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The second integral homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$

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Abstract. In this article, we explore the second integral homology, or Schur multiplier, of the special linear group $\mathrm{SL}_2(\mathbb{Z}[1/n])$ for a positive integer n. We definitively calculate the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$ when n is divisible by one of the primes 2, 3, 5, 7 or 13. For a general $n > 1$, we offer a partial description by placing the homology group within an exact sequence, and we investigate its rank. Finally, we propose a conjectural structure for $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$ when n is not divisible by any of those specific primes.

Introduction

The (co)homology groups of $\mathrm{SL}_2(\mathbb{Z}[1/n])$ are of considerable interest and importance, finding applications in diverse fields of mathematics such as number theory, algebraic K-theory, hyperbolic geometry and the theory of modular and automorphic forms. These groups offer valuable insights into the arithmetic properties of the ring $\mathbb{Z}[1/n]$ and related rings.

Since $\mathrm{SL}_2(\mathbb{Z}[1/n])$ is an arithmetic group, its (co)homology groups are known to be finitely generated. However, the exact determination of their group structure remains a challenging and important problem, which has been the subject of many research articles (see, for example, [1, 6, 11–13, 17, 22]).

The study of (co)homology groups of $\mathrm{SL}_2(\mathbb{Z}[1/n])$ has seen considerable progress. Adem and Naffah [1] fully computed these groups for $\mathrm{SL}_2(\mathbb{Z}[1/p])$, where p is a prime number. Bui and Ellis [17] used computational methods to calculate homology groups for $n \leq 50$ (with few exceptions). Hutchinson [6] later determined the second homology when 6 divides n. More recently, [11] and [13], independently and with different methods, have achieved complete calculations of the first homology groups for arbitrary n (see Theorem 2.1 in Section 2).

The primary focus of this article is the investigation of the second integral homology of the group $\mathrm{SL}_2(\mathbb{Z}[1/n])$. We may always assume that n is a square-free integer. Building on Hutchinson's insights, incorporating new ideas and drawing upon the results outlined in the preceding paragraph, we present a structural theorem for $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$

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provided n is divided by one of the primes 2, 3, 5, 7 or 13. We state this as the next theorem (see Theorem 6.8).

Theorem A. *Let n be a square-free positive integer and let* $\mathrm{scpd}(n, 2730)$ *be the smallest common prime divisor of n and* $2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.

(i) *If* $\mathrm{scpd}(n, 2730) = 2$, *i.e., if n is even, then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \bigoplus_{p\,|\,(n/2)} \mathbb{Z}/(p-1).$$

(ii) *If* $\mathrm{scpd}(n, 2730) = 3$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \bigoplus_{p\,|\,(n/3)} \mathbb{Z}/(p-1).$$

(iii) *If* $\mathrm{scpd}(n, 2730) = 5$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \begin{cases} \mathbb{Z}/2 \oplus \bigoplus_{p|(n/5)} \mathbb{Z}/(p-1) & \text{if } p \equiv 1 \pmod 4 \text{ for all } p\,|\,(n/5), \\ \mathbb{Z}/4 \oplus \mathbb{Z}/((q-1)/2) \oplus \bigoplus_{p|(n/5q)} \mathbb{Z}/(p-1) & \text{if } q \equiv 3 \pmod 4 \text{ for some } q\,|\,(n/5). \end{cases}$$

(iv) *If* $\mathrm{scpd}(n, 2730) = 7$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \bigoplus_{p\,|\,(n/7)} \mathbb{Z}/(p-1).$$

(v) *If* $\mathrm{scpd}(n, 2730) = 13$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \begin{cases} \mathbb{Z}/6 \oplus \bigoplus_{p\,|\,(n/13)} \mathbb{Z}/(p-1) & \text{if } p \equiv 1 \pmod 4 \text{ for all } p\,|\,(n/13), \\ \mathbb{Z}/12 \oplus \mathbb{Z}/((q-1)/2) \oplus \bigoplus_{p|(n/13q)} \mathbb{Z}/(p-1) & \text{if } q \equiv 3 \pmod 4 \text{ for some } q\,|\,(n/13). \end{cases}$$

It is a known fact, proved by Adem and Naffah in [1], that the second homology group $H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathbb{Z})$, where p is a prime, is of rank one precisely when p is one of the primes 2, 3, 5, 7 or 13. More precisely, if

$$r_p := \mathrm{rank}\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathbb{Z}),$$

then $r_p = 1$ if and only if $p = 2, 3, 5, 7$ or 13 (see Theorem 2.3 below).

For any square-free integer n, we establish the following theorem (see Theorem 6.6) that provides insights into the structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$. Notably, certain parts of Theorem A are derived directly from this broader theorem.

Theorem B. *Let n be a square-free positive integer with prime decomposition $n = p_1 \cdots p_l$ and let $r_{p_1} \leq \cdots \leq r_{p_l}$. If $r_{p_i} = r_{p_{i+1}}$, we assume that $p_i < p_{i+1}$. Then we have the exact sequence*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/p_1]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \bigoplus_{i=2}^{l} \mathbb{Z}/(p_i - 1) \to 0.$$

Theorem B follows from a careful analysis of the Mayer–Vietoris exact sequence applied to the amalgamated product

$$\mathrm{SL}_2(\mathbb{Z}[1/pn]) \simeq \mathrm{SL}_2(\mathbb{Z}[1/n]) *_{\Gamma_0(n,p)} \mathrm{SL}_2(\mathbb{Z}[1/n]),$$

where p is a prime not dividing n and $\Gamma_0(n, p)$ is the following subgroup of $\mathrm{SL}_2(\mathbb{Z}[1/n])$:

$$\Gamma_0(n, p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) : p \mid c \right\}.$$

For more on this isomorphism, we refer the reader to p. 80 of [15] (see also Theorem 1.5 in the next section).

Understanding the first and the second homology of $\Gamma_0(n, p)$ is essential for proving Theorem B. Specifically, the structure of the first homology, $H_1(\Gamma_0(n, p), \mathbb{Z})$, is determined and detailed as follows (see Theorem 4.2).

Theorem C. *Let n be a natural number greater than one, p a prime not dividing n and $d := \gcd\{m^2 - 1 : m \mid n\}$.*

(i) *If $p > 3$ and $p \nmid d$, then*

$$H_1(\Gamma_0(n, p), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{Z}/(p - 1).$$

(ii) *If $p = 3$ and $d = 3t$, where $3 \nmid t$ (e.g., when $2 \mid n$ or $5 \mid n$), then*

$$H_1(\Gamma_0(n, 3), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3.$$

(iii) *If $p = 2$ and $d = 8t$, where $2 \nmid t$ (e.g., when $3 \mid n$ or $5 \mid n$), then*

$$H_1(\Gamma_0(n, 2), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

We conclude the introduction by outlining the structure of the present article. In Section 1, we review established results concerning the group $\mathrm{SL}_2(\mathbb{Z}[1/n])$, specifically its congruence subgroup property and its description as an amalgamated product. Section 2 reviews necessary results on the first and second homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$. Section 3 provides a brief overview of the Mayer–Vietoris exact sequence associated with the amalgamated product decomposition of $\mathrm{SL}_2(\mathbb{Z}[1/pn])$, where p is a prime not dividing n. Section 4 examines the first homology of $\Gamma_0(n, p)$ and presents the proof of Theorem C. In Section 5, we demonstrate that the inclusion $\Gamma_0(n, p) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/n])$ induces a surjective map on second homology. Section 6 then provides the proofs of Theorems B and A. Finally, Section 7 investigates the rank of the second homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$ and concludes the article by proposing a conjectural structure for the second homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$ when n is not divisible by any of the primes 2, 3, 5, 7 or 13.

1. The special linear group of degree two over $\mathbb{Z}[1/n]$

For any nonzero integer n, $\mathbb{Z}[1/n]$ is the following subring of $\mathbb{Q}$:

$$\mathbb{Z}[1/n] := \{a/n^r : a \in \mathbb{Z}, r \in \mathbb{Z}^{\geq 0}\}.$$

This is a Euclidean domain. If $m \mid n$, then $\mathbb{Z}[1/m] \subseteq \mathbb{Z}[1/n]$. Moreover, for any $k \neq 0$ and $r \geq 1$, $\mathbb{Z}[\pm 1/k^r m] = \mathbb{Z}[1/km]$. Thus, if $n = \pm p_1^{m_1} \cdots p_s^{m_s}$ is the prime factorization of n, then

$$\mathbb{Z}[1/n] = \mathbb{Z}[1/p_1 \cdots p_s].$$

Therefore, to study $\mathbb{Z}[1/n]$, we are allowed to always assume that n is a square-free positive integer.

For a commutative ring A, let $\mathrm{E}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by the elementary matrices

$$E_{12}(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad E_{21}(a) := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad a \in A.$$

Clearly, $\mathrm{E}_2(A) \subseteq \mathrm{SL}_2(A)$. For examples of rings for which this is a proper subgroup, see Theorem 6.1 in [5].

We say that A is a GE_2-*ring* if $\mathrm{E}_2(A) = \mathrm{SL}_2(A)$. It is known that semilocal rings and Euclidean domains are GE_2-rings (see p. 245 of [16] and Section 2 of [5]). Hence, $\mathbb{Z}[1/n]$ is a GE_2-ring. Indeed, it can be shown that for any $n > 1$, $\mathrm{SL}_2(\mathbb{Z}[1/n])$ is generated by the matrices $E_{21}(1)$ and $E_{12}(-1/n)$ (see p. 204 of [8] and Lemma 1.1 in [13]).

The next theorem is a special case of a general result due to Vaserstein and Liehl (see [18] and [7]).

Theorem 1.1 (Vaserstein, Liehl). *Let $n > 1$ be an integer and let I_1 and I_2 be nonzero ideals of $\mathbb{Z}[1/n]$. Let*

$$\widetilde{\Gamma}(I_1, I_2) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) : b \in I_1, c \in I_2, a-1, d-1 \in I_1 I_2\right\}.$$

Then $\widetilde{\Gamma}(I_1, I_2)$ is generated by elementary matrices $E_{12}(x)$, $x \in I_1$, and $E_{21}(y)$, $y \in I_2$, and it is of finite index in $\mathrm{SL}_2(\mathbb{Z}[1/n])$.

Proof. See Theorem on p. 321 of [18] and Section 4 of [7]. ∎

Lemma 1.2. *Let I be an ideal of a Euclidean domain A and let $\pi\colon A \to A/I$ be the quotient map of rings. Then the natural map $\pi_*\colon \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/I)$ is surjective.*

Proof. If $I = (0)$, there is nothing to prove. Thus, let I be a nontrivial ideal of A. Then A/I is semi-local and thus it is a GE_2-ring. It follows from this that $\mathrm{SL}_2(A/I)$ is generated by elementary matrices $E_{12}(\bar{a})$ and $E_{21}(\bar{a})$, $a \in A$. Since

$$\pi_*(E_{ij}(a)) = E_{ij}(\bar{a}),$$

the map π_* is surjective. ∎

Let I be an ideal of a Euclidean domain A. Let $\Gamma(A,I)$ be the kernel of the surjective map $\pi_*\colon \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/I)$. Hence,

$$\Gamma(A,I) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A) : b, c, a-1, d-1 \in I \right\}.$$

Subgroups of the form $\Gamma(A,I)$, for some nontrivial ideal I, are called *principal congruence subgroups* of $\mathrm{SL}_2(A)$. A subgroup of $\mathrm{SL}_2(A)$ is called a *congruence subgroup* if it contains a principal congruence subgroup.

Lemma 1.3. *Let I and J be two coprime ideals of a Euclidean domain A. Then the composite $\Gamma(A,I) \to \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/J)$ is surjective.*

Proof. See Lemma 5.12 in [6]. ∎

Let A be a Euclidean domain such that for any nontrivial ideal I, A/I is a finite ring. Then any principal congruence subgroup of $\mathrm{SL}_2(A)$ is of finite index. We say that $\mathrm{SL}_2(A)$ has the *congruence subgroup property* if any finite-index subgroup of $\mathrm{SL}_2(A)$ is a congruence subgroup.

The next theorem is a special case of a general result due to Serre (see [14] and [8]) and shows that, for any $n > 1$, $\mathrm{SL}_2(\mathbb{Z}[1/n])$ has the congruence subgroup property.

Theorem 1.4 (Congruence subgroup property). *Let Γ be a noncentral normal subgroup of $\mathrm{SL}_2(\mathbb{Z}[1/n])$, where $n > 1$. Then Γ contains a subgroup of the form $\Gamma(\mathbb{Z}[1/n], I)$, for some nontrivial ideal I of $\mathbb{Z}[1/n]$. In particular, Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z}[1/n])$.*

Proof. This is a special case of Proposition 2 in [14]. ∎

Let n be a natural number and p a prime such that $p \nmid n$. Let

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}[1/n])^{\pi_p} &:= \pi_p^{-1} \mathrm{SL}_2(\mathbb{Z}[1/n]) \pi_p \\ &= \left\{ \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) \right\}, \end{aligned}$$

where $\pi_p := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$. It is easy to see that

$$\mathrm{SL}_2(\mathbb{Z}[1/n]) \cap \mathrm{SL}_2(\mathbb{Z}[1/n])^{\pi_p} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) : p \mid c \right\}.$$

In the introduction we denoted this group by $\Gamma_0(n,p)$. Now, consider the following inclusions:

$$\begin{aligned} & i_1 : \Gamma_0(n,p) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/n]), && i_2 : \Gamma_0(n,p) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/n])^{\pi_p}, \\ & j_1 : \mathrm{SL}_2(\mathbb{Z}[1/n]) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/pn]), && j_2 : \mathrm{SL}_2(\mathbb{Z}[1/n])^{\pi_p} \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/pn]). \end{aligned}$$

As established in [15], the theory of trees provides a proof for the following well-known result.

Theorem 1.5. *Let n be a natural number and p a prime such that $p \nmid n$. Then*

$$\mathrm{SL}_2(\mathbb{Z}[1/pn]) \simeq \mathrm{SL}_2(\mathbb{Z}[1/n]) *_{\Gamma_0(n,p)} \mathrm{SL}_2(\mathbb{Z}[1/n])^{\pi_p}.$$

Proof. See p. 80 of [15]. ∎

If we replace i_2 and j_2 by the injective maps

$$i_2' : \Gamma_0(n,p) \to \mathrm{SL}_2(\mathbb{Z}[1/n]) \quad \text{and} \quad j_2' : \mathrm{SL}_2(\mathbb{Z}[1/n]) \to \mathrm{SL}_2(\mathbb{Z}[1/pn]),$$

which are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix},$$

respectively, then the above isomorphism finds the following form:

$$\mathrm{SL}_2(\mathbb{Z}[1/pn]) \simeq \mathrm{SL}_2(\mathbb{Z}[1/n]) *_{\Gamma_0(n,p)} \mathrm{SL}_2(\mathbb{Z}[1/n]).$$

For a prime p, let $\mathbb{F}_p := \mathbb{Z}/p$ be the prime field with p elements. If $p \nmid n$, then the natural map $\mathbb{Z}[1/n] \to \mathbb{F}_p$, $a/n^r \mapsto \bar{a}/\bar{n}^r$, induces the natural surjective homomorphisms

$$\mathrm{SL}_2(\mathbb{Z}[1/n]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_p), \quad \Gamma_0(n,p) \twoheadrightarrow \mathrm{B}(\mathbb{F}_p),$$

where

$$\mathrm{B}(\mathbb{F}_p) := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\} \subseteq \mathrm{SL}_2(\mathbb{F}_p).$$

Thus, we have the morphism of extensions

$$\begin{tikzcd} 1 \arrow[r] & \Gamma(n,p) \arrow[r] \arrow[d, equal] & \Gamma_0(n,p) \arrow[r] \arrow[d, "i_1"] & \mathrm{B}(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \Gamma(n,p) \arrow[r] & \mathrm{SL}_2(\mathbb{Z}[1/n]) \arrow[r] & \mathrm{SL}_2(\mathbb{F}_p) \arrow[r] & 1, \end{tikzcd} \tag{1.1}$$

where

$$\Gamma(n,p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) : p \mid b, c, a-1, d-1 \right\} = \Gamma(\mathbb{Z}[1/n], \langle p \rangle).$$

Observe that

$$[\mathrm{SL}_2(\mathbb{Z}[1/n]) : \Gamma_0(n,p)] = p+1 \quad \text{and} \quad [\Gamma_0(n,p) : \Gamma(n,p)] = p(p-1),$$

and thus

$$[\mathrm{SL}_2(\mathbb{Z}[1/n]) : \Gamma(n,p)] = p(p^2-1). \tag{1.2}$$

Let

$$\Gamma(n,p^k) := \Gamma(\mathbb{Z}[1/n], \langle p^k \rangle).$$

Lemma 1.6. *If p is a prime and n is not divisible by p, then*

$$[\Gamma(n,p) : \Gamma(n,p^k)] = p^{3(k-1)}.$$

Proof. See Corollary 5.11 in [6]. ∎

2. The first and second homology groups of $\mathrm{SL}_2(\mathbb{Z}[1/n])$

Understanding the exact structure of the finitely generated groups $H_k(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$ is essential for various applications. While $H_0(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$ is simply $\mathbb{Z}$, the higher homology groups are more complex. The structural theorem for $H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$, proved independently and with different methods in [11] and [13], will be a fundamental tool in this work.

Theorem 2.1. *If $n > 1$ is an integer, then*

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \begin{cases} 0 & \text{if } 2 \mid n,\ 3 \mid n, \\ \mathbb{Z}/3 & \text{if } 2 \mid n,\ 3 \nmid n, \\ \mathbb{Z}/4 & \text{if } 2 \nmid n,\ 3 \mid n, \\ \mathbb{Z}/12 & \text{if } 2 \nmid n,\ 3 \nmid n, \end{cases}$$

which is induced by the map $\mathbb{Z}[1/n] \to \mathrm{SL}_2(\mathbb{Z}[1/n])^{\mathrm{ab}}$, $a \mapsto \overline{E_{12}(a)}$.

Proof. See Proposition 4.4 in [11] or Theorem 1.2 in [13]. ∎

The proof of the above theorem in [11] relies partly on the following result, which will also be essential in some of the proofs presented in this article.

Proposition 2.2. *Let A be a commutative local ring with maximal ideal $\mathfrak{m}_A$. Then*

$$H_1(\mathrm{SL}_2(A),\mathbb{Z}) \simeq \begin{cases} A/\mathfrak{m}_A^2 & \text{if } |A/\mathfrak{m}_A| = 2, \\ A/\mathfrak{m}_A & \text{if } |A/\mathfrak{m}_A| = 3, \\ 0 & \text{if } |A/\mathfrak{m}_A| \geq 4. \end{cases}$$

Proof. See Proposition 4.1 in [11]. ∎

This paper focuses on determining the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$. Our proofs rely on two key results concerning this group, which are already established in the literature.

Theorem 2.3 (Adem–Naffah, [1]). *If p is a prime number, then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } p = 2,3, \\ \mathbb{Z}^{(p-7)/6} \oplus \mathbb{Z}/6 & \text{if } p \equiv 1 \pmod{12}, \\ \mathbb{Z}^{(p+1)/6} \oplus \mathbb{Z}/2 & \text{if } p \equiv 5 \pmod{12}, \\ \mathbb{Z}^{(p-1)/6} \oplus \mathbb{Z}/3 & \text{if } p \equiv 7 \pmod{12}, \\ \mathbb{Z}^{(p+7)/6} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. Adem and Naffah gave a complete description of the structure of the cohomology groups $H^k(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathbb{Z})$, for any $k \geq 0$ (see pp. 7-9 of [1]). Using this and Lemma 4.3 in [10], one can calculate the homology groups $H_k(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathbb{Z})$. In particular, we obtain the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathbb{Z})$, as claimed in this theorem. ∎

Theorem 2.4 (Hutchinson). *Let n be a square-free integer such that $6 \mid n$. Then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \bigoplus_{p \mid n} \mathbb{Z}/(p-1).$$

More generally, if $m \mid n$ and $6 \mid m$, then we have the split exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \bigoplus_{p \mid (n/m)} \mathbb{F}_p^\times \to 1.$$

Proof. See Theorems 6.10 and 6.12 in [6]. ∎

For the map $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \bigoplus_{p \mid (n/m)} \mathbb{F}_p^\times$, appearing in Theorem 2.4, we refer the reader to Section 6.

This article presents a generalization of the results of Hutchinson, Adem–Naffah, and Bui–Ellis, determining the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$ for any $n \in \mathbb{N}$ that has a prime factor within the set $\{2, 3, 5, 7, 13\}$.

3. Mayer–Vietoris exact sequence

Let n be an integer and let p be a prime such that $p \nmid n$. By the Mayer–Vietoris exact sequence (Corollary 7.7 in Section 7, Chapter II, of [3]) applied to the amalgamated isomorphism of Theorem 1.5, we have the exact sequence

$$\begin{aligned} H_2(\Gamma_0(n,p), \mathbb{Z}) &\xrightarrow{\alpha_2} H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \\ &\xrightarrow{\beta_2} H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \longrightarrow H_1(\Gamma_0(n,p), \mathbb{Z}) \\ &\xrightarrow{\alpha_1} H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \\ &\xrightarrow{\beta_1} H_1(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to 0. \end{aligned}$$

Here,

$$\alpha_k(x) = (i_{1*}(x), i'_{2*}(x)) \quad \text{and} \quad \beta_k(y,z) = j'_{2*}(z) - j_{1*}(y).$$

Based on this and Theorem 2.1, β_1 is given by

$$(\bar{a}, \bar{b}) \mapsto \overline{pb - a}.$$

Moreover, the Lyndon/Hochschild–Serre spectral sequence of the morphism of extensions (1.1) gives us the commutative diagram with exact rows

$$\begin{array}{ccccccc} H_2(\mathrm{B}(\mathbb{F}_p),\mathbb{Z}) & \longrightarrow & H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_p)} & \longrightarrow & H_1(\Gamma_0(n,p),\mathbb{Z}) & \twoheadrightarrow & H_1(\mathrm{B}(\mathbb{F}_p),\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) & \longrightarrow & H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_p)} & \longrightarrow & H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) & \twoheadrightarrow & H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) \end{array}$$

(see Corollary 6.4 in Chapter VII of [3]). It is known that $H_2(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z})=0$ (see Theorem 3.9 in [2]). Moreover, by Proposition 2.2, we have

$$H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z})\simeq(\mathbb{F}_p)_{\mathbb{F}_p^\times}=\begin{cases}\mathbb{F}_2 & \text{if } p=2,\\ \mathbb{F}_3 & \text{if } p=3,\\ 0 & \text{if } p>3,\end{cases}$$

where $\mathbb{F}_p^\times$ acts on $\mathbb{F}_p$ by the formula $a.x:=a^2x$. The homology groups of $\mathrm{B}(\mathbb{F}_p)$ can be obtained by studying the Lyndon/Hochschild-Serre spectral sequence of the split extension

$$0\to\mathbb{F}_p\xrightarrow{i}\mathrm{B}(\mathbb{F}_p)\xrightarrow{pr}\mathbb{F}_p^\times\to 1,$$

where $i(x)=\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$, $pr\left(\left(\begin{smallmatrix}a&b\\0&a^{-1}\end{smallmatrix}\right)\right)=a$ and a section $s\colon\mathbb{F}_p^\times\to\mathrm{B}(\mathbb{F}_p)$ can be given by $a\mapsto\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)$. From this, we obtain the isomorphisms

$$H_1(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})\simeq\mathbb{F}_p^\times\oplus(\mathbb{F}_p)_{\mathbb{F}_p^\times}\quad\text{and}\quad H_2(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})\simeq H_1(\mathbb{F}_p^\times,\mathbb{F}_p).$$

Hence,

$$H_1(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})\simeq\begin{cases}\mathbb{F}_2 & \text{if } p=2,\\ \mathbb{F}_3^\times\oplus\mathbb{F}_3 & \text{if } p=3,\\ \mathbb{F}_p^\times & \text{if } p>3,\end{cases}\quad\text{and}\quad H_2(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})=0$$

(for the calculation of the second homology we used the calculation of the homology of finite cyclic groups given in pp. 58–59 of [3]). Thus, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}0&\longrightarrow&H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_p)}&\longrightarrow&H_1(\Gamma_0(n,p),\mathbb{Z})&\longrightarrow&H_1(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})&\longrightarrow&0\\ &&\downarrow&&\downarrow&&\downarrow&&\\ 0&\longrightarrow&H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_p)}&\longrightarrow&H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})&\longrightarrow&H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z})&\longrightarrow&0.\end{array}\tag{3.1}$$

We break the study of the Mayer–Vietoris exact sequence and the above diagram into three cases.

Case (i) $p>3$. Since $p\nmid n$, by Theorem 2.1,

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})\simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/pn]),\mathbb{Z}).$$

Thus, we have the Mayer–Vietoris exact sequence

$$\begin{aligned}H_2(\Gamma_0(n,p),\mathbb{Z})&\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})\oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})\\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]),\mathbb{Z})\to H_1(\Gamma_0(n,p),\mathbb{Z})\to H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})\to 0.\end{aligned}$$

Since $p>3$,

$$H_1(\mathrm{B}(\mathbb{F}_p),\mathbb{Z})\simeq\mathbb{F}_p^\times\quad\text{and}\quad H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z})=0.$$

Hence, the diagram (3.1) finds the following form:

$$\begin{tikzcd} 0 \arrow[r] & H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_p)} \arrow[r] \arrow[d, two heads] & H_1(\Gamma_0(n,p),\mathbb{Z}) \arrow[r] \arrow[d, two heads, "i_{1*}"] & \mathbb{F}_p^\times \arrow[r] & 1 \\ & H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_p)} \arrow[r, two heads, "\simeq"] & H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}). & & \end{tikzcd} \tag{3.2}$$

Case (ii) $p=3$. This case has two parts.

Case (ii-a) $2\mid n$.

In this case, by Theorem 2.1, we have

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}/3 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{Z}[1/3n]),\mathbb{Z}) = 0.$$

Thus, we have the Mayer–Vietoris exact sequence

$$\begin{aligned} H_2(\Gamma_0(n,3),\mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/3n]),\mathbb{Z}) \to H_1(\Gamma_0(n,3),\mathbb{Z}) \to \mathbb{Z}/3 \oplus \mathbb{Z}/3 \to 0. \end{aligned}$$

Since

$$H_1(\mathrm{B}(\mathbb{F}_3),\mathbb{Z}) \simeq \mathbb{F}_3^\times \oplus \mathbb{F}_3 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{F}_3),\mathbb{Z}) \simeq \mathbb{F}_3,$$

and the map

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{F}_3),\mathbb{Z})$$

is an isomorphism, we have

$$H_1(\Gamma(n,3),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_3)} = 0.$$

Hence, the diagram (3.1) becomes

$$\begin{tikzcd} 0 \arrow[r] & H_1(\Gamma(n,3),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_3)} \arrow[r] & H_1(\Gamma_0(n,3),\mathbb{Z}) \arrow[r] \arrow[d, two heads, "i_{1*}"] & \mathbb{F}_3^\times \oplus \mathbb{F}_3 \arrow[r] \arrow[d] & 0 \\ & & H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \arrow[r, "\simeq"] & \mathbb{F}_3. & \end{tikzcd} \tag{3.3}$$

Case (ii-b) $2\nmid n$.

In this case,

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}/12 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{Z}[1/3n]),\mathbb{Z}) \simeq \mathbb{Z}/4$$

and, thus, we have the Mayer–Vietoris exact sequence

$$\begin{aligned} H_2(\Gamma_0(n,3),\mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/3n]),\mathbb{Z}) \to H_1(\Gamma_0(n,3),\mathbb{Z}) \to \mathbb{Z}/12 \oplus \mathbb{Z}/12 \to \mathbb{Z}/4 \to 0. \end{aligned}$$

Since

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{F}_3),\mathbb{Z}) \simeq \mathbb{Z}/3$$

is surjective, we have the isomorphism

$$H_1(\Gamma(n,3),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_3)} \simeq \mathbb{Z}/4.$$

Hence, the diagram (3.1) finds the following form:

$$\begin{tikzcd} 0 \arrow[r] & H_1(\Gamma(n,3),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_3)} \arrow[r] \arrow[d, two heads] & H_1(\Gamma_0(n,3),\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{F}_3^\times \oplus \mathbb{F}_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}/4 \arrow[r] & H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \arrow[r] & \mathbb{F}_3 \arrow[r] & 0. \end{tikzcd} \tag{3.4}$$

Case (iii) $p=2$. This case has two parts.

Case (iii-a) $3 \mid n$.

In this case,

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}/4 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{Z}[1/2n]),\mathbb{Z}) = 0,$$

and, thus, we have the Mayer–Vietoris exact sequence

$$\begin{aligned} H_2(\Gamma_0(n,2),\mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/2n]),\mathbb{Z}) \to H_1(\Gamma_0(n,2),\mathbb{Z}) \to \mathbb{Z}/4 \oplus \mathbb{Z}/4 \to 0. \end{aligned}$$

Since

$$H_1(\mathrm{B}(\mathbb{F}_2),\mathbb{Z}) \simeq \mathbb{F}_2 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{F}_2),\mathbb{Z}) \simeq \mathbb{F}_2,$$

and the map $H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{F}_2),\mathbb{Z})$ is surjective, we have

$$H_1(\Gamma(n,2),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_2)} \simeq \mathbb{Z}/2.$$

Hence, the diagram (3.1) finds the following form:

$$\begin{tikzcd} 0 \arrow[r] & H_1(\Gamma(n,2),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_2)} \arrow[r] \arrow[d] & H_1(\Gamma_0(n,2),\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{F}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{F}_2 \arrow[r] & 0. \end{tikzcd} \tag{3.5}$$

Case (iii-b) $3 \nmid n$.

In this case,

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}/12 \quad \text{and} \quad H_1(\mathrm{SL}_2(\mathbb{Z}[1/2n]),\mathbb{Z}) \simeq \mathbb{Z}/3,$$

and, thus, we have the Mayer–Vietoris exact sequence

$$\begin{aligned} H_2(\Gamma_0(n,2),\mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/2n]),\mathbb{Z}) \to H_1(\Gamma_0(n,2),\mathbb{Z}) \to \mathbb{Z}/12 \oplus \mathbb{Z}/12 \to \mathbb{Z}/3 \to 0. \end{aligned}$$

Since $H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{F}_2),\mathbb{Z})$ is surjective, we have

$$H_1(\Gamma(n,2),\mathbb{Z})_{\mathrm{SL}_2(\mathbb{F}_2)} \simeq \mathbb{Z}/6.$$

Hence, the diagram (3.1) is of the following form:

$$\begin{tikzcd} 0 \arrow[r] & H_1(\Gamma(n,2),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_2)} \arrow[r] \arrow[d, two heads] & H_1(\Gamma_0(n,2),\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{F}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}/6 \arrow[r] & \mathbb{Z}/12 \arrow[r] & \mathbb{F}_2 \arrow[r] & 0. \end{tikzcd} \tag{3.6}$$

4. The first homology of $\Gamma_0(n, p)$

We denote the natural inclusion $\Gamma(n,p) \to \mathrm{SL}_2(\mathbb{Z}[1/n])$ by i.

Theorem 4.1. *Let $n > 1$ be a square-free integer and p a prime such that $p \nmid n$. Then the kernel of the natural map*

$$i_* : H_1(\Gamma(n,p), \mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$$

is a p-group.

Proof. Let $A := \mathbb{Z}[1/n]$, $\mathfrak{p} := \langle p \rangle$, $\mathcal{K} := \ker(i_*)$ and $\mathcal{G} := \mathrm{im}(i_*)$. Observe that $\Gamma(A, \mathfrak{p}) = \Gamma(n,p)$. By these notations, we have

$$|\mathcal{K}| = |H_1(\Gamma(A,\mathfrak{p}), \mathbb{Z})|/|\mathcal{G}|.$$

Since $[\Gamma(A,\mathfrak{p}), \Gamma(A,\mathfrak{p})]$ is a noncentral normal subgroup of $\mathrm{SL}_2(A)$, by Theorem 1.4, the group $[\Gamma(A,\mathfrak{p}), \Gamma(A,\mathfrak{p})]$ contains a subgroup of the form $\Gamma(A, I)$, for some nontrivial ideal I of A. Note that $I \subseteq \mathfrak{p}$. In fact, if $a \in I$, then $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(A, I) \subseteq \Gamma(A, \mathfrak{p})$. It follows from this that $a \in \mathfrak{p}$. Since $\Gamma(A, 2^2 3^2 I) \subseteq \Gamma(A, I)$, we may assume that $I = \langle 2^2 3^2 y \rangle$, for some $y \in A$.

Let $I = \langle p_1^{r_1} \cdots p_m^{r_m} \rangle$, where $p_1 = p$ and $p_2 < p_3 < \cdots < p_m$ are primes, and so irreducible, elements of A. Let $k := r_1$ and $J := \langle p_2^{r_2} \cdots p_m^{r_m} \rangle$. Then $I = \mathfrak{p}^k J$, where $k \geq 1$. Note that $\mathfrak{p}^k$ and J are coprime. Since $\Gamma(A, \mathfrak{p}^{k+1}) \subseteq \Gamma(A, \mathfrak{p}^k)$, we may assume without loss that $k \geq 2$. Now, by Lemma 1.3, the map

$$\Gamma(A, \mathfrak{p}) \to \mathrm{SL}_2(A/J)$$

is surjective. From this, we obtain the surjective map

$$[\Gamma(A,\mathfrak{p}), \Gamma(A,\mathfrak{p})] \to [\mathrm{SL}_2(A/J), \mathrm{SL}_2(A/J)].$$

Moreover, the inclusion $I \subseteq J$ gives us the inclusion $\Gamma(A, I) \subseteq \Gamma(A, J)$. From this, we obtain the surjective map

$$[\Gamma(A,\mathfrak{p}), \Gamma(A,\mathfrak{p})]/\Gamma(A, I) \to [\mathrm{SL}_2(A/J), \mathrm{SL}_2(A/J)]$$

and thus

$$|[\mathrm{SL}_2(A/J), \mathrm{SL}_2(A/J)]| \;\big|\; [[\Gamma(A,\mathfrak{p}), \Gamma(A,\mathfrak{p})] : \Gamma(A, I)]. \tag{4.1}$$

From the isomorphism

$$H_1(\mathrm{SL}_2(A/J), \mathbb{Z}) \simeq \frac{\mathrm{SL}_2(A/J)}{[\mathrm{SL}_2(A/J), \mathrm{SL}_2(A/J)]},$$

we have

$$|[\mathrm{SL}_2(A/J), \mathrm{SL}_2(A/J)]| = |\mathrm{SL}_2(A/J)|/|H_1(\mathrm{SL}_2(A/J), \mathbb{Z})|.$$

It follows from this and (4.1) that

$$|\mathrm{SL}_2(A/J)|/|H_1(\mathrm{SL}_2(A/J),\mathbb{Z})| \;\big|\; [[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]:\Gamma(A,I)].$$

Let $m \in \mathbb{N}$ be such that

$$[[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]:\Gamma(A,I)] = \frac{m|\mathrm{SL}_2(A/J)|}{|H_1(\mathrm{SL}_2(A/J),\mathbb{Z})|}.$$

Since $\mathfrak{p}^k$ and J are coprime ideals of A and $I = \mathfrak{p}^k J$, we have $A/I \simeq A/\mathfrak{p}^k \times A/J$ and, thus,

$$\mathrm{SL}_2(A/I) \simeq \mathrm{SL}_2(A/\mathfrak{p}^k) \times \mathrm{SL}_2(A/J). \tag{4.2}$$

By Theorem 1.4, the index $[\mathrm{SL}_2(A):[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]]$ is finite and so we have

$$\begin{aligned}
[\mathrm{SL}_2(A):[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]] &= \frac{[\mathrm{SL}_2(A):\Gamma(A,I)]}{[[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]:\Gamma(A,I)]}\\
&= \frac{|\mathrm{SL}_2(A/I)|}{[[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]:\Gamma(A,I)]} = \frac{|\mathrm{SL}_2(A/\mathfrak{p}^k)|\,|\mathrm{SL}_2(A/J)|}{[[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]:\Gamma(A,I)]}\\
&= \frac{|H_1(\mathrm{SL}_2(A/J),\mathbb{Z})|\,|\mathrm{SL}_2(A/\mathfrak{p}^k)|}{m}.
\end{aligned}$$

By (4.2),

$$H_1(\mathrm{SL}_2(A/I),\mathbb{Z}) \simeq H_1(\mathrm{SL}_2(A/J),\mathbb{Z}) \oplus H_1(\mathrm{SL}_2(A/\mathfrak{p}^k),\mathbb{Z}).$$

Hence,

$$\begin{aligned}
\frac{|H_1(\mathrm{SL}_2(A/I),\mathbb{Z})|\,|\mathrm{SL}_2(A/\mathfrak{p}^k)|}{|H_1(\mathrm{SL}_2(A/\mathfrak{p}^k),\mathbb{Z})|} &= m[\mathrm{SL}_2(A):[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]]\\
&= m[\mathrm{SL}_2(A):\Gamma(A,\mathfrak{p})][\Gamma(A,\mathfrak{p}):[\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})]]\\
&\overset{(1.2)}{=} mp(p^2-1)|H_1(\Gamma(A,\mathfrak{p}),\mathbb{Z})| = mp(p^2-1)|\mathcal{K}||\mathcal{G}|.
\end{aligned}$$

Now we prove that

$$H_1(\mathrm{SL}_2(A/I),\mathbb{Z}) \simeq H_1(\mathrm{SL}_2(A),\mathbb{Z}). \tag{4.3}$$

Since

$$\Gamma(A,I) \subseteq [\Gamma(A,\mathfrak{p}),\Gamma(A,\mathfrak{p})] \subseteq [\mathrm{SL}_2(A),\mathrm{SL}_2(A)],$$

we have the surjective map

$$\phi : \mathrm{SL}_2(A/I) \simeq \mathrm{SL}_2(A)/\Gamma(A,I) \twoheadrightarrow \mathrm{SL}_2(A)/[\mathrm{SL}_2(A),\mathrm{SL}_2(A)] = H_1(\mathrm{SL}_2(A),\mathbb{Z}).$$

This gives us the surjective map

$$\phi_* : H_1(\mathrm{SL}_2(A/I),\mathbb{Z}) \to H_1(\mathrm{SL}_2(A),\mathbb{Z}).$$

On the other hand, we have the natural surjective map

$$\pi_* : H_1(\mathrm{SL}_2(A), \mathbb{Z}) \twoheadrightarrow H_1(\mathrm{SL}_2(A/I), \mathbb{Z}).$$

It is straightforward to check that the maps ϕ_* and π_* are inverse of each other. This proves (4.3).

Let $l = |H_1(\mathrm{SL}_2(A), \mathbb{Z})|$. By Theorem 2.1 we have

$$H_1(\mathrm{SL}_2(A/\mathfrak{p}^k), \mathbb{Z}) \simeq \begin{cases} 0 & \text{if } p > 3, \\ \mathbb{Z}/3 & \text{if } p = 3, \\ \mathbb{Z}/4 & \text{if } p = 2. \end{cases}$$

Hence,

$$|H_1(\mathrm{SL}_2(A/\mathfrak{p}^k), \mathbb{Z})| = p^\alpha, \quad \text{for some } \alpha \in \{0, 1, 2\}.$$

Now, from the above, we have

$$l|\mathrm{SL}_2(A/\mathfrak{p}^k)| = mp(p^2-1)|\mathcal{K}||\mathcal{G}||H_1(\mathrm{SL}_2(A/\mathfrak{p}^k), \mathbb{Z})| = mp(p^2-1)|\mathcal{K}||\mathcal{G}|p^\alpha.$$

Using diagrams (3.2), (3.3), (3.4), (3.5), (3.6) and Theorem 2.1 and Proposition 2.2, we easily get

$$|\mathcal{G}| = \begin{cases} l & \text{if } p > 3, \\ l/3 & \text{if } p = 3, \\ l/2 & \text{if } p = 2. \end{cases}$$

Thus, $l = p^r|\mathcal{G}|$, where $r \in \{0, 1\}$. Now, since

$$|\mathrm{SL}_2(A/\mathfrak{p}^k)| = (p^2-1)\, p^{3k-2}$$

(see p. 248 of [6]), we have

$$l(p^2-1)p^{3k-2} = mp(p^2-1)|\mathcal{K}||\mathcal{G}|p^\alpha$$

and hence

$$p^{3k-3+r} = m|\mathcal{K}|p^\alpha.$$

It follows from this that $|\mathcal{K}|$ is a power of p. This completes the proof of the theorem. ∎

The next theorem is Theorem C of the introduction.

Theorem 4.2. *Let $n > 1$ be a square-free natural number, p a prime such that $p \nmid n$ and $d := \gcd\{m^2 - 1 : m \mid n\}$.*

(i) *If $p > 3$ and $p \nmid d$, then*

$$H_1(\Gamma_0(n, p), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{F}_p^\times.$$

(ii) *If $p = 3$ and $d = 3t$, where $3 \nmid t$ (e.g., when $2 \mid n$ or $5 \mid n$), then*

$$H_1(\Gamma_0(n, 3), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{F}_3^\times \oplus \mathbb{Z}/3.$$

(iii) *If $p = 2$ and $d = 8t$, where $2 \nmid t$ (e.g., when $3 \mid n$ or $5 \mid n$), then*

$$H_1(\Gamma_0(n, 2), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \oplus \mathbb{F}_2 \oplus \mathbb{Z}/4.$$

Proof. Let $A := \mathbb{Z}[1/n]$. The map $\Gamma_0(n,p) \to \mathbb{F}_p^\times$, given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \bar{a}$, is a surjective homomorphism of groups. We denote its kernel by $\Gamma_1(n,p)$. Thus, we have the group extension

$$1 \to \Gamma_1(n,p) \to \Gamma_0(n,p) \to \mathbb{F}_p^\times \to 1,$$

where

$$\Gamma_1(n,p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A) : p \mid a-1, d-1, c \right\}.$$

Note that $\Gamma_1(n,p) = \widetilde{\Gamma}(A,\mathfrak{p})$, where $\mathfrak{p} = \langle p \rangle$, and

$$\Gamma(n,p) \subseteq \Gamma_1(n,p) \subseteq \Gamma_0(n,p) \subseteq \mathrm{SL}_2(A).$$

By Theorem 1.1 (for $I_1 = A$ and $I_2 = \mathfrak{p}$), $\Gamma_1(n,p)$ is generated by the matrices $E_{12}(x)$ and $E_{21}(py)$, with $x, y \in A$. From the morphism of extensions

$$\begin{tikzcd}
1 \arrow[r] & \Gamma(n,p) \arrow[r] \arrow[d, hook] & \Gamma_0(n,p) \arrow[r] \arrow[d, equal] & \mathrm{B}(\mathbb{F}_p) \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & \Gamma_1(n,p) \arrow[r] & \Gamma_0(n,p) \arrow[r] & \mathbb{F}_p^\times \arrow[r] & 1,
\end{tikzcd}$$

we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_p)} \arrow[r] \arrow[d] & H_1(\Gamma_0(n,p),\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathbb{F}_p^\times \oplus (\mathbb{F}_p)_{\mathbb{F}_p^\times} \arrow[r] \arrow[d, "p_*"] & 1 \\
0 \arrow[r] & H_1(\Gamma_1(n,p),\mathbb{Z})_{\mathbb{F}_p^\times} \arrow[r] & H_1(\Gamma_0(n,p),\mathbb{Z}) \arrow[r] & \mathbb{F}_p^\times \arrow[r] & 1.
\end{tikzcd}$$

Since

$$H_1(\Gamma(n,p),\mathbb{Z})_{\mathrm{B}(\mathbb{F}_p)} \simeq \Gamma(n,p)/[\Gamma(n,p),\Gamma_0(n,p)]$$

and

$$H_1(\Gamma_1(n,p),\mathbb{Z})_{\mathbb{F}_p^\times} \simeq \Gamma_1(n,p)/[\Gamma_1(n,p),\Gamma_0(n,p)],$$

from the snake lemma applied to the above diagram, we obtain the exact sequence

$$1 \to \frac{\Gamma(n,p)}{[\Gamma(n,p),\Gamma_0(n,p)]} \to \frac{\Gamma_1(n,p)}{[\Gamma_1(n,p),\Gamma_0(n,p)]} \to (\mathbb{F}_p)_{\mathbb{F}_p^\times} \to 0.$$

Again, applying the snake lemma to the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \dfrac{\Gamma(n,p)}{[\Gamma(n,p),\Gamma_0(n,p)]} \arrow[r] \arrow[d] & \dfrac{\Gamma_1(n,p)}{[\Gamma_1(n,p),\Gamma_0(n,p)]} \arrow[r] \arrow[d, two heads] & (\mathbb{F}_p)_{\mathbb{F}_p^\times} \arrow[r] & 0 \\
& H_1(\mathrm{SL}_2(A),\mathbb{Z}) \arrow[r, equal] & H_1(\mathrm{SL}_2(A),\mathbb{Z}), & &
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to \mathcal{K} \to \mathcal{L} \to (\mathbb{F}_p)_{\mathbb{F}_p^\times},$$

where $\mathcal{K}$ and $\mathcal{L}$ are the kernels of the left and right vertical maps, respectively. By Theorem 4.1, $\mathcal{K}$ is a p-group. Thus, by the above exact sequence, $\mathcal{L}$ is also a p-group. The natural map

$$\tau : A \times A \to \frac{\Gamma_1(n,p)}{[\Gamma_1(n,p), \Gamma_0(n,p)]},$$

defined by

$$(a,0) \mapsto \overline{E_{12}(a)} \quad \text{and} \quad (0,a) \mapsto \overline{E_{21}(pa)},$$

is a surjective map. Since $d = \gcd\{m^2-1 : m \mid n\}$, there are $r_m \in A$ such that

$$d = \sum_{m|n} r_m(m^2-1).$$

Thus

$$\begin{aligned}\tau(-d,0) &= \overline{E_{12}(-d)} = \overline{E_{12}\Big(\sum_{m|n} r_m(1-m^2)\Big)} = \prod_{m|n} \overline{E_{12}(r_m(1-m^2))}\\ &= \prod_{m|n} \overline{[E_{12}(r_m), D(m)]} = 1,\end{aligned}$$

where

$$D(m) := \begin{pmatrix} m & 0 \\ 1 & 1/m \end{pmatrix} \in \Gamma_0(n,p).$$

Similarly,

$$\tau(0,-d) = \prod_{m|n} \overline{[D(m), E_{21}(pr_m)]} = 1.$$

Thus, we have the surjective map

$$\bar{\tau} : A/\langle d\rangle \times A/\langle d\rangle \to \frac{\Gamma_1(n,p)}{[\Gamma_1(n,p), \Gamma_0(n,p)]}.$$

Since $A/\langle d\rangle \simeq \mathbb{Z}/d$, we have that $|\Gamma_1(n,p)/[\Gamma_1(n,p), \Gamma_0(n,p)]|$ divides d^2. Hence

$$l \cdot |\mathcal{L}| \,\big|\, d^2, \tag{4.4}$$

where $l = |H_1(\mathrm{SL}_2(A), \mathbb{Z})|$.

(i) Let $p > 3$. Then, on the one hand, $\mathcal{L}$ is a p-group and, on the other hand, $|\mathcal{L}|$ divides d^2. Since $p \nmid d$, $\mathcal{L}$ must be trivial. It follows from this that

$$\frac{\Gamma_1(n,p)}{[\Gamma_1(n,p), \Gamma_0(n,p)]} \simeq H_1(\mathrm{SL}_2(A), \mathbb{Z}).$$

Now, from the commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \frac{\Gamma_1(n,p)}{[\Gamma_1(n,p), \Gamma_0(n,p)]} \arrow[r] \arrow[d, "\simeq"] & H_1(\Gamma_0(n,p), \mathbb{Z}) \arrow[r] \arrow[d, two heads] & \mathbb{F}_p^\times \arrow[r] & 1 \\ & H_1(\mathrm{SL}_2(A), \mathbb{Z}) \arrow[r, equal] & H_1(\mathrm{SL}_2(A), \mathbb{Z}), & & \end{tikzcd}$$

we see that the first row splits. This completes the proof of the first item.

(ii) Let $p=3$. Since $3 \nmid n$,

$$l=|H_1(\mathrm{SL}_2(A),\mathbb{Z})|=\begin{cases}3 & \text{if } 2\mid n\\ 12 & \text{if } 2\nmid n\end{cases}=2^r3, \quad r=0,2$$

(see Theorem 2.1). Thus, by (4.4), we get

$$2^r3|\mathcal{L}| \,\Big|\, d^2=3^2t^2.$$

Since $\mathcal{L}$ is a 3-group (Theorem 4.1) and $3\nmid t$, we have that $|\mathcal{L}|$ divides 3. Under the natural map

$$\frac{\Gamma_1(n,3)}{[\Gamma_1(n,3),\Gamma_0(n,3)]}\xrightarrow{i_{1*}} H_1(\mathrm{SL}_2(A),\mathbb{Z}),$$

$\overline{E_{21}(l)}$ maps to zero. In fact, in $H_1(\mathrm{SL}_2(A),\mathbb{Z})$, we have

$$\overline{E_{21}(l)}=\overline{wE_{21}(l)w^{-1}}=\overline{E_{12}(-l)}=1,$$

where $w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (see Theorem 2.1). But, under the map

$$\frac{\Gamma_1(n,3)}{[\Gamma_1(n,3),\Gamma_0(n,3)]}\xrightarrow{(i_{1*},i'_{2*})} H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus H_1(\mathrm{SL}_2(A),\mathbb{Z}),$$

we have

$$(i_{1*},i'_{2*})(\overline{E_{21}(l)}))=(\overline{E_{21}(l)},\overline{E_{21}(l/3)})=(1,\overline{E_{21}(l/3)})=(1,\overline{E_{12}(l/3)})\neq 1.$$

Thus $\overline{E_{21}(l)})$ is a nonzero element of $\Gamma_1(n,3)/[\Gamma_1(n,3),\Gamma_0(n,3)]$ that belongs to $\mathcal{L}$. Thus, $\mathcal{L}$ is nontrivial and hence $\mathcal{L}\simeq\mathbb{Z}/3$. Therefore, we have the exact sequence

$$0\to\mathbb{Z}/3\to\frac{\Gamma_1(n,3)}{[\Gamma_1(n,3),\Gamma_0(n,3)]}\to H_1(\mathrm{SL}_2(A),\mathbb{Z})\to 0.$$

Using Theorem 2.1, it is straightforward to check that the sequence

$$\begin{aligned}\frac{\Gamma_1(n,3)}{[\Gamma_1(n,3),\Gamma_0(n,3)]}&\xrightarrow{(i_{1*},i'_{2*})} H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus H_1(\mathrm{SL}_2(A),\mathbb{Z})\\ &\longrightarrow H_1(\mathrm{SL}_2(\mathbb{Z}[1/3n]),\mathbb{Z})\to 0\end{aligned}$$

is exact. Thus, the above exact sequence splits and hence

$$\frac{\Gamma_1(n,3)}{[\Gamma_1(n,3),\Gamma_0(n,3)]}\simeq H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus\mathbb{Z}/3.$$

Therefore,

$$H_1(\Gamma_0(n,3),\mathbb{Z})\simeq\mathbb{F}_3^\times\oplus H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus\mathbb{Z}/3.$$

This completes the proof of (ii).

(iii) Let $p=2$. Note that, in this case, $\Gamma_1(n,2)=\Gamma_0(n,2)$. Then

$$H_1(\Gamma_0(n,2),\mathbb{Z})\simeq\frac{\Gamma_1(n,2)}{[\Gamma_1(n,2),\Gamma_0(n,2)]}.$$

Since $2\nmid n$,

$$l=|H_1(\mathrm{SL}_2(A),\mathbb{Z})|=\begin{cases}4 & \text{if } 3\mid n\\ 12 & \text{if } 3\nmid n\end{cases}=2^2 3^r,\quad r=0,1$$

(see Theorem 2.1). Thus, by (4.4), we get

$$2^2 3^r|\mathcal{L}|\,\big|\,d^2=(8t)^2.$$

Since $\mathcal{L}$ is a 2-group and $2\nmid t$, $|\mathcal{L}|\,\big|\,16$. In $\Gamma_0(n,2)$ we have

$$-I_2=E_{21}(-2)E_{12}(1)E_{21}(-2)E_{12}(1).$$

Hence, in $H_1(\Gamma_0(n,2),\mathbb{Z})=\Gamma_0(n,2)/[\Gamma_0(n,2),\Gamma_0(n,2)]$, we have

$$1=\overline{I_2}=(\overline{-I_2})^2=\overline{E_{21}(-8)}\ \overline{E_{12}(4)}.$$

This implies that, under the map

$$\mathbb{Z}/8\times\mathbb{Z}/8\to\frac{\Gamma_0(n,2)}{[\Gamma_0(n,2),\Gamma_0(n,2)]},$$

the element $(4,-4)$ maps to zero. It follows from this that $|\mathcal{L}|\,\big|\,8$. Since we have the exact sequence

$$\frac{\Gamma_0(n,2)}{[\Gamma_0(n,2),\Gamma_0(n,2)]}\to H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus H_1(\mathrm{SL}_2(A),\mathbb{Z})\to H_1(\mathrm{SL}_2(\mathbb{Z}[1/2n]),\mathbb{Z})\to 0,$$

using Theorem 2.1, we get the following estimate on the order of $H_1(\Gamma_0(n,2),\mathbb{Z})$:

$$16\le|H_1(\Gamma_0(n,2),\mathbb{Z})|\le 32.$$

But the element $\overline{E_{21}(8)}=\overline{E_{12}(4)}$ of $\Gamma_0(n,2)/[\Gamma_0(n,2),\Gamma_0(n,2)]$ is of order 2 and, under the above map, goes to zero. This is a nontrivial element of $H_1(\Gamma_0(n,2),\mathbb{Z})$ and thus

$$|H_1(\Gamma_0(n,2),\mathbb{Z})|=32.$$

Now, as in case of (ii), we can show that

$$H_1(\Gamma_0(n,2),\mathbb{Z})\simeq H_1(\mathrm{SL}_2(A),\mathbb{Z})\oplus\mathbb{F}_2\oplus\mathbb{Z}/4.$$

This completes the proof of (iii) and the proof of the theorem. ∎

Remark 4.3. We believe that the theorem holds even after removing the conditions placed on d (the divisibility restrictions in each case). However, the current theorem, with those conditions, is sufficient to prove Theorem 6.6, which relates to the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$.

5. The second homology of $\Gamma_0(n, p)$

We will need the following lemma. Here, for a finite abelian group N, $N_{(p)}$ denotes the p-Sylow subgroup of N.

Lemma 5.1. *For any* $\mathrm{SL}_2(\mathbb{F}_p)$*-module* M *and any integer* $m \geq 1$, *we have the isomorphism*

$$H_m(\mathrm{B}(\mathbb{F}_p), M)_{(p)} \simeq H_m(\mathrm{SL}_2(\mathbb{F}_p), M)_{(p)}.$$

Proof. See Lemma 5.15 in [6]. ■

Theorem 5.2. *Let* $n > 1$ *be a square-free integer and* p *a prime such that* $p \nmid n$. *Then the natural maps*

$$i_{1*} : H_2(\Gamma_0(n, p), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}),$$
$$i'_{2*} : H_2(\Gamma_0(n, p), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}),$$

are surjective.

Proof. We first prove the claim for i_{1*}. The morphism of extensions (1.1) gives us the morphism of spectral sequences

$$\begin{array}{ccc} E'^2_{r,s} = H_r(\mathrm{B}(\mathbb{F}_p), H_s(\Gamma(n, p), \mathbb{Z})) & \Longrightarrow & H_{r+s}(\Gamma_0(n, p), \mathbb{Z}) \\ \downarrow & & \downarrow \\ E^2_{r,s} = H_r(\mathrm{SL}_2(\mathbb{F}_p), H_s(\Gamma(n, p), \mathbb{Z})) & \Longrightarrow & H_{r+s}(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}). \end{array}$$

Since

$$H_2(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z}) = 0 \quad \text{and} \quad H_2(\mathrm{B}(\mathbb{F}_p), \mathbb{Z}) = 0$$

(see Section 3), we have

$$E^2_{2,0} = 0 \quad \text{and} \quad E'^2_{2,0} = 0.$$

By an easy analysis of these spectral sequences, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} H_2(\Gamma(n, p), \mathbb{Z}) & \longrightarrow & H_2(\Gamma_0(n, p), \mathbb{Z}) & \longrightarrow & E'^\infty_{1,1} & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ H_2(\Gamma(n, p), \mathbb{Z}) & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) & \longrightarrow & E^\infty_{1,1} & \longrightarrow & 0. \end{array}$$

So, to prove the claim, we may show that the map $E'^\infty_{1,1} \to E^\infty_{1,1}$ is surjective. Again, from the above morphism of spectral sequences, we obtain the commutative diagram with exact rows

$$\begin{array}{ccc} H_1(\mathrm{B}(\mathbb{F}_p), H_1(\Gamma(n, p), \mathbb{Z})) = E'^2_{1,1} & \twoheadrightarrow & E'^\infty_{1,1} \\ \downarrow & & \downarrow \\ H_1(\mathrm{SL}_2(\mathbb{F}_p), H_1(\Gamma(n, p), \mathbb{Z})) = E^2_{1,1} & \twoheadrightarrow & E^\infty_{1,1}. \end{array}$$

So, to prove the surjectivity of the right vertical map, it is sufficient to prove the surjectivity of the left vertical map. Let $\mathcal{K}_p$ and $\mathcal{G}_p$ be the kernel and the image of the natural map

$$i_* : H_1(\Gamma(n,p),\mathbb{Z}) \to H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}),$$

respectively. So, we have the exact sequence

$$0 \to \mathcal{K}_p \to H_1(\Gamma(n,p),\mathbb{Z}) \to \mathcal{G}_p \to 0.$$

By Theorem 4.1, $\mathcal{K}_p$ is a p-group. By studying the diagrams (3.2), (3.3), (3.4), (3.5) and (3.6), we see that

$$\mathcal{G}_p \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}), \quad \text{for } p > 3,$$

and

$$\mathcal{G}_3 \simeq \begin{cases} 0 & \text{if } 2 \mid n, \\ \mathbb{Z}/4 & \text{if } 2 \nmid n, \end{cases} \quad \mathcal{G}_2 \simeq \begin{cases} \mathbb{Z}/2 & \text{if } 3 \mid n, \\ \mathbb{Z}/6 & \text{if } 3 \nmid n. \end{cases}$$

It follows from these that

$$H_1(\Gamma(n,p),\mathbb{Z}) \simeq \begin{cases} \mathcal{K}_p \oplus \mathcal{G}_p & \text{if } p > 3, \\ \mathcal{K}_3 & \text{if } p = 3, 2 \mid n, \\ \mathcal{K}_3 \oplus \mathbb{Z}/4 & \text{if } p = 3, 2 \nmid n, \\ \mathcal{K}_2' \oplus \mathbb{Z}/3 & \text{if } p = 2, 3 \nmid n, \\ \mathcal{K}_2' & \text{if } p = 2, 3 \mid n, \end{cases}$$

where $\mathcal{K}_2'$ is a 2-group. All these show that, for any prime p,

$$H_1(\Gamma(n,p),\mathbb{Z}) \simeq \mathcal{K}_p'' \oplus \mathcal{G}_p'',$$

where $\mathcal{K}_p''$ is a p-group and $\mathcal{G}_p''$ is a subgroup of $H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$ such that $p \nmid |\mathcal{G}_p''|$. Now we are ready to study the map

$$H_1(\mathrm{B}(\mathbb{F}_p), H_1(\Gamma(n,p),\mathbb{Z})) \to H_1(\mathrm{SL}_2(\mathbb{F}_p), H_1(\Gamma(n,p),\mathbb{Z})). \tag{5.1}$$

From this and the above isomorphism, we obtain the map

$$H_1(\mathrm{B}(\mathbb{F}_p),\mathcal{K}_p'') \oplus H_1(\mathrm{B}(\mathbb{F}_p),\mathcal{G}_p'') \to H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathcal{K}_p'') \oplus H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathcal{G}_p'').$$

The induced map

$$H_1(\mathrm{B}(\mathbb{F}_p),\mathcal{K}_p'') \to H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathcal{K}_p'')$$

is a map of p-groups (see Corollary 11.8.12 in [19]), so by Lemma 5.1 it is an isomorphism. Since $\mathrm{SL}_2(\mathbb{F}_p)$ acts trivially on the group $H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$, it acts trivially on $\mathcal{G}_p''$. Now, by the universal coefficient theorem, we have

$$H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathcal{G}_p'') \simeq H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{G}_p''.$$

By Proposition 2.2,

$$H_1(\mathrm{SL}_2(\mathbb{F}_p),\mathbb{Z}) \simeq \begin{cases} 0 & \text{if } p > 3, \\ \mathbb{Z}/3 & \text{if } p = 3, \\ \mathbb{Z}/2 & \text{if } p = 2. \end{cases}$$

This shows that the order of $H_1(\mathrm{SL}_2(\mathbb{F}_p), \mathbb{Z})$ and the order of $\mathscr{G}''_p$ are coprime. Hence,

$$H_1(\mathrm{SL}_2(\mathbb{F}_p), \mathscr{G}''_p) = 0.$$

All these imply that the map (5.1) is surjective. This completes the proof of the surjectivity of the map i_{1*}.

Now consider the map i'_{2*}. We recall that $i'_2 \colon \Gamma_0(n, p) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/n])$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix}.$$

Let $\Gamma'_0(n, p)$ be the image of i'_2. Thus

$$\Gamma'_0(n, p) := \mathrm{im}(i'_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/n]) : p \mid b \right\}.$$

Let i''_2 denotes the inclusion $\Gamma'_0(n, p) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[1/n])$. So, to prove the surjectivity of i'_{2*}, it is sufficient to prove the surjectivity of i''_{2*}.

Now the proof of the surjectivity of i''_{2*} follows a similar pass as the proof of the surjectivity of i_{1*}. Here, we have to study the Lyndon/Hochschild–Serre spectral sequence associated to the morphism of extensions

$$\begin{tikzcd} 1 \arrow[r] & \Gamma(n, p) \arrow[r] \arrow[d, equal] & \Gamma'_0(n, p) \arrow[r, "\pi'"] \arrow[d, "i''_2"] & \mathrm{B}'(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \Gamma(n, p) \arrow[r] & \mathrm{SL}_2(\mathbb{Z}[1/n]) \arrow[r] & \mathrm{SL}_2(\mathbb{F}_p) \arrow[r] & 1, \end{tikzcd}$$

where

$$\mathrm{B}'(\mathbb{F}_p) := \left\{ \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p) : x \in \mathbb{F}_p^\times, y \in \mathbb{F}_p \right\}$$

and

$$\pi'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & \bar{a}^{-1} \end{pmatrix}.$$

∎

6. The second homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$

Proposition 6.1. *Let $n > 1$ be a square-free integer, p a prime such that $p \nmid n$, and $d := \gcd\{m^2 - 1 : m \mid n\}$.*

(i) *If $p > 3$ and $p \nmid d$, then we have the exact sequence*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to \mathbb{F}_p^\times \to 1.$$

(ii) *If $p = 3$ and $d = 3t$, where $3 \nmid t$ (e.g., when $2 \mid n$ or $5 \mid n$), then we have the exact sequence*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/3n]), \mathbb{Z}) \to \mathbb{F}_3^\times \to 1.$$

(iii) *If $p = 2$ and $d = 8t$, where $2 \nmid t$ (e.g., when $3 \mid n$ or $5 \mid n$), then we have the exact sequence*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/2n]), \mathbb{Z}) \to \mathbb{F}_2 \to 0.$$

Proof. These statements follow from the Mayer–Vietoris exact sequence (see Section 3), Theorem 4.2 and Theorem 5.2. ∎

Let $n > 1$ be an integer and let p be a prime such that $p \nmid n$. Let δ_p be the composition

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to H_1(\Gamma_0(n, p), \mathbb{Z}) \twoheadrightarrow H_1(\mathrm{B}(\mathbb{F}_p), \mathbb{Z}) \twoheadrightarrow \mathbb{F}_p^\times.$$

Lemma 6.2. *If $n > 1$ is a square-free integer and $p > 3$ is a prime such that $p \nmid n$, then the map*

$$\delta_p : H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to \mathbb{F}_p^\times$$

is surjective.

Proof. By Theorem 2.1,

$$H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq H_1(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}).$$

Thus, from the Mayer–Vietoris exact sequence, we get the exact sequence

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to H_1(\Gamma_0(n, p), \mathbb{Z}) \xrightarrow{i_{1*}} H_1(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to 0.$$

By applying the snake lemma to the diagram (3.2), we obtain the exact sequence

$$1 \to \frac{[\Gamma(n, p), \mathrm{SL}_2(\mathbb{Z}[1/n])]}{[\Gamma(n, p), \Gamma_0(n, p)]} \to \ker(i_{1*}) \to \mathbb{F}_p^\times \to 1.$$

Now, the composition

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \twoheadrightarrow \ker(i_{1*}) \twoheadrightarrow \mathbb{F}_p^\times$$

is surjective and coincides with the above composition. ∎

Remark 6.3. We believe that the above lemma is correct for $p = 3$ and $p = 2$ (with $\mathbb{F}_2^\times$ replaced by $\mathbb{F}_2$ in the case $p = 2$). Since this result was not required, we did not attempt a proof.

If $p_1, \ldots, p_k$, $p_i > 2$, are distinct primes such that $p_i \nmid n$, then the maps δ_{p_i} induce the natural map

$$\delta_{p_1,\ldots,p_k} := (\delta_{p_i})_{i=1}^k : H_2(\mathrm{SL}_2(\mathbb{Z}[1/(p_1 \cdots p_k n)]), \mathbb{Z}) \to \bigoplus_{i=1}^k \mathbb{F}_{p_i}^\times.$$

Let F be a field and let $K_2(F)$ be the second K-group of F. It is known that

$$K_2(F) \simeq H_2(\mathrm{SL}(F), \mathbb{Z}).$$

By a theorem of Matsumoto (see Theorem 11.1 in [9]), we have an isomorphism

$$K_2(F) \simeq (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle a \otimes (1 - a) : a \in F \backslash \{0, 1\}\rangle.$$

We denote the image of $\overline{a \otimes b}$, in $K_2(F)$, by $\{a, b\}$.

Let A be a Euclidean domain with quotient field F. If $\mathfrak{p} = \langle \pi \rangle \in A$ is a nonzero prime ideal of A, then $A_\mathfrak{p}$ is a discrete valuation ring. Let $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the residue field of $A_\mathfrak{p}$. Since A is a Euclidean domain, $k(\mathfrak{p}) \simeq A/\mathfrak{p}$. The ring $A_\mathfrak{p}$ induces a discrete valuation

$$v_\mathfrak{p} : F^\times \to \mathbb{Z}$$

on F. This valuation defines the following *tame symbol* on $K_2(F)$:

$$\tau_\mathfrak{p} : K_2(F) \to k(\mathfrak{p})^\times, \quad \{a,b\} \mapsto (-1)^{v_p(a)v_p(b)} \overline{\left(\frac{b^{v_p(a)}}{a^{v_p(b)}}\right)}$$

(see p. 98 of [9] or Lemma 6.3 in [21]). The next result is proved by Hutchinson.

Proposition 6.4 (Hutchinson). *Let $n > 1$ be an integer and $p > 2$ a prime such that $p \nmid n$. Then the map δ_p coincides with the composition*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/pn]), \mathbb{Z}) \to H_2(\mathrm{SL}(\mathbb{Z}[1/pn]), \mathbb{Z}) \to H_2(\mathrm{SL}(\mathbb{Q}), \mathbb{Z}) \simeq K_2(\mathbb{Q}) \xrightarrow{\tau_p} \mathbb{F}_p^\times,$$

where τ_p is the tame symbol on $K_2(\mathbb{Q})$ induced by the prime ideal $\langle p \rangle$ of $\mathbb{Z}$.

Proof. See Proposition 5.6 in [6]. ∎

For a prime p, let

$$r_p := \operatorname{rank} H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathbb{Z}).$$

By Theorem 2.3 (of Adem–Naffah), we have

$$r_p := \begin{cases} 1 & \text{if } p = 2, 3, \\ (p-7)/6 & \text{if } p \equiv 1 \pmod{12}, \\ (p+1)/6 & \text{if } p \equiv 5 \pmod{12}, \\ (p-1)/6 & \text{if } p \equiv 7 \pmod{12}, \\ (p+7)/6 & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Observe that r_p is always odd. Clearly, this is true for $p = 2, 3$. If $p \equiv 1 \pmod{12}$, then $p - 7 \equiv -6 \pmod{12}$. From this, we have $p - 7 = 12k - 6$ and thus

$$r_p = (p-7)/6 = 2k - 1$$

is odd. A parallel argument proves the results for the remaining cases.

Example 6.5. The primes 2, 3, 5, 7 and 13 are the only primes for which the value of r_p is 1. Beyond these, for any odd integer $l > 1$, there can be no more than four primes with r_p equal to l.

The following theorem (Theorem B from the introduction) will now be demonstrated.

Theorem 6.6. *Let $n = p_1 \cdots p_l$, $l > 1$, where the p_i's are distinct primes such that $r_{p_1} \leq \cdots \leq r_{p_l}$. When $r_{p_i} = r_{p_{i+1}}$, for some i, we assume that $p_i < p_{i+1}$. Then we have the exact sequence*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/p_1]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \bigoplus_{i=2}^{l} \mathbb{F}_{p_i}^\times \to 1.$$

Proof. The proof is by induction on l. If $l = 2$, then the claim follows from Proposition 6.1. So, let $l \geq 3$. Thus, clearly

$$p_l \nmid \gcd\{m^2 - 1 : m \mid p_1 p_2 \cdots p_{l-1}\}$$

and so, by Proposition 6.1, we have the exact sequence

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/(p_1 \cdots p_{l-1})]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \mathbb{F}_{p_l}^\times \to 1.$$

Now, the claim follows by induction and applying the snake lemma to the commutative diagram with exact rows

$$\begin{tikzcd} & H_2(\mathrm{SL}_2(\mathbb{Z}[1/(p_1 \cdots p_{l-1})]), \mathbb{Z}) \arrow[r] \arrow[d, "\delta_{p_2,\dots,p_{l-1}}"] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \arrow[r] \arrow[d, "\delta_{p_2,\dots,p_l}"] & \mathbb{F}_{p_l}^\times \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \bigoplus_{i=2}^{l-1} \mathbb{F}_{p_i}^\times \arrow[r] & \bigoplus_{i=2}^{l} \mathbb{F}_{p_i}^\times \arrow[r] & \mathbb{F}_{p_l}^\times \arrow[r] & 1. \end{tikzcd}$$

∎

Proposition 6.7. *Let m be a square-free integer that is divisible by at least one of the primes 2, 3, 5, 7 or 13. If m divides a square-free integer n, then the natural map*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})$$

is injective. Moreover, the left homology group is a direct summand of the right homology group if m is equal to one of the primes 2, 3 or 7.

Proof. Firstly, we assume that m is a prime. Thus, m must be one of the primes 2, 3, 5, 7 or 13. Let us denote m by q. In this case, we start our argument by showing that the natural map

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z})$$

is injective. If $q = 2, 3$, then $\mathbb{Z}[1/6q] = \mathbb{Z}[1/6]$. By Proposition 6.1, we have the exact sequences

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \to \mathbb{F}_3^\times \to 1,$$
$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \to \mathbb{F}_2 \to 0.$$

By the calculation of Bui–Ellis in Table 1 of [17], we know that

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2.$$

Since

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad H_2(\mathrm{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \simeq \mathbb{Z}$$

(see Theorem 2.3), the left maps in the above exact sequences are injective. In fact, we have

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \simeq H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \oplus \mathbb{F}_3^\times,$$
$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/6]), \mathbb{Z}) \simeq H_2(\mathrm{SL}_2(\mathbb{Z}[1/3]), \mathbb{Z}) \oplus \mathbb{F}_2.$$

This proves the claim for $q = 2, 3$. So let $q \in \{5, 7, 13\}$. Observe that for these primes, by Theorem 2.3, we have

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/((q-1)/2).$$

Moreover, by Theorem 2.4 (of Hutchinson),

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_q^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/(q-1).$$

More precisely, we have the split exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \to \mathbb{F}_3^\times \oplus \mathbb{F}_q^\times \to 1.$$

On the other hand, by Proposition 6.1, we have the exact sequences

$$\begin{aligned} H_2(\mathrm{SL}_2(\mathbb{Z}[1/3q]), \mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \to \mathbb{F}_2 \to 0, \\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/3q]), \mathbb{Z}) \to \mathbb{F}_3^\times \to 0. \end{aligned}$$

Combining these two (as in Theorem 6.6), we obtain the exact sequence

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \to \mathbb{F}_2 \oplus \mathbb{F}_3^\times \to 0.$$

Now, using the above exact sequence of Hutchinson, the sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \to \mathbb{F}_2 \oplus \mathbb{F}_3^\times \to 0$$

is exact and it splits only if $q = 7$. Finally, the general claim (for the case $m = q$ a prime) follows from the commutative diagram

$$\begin{tikzcd} H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \arrow[r] \arrow[d, hook] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \arrow[d] \\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z}) \arrow[r, hook] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/6n]), \mathbb{Z}) \end{tikzcd}$$

and the fact that $H_2(\mathrm{SL}_2(\mathbb{Z}[1/6q]), \mathbb{Z})$ is a direct summand of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/6n]), \mathbb{Z})$ (this follows from Theorem 2.4).

Now, let q be the smallest prime among $2, 3, 5, 7, 13$ that divides m. Let $m = qp_1 \cdots p_h$ and $n = mp_{h+1} \cdots p_l = qp_1 \cdots p_l$. By Theorem 6.6 and what we have just proved, we get the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \arrow[r] \arrow[d, equal] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Z}) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{h} \mathbb{F}_{p_i}^\times \arrow[r] \arrow[d, hook] & 1 \\ 0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/q]), \mathbb{Z}) \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \arrow[r] & \bigoplus_{i=1}^{l} \mathbb{F}_{p_i}^\times \arrow[r] & 1. \end{tikzcd}$$

The general injectivity claim of the theorem follows from the snake lemma applied to the above diagram. ∎

The next theorem, Theorem A from the introduction, represents a significant generalization of Hutchinson's Theorem 2.4.

Theorem 6.8. *Let n be a square-free positive integer and let* $\mathrm{scpd}(n, 2730)$ *be the smallest common prime divisor of n and $2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.*

(i) *If* $\mathrm{scpd}(n, 2730) = 2$, *i.e., if n is even, then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \bigoplus_{p \mid (n/2)} \mathbb{F}_p^\times.$$

(ii) *If* $\mathrm{scpd}(n, 2730) = 3$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \bigoplus_{p \mid (n/3)} \mathbb{F}_p^\times.$$

(iii) *If* $\mathrm{scpd}(n, 2730) = 5$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \begin{cases} \mathbb{Z}/2 \oplus \bigoplus_{p \mid (n/5)} \mathbb{F}_p^\times & \text{if } p \equiv 1 \pmod 4 \text{ for all } p \mid (n/5), \\ \mathbb{Z}/4 \oplus \mathbb{Z}/((q-1)/2) \oplus \bigoplus_{p \mid (n/5q)} \mathbb{F}_p^\times & \text{if } q \equiv 3 \pmod 4 \text{ for some } q \mid (n/5). \end{cases}$$

(iv) *If* $\mathrm{scpd}(n, 2730) = 7$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \bigoplus_{p \mid (n/7)} \mathbb{F}_p^\times.$$

(v) *If* $\mathrm{scpd}(n, 2730) = 13$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \begin{cases} \mathbb{Z}/6 \oplus \bigoplus_{p \mid (n/13)} \mathbb{F}_p^\times & \text{if } p \equiv 1 \pmod 4 \text{ for all } p \mid (n/13), \\ \mathbb{Z}/12 \oplus \mathbb{Z}/((q-1)/2) \oplus \bigoplus_{p \mid (n/13q)} \mathbb{F}_p^\times & \text{if } q \equiv 3 \pmod 4 \text{ for some } q \mid (n/13). \end{cases}$$

Proof. Let $n = p_1 \cdots p_l$ be the prime decomposition of n such that $r_{p_1} \leq \cdots \leq r_{p_l}$. Moreover, if $r_{p_i} = r_{p_{i+1}}$, for some i, we assume that $p_i < p_{i+1}$.

(i) Since n is even, we put $p_1 = 2$. By Theorem 6.6 and Proposition 6.7, we get the exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \to \bigoplus_{i=2}^{l} \mathbb{F}_{p_i}^\times \to 1. \tag{6.1}$$

Again, by Proposition 6.7, the exact sequence (6.1) splits. This, together with the isomorphism $H_2(\mathrm{SL}_2(\mathbb{Z}[1/2]), \mathbb{Z}) \simeq \mathbb{Z}$ (Theorem 2.3), imply the first item.

(ii) Since $\mathrm{scpd}(n, 2730) = 3$, we have $p_1 = 3$. Now, a similar argument as in (i) proves the second item.

(iv) If $\mathrm{scpd}(n, 2730) = 7$, then we have $p_1 = 7$. Again, by a similar argument as in (i), we can prove the fourth item.

(iii) Now, let $\mathrm{scpd}(n, 2730) = 5$. Then $p_1 = 5$. If p is an odd prime which divides n, let

$$p - 1 = 2^{s_p} m_p,$$

where $s_p \geq 1$ and m_p is odd. Note that $p \equiv 1 \pmod 4$ if and only if $s_p > 1$ and $p \equiv 3 \pmod 4$ if and only if $s_p = 1$.

First, let us assume that $l = 2$, i.e., $n = 5p_2$. By Theorem 6.6 and Proposition 6.7, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]), \mathbb{Z}) & \xrightarrow{i_*} & H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z}) & \longrightarrow & \mathbb{F}_{p_2}^\times & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/30]), \mathbb{Z}) & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/30p_2, \mathbb{Z}) & \longrightarrow & \mathbb{F}_{p_2}^\times & \longrightarrow & 1. \end{array} \tag{6.2}$$

By Theorem 2.4, the lower exact sequence splits. Note that, by Theorem 2.3,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2.$$

It follows from this that the free part of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z})$ splits naturally. Observe that in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]), \mathbb{Z}) & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/30]), \mathbb{Z}) \end{array}$$

the upper map is given by $(a, \bar{b}) \mapsto (a, 0, 2\bar{b})$.

Let $s_{p_2} > 1$. Note that

$$\mathbb{F}_{p_2}^\times \simeq \mathbb{Z}/(p_2 - 1) \simeq \mathbb{Z}/2^{s_{p_2}} \oplus \mathbb{Z}/m_{p_2}.$$

If the upper exact sequence of the diagram (6.2) does not split, then

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2^{s_{p_2}+1} \oplus \mathbb{Z}/m_{p_2}$$

(see Calculation 3.3.2 and Theorem 3.4.3 in [20]). This means that $H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z})$ has a copy of $\mathbb{Z}/2^{s_{p_2}+1}$ as direct summand. Moreover, $H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z})$ injects into $H_2(\mathrm{SL}_2(\mathbb{Z}[1/30p_2]), \mathbb{Z})$. But this later group does not has any element of order $s_{p_2} + 1$. This is a contradiction and therefore the upper row of (6.2) must split. Therefore,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{F}_{p_2}^\times.$$

Now, let $s_{p_2} = 1$. If the upper exact sequence of the diagram (6.2) splits, then

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{F}_{p_2}^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/m_{p_2}.$$

But this contradicts the surjectivity of the map

$$\delta_5 : H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]),\mathbb{Z}) \twoheadrightarrow \mathbb{F}_5^\times \simeq \mathbb{Z}/4$$

of Lemma 6.2. Observe that δ_5 maps the free part of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]),\mathbb{Z})$ to zero. Therefore the first row of the diagram (6.2) does not split and hence,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/5p_2]),\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/m_{p_2} \simeq \mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/((p_2-1)/2).$$

This proves the claim of item (iii) for $l=2$.

Now, let $l>2$ and consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]),\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_{l-1})]),\mathbb{Z}) \arrow[r] \arrow[d] & \bigoplus_{i=2}^{l-1}\mathbb{F}_{p_i}^\times \arrow[r] \arrow[d, hook] & 1 \\
0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]),\mathbb{Z}) \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \arrow[r] & \bigoplus_{i=2}^{l}\mathbb{F}_{p_i}^\times \arrow[r] & 1.
\end{tikzcd}$$

If $s_{p_j}>1$, for all $2\le j\le l$, then, by induction, the first row splits. Let η be the composition

$$\bigoplus_{i=2}^{l-1}\mathbb{F}_{p_i}^\times \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_{l-1})]),\mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$$

and consider the exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/5]),\mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_l)]),\mathbb{Z})/\mathrm{im}(\eta) \to \mathbb{F}_{p_l}^\times \to 1.$$

Then, as in case $l=2$, one can show that the above exact sequence splits. Therefore

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}/2\oplus\bigoplus_{i=2}^{l-1}\mathbb{F}_{p_i}^\times.$$

Now, let $s_{p_j}=1$, for some j. We may assume that $s_{p_2}=1$. Again, by induction on the first row of the above diagram, we have the decomposition

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_{l-1})]),\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/((p_2-1)/2)\oplus\bigoplus_{i=3}^{l-1}\mathbb{F}_{p_i}^\times.$$

By Theorem 6.6 and Proposition 6.7, we have the exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_{l-1})]),\mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \to \mathbb{F}_{p_l}^\times \to 1.$$

Let θ be the composition

$$\bigoplus_{i=3}^{l-1}\mathbb{F}_{p_i}^\times \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/(5p_2\cdots p_{l-1})]),\mathbb{Z}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}).$$

From this and the above exact sequence, we obtain the exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/((p_2-1)/2) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})/\mathrm{im}(\theta) \to \mathbb{F}_{p_l}^{\times} \to 1.$$

Note that $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z})/\mathrm{im}(\theta)$ injects into

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/(30p_2 \cdots p_l)]), \mathbb{Z})/\mathrm{im}(\theta) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{F}_{p_2}^{\times} \oplus \mathbb{F}_{p_l}^{\times}.$$

But this is only possible if the above exact sequence splits, i.e.,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/((p_2-1)/2) \oplus \bigoplus_{i=3}^{l} \mathbb{F}_{p_i}^{\times}.$$

This completes the proof of the item (iii) of the theorem.

(v) Let $\mathrm{scpd}(n, 2730) = 13$. The proof of this part is similar to the proof of item (iii), but is more involved. Take $p_1 = 13$. First, let us assume that $l = 2$, i.e., $n = 13p_2$. By Theorem 6.6 and Proposition 6.7, we get the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/13]), \mathbb{Z}) \arrow[r, "i_*"] \arrow[d, hook] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{F}_{p_2}^{\times} \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/78]), \mathbb{Z}) \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/78p_2, \mathbb{Z}) \arrow[r] & \mathbb{F}_{p_2}^{\times} \arrow[r] & 1. \end{tikzcd} \tag{6.3}$$

Note that $78 = 6 \cdot 13$. By Theorem 2.4, the lower exact sequence splits and by Theorem 2.3, we have

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/13]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/6 \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3.$$

Similar to the case $p_1 = 5$, the free part of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z})$ splits naturally and in the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z}/6 \arrow[r] \arrow[d, "\simeq"] & \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/12 \arrow[d, "\simeq"] \\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/13]), \mathbb{Z}) \arrow[r] & H_2(\mathrm{SL}_2(\mathbb{Z}[1/78]), \mathbb{Z}) \end{tikzcd}$$

the upper map is given by $(a, \bar{b}) \mapsto (a, 0, 2\bar{b})$. Let

$$p_2 - 1 = 2^{s_{p_2}} m_{p_2} = 2^{s_{p_2}} 3^t m'_{p_2},$$

where $m_{p_2} = 3^t m'_{p_2}$, $t \geq 0$ and $3 \nmid m'_{p_2}$. Then

$$\mathbb{F}_{p_2}^{\times} \simeq \mathbb{Z}/(p_2 - 1) \simeq \mathbb{Z}/2^{s_{p_2}} \oplus \mathbb{Z}/3^t \oplus \mathbb{Z}/m'_{p_2}.$$

Let $s_{p_2} > 1$. If the upper exact sequence of the diagram (6.3) does not split, then $H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z})$ has either a copy of $\mathbb{Z}/2^{s_{p_2}+1}$ or a copy of $\mathbb{Z}/3^{t+1}$, in case $t \geq 1$, as direct summand (see Theorem 3.4.3, Calculation 3.3.2, and Exercise 3.4.1 in [20]). But

$H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z})$ injects into $H_2(\mathrm{SL}_2(\mathbb{Z}[1/78p_2]), \mathbb{Z})$ and this latter group does not have any element of order $s_{p_2} + 1$ or $t + 1$, in case $t \geq 1$, since

$$\begin{aligned} H_2(\mathrm{SL}_2(\mathbb{Z}[1/78p_2]), \mathbb{Z}) &\simeq \mathbb{Z} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_{13}^\times \oplus \mathbb{F}_{p_2}^\times \\ &\simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/2^{s_{p_2}} \oplus \mathbb{Z}/3^t \oplus \mathbb{Z}/m'_{p_2}. \end{aligned}$$

This is a contradiction and therefore the upper row of (6.3) must split. Therefore,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{F}_{p_2}^\times.$$

Now, let $s_{p_2} = 1$. If the upper exact sequence of the diagram (6.3) splits, then

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{F}_{p_2}^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/m_{p_2}.$$

But this contradicts the surjectivity of the map

$$\delta_{13} : H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z}) \twoheadrightarrow \mathbb{F}_{13}^\times \simeq \mathbb{Z}/4 \oplus \mathbb{Z}/3$$

of Lemma 6.2. Therefore the first row of the diagram (6.3) does not split. If $t \geq 1$, then, as above, we can show that $H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z})$ cannot have a copy of $\mathbb{Z}/3^{t+1}$. Thus, the only remaining case is the isomorphism

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/13p_2]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/m_{p_2} \simeq \mathbb{Z} \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/((p_2 - 1)/2).$$

This proves the claim of item (v) for $l = 2$. The case $l > 2$ can be done as in the proof of (iii). This completes the proof of the theorem. ∎

Remark 6.9. By Theorem 6.8, we have

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/46]), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/22.$$

However, Table 1 in [17] states the isomorphism $H_2(\mathrm{SL}_2(\mathbb{Z}[1/46]), \mathbb{Z}) \simeq \mathbb{Z}/22$. This is a notational error in [17]; the correct isomorphism is confirmed in Table 3.1 on p. 27 of [4].

7. On the rank of the second homology of $\mathrm{SL}_2(\mathbb{Z}[1/n])$

For any $n > 1$, let

$$r_n := \operatorname{rank} H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Z}).$$

We have already seen the value of r_p for any prime p.

Proposition 7.1. *Let $n > 1$ be a square-free integer. If one of the primes 2, 3, 5, 7 or 13 divides n, then $r_n = 1$.*

Proof. This follows from Theorem 6.6, Proposition 6.7 and the fact that, for a prime p, $r_p = 1$ if and only if $p = 2, 3, 5, 7$ or 13. But this also follows from the next proposition, which can be proved much easier. ∎

Proposition 7.2. *For any square-free integer $n > 1$,*

$$1 \leq r_n \leq \min\{r_p : p \text{ prime},\ p \mid n\}.$$

Proof. It follows from Theorem 6.6 that

$$r_n \leq \min\{r_p : p \text{ prime},\ p \mid n\}.$$

But here we give a much easier proof of this fact. We may assume that $n = p_1 \cdots p_l$, where the p_i's are distinct primes with $r_{p_1} \leq r_{p_2} \leq \cdots \leq r_{p_l}$. Let $m = p_1 \cdots p_{l-1}$. From the diagram (1.1), we obtain the morphism of Lyndon/Hochschild–Serre spectral sequences

$$\begin{array}{ccc} E^2_{r,s} = H_r(\mathrm{B}(\mathbb{F}_p), H_s(\Gamma(m, p_l), \mathbb{Q})) & \Longrightarrow & H_{r+s}(\Gamma_0(m, p_l), \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathcal{E}^2_{r,s} = H_r(\mathrm{SL}_2(\mathbb{F}_p), H_s(\Gamma(m, p_l), \mathbb{Q})) & \Longrightarrow & H_{r+s}(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Q}). \end{array}$$

Since $E^2_{r,s} = 0$ and $\mathcal{E}^2_{r,s} = 0$, for $r > 0$ (see Corollary 10.2 in Section 10, Chapter III, of [3]), and

$$E^2_{0,s} \to \mathcal{E}^2_{0,s}$$

is surjective, we see that, for any $k \geq 0$, the map

$$H_k(\Gamma_0(m, p_l), \mathbb{Q}) \to H_k(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Q}) \tag{7.1}$$

is surjective. Since $H_1(\Gamma_0(m, p_l), \mathbb{Z})$ is a finite group,

$$H_1(\Gamma_0(m, p_l), \mathbb{Q}) = 0.$$

Now, by the Mayer–Vietoris exact sequence

$$\begin{aligned} H_2(\Gamma_0(m, p_l), \mathbb{Q}) &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Q}) \oplus H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Q}) \\ &\to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Q}) \to 0 \end{aligned}$$

(see Section 3) and the surjective map (7.1), we see that

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/m]), \mathbb{Q}) \to H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Q}) \tag{7.2}$$

is surjective. Thus, by induction on l, we have

$$r_n \leq r_{p_1} = \min\{r_p : p \text{ prime},\ p \mid n\}.$$

Now, let $p \mid n$ and consider the commutative diagram

$$\begin{array}{ccc} H_2(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathbb{Q}) & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]), \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_2(\mathrm{SL}_2(\mathbb{Z}[1/6p]), \mathbb{Q}) & \longrightarrow & H_2(\mathrm{SL}_2(\mathbb{Z}[1/6n]), \mathbb{Q}). \end{array}$$

By Theorem 2.4, $r_{6n} = 1$. Thus, the lower horizontal map in the above diagram is bijective, i.e.,

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/6p]), \mathbb{Q}) \simeq \mathbb{Q} \simeq H_2(\mathrm{SL}_2(\mathbb{Z}[1/6n]), \mathbb{Q}).$$

Moreover, by (7.2), the vertical maps in the above diagram are surjective. It follows from these that the upper horizontal map is not trivial. Therefore, $r_n \geq 1$. ∎

We strongly suspect that the value of r_n in the above theorem is equal to its upper bound. At this time, we lack a definitive proof (or disproof) of this claim. However, building upon the results of this paper, we propose the following conjecture over the group structure of $H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})$ when n is an integer not divisible by any of the primes 2, 3, 5, 7 or 13.

Conjecture 7.3. *Let $n = p_1 \cdots p_l$, $l > 1$, be the prime decomposition of n such that $1 < r_{p_1} \le \cdots \le r_{p_l}$. If $r_{p_j} = r_{p_{j+1}}$, for some j, we assume that $p_j < p_{j+1}$.*

(i) *If* $p_1 \equiv 11 \pmod{12}$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}^{r_{p_1}} \oplus \bigoplus_{j=2}^{l} \mathbb{F}_{p_j}^{\times}.$$

(ii) *If* $p_1 \equiv 5 \pmod{12}$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}^{r_{p_1}} \oplus \begin{cases} \mathbb{Z}/2 \oplus \bigoplus_{j=2}^{l} \mathbb{F}_{p_j}^{\times} & \text{if } p_j \equiv 1 \pmod 4 \text{ for all } 2 \le j \le l, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/((q_i-1)/2) \oplus \bigoplus_{\substack{j=2\\ j\neq i}}^{l} \mathbb{F}_{p_j}^{\times} & \text{if } p_i \equiv 3 \pmod 4 \text{ for some } 2 \le i \le l. \end{cases}$$

(iii) *If* $p_1 \equiv 7 \pmod{12}$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z}) \simeq \mathbb{Z}^{r_{p_1}} \oplus \mathbb{Z}/3 \oplus \bigoplus_{j=2}^{l} \mathbb{F}_{p_j}^{\times}.$$

(iv) *If* $p_1 \equiv 1 \pmod{12}$, *then*

$$H_2(\mathrm{SL}_2(\mathbb{Z}[1/n]),\mathbb{Z})` \simeq \mathbb{Z}^{r_{p_1}} \oplus \begin{cases} \mathbb{Z}/6 \oplus \bigoplus_{j=2}^{l} \mathbb{F}_{p_j}^{\times} & \text{if } p_j \equiv 1 \pmod 4 \text{ for all } 2 \le j \le l, \\ \mathbb{Z}/12 \oplus \mathbb{Z}/((q_i-1)/2) \oplus \bigoplus_{\substack{j=2\\ j\neq i}}^{l} \mathbb{F}_{p_j}^{\times} & \text{if } p_i \equiv 3 \pmod 4 \text{ for some } 2 \le i \le l. \end{cases}$$

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