of convergence of eigenvalues and weak convergence of eigenfunctions.

The role of the one-dimensional Dirichlet Laplacian H_0 from the previous subsection is played now by the self-adjoint Dirichlet Laplacian in $L_2(\Omega)$

$$\mathsf{A}_0 u := -\Delta u, \quad \operatorname{dom} \mathsf{A}_0 = \mathring{H}^1(\Omega) \cap H^2(\Omega).$$

Our second main result concerns the case of non-resonant potentials. Let us introduce the parameter

$$C_\Omega := \inf_{u \in \mathring{H}^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\displaystyle\int_\Omega |u(x)|^2 \,(\operatorname{dist}(x,\Sigma))^{-2} \,\mathrm{d}x}. \tag{1.4}$$

By Theorem 3.2.1, and the discussion below equation (3.2.4) in [6], we have $C_\Omega = 1/4$ for bounded smooth convex domains, while for general bounded C^∞-smooth domains, we have $C_\Omega \in (0, 1/4]$.

Theorem B (The non-resonant case). *Assume that $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ is such that the condition $\max_{t\in\mathbb{R}_+}(-t^2 V(t)) < C_\Omega$ holds. Then the family of operators A_ε converges to A_0 in strong resolvent sense as $\varepsilon \to 0$.*

It follows from the Hardy inequality on an interval (see equation 3.3.4 in [6]) that any potential satisfying the assumption in Theorem B is non-resonant, since the Schrödinger operator in Remark 1.2 (a) is strictly positive and thus cannot have eigenvalue zero. Any non-negative potential satisfies the assumption in the above theorem. Moreover, a small negative part is allowed. The assumption imposes no restriction on the positive part of the potential. The proof of this theorem relies on the same circle of ideas as those of the proof of Theorem A. In the construction of the identification operators, we use the derivative of the non-resonant solution instead of the non-resonant solution itself, as we do in the proof of Theorem A with the resonant solution. By analogy with the one-dimensional case, we expect that A_ε converge as $\varepsilon \to 0$ to A_0 in strong resolvent sense for general non-resonant potential V. However, we have not been able to find a proof for this claim.

In the proofs of both Theorems A and B, the use of identification operators with cut-off functions based on the solution ψ_0 leads to cancellation of 'bad' terms; see Lemmata 3.3 and 4.4. Identification operators with the same mapping properties, but with other cut-off functions, would not lead to such a cancellation.

Remark 1.4 (No uniform ellipticity). The condition on V in the non-resonant case implies that there is a constant $c > 0$ such that $\|(\mathsf{A}_\varepsilon + \mathrm{i})^{-1} v\|_{H^1(\Omega)} \le c\|v\|_{L_2(\Omega)}$ for all $v \in L_2(\Omega)$ and all ε small enough (see Lemma 4.6). If the latter estimate does not hold, then we can show that the family of sesquilinear forms associated with the operators A_ε is not uniformly elliptic (see Lemma 4.7), a concept called "equi-elliptic" in [26].

2. Preliminaries

All operators and forms act in the Hilbert space $L_2(\Omega)$; we denote its norm simply by $\|u\| := (\int_\Omega |u(x)|^2\,\mathrm{d}x)^{1/2}$. The L_2-norms of subsets $\Omega' \subset \Omega$ and similar norms are typically indicated by a corresponding subscript such as $\|u\|_{L_2(\Omega')}$.

2.1. Tubular coordinates

In this subsection, we briefly recall main properties of tubular coordinates. For any $t > 0$, we will use the notation $\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Sigma) < t\} \subset \Omega$, where $\operatorname{dist}(\cdot, \Sigma)$ is the Euclidean distance to Σ. By Theorem 5.25 in [22], there exists a sufficiently small $\delta > 0$ such that the mapping

$$\Phi : \Sigma \times (0, \delta) \to \mathbb{R}^d, \quad \Phi(s,t) := s - t\nu(s), \tag{2.1}$$

is a diffeomorphism onto Ω_δ. This mapping defines coordinates (s,t) in Ω on the tubular neighbourhood Ω_δ of Σ. The metric G induced on $\Sigma \times (0,\delta)$ by this embedding is

$$G = g \circ (\mathsf{I}_s - t\,\mathsf{L}_s)^2 + \mathrm{d}t^2, \tag{2.2}$$

where $\mathsf{I}_s\colon T_s\Sigma \to T_s\Sigma$ is the identity map, and g is the metric on Σ induced by the embedding into $\mathbb{R}^d$. The volume form associated with the metric G on $\Sigma \times (0,\delta)$ is given by

$$|\det G|^{1/2}\,\mathrm{d}s\,\mathrm{d}t = \varphi(s,t)\,|\det g|^{1/2}\,\mathrm{d}s\,\mathrm{d}t = \varphi(s,t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t,$$

where

$$\varphi : \Sigma \times (0,\delta) \to \mathbb{R}, \quad \varphi(s,t) = |\det(\mathsf{I}_s - t\,\mathsf{L}_s)| = 1 - (d-1)H(s)t + p(s,t)t^2 \tag{2.3}$$

and where p is a polynomial in t with C^∞-smooth coefficients depending on s. We will also make use of the following constant:

$$\rho := \min_{(s,t)\in\Sigma\times(0,\delta)} \varphi(s,t) > 0. \tag{2.4}$$

Note that $\rho > 0$ is automatically fulfilled as Σ is compact, φ is continuous, and Φ is a diffeomorphism. Let us choose an orthonormal local coordinate system $(e_1(s), \dots, e_{d-1}(s))$ on Σ at $s \in \Sigma$. By (2.2), the matrix $(G_{jk})_{j,k=1}^d$ of the metric G in the local coordinate system $(e_1(s), \dots, e_{d-1}(s), \nu(s))$ on Ω_δ at $x = \Phi(s,t)$ has block structure and, in particular, $G_{jd} = G_{dj} = \delta_{jd}$ for all $j \in \{1, 2, \dots, d\}$; cf. Lemma 2.3 in [23].

Let us define the following unitary map:

$$\mathsf{U} : L_2(\Omega_\delta) \to L_2(\Sigma \times (0,\delta); \varphi(s,t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t), \quad (\mathsf{U}u)(s,t) = u(\Phi(s,t)).$$

For any $u, v \in H^1(\Omega_\delta)$, and with the notation $\tilde{u} := \mathsf{U}u$ and $\tilde{v} := \mathsf{U}v$, we obtain

$$\begin{aligned}\int_{\Omega_\delta} \nabla u\,\overline{\nabla v}\,\mathrm{d}x &= \int_0^\delta \int_\Sigma \sum_{j,k=1}^d G^{jk}\partial_j\tilde{u}\,\overline{\partial_k\tilde{v}}\,\varphi(s,t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t \\ &= \int_0^\delta \int_\Sigma \Big(\sum_{j,k=1}^{d-1} G^{jk}\partial_j\tilde{u}\,\overline{\partial_k\tilde{v}} + \partial_d\tilde{u}\,\overline{\partial_d\tilde{v}}\Big)\varphi(s,t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t, \end{aligned} \tag{2.5}$$

where the derivatives ∂_j for $j = 1, \ldots, d-1$ on Σ correspond to the choice of local coordinate system and where $(G^{jk})_{j,k=1}^d$ stands for the inverse to the matrix $(G_{jk})_{j,k=1}^d$. We also write ∂_t for ∂_d if we need to stress that the derivative is with respect to the d-th variable t.

2.2. Quadratic forms and operators

The self-adjoint Robin Laplacian A defined in (1.2) with mean curvature entering the boundary condition is associated with the following closed, densely defined, symmetric, and lower-semibounded quadratic form:

$$\mathfrak{a}[u] := \|\nabla u\|^2_{L_2(\Omega;\mathbb{C}^d)} + \frac{d-1}{2}\int_\Sigma H(s)\,|u(s)|^2\,\mathrm{d}\sigma(s), \quad \operatorname{dom}\mathfrak{a} := H^1(\Omega),$$

in the Hilbert space $L_2(\Omega)$. Assume that $\varepsilon > 0$ is so small such that $\operatorname{supp} V \subset [0, \delta\varepsilon^{-1})$ holds. Then the Schrödinger operator A_ε defined in (1.3) is associated with the closed, densely defined, symmetric, and lower-semibounded quadratic form

$$\mathfrak{a}_\varepsilon[u] := \|\nabla u\|^2_{L_2(\Omega;\mathbb{C}^d)} + \frac{1}{\varepsilon^2}\int_0^\delta\int_\Sigma V\Big(\frac{t}{\varepsilon}\Big)|u(\Phi(s,t))|^2\,\varphi(s,t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t,$$
$$\operatorname{dom}\mathfrak{a}_\varepsilon := \mathring{H}^1(\Omega)$$

in the Hilbert space $L_2(\Omega)$. Finally, the Dirichlet Laplacian A_0 is associated with the closed, non-negative, densely defined quadratic form in $L_2(\Omega)$ defined by

$$\mathfrak{a}_0[u] := \|\nabla u\|^2_{L_2(\Omega;\mathbb{C}^d)}, \quad \operatorname{dom}\mathfrak{a}_0 := \mathring{H}^1(\Omega).$$

Let us denote the resolvents of A_ε, A, and A_0 (at the point $\lambda = \mathrm{i}$) by

$$\mathsf{R}_\varepsilon := (\mathsf{A}_\varepsilon - \mathrm{i})^{-1}, \quad \mathsf{R} := (\mathsf{A} - \mathrm{i})^{-1}, \quad \text{and} \quad \mathsf{R}_0 := (\mathsf{A}_0 - \mathrm{i})^{-1}.$$

Note also that by elliptic regularity (Theorem 4.18 in [25]), for any function $u \in C_c^\infty(\Omega)$, we have $\mathsf{R}_\varepsilon^* u, \mathsf{R}u, \mathsf{R}_0 u \in C^\infty(\overline{\Omega})$. This observation will be used in the proofs of Theorems A and B.

3. Convergence for resonant potentials

We split the proof of Theorem A into several steps. We first define some auxiliary boundary mappings which will be used in a convenient representation for the difference of the sesquilinear forms of A and A_ε. In this representation, we also use an identification operator defined in a second step and mapping functions from the form domain of A into the form domain of A_ε (the latter requires a Dirichlet boundary condition on $\partial\Omega$). The identification operator is basically multiplication with the scaled resonant solution ψ_0 of the initial-value problem (1.1).

3.1. Auxiliary boundary mappings

In this subsection we prove an auxiliary estimate in the neighbourhood of the boundary. Let us define for $t \in [0, \delta)$ the mappings

$$\text{(3.1a)} \qquad \Gamma_t : \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\bar{\Omega}) \to L_2(\Sigma), \quad (\Gamma_t v)(s) := \tilde{v}(s,t),$$

$$\text{(3.1b)} \qquad \Upsilon_t : \operatorname{dom} \mathsf{A} \cap C^\infty(\bar{\Omega}) \to L_2(\Sigma),$$
$$(\Upsilon_t u)(s) := 2\varphi(s,t)(\partial_t \tilde{u})(s,t) + \partial_t \varphi(s,t)\tilde{u}(s,t),$$

where the function φ is as in (2.3) and where we employ the notation

$$\text{(3.2)} \qquad \tilde{u} = \mathsf{U}(u \restriction_{\Omega_\delta}) = u \circ \Phi \quad \text{and} \quad \tilde{v} = \mathsf{U}(v \restriction_{\Omega_\delta}) = v \circ \Phi$$

for $u \in \operatorname{dom} \mathsf{A} \cap C^\infty(\bar{\Omega})$ and $v \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\bar{\Omega})$. Note that the functions $\tilde{u}$ and $\tilde{v}$ are smooth ($\tilde{u}, \tilde{v} \in C^\infty(\Sigma \times [0,\delta))$). Moreover, we have $\tilde{v}(s,0) = 0$ for all $s \in \Sigma$.

The auxiliary mappings Γ_t and Υ_t will appear in an expression for the difference of the limit and approximating sesquilinear forms, cf. Lemma 3.3 below.

For an open set $\Omega' \subset \mathbb{R}^d$, we define the following norms in the Sobolev spaces $H^1(\Omega')$ and $H^2(\Omega')$:

$$\|u\|^2_{H^1(\Omega')} := \int_{\Omega'} \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x, \quad \|u\|^2_{H^2(\Omega')} := \int_{\Omega'} \left(|D^2 u|^2 + |\nabla u|^2 + |u|^2\right) \mathrm{d}x,$$

where $|D^2 u|$ stands for the Hilbert–Schmidt norm of the Hessian of u.

We now estimate one of the auxiliary mappings

Lemma 3.1. *Let the mapping Υ_t be defined as in (3.1b) and let the operator A be as in (1.2). Then, there exists a constant $c > 0$ such that for any $t \in [0,\delta)$,*

$$\|\Upsilon_t u\|_{L_2(\Sigma)} \le c\sqrt{t}\, \|u\|_{H^2(\Omega_t)}$$

holds for all $u \in \operatorname{dom} \mathsf{A} \cap C^\infty(\bar{\Omega})$.

Proof. For $u \in \operatorname{dom} \mathsf{A} \cap C^\infty(\bar{\Omega})$, we use again the notation (3.2). Combining the boundary condition (1.2) together with the identities $\varphi(s,0) = 1$ and $\partial_t \varphi(s,0) = -(d-1)H(s)$ for $s \in \Sigma$, we see that

$$\begin{aligned}(\Upsilon_0 u)(s) &= 2(\partial_t \tilde{u})(s,0) - (d-1)H(s)\tilde{u}(s,0) \\ &= -2\left(\partial_\nu u\restriction_\Sigma + \frac{(d-1)H}{2} u\restriction_\Sigma\right)(\Phi(s,0)) = 0.\end{aligned}$$

By the fundamental theorem of calculus and (3.1b), we obtain

$$\begin{aligned}(\Upsilon_t u)(s) &= \int_0^t \frac{\partial((\Upsilon_t u)(s))}{\partial t}\Big|_{t=t'} \mathrm{d}t' \\ &= \int_0^t \left(2\varphi(s,t')\,\partial_t^2 \tilde{u}(s,t') + 3\partial_t\varphi(s,t')\,\partial_t \tilde{u}(s,t') + \partial_t^2\varphi(s,t')\,\tilde{u}(s,t')\right) \mathrm{d}t'\end{aligned}$$

for any $t \in [0, \delta)$ and any $s \in \Sigma$. In view of (2.3) there exists a constant $C > 0$ such that $|\varphi(s,t)|$, $|\partial_t \varphi(s,t)|$, $|\partial_t^2 \varphi(s,t)| \le C$ for all $s \in \Sigma$ and $t \in [0, \delta)$. Applying the Cauchy–Schwarz inequality, we obtain

$$|(\Upsilon_t u)(s)| \le 3\sqrt{3} C \sqrt{t} \Big(\int_0^t \big(|\partial_t^2 \tilde{u}(s,t')|^2 + |\partial_t \tilde{u}(s,t')|^2 + |\tilde{u}(s,t')|^2 \big) \, \mathrm{d}t' \Big)^{1/2}. \tag{3.3}$$

We have

$$\partial_t \tilde{u}(s,t) = -\langle \nabla u(\Phi(s,t)), \nu(s) \rangle_{\mathbb{R}^d} \quad \text{and} \quad \partial_t^2 \tilde{u}(s,t) = \langle D^2 u(\Phi(s,t)) \nu(s), \nu(s) \rangle_{\mathbb{R}^d},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ stands for the standard inner product in $\mathbb{R}^d$. Hence, we obtain

$$\begin{aligned}
&\|\Upsilon_t u\|^2_{L_2(\Sigma)} \\
&\le 27 C^2 t \int_\Sigma \int_0^t \big(|D^2 u(\Phi(s,t'))|^2 + |\nabla u(\Phi(s,t'))|^2 + |u(\Phi(s,t'))|^2 \big) \, \mathrm{d}t' \, \mathrm{d}\sigma(s) \\
&\le \frac{27 C^2 t}{\rho} \int_\Sigma \int_0^t \big(|D^2 u(\Phi(s,t'))|^2 + |\nabla u(\Phi(s,t'))|^2 + |u(\Phi(s,t'))|^2 \big) \varphi(s,t') \, \mathrm{d}t' \, \mathrm{d}\sigma(s) \\
&\le \frac{27 C^2 t}{\rho} \|u\|^2_{H^2(\Omega_t)},
\end{aligned}$$

where the constant ρ is as in (2.4). Hence, the inequality in the formulation of the lemma holds with $c = (3\sqrt{3} C)/\sqrt{\rho}$. ■

3.2. The identification operator and an expression for the form difference

For $\varepsilon > 0$, we define the self-adjoint bounded multiplication operator

$$\mathsf{J}_\varepsilon : L_2(\Omega) \to L_2(\Omega), \quad (\mathsf{J}_\varepsilon u)(x) := \psi_0\Big(\frac{\operatorname{dist}(x, \Sigma)}{\varepsilon} \Big) u(x),$$

where $\psi_0 \in C^\infty(\overline{\mathbb{R}}_+)$ is the bounded solution of the initial-value problem (1.1) satisfying $\psi_0(t) = 1$ for all $t > a$.

It follows from C^∞-smoothness of the mapping $\operatorname{dist}(\cdot, \Sigma) \colon \Omega_\delta \to \mathbb{R}_+$ and of the function ψ_0 that, for all sufficiently small $\varepsilon > 0$, it holds that $\operatorname{ran}(\mathsf{J}_\varepsilon \restriction_{H^1(\Omega)}) \subset \mathring{H}^1(\Omega)$ and $\operatorname{ran}(\mathsf{J}_\varepsilon \restriction_{C^\infty(\overline{\Omega})}) \subset C^\infty(\overline{\Omega})$, where for the first-mentioned property we took into account $\psi_0(0) = 0$.

We first compare the identification operator J_ε with the identity I.

Lemma 3.2. *For any $u \in L_2(\Omega)$, we have*

$$\|(\mathsf{J}_\varepsilon - \mathsf{I})u\| \le \|\psi_0 - 1\|_\infty \|u\|_{L_2(\Omega_{a\varepsilon})} \to 0, \quad \text{as } \varepsilon \to 0.$$

In other words, J_ε converges to the identity operator in strong operator sense as $\varepsilon \to 0$.

Proof. We actually have

$$\|(\mathsf{J}_\varepsilon - \mathsf{I})u\|^2 = \int_\Omega \Big| \psi_0\Big(\frac{\operatorname{dist}(x, \Sigma)}{\varepsilon} \Big) - 1 \Big|^2 |u(x)|^2 \, \mathrm{d}x,$$

from which the desired inequality follows. ■

We now see the reason for defining the auxiliary boundary mappings and the choice of identification operator.

Lemma 3.3. *Assume that $\varepsilon < a^{-1}\delta$. Then we have*

$$\mathfrak{a}[u, \mathsf{J}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v] = \frac{1}{\varepsilon}\int_0^{a\varepsilon} \psi_0'\Big(\frac{t}{\varepsilon}\Big)\,\langle \Upsilon_t u, \Gamma_t v\rangle_{L_2(\Sigma)}\,\mathrm{d}t$$

for $u \in \operatorname{dom}\mathsf{A} \cap C^\infty(\overline{\Omega})$ and $v \in \operatorname{dom}\mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$.

Proof. Recall that $\delta > 0$ is chosen, as in Subsection 2.1, so that the mapping Φ in (2.1) is a diffeomorphism from $\Sigma \times (0, \delta)$ onto δ-neighbourhood Ω_δ of the boundary $\partial\Omega$. Under the assumption $\varepsilon < a^{-1}\delta$, the tubular coordinates (2.1) can be used. In particular, using the notation (3.2), we have $\tilde{u}, \tilde{v} \in L_2(\Sigma \times (0, \delta); \varphi(s, t)\,\mathrm{d}\sigma(s)\,\mathrm{d}t)$, and these functions are C^∞-smooth.

Clearly, the contributions of $\mathfrak{a}[u, \mathsf{J}_\varepsilon v]$ and $\mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v]$ outside the tubular neighbourhood $\Omega_{a\varepsilon}$ cancel. Using equation (2.5), we also observe that the contribution of the gradient terms corresponding to derivatives in the direction tangential to Σ cancel too. We end up with the following formula:

$$\begin{aligned}
\mathfrak{a}[u, \mathsf{J}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v] &= \int_\Sigma \int_0^{a\varepsilon} \partial_t \tilde{u}(s,t)\partial_t\Big(\psi_0\Big(\frac{t}{\varepsilon}\Big)\,\overline{\tilde{v}(s,t)}\Big)\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad - \int_\Sigma \int_0^{a\varepsilon} \partial_t\Big(\psi_0\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\Big)\,\overline{\partial_t \tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad - \frac{1}{\varepsilon^2}\int_\Sigma \int_0^{a\varepsilon} V\Big(\frac{t}{\varepsilon}\Big)\,\psi_0\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&= \frac{1}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \partial_t \tilde{u}(s,t)\,\psi_0'\Big(\frac{t}{\varepsilon}\Big)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad - \frac{1}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \psi_0'\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\partial_t \tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad - \frac{1}{\varepsilon^2}\int_\Sigma \int_0^{a\varepsilon} V\Big(\frac{t}{\varepsilon}\Big)\,\psi_0\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s),
\end{aligned}$$

where in the second step two terms cancelled upon using the product rule for differentiation. After integration by parts in the second term on the right-hand side of the above formula, we arrive at

$$\begin{aligned}
\mathfrak{a}[u, \mathsf{J}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v] &= \frac{1}{\varepsilon^2}\int_\Sigma \int_0^{a\varepsilon} \psi_0''\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad + \frac{1}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \psi_0'\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\partial_t\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad + \frac{2}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \psi_0'\Big(\frac{t}{\varepsilon}\Big)\,\partial_t\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\quad - \frac{1}{\varepsilon^2}\int_\Sigma \int_0^{a\varepsilon} V\Big(\frac{t}{\varepsilon}\Big)\,\psi_0\Big(\frac{t}{\varepsilon}\Big)\,\tilde{u}(s,t)\,\overline{\tilde{v}(s,t)}\,\varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s),
\end{aligned} \tag{3.4}$$

where the boundary term at $t = 0$ vanishes due to $\tilde{v}(s, 0) = 0$ while the boundary term at $t = a\varepsilon$ vanishes due to $\psi_0'(a) = 0$. Using that ψ_0 satisfies the differential equation $-\psi_0'' + V\psi_0 = 0$ we note that the first and the last terms on the right hand side in the above formula cancel each other. The remaining two integrals just give the desired expression involving Γ_t and Υ_t. ∎

We now estimate the expression of Lemma 3.3.

Lemma 3.4. *Assume that $\varepsilon < a^{-1}\delta$. Then we have*

$$\big|\mathfrak{a}[u, \mathsf{J}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v]\big| \le \hat{c}\, \|u\|_{H^2(\Omega_{a\varepsilon})}\, \|v\|_{L_2(\Omega_{a\varepsilon})}$$

for $u \in \operatorname{dom} \mathsf{A} \cap C^\infty(\bar{\Omega})$ and $v \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\bar{\Omega})$, where $\hat{c}$ is given in (3.5) below.

Proof. Using the Cauchy–Schwarz inequality (twice), Lemma 3.1 and Lemma 3.3, we obtain

$$\begin{aligned}
|\mathfrak{a}[u, \mathsf{J}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon u, v]| &\le \frac{\|\psi_0'\|_\infty}{\varepsilon} \int_0^{a\varepsilon} \|\Upsilon_t u\|_{L_2(\Sigma)}\, \|\Gamma_t v\|_{L_2(\Sigma)}\, \mathrm{d}t \\
&\le \frac{c\|\psi_0'\|_\infty \sqrt{a}}{\sqrt{\varepsilon}}\, \|u\|_{H^2(\Omega_{a\varepsilon})} \int_0^{a\varepsilon} \|\Gamma_t v\|_{L_2(\Sigma)}\, \mathrm{d}t \\
&\le ca\|\psi_0'\|_\infty\, \|u\|_{H^2(\Omega_{a\varepsilon})} \Big(\int_0^{a\varepsilon} \int_\Sigma |\tilde{v}(s,t)|^2\, \mathrm{d}\sigma(s)\, \mathrm{d}t \Big)^{1/2} \\
&\le \hat{c}\, \|u\|_{H^2(\Omega_{a\varepsilon})}\, \|v\|_{L_2(\Omega_{a\varepsilon})}, \quad \text{where } \hat{c} := \frac{ca\|\psi_0'\|_\infty}{\sqrt{\rho}}, \qquad (3.5)
\end{aligned}$$

using also (2.4) for the last estimate. ∎

3.3. Proof of Theorem A

Proof of Theorem A. For any $u, v \in C_{\mathrm{c}}^\infty(\Omega)$, we obtain

$$\begin{aligned}
\langle (\mathsf{R}_\varepsilon - \mathsf{R})u, v\rangle &= \langle u, \mathsf{J}_\varepsilon \mathsf{R}_\varepsilon^* v\rangle - \langle \mathsf{J}_\varepsilon \mathsf{R} u, v\rangle + \langle u, (\mathsf{I} - \mathsf{J}_\varepsilon)\mathsf{R}_\varepsilon^* v\rangle - \langle (\mathsf{I} - \mathsf{J}_\varepsilon)\mathsf{R} u, v\rangle \\
&= \langle (\mathsf{A} - \mathrm{i})\mathsf{R} u, \mathsf{J}_\varepsilon \mathsf{R}_\varepsilon^* v\rangle - \langle \mathsf{J}_\varepsilon \mathsf{R} u, (\mathsf{A}_\varepsilon + \mathrm{i})\mathsf{R}_\varepsilon^* v\rangle \\
&\quad + \langle u, (\mathsf{I} - \mathsf{J}_\varepsilon)\mathsf{R}_\varepsilon^* v\rangle - \langle (\mathsf{I} - \mathsf{J}_\varepsilon)\mathsf{R} u, v\rangle \\
&= \mathfrak{a}[\mathsf{R} u, \mathsf{J}_\varepsilon \mathsf{R}_\varepsilon^* v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon \mathsf{R} u, \mathsf{R}_\varepsilon^* v] + \langle (\mathsf{I} - \mathsf{J}_\varepsilon)u, \mathsf{R}_\varepsilon^* v\rangle - \langle (\mathsf{I} - \mathsf{J}_\varepsilon)\mathsf{R} u, v\rangle,
\end{aligned}$$

using first representation theorem for the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{a}_\varepsilon$ associated with A and A_ε, respectively, and using also the self-adjointness of $\mathsf{I} - \mathsf{J}_\varepsilon$ for the last equality. Moreover, we have $\|\mathsf{R} u\| \le \|u\|$ and $\|\mathsf{R}_\varepsilon^* v\| \le \|v\|$, as R and R_ε^* are the resolvents at the points $\pm\mathrm{i}$ and A and A_ε are self-adjoint, respectively. Hence we conclude (using Cauchy–Schwarz)

$$\begin{aligned}
&|\langle (\mathsf{R}_\varepsilon - \mathsf{R})u, v\rangle| \\
&\quad \le |\mathfrak{a}[\mathsf{R} u, \mathsf{J}_\varepsilon \mathsf{R}_\varepsilon^* v] - \mathfrak{a}_\varepsilon[\mathsf{J}_\varepsilon \mathsf{R} u, \mathsf{R}_\varepsilon^* v]| + \|(\mathsf{J}_\varepsilon - \mathsf{I})u\|\, \|\mathsf{R}_\varepsilon^* v\| + \|(\mathsf{J}_\varepsilon - \mathsf{I})\mathsf{R} u\|\, \|v\| \\
&\quad \le \big(\hat{c}\|\mathsf{R} u\|_{H^2(\Omega_{a\varepsilon})} + \|\psi_0 - 1\|_\infty (\|u\|_{L_2(\Omega_{a\varepsilon})} + \|\mathsf{R} u\|_{L_2(\Omega_{a\varepsilon})})\big)\, \|v\|
\end{aligned}$$

using also Lemmas 3.2 and 3.4 for the second estimate (note that $\mathsf{R}u, \mathsf{R}_\varepsilon^* v \in C^\infty(\overline{\Omega})$). From the characterisation of the dual of a Hilbert space on the dense subset $C_c^\infty(\Omega)$ of $L_2(\Omega)$, we obtain

$$\begin{aligned}\|\mathsf{R}_\varepsilon u - \mathsf{R}u\| &= \sup_{\substack{v \in C_c^\infty(\Omega)\\ \|v\|=1}} |\langle (\mathsf{R}_\varepsilon - \mathsf{R})u, v\rangle| \\ &\le \hat{c}\, \|\mathsf{R}u\|_{H^2(\Omega_{a\varepsilon})} + \|\psi_0 - 1\|_\infty \big(\|u\|_{L_2(\Omega_{a\varepsilon})} + \|\mathsf{R}u\|_{L_2(\Omega_{a\varepsilon})}\big).\end{aligned}$$

Now all norms on $\Omega_{a\varepsilon}$ converge to 0 by Lebesgue's convergence theorem for $u \in C_c^\infty(\Omega)$. By density of $C_c^\infty(\Omega)$ in $L_2(\Omega)$, we conclude that A_ε converges to A in strong resolvent sense. ∎

4. Convergence for non-resonant potentials

4.1. Auxiliary boundary mappings, the identification operator and some related estimates

In this subsection, we provide another lemma needed in the proof of Theorem B. Recall that the mapping Γ_t is defined in (3.1).

Lemma 4.1. *We have*

$$\|\Gamma_t u\|_{L_2(\Sigma)} \le \sqrt{\frac{t}{\rho}}\, \|\nabla u\|_{L_2(\Omega_t;\mathbb{C}^d)}$$

for all $u \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$ *and* $t \in (0, \delta)$.

Proof. Let $t \in (0, \delta)$ and $u \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$. By the fundamental theorem of calculus, we obtain, in view of $\Gamma_0 u = 0$, that

$$(\Gamma_t u)(s) = \int_0^t \partial_t \tilde{u}(s, t')\, \mathrm{d}t'$$

for any $t \in [0, \delta)$ and $s \in \Sigma$ (recall that $\tilde{u} = u \circ \Phi$ as in (3.2)). Using the Cauchy–Schwarz inequality, we obtain

$$|(\Gamma_t u)(s)| \le \sqrt{t} \Big(\int_0^t |\partial_t \tilde{u}(s, t')|^2\, \mathrm{d}t' \Big)^{1/2}.$$

As $\partial_t \tilde{u}(s, t) = -\langle \nabla u(\Phi(s, t)), \nu(s)\rangle_{\mathbb{R}^d}$, we deduce (using Cauchy–Schwarz again)

$$\begin{aligned}&\|\Gamma_t u\|^2_{L_2(\Sigma)} \le t \int_\Sigma \int_0^t |\nabla u(\Phi(s, t'))|^2\, \mathrm{d}t'\, \mathrm{d}\sigma(s) \\ &\le \frac{t}{\rho} \int_\Sigma \int_0^t |\nabla u(\Phi(s, t'))|^2 \varphi(s, t')\, \mathrm{d}t'\, \mathrm{d}\sigma(s) = \frac{t}{\rho} \int_{\Omega_t} |\nabla u|^2\, \mathrm{d}x = \frac{t}{\rho} \|\nabla u\|^2_{L_2(\Omega_t;\mathbb{C}^d)}.\end{aligned}$$ ∎

We also need the following modified boundary mapping $\widetilde{\Gamma}_t \colon \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega}) \to L_2(\Sigma)$ defined, for all $t \in (0, \delta)$, by

$$(\widetilde{\Gamma}_t v)(s) := \varphi(s, t)\, \tilde{v}(s, t) = \varphi(s, t)\, v(\Phi(s, t)). \tag{4.1}$$

Corollary 4.2. *We have*

$$\|\widetilde{\Gamma}_t v\|_{L_2(\Sigma)} \le \|\varphi\|_\infty \sqrt{\frac{t}{\rho}}\, \|\nabla v\|_{L_2(\Omega_t;\mathbb{C}^d)}$$

for all $v \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$ *and* $t \in (0,\delta)$.

Proof. We just estimate $0 < \varphi(s,t) \le \|\varphi\|_\infty$ in the first step and then use Lemma 4.1. ∎

Recall that we fix $a > 0$ such that $\operatorname{supp} V \subset [0,a]$. Let ψ_0 be a non-resonant solution of (1.1) normalised so that $\psi_0'(t) = 1$ for all $t > a$; see Remark 1.2(b).

For $\varepsilon > 0$, we define the self-adjoint bounded multiplication operator

$$\mathsf{K}_\varepsilon : L_2(\Omega) \to L_2(\Omega), \quad (\mathsf{K}_\varepsilon u)(x) = \psi_0'\Big(\frac{\operatorname{dist}(x,\Sigma)}{\varepsilon}\Big) u(x).$$

The only difference with J_ε is that we use ψ_0' (with $\psi_0' = 1$ outside $(0,a)$) instead of ψ_0. We obtain, as in the proof of Lemma 3.2, the following.

Lemma 4.3. *For any* $u \in L_2(\Omega)$, *we have*

$$\|(\mathsf{K}_\varepsilon - \mathsf{I})u\| \le \|\psi_0' - 1\|_\infty \|u\|_{L_2(\Omega_{a\varepsilon})} \to 0, \quad \text{as } \varepsilon \to 0.$$

In other words, K_ε *converges to the identity operator in strong operator sense as* $\varepsilon \to 0$.

For the form difference, we have an expression similar to that in Lemma 3.3.

Lemma 4.4. *Assume that* $\varepsilon < a^{-1}\delta$. *Then we have*

$$\begin{aligned}\mathfrak{a}_0[u,\mathsf{K}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon u, v] &= \frac{1}{\varepsilon}\int_0^{a\varepsilon} \psi_0''\Big(\frac{t}{\varepsilon}\Big) \langle \Upsilon_t u, \Gamma_t v\rangle_{L_2(\Sigma)}\,\mathrm{d}t \\ &\quad + \frac{1}{\varepsilon^2}\int_0^{a\varepsilon} (V'\psi_0)\Big(\frac{t}{\varepsilon}\Big) \langle \Gamma_t u, \widetilde{\Gamma}_t v\rangle_{L_2(\Sigma)}\,\mathrm{d}t\end{aligned}$$

for $u \in \operatorname{dom} \mathsf{A}_0 \cap C^\infty(\overline{\Omega})$ *and* $v \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$.

Proof. The proof is exactly the same as the proof of Lemma 3.3, except that by differentiating the identity $-\psi_0'' + V\psi_0 = 0$, we obtain $-\psi_0''' + V'\psi_0 + V\psi_0' = 0$. In particular, the first and fourth terms in (3.4) do not cancel, as now $\psi_0''' - V\psi_0' = V'\psi_0$ remains. In particular, we have

$$\begin{aligned}\mathfrak{a}_0[u,\mathsf{K}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon u, v] &= \frac{1}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \psi_0''\Big(\frac{t}{\varepsilon}\Big)\, \tilde{u}(s,t)\, \overline{\tilde{v}(s,t)}\, \partial_t \varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\ &\quad + \frac{2}{\varepsilon}\int_\Sigma \int_0^{a\varepsilon} \psi_0''\Big(\frac{t}{\varepsilon}\Big)\, \partial_t\tilde{u}(s,t)\, \overline{\tilde{v}(s,t)}\, \varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s) \\ &\quad + \frac{1}{\varepsilon^2}\int_\Sigma \int_0^{a\varepsilon} (\psi_0''' - V\psi_0')\Big(\frac{t}{\varepsilon}\Big)\, \tilde{u}(s,t)\, \overline{\tilde{v}(s,t)}\, \varphi(s,t)\,\mathrm{d}t\,\mathrm{d}\sigma(s),\end{aligned}$$

from which the desired formula follows. Here the boundary term at $t = 0$ vanishes due to $\tilde{v}(s,0) = 0$, while the boundary term at $t = a\varepsilon$ vanishes due to $\psi_0''(a) = 0$. ∎

As before, we now estimate the expression of Lemma 4.4.

Lemma 4.5. *Assume that* $\varepsilon < a^{-1}\delta$*. Then we have*

$$\left|\mathfrak{a}_0[u, \mathsf{K}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon u, v]\right| \le \hat{c}_0 \, \|u\|_{H^1(\Omega_{a\varepsilon})} \, \|v\|_{H^1(\Omega_{a\varepsilon})}$$

for $u \in \operatorname{dom} \mathsf{A}_0 \cap C^\infty(\overline{\Omega})$ *and* $v \in \operatorname{dom} \mathsf{A}_\varepsilon \cap C^\infty(\overline{\Omega})$*, where* $\hat{c}_0$ *is given in* (4.2) *below.*

Proof. We estimate the first term in Lemma 4.4 as: follows

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_0^{a\varepsilon} \psi_0''\Big(\frac{t}{\varepsilon}\Big)\, \langle \Upsilon_t u, \Gamma_t v\rangle_{L_2(\Sigma)}\, \mathrm{d}t\right| \\
&\qquad \le \frac{\|\psi_0''\|_\infty}{\varepsilon}\Big(\int_0^{a\varepsilon} \|\Upsilon_t u\|^2_{L_2(\Sigma)}\, \mathrm{d}t\Big)^{1/2}\Big(\int_0^{a\varepsilon} \|\Gamma_t v\|^2_{L_2(\Sigma)}\, \mathrm{d}t\Big)^{1/2} \\
&\qquad \le \frac{a\|\psi_0''\|_\infty \|v\|_{H^1(\Omega_{a\varepsilon})}}{\sqrt{2\rho}} \\
&\qquad\quad \times \Big(\int_0^{a\varepsilon}\int_\Sigma \big(8\varphi^2(s,t)\,|\partial_t\tilde{u}(s,t)|^2 + 2(\partial_t\varphi(s,t))^2\,|\tilde{u}(s,t)|^2\big)\, \mathrm{d}\sigma(s)\, \mathrm{d}t\Big)^{1/2} \\
&\qquad \le \frac{2a\|\psi_0''\|_\infty \|v\|_{H^1(\Omega_{a\varepsilon})}}{\rho}\, \max\{\|\varphi\|_\infty, \|\partial_t\varphi\|_\infty\} \\
&\qquad\quad \times \Big(\int_0^{a\varepsilon}\int_\Sigma [|\partial_t\tilde{u}(s,t)|^2 + |\tilde{u}(s,t)|^2]\,\varphi(s,t)\, \mathrm{d}\sigma(s)\, \mathrm{d}t\Big)^{1/2} \\
&\qquad \le \hat{c}'\, \|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})},
\end{aligned}$$

where

$$\hat{c}' := \frac{2a\|\psi_0''\|_\infty}{\rho}\, \max\big\{\|\varphi\|_\infty, \|\partial_t\varphi\|_\infty\big\},$$

and where we used the Cauchy–Schwarz inequality in the first estimate, and Lemma 4.1 in the second. Employing the above estimate, we get

$$\begin{aligned}
&|\mathfrak{a}_0[u, \mathsf{K}_\varepsilon v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon u, v]| \\
&\quad \le \hat{c}'\, \|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})} + \frac{\|V'\|_\infty\, \|\psi_0\|_{L_\infty((0,a))}}{\varepsilon^2}\int_0^{a\varepsilon} \|\Gamma_t u\|_{L_2(\Sigma)}\, \|\widetilde{\Gamma}_t v\|_{L_2(\Sigma)}\, \mathrm{d}t \\
&\quad \le \hat{c}'\, \|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})} \\
&\qquad + \frac{\|V'\|_\infty \|\psi_0\|_{L_\infty((0,a))}\|\varphi\|_\infty}{\rho\varepsilon^2}\int_0^{a\varepsilon} t\, \mathrm{d}t\, \|\nabla u\|_{L_2(\Omega_{a\varepsilon};\mathbb{C}^d)}\, \|\nabla v\|_{L_2(\Omega_{a\varepsilon};\mathbb{C}^d)} \\
&\quad = \hat{c}'\, \|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})} \\
&\qquad + \frac{a^2\|V'\|_\infty \|\psi_0\|_{L_\infty((0,a))}\, \|\varphi\|_\infty}{2\rho}\, \|\nabla u\|_{L_2(\Omega_{a\varepsilon};\mathbb{C}^d)}\, \|\nabla v\|_{L_2(\Omega_{a\varepsilon};\mathbb{C}^d)} \\
&\quad \le \Big(\hat{c}' + \frac{a^2\|V'\|_\infty \|\psi_0\|_{L_\infty((0,a))}\, \|\varphi\|_\infty}{2\rho}\Big)\|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})} \\
&\quad \le \hat{c}_0\, \|u\|_{H^1(\Omega_{a\varepsilon})}\, \|v\|_{H^1(\Omega_{a\varepsilon})},
\end{aligned}$$

where

$$\hat{c}_0 := \hat{c}' + \frac{a^2 \|V'\|_\infty \|\psi_0\|_{L_\infty((0,a))} \|\varphi\|_\infty}{2\rho}, \tag{4.2}$$

using Lemma 4.1 and Corollary 4.2 in the second estimate. ■

4.2. Proof of Theorem B

Before providing the proof of Theorem B, we need one more estimate: here, it is essential that the potential satisfies an assumption on the negative part.

Lemma 4.6. *Let the constant $C_\Omega > 0$ be defined as in (1.4). Under the assumption*

$$C_V := \max_{t \in \mathbb{R}_+} (-V(t) t^2) < C_\Omega,$$

we have

$$\|\mathsf{R}_\varepsilon^* v\|_{H^1(\Omega)} \le \sqrt{c_{\Omega,V} + 1} \|v\|$$

for all $v \in L_2(\Omega)$ and the constant $c_{\Omega,V} := \frac{1}{2}(1 - C_V/C_\Omega)^{-1} \ge 1/2$.

Proof. The estimate $\|\mathsf{R}_\varepsilon^* v\|_{L_2(\Omega)} \le \|v\|$ is clear by spectral calculus. Under the assumption we stated on the potential V, we have the inequality

$$V_\varepsilon(x) \ge -\frac{C_V}{(\operatorname{dist}(x, \Sigma))^2},$$

for all $\varepsilon > 0$ and all $x \in \Omega$. Now we have, by the definition of the constant C_Ω, that for all $\hat{v} \in \operatorname{dom} \mathsf{A}_\varepsilon$,

$$\begin{aligned} \|\nabla \hat{v}\|^2_{L_2(\Omega;\mathbb{C}^d)} &\le 2c_{\Omega,V} (\|\nabla \hat{v}\|^2_{L_2(\Omega;\mathbb{C}^d)} + \langle V_\varepsilon \hat{v}, \hat{v} \rangle) = 2c_{\Omega,V}\, \mathfrak{a}_\varepsilon[\hat{v}] \\ &= 2c_{\Omega,V} \langle \mathsf{A}_\varepsilon \hat{v}, \hat{v} \rangle \le c_{\Omega,V} (\|\mathsf{A}_\varepsilon \hat{v}\|^2 + \|\hat{v}\|^2) = c_{\Omega,V} \|(\mathsf{A}_\varepsilon + \mathrm{i})\hat{v}\|^2 \end{aligned}$$

and using the Cauchy–Schwarz inequality combined with the standard inequality $2ab \le a^2 + b^2$ for $a, b \ge 0$ in the penultimate step. The desired estimate follows by setting $\hat{v} = \mathsf{R}_\varepsilon^* v$. ■

Proof of Theorem B. For any $u, v \in C_c^\infty(\Omega)$, we obtain, as in the proof of Theorem A, the decomposition

$$\begin{aligned} &\langle (\mathsf{R}_\varepsilon - \mathsf{R}_0) u, v \rangle \\ &\quad = \mathfrak{a}_0[\mathsf{R}_0 u, \mathsf{K}_\varepsilon \mathsf{R}_\varepsilon^* v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon \mathsf{R}_0 u, \mathsf{R}_\varepsilon^* v] + \langle u, (\mathsf{I} - \mathsf{K}_\varepsilon) \mathsf{R}_\varepsilon^* v \rangle - \langle (\mathsf{I} - \mathsf{K}_\varepsilon) \mathsf{R}_0 u, v \rangle \\ &\quad = \mathfrak{a}_0[\mathsf{R}_0 u, \mathsf{K}_\varepsilon \mathsf{R}_\varepsilon^* v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon \mathsf{R}_0 u, \mathsf{R}_\varepsilon^* v] + \langle (\mathsf{I} - \mathsf{K}_\varepsilon) u, \mathsf{R}_\varepsilon^* v \rangle - \langle (\mathsf{I} - \mathsf{K}_\varepsilon) \mathsf{R}_0 u, v \rangle, \end{aligned}$$

where we used again that $\mathsf{I} - \mathsf{K}_\varepsilon$ is self-adjoint. We now have (using Cauchy–Schwarz)

$$\begin{aligned} &|\langle (\mathsf{R}_\varepsilon - \mathsf{R}_0) u, v \rangle| \\ &\quad \le \left| \mathfrak{a}_0[\mathsf{R}_0 u, \mathsf{K}_\varepsilon \mathsf{R}_\varepsilon^* v] - \mathfrak{a}_\varepsilon[\mathsf{K}_\varepsilon \mathsf{R}_0 u, \mathsf{R}_\varepsilon^* v] \right| + \|(\mathsf{I} - \mathsf{K}_\varepsilon) u\| \|\mathsf{R}_\varepsilon^* v\| + \|(\mathsf{I} - \mathsf{K}_\varepsilon) \mathsf{R}_0 u\| \|v\| \\ &\quad \le \hat{c}_0 \|\mathsf{R}_0 u\|_{H^1(\Omega_{a\varepsilon})} \|\mathsf{R}_\varepsilon^* v\|_{H^1(\Omega_{a\varepsilon})} \\ &\qquad + \|\psi_0' - 1\|_\infty \big(\|u\|_{L_2(\Omega_{a\varepsilon})} \|\mathsf{R}_\varepsilon^* v\| + \|\mathsf{R}_0 u\|_{L_2(\Omega_{a\varepsilon})} \|v\| \big) \\ &\quad \le \big(\sqrt{c_{\Omega,V} + 1}\, \hat{c}_0 \|\mathsf{R}_0 u\|_{H^1(\Omega_{a\varepsilon})} + \|\psi_0' - 1\|_\infty (\|u\|_{L_2(\Omega_{a\varepsilon})} + \|\mathsf{R}_0 u\|_{L_2(\Omega_{a\varepsilon})}) \big) \|v\| \end{aligned}$$

using Lemmas 4.3 and 4.5 in the second estimate, and $\|\mathsf{R}_\varepsilon^* v\|_{H^1(\Omega)} \le \sqrt{c_{\Omega,V}+1}\|v\|$, respectively, Lemma 4.6, in the last one. We conclude that

$$\begin{aligned}\|\mathsf{R}_\varepsilon u - \mathsf{R}_0 u\| &= \sup_{\substack{v\in C_c^\infty(\Omega)\\ \|v\|=1}} \left|\langle(\mathsf{R}_\varepsilon - \mathsf{R}_0)u, v\rangle\right| \\ &\le \sqrt{c_{\Omega,V}+1}\,\hat{c}_0\,\|\mathsf{R}_0 u\|_{H^1(\Omega_{a\varepsilon})} + \|\psi_0' - 1\|_\infty\left(\|u\|_{L_2(\Omega_{a\varepsilon})} + \|\mathsf{R}_0 u\|_{L_2(\Omega_{a\varepsilon})}\right).\end{aligned}$$

As before, all norms on $\Omega_{a\varepsilon}$ converge to 0 by Lebesgue's convergence theorem for $u\in C_c^\infty(\Omega)$. as $\varepsilon\to 0$. By density of $C_c^\infty(\Omega)$ in the respective spaces, we conclude that A_ε converges to A_0 in the strong resolvent sense. ∎

4.3. No uniform ellipticity

We finally show that if the estimate in Lemma 4.6 with any constant is not true, then $(\mathfrak{a}_\varepsilon)_\varepsilon$ is not uniformly elliptic:

Lemma 4.7. *If the estimate in Lemma 4.6 does not hold for any constant, then $(\mathfrak{a}_\varepsilon)_\varepsilon$ is not uniformly elliptic, i.e.,*

$$\exists\alpha>0, \exists\omega\in\mathbb{R}, \exists\varepsilon_0>0, \forall\varepsilon\in(0,\varepsilon_0), \forall\hat{v}\in \mathring{H}^1(\Omega):\ \alpha\|\hat{v}\|^2_{H^1(\Omega)} \le \mathfrak{a}_\varepsilon[\hat{v}] + \omega\|\hat{v}\|^2$$

does not hold.

Proof. Without loss of generality, we can assume that $\omega>0$ in the definition of uniform ellipticity. If $(\mathfrak{a}_\varepsilon)_\varepsilon$ was uniformly elliptic, then for any $\hat{v}\in\operatorname{dom}\mathsf{A}_\varepsilon$,

$$\begin{aligned}\|\nabla\hat{v}\|^2_{L_2(\Omega;\mathbb{C}^d)} &\le \frac{1}{\alpha}\left(\mathfrak{a}_\varepsilon[\hat{v}] + \omega\|\hat{v}\|^2\right) \\ &\le \frac{1}{\alpha}\left(\|(\mathsf{A}_\varepsilon+\mathrm{i})\hat{v}\|^2 + \omega\|\hat{v}\|^2\right) \le \frac{1+\omega}{\alpha}\,\|(\mathsf{A}_\varepsilon+\mathrm{i})\hat{v}\|^2\end{aligned}$$

as in the proof of Lemma 4.6. In particular, we would have

$$\|\hat{v}\|^2_{H^1(\Omega)} \le \left(\frac{1+\omega}{\alpha}+1\right)\|(\mathsf{A}_\varepsilon+\mathrm{i})\hat{v}\|^2$$

and the claim of Lemma 4.6 would follow (with another constant). ∎

5. Absence of norm resolvent convergence

The aim of this section is to construct a counterexample to norm resolvent convergence of the operators A_ε to the operator A in the case of resonant potentials, and thus to justify that we can only prove strong resolvent convergence in this setting. This counterexample relies on the analysis of convergence on the unit disk $\mathcal{B}\subset\mathbb{R}^2$. The model on the disk admits separation of variables in polar coordinates, and the analysis significantly simplifies. We expect that also for more general domains one cannot hope for norm resolvent convergence of A_ε to A.

In order to construct the counterexample, we need to restrict further the class of resonant potentials. This restriction is clarified in the following hypothesis.

Hypothesis 5.1. Assume that the resonant potential $V \in C_c^\infty(\overline{\mathbb{R}}_+)$ (in the sense of Definition 1.1) is such that the self-adjoint one-dimensional Schrödinger operator with domain $\mathring{H}^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$ acting as $\psi \mapsto -\psi'' + V\psi$ in the Hilbert space $L_2(\mathbb{R}_+)$ has at least one negative eigenvalue. We denote by $\mu < 0$ the lowest eigenvalue of this Schrödinger operator, and by $f_\mu \in \mathring{H}^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$, the corresponding real-valued eigenfunction.

Example 5.2. Let $V \in C_c^\infty(\overline{\mathbb{R}}_+)$, such that $\operatorname{supp} V \subset [0, a]$, with $a > 0$, be a non-positive resonant potential in the sense of Definition 1.1; i.e., the self-adjoint Schrödinger operator in $L_2((0,a))$ corresponding to the quadratic form $f \mapsto \int_0^a (|f'|^2 + V|f|^2)\,\mathrm{d}x$ with domain $\{f \in H^1((0,a)) \colon f(0) = 0\}$ has eigenvalue zero. Recall that there exists a sequence of real numbers $\{\alpha_n\}_{n\in\mathbb{N}}$, $1 = \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \ldots$ such that $\alpha_n \to \infty$, for which the multiple $\alpha_n V$ of the potential V is resonant for all $n \in \mathbb{N}$. It remains to note that for all $n \in \mathbb{N}$ sufficiently large the resonant potential $\alpha_n V$ necessarily satisfies Hypothesis 5.1. Thus, the family of resonant potentials satisfying the above hypothesis is non-void.

The quadratic form of the operator A_ε, in the case of the unit disk, can be written in polar coordinates:

$$\mathfrak{a}_\varepsilon[u] = \int_0^1 \int_0^{2\pi} \Big(|\partial_r u|^2 + \frac{|\partial_\theta u|^2}{r^2} + \frac{1}{\varepsilon^2} V\Big(\frac{1-r}{\varepsilon}\Big)|u|^2\Big)\, r\,\mathrm{d}\theta\,\mathrm{d}r,$$

where the form domain remains the Sobolev space $\mathring{H}^1(\mathcal{B})$. For $m \in \mathbb{Z}$, consider the quadratic form of the fibre operator:

$$\begin{aligned}\mathfrak{a}_\varepsilon^{(m)}[f] &:= \int_0^1 \Big(|f'(r)|^2 + \frac{m^2}{r^2}|f(r)|^2 + \frac{1}{\varepsilon^2} V\Big(\frac{1-r}{\varepsilon}\Big)|f(r)|^2\Big)\, r\,\mathrm{d}r,\\ \operatorname{dom} \mathfrak{a}_\varepsilon^{(m)} &:= \{f \in L_2((0,1); r\,\mathrm{d}r) : \mathfrak{a}_\varepsilon^{(m)}[f] < \infty\}.\end{aligned}$$

The symmetric quadratic form $\mathfrak{a}_\varepsilon^{(m)}$ is closed, densely defined, and lower-semibounded in $L^2((0,1); r\,\mathrm{d}r)$ for any $m \in \mathbb{Z}$. The mentioned properties of $\mathfrak{a}_\varepsilon^{(m)}$ follow immediately from the perturbation result (Theorem 1.33 in Chapter VI of [19]) and the fact that this form can be represented as a sum of a bounded quadratic form

$$f \mapsto \frac{1}{\varepsilon^2} \int_0^1 V\Big(\frac{1-r}{\varepsilon}\Big)|f(r)|^2 r\,\mathrm{d}r$$

on $L_2((0,1); r\,\mathrm{d}r)$ and the quadratic for the fibre operator of the Dirichlet Laplacian on the disk, for which these properties are well known. Let us denote by $\mathsf{A}_\varepsilon^{(m)}$ the self-adjoint fibre operator in $L_2((0,1); r\,\mathrm{d}r)$ associated with the quadratic form $\mathfrak{a}_\varepsilon^{(m)}$. Using standard procedure based on separation of variables, we infer the following unitary equivalence:

$$\mathsf{A}_\varepsilon \cong \bigoplus_{m\in\mathbb{Z}} \mathsf{A}_\varepsilon^{(m)}. \tag{5.1}$$

In particular, we get, as a direct consequence,

$$\sigma(\mathsf{A}_\varepsilon) = \overline{\bigcup_{m\in\mathbb{Z}} \sigma(\mathsf{A}_\varepsilon^{(m)})}. \tag{5.2}$$

The spectrum of the fibre operator $\mathsf{A}_\varepsilon^{(m)}$ is clearly purely discrete; let us denote by $\lambda_1^{(m)}(\varepsilon)$ the lowest eigenvalue of $\mathsf{A}_\varepsilon^{(m)}$.

The following lemma is essential in the construction of the counterexample. Its proof is outsourced to Appendix A.

Lemma 5.3. *For any $m \in \mathbb{Z}$, the following properties hold.*

(a) $\lambda_1^{(m)}(\cdot)$ *is a continuous function.*

(b) $\lim_{\varepsilon\to 0} \lambda_1^{(m)}(\varepsilon) = -\infty$ *for any V satisfying Hypothesis 5.1.*

(c) $\lambda_1^{(m)}(\varepsilon) \geq 0$ *if* $\varepsilon \geq \frac{1}{|m|}\sqrt{\|V\|_\infty}$.

The next proposition provides a counterexample based on the disk. The proposed technique can be also used to construct counterexamples for domains other than the disk.

Proposition 5.4. *For the unit disk and a resonant potential V satisfying Hypothesis 5.1, the family of operators A_ε does not converge in norm resolvent sense to the operator A.*

Proof. Recall that the Robin Laplacian A is bounded from below. Choose $\beta < 0$ such that $\beta < \inf\sigma(\mathsf{A})$. By Lemma 5.3, we can find $m_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that $\lambda_1^{(m_1)}(\varepsilon_1) = \beta$. By item (c) of the same lemma, we can choose an integer $m_2 > m_1$ such that $\lambda_1^{(m_2)}(\varepsilon_1) \geq 0$. Hence, by items (a) and (b) of Lemma 5.3, we can find $\varepsilon_2 \in (0, \varepsilon_1)$ such that $\lambda_1^{(m_2)}(\varepsilon_2) = \beta$. Analogously, we can find $m_3 > m_2$ and $\varepsilon_3 \in (0, \varepsilon_2)$ such that $\lambda_1^{(m_3)}(\varepsilon_3) = \beta$. Thus, repeating the construction, we conclude that there exist a sequence of real numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > \cdots > 0$ and integers $m_1 < m_2 < \cdots < m_k < \cdots < +\infty$ such that $\lambda_1^{(m_k)}(\varepsilon_k) = \beta$ for all $k \in \mathbb{N}$. Moreover, it follows from item (c) of Lemma 5.3 that $\varepsilon_k \leq \frac{1}{m_k}\sqrt{\|V\|_\infty} \to 0$ as $k \to \infty$.

Suppose for the moment that A_ε converges to A in the norm resolvent sense as $\varepsilon \to 0^+$. Then also $\mathsf{A}_{\varepsilon_k}$ converges to A in norm resolvent sense as $k \to \infty$. By Satz 9.24 (i) in [34], we would get that the spectrum of the operator $\mathsf{A}_{\varepsilon_k}$ must converge to the spectrum of the operator A as $k \to \infty$. This consequence of the norm resolvent convergence, combined with (5.2), contradicts the choice of the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, since $\beta < \inf\sigma(\mathsf{A})$ is in the spectrum of $\mathsf{A}_{\varepsilon_k}$ for all $k \in \mathbb{N}$. ∎

A. Proof of Lemma 5.3

Proof of Lemma 5.3. (a) Let $\varepsilon_0 \in (0, \infty)$. It is straightforward to see that

$$\lambda_1^{(m)}(\varepsilon) \geq -\frac{4}{\varepsilon_0^2}\|V\|_\infty \quad \text{for all } \varepsilon \in (\varepsilon_0/2, 2\varepsilon_0).$$

In other words, the lowest eigenvalue $\lambda_1^{(m)}(\varepsilon)$ is uniformly bounded from below for $\varepsilon \in (\varepsilon_0/2, 2\varepsilon_0)$. Thus, in view of Satz 9.24 in [34], continuity of $\varepsilon \mapsto \lambda_1^{(m)}(\varepsilon)$ for any $m \in \mathbb{Z}$ would immediately follow if we show that the operators $\mathsf{A}_\varepsilon^{(m)}$ converge in norm resolvent sense to $\mathsf{A}_{\varepsilon_0}^{(m)}$ as $\varepsilon \to \varepsilon_0$. To this aim, notice that for any $\varepsilon \in (\varepsilon_0/2, 2\varepsilon_0)$,

$$\begin{aligned}
&\int_0^1 \left| \frac{1}{\varepsilon_0^2} V\Big(\frac{1-r}{\varepsilon_0}\Big) - \frac{1}{\varepsilon^2} V\Big(\frac{1-r}{\varepsilon}\Big)\right| |f(r)|^2\, r\,\mathrm{d}r \\
&\qquad\qquad \le \sup_{r\in(0,1)} \left| \frac{1}{\varepsilon_0^2} V\Big(\frac{1-r}{\varepsilon_0}\Big) - \frac{1}{\varepsilon^2} V\Big(\frac{1-r}{\varepsilon}\Big)\right| \int_0^1 |f(r)|^2\, r\,\mathrm{d}r \\
&\qquad\qquad \le \frac{16}{\varepsilon_0^4} (\varepsilon_0 \|V\|_\infty + \|V'\|_\infty)\, |\varepsilon - \varepsilon_0| \int_0^1 |f(r)|^2\, r\,\mathrm{d}r.
\end{aligned}$$

Thus, it follows that

$$\left|\mathfrak{a}_\varepsilon^{(m)}[f] - \mathfrak{a}_{\varepsilon_0}^{(m)}[f]\right| \le C\,|\varepsilon - \varepsilon_0| \int_0^1 |f(r)|^2\, r\,\mathrm{d}r$$

for any $\varepsilon \in (\varepsilon_0/2, 2\varepsilon_0)$ and all $f \in \operatorname{dom} \mathfrak{a}_\varepsilon^{(m)} = \operatorname{dom} \mathfrak{a}_{\varepsilon_0}^{(m)}$ with constant $C = C(V, \varepsilon_0) = (16\varepsilon_0^{-4})(\varepsilon_0\|V\|_\infty + \|V'\|_\infty) > 0$. Hence, the norm resolvent convergence of $\mathsf{A}_\varepsilon^{(m)}$ to $\mathsf{A}_{\varepsilon_0}^{(m)}$ as $\varepsilon \to \varepsilon_0$ is a consequence of Theorem 3.4 in Chapter VI of [19].

(b) Let the cut-off function $\chi \in C_c^\infty((0,1])$ be such that $0 \le \chi \le 1$, $\chi(r) = 1$ for $r \in [3/4, 1]$, and $\chi(r) = 0$ for all $r \in (0, 1/2]$. As a trial function for the quadratic form $\mathfrak{a}_\varepsilon^{(m)}$, we use

$$g_\varepsilon(r) := \chi(r)\, f_\mu\Big(\frac{1-r}{\varepsilon}\Big), \quad r \in (0,1),$$

where $f_\mu \in \mathring{H}^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$ satisfies the differential equation $-f_\mu'' + Vf_\mu = \mu f_\mu$ and where $\mu < 0$ is as in Hypothesis 5.1. Since $\operatorname{supp} V \subset [0,a]$, we obtain that for $t > a$,

$$f_\mu(t) = C_\mu\, e^{-t\sqrt{-\mu}} \tag{A.1}$$

for some $C_\mu \in \mathbb{R} \setminus \{0\}$. By a direct computation, we obtain for the square of the weighted L_2-norm of g_ε the following asymptotic expansion:

$$\begin{aligned}
\int_0^1 |g_\varepsilon(r)|^2\, r\,\mathrm{d}r &= \varepsilon \int_0^{1/\varepsilon} \chi^2(1-\varepsilon t)\, |f_\mu(t)|^2 (1-\varepsilon t)\,\mathrm{d}t \\
&= \varepsilon \int_0^\infty |f_\mu(t)|^2\,\mathrm{d}t - \varepsilon \int_{1/\varepsilon}^\infty |f_\mu(t)|^2\,\mathrm{d}t - \varepsilon^2 \int_0^{1/(4\varepsilon)} |f_\mu(t)|^2\, t\,\mathrm{d}t \\
&\quad + \varepsilon \int_{1/(4\varepsilon)}^{1/\varepsilon} ((1-\varepsilon t)\,\chi^2(1-\varepsilon t) - 1)\, |f_\mu(t)|^2\,\mathrm{d}t \\
&= \varepsilon \int_0^\infty |f_\mu(t)|^2\,\mathrm{d}t + o(\varepsilon), \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{A.2}$$

where in the first step we perform the change of variable $r = 1 - \varepsilon t$, in the second step we decompose the integral term into the sum of four integral terms via an identical transform based on the properties of χ, and in the last step we used that $|(1-\varepsilon t)\chi^2(1-\varepsilon t) - 1| \le 1$ for all $t \in (1/(4\varepsilon), 1/\varepsilon)$ and that

$$\lim_{\varepsilon\to 0} \int_{1/\varepsilon}^\infty |f_\mu(t)|^2\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_{1/(4\varepsilon)}^{1/\varepsilon} |f_\mu(t)|^2\,\mathrm{d}t = 0, \qquad \int_0^\infty |f_\mu(t)|^2 t\,\mathrm{d}t < \infty,$$

where the last integral is finite due to (A.1).

Without loss of generality, we may assume in the rest of the argument that $\varepsilon < 1/16$. For the quadratic form $\mathfrak{a}_\varepsilon^{(m)}$ of the fibre operator evaluated on the trial function g_ε, we obtain using the properties of the cut-off function χ and the substitution $r = 1 - t\varepsilon$ that

$$\begin{aligned}
\mathfrak{a}_\varepsilon^{(m)}[g_\varepsilon] &= \int_0^1 \Big[\Big(\chi'(r) f_\mu\Big(\frac{1-r}{\varepsilon}\Big) - \frac{1}{\varepsilon}\,\chi(r) f_\mu'\Big(\frac{1-r}{\varepsilon}\Big)\Big)^2 + \frac{m^2}{r^2}\,\chi^2(r) f_\mu^2\Big(\frac{1-r}{\varepsilon}\Big) \\
&\quad + \frac{1}{\varepsilon^2}\,V\Big(\frac{1-r}{\varepsilon}\Big)\chi^2(r)\, f_\mu^2\Big(\frac{1-r}{\varepsilon}\Big)\Big]\, r\,\mathrm{d}r \\
&\le \varepsilon \int_0^{1/\varepsilon} \Big[\Big(\chi'(1-t\varepsilon) f_\mu(t) - \frac{1}{\varepsilon}\,\chi(1-\varepsilon t) f_\mu'(t)\Big)^2 + 4m^2\chi^2(1-\varepsilon t)\,|f_\mu(t)|^2 \\
&\quad + \frac{1}{\varepsilon^2}\,V(t)\chi^2(1-\varepsilon t)\,|f_\mu(t)|^2\Big](1-\varepsilon t)\,\mathrm{d}t = I(\varepsilon) + J(\varepsilon),
\end{aligned} \tag{A.3}$$

where the terms $I(\varepsilon)$ and $J(\varepsilon)$ are defined by

$$\begin{aligned}
I(\varepsilon) &:= \varepsilon \int_0^{1/(\sqrt{\varepsilon})} \Big[\frac{1}{\varepsilon^2}\,|f_\mu'(t)|^2 + 4m^2\,|f_\mu(t)|^2 + \frac{1}{\varepsilon^2}\,V(t)|f_\mu(t)|^2\Big](1-\varepsilon t)\,\mathrm{d}t, \\
J(\varepsilon) &:= \varepsilon \int_{1/(\sqrt{\varepsilon})}^{1/\varepsilon} \Big[\Big(\chi'(1-t\varepsilon) f_\mu(t) - \frac{1}{\varepsilon}\,\chi(1-\varepsilon t) f_\mu'(t)\Big)^2 + 4m^2\chi^2(1-\varepsilon t)\,|f_\mu(t)|^2 \\
&\quad + \frac{1}{\varepsilon^2}\,V(t)\chi^2(1-\varepsilon t)\,|f_\mu(t)|^2\Big](1-\varepsilon t)\,\mathrm{d}t.
\end{aligned}$$

Using that, as $\varepsilon \to 0$,

$$\begin{aligned}
\int_0^{1/(\sqrt{\varepsilon})} \big[|f_\mu'(t)|^2 + V(t)\,|f_\mu(t)|^2\big]\,\mathrm{d}t &\to \int_0^\infty \big[|f_\mu'(t)|^2 + V(t)|f_\mu(t)|^2\big]\,\mathrm{d}t \\
&= \mu \int_0^\infty |f_\mu(t)|^2\,\mathrm{d}t,
\end{aligned}$$

we obtain

$$\begin{aligned}
I(\varepsilon) &= \frac{1}{\varepsilon}\int_0^{1/(\sqrt{\varepsilon})} \big[|f_\mu'(t)|^2 + 4m^2\varepsilon^2\,|f_\mu(t)|^2 + V(t)\,|f_\mu(t)|^2\big]\,\mathrm{d}t \\
&\quad - \int_0^{1/(\sqrt{\varepsilon})} \big[|f_\mu'(t)|^2 + 4m^2\varepsilon^2|f_\mu(t)|^2 + V(t)|f_\mu(t)|^2\big]\,t\,\mathrm{d}t \\
&= \frac{\mu}{\varepsilon}\int_0^\infty |f_\mu(t)|^2\,\mathrm{d}t + o(\varepsilon^{-1}), \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{A.4}$$

Moreover, we conclude, by applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$ (valid for any $a, b > 0$) and the properties of χ, that

$$\begin{aligned}
|J(\varepsilon)| &\le \varepsilon \int_{1/\sqrt{\varepsilon}}^{1/\varepsilon} \Big[2\|\chi'\|_\infty^2 |f_\mu(t)|^2 + \frac{2}{\varepsilon^2}|f_\mu'(t)|^2 + 4m^2|f_\mu(t)|^2 + \frac{1}{\varepsilon^2}\|V\|_\infty |f_\mu(t)|^2\Big]\,\mathrm{d}t \\
&= o(\varepsilon^{-1}).
\end{aligned} \tag{A.5}$$

Plugging (A.4) and (A.5) into (A.3), we end up with the asymptotic expansion

$$\mathfrak{a}_\varepsilon^{(m)}[g_\varepsilon] = \frac{\mu}{\varepsilon}\int_0^\infty |f_\mu(t)|^2\,\mathrm{d}t + o(\varepsilon^{-1}), \quad \text{as } \varepsilon \to 0. \tag{A.6}$$

Finally, combining (A.2) and (A.6) with the min-max principle we arrive at

$$\lambda_1^{(m)}(\varepsilon) \leq \frac{\mathfrak{a}_\varepsilon^{(m)}[g_\varepsilon]}{\int_0^1 |g_\varepsilon(r)|^2\, r\,\mathrm{d}r} = \frac{\mu}{\varepsilon^2} + o(\varepsilon^{-2}), \quad \text{as } \varepsilon \to 0.$$

The claim then follows from the fact that $\mu < 0$.

(c) The statement is a consequence of the representation of the quadratic form $\mathfrak{a}_\varepsilon^{(m)}$ and the fact that, under the assumption $\varepsilon \geq (1/|m|)\sqrt{\|V\|_\infty}$, the function on $(0,1)$ given by

$$r \mapsto \frac{m^2}{r^2} + \frac{1}{\varepsilon^2} V\Big(\frac{1-r}{\varepsilon}\Big)$$

is non-negative. ∎

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