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Irreducible representations of untwisted affine Kac–Moody algebras

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Abstract. In this paper, we construct a class of new irreducible modules over untwisted affine Kac–Moody algebras $\tilde{\mathfrak{g}}$, generalizing and including both highest weight modules and Whittaker modules. These modules allow us to obtain a complete classification of irreducible $\tilde{\mathfrak{g}}$-modules on which the action of each root vector in $\tilde{\mathfrak{n}}_+$ is locally finite, where $\tilde{\mathfrak{n}}_+$ is the locally nilpotent subalgebra (or positive part) of $\tilde{\mathfrak{g}}$. The necessary and sufficient conditions for two such irreducible $\tilde{\mathfrak{g}}$-modules to be isomorphic are also determined. Based on the first part of the paper, we develop the so-called "shifting technique", which enables us to obtain a necessary and sufficient condition for the tensor product of irreducible integrable loop $\tilde{\mathfrak{g}}$-modules and irreducible integrable highest weight $\tilde{\mathfrak{g}}$-modules to be simple. This tensor product problem was first studied and partly solved by Chari and Pressley in 1987, and remained open since then. Our result also recovers the result for $\widehat{\mathfrak{sl}}_2$ by Adamović (1997).

1. Introduction

Affine Lie algebras are the most extensively studied and the most useful among infinite dimensional Kac–Moody algebras. Their representation theory is as rich as that of finite dimensional simple Lie algebras, but quite different from it. A key difference is that affine Lie algebras possess irreducible weight modules with nonzero weight spaces of both finite and infinite dimension, which cannot occur for finite dimensional simple Lie algebras. In the first part of the present paper, we will construct irreducible modules over untwisted affine Kac–Moody algebras that do not have counterpart for finite dimensional simple Lie algebras.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. We use $\tilde{\mathfrak{g}}$ to denote the corresponding untwisted affine Kac–Moody algebra, and let $\widehat{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$.

The integrable highest weight modules were the first class of representations over affine Kac–Moody algebras that have been extensively studied, see [7,31,37] for detailed results and further bibliography. In [8], Chari classified all irreducible integrable weight modules with finite dimensional weight spaces over the untwisted affine Lie algebras.

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V. Chari and A. Pressley [11] then extended this classification to all affine Lie algebras. The results of [8] and [11] state that an irreducible integrable weight module with finite dimensional weight spaces is either a highest weight module or a loop module; in particular, the study of loop modules was initiated. Some more general results on irreducible weight modules over untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ were obtained in [35].

In the 1990's, V. Futorny began a comprehensive program studying weight modules over arbitrary affine Lie algebras by taking non-standard partitions of the root system; that is, partitions which are not equivalent under the Weyl group to the standard partition into positive and negative roots (see [16, 20]). For affine Lie algebras, there are only finitely many equivalence classes of such non-standard partitions (see [22]). Corresponding to each partition is a Borel subalgebra, and one can form representations induced from one-dimensional modules for these Borel subalgebras. These modules, often referred to as Verma-type modules or imaginary Verma modules, were first studied by Jakobsen and Kac [30], and then by Futorny [21, 22]. Results on the structure of Verma-type modules can also be found in [20], and for a nice exposition of Futorny's program, see [23].

A classification of irreducible nonzero level modules with finite dimensional weight spaces for affine Lie algebras was obtained in [26], while such a classification for zero level was claimed in [17]. Naturally, the next important task is to study irreducible weight modules with infinite dimensional weight spaces and irreducible non-weight modules. The first examples of irreducible weight modules with infinite dimensional weight spaces were given by Chari and Pressley in [10] by taking the tensor product of irreducible integrable highest weight modules and integrable loop modules over affine Lie algebras. Besides these irreducible weight modules, a class of irreducible weight modules over affine Lie algebras with infinite dimensional weight spaces was constructed in [6]. Another class of such irreducible weight modules over the affine Lie algebra $A_1^{(1)}$ was given in [24]. A complete classification for all irreducible (weight and non-weight) modules over affine Lie algebras with locally nilpotent action of the nilpotent radical was obtained in [36]. In the paper [15], some irreducible non-weight modules, called imaginary Whittaker modules, were constructed. The structure of the usual Whittaker modules over the affine Kac–Moody algebras in the case of $A_N^{(1)}$, $B_N^{(1)}$, $C_N^{(1)}$, and $D_N^{(1)}$ was completely determined in [4, 13, 27], while this study for other affine Kac–Moody algebras is still open. Note that both highest weight modules and Whittaker modules can be regarded as special cases of smooth modules (also called restricted modules). In quite recent papers [25, 38], a class of new smooth modules was constructed, and studied thoroughly for the case $A_1^{(1)}$.

The first goal of the present paper is to classify irreducible modules over untwisted affine Kac–Moody algebras $\tilde{\mathfrak{g}}$ on which the action of each weight vector in $\tilde{\mathfrak{n}}_+$ is locally finite (not necessarily locally nilpotent), where $\tilde{\mathfrak{n}}_+$ is the locally nilpotent subalgebra (or positive part) of $\tilde{\mathfrak{g}}$. To obtain this classification, we construct some new irreducible $\tilde{\mathfrak{g}}$-modules (see Theorem 3.4), among which many are weight modules.

The modules we constructed and studied above are closely related to the highest weight modules and loop modules. In 1987, Chari and Pressley [10] first gave some sufficient conditions for the tensor product of irreducible loop $\tilde{\mathfrak{g}}$-modules $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ and irreducible integrable highest weight $\tilde{\mathfrak{g}}$-modules $\widetilde{V}(\Lambda + k\Lambda_0)$ to be simple. Then other sufficient conditions were obtained in [1–3, 12], and a sufficient and necessary condition was given by Adamović for $\widehat{\mathfrak{sl}}_2$ of one case in [2]. The irreducibility of these tensor product modules remains unclear in general until now. Our second goal is to solve this problem using methods completely different from those of Chari–Pressley and Adamović. From our result, we recover Adamović's result for $\widehat{\mathfrak{sl}}_2$ in [2]. Finally, we present examples where the methods in [10] and [2] are not applicable.

The paper is organized as follows. In Section 2, we recall some basic notation and results for later use. In Section 3, using evaluation modules $E(\boldsymbol{\lambda}, \boldsymbol{a})$ (such modules were classified in [11], and a more general case was obtained in [34]) and irreducible smooth $\hat{\mathfrak{g}}$-modules L (including highest weight modules and Whittaker modules), we construct a class of new modules $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L)[d] := \mathrm{Ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}}(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L)$ over the untwisted affine Kac–Moody algebras $\tilde{\mathfrak{g}}$, where d is the degree derivation in $\tilde{\mathfrak{g}}$. We study the submodule structure of these modules and in particular, show that the $\tilde{\mathfrak{g}}$-module $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L)[d]$ is irreducible if $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is nontrivial irreducible and L is irreducible (see Theorem 3.4). In Section 4, we study properties of irreducible $\tilde{\mathfrak{g}}$-modules V on which the action of each root vector in $\tilde{\mathfrak{n}}_+$ is locally finite (not necessarily locally nilpotent). In Section 5, using our results in Sections 3 and 4, we classify all irreducible $\tilde{\mathfrak{g}}$-modules on which the action of each root vector in $\tilde{\mathfrak{n}}_+$ is locally finite. They are precisely irreducible highest weight modules, irreducible Whittaker modules, the irreducible modules $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\gamma))[d]$ and $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes W(\eta))[d]$, with $E(\boldsymbol{\lambda}, \boldsymbol{a})$ being finite dimensional (see Theorem 5.3), where $V(\gamma)$ is an irreducible highest weight module over $\hat{\mathfrak{g}}$ and $W(\eta)$ is an irreducible Whittaker module of $\hat{\mathfrak{g}}$. The last two classes are special examples of the irreducible modules constructed in Section 3. As a byproduct, we see that the action of $\tilde{\mathfrak{n}}_+$ on these irreducible $\tilde{\mathfrak{g}}$-modules is also locally finite. In Section 6, we determine the necessary and sufficient conditions for two such irreducible $\tilde{\mathfrak{g}}$-modules to be isomorphic (see Theorem 6.3). In Section 7, we use the "shifting technique" to obtain a necessary and sufficient condition for the tensor product of irreducible integrable loop modules $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ and irreducible integrable highest weight modules $\widetilde{V}(\Lambda + k\Lambda_0)$ to be simple for untwisted affine Kac–Moody algebras (Theorems 7.9 and 7.12). As an application, we recover Adamović's result for $\widehat{\mathfrak{sl}}_2$, see Corollary 7.10.

Throughout this paper, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers and complex numbers, respectively. All vector spaces and Lie algebras are over $\mathbb{C}$, and all tensor products are taken over $\mathbb{C}$ unless otherwise noted. For a Lie algebra $\mathscr{G}$, we denote its universal enveloping algebra by $U(\mathscr{G})$.

2. Notation

In this section, we will recall standard notions, basically from [31] and [5].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type X_l over $\mathbb{C}$. For this $\mathfrak{g}$, we fix a Cartan subalgebra $\mathfrak{h}$, then a root system Δ with respect to $\mathfrak{h}$ and a base Π of Δ. Denote by Δ_+ and Δ_- the sets of positive and negative roots, respectively. Then $\mathfrak{g}$ admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm$ are corresponding maximal nilpotent subalgebras with respect to Π. Let θ be the longest positive root in Δ_+.

The affine Lie algebra $\tilde{\mathfrak{g}}$ associated with $\mathfrak{g}$ is

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Its Lie algebra structure is defined by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + (x|y) m \delta_{m+n,0} K, \quad [d, x \otimes t^m] = mx \otimes t^m,$$

where K is central in $\tilde{\mathfrak{g}}$, and $(x|y)$ is the Killing form of $\mathfrak{g}$.

Now we can describe the root system $\widetilde{\Delta}$ and the root space decomposition of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ as follows:

$$\widetilde{\Delta} = \{\alpha + k\delta : k \in \mathbb{Z}, \alpha \in \Delta\} \cup \{j\delta \mid j \in \mathbb{Z} \setminus \{0\}\};$$

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \Big(\bigoplus_{\gamma \in \widetilde{\Delta}} \tilde{\mathfrak{g}}_\gamma\Big) \quad \text{with} \quad \tilde{\mathfrak{g}}_{\alpha+k\delta} = \mathfrak{g}_\alpha \otimes t^k,\ \tilde{\mathfrak{g}}_{j\delta} = \mathfrak{h} \otimes t^j,$$

where δ is the standard imaginary root. The set of positive roots of $\tilde{\mathfrak{g}}$ is

$$\widetilde{\Delta}_+ = \Delta_+ \cup \{n\delta \mid n \in \mathbb{N}\} \cup \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{N}\}.$$

Set $\widetilde{\Delta}_- = -\widetilde{\Delta}_+$. Now we have

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{\gamma \in \widetilde{\Delta}_\pm} \tilde{\mathfrak{g}}_\gamma,$$

and the standard triangular decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-,$$

where

$$\tilde{\mathfrak{n}}_+ = (\mathfrak{g} \otimes t\mathbb{C}[t]) \oplus \mathfrak{n}_+ \quad \text{and} \quad \tilde{\mathfrak{n}}_- = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus \mathfrak{n}_-.$$

For the subalgebra $\hat{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$, we can define similar notation just by replacing $\tilde{\mathfrak{h}}$ with $\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K$ and obtain a standard triangular decomposition $\hat{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-$. Throughout the paper, we will fix these triangular decompositions and define highest weight modules and Whittaker modules relative to them. Let $\mathcal{G} = \tilde{\mathfrak{g}}$ and $\mathcal{H} = \tilde{\mathfrak{h}}$ (or sometimes $\mathcal{G} = \hat{\mathfrak{g}}$ and $\mathcal{H} = \hat{\mathfrak{h}}$).

Definition 2.1. Let $\gamma\colon \mathcal{H} \to \mathbb{C}$ be a linear map and let V be a $\mathcal{G}$-module.

(1) A nonzero vector $v \in V$ is called a *highest weight vector with highest weight* γ if $xv = \gamma(x)v$ for all $x \in \mathcal{H}$, and $\tilde{\mathfrak{n}}_+ v = 0$.

(2) The module V is called a *highest weight module with highest weight* γ *over* $\mathcal{G}$ if it is generated by a highest weight vector with highest weight γ.

It is well known that irreducible highest weight modules are determined uniquely by its highest weight. For $\mathcal{G} = \tilde{\mathfrak{g}}$ and $\gamma \in \mathrm{Hom}(\tilde{\mathfrak{h}}, \mathbb{C})$, we denote by $\widetilde{V}(\gamma)$ the irreducible highest weight $\tilde{\mathfrak{g}}$-module with highest weight γ, and similarly, for $\mathcal{G} = \hat{\mathfrak{g}}$ and $\gamma \in \mathrm{Hom}(\hat{\mathfrak{h}}, \mathbb{C})$, we denote the irreducible highest weight module by $\widehat{V}(\gamma)$.

Definition 2.2. Let $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ be a nonzero Lie algebra homomorphism and let W be a $\mathcal{G}$-module.

(1) A nonzero vector $v \in W$ is called a *Whittaker vector of type* η if $xv = \eta(x)v$ for all $x \in \tilde{\mathfrak{n}}_+$.

(2) W is called a *type* η *Whittaker module for* $\mathcal{G}$ if it is generated by a type η Whittaker vector.

We say that the Lie algebra homomorphism $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ for affine Lie algebras $\tilde{\mathfrak{g}}$ is *nondegenerate* if $\eta(e_i) \neq 0$ for each Chevalley generator e_i of $\tilde{\mathfrak{n}}_+$ (see [31]). We denote by $W(\eta)$ any one of the irreducible Whittaker module of type η in this case (it may not be unique).

The structure of irreducible Whittaker modules $W(\eta)$ over the affine Lie algebra $A_1^{(1)}$ was thoroughly studied in [4], while it is not determined for other affine Lie algebras.

Let V be a module over a Lie algebra $\mathcal{L}$ and let $x \in \mathcal{L}$. Then V is called locally finite if any element in V is contained in a finite dimensional $\mathfrak{g}$-submodule of V, and similarly, we say the action of x on V is locally finite if V is a locally finite module over the 1-dimensional Lie algebra $\mathbb{C}x$. In the following sections, we will determine the irreducible $\tilde{\mathfrak{g}}$-modules such that the actions of all root vectors in $\tilde{\mathfrak{n}}_+$ are locally finite.

3. Construction of irreducible modules

In this section, we construct a class of new modules over untwisted affine Lie algebras $\tilde{\mathfrak{g}}$ and study their properties.

For $m \in \mathbb{N}$, let $\widetilde{\mathbb{C}}^m$ be the set of all vectors $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{C}^m$ with all $a_1, \ldots, a_m$ being nonzero and pairwise distinct. Take $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$ and set $f_{\boldsymbol{a}}(t) = (t - a_1)\cdots(t - a_m)$. It is well known that the quotient Lie algebra

$$\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g}) = \frac{\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]}{\mathfrak{g} \otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm 1}]} = \mathfrak{g} \otimes (\mathbb{C}[t^{\pm 1}]/\langle f_{\boldsymbol{a}}(t)\rangle) \tag{3.1}$$

is the direct sum of m copies of $\mathfrak{g}$ and hence is semisimple. Actually,

$$\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g}) = \bigoplus_{i=1}^{m} \mathcal{L}_{\boldsymbol{a}}^{(i)}(\mathfrak{g}),$$

where

$$\mathcal{L}_{\boldsymbol{a}}^{(i)}(\mathfrak{g}) = \mathfrak{g} \otimes \Big(\frac{f_{\boldsymbol{a}}(t)}{(t - a_i)\prod_{j\neq i}(a_i - a_j)}\Big) \simeq \mathfrak{g}. \tag{3.2}$$

This is because

$$\Big(\frac{f_{\boldsymbol{a}}(t)}{(t - a_i)\prod_{j\neq i}(a_i - a_j)}\Big)^2 \equiv \frac{f_{\boldsymbol{a}}(t)}{(t - a_i)\prod_{j\neq i}(a_i - a_j)} \mod (f_{\boldsymbol{a}}(t)).$$

We note that $\mathcal{L}_{\boldsymbol{a}}$ inherits a standard triangular decomposition and root system from $\mathfrak{g}$. In particular, $\mathfrak{h}_{\boldsymbol{a}} = \mathfrak{h} \otimes (\mathbb{C}[t^{\pm 1}]/(f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm 1}]))$ is a Cartan subalgebra of $\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g})$. Within the quotient Lie algebra $\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g})$, we have the following.

Lemma 3.1. *For any $k \in \mathbb{N}$, we have the following canonical isomorphism of vector spaces*:

$$\frac{\mathfrak{g} \otimes t^k\mathbb{C}[t]}{(\mathfrak{g} \otimes t^k\mathbb{C}[t]) \cap (\mathfrak{g} \otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm 1}])} \cong \frac{\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]}{\mathfrak{g} \otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm 1}]}. \tag{3.3}$$

Proof. This is the composition of the following obvious isomorphisms:

$$\frac{\mathfrak{g}\otimes t^k\mathbb{C}[t]}{(\mathfrak{g}\otimes t^k\mathbb{C}[t])\cap(\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm1}])}=\frac{\mathfrak{g}\otimes t^k\mathbb{C}[t]}{(\mathfrak{g}\otimes t^k\mathbb{C}[t])\cap(\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t])}$$
$$\cong\frac{\mathfrak{g}\otimes t^k\mathbb{C}[t]+\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t]}{\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t]}\cong\frac{\mathfrak{g}\otimes\mathbb{C}[t]}{\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t]}\cong\frac{\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]}{\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm1}]}.\qquad\blacksquare$$

For $\boldsymbol{a}=(a_1,\dots,a_m)\in\widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_m)\in(\mathfrak{h}^*)^m$, where $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$, let us first recall from [9] the evaluation module $E(\boldsymbol{\lambda},\boldsymbol{a})$ over $\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]$. Let $V(\lambda_i)$ be the irreducible highest weight module over the finite dimensional simple Lie algebra $\mathfrak{g}$ with highest weight λ_i. The action on the module $E(\boldsymbol{\lambda},\boldsymbol{a})=\bigotimes_{i=1}^m V(\lambda_i)$ is as follows:

$$(x\otimes t^k)\Big(\bigotimes_{i=1}^m v_i\Big)=\sum_{i=1}^m a_i^k(v_1\otimes\cdots\otimes xv_i\otimes\cdots\otimes v_m),\tag{3.4}$$

for any $x\in\mathfrak{g}$ and $k\in\mathbb{Z}$. Then $E(\boldsymbol{\lambda},\boldsymbol{a})$ is an irreducible module over the Lie algebra $\hat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]\oplus\mathbb{C}K$ with trivial action of K.

Note that $(\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm1}])\oplus\mathbb{C}K\subseteq\mathrm{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a}))$, so we can regard $E(\boldsymbol{\lambda},\boldsymbol{a})$ as a module over $\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g})\cong\hat{\mathfrak{g}}/\big((\mathfrak{g}\otimes f_{\boldsymbol{a}}(t)\mathbb{C}[t^{\pm1}])\oplus\mathbb{C}K\big)$. Thus we can consider the weight vectors and weight spaces of $E(\boldsymbol{\lambda},\boldsymbol{a})$ with respect to the Cartan subalgebra $\mathfrak{h}_{\boldsymbol{a}}$ of $\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g})$. Identify $\mathfrak{h}_{\boldsymbol{a}}^*$ with $(\mathfrak{h}^*)^m$ via (3.1) and (3.2). For example, a weight vector $v\in E(\boldsymbol{\lambda},\boldsymbol{a})$ of weight $\boldsymbol{\mu}=(\mu_1,\dots,\mu_m)\in(\mathfrak{h}^*)^m$ satisfies

$$(h\otimes f(t))v=\sum_{i=1}^m\mu_i(h_i)f(a_i)v.$$

We will use these notation freely in the rest of the paper.

Remark 3.2. We notice that $(\mathfrak{g}\otimes f_{\boldsymbol{a}}(t))\oplus\mathbb{C}K\neq\mathrm{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a}))$ in general. Indeed, if we take $\bar{\boldsymbol{\lambda}}$ and $\bar{\boldsymbol{a}}$ to be the vectors obtained by removing the entries λ_i and a_i with $\lambda_i=0$ from $\boldsymbol{\lambda}$ and $\boldsymbol{a}$, respectively, then $E(\boldsymbol{\lambda},\boldsymbol{a})=E(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{a}})$ and $\mathrm{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a}))=\mathrm{Ann}(E(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{a}}))=(\mathfrak{g}\otimes f_{\bar{\boldsymbol{a}}}(t))\oplus\mathbb{C}K$. The first equation is obvious, so to obtain the second equation, one need only to show $\mathrm{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a}))=(\mathfrak{g}\otimes f_{\boldsymbol{a}}(t))\oplus\mathbb{C}K$ provided all $\lambda_i\neq0, i=1,\dots,m$. This can be verified by the fact that any simple ideal of $\mathcal{L}_{\boldsymbol{a}}(\mathfrak{g})$ is one of $\mathcal{L}_{\boldsymbol{a}}^{(i)}(\mathfrak{g})$ in (3.2), which does not annihilate $V(\lambda_i)$; consequently, neither does it annihilate $E(\boldsymbol{\lambda},\boldsymbol{a})$ when $\lambda_i\neq0$. This description of $\mathrm{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a}))$ will be used in the proof of Lemma 6.2.

Let us recall the smooth modules over affine Kac–Moody algebras from [32, 33]. A module L over $\hat{\mathfrak{g}}$ or $\tilde{\mathfrak{g}}$ is called a *smooth module* if for any $v\in L$ we have $(\mathfrak{g}\otimes t^k\mathbb{C}[t])v=0$ for a sufficiently large $k\in\mathbb{N}$.

Lemma 3.3. *Let L be a smooth module over $\hat{\mathfrak{g}}$. Let $\boldsymbol{a}\in\widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda}\in(\mathfrak{h}^*)^m$. Then any $\hat{\mathfrak{g}}$-submodule of $E(\boldsymbol{\lambda},\boldsymbol{a})\otimes L$ is of the form $E(\boldsymbol{\lambda},\boldsymbol{a})\otimes N$ for some $\hat{\mathfrak{g}}$-submodule N of L.*

Proof. Let W be a nonzero $\hat{\mathfrak{g}}$-submodule of $E(\boldsymbol{\lambda},\boldsymbol{a})\otimes L$ and take any nonzero element $w\in W$. We can write $w=\sum_{i=1}^n v_i\otimes u_i$ for some nonzero $v_i\in E(\boldsymbol{\lambda},\boldsymbol{a})$ and nonzero $u_i\in L$. Fix one such expression for w with n minimal. We see that $v_1,v_2,\dots,v_n$ are linearly independent. There exists $k \in \mathbb{N}$ such that $(\mathfrak{g} \otimes t^j)u_i = 0$ for all $j \geq k$ and $i = 1, \ldots, n$. Thus for any $x \in \mathfrak{g} \otimes t^k\mathbb{C}[t]$, we have $xw = \sum_{i=1}^{n}(xv_i) \otimes u_i \in W$. From Lemma 3.1, we deduce that

$$\sum_{i=1}^{n}(xv_i) \otimes u_i \in W, \quad \forall x \in U(\widehat{\mathfrak{g}}). \tag{3.5}$$

It is well known that any endomorphism of an irreducible module over a countably generated associative $\mathbb{C}$-algebra is a scalar (see Proposition 2.6.5 and Corollary 2.6.6 in [18]). Thus $\mathrm{Hom}_{U(\widehat{\mathfrak{g}})}(E(\boldsymbol{\lambda}, \boldsymbol{a}), E(\boldsymbol{\lambda}, \boldsymbol{a})) \cong \mathbb{C}$, since $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is irreducible over the finite dimensional Lie algebra $\mathcal{L}_{\boldsymbol{a}}(g)$. Note that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is a faithful irreducible module over $U(\widehat{\mathfrak{g}})/\mathrm{Ann}_{U(\widehat{\mathfrak{g}})}(E(\boldsymbol{\lambda}, \boldsymbol{a}))$. From the Jacobson density theorem (see p. 197 of [29]) or Proposition 2.6.5 in [18], we know that $U(\widehat{\mathfrak{g}})/\mathrm{Ann}_{U(\widehat{\mathfrak{g}})}(E(\boldsymbol{\lambda}, \boldsymbol{a}))$ is isomorphic to a dense ring of endomorphisms of the $\mathbb{C}$-vector space $E(\boldsymbol{\lambda}, \boldsymbol{a})$. Then there exists $x_i \in U(\widehat{\mathfrak{g}})$ such that $x_i v_j = \delta_{i,j} v_j$ for all $i, j = 1, \ldots, n$. Taking $x = x_i$ in (3.5), we deduce that $v_i \otimes u_i \in W$. Again using (3.5) for $n = 1$, we get $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_i \subseteq W$ for $i = 1, \ldots, n$. Thus we deduce that if $\sum_{i=1}^{n} v_i \otimes u_i \in W$ for linearly independent $v_i \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, then $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_i \subseteq W$ for each i.

Let

$$N = \{u \in L \mid v \otimes u \in W \text{ for some nonzero } v \in E(\boldsymbol{\lambda}, \boldsymbol{a})\}.$$

From the above arguments, we know that N is a nonzero subspace of L and that $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes N \subset W$. Then for any nonzero $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, $u \in N$ and $x \in \widehat{\mathfrak{g}}$, we deduce that $x(v \otimes u) = (xv) \otimes u + v \otimes (xu) \in W$. Since $(xv) \otimes u \in W$, we have $v \otimes (xu) \in W$. Thus $xu \in N$. In other words, N is a $\widehat{\mathfrak{g}}$-submodule of L. Combining this with the result established in the previous paragraph we see that $W = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes N$. Noticing that the action of K is central, the lemma follows immediately. ∎

For any $\widehat{\mathfrak{g}}$-module M, we can define an induced $\tilde{\mathfrak{g}}$-module

$$M[d] = \mathrm{Ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} M.$$

It is easy to see that $M[d] = \mathbb{C}[d] \otimes M$ as vector spaces. We will simply write $d^n v = d^n \otimes v$ for all $n \in \mathbb{Z}_+$ and $v \in M$. Now we can prove the following.

Theorem 3.4. *Let $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$. Let L be a smooth module over $\widehat{\mathfrak{g}}$. Then any $\tilde{\mathfrak{g}}$-submodule of $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L)[d]$ is of the form $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes N)[d]$ for some $\widehat{\mathfrak{g}}$-submodule N of L.*

Proof. Denote $M = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L$ for short. Let $M^{(n)} = \sum_{i=0}^{n} d^i \otimes M$ for all $n \in \mathbb{Z}_+$ and $M^{(n)} = 0$ for $n < 0$. Analogous to Lemma 3.3, we have

$$(x \otimes t^k)(d^n v) = d^n (x \otimes t^k) v - [d^n, x \otimes t^k] v,$$

for all $x \in \mathfrak{g}$, $v \in M$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, which induces a $\widehat{\mathfrak{g}}$-module isomorphism

$$M^{(n)}/M^{(n-1)} \to M, \quad d^n v \mapsto v.$$

Let W be a nonzero submodule of $M[d]$ and take a nonzero $w \in W$. Suppose that $w \in W \cap M^{(n)}$. Then we can write w as

$$w = \sum_{j=0}^{n} d^j \sum_{i=1}^{l_j} (v_{j,i} \otimes u_{j,i}),$$

where $v_{j,i} \in E(\boldsymbol{\lambda}, \boldsymbol{a}), u_{j,i} \in L, l_j \in \mathbb{N}, j = 1, \ldots, n$ such that $v_{j,1}, \ldots, v_{j,l_j}$ are linearly independent weight vectors for each j. We may assume that $v_{j,i}$ has weight $\boldsymbol{\mu}_{j,i} \in (\mathfrak{h}^*)^m$. We note that there exists $k' \in \mathbb{N}$ such that $(\mathfrak{g} \otimes t^{k'}\mathbb{C}[t])u_{j,i} = 0$ for all $i = 1, \ldots, l_j$ and $j = 1, \ldots, n$.

Claim. $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_{j,i})[d] \subseteq W$ *for all* i, j.

We prove this by induction on n and l_n. If $n = 0$ and $l_0 = 1$, the result is clear from (3.3) and (3.5). Now we fix some $n \in \mathbb{Z}_+$ and $l_n \in \mathbb{N}$. Suppose that the claim holds for smaller n or for the same n and smaller l_n.

Case 1. $l_n = 1$.

We may assume $n \geq 1$. Take any $x \in U(\mathfrak{g} \otimes t^{k'}\mathbb{C}[t])$ such that $x v_{n,1} = v_{\boldsymbol{\lambda}}$, the nonzero highest weight vector in $E(\boldsymbol{\lambda}, \boldsymbol{a})$. Then we have $x u_{j,i} = 0$ for all $i = 1, \ldots, l_j$ and $j = 1, \ldots, n$ and $xw \equiv d^n(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) \mod M^{(n-1)}$. By replacing w with xw (noticing that $u_{n,1}$ has not been changed), we may assume that

$$w = d^n(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) + \sum_{j=1}^{n-1} d^j \sum_{i=1}^{l_j} (v_{j,i} \otimes u_{j,i}),$$

with $\boldsymbol{\mu}_{n,1} = \boldsymbol{\lambda}$.

Without loss of generality, we may assume that either $v_{\boldsymbol{\lambda}}, v_{n-1,1}, \ldots, v_{n-1,l_{n-1}}$ are linearly independent, or $v_{n-1,1} = v_{\boldsymbol{\lambda}}$ (i.e., $\boldsymbol{\mu}_{n-1,1} = \boldsymbol{\lambda}$). For any $h \in \mathfrak{h}$ and $k \geq k'$, we can compute that

$$\begin{aligned}
(h \otimes t^k)w &= (h \otimes t^k)d^n(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) + (h \otimes t^k) \sum_{j=1}^{n-1} d^j \sum_{i=1}^{l_j} (v_{j,i} \otimes u_{j,i}) \\
&\equiv d^n(h \otimes t^k)(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) - [d^n, h \otimes t^k](v_{\boldsymbol{\lambda}} \otimes u_{n,1}) \\
&\quad + d^{n-1} \sum_{i=1}^{l_{n-1}} (h \otimes t^k)(v_{n-1,i} \otimes u_{n-1,i}) \mod M^{(n-2)} \\
&\equiv d^n \boldsymbol{\lambda}(h \otimes t^k)(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) - nkd^{n-1}\boldsymbol{\lambda}(h \otimes t^k)(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) \\
&\quad + d^{n-1} \sum_{i=1}^{l_{n-1}} \boldsymbol{\mu}_{n-1,i}(h \otimes t^k)(v_{n-1,i} \otimes u_{n-1,i}) \mod M^{(n-2)},
\end{aligned}$$

where we have used the notation $\boldsymbol{\lambda}(h \otimes t^k) = \sum_{i=1}^{m} a_i^k \lambda_i(h)$ for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (\mathfrak{h}^*)^m$, and similarly for $\boldsymbol{\mu}_{n-1,i}(h \otimes t^k)$.

Then W contains the following vector:

$$w' = (h \otimes t^k)w - \boldsymbol{\lambda}(h \otimes t^k)w \equiv -nkd^{n-1}\boldsymbol{\lambda}(h \otimes t^k)(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) + d^{n-1}\sum_{i=1}^{l_{n-1}}(\boldsymbol{\mu}_{n-1,i} - \boldsymbol{\lambda})(h \otimes t^k)(v_{n-1,i} \otimes u_{n-1,i}) \mod M^{(n-2)}.$$

for all $k > k'$ and $h \in \mathfrak{h}$. Since $\boldsymbol{\lambda} \neq 0$, there exist $h \in \mathfrak{h}$ and $k \geq k'$ such that $\boldsymbol{\lambda}(h \otimes t^k) = \sum_{j=1}^{m} a_i^k \lambda_i(h) \neq 0$. It is easy to check that $w' \neq 0$ for this $h \otimes t^k$, and hence $w' \in W \cap M^{(n-1)} \setminus \{0\}$. By the induction hypothesis, we deduce that $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_{n,1})[d] \subseteq W$. Consequently, the claim follows in this case.

Case 2. $l_n \geq 2$.

Since $v_{n,1}, \ldots, v_{n,l_n}$ are linearly independent and $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is irreducible, using the density theorem and Lemma 3.1, we can choose an element $x \in U(\mathfrak{g} \otimes t^{k'}\mathbb{C}[t])$ such that $xv_{n,i} = \delta_{i,1}v_{\boldsymbol{\lambda}}$. Then W contains

$$xw \equiv d^n(v_{\boldsymbol{\lambda}} \otimes u_{n,1}) \mod M^{(n-1)}.$$

From Case 1, we deduce that $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_{n,1})[d] \subseteq W$. Using the induction hypothesis, we obtain the claim.

From the above claim, we see that

$$N = \{u \in L \mid (E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u)[d] \subseteq W\}$$

is a nonzero subspace of L. It is easy to see that N is a $\hat{\mathfrak{g}}$-submodule of L. Using the above claim, we can easily prove that $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes N)[d] = W$. This completes the proof of the theorem. ∎

Now we can have two important consequences from the above theorem.

Corollary 3.5. *Let L be an irreducible smooth $\hat{\mathfrak{g}}$-module. Let $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$. Then the $\tilde{\mathfrak{g}}$-module $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L)[d]$ is irreducible.*

Taking L to be the 1-dimensional trivial $\hat{\mathfrak{g}}$-module in Corollary 3.5, we obtain the following.

Corollary 3.6. *Let $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$. Then $E(\boldsymbol{\lambda}, \boldsymbol{a})[d]$ is an irreducible $\tilde{\mathfrak{g}}$-module.*

4. Properties of locally finite modules

Recall that $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is the standard triangular decomposition of the finite dimensional simple Lie algebra $\mathfrak{g}$ with respect to a fixed Cartan subalgebra $\mathfrak{h}$, where $\mathfrak{n}_\pm = \sum_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$. We consider the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ and its subalgebra $\tilde{\mathfrak{n}}_+$ (see Section 2). We identify $\mathfrak{g} \otimes 1$ with $\mathfrak{g}$. For any $f(t) \in \mathbb{C}[t]$, we can define an ideal of $\tilde{\mathfrak{n}}_+$:

$$\mathcal{I}(f) = \mathfrak{n}_+ \otimes \langle f(t) \rangle + (\mathfrak{h} + \mathfrak{n}_-) \otimes \langle tf(t) \rangle,$$

where $\langle f(t) \rangle$ denotes the ideal generated by $f(t)$ in $\mathbb{C}[t]$, and similar for $\langle tf(t) \rangle$.

It is easy to see $\mathcal{I}(f) = \tilde{\mathfrak{n}}_+ \cap (\mathfrak{g} \otimes f\mathbb{C}[t^{\pm 1}])$ if $f(0) \neq 0$. For any $a \neq 0$, we have the following canonical Lie algebra isomorphism:

$$\tilde{\mathfrak{n}}_+/\mathcal{I}(t-a) \to \mathfrak{g}, \quad x \otimes g(t) \mapsto g(a)x, \tag{4.1}$$

for all $x \otimes g(t) \in \tilde{\mathfrak{n}}_+$, with $x \in \mathfrak{g}$ and $g(t) \in \mathbb{C}[t]$.

Theorem 4.1. *Let V be a $\tilde{\mathfrak{g}}$-module such that each root vector in $\tilde{\mathfrak{n}}_+$ acts locally finitely on V. Then V has a finite dimensional nonzero $\tilde{\mathfrak{n}}_+$-submodule W such that $\mathcal{I}(f(t))W = 0$ for some nonzero $f(t) \in \mathbb{C}[t]$.*

Proof. Recall that δ is the standard imaginary root of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$. Then $\tilde{\mathfrak{g}}_\delta = \mathfrak{h} \otimes t$ is a finite dimensional abelian subalgebra of $\tilde{\mathfrak{g}}$. Since $\tilde{\mathfrak{g}}_\delta$ is locally finite on V, there is a nonzero vector $v \in V$ such that $(h \otimes t)v \in \mathbb{C}v$ for all $h \otimes t \in \tilde{\mathfrak{g}}_\delta$.

Take any $\alpha \in \Delta$ which is considered as a subset of $\widetilde{\Delta}$. We have

$$\alpha(h)x \otimes t^{k+1}v = [h \otimes t, x \otimes t^k]v = (h \otimes t)(x \otimes t^k)v - (x \otimes t^k)(h \otimes t)v,$$

for all $h \in \mathfrak{h}$ and $x \in \tilde{\mathfrak{g}}_\alpha$, which implies $\tilde{\mathfrak{g}}_{\alpha+(k+1)\delta} \subseteq \tilde{\mathfrak{g}}_\delta \tilde{\mathfrak{g}}_{\alpha+k\delta} v + \tilde{\mathfrak{g}}_{\alpha+k\delta} v$ for all $k \in \mathbb{Z}_+$. Inductively, we have

$$\tilde{\mathfrak{g}}_{k\delta+\alpha} v \subseteq \sum_{i=0}^{k} \tilde{\mathfrak{g}}_\delta^i \, \tilde{\mathfrak{g}}_\alpha v \quad \text{for all } k \in \mathbb{Z}_+.$$

Since $\dim \sum_{k=0}^{\infty} \tilde{\mathfrak{g}}_\delta^k \, \tilde{\mathfrak{g}}_\alpha v < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=0}^{\infty} \tilde{\mathfrak{g}}_\delta^k \, \tilde{\mathfrak{g}}_\alpha v = \sum_{k=0}^{k_0} \tilde{\mathfrak{g}}_\delta^k \, \tilde{\mathfrak{g}}_\alpha v$$

is a finite dimensional space, i.e.,

$$\dim\Big(\sum_{k \in \mathbb{Z}_+} \tilde{\mathfrak{g}}_{k\delta+\alpha} v\Big) = \dim\Big(\sum_{k=0}^{k_0} \tilde{\mathfrak{g}}_{k\delta+\alpha} v\Big) < \infty.$$

Let $0 \neq x_\alpha \in \tilde{\mathfrak{g}}_\alpha$. Then there is a nonzero $g(t) \in \mathbb{C}[t]$ such that $(x_\alpha \otimes g(t))v = 0$. Note that we may assume that $g(t)$ is divisible by t when $\alpha \in \Delta_-$. Set

$$\mathcal{I}_\alpha = \{g(t) \in \mathbb{C}[t] \mid (x_\alpha \otimes g(t))v = 0 \text{ and } t \text{ divides } g(t) \text{ if } \alpha \in \Delta_-\},$$

which is nonzero. If $g(t) \in \mathcal{I}_\alpha$, for any $h \in \mathfrak{h}$ we have

$$\begin{aligned} 0 &= (h \otimes t)(x_\alpha \otimes g(t))v = ([h, x_\alpha] \otimes tg(t))v + (x_\alpha \otimes g(t))(h \otimes t)v \\ &= \alpha(h)(x_\alpha \otimes tg(t))v. \end{aligned}$$

Hence $tg(t) \in \mathcal{I}_\alpha$, i.e., $\mathcal{I}_\alpha$ is an ideal of $\mathbb{C}[t]$. Assume $\mathcal{I}_\alpha = \langle f_\alpha(t) \rangle$ for some nonzero polynomial $f_\alpha(t) \in \mathbb{C}[t]$. Setting $\tilde{f}(t) = \operatorname{lcm}\{f_\alpha(t) : \alpha \in \Delta\} \in \mathbb{C}[t]$, the least common multiple of $f_\alpha(t), \alpha \in \Delta$, we have $(\mathfrak{g}_\alpha \otimes \langle \tilde{f}(x) \rangle)v = 0$ for all $\alpha \in \Delta$. It follows that

$$(\mathfrak{h} \otimes \langle \tilde{f}^2(t) \rangle)v = \sum_{\alpha \in \Delta} \big[\tilde{\mathfrak{g}}_\alpha \otimes \langle \tilde{f}(t) \rangle, \tilde{\mathfrak{g}}_{-\alpha} \otimes \langle \tilde{f}(t) \rangle\big]v = 0.$$

Then there exists nonzero $f \in \mathbb{C}[t]$ such that $\mathcal{I}(f)v = 0$. Noting that t divides $f(t)$, so we may assume that $\deg(f^2(t)) = p + 1 > 1$. Our result follows from:

Claim. *$U(\tilde{\mathfrak{n}}_+)v$ is a finite dimensional $\tilde{\mathfrak{n}}_+$-module.*

We know $\mathcal{I}(f)U(\tilde{\mathfrak{n}}_+)v = 0$ since $\mathcal{I}(f)$ is an ideal of $\tilde{\mathfrak{n}}_+$. Take nonzero $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Delta_+$, and let $\{h_1, \ldots, h_l\}$ be a basis of $\mathfrak{h}$. Denote $\deg(x \otimes t^i) = i$ for all $x \in \mathfrak{g}$ and $i \in \mathbb{Z}_+$. Let

$$\{x_i \mid i \in \mathbb{N}\} = \{e_\alpha, e_\alpha \otimes t^i, f_\alpha \otimes t^i, h_j \otimes t^i \mid \alpha \in \Delta_+, j = 1, \ldots, l, i \in \mathbb{N}\}$$

be such that $\deg(x_i) \leq \deg(x_{i+1})$ for all $i \in \mathbb{N}$. Now take s such that $\deg(x_s) = p$ and $\deg(x_{s+1}) = p + 1$. We know that

$$W = \sum_{r_i \in \mathbb{Z}_+} \mathbb{C} x_1^{r_1} \cdots x_s^{r_s} v$$

is finite dimensional since the actions of all x_i are locally finite. For $q \in \mathbb{Z}_+$, let

$$W_q = \operatorname{span}\{x_1^{r_1} \cdots x_s^{r_s} v \mid r_1 + r_2 + \cdots + r_s \leq q\}.$$

We know that $W_q = W$ for sufficiently large q. We will show that $\tilde{\mathfrak{n}}_+ W_q \subset W_{q+1}$. Let

$$w = x_1^{r_1} \cdots x_s^{r_s} v \in W_q, \quad r_i \in \mathbb{Z}_+. \tag{4.2}$$

It is enough to show that $x_j w \in W_{q+1}$ for any $j \in \mathbb{N}$. We will do this by induction on j and then on q. For $j = 1$ or $q = 0$, it is clear that $x_j w \in \mathbb{C} x_j v \subseteq W_{q+1}$, since any $x_j v$ is a linear combination of $x_1 v, \ldots, x_s v$ (because of the definition of $f(t)$ and p).

Suppose $x_j w \in W_{q+1}$ for any w given in (4.2) and any $j \in \mathbb{N}$ if $q < k$. Now consider one such $w \in W_q$ given in (4.2) with $q = k$.

We write $w = x_{i_1}^{r_1} \cdots x_{i_l}^{r_l} v$, where $i_1 < i_2 < \cdots < i_l \leq s$, $r_1, r_2, \ldots, r_l \in \mathbb{N}$ and $r_1 + r_2 + \cdots + r_l = q$. Now we use induction on j. If $j \leq i_1$, it is clear that $x_j w \in W_{q+1}$. Now suppose $j > i_1$. We have

$$x_j w = x_j x_{i_1} x_{i_1}^{r_1 - 1} \cdots x_{i_l}^{r_l} v = [x_j, x_{i_1}] x_{i_1}^{r_1 - 1} \cdots x_{i_l}^{r_l} v + x_{i_1} x_j x_{i_1}^{r_1 - 1} \cdots x_{i_l}^{r_l} v. \tag{4.3}$$

Since $[x_j, x_{i_1}] W_{q-1} \subseteq \operatorname{span}\{x_i \mid i \in \mathbb{N}\} W_{q-1}$, by the induction hypotheses we know that the first term in (4.3) is $[x_j, x_{i_1}] x_{i_1}^{r_1-1} \cdots x_{i_l}^{r_l} v \in W_q$. For the second term, we see that $x_j x_{i_1}^{r_1-1} \cdots x_{i_l}^{r_l} v \in W_q$, by the induction hypothesis on q. Then by the induction hypothesis on j, we know that the second term in (4.3) is $x_{i_1} x_j x_{i_1}^{r_1-1} \cdots x_{i_l}^{r_l} v \in x_{i_1} W_q \subset W_{q+1}$ since $i_1 < j$. Thus $x_j W_q \subset W_{q+1}$ for all j and all q. So W is an $\tilde{\mathfrak{n}}_+$-module, as desired. ■

It is not hard to verify that

$$[\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+] = \sum_{\alpha \in \Delta_+ \setminus \Pi} \mathfrak{g}_\alpha \otimes \mathbb{C}[t] + \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha \otimes \langle t \rangle + \sum_{\alpha \in \Delta_+ \setminus \{\theta\}} \mathfrak{g}_{-\alpha} \otimes \langle t \rangle$$
$$+ \mathfrak{g}_{-\theta} \otimes \langle t^2 \rangle + \mathfrak{h} \otimes \langle t \rangle,$$

where Π is the base of the root system Δ of $\mathfrak{g}$, and

$$\mathcal{I}(t) = \mathfrak{n}_+ \otimes \langle t \rangle + (\mathfrak{n}_- + \mathfrak{h}) \otimes \langle t^2 \rangle \subseteq [\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+].$$

Lemma 4.2. *Let $a \in \mathbb{C}$, $n \in \mathbb{N}$, and let S be a nontrivial finite dimensional irreducible $\tilde{\mathfrak{n}}_+$-module with $\mathcal{I}((t-a)^n)S = 0$.*

(i) *If $a \neq 0$, then $\mathcal{I}(t-a) = \mathrm{Ann}(S)$.*

(ii) *If $a = 0$, then S is 1-dimensional and $[\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+] \subseteq \mathrm{Ann}(S)$.*

Proof. Denote $\mathcal{L} = \tilde{\mathfrak{n}}_+/\mathcal{I}((t-a)^n)$ for short. We may consider S as an $\mathcal{L}$-module. Let $\mathrm{Rad}(\mathcal{L})$ be the radical of $\mathcal{L}$. Then $\mathrm{Rad}(\mathcal{L})$ is nilpotent, and by Proposition 19.1 in [28], we see that $\mathrm{Rad}(\mathcal{L})/\mathrm{Ann}(S)$ is contained in the center of $\mathcal{L}/\mathrm{Ann}(S)$, which implies that any element in $\mathrm{Rad}(\mathcal{L})$ acts on S as a scalar. Then we have $\mathrm{Rad}(\mathcal{L}) \cap [\mathcal{L}, \mathcal{L}] \subseteq \mathrm{Ann}(S)$ since the trace of any element in $[\mathcal{L}, \mathcal{L}]$ on S is 0. We know that $\mathcal{I}(t-a)/\mathcal{I}((t-a)^n) \subseteq \mathrm{Rad}(\mathcal{L})$. By computing $[\mathcal{L}, \mathcal{L}] \cap \mathcal{I}(t-a)$, we deduce that $\mathrm{Ann}(S)$ contains the following subspace:

$$\sum_{\alpha \in \Delta_+ \setminus \Pi} \mathfrak{g}_\alpha \otimes \langle t-a \rangle + \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha \otimes \langle t(t-a) \rangle$$
$$+ \sum_{\alpha \in \Delta_+ \setminus \{\theta\}} \mathfrak{g}_{-\alpha} \otimes \langle t(t-a) \rangle + \mathfrak{g}_{-\theta} \otimes \langle t^2(t-a) \rangle + \mathfrak{h} \otimes \langle t(t-a) \rangle.$$

(i) Suppose $a \neq 0$. Note that

$$\langle t(t-a) \rangle + \langle (t-a)^n \rangle = \langle (t-a) \rangle.$$

Combining this with the fact $\mathcal{I}((t-a)^n) \subset \mathrm{Ann}(S)$, we see that $\mathcal{I}(t-a) \subseteq \mathrm{Ann}(S)$. Since $\tilde{\mathfrak{n}}_+/\mathcal{I}(t-a) \cong \mathfrak{g}$ is simple, we must have $\mathcal{I}(t-a) = \mathrm{Ann}(S)$.

(ii) If $a = 0$, we know that $\mathcal{L}$ is nilpotent and finite dimensional. So S has to be 1-dimensional, and $[\mathcal{L}, \mathcal{L}]$ acts trivially on S, i.e., $[\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+] \subseteq \mathrm{Ann}(S)$. ∎

Take any $\boldsymbol{a} = (a_1, \ldots, a_m) \in \widetilde{\mathbb{C}}^m$ and let $f_{\boldsymbol{a}}(t) = \prod_{i=1}^m (t - a_i)$. There is a canonical Lie algebra isomorphism

$$\pi_{\boldsymbol{a}} : (\tilde{\mathfrak{n}}_+/\mathcal{I}(t f_{\boldsymbol{a}})) \to \tilde{\mathfrak{n}}_+/\mathcal{I}(t) \oplus \bigoplus_{i=1}^m \tilde{\mathfrak{n}}_+/\mathcal{I}(t-a_i), \tag{4.4}$$

defined by mapping $x \otimes g(t)$ to $(x \otimes g(t), \ldots, x \otimes g(t))$. Here and later, we continue to denote by $x \otimes g(t)$ its image in $\tilde{\mathfrak{n}}_+/\mathcal{I}$ for any $x \otimes g(t) \in \tilde{\mathfrak{n}}_+$ and an ideal $\mathcal{I} \subseteq \tilde{\mathfrak{n}}_+$ for convenience. Using (4.1), we further have the Lie algebra isomorphism

$$\begin{aligned} \tilde{\mathfrak{n}}_+/\mathcal{I}(t f_{\boldsymbol{a}}) &\to (\tilde{\mathfrak{n}}_+/\mathcal{I}(t)) \oplus \bigoplus_{i=1}^m \mathfrak{g}, \\ x \otimes g(t) &\mapsto (x \otimes g(t), g(a_1)x, \ldots, g(a_m)x), \quad \forall x \in \mathfrak{g},\ g(t) \in \mathbb{C}[t]. \end{aligned} \tag{4.5}$$

Given any Lie algebra homomorphism $\eta : \tilde{\mathfrak{n}}_+ \to \mathbb{C}$, we can define a 1-dimensional $\tilde{\mathfrak{n}}_+$-module $\mathbb{C}v_\eta$ by $xv_\eta = \eta(x)v_\eta$ for all $x \in \tilde{\mathfrak{n}}_+$. Note that $\eta([\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+]) = 0$, and hence η is completely determined by its values on $\sum_{\alpha \in \Pi} \mathfrak{g}_\alpha + \mathfrak{g}_{-\theta} \otimes t$.

For any $\mathfrak{g}$-modules $L_1, \dots, L_m$, using the isomorphism in (4.5) and noticing that $\mathcal{J}(t) \subseteq [\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+]$, we can make $L_1 \otimes \cdots \otimes L_m \otimes \mathbb{C}v_\eta$ into an $\tilde{\mathfrak{n}}_+/\mathcal{J}(tf_{\boldsymbol{a}})$-module via:

$$(x \otimes g(t))v_1 \otimes \cdots \otimes v_m \otimes v_\eta = \eta(x \otimes g(t))v_1 \otimes \cdots \otimes v_m \otimes v_\eta \\ + \sum_{i=1}^{m} g(a_i)v_1 \otimes \cdots \otimes xv_i \otimes \cdots \otimes v_m \otimes v_\eta, \quad \forall v_i \in L_i, x \otimes g(t) \in \tilde{\mathfrak{n}}_+.$$

We denote the resulting $\tilde{\mathfrak{n}}_+$-module by $S(\eta; L_1, \dots, L_m; a_1, \dots, a_m)$.

Let $L(\lambda_i)$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda_i \in \mathfrak{h}^*$ for $i = 1, \dots, m$. Then we denote $S(\eta; L(\lambda_1), \dots, L(\lambda_m); a_1, \dots, a_m)$ by $S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ for short, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ and $\boldsymbol{a} = (a_1, \dots, a_m)$. It is not hard to prove that $S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ is an irreducible $\tilde{\mathfrak{n}}_+$-module as well as an irreducible $\tilde{\mathfrak{n}}_+/\mathcal{J}(tf_{\boldsymbol{a}})$-module. If all $\lambda_1, \dots, \lambda_m$ are dominant, then $S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional. We also note that when some $\lambda_i = 0$, this $L(\lambda_i)$ is a 1-dimensional trivial $\mathfrak{g}$-module and hence makes no contribution to the $\tilde{\mathfrak{n}}_+$-module $S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$.

Lemma 4.3. *Let $f_{\boldsymbol{a}}(t) = \prod_{i=1}^{m}(t - a_i)$ and $\tilde{f}_{\boldsymbol{a}}(t) = \prod_{i=0}^{m}(t - a_i)^{n_i}$, where $m \in \mathbb{N}$, $a_0 = 0$, $n_0 \in \mathbb{Z}_+$ and the other $n_i \in \mathbb{N}$, and with $\boldsymbol{a} = (a_1, \dots, a_m) \in \widetilde{\mathbb{C}}^m$. Let S be an irreducible finite dimensional $\tilde{\mathfrak{n}}_+$-module with $\mathcal{J}(\tilde{f}_{\boldsymbol{a}})S = 0$. Then $S \cong S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ for some Lie algebra homomorphism $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ with each $\lambda_i \in \mathfrak{h}^*$ being dominant. Moreover,*

(i) *if $n_0 = 0$, then $\eta = 0$ and $\mathcal{J}(f_{\boldsymbol{a}}) \subseteq \operatorname{Ann}(S)$;*

(ii) *if $n_0 \neq 0$, we have*

$$\sum_{\alpha \in \Delta_+ \setminus \Pi} \mathfrak{g}_\alpha \otimes \langle f_{\boldsymbol{a}} \rangle + \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha \otimes \langle tf_{\boldsymbol{a}} \rangle + \sum_{\alpha \in \Delta_+ \setminus \{\theta\}} \mathfrak{g}_{-\alpha} \otimes \langle tf_{\boldsymbol{a}} \rangle \\ + \mathfrak{g}_{-\theta} \otimes \langle t^2 f_{\boldsymbol{a}} \rangle + \mathfrak{h} \otimes \langle tf_{\boldsymbol{a}} \rangle \subseteq \operatorname{Ann}(S).$$

Proof. Denote $\mathcal{L} = \tilde{\mathfrak{n}}_+/\mathcal{J}(\tilde{f}_{\boldsymbol{a}})$. We regard S as an $\mathcal{L}$-module. In a similar Lie algebra isomorphism as in (4.4),

$$\tilde{\pi}_{\boldsymbol{a}} : \tilde{\mathfrak{n}}_+/\mathcal{J}(\tilde{f}_{\boldsymbol{a}}) \to \bigoplus_{i=0}^{m} \tilde{\mathfrak{n}}_+/\mathcal{J}((t - a_i)^{n_i}),$$

let $\mathcal{L}_i = \tilde{\mathfrak{n}}_+/\mathcal{J}((t - a_i)^{n_i})$. Hence S can be viewed as an $\mathcal{L}_i$-module for each i. We choose a nonzero irreducible $\mathcal{L}_i$-submodule of S, say L_i.

First suppose $i \neq 0$. By Lemma 4.2, we see that $\mathcal{J}(t - a_i)L_i = 0$. So L_i is an irreducible module over the simple Lie algebra $\mathfrak{g} \cong \tilde{\mathfrak{n}}_+/\mathcal{J}(t - a_i)$. We see that $L_i \cong L(\lambda_i)$ for some weight of $\mathfrak{g}$ with respect to $\mathfrak{h}$, i.e., $(x \otimes t^k)v = a_i^k xv$ for any $v \in L_i$, $x \in \mathfrak{g}$ and $k \in \mathbb{N}$. Moreover, λ_i is dominant, and $\lambda_i = 0$ if further L_i is trivial.

Now suppose that $i = 0$ and $n_0 \neq 0$. Then $\mathcal{L}_0$ is nilpotent; furthermore, L_0 is 1-dimensional and there exist a Lie algebra homomorphism $\eta_0\colon \mathcal{L}_0 \to \mathbb{C}$ and $w \in L_0$ such that $xw = \eta_0(x)w$ for all $x \in \mathcal{L}_0$. Again by Lemma 4.2, we get $[\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+]L_0 = 0$. Recalling that $\mathcal{J}(t) \subseteq [\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+]$, so we obtain $\mathcal{J}(t)L_0 = 0$.

If $n_0 = 0$, then $\mathcal{L}_0 = 0$ and hence $\eta = 0$.

From Lemma 2.7 in [35], we know that $S \cong S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$. Part (i) follows easily. For part (ii), computing the left-hand side of $[\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_+] \cap \mathcal{I}(f_{\boldsymbol{a}}) \subset \mathrm{Ann}(S)$, we deduce the statement in (ii). The proof is complete. ∎

5. Classification of $\tilde{\mathfrak{n}}_+$-locally finite irreducible modules

In this section, we will obtain the complete classification of irreducible $\tilde{\mathfrak{g}}$-modules on which each root vector in $\tilde{\mathfrak{n}}_+$ acts locally finitely.

Let $m \in \mathbb{N}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (\mathfrak{h}^*)^m$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_m) \in \widetilde{\mathbb{C}}^m$, and let $\eta: \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ be a Lie algebra homomorphism. We shall denote by $W(\eta)$ an irreducible Whittaker $\hat{\mathfrak{g}}$-module with Whittaker function η if $\eta \neq 0$. For any $\gamma \in \hat{\mathfrak{h}}^*$, we denote by $\hat{V}(\gamma)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module with highest weight γ.

Let $S = S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ be as defined in Section 4, and let $f_{\boldsymbol{a}} = \prod_{i=1}^m (t - a_i)$. Then $\mathcal{I}(tf_{\boldsymbol{a}})S = 0$, and $\mathcal{I}(f_{\boldsymbol{a}})S = 0$ if additionally $\eta = 0$. Recall from (4.4) that

$$\tilde{\mathfrak{n}}_+/\mathcal{I}(tf_{\boldsymbol{a}}) \cong (\tilde{\mathfrak{n}}_+/\mathcal{I}(t)) \oplus (\tilde{\mathfrak{n}}_+/\mathcal{I}(f_a)) \cong (\tilde{\mathfrak{n}}_+/\mathcal{I}(t)) \oplus \bigoplus_{i=1}^m \tilde{\mathfrak{n}}_+/\mathcal{I}(t - a_i).$$

Let π_i be the projection from $\tilde{\mathfrak{n}}_+/\mathcal{I}(f_{\boldsymbol{a}})$ to $\mathfrak{g}^{(i)} = \tilde{\mathfrak{n}}_+/\mathcal{I}(t - a_i)$, which is isomorphic to $\mathfrak{g}$ for all $i = 1, \ldots, m$. We identify $\tilde{\mathfrak{n}}_+/\mathcal{I}(f_{\boldsymbol{a}})$, which can be viewed as an ideal of $\tilde{\mathfrak{n}}_+/\mathcal{I}(tf_{\boldsymbol{a}})$, with $\mathfrak{g}^m$. Then regarded as an $\tilde{\mathfrak{n}}_+/\mathcal{I}(f_{\boldsymbol{a}})$-module, S is a sum of irreducible highest weight modules with the same highest weight $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$.

We consider the induced $\tilde{\mathfrak{g}}$-module

$$\widetilde{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = \mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\tilde{\mathfrak{g}}} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{n}}_+)} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}),$$

and the induced $\hat{\mathfrak{g}}$-module

$$\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = \mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\hat{\mathfrak{g}}} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = U(\hat{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{n}}_+)} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}).$$

Clearly,

$$\widetilde{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = \widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a})[d] = \mathbb{C}[d] \otimes \widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}).$$

Recall the 1-dimensional $\tilde{\mathfrak{n}}_+$-module $\mathbb{C}v_\eta$ defined by $xv_\eta = \eta(x)v_\eta$ for all $x \in \tilde{\mathfrak{n}}_+$. Then the induced $\hat{\mathfrak{g}}$-module

$$\widehat{W}(\eta) = \mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}v_\eta,$$

is the universal Whittaker module with Whittaker function η. Recall that the irreducible $\hat{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is defined in Section 3 (after Lemma 3.1).

Lemma 5.1. *As $\hat{\mathfrak{g}}$-modules, we have $\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) \cong E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta)$.*

Proof. Regarding $E(\boldsymbol{\lambda}, \boldsymbol{a})$ as an $\tilde{\mathfrak{n}}_+$-module, we have the canonical $\tilde{\mathfrak{n}}_+$-module isomorphism $\varphi: S(0, \boldsymbol{\lambda}, \boldsymbol{a}) \to E(\boldsymbol{\lambda}, \boldsymbol{a})$. We get the canonical $\tilde{\mathfrak{n}}_+$-module homomorphism

$$\begin{aligned} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = S(0, \boldsymbol{\lambda}, \boldsymbol{a}) \otimes v_\eta &\to E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta), \\ v \otimes v_\eta &\mapsto \varphi(v) \otimes v_\eta, \quad \forall v \in S(0, \boldsymbol{\lambda}, \boldsymbol{a}), \end{aligned}$$

whose restriction is the canonical $\tilde{\mathfrak{n}}_+$-module isomorphism

$$S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) \to E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes v_\eta.$$

From the universal property, we obtain an induced $\widehat{\mathfrak{g}}$-module homomorphism

$$\widehat{\varphi} : \widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = \operatorname{Ind}_{\tilde{\mathfrak{n}}_+}^{\widehat{\mathfrak{g}}} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) \to E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes (\operatorname{Ind}_{\tilde{\mathfrak{n}}_+}^{\widehat{\mathfrak{g}}} \mathbb{C} v_\eta).$$

Note that $\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = U(\widehat{\mathfrak{b}}_-) \otimes S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ and

$$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes U(\widehat{\mathfrak{b}}_-) \otimes \mathbb{C} v_\eta$$

as vector spaces, where $\widehat{\mathfrak{b}}_- = \widehat{\mathfrak{h}} + \tilde{\mathfrak{n}}_-$. Take a basis $\{b_1, b_2, \dots\}$ of $\widehat{\mathfrak{b}}_-$. We have a PBW basis

$$\{b_n^{r_n} \cdots b_1^{r_1} \mid n \in \mathbb{Z}_+, r_1, \dots, r_n \in \mathbb{Z}_+\}$$

for $U(\widehat{\mathfrak{b}}_-)$. For any $l \in \mathbb{Z}_+$, we denote

$$U(\widehat{\mathfrak{b}}_-)_l = \sum_{\substack{n, r_i \in \mathbb{Z}_+, \\ r_1 + \cdots + r_n = l}} \mathbb{C} b_n^{r_n} \cdots b_1^{r_1}.$$

By induction on $l \in \mathbb{Z}_+$, we can easily deduce the following.

Claim 1. *For any $x \in U(\widehat{\mathfrak{b}}_-)_l, l \in \mathbb{Z}_+$ and $v \in S(0, \boldsymbol{\lambda}, \boldsymbol{a})$,*

$$\widehat{\varphi}\big(x(v \otimes v_\eta)\big) = x(\varphi(v) \otimes v_\eta) \equiv \varphi(v) \otimes (x v_\eta) \mod E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \sum_{i=0}^{l-1} U(\widehat{\mathfrak{b}}_-)_i v_\eta.$$

Noticing that φ is an isomorphism, we see $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \sum_{i=0}^{l-1} U(\widehat{\mathfrak{b}}_-)_i v_\eta \subseteq \widehat{\varphi}(\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}))$ implies $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \sum_{i=0}^{l} U(\widehat{\mathfrak{b}}_-)_i v_\eta \subseteq \widehat{\varphi}(\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}))$ from the claim. Then by induction we can obtain that $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta) \subseteq \widehat{\varphi}(\widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}))$, that is, $\widehat{\varphi}$ is surjective.

Now we prove that $\widehat{\varphi}$ is injective. For any nonzero $w \in \widehat{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$, we write

$$w = \sum_{i=0}^{k} x_i (v_i \otimes v_\eta) \mod \sum_{i=0}^{l-1} U(\widehat{\mathfrak{b}}_-)_i (S(0, \boldsymbol{\lambda}, \boldsymbol{a}) \otimes v_\eta),$$

where $x_i \in U(\widehat{\mathfrak{b}}_-)_l$ and $v_i \in S(0, \boldsymbol{\lambda}, \boldsymbol{a})$. We may assume that the v_i's are linearly independent and each $x_i \neq 0$. Using Claim 1, we see that

$$\widehat{\varphi}(w) \equiv \sum_{i=0}^{k} \varphi(v_i) \otimes (x_i v_\eta) \mod E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \sum_{i=0}^{l-1} U(\widehat{\mathfrak{b}}_-)_i v_\eta.$$

which is nonzero since the $\varphi(v_i)$'s are linearly independent and each $x_i \neq 0$. Thus $\widehat{\varphi}$ is injective. We conclude that $\widehat{\varphi}$ is a $\widehat{\mathfrak{g}}$-module isomorphism. ∎

As a result of Lemma 5.1, we have a $\tilde{\mathfrak{g}}$-module isomorphism

$$\widetilde{\varphi} : \widetilde{M}(\eta,\boldsymbol{\lambda},\boldsymbol{a}) = \mathrm{Ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}}\widehat{M}(\eta,\boldsymbol{\lambda},\boldsymbol{a}) \longrightarrow \mathrm{Ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}}\big(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \widehat{W}(\eta)\big).$$

Note that

$$\mathrm{Ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}}(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \widehat{W}(\eta)) = (E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \widehat{W}(\eta))[d].$$

Thanks to Theorem 3.4, any $\tilde{\mathfrak{g}}$-submodule of $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \widehat{W}(\eta))[d]$ is of the form $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes N)[d]$ for some $\hat{\mathfrak{g}}$-submodule N of $\widehat{W}(\eta)$. Thus any irreducible $\tilde{\mathfrak{g}}$-quotient module of $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \widehat{W}(\eta))[d]$ is of the form $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes W(\eta))[d]$ for some irreducible $\hat{\mathfrak{g}}$-quotient module $W(\eta)$ of $\widehat{W}(\eta)$. Now we can characterize all irreducible quotients of $\hat{\mathfrak{g}}$-modules $\widehat{W}(\eta)$.

Lemma 5.2. *Let $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ be a Lie algebra homomorphism, and let $\widehat{W}(\eta)$ be the universal Whittaker module over $\hat{\mathfrak{g}}$.*

(1) *If $\eta \neq 0$, then any irreducible quotient module of $\widehat{W}(\eta)$ is an irreducible Whittaker $\hat{\mathfrak{g}}$-module of type η.*

(2) *If $\eta = 0$, then any irreducible quotient module of $\widehat{W}(\eta)$ is an irreducible highest weight module $\hat{V}(\gamma)$ over $\hat{\mathfrak{g}}$ with some highest weight $\gamma \in \hat{\mathfrak{h}}^*$.*

Proof. Part (1) is clear by definition. Now suppose $\eta = 0$, and W is an irreducible quotient $\hat{\mathfrak{g}}$-module of $\widehat{W}(0)$. Consider the induced $\tilde{\mathfrak{g}}$-module $\widetilde{W} = W[d] = \mathrm{ind}_{\hat{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} W$.

If $\widetilde{W}$ is irreducible, using Theorem 1 in [36] we know that $\widetilde{W}$ is an irreducible highest weight $\tilde{\mathfrak{g}}$-module, which is a contradiction to the fact that the action of d is free on $\widetilde{W}$.

So $\widetilde{W}$ is not irreducible. Let Y be a nonzero proper $\tilde{\mathfrak{g}}$-submodule of $\widetilde{W}$. It is obvious that $Y \cap W = 0$, since otherwise

$$\widetilde{W} = U(\tilde{\mathfrak{g}})W = U(\tilde{\mathfrak{g}})U(\hat{\mathfrak{g}})(Y \cap W) \subseteq Y.$$

Because W is irreducible over $\hat{\mathfrak{g}}$, then $\widetilde{W}$ is generated by one vector, and it has a maximal submodule over $\tilde{\mathfrak{g}}$. Take a maximal $\tilde{\mathfrak{g}}$-submodule X of $\widetilde{W}$ and choose any

$$w = d^n \otimes w_n + d^{n-1} \otimes w_{n-1} + \cdots + w_0 \in X, \quad w_i \in W,\ w_n \neq 0$$

such that n is minimal. For any $x \in U(\hat{\mathfrak{g}})$, $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$, by computing $d^l x w \in X$ and noticing that W is an irreducible $\hat{\mathfrak{g}}$-module, we see that for any $v \in W$ there is an element $d^{n+l} \otimes v + \sum_{i=0}^{n+l-1} d^i \otimes v_i \in X$ for some $v_i \in W$, and hence

$$X + W \oplus (d \otimes W) \oplus \cdots \oplus (d^n \otimes W) = \widetilde{W}.$$

In particular,

$$\widetilde{W}/X \cong \frac{W \oplus (d \otimes W) \oplus \cdots \oplus (d^n \otimes W)}{X \cap \big(W \oplus (d \otimes W) \oplus \cdots \oplus (d^n \otimes W)\big)}$$

is a nontrivial irreducible $\tilde{\mathfrak{g}}$-module. Using Theorem 1 in [36], we know that $n = 1$ and that $\widetilde{W}/X$ is a highest weight $\tilde{\mathfrak{g}}$-module. If we consider $\widetilde{W}/X$ as a $\hat{\mathfrak{g}}$-module, we have $\widetilde{W}/X \cong W$. Hence W is a highest weight $\hat{\mathfrak{g}}$-module. Part (2) follows. ∎

Using the above two lemmas, we can determine all irreducible $\tilde{\mathfrak{g}}$-modules such that the actions of root vectors in $\tilde{\mathfrak{n}}_+$ are locally finite.

Theorem 5.3. *Let V be an irreducible $\tilde{\mathfrak{g}}$-module on which the action of each root vector in $\tilde{\mathfrak{n}}_+$ is locally finite. Then V is isomorphic to one of the following $\tilde{\mathfrak{g}}$-modules*:

(1) *an irreducible highest weight $\tilde{\mathfrak{g}}$-module $\widetilde{V}(\gamma)$ for some $\gamma \in \tilde{\mathfrak{h}}^*$;*

(2) *an irreducible Whittaker $\tilde{\mathfrak{g}}$-module $W(\eta)$ for a nonzero Lie algebra homomorphism $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$;*

(3) *$(\hat{V}(\gamma) \otimes E(\boldsymbol{\lambda}, \boldsymbol{a}))[d]$ for some $\gamma \in \hat{\mathfrak{h}}^*$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$ with all λ_i dominant, where $m \in \mathbb{N}$ and $\hat{V}(\gamma)$ is the irreducible highest weight module over $\hat{\mathfrak{g}}$;*

(4) *$(W(\eta) \otimes E(\boldsymbol{\lambda}, \boldsymbol{a}))[d]$ for a nonzero Lie algebra homomorphism $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$ with all λ_i dominant, where $m \in \mathbb{N}$ and $W(\eta)$ is an irreducible Whittaker module of type η over $\hat{\mathfrak{g}}$.*

Proof. By Theorem 4.1, V has a finite dimensional $\tilde{\mathfrak{n}}_+$-submodule and hence an irreducible finite dimensional $\tilde{\mathfrak{n}}_+$-submodule, say S. If this S is one-dimensional, then $\mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\hat{\mathfrak{g}}} S$ is a universal Whittaker $\hat{\mathfrak{g}}$-module, furthermore V is a simple quotient of $(\mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\hat{\mathfrak{g}}} S)[d]$. We see that V is a highest weigh $\tilde{\mathfrak{g}}$-module or an irreducible Whittaker $\tilde{\mathfrak{g}}$-module by Lemma 5.2 and Theorem 3.4.

Now suppose that $\dim S > 1$. By Theorem 4.1 and Lemmas 4.2 and 4.3, this irreducible $\tilde{\mathfrak{n}}_+$-submodule is isomorphic to $S(\eta, \boldsymbol{\lambda}, \boldsymbol{a})$ for a Lie algebra homomorphism $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ with all λ_i dominant and at least one nonzero. Then we know that V is an irreducible quotient of the $\tilde{\mathfrak{g}}$-module

$$\widetilde{M}(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = \mathrm{Ind}_{\tilde{\mathfrak{n}}_+}^{\tilde{\mathfrak{g}}} S(\eta, \boldsymbol{\lambda}, \boldsymbol{a}) = (E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \widehat{W}(\eta))[d].$$

The theorem follows from Theorem 3.4, and Lemmas 5.1 and 5.2. ∎

6. Isomorphism classes of irreducible $\tilde{\mathfrak{g}}$-modules

In this section, we determine the necessary and sufficient conditions for two irreducible modules of the form $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \hat{V}(\gamma))[d]$ or $(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes W(\eta))[d]$ to be isomorphic, where $\eta\colon \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ is a nonzero Lie algebra homomorphism, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in (\mathfrak{h}^*)^m \setminus \{0\}$ and $\gamma \in \hat{\mathfrak{h}}^*$. Recall that $\hat{V}(\gamma)$ is the irreducible highest weight $\hat{\mathfrak{g}}$-module of highest weight γ and $W(\eta)$ is an irreducible Whittaker $\hat{\mathfrak{g}}$-module of type η.

Lemma 6.1. *Let M and M' be irreducible $\hat{\mathfrak{g}}$-modules. Then $M[d] \cong M'[d]$ as $\tilde{\mathfrak{g}}$-modules if and only if $M \cong M'$ as $\hat{\mathfrak{g}}$-modules.*

Proof. The sufficiency is clear. We need only to prove the necessity. Let $\widetilde{\varphi}\colon M[d] \to M'[d]$ be a $\tilde{\mathfrak{g}}$-module isomorphism. We can define a map $\varphi\colon M \to M'$ as follows. Take any nonzero $u \in M$, there exist unique $l_u \in \mathbb{Z}_+$ and $v_i \in M'$ with $v_{l_u} \neq 0$ such that $\widetilde{\varphi}(u) = \sum_{i=0}^{l_u} d^i v_i$. Define $\varphi(u) = v_{l_u}$. Note that for any $x \in \hat{\mathfrak{g}}$, we have

$$\widetilde{\varphi}(xu) = x\widetilde{\varphi}(u) = \sum_{i=0}^{l_u} x(d^i v_i) = d^{l_u} x v_{l_u} + \sum_{i=0}^{l_u - 1} d^i x v_i + \sum_{i=0}^{l_u} [x, d^i] v_i. \tag{6.1}$$

This, together with the fact that $M = \mathcal{U}(\widehat{\mathfrak{g}})u$, implies that $l_{u'} \le l_u$ for all $u' \in M$ and hence $l_u = l_{u'}$ for all nonzero $u, u' \in M$. Then the linearity of φ_i follows from its definition and (6.1) yields $\varphi(xu) = xv_{l_u} = x\varphi(u)$. As a result, φ is a nonzero $\widehat{\mathfrak{g}}$-module homomorphism and hence an isomorphism between $\widehat{\mathfrak{g}}$-modules M and M'. ∎

Lemma 6.2. *Let $m, m' \in \mathbb{N}$, $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$, $\boldsymbol{\lambda}' \in (\mathfrak{h}^*)^{m'}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $\boldsymbol{a}' \in \widetilde{\mathbb{C}}^{m'}$, and let L, L' be smooth $\widehat{\mathfrak{g}}$-modules. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \cong E(\boldsymbol{\lambda}', \boldsymbol{a}') \otimes L'$ if and only if $E(\boldsymbol{\lambda}, \boldsymbol{a}) \simeq E(\boldsymbol{\lambda}', \boldsymbol{a}')$ and $L \cong L'$.*

Proof. The sufficiency is clear. We need only to prove the necessity. We know that both $E(\boldsymbol{\lambda}, \boldsymbol{a})$ and $E(\boldsymbol{\lambda}', \boldsymbol{a}')$ are irreducible $\widehat{\mathfrak{g}}$-modules. Let

$$\widetilde{\varphi} : E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \to E(\boldsymbol{\lambda}', \boldsymbol{a}') \otimes L'$$

be a $\widehat{\mathfrak{g}}$-module isomorphism. Fix nonzero $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$ and nonzero $w \in L$. Let

$$\widetilde{\varphi}(v \otimes w) = \sum_{i=1}^{s} v_i \otimes w_i,$$

where $v_i \in E(\boldsymbol{\lambda}', \boldsymbol{a}')$ and where the $w_i \in L'$ are linearly independent. We may also assume that each $v_i \neq 0$. Choose any $k \in \mathbb{N}$ such that

$$(\mathfrak{g} \otimes t^k\mathbb{C}[t])w = (\mathfrak{g} \otimes t^k\mathbb{C}[t])w_i = 0, \quad \text{for all } i.$$

Note that

$$\widetilde{\varphi}(xv \otimes w) = x\widetilde{\varphi}(v \otimes w) = \sum_{i=1}^{s} xv_i \otimes w_i, \quad \forall x \in \mathfrak{g} \otimes t^k\mathbb{C}[t]. \tag{6.2}$$

Combining this with the fact that $E(\boldsymbol{\lambda}, \boldsymbol{a}) = U(\widehat{\mathfrak{g}})v$ and using Lemma 3.1, we see that, for any $u \in E(\boldsymbol{\lambda}, \boldsymbol{a})$,

$$\widetilde{\varphi}(u \otimes w) = \sum_{i=1}^{s} u_i \otimes w_i$$

where $u_i \in E(\boldsymbol{\lambda}', \boldsymbol{a}')$. Now, we define nonzero vector space homomorphisms $\varphi_i : E(\boldsymbol{\lambda}, \boldsymbol{a}) \to E(\boldsymbol{\lambda}', \boldsymbol{a}')$ by $\varphi_i(u) = u_i$ for all $u \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, and we have

$$\widetilde{\varphi}(u \otimes w) = \sum_{i=1}^{s} \varphi_i(u) \otimes w_i, \quad \forall u \in E(\boldsymbol{\lambda}, \boldsymbol{a}).$$

Note that (6.2) with v replaced by u indicates that $\varphi_i(xu) = x\varphi_i(u)$ for $x \in \mathfrak{g} \otimes t^k\mathbb{C}[t]$ and $u \in E(\boldsymbol{\lambda}, \boldsymbol{a})$. The irreducibility of $E(\boldsymbol{\lambda}', \boldsymbol{a}')$ and Lemma 3.1 show that all φ_i are surjective, i.e., $\varphi_i(E(\boldsymbol{\lambda}, \boldsymbol{a})) = E(\boldsymbol{\lambda}', \boldsymbol{a}')$.

Recall $f_{\boldsymbol{a}}(t) = \prod_{j=1}^{m}(t - a_j)$, and let $f_{\boldsymbol{\lambda},\boldsymbol{a}}(t) = \prod_{j=1,\lambda_j \neq 0}^{m}(t - a_j)$. Define similarly $f_{\boldsymbol{a}'}$ and $f_{\boldsymbol{\lambda}',\boldsymbol{a}'}$. Now we have $0 = \varphi_i(xu) = x\varphi_i(u)$ for all $u \in E(\boldsymbol{\lambda}, \boldsymbol{a})$ and $x \in f_{\boldsymbol{\lambda},\boldsymbol{a}}(t)t^k\mathbb{C}[t]$. Since φ_i is surjective, we get $xE(\boldsymbol{\lambda}', \boldsymbol{a}') = 0$ for all $x \in f_{\boldsymbol{\lambda},\boldsymbol{a}}(t)t^k\mathbb{C}[t]$, yielding from Remark 3.2 that

$$f_{\boldsymbol{\lambda},\boldsymbol{a}}(t)t^k \in \mathrm{Ann}(E(\boldsymbol{\lambda}', \boldsymbol{a}')) = f_{\boldsymbol{\lambda}',\boldsymbol{a}'}(t)\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K.$$

Thus we get $f_{\boldsymbol{\lambda}',\boldsymbol{a}'}$ divides $f_{\boldsymbol{\lambda},\boldsymbol{a}}$, and symmetrically $f_{\boldsymbol{\lambda},\boldsymbol{a}}$ divides $f_{\boldsymbol{\lambda},\boldsymbol{a}'}$. In particular,

$$\begin{aligned} f_{\boldsymbol{\lambda},\boldsymbol{a}} &= f_{\boldsymbol{\lambda},\boldsymbol{a}'}, \quad \operatorname{Ann}(E(\boldsymbol{\lambda},\boldsymbol{a})) = \operatorname{Ann}(E(\boldsymbol{\lambda}',\boldsymbol{a}')) \\ \text{and} \quad \varphi_i(xu) &= 0 = x\varphi_i(u) \quad \text{for } x \in \mathfrak{g} \otimes f_{\boldsymbol{\lambda},\boldsymbol{a}}\mathbb{C}[t^{\pm 1}]. \end{aligned}$$

Since

$$\mathfrak{g} \otimes f_{\boldsymbol{\lambda},\boldsymbol{a}}\mathbb{C}[t^{\pm 1}] + \mathfrak{g} \otimes t^k\mathbb{C}[t] = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}],$$

we obtain that $\varphi_i(xu) = x\varphi_i(u)$ for $x \in \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Hence each φ_i is a $\widehat{\mathfrak{g}}$-module homomorphism. Since $E(\boldsymbol{\lambda},\boldsymbol{a})$ is a simple module, $\varphi_i \neq 0$ is also injective. Consequently, each φ_i is a $\widehat{\mathfrak{g}}$-module isomorphism. Thus $E(\boldsymbol{\lambda},\boldsymbol{a}) \simeq E(\boldsymbol{\lambda}',\boldsymbol{a}')$.

It is well known that any endomorphism of a simple module over a countably generated associative $\mathbb{C}$-algebra is a multiplication by a scalar (see, for instance, Proposition 2.6.5 and Corollary 2.6.6 in [18]). There exist $a_i \in \mathbb{C}^*$, $i = 1, \ldots, s$, such that $\varphi_i = a_i\varphi_1$. Let $w' = \sum_{i=1}^{s} a_i w_i$. Then

$$\widetilde{\varphi}(u \otimes w) = \varphi_1(u) \otimes w', \quad \forall u \in E(\boldsymbol{\lambda},\boldsymbol{a}).$$

Applying $x \in \widehat{\mathfrak{g}}$ to both sides, we deduce that

$$\begin{aligned} \widetilde{\varphi}(x(u \otimes w)) &= \widetilde{\varphi}(xu \otimes w) + \widetilde{\varphi}(u \otimes xw) = \varphi_1(xu) \otimes w' + \widetilde{\varphi}(u \otimes xw) \\ &= (x\varphi_1(u)) \otimes w' + \widetilde{\varphi}(u \otimes xw), \\ x\widetilde{\varphi}(u \otimes w) &= (x\varphi_1(u)) \otimes w' + \varphi_1(u) \otimes xw' \end{aligned}$$

i.e., $\widetilde{\varphi}(u \otimes xw) = \varphi_1(u) \otimes xw'$. Thus we have a vector space homomorphism $\varphi' : L \to L'$ such that

$$\widetilde{\varphi}(u \otimes y) = \varphi_1(u) \otimes \varphi'(y), \quad \forall u \in E(\boldsymbol{\lambda},\boldsymbol{a}),\ y \in L.$$

Applying $x \in \widehat{\mathfrak{g}}$ to both sides, we deduce that $\varphi'(xy) = x\varphi'(y)$ for all $x \in \widehat{\mathfrak{g}}$ and all $y \in L$. That is, $\varphi' : L \to L'$ is a nonzero $\widehat{\mathfrak{g}}$-module homomorphism. Since $\widetilde{\varphi}$ is an isomorphism of $\tilde{\mathfrak{g}}$-modules, we obtain that φ' is an isomorphism of $\widehat{\mathfrak{g}}$-modules, i.e., $L \simeq L'$. ■

The necessary and sufficient condition for the isomorphism $E(\boldsymbol{\lambda},\boldsymbol{a}) \simeq E(\boldsymbol{\lambda}',\boldsymbol{a}')$ is well known, namely, $E(\boldsymbol{\lambda},\boldsymbol{a}) \simeq E(\boldsymbol{\lambda}',\boldsymbol{a}')$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, and $\boldsymbol{a}$ are proportional to $\boldsymbol{a}'$ after reindexing $a_1, \ldots, a_m$ and $\lambda_1, \ldots, \lambda_m$ simultaneously if necessary, see Theorem 1.3 (d) in [10]. Combining Lemma 6.1, Lemma 6.2 and highest weight representation theory of $\widehat{\mathfrak{g}}$, we can deduce the isomorphisms among the irreducible $\tilde{\mathfrak{g}}$-modules $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \hat{V}(\gamma))[d]$ and $(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes W(\eta))[d]$.

Theorem 6.3. *Let $m, m' \in \mathbb{N}$, $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m \setminus \{0\}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $\boldsymbol{\lambda}' \in (\mathfrak{h}^*)^{m'}$, $\boldsymbol{a}' \in \widetilde{\mathbb{C}}^{m'}$ and $\gamma, \gamma' \in \widehat{\mathfrak{h}}^*$. Let $\eta, \eta' : \tilde{\mathfrak{n}}_+ \to \mathbb{C}$ be nonzero Lie algebra homomorphisms.*

(1) *$(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \hat{V}(\gamma))[d]$ and $(E(\boldsymbol{\lambda}',\boldsymbol{a}') \otimes W(\eta))[d]$ are never isomorphic;*

(2) *$(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes \hat{V}(\gamma))[d] \simeq (E(\boldsymbol{\lambda}',\boldsymbol{a}') \otimes V(\gamma'))[d]$ as $\tilde{\mathfrak{g}}$-modules if and only if $E(\boldsymbol{\lambda},\boldsymbol{a}) \simeq E(\boldsymbol{\lambda}',\boldsymbol{a}')$ as $\widehat{\mathfrak{g}}$-modules and $\gamma = \gamma'$;*

(3) *$(E(\boldsymbol{\lambda},\boldsymbol{a}) \otimes W(\eta))[d] \simeq (E(\boldsymbol{\lambda}',\boldsymbol{a}') \otimes W(\eta'))[d]$ as $\tilde{\mathfrak{g}}$-modules if and only if $E(\boldsymbol{\lambda},\boldsymbol{a}) \simeq E(\boldsymbol{\lambda}',\boldsymbol{a}')$ and $W(\eta) \cong W(\eta')$ as $\widehat{\mathfrak{g}}$-modules.*

7. Tensor products of highest weight modules and loop modules over untwisted affine Kac–Moody algebras

In this last section, we will use the "shifting technique" to obtain a necessary and sufficient condition for the tensor product of irreducible integrable loop modules and irreducible integrable highest weight modules over untwisted affine Kac–Moody algebras $\tilde{\mathfrak{g}}$ to be simple. This problem was started by Chari and Pressley in 1987 in [10], and further studied by them and other authors later, for example [1–3]. We will start with a slightly more general setting.

For any $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in (\mathfrak{h}^*)^m$, we have the irreducible $\hat{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a})$. For any $b \in \mathbb{C}$, recall from [19] or [10] the $\tilde{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \mathbb{C}[t, t^{-1}]$, with actions

$$(7.1)\qquad \begin{aligned} (x \otimes t^k)(v \otimes t^n) &= ((x \otimes t^k)v) \otimes t^{n+k}, \\ K(v \otimes t^n) = 0, &\quad d(v \otimes t^n) = (b+n)(v \otimes t^n), \end{aligned}$$

for all $k, n \in \mathbb{Z}$, $x \in \mathfrak{g}$, and $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$. Note that the $\tilde{\mathfrak{g}}$-modules $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ are integrable weight $\tilde{\mathfrak{g}}$-modules if all λ_i are dominant.

Let us recall from [10] the graded associative algebra homomorphism χ defined by

$$(7.2)\qquad \begin{aligned} \chi : U(\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]) &\to \mathbb{C}[t, t^{-1}], \\ h \otimes t^n &\mapsto \Big(\sum_{i=1}^{m} \lambda_i(h)\, a_i^n \Big) t^n. \end{aligned}$$

By Theorem 1.3 (c) in [10], we see that $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible if and only if χ is surjective. In other words, for all $k \in \mathbb{Z}$, there exist $h_k \in U(\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}])$ with $[d, h_k] = kh_k$ such that $\chi(h_k) \neq 0$. Although the proof in [10] is for only finite dimensional $E(\boldsymbol{\lambda}, \boldsymbol{a})$, it is valid even for any infinite dimensional $E(\boldsymbol{\lambda}, \boldsymbol{a})$.

From now on, we will fix an irreducible $\tilde{\mathfrak{g}}$-module such that

$$(7.3)\qquad L = \sum_{i \in \mathbb{Z}_+} L_{-i}, \quad \text{where } L_{-i} = \{v \in L \mid dv = (\zeta - i)v\}$$

for some $\zeta \in \mathbb{C}$ with $L_0 \neq 0$. We call ζ the highest degree of L. Note that irreducible highest weight $\tilde{\mathfrak{g}}$-modules are such modules. There are certainly a lot of other such irreducible modules. Similarly, we have

$$U(\tilde{\mathfrak{g}}) = \sum_{i \in \mathbb{Z}} U(\tilde{\mathfrak{g}})_i, \quad \text{where } U(\tilde{\mathfrak{g}})_i = \{x \in U(\tilde{\mathfrak{g}}) \mid [d, x] = ix\}.$$

Any vector $x \in U(\tilde{\mathfrak{g}})_i$ is called homogeneous of degree i, denoted by $\deg(x) = i$. Clearly $U(\tilde{\mathfrak{g}})_i L_j \subseteq L_{i+j}$, where we make the convention that $L_j = 0$ for $j \in \mathbb{N}$.

Lemma 7.1. *The above irreducible $\tilde{\mathfrak{g}}$-module L is also irreducible as a $\hat{\mathfrak{g}}$-module.*

Proof. Let W be a nonzero $\hat{\mathfrak{g}}$-submodule of L. Take any nonzero $w \in W$ and write $w = \sum_{i=0}^{n} u_i$, with $u_i \in L_{-i}$ and $u_n \neq 0$. Since L is simple as a $\tilde{\mathfrak{g}}$-module, there exist $x_1, \ldots, x_s \in \mathfrak{g}$ and $j_1, \ldots, j_s \in \mathbb{Z}_+$ with $j_1 + \cdots + j_s = n$ such that

$$w_0 = (x_1 \otimes t^{j_1}) \cdots (x_s \otimes t^{j_s}) w \in L_0 \setminus \{0\}.$$

Then the $\hat{\mathfrak{g}}$-submodule generated by w_0 is a $\tilde{\mathfrak{g}}$-submodule of L. Hence $W = L$. Consequently, L is irreducible as a $\hat{\mathfrak{g}}$-module. ∎

We will use the "shifting technique" established in [14] to study the tensor product $\tilde{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$. Define a new action of $\tilde{\mathfrak{g}}$ on the vector space $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \otimes \mathbb{C}[t^{\pm 1}]$ by

$$\begin{aligned}(x \otimes t^k)(v \otimes u \otimes t^n) &= \big((x \otimes t^k)v \otimes u + v \otimes (x \otimes t^k)u\big) \otimes t^{n+k},\\ d(v \otimes u \otimes t^n) &= (\zeta + b + n)(v \otimes u \otimes t^n),\\ K(v \otimes u \otimes t^n) &= v \otimes Ku \otimes t^n,\end{aligned} \tag{7.4}$$

where $b \in \mathbb{C}$, $u \in L$, $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, $x \in \mathfrak{g}$ and $k, n \in \mathbb{Z}$. Denote this $\tilde{\mathfrak{g}}$-module as $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$. On the other hand, we can form the tensor product $\tilde{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$. Actually, we have the $\tilde{\mathfrak{g}}$-module isomorphism $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L) \cong E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$ given by

$$v \otimes u \otimes t^n \mapsto (v \otimes t^{n+i}) \otimes u, \quad \forall u \in L_{-i}, v \in E(\boldsymbol{\lambda}, \boldsymbol{a}), n \in \mathbb{Z}.$$

In the rest of this section, we will consider the module $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$, instead of $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$, to deduce its irreducibility. For any $p, k \in \mathbb{N}$, define

$$U_k\big(\mathfrak{h} \otimes (t^p\mathbb{C}[t])\big) = \{x \in U\big(\mathfrak{h} \otimes (t^p\mathbb{C}[t])\big) \mid [d, x] = kx\}.$$

We first prove a new property of the function χ defined in (7.2).

Lemma 7.2. *Suppose that the function χ defined in (7.2) is surjective. Then for any $p \in \mathbb{N}$, we have $\chi(U_k(\mathfrak{h} \otimes (t^p\mathbb{C}[t]))) \neq 0$ for all sufficiently large $k \in \mathbb{N}$.*

Proof. Let
$$S_p = \{k \in \mathbb{N} \mid \chi(U_k(\mathfrak{h} \otimes (t^p\mathbb{C}[t])) \neq 0\},$$
and let $\langle S_p \rangle$ be the subgroup of $\mathbb{Z}$ generated by S_p. It is enough to show that $\langle S_p \rangle = \mathbb{Z}$. Otherwise suppose that $\langle S_p \rangle = n_1\mathbb{Z} \neq \mathbb{Z}$, where $n_1 > 1$. Then $\chi(\mathfrak{h} \otimes t^{q+n_1k}) = 0$ for all $k \geq p$ and all $q = 1, 2, \ldots, n_1 - 1$, i.e.,

$$\sum_{i=1}^{m} \lambda_i(h)\, a_i^q (a_i^{n_1})^k = 0, \quad \forall h \in \mathfrak{h}, k > p.$$

Note that although $a_i \neq a_j$ for $i \neq j$, we can have $a_i^{n_1} = a_j^{n_1}$ for some $i \neq j$. Let now $S_1, S_2, \ldots, S_s$ be the partition of $\{1, 2, \ldots, m\}$ so that

$$i, j \in S_k \iff a_i^{n_1} = a_j^{n_1}.$$

We deduce that

$$\sum_{i \in S_j} \lambda_i(h)\, a_i^q = 0, \quad \forall h \in \mathfrak{h}, q = 1, 2, \ldots, n_1 - 1.$$

Equivalently,
$$\sum_{i \in S_j} \lambda_i(h)\, a_i^n = 0, \quad \forall h \in \mathfrak{h},\ n \notin n_1\mathbb{Z}.$$

We see that χ is not surjective, contradicting the assumption. The lemma is true. ∎

Lemma 7.3. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$, and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Let W be a nonzero $\tilde{\mathfrak{g}}$-submodule of $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \otimes t^n \subseteq W$ for all sufficiently large $n \in \mathbb{Z}$.*

Proof. Denote $\mathcal{E} = \mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$ and $\mathcal{E}_n = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \otimes t^n$ for convenience. Elements in $\mathcal{E}_n$ are called homogeneous. Denote $f_{\boldsymbol{a}}(t) = \prod_{i=1}^m (t - a_i)$ as before. Write $W = \sum_{n \in \mathbb{Z}} W_n \otimes t^n$, where $W_n \subseteq E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L$ for all $n \in \mathbb{Z}$.

Choose any $n' \in \mathbb{Z}$ such that $W_{n'} \neq 0$ and take a nonzero $w \in W_{n'}$. We can write $w = \sum_{i=1}^r w_i \otimes u_i$ with $0 \neq w_i \in E(\boldsymbol{\lambda}, \boldsymbol{a})$ and $u_1, \dots, u_r \in L$ such that $u_1, \dots, u_r$ are linearly independent. There is $p \in \mathbb{Z}_+$ such that $(\mathfrak{g} \otimes t^p \mathbb{C}[t]) u_i = 0$ for all i.

Regard $E(\boldsymbol{\lambda}, \boldsymbol{a})$ as a weight module over $\mathcal{L}_{\boldsymbol{a}}$ (defined in (3.1)) with respect to the Cartan subalgebra

$$\mathfrak{h}_{\boldsymbol{a}} = \big(\mathfrak{h} \otimes t^p \mathbb{C}[t]\big) / \big((\mathfrak{h} \otimes t^p \mathbb{C}[t]) \cap (\mathfrak{h} \otimes f_{\boldsymbol{a}} \mathbb{C}[t])\big).$$

By (3.2) and Lemma 3.1, we see that

$$\big(\mathfrak{n}_+ \otimes t^p \mathbb{C}[t]\big) / \big((\mathfrak{n}_+ \otimes t^p \mathbb{C}[t]) \cap (\mathfrak{n}_+ \otimes f_{\boldsymbol{a}} \mathbb{C}[t])\big)$$

is just the positive part (nilpotent radical) of $\mathcal{L}_a$ with respect to $\mathfrak{h}_{\boldsymbol{a}}$.

Recall that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is a highest weight module over the semisimple Lie algebra $\mathcal{L}_{\boldsymbol{a}}$ with highest weight $\boldsymbol{\lambda}$ (see the comments before Remark 3.2). By $v_{\boldsymbol{\lambda}}$ we denote the highest weight vector of $E(\boldsymbol{\lambda}, \boldsymbol{a})$. For any weight $\boldsymbol{\mu} \in (\mathfrak{h}^*)^m$, we say that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ if $\lambda_i - \mu_i$ is dominant for all $i = 1, \dots, m$. For any element $v = v_1 + \dots + v_s \in E(\boldsymbol{\lambda}, \boldsymbol{a})$ such that the v_i's are nonzero weight vectors of distinct weights $\boldsymbol{\mu}_i$, denote $\mathrm{wt}(v) = \min\{\boldsymbol{\mu}_i, i = 1, \dots, s\}$.

If there is w_i so that $\mathrm{wt}(w_i) \neq \boldsymbol{\lambda}$, say, $\mathrm{wt}(w_1) < \boldsymbol{\lambda}$, then w_1 cannot be annihilated by the whole positive part of $\mathcal{L}_{\boldsymbol{a}}$. Hence $x w_1 \neq 0$ for some homogeneous $x \in \mathfrak{n}_+ \otimes t^p \mathbb{C}[t]$. Then we have

$$x(w \otimes t^{n'}) = x\Big(\sum_{i=1}^r w_i \otimes u_i \otimes t^{n'}\Big) = \sum_{i=1}^r x w_i \otimes u_i \otimes t^{n' + \deg(x)},$$

which is nonzero by the linear independence of $u_1, \dots, u_r$. It is clear that $\mathrm{wt}(x w_i) > \mathrm{wt}(w_i)$ for $i = 1, 2, \dots, r$. Repeating this process if necessary, we can obtain a nonzero element $w' = \sum_{i=1}^r v_i' \otimes u_i \otimes t^{n''} \in W$ with either $\mathrm{wt}(v_i') = \boldsymbol{\lambda}$ or $v_i' = 0$ for all $i = 1, \dots, r$. Then $w' = v_{\boldsymbol{\lambda}} \otimes u \otimes t^{n''}$ for some nonzero $u \in L$. By replacing w with w' and n' with n'', we may assume that $w = v_{\boldsymbol{\lambda}} \otimes u \in W_{n'}$.

Claim 1. *$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u \subseteq W_n$ for all sufficiently large n.*

From Lemma 7.2, we can find $k' \in \mathbb{N}$ such that there exist $h_k \in U_k(\mathfrak{h} \otimes (t^p \mathbb{C}[t]))$ with $\chi(h_k) \neq 0$ for all $k > k'$. By (3.3), we can find homogeneous $x_1, x_2, \dots, x_q \in U(\mathfrak{g} \otimes t^p \mathbb{C}[t]) \setminus \mathbb{C}$ such that $E(\boldsymbol{\lambda}, \boldsymbol{a}) = \mathrm{span}\{x_i v_{\boldsymbol{\lambda}} \mid i = 1, 2, \dots, q\}$, which is finite dimensional. Let $\deg(x_i) = l_i$ and $l = \max\{l_i, 1 \leq i \leq q\}$. Then for any $k \geq k' + l$, we have

$$x_i h_{k - l_i}(v_{\boldsymbol{\lambda}} \otimes u \otimes t^{n'}) = \chi(h_{k - l_i}) x_i v_{\boldsymbol{\lambda}} \otimes u \otimes t^{n' + k},$$

which implies that $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u \otimes t^k \in W$ for all $k \geq n' + k' + l$. The claim follows.

Claim 2. *$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \subseteq W_n$ for all sufficiently large n.*

By Claim 1, we suppose that $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u \otimes t^n \subseteq W$ for all $n \geq n_1$ where $n_1 \in \mathbb{Z}$ is fixed. Let $u \in \bigoplus_{i=0}^{m'} L_{-i}$ with minimal $m' \in \mathbb{Z}_+$. Since L is simple as a $\tilde{\mathfrak{g}}$-module, there exist $y_1, \ldots, y_s \in \mathfrak{g}$ and $j_1, \ldots, j_s \in \mathbb{Z}_+$, with $j_1 + \cdots + j_s = m'$, such that

$$w_0 = (y_s \otimes t^{j_s}) \cdots (y_1 \otimes t^{j_1}) u \in L_0 \setminus \{0\}.$$

By induction on r, we see that

$$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes (y_r \otimes t^{j_r}) \cdots (y_1 \otimes t^{j_1}) u \otimes t^{n+j_1+\cdots+j_r} \subseteq W, \quad \forall n \geq n_1.$$

So

$$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes w_0 \otimes t^{n+m'} \subseteq W, \quad \forall n \geq n_1.$$

Let

$$K = \{v \in L \mid E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes v \otimes t^{n+m'} \subseteq W, \forall n \geq n_1\}.$$

It is enough to prove that $L_{-l} \subset K$ by induction on $l \in \mathbb{Z}_+$. This is clear for $l = 0$. From Lemma 7.1, we know that

$$\sum_{k=1}^{l+1} (\mathfrak{g} \otimes t^{-k}) L_{k-l-1} = L_{-l-1}.$$

One can easily deduce that

$$\begin{aligned} E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L_{-l-1} \otimes t^n &= E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \Big(\sum_{k=1}^{l+1} (\mathfrak{g} \otimes t^{-k}) L_{k-l-1} \Big) \otimes t^n \\ &\subseteq W + \sum_{k=1}^{l+1} (\mathfrak{g} \otimes t^{-k}) (E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L_{k-l-1} \otimes t^{n+k}) \subset W, \quad \forall n \geq n_1 + m'. \end{aligned}$$

Thus $L_{-l} \subset K$ for all $l \in \mathbb{Z}_+$, i.e., $K = L$. Claim 2 follows. Hence the lemma is true. ■

The following corollary will be used later in this paper.

Corollary 7.4. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Let W be a submodule of $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$, $n_0 \in \mathbb{Z}$ and $u_0 \in L_0 \setminus \{0\}$. If $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_0 \otimes t^n \subseteq W$ for all $n \geq n_0$, then $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L \otimes t^n \subseteq W$ for all $n \geq n_0$.*

Proof. Taking $u = u_0$ and $n_1 = n_0$ in the proof of Claim 2 of Lemma 7.3, we can deduce the result. ■

Noticing that

$$\tilde{\mathfrak{n}}_- = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus \mathfrak{n}_-,$$

we have

$$U(\tilde{\mathfrak{n}}_-) = U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{n}_-).$$

Fix any nonzero $u_0 \in L_0$. Then we can define a linear map

$$\varphi : E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes U(\tilde{\mathfrak{n}}_-) \to E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L_0, \tag{7.5}$$

by

$$\varphi(v \otimes xy) = (x \circ v) \otimes yu_0,$$

for all $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, $x \in U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ and $y \in U(\mathfrak{n}_-)$, where the action $\circ$ is defined by $1 \circ v = v$ and

$$(x_1 x_2 \cdots x_s) \circ v = (-1)^s x_s \cdots x_2 x_1 v, \quad \forall x_i \in \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

It is easy to see that φ is well-defined, and

$$\varphi(xv \otimes y + v \otimes xy) = 0, \quad \forall x \in \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}], y \in U(\tilde{\mathfrak{n}}_-), \text{ and } v \in E(\boldsymbol{\lambda}, \boldsymbol{a}).$$

Both $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes U(\tilde{\mathfrak{n}}_-)$ and $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L_0$ are tensor products of $U(\mathfrak{n}_-)$-modules in the usual way, and we have the following.

Lemma 7.5. *The linear map φ defined in (7.5) is a homomorphism of $U(\mathfrak{n}_-)$-modules.*

Proof. Take any $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, $x_i \in \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$, for $i = 1, \ldots, s$, $y \in U(\mathfrak{n}_-)$ and $x' \in \mathfrak{n}_-$. We have

$$\varphi(x'(v \otimes x_1 \cdots x_s y)) = \varphi(x'v \otimes x_1 \cdots x_s y) + \varphi(v \otimes x'x_1 \cdots x_s y).$$

Noticing that

$$\varphi(x'v \otimes x_1 \cdots x_s y) = (-1)^s x_s \cdots x_1 x'v \otimes yu_0$$

and

$$\begin{aligned}
\varphi(v \otimes x'x_1 \cdots x_s y) &= \varphi(v \otimes x_1 \cdots x_s x'y + v \otimes \sum_{i=1}^{s} x_1 \cdots [x', x_i] \cdots x_s y) \\
&= (-1)^s x_s \cdots x_1 v \otimes x'yu_0 + (-1)^s \sum_{i=1}^{s} x_s \cdots [x', x_i] \cdots x_1 v \otimes yu_0 \\
&= (-1)^s x_s \cdots x_1 v \otimes x'yu_0 + (-1)^s [x', x_s \cdots x_1] v \otimes yu_0,
\end{aligned}$$

we get that

$$\begin{aligned}
\varphi(x'(v \otimes x_1 \cdots x_s y)) &= (-1)^s x_s \cdots x_1 v \otimes x'yu_0 + (-1)^s x'x_s \cdots x_1 v \otimes yu_0 \\
&= (-1)^s x'(x_s \cdots x_1 v \otimes yu_0) = x'\varphi(v \otimes x_1 \cdots x_s y),
\end{aligned}$$

as desired. ∎

Theorem 7.6. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Let $L = \widetilde{V}(\gamma)$ be an irreducible highest weight $\tilde{\mathfrak{g}}$-module with highest weight vector u_0 of highest weight $\gamma \in \tilde{\mathfrak{h}}^*$. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$ is irreducible if and only if $\varphi(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_0)) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes L_0$.*

Proof. Consider $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$ instead of $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes L$. For convenience, denote $E = E(\boldsymbol{\lambda}, \boldsymbol{a})$, $\mathcal{E} = \mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, L)$, and $J = \mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_0)$. Let $W = \sum_{k \in \mathbb{Z}} W_k \otimes t^k$ be a nonzero submodule of $\mathcal{E}$. By Lemma 7.3, there is $n \in \mathbb{Z}$ such that $E \otimes L \otimes t^i \subseteq W$ for all $i \geq n + 1$.

Claim 1. *For any* $v \in E$, $x \in U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ *and* $y \in U(\mathfrak{n}_-)$, *we have* $v \otimes xyu_0 \equiv (x \circ v) \otimes yu_0 = \varphi(v \otimes xy) \mod W_n$.

By linearity, it suffices to prove the claim for $x = (x_1 \otimes t^{-i_1}) \cdots (x_s \otimes t^{-i_s})$ with $x_j \in \mathfrak{g}$ and $i_j \in \mathbb{N}$. If $s = 0$, the result is trivial. Now suppose the result is true for all $v \in E$, $x = (x_1 \otimes t^{-i_1}) \cdots (x_s \otimes t^{-i_s})$ and $y \in U(\mathfrak{n}_-)$. Set $u = yu_0$. Then for any $x_0 \in \mathfrak{g}$ and $i_0 \in \mathbb{N}$, we have that

$$(x_0 \otimes t^{-i_0})(v \otimes xu \otimes t^{n+i_0}) = (v \otimes (x_0 \otimes t^{-i_0})xu + (x_0 \otimes t^{-i_0})v \otimes xu) \otimes t^n \in W.$$

Hence, by the inductive hypothesis on s, we have

$$\begin{aligned} v \otimes (x_0 \otimes t^{-i_0})xyu_0 &\equiv -(x_0 \otimes t^{-i_0})v \otimes xyu_0 && \mod W_n \\ &\equiv -x \circ (x_0 \otimes t^{-i_0})v \otimes yu_0 && \mod W_n \\ &\equiv ((x_0 \otimes t^{-i_0})x) \circ v \otimes yu_0 && \mod W_n. \end{aligned}$$

The claim follows by induction.

First suppose that $\varphi(E \otimes J) = E \otimes L_0$. Then for any $v \in E(\boldsymbol{\lambda}, \boldsymbol{a})$, we can find $x_i \in J$ and $v_i \in E$ such that $\varphi(\sum_{i=1}^{s} v_i \otimes x_i) = v \otimes u_0$. By Claim 1, we have

$$v \otimes u_0 = \sum_{i=1}^{s} \varphi(v_i \otimes x_i) \equiv \sum_{i=1}^{s} v_i \otimes x_i u_0 = 0 \mod W_n,$$

that is, $E \otimes u_0 \in W_n$. By Corollary 7.4, we obtain that $W_n = E \otimes L$, and by induction on n, we conclude that $W_i = E \otimes L$ for all $i \in \mathbb{Z}$, that is $W = \mathcal{E}$. Therefore $\mathcal{E}$ is irreducible.

Now suppose that $\varphi(E \otimes J) \neq E \otimes L_0$. Then we have an induced linear map

$$\bar{\varphi} : E \otimes U(\tilde{\mathfrak{n}}_-) \to (E \otimes L_0)/\varphi(E \otimes J), \quad v \otimes x \mapsto \overline{\varphi(v \otimes x)}.$$

Clearly $\bar{\varphi}(E \otimes J) = 0$. Noticing that $L = U(\tilde{\mathfrak{n}}_-)u_0 \cong U(\tilde{\mathfrak{n}}_-)/J$, we get the induced linear map

$$\widehat{\varphi} : E \otimes L \to (E \otimes L_0)/\varphi(E \otimes J), \quad v \otimes xu_0 \mapsto \overline{\varphi(v \otimes x)}.$$

Without loss of generality, we may suppose that W is the submodule of $\mathcal{E}$ generated by $E \otimes L \otimes t^i$, $i \geq n+1$. Then any element in $W_n \otimes t^n$ is a sum of elements of the form

$$x(v \otimes yu_0 \otimes t^{n+i}), \quad \text{with } x \in U(\hat{\mathfrak{g}})_{-i}, v \in E, y \in U(\tilde{\mathfrak{n}}_-), i \in \mathbb{N},$$

and, by the PBW theorem, is a sum of elements of the form

$$x(v \otimes yu_0 \otimes t^{n+i}), \quad \text{with } x \in \tilde{\mathfrak{n}}_-, \deg(x) = -i, v \in E, y \in U(\tilde{\mathfrak{n}}_-), i \in \mathbb{N}.$$

While in this case

$$x(v \otimes yu_0 \otimes t^{n+i}) = (xv \otimes yu_0 + v \otimes xyu_0) \otimes t^n$$

and

$$\widehat{\varphi}(xv \otimes yu_0 + v \otimes xyu_0) = \overline{\varphi(xv \otimes y + v \otimes xy)} = 0.$$

As a result, we obtain that $\widehat{\varphi}(W_n) = 0$, or equivalently, $W_n \subseteq \ker \widehat{\varphi} \neq E \otimes L$, since $\widehat{\varphi}$ is surjective and nonzero. Thus W is a proper submodule of $\mathcal{E}$, which has to be reducible. ∎

The above result is not easy to use, since we generally do not know what $\mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_0)$ is for a non-integrable irreducible highest weight $\tilde{\mathfrak{g}}$-module $\widetilde{V}(\gamma)$ with highest weight $\gamma \in \tilde{\mathfrak{h}}^*$. Fortunately, our goal is to handle the case when γ is dominant.

From Proposition 9.9 in [31], we know that there are a lot of irreducible highest weight modules $\widetilde{V}(\gamma)$ that are also Verma modules over $\tilde{\mathfrak{g}}$. For these irreducible Verma modules $\widetilde{V}(\gamma)$, we know that $J = \mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_0) = 0$, where u_0 is a highest weight vector. Applying Theorem 7.6, we obtain the following.

Corollary 7.7. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Let $\widetilde{V}(\gamma)$ be an irreducible Verma module over $\tilde{\mathfrak{g}}$. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \widetilde{V}(\gamma)$ is always reducible.*

On the other hand, we do have a clear description for $\mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_0)$ if γ is dominant. Next we will concentrate on this very important case, which is of most interest to us.

Recall from Section 2 the notation of a root system relative to the triangular decomposition of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ given there. Let $\alpha_i, i = 1, \ldots, l$, be the simple roots of $\mathfrak{g}$, let θ be the longest root of $\mathfrak{g}$, and let $\check{\alpha}_i$ and $\check{\theta}$ be the corresponding coroots. Let $\alpha_0 = \delta - \theta$. Then $\alpha_i, i = 0, 1, \ldots, l$, are the simple roots of $\tilde{\mathfrak{g}}$. It is well known that $f_0 = e_\theta \otimes t^{-1}$ and $\check{\alpha}_0 = K - \check{\theta}$. Any weight $\gamma \in \tilde{\mathfrak{h}}^*$ is of the form $\gamma = \Lambda + k\Lambda_0$, where $\Lambda \in \mathfrak{h}^*$ is a weight of $\mathfrak{g}$, $k \in \mathbb{C}$, and Λ_0 is the weight defined by $\langle \Lambda_0, \check{\alpha}_i \rangle = \delta_{i,0}$, $\langle \Lambda_0, d \rangle = 0$. Denote by $\widetilde{V}(\gamma)$ the highest weight $\tilde{\mathfrak{g}}$-module with highest weight γ, and denote by $V(\Lambda)$ the $\mathfrak{g}$-submodule of $\widetilde{V}(\gamma)$ generated by the highest weight vector (that is, a highest weight $\mathfrak{g}$-module with highest weight Λ).

It is well known that $\gamma = \Lambda + k\Lambda_0$ is dominant if and only if Λ is a dominant weight relative to $\mathfrak{g}$ and $k \in \mathbb{N}$ with $k \geq \Lambda(\check{\theta})$.

From now on, we assume that $L = \widetilde{V}(\gamma)$, the irreducible highest weight $\tilde{\mathfrak{g}}$-module with dominant highest weight $\gamma = \Lambda + k\Lambda_0$. It is easy to see that $L_0 = V(\Lambda)$ is just the finite dimensional irreducible highest weight $\mathfrak{g}$-module with highest weight Λ. Moreover, $\tilde{V}(\gamma)$ and $V(\Lambda)$ have common highest weight vector, and we choose one of them, say u_Λ, and define the function φ as in (7.5) replacing u_0 with u_Λ. Denote $\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, \gamma) = \mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, \tilde{V}(\gamma))$ for convenience. We have noticed the $\tilde{\mathfrak{g}}$-module isomorphism

$$\mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, \gamma) \cong E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \tilde{V}(\gamma).$$

Theorem 7.8. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Suppose that $\Lambda + k\Lambda_0 \in \tilde{\mathfrak{h}}^*$ is dominant. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \tilde{V}(\Lambda + k\Lambda_0)$ is irreducible if and only if*

$$\varphi\big(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes U(\mathfrak{n}_-)(e_\theta \otimes t^{-1})^{k - \langle \Lambda | \check{\theta} \rangle + 1}\big) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda). \tag{7.6}$$

Proof. Let u_Λ be a common highest weight vector of $\widetilde{V}(\Lambda + k\Lambda_0)$ and $V(\Lambda)$. It is well known that $J = \mathrm{Ann}_{U(\tilde{\mathfrak{n}}_-)}(u_\Lambda)$ is the left ideal of $U(\tilde{\mathfrak{n}}_-)$ generated by $f_i^{\langle \Lambda + k\Lambda_0 | \check{\alpha}_i \rangle + 1}$ for $i = 0, 1, \ldots, l$. Then

$$\varphi(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes J) = \varphi\Big(E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \sum_{i=0}^{l} U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])\, U(\mathfrak{n}_-) f_i^{\langle \Lambda + k\Lambda_0 | \check{\alpha}_i \rangle + 1}\Big).$$

Note that

$$\varphi\Big(E(\boldsymbol{\lambda},\boldsymbol{a})\otimes\sum_{i=1}^{l}U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\,U(\mathfrak{n}_-)\,f_i^{\langle\Lambda+k\Lambda_0|\check{\alpha}_i\rangle+1}\Big)$$
$$=\sum_{i=1}^{l}U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\circ E(\boldsymbol{\lambda},\boldsymbol{a})\otimes U(\mathfrak{n}_-)\,f_i^{\langle\Lambda|\check{\alpha}_i\rangle+1}u_0=0.$$

We have

$$\begin{aligned}\varphi(E(\boldsymbol{\lambda},\boldsymbol{a})\otimes J)&=\varphi\big(E(\boldsymbol{\lambda},\boldsymbol{a})\otimes U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\,U(\mathfrak{n}_-)\,f_0^{\langle\Lambda+k\Lambda_0|\check{\alpha}_0\rangle+1}\big)\\&=\varphi\big(U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\circ E(\boldsymbol{\lambda},\boldsymbol{a})\otimes U(\mathfrak{n}_-)\,f_0^{k-\langle\Lambda|\check{\theta}\rangle+1}\big)\\&=\varphi(E(\boldsymbol{\lambda},\boldsymbol{a})\otimes U(\mathfrak{n}_-)\,f_0^{k-\langle\Lambda|\check{\theta}\rangle+1}).\end{aligned}$$

Now the statement follows from Theorem 7.6. ■

The condition in Theorem 7.8 is still not very easy to verify. Now we can give a further criterion which we can compute explicitly.

Theorem 7.9. *Let $m\in\mathbb{N}$, $\boldsymbol{a}\in\widetilde{\mathbb{C}}^m$, $b\in\mathbb{C}$ and $\boldsymbol{\lambda}\in(\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda},\boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda},\boldsymbol{a},b)$ is irreducible. Suppose that $\widetilde{V}(\Lambda+k\Lambda_0)$ is an irreducible integrable highest weight $\tilde{\mathfrak{g}}$-module, where $\Lambda\in\mathfrak{h}^*$ and $k\in\mathbb{Z}_+$. Let u_Λ be the common highest weight vector of $\widetilde{V}(\Lambda+k\Lambda_0)$ and $V(\Lambda)$. Then $E(\boldsymbol{\lambda},\boldsymbol{a},b)\otimes\widetilde{V}(\Lambda+k\Lambda_0)$ is irreducible if and only if*

$$U(\mathfrak{n}_-)\big((e_\theta\otimes t^{-1})^{k-\langle\Lambda|\check{\theta}\rangle+1}E(\boldsymbol{\lambda},\boldsymbol{a})\otimes u_\Lambda\big)=E(\boldsymbol{\lambda},\boldsymbol{a})\otimes V(\Lambda).\tag{7.7}$$

Proof. As before, denote $\gamma=\Lambda+k\Lambda_0$ and we consider $\mathcal{E}=\mathcal{E}(\boldsymbol{\lambda},\boldsymbol{a},b,\gamma)$ instead of $E(\boldsymbol{\lambda},\boldsymbol{a},b)\otimes\tilde{V}(\Lambda+k\Lambda_0)$. Denote $E=E(\boldsymbol{\lambda},\boldsymbol{a})$, $f_0=e_\theta\otimes t^{-1}$ and $k_0=k-\langle\Lambda|\check{\theta}\rangle+1$ for short.

First suppose that

$$U(\mathfrak{n}_-)(f_0^{k_0}E\otimes u_\Lambda)=E\otimes V(\Lambda).$$

Let W be a nonzero submodule of $\mathcal{E}$. As in the proof of Theorem 7.6, we have that $W=\sum_{i\in\mathbb{Z}}W_i\otimes t^i$, $W_i\subseteq E\otimes\widetilde{V}(\gamma)$, and by Lemma 7.3, there is $n\in\mathbb{Z}$ such that

$$W_i=E\otimes\widetilde{V}(\gamma),\quad\text{for all } i\geq n+1.$$

By Claim 1 of Theorem 7.6, we have that, for any $v\in E$,

$$(-1)^{k_0}f_0^{k_0}v\otimes u_\Lambda=\varphi(v\otimes f_0^{k_0})\equiv v\otimes f_0^{k_0}u_\Lambda=0 \mod W_n.$$

So $f_0^{k_0}E\otimes u_\Lambda\subseteq W_n$, and hence

$$E\otimes V(\Lambda)=U(\mathfrak{n}_-)(f_0^{k_0}E\otimes u_\Lambda)\subseteq W_n,$$

which implies that $E\otimes\widetilde{V}(\gamma)=W_n$ by Corollary 7.4. By induction, we can obtain that $W_i=E\otimes\widetilde{V}(\gamma)$ for all $i\in\mathbb{Z}$. Therefore $\mathcal{E}$ is irreducible.

To complete the proof, it suffices to show the following claim, thanks to Theorem 7.8 and the fact that $U(\mathfrak{n}_-)(f_0^{k_0}E\otimes u_\Lambda)\subseteq E(\boldsymbol{\lambda},\boldsymbol{a})\otimes V(\Lambda)$.

Claim 1. $\varphi(E \otimes U(\mathfrak{n}_-) f_0^{k_0}) \subseteq U(\mathfrak{n}_-)(f_0^{k_0} E \otimes u_\Lambda)$.

Take any $x = x_1 x_2 \cdots x_r \in U(\mathfrak{n}_-)$ with $x_i \in \mathfrak{n}_-$. We will prove

$$\varphi(E \otimes x_1 \cdots x_r f_0^{k_0}) \subseteq U(\mathfrak{n}_-)(f_0^{k_0} E \otimes u_\Lambda) \tag{7.8}$$

by induction on r. The assertion is obvious for $r = 0$. Now suppose that (7.8) holds for $r \in \mathbb{Z}_+$. Then for any $v \in E$ and $x_0 \in \mathfrak{n}_-$, we have

$$\varphi(v \otimes x_0 x f_0^{k_0}) = \varphi(x_0(v \otimes x f_0^{k_0}) - x_0 v \otimes x f_0^{k_0}) = x_0 \varphi(v \otimes x f_0^{k_0}) - \varphi(x_0 v \otimes x f_0^{k_0}),$$

since φ is a $U(\mathfrak{n}_-)$-module homomorphism. Then by induction hypothesis, we have

$$\varphi(v \otimes x f_0^{k_0}), \varphi(x_0 v \otimes x f_0^{k_0}) \in U(\mathfrak{n}_-)(f_0^{k_0} E \otimes u_\Lambda).$$

Hence $\varphi(v \otimes x_0 x f_0^{k_0}) \in U(\mathfrak{n}_-)(f_0^{k_0} E \otimes u_\Lambda)$. The claim follows. ■

It is interesting to note that the left-hand side of (7.7) is a $\mathfrak{g}$-module (even if the equality in (7.7) does not hold), since $[\mathfrak{n}_+, e_\theta \otimes t^{-1}] = 0$, while the vector space inside $\varphi(\cdot)$ on the left-hand side of (7.6) is not a $\mathfrak{g}$-module in general.

From Theorem 7.9, the condition (7.7) is computable since everything is within the finite dimensional $\mathfrak{g}$-module

$$E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda) = V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_m) \otimes V(\Lambda).$$

Now the main task is to compute the highest weight vectors with respect to $\mathfrak{g}$ in the $\mathfrak{g}$-submodule $U(\mathfrak{n}_-)((e_\theta \otimes t^{-1})^{k - \langle \Lambda | \check{\theta} \rangle + 1} E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_\Lambda)$.

As an application of the above theorem, we can obtain the irreducibility for the tensor products of integrable loop modules and integrable highest weight modules for $\widetilde{\mathfrak{sl}}_2$ explicitly, recovering the result of Adamović [2]. Choose a standard basis $\{e, f, h\}$ of $\mathfrak{sl}_2$ with a Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$, and let ε be the fundamental weight, i.e., $\varepsilon \in \mathfrak{h}^*$ is given by $\varepsilon(h) = 1$.

Corollary 7.10. *For any $a, b \in \mathbb{C}$ and $i, j, k \in \mathbb{Z}_+$ with $a \neq 0$ and $k \geq i$, the $\widetilde{\mathfrak{sl}}_2$-module $E(j\varepsilon, a, b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$ is irreducible if and only if $i \leq k < j$.*

Proof. We know that $\mathfrak{g} = \mathfrak{sl}_2 = \operatorname{span}\{e, f, h\}$, $e_0 = f \otimes t$ and $f_0 = e \otimes t^{-1}$. Denote by v and u the highest weight vectors of the $\mathfrak{g}$-module $V(j\varepsilon)$ and the $\tilde{\mathfrak{g}}$-module $\widetilde{V}(i\varepsilon + k\Lambda_0)$, respectively. Denote, for convenience,

$$W = V(j\varepsilon) \otimes V(i\varepsilon) \quad \text{and} \quad W' = U(\mathfrak{n}_-)(f_0^{k-i+1} E(j\varepsilon, a) \otimes u).$$

By Theorem 7.9, we see that $E(j\varepsilon, a, b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$ is irreducible if and only if $W' = W$ if and only if $\mathbb{C}[f](f_0^{k-i+1} E(j\varepsilon, a) \otimes u) = \mathbb{C}[f](e^{k-i+1} V(j\varepsilon) \otimes u)$ contains all highest weight vectors of $V(j\varepsilon) \otimes V(i\varepsilon)$.

First note that $e^{k-i+1} E(j\varepsilon, a) = 0$ and $W' \neq W$ if $j < k - i + 1$.

Next we suppose $j \geq k - i + 1$. In this case, we have

$$f_0^{k-i+1} E(j\varepsilon, a) = \operatorname{span}\{v, f v, \ldots, f^{j-(k-i+1)} v\}.$$

Let W'_p (respectively, W_p) be the weight space of W' (respectively, W) of weight $p\varepsilon$. For $s = 0, 1, \ldots, \min\{i, j\}$, we can deduce that

$$\dim W'_{i+j-2s} = \begin{cases} s+1, & \text{if } s \le j-(k-i+1), \\ j-k+i, & \text{if } s > j-(k-i+1). \end{cases}$$

On the other hand, by the Clebsch–Gordan formula, we have

$$W = V((i+j)\varepsilon) \oplus V((i+j-2)\varepsilon) \oplus \cdots \oplus V(|j-i|\varepsilon).$$

It is clear that $W_{i+j} \oplus \cdots \oplus W_{|i-j|}$ contains all highest weight vector of W and that $\dim W_{i+j-2s} = s+1$ for all $s = 0, 1, \ldots, \min\{i, j\}$.

Then $W' = W$ if and only if $W'_{i+j-2s} = W_{i+j-2s}$ for all $0 \le s \le \min\{i, j\}$, if and only if $j-(k-i+1) \ge \min\{i, j\}$, if and only if $k+1 \le \max\{i, j\}$. Noticing that $\widetilde{V}(i\varepsilon + k\Lambda_0)$ being dominant requires $k \ge i$, we can conclude that $E(j\varepsilon, a, b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$ is irreducible if and only if $i \le k < j$. ∎

Retain the notation of Theorem 7.9. In particular, $\gamma = \Lambda + k\Lambda_0$. Denote by E_μ the weight space of the $\mathfrak{g}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda)$ with weight $\mu \in \mathfrak{h}^*$, then μ can be written as a sum $\mu = \mu_1 + \cdots + \mu_m + \mu_{m+1}$, where μ_i is a weight of the $\mathfrak{g}$-module $V(\lambda_i)$ for $1 \le i \le m$ and u_{m+1} is a weight of $V(\Lambda)$. For any root α of $\mathfrak{g}$, let $E^{(\alpha)}$ be the sum of E_μ for all weights μ of the $\mathfrak{g}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda)$ satisfying $k + \langle \mu, \check{\alpha} \rangle < 0$.

Proposition 7.11. *Let notation be as in Theorem 7.9 and let $n \in \mathbb{Z}$. Let $W = \sum_{i\in\mathbb{Z}} W_i \otimes t^i$ be a submodule of $\mathcal{E} = \mathcal{E}(\boldsymbol{\lambda}, \boldsymbol{a}, b, \Lambda + k_0\Lambda_0)$ such that each W_i is a subspace of $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \tilde{V}(\Lambda + k_0\Lambda)$ and $W_i = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes \tilde{V}(\Lambda + k_0\Lambda)$ for all $i > n$. Then $E^{(\alpha)} \subseteq W_n$ for any root α of $\mathfrak{g}$.*

Proof. Suppose $E_\mu \not\subseteq W_n$ for some weight μ of the $\mathfrak{g}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda)$, and take any $v \in E_\mu \setminus W_n$.

For any root α of $\mathfrak{g}$, take a nonzero $x \in \mathfrak{g}_\alpha$. We have

$$(x \otimes t)(v \otimes t^n) \in W_{n+1}.$$

That is, the image of $v \otimes t^n$ in $\mathcal{E}/W$ is a highest weight vector of the integrable module over the Lie algebra $\mathfrak{sl}_2$ spanned by $x \otimes t$, $y \otimes t^{-1}$ and $\check{\alpha} + K$, where y is a nonzero root vector in $\mathfrak{g}_{-\alpha}$. Note that the weight of $v \otimes t^n$ is $\mu + k\Lambda_0 + (b+n)\delta$ with respect to $\tilde{\mathfrak{h}}$. So we must have

$$\langle \mu + k\Lambda_0 + (b+n)\delta, \check{\alpha} + K \rangle \ge 0,$$

or equivalently

$$k + \langle \mu, \check{\alpha} \rangle \ge 0, \quad \forall \alpha \in \Delta.$$

The result hence follows. ∎

By the above proposition, we see that

$$U(\mathfrak{g})\Big(\sum_{\alpha\in\Delta} E^{(\alpha)}\Big) \subseteq W_n.$$

Using this and the first part of the proof of Theorem 7.9, we obtain the following result.

Theorem 7.12. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Suppose that $\widetilde{V}(\Lambda + k\Lambda_0)$ is an irreducible integrable highest weight $\tilde{\mathfrak{g}}$-module, where $\Lambda \in \mathfrak{h}^*$ and $k \in \mathbb{Z}_+$. Let u_Λ be the common highest weight vector of $\widetilde{V}(\Lambda + k\Lambda_0)$ and $V(\Lambda)$. Then $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \widetilde{V}(\Lambda + k\Lambda_0)$ is irreducible if and only if*

$$U(\mathfrak{g})\Big(\sum_{\alpha \in \Delta} E^{(\alpha)} + (e_\theta \otimes t^{-1})^{k+1-\langle \Lambda, \check{\theta} \rangle} E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes u_\Lambda \Big) = E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda).$$

The following consequence is easy to see.

Corollary 7.13. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in (\mathfrak{h}^*)^m$. Under the conditions in Theorem 7.9, the following hold:*

(1) *If $k \geq \langle \lambda_1 + \lambda_2 + \cdots + \lambda_m + \Lambda, \check{\theta} \rangle$, then $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \widetilde{V}(\Lambda + k\Lambda_0)$ is not irreducible.*

(2) *If $k < \langle \mu, \check{\theta} \rangle$ for all the highest weights μ of all irreducible $\mathfrak{g}$-submodules of $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda)$, then the $\tilde{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \widetilde{V}(\Lambda + k\Lambda_0)$ is irreducible.*

Proof. If $k \geq \langle \lambda_1 + \lambda_2 + \cdots + \lambda_m + \Lambda, \check{\theta} \rangle$, then the left-hand side of (7.7) is 0. So $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes \widetilde{V}(\Lambda + k\Lambda_0)$ is not irreducible. Part (1) follows.

For part (2), we see that $E_{\boldsymbol{\mu}} \subseteq E^{(-\theta)}$, and the result follows from Theorem 7.12. ∎

Corollary 7.14. *Let $m \in \mathbb{N}$, $\boldsymbol{a} \in \widetilde{\mathbb{C}}^m$, $b \in \mathbb{C}$ and $\boldsymbol{\lambda} \in (\mathfrak{h}^*)^m$ be such that $E(\boldsymbol{\lambda}, \boldsymbol{a})$ is finite dimensional and $E(\boldsymbol{\lambda}, \boldsymbol{a}, b)$ is irreducible. Suppose further that $\widetilde{V}(\Lambda + k\Lambda_0)$ is an irreducible integrable highest weight $\tilde{\mathfrak{g}}$-module, where $\Lambda \in \mathfrak{h}^*$ and $k \in \mathbb{Z}_+$. Then there exists $\tau(\boldsymbol{\lambda}, \boldsymbol{a}, \Lambda) \in \mathbb{Z}_+$, depending on $\boldsymbol{\lambda}$, $\boldsymbol{a}$ and Λ, such that the $\tilde{\mathfrak{g}}$-module $E(\boldsymbol{\lambda}, \boldsymbol{a}, b) \otimes V(\Lambda + k\Lambda_0)$, where $k \in \mathbb{Z}_+$ with $k \geq \langle \Lambda, \check{\theta} \rangle$, is irreducible if and only if $k < \tau(\boldsymbol{\lambda}, \boldsymbol{a}, \Lambda)$.*

Proof. By Theorem 7.9, it is sufficient to sow that

$$(e_\theta \otimes t^{-1})^j E(\boldsymbol{\lambda}, \boldsymbol{a}) \subseteq (e_\theta \otimes t^{-1})^{j'} E(\boldsymbol{\lambda}, \boldsymbol{a}), \quad \forall j > j'.$$

This can be seen from the fact that

$$(e_\theta \otimes t^{-1})^{j-j'} E(\boldsymbol{\lambda}, \boldsymbol{a}) \subseteq E(\boldsymbol{\lambda}, \boldsymbol{a}), \quad \forall j > j'.$$ ∎

From Corollary 7.13, we see that $\min\{\langle \mu, \check{\theta} \rangle \mid \mu\}$ is the highest weight of an irreducible $\mathfrak{g}$-submodules of $E(\boldsymbol{\lambda}, \boldsymbol{a}) \otimes V(\Lambda)\} \leq \tau(\boldsymbol{\lambda}, \boldsymbol{a}, \Lambda) < \langle \lambda_1 + \cdots + \lambda_m + \Lambda, \check{\theta} \rangle$.

We will give another explicit example of tensor product modules for $\widetilde{\mathfrak{sl}}_2$, which is not covered in [2]. We can determine their irreducibility using our methods developed in this section.

Example. Let $m = 2$, $\boldsymbol{\lambda} = (\varepsilon, 2\varepsilon)$, $\mathbf{a} = (a_1, a_2) \in \widetilde{\mathbb{C}}^2$, and $b \in \mathbb{C}$. We have the $\widetilde{\mathfrak{sl}}_2$-module $E((\varepsilon, 2\varepsilon); (a_1, a_2); b)$ and the highest weight $\widetilde{\mathfrak{sl}}_2$-module $\widetilde{V}(i\varepsilon + k\Lambda_0)$ with nonnegative integers $k \geq i$. Now we consider the $\widetilde{\mathfrak{sl}}_2$-module $E((\varepsilon, 2\varepsilon); (a_1, a_2); b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$.

Note that the function χ defined in (7.2) is onto (see Theorem 1.3 in [10]).

Let v_1, v_2 and u be the highest weight vectors of the $\mathfrak{sl}_2$-modules $V(\varepsilon)$, $V(2\varepsilon)$ (in the tensor product of $E((\varepsilon, 2\varepsilon); (a_1, a_2)))$ and the $\widetilde{\mathfrak{sl}}_2$-module $\widetilde{V}(i\varepsilon + k\Lambda_0)$, respectively. Then u is also the highest weight vector of the $\mathfrak{sl}_2$-submodule $V(i\varepsilon)$ in the $\widetilde{\mathfrak{sl}}_2$-module $\widetilde{V}(i\varepsilon + k\Lambda_0)$.

For convenience, denote

$$W = E((\varepsilon, 2\varepsilon); (a_1, a_2)) \otimes V(i\varepsilon) \cong (V(3\varepsilon) \oplus V(\varepsilon)) \otimes V(i\varepsilon)$$

and

$$W^{(j)} = \mathbb{C}[f]((e \otimes t^{-1})^j (V(\varepsilon) \otimes V(2\varepsilon)) \otimes u), \quad \forall j \in \mathbb{Z}_+.$$

Then the $\tilde{\mathfrak{sl}}_2$-module $E((\varepsilon, 2\varepsilon); (a_1, a_2); b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$ is irreducible if and only if $W^{(k-i+1)} = W$. To compute $W^{(j)}$, we need the following formulas:

$$\begin{aligned} &efv_1 = v_1, \quad efv_2 = 2v_2, \quad ef^2v_2 = 2fv_2, \\ &f^2v_1 = f^3v_2 = 0, \quad efu = iu. \end{aligned}$$

It is not hard to calculate that

$$\begin{aligned} &(e \otimes t^{-1})(V(\varepsilon) \otimes V(2\varepsilon)) \\ &\qquad = \operatorname{span}\{v_1 \otimes v_2,\ v_1 \otimes fv_2,\ fv_1 \otimes v_2,\ a_2v_1 \otimes f^2v_2 + 2a_1fv_1 \otimes fv_2\}; \\ &(e \otimes t^{-1})^2(V(\varepsilon) \otimes V(2\varepsilon)) = \operatorname{span}\{v_1 \otimes v_2,\ a_2v_1 \otimes fv_2 + a_1fv_1 \otimes v_2\}; \\ &(e \otimes t^{-1})^3(V(\varepsilon) \otimes V(2\varepsilon)) = \mathbb{C}v_1 \otimes v_2. \end{aligned}$$

Case 1. $i = 0$.

In this case, it is obvious that $W^{(2)} = W$ and $W^{(3)} \neq W$ and hence we have that $E((\varepsilon, 2\varepsilon); (a_1, a_2), b) \otimes \widetilde{V}(\Lambda_0)$ is an irreducible $\tilde{\mathfrak{sl}}_2$-module, while $E((\varepsilon, 2\varepsilon); (a_1, a_2), b) \otimes \widetilde{V}(2\Lambda_0)$ is not irreducible. We conclude that $\tau((\varepsilon, 2\varepsilon); (a_1, a_2); 0) = 2$ for all distinct $a_1, a_2 \in \mathbb{C}^*$.

Case 2. $i = 1$.

The vector space

$$\begin{aligned} \mathbb{C}f^2(v_1 \otimes v_2 \otimes u) + \mathbb{C}f(v_1 \otimes fv_2 \otimes u) + \mathbb{C}f(fv_1 \otimes v_2 \otimes u) \\ + \mathbb{C}(a_2v_1 \otimes f^2v_2 + 2a_1fv_1 \otimes fv_2) \end{aligned}$$

contains the following linearly independent weight-0 vectors:

$$v_1 \otimes f^2v_2 \otimes u, \quad fv_1 \otimes fv_2 \otimes u, \quad fv_1 \otimes v_2 \otimes fu \quad \text{and} \quad v_1 \otimes fv_2 \otimes fu.$$

We obtain $4 \le \dim W_0^{(1)} \le \dim W_0 = 4$. Hence $W_0 = W_0^{(1)}$ and $W^{(1)} = W$, and the $\tilde{\mathfrak{sl}}_2$-module $E((\varepsilon, 2\varepsilon); (a_1, a_2); b) \otimes \widetilde{V}(\varepsilon + \Lambda_0)$ is irreducible. On the other hand, we can easily see that $\dim W_2^{(2)} \le 2$ while $\dim W_2 = 3$. So $W^{(2)} \neq W$, and the $\tilde{\mathfrak{sl}}_2$-module $E((\varepsilon, 2\varepsilon); (a_1, a_2); b) \otimes \widetilde{V}(\varepsilon + 2\Lambda_0)$ is not irreducible. Therefore, we obtain again that $\tau((\varepsilon, 2\varepsilon); (a_1, a_2); \varepsilon) = 2$ for all distinct $a_1, a_2 \in \mathbb{C}^*$.

Case 3. $i \ge 2$.

In this case, as $\mathfrak{g}$-modules we see that

$$\begin{aligned} W = V(\varepsilon) \otimes V(2\varepsilon) \otimes V(i\varepsilon) = V((i+3)\varepsilon) \otimes V((i+1)\varepsilon) \otimes \cdots \otimes V(|i-3|\varepsilon) \\ \otimes V((i+1)\varepsilon) \otimes V((i-1)\varepsilon) \otimes \cdots \otimes V((i-1)\varepsilon). \end{aligned}$$

So $\dim W_{i-1}^{(1)} \leq 4$ and $\dim W_{i-1} = 5$. Thus $W^{(1)} \neq W$, and $E((\varepsilon, 2\varepsilon); (a_1, a_2); b) \otimes \widetilde{V}(i\varepsilon + k\Lambda_0)$ is not irreducible for $k = i$ and hence for all $k \geq i$. We may say that $\tau((\varepsilon, 2\varepsilon); (a_1, a_2); i\varepsilon) = i$ for all distinct $a_1, a_2 \in \mathbb{C}^*$. ∎

Note that the irreducible modules for $i = 0, k = 1$ in the above example do not satisfy the conditions in Theorem 2.2 of [10]:

$$a_1\lambda_1 + a_2\lambda_2 + \cdots + a_m\lambda_m \neq 0, \quad \text{or} \quad (m+1)k < \langle \lambda_1 + \lambda_2 + \cdots + \lambda_m + \Lambda, \check{\theta} \rangle.$$

It would be nice to have a formula for $\tau(\boldsymbol{\lambda}, \boldsymbol{a}, \Lambda)$ defined in Corollary 7.14 in terms of $\boldsymbol{\lambda}$, $\boldsymbol{a}$ and Λ.

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References

[1] Adamović, D.: New irreducible modules for affine Lie algebras at the critical level. *Internat. Math. Res. Notices* (1996), no. 6, 253–261. Zbl 0858.17019 MR 1386077

[2] Adamović, D.: Vertex operator algebras and irreducibility of certain modules for affine Lie algebras. *Math. Res. Lett.* **4** (1997), no. 6, 809–821. Zbl 0895.17025 MR 1492122

[3] Adamović, D.: An application of $U(\mathfrak{g})$-bimodules to representation theory of affine Lie algebras. *Algebr. Represent. Theory* **7** (2004), no. 4, 457–469. Zbl 1092.17016 MR 2096671

[4] Adamović, D., Lü, R. and Zhao, K.: Whittaker modules for the affine Lie algebra $A_1^{(1)}$. *Adv. Math.* **289** (2016), 438–479. Zbl 1369.17018 MR 3439693

[5] Batra, P. and Mazorchuk, V.: Blocks and modules for Whittaker pairs. *J. Pure Appl. Algebra* **215** (2011), no. 7, 1552–1568. Zbl 1228.17008 MR 2771629

[6] Bekkert, V., Benkart, G., Futorny, V. and Kashuba, I.: New irreducible modules for Heisenberg and affine Lie algebras. *J. Algebra* **373** (2013), 284–298. Zbl 1306.17009 MR 2995027

[7] Carter, R. W.: *Lie algebras of finite and affine type*. Cambridge Stud. Adv. Math. 96, Cambridge University Press, Cambridge, 2005. Zbl 1110.17001 MR 2188930

[8] Chari, V.: Integrable representations of affine Lie-algebras. *Invent. Math.* **85** (1986), no. 2, 317–335. Zbl 0603.17011 MR 0846931

[9] Chari, V. and Pressley, A.: New unitary representations of loop groups. *Math. Ann.* **275** (1986), no. 1, 87–104. Zbl 0603.17012 MR 0849057

[10] Chari, V. and Pressley, A.: A new family of irreducible, integrable modules for affine Lie algebras. *Math. Ann.* **277** (1987), no. 3, 543–562. Zbl 0608.17009 MR 0891591

[11] Chari, V. and Pressley, A.: Integrable representations of twisted affine Lie algebras. *J. Algebra* **113** (1988), no. 2, 438–464. Zbl 0661.17023 MR 0929772

[12] Chari, V. and Pressley, A.: An application of Lie superalgebras to affine Lie algebras. *J. Algebra* **135** (1990), no. 1, 203–216. Zbl 0717.17020 MR 1076086

[13] Chen, H., Ge, L., Li, Z. and Wang, L.: Classical Whittaker modules for the affine Kac–Moody algebras $A_N^{(1)}$. *Adv. Math.* **454** (2024), article no. 109874, 60 pp. Zbl 1555.17023 MR 4782893

[14] Chen, H., Guo, X. and Zhao, K.: Tensor product weight modules over the Virasoro algebra. *J. Lond. Math. Soc. (2)* **88** (2013), no. 3, 829–844. Zbl 1311.17004 MR 3145133

[15] Christodoulopoulou, K.: Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras. *J. Algebra* **320** (2008), no. 7, 2871–2890. Zbl 1221.17009 MR 2442000

[16] Dimitrov, I., Futorny, V. and Grantcharov, D.: Parabolic sets of roots. In *Groups, rings and group rings*, pp. 61–73. Contemp. Math. 499, American Mathematical Society, Providence, RI, 2009. Zbl 1229.17008 MR 2581926

[17] Dimitrov, I. and Grantcharov, D.: Classification of simple weight modules over affine Lie algebras. Preprint 2009, arXiv:0910.0688v1.

[18] Dixmier, J.: *Enveloping algebras*. Revised reprint of the 1977 translation. Grad. Stud. Math. 11, American Mathematical Society, Providence, RI, 1996. Zbl 0867.17001 MR 1393197

[19] Eswara Rao, S.: Classification of loop modules with finite-dimensional weight spaces. *Math. Ann.* **305** (1996), no. 4, 651–663. Zbl 0852.17022 MR 1399709

[20] Futorny, V. M.: The parabolic subsets of root system and corresponding representations of affine Lie algebras. In *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, pp. 45–52. Contemp. Math. 131, American Mathematical Society, Providence, RI, 1992. Zbl 0774.17030 MR 1175820

[21] Futorny, V. M.: Irreducible non-dense $A_1^{(1)}$-modules. *Pacific J. Math.* **172** (1996), no. 1, 83–99. Zbl 0863.17021 MR 1379287

[22] Futorny, V. M.: *Representations of affine Lie algebras*. Queen's Papers Pure Appl. Math. 106, Queen's University, Kingston, ON, 1997. Zbl 0903.17011 MR 1627814

[23] Futorny, V.: Representations of affine Lie superalgebras. In *Groups, rings and group rings*, pp. 163–172. Lect. Notes Pure Appl. Math. 248, Chapman & Hall/CRC, Boca Raton, FL, 2006. Zbl 1106.17002 MR 2226191

[24] Futorny, V., Grantcharov, D. and Martins, R. A.: Localization of free field realizations of affine Lie algebras. *Lett. Math. Phys.* **105** (2015), no. 4, 483–502. Zbl 1390.17034 MR 3323663

[25] Futorny, V., Guo, X., Xue, Y. and Zhao, K.: Smooth representations of affine Kac–Moody algebras. *Adv. Math.* **481** (2025), article no. 110559, 34 pp. Zbl 08111017 MR 4966333

[26] Futorny, V. and Tsylke, A.: Classification of irreducible nonzero level modules with finite-dimensional weight spaces for affine Lie algebras. *J. Algebra* **238** (2001), no. 2, 426–441. Zbl 1017.17008 MR 1823767

[27] Ge, L. and Li, Z.: Classical Whittaker modules for the classical affine Kac–Moody algebras. *J. Algebra* **644** (2024), 23–63. Zbl 1547.17046 MR 4695614

[28] Humphreys, J. E.: *Introduction to Lie algebras and representation theory*. Grad. Texts in Math. 9, Springer, New York-Berlin, 1978. Zbl 0447.17001 MR 0499562

[29] Jacobson, N.: *Basic algebra. II*. W. H. Freeman, San Francisco, CA, 1980. Zbl 0441.16001 MR 0571884

[30] Jakobsen, H. P. and Kac, V. G.: A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. In *Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984)*, pp. 1–20. Lecture Notes in Phys. 226, Springer, Berlin, 1985. Zbl 0581.17009 MR 0802097

[31] Kac, V. G.: *Infinite-dimensional Lie algebras*. Third edition. Cambridge University Press, Cambridge, 1990. Zbl 0716.17022 MR 1104219

[32] Kazhdan, D. and Lusztig, G.: Tensor structures arising from affine Lie algebras. I. *J. Amer. Math. Soc.* **6** (1993), no. 4, 905–947. Zbl 0786.17017 MR 1186962

[33] Kazhdan, D. and Lusztig, G.: Tensor structures arising from affine Lie algebras. II. *J. Amer. Math. Soc.* **6** (1993), no. 4, 949–1011. Zbl 0786.17017 MR 1186962

[34] Lau, M.: Representations of multiloop algebras. *Pacific J. Math.* **245** (2010), no. 1, 167–184. Zbl 1237.17012 MR 2602688

[35] Li, H.: On certain categories of modules for affine Lie algebras. *Math. Z.* **248** (2004), no. 3, 635–664. Zbl 1107.17014 MR 2097377

[36] Mazorchuk, V. and Zhao, K.: Characterization of simple highest weight modules. *Canad. Math. Bull.* **56** (2013), no. 3, 606–614. Zbl 1327.17006 MR 3078230

[37] Moody, R. V. and Pianzola, A.: *Lie algebras with triangular decompositions*. Canadian Math. Soc. Ser. Monogr. Advanced Texts, John Wiley & Sons, New York, 1995. Zbl 0874.17026 MR 1323858

[38] Nguyen, K., Xue, Y. and Zhao, K.: A new class of simple smooth modules over the affine algebra $A_1^{(1)}$. *J. Algebra* **658** (2024), 728–747. Zbl 1551.17011 MR 4770985

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